Bounded Poincaré operators for BGG complexes

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1. Review: BGG construction

2. Bounded Poincaré operators

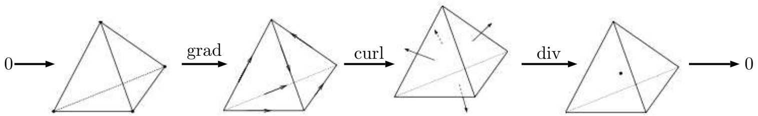
Basic homological algebra

$$\cdots \longrightarrow V^{i-1} \xrightarrow{d^{i-1}} V^i \xrightarrow{d^i} V^{i+1} \longrightarrow \cdots$$

V^i: vector spaces, d^i: linear (or nonlinear) operators

- complex: $d^i V^i \subset V^{i+1}$, $d^{i+1} \circ d^i = 0$, $\forall i$, (implies $\mathcal{R}(d^{i-1}) \subset \mathcal{N}(d^i)$)
- exact: $\mathcal{N}(d^i) = \mathcal{R}(d^{i-1})$,
- cohomology (when d is linear): $\mathscr{H}^i := \mathcal{N}(d^i)/\mathcal{R}(d^{i-1})$.

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0.$$



Raviart-Thomas, Nédélec in numerical analysis, Whitney forms for topology.
Finite element exterior calculus (FEEC): cohomological framework for studying numerical methods. (c.f., Arnold, Falk, Winther 2006, Arnold 2018)

A cohomological approach: elasticity complex

displacement formulation

Kröner's continuum description of dislocations/defects, internal stress

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\ \mathrm{def}\ } C^\infty \otimes \mathbb{S} \xrightarrow{\ \mathrm{inc}\ } C^\infty \otimes \mathbb{S} \xrightarrow{\ \mathrm{div}\ } C^\infty \otimes \mathbb{V} \longrightarrow 0$$

displacement *strain (metric)* *stress (curvature)* *force*

intrinsic elasticity (Ciarlet et al.)

Hellinger-Reissner principle of elasticity

$$\mathbb{V} := \mathbb{R}^3 \text{ vectors}, \quad \mathbb{S} := \mathbb{R}^{3\times 3}_{\mathrm{sym}} \text{ symmetric matrices}$$

$$\mathrm{def}\, u := 1/2(\nabla u + \nabla u^T), \quad (\mathrm{def}\, u)_{ij} = 1/2(\partial_i u_j + \partial_j u_i).$$

$$\mathrm{inc}\, g := \nabla \times g \times \nabla, \quad (\mathrm{inc}\, g)^{ij} = \epsilon^{ikl}\epsilon^{jst}\partial_k \partial_s g_{lt}.$$

$$\mathrm{div}\, v := \nabla \cdot v, \quad (\mathrm{div}\, v)_i = \partial^j u_{ij}.$$

g metric $\Rightarrow$ $\mathrm{inc}\, g$ linearized Einstein tensor ($\backsimeq \mathrm{Riem} \backsimeq \mathrm{Ric}$ in 3D)

$\mathrm{inc} \circ \mathrm{def} = 0$: Saint-Venant compatibility

$\mathrm{div} \circ \mathrm{inc} = 0$: Bianchi identity

Bernstein-Gelfand-Gelfand (BGG) construction:
Eastwood 1999, Čap, Slovák, Souček 2001, Arnold, Falk, Winther 2006, Arnold, Hu 2021, Čap, Hu 2022.

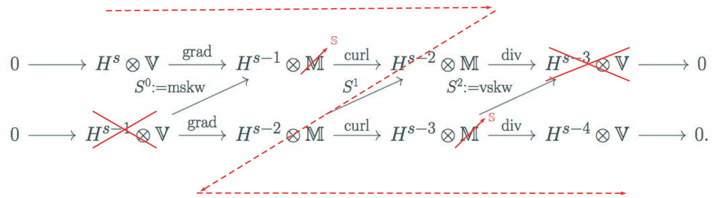
Continuous level

$$0 \longrightarrow H^2 \xrightarrow{\partial_x^2} L^2 \longrightarrow 0.$$

$$\begin{tikzcd}
0 \arrow[r] & H^2 \arrow[r, "\partial_x"] & H^1 \arrow[r] & 0 \\
0 \arrow[r] & H^1 \arrow[r, "\partial_x"] \arrow[ur, "I"] & L^2 \arrow[r] & 0.
\end{tikzcd}$$

- two de-Rham complexes with different continuity,
- cohomology: $\mathcal{N}(\partial_x^2) \cong \mathcal{N}(\partial_x) \oplus \mathcal{N}(\partial_x)$, ∂_x^2 is onto.

Algebraic and analytic construction (Arnold, Hu 2021): derive elasticity from deRham



$S^1 u := u^T - \mathrm{tr}(u) I.$

key: Sobolev complexes ($\forall s \in \mathbb{R}$), match indices, commuting diagrams, injectivity & surjective.

output: elasticity complex

$$0 \longrightarrow H^s \otimes \mathbb{V} \xrightarrow{\text{def}} H^{s-1} \otimes \mathbb{S} \xrightarrow{\text{curl}\,\mathrm{T}\,\text{curl}} H^{s-3} \otimes \mathbb{S} \xrightarrow{\text{div}} H^{s-4} \otimes \mathbb{V} \longrightarrow 0.$$

Theorem

The cohomology is isomorphic to the smooth de Rham cohomology:

$$\mathcal{N}(\mathscr{D}^i) = \mathcal{R}(\mathscr{D}^{i-1}) \oplus \mathscr{H}^i_\infty, \quad \mathscr{H}^i_\infty \simeq \mathcal{H}^i_{\mathrm{deRham}} \otimes (\mathbb{V} \times \mathbb{V})$$

Proof: Homological algebra + results for de Rham by Costabel & McIntosh.

Corollary: finite dimensional cohomology $\Longrightarrow$ operators have closed range.

Consequences:

- analytic results (Poincaré inequality, Hodge decomposition, compactness etc.)
  e.g., Korn inequality $\|u\|_1 \leq C\|\operatorname{def} u\|, \;\; u \perp \mathcal{N}(\operatorname{def})$.
- explicit representatives of elasticity cohomology.

A general picture

- input: $(Z^\bullet, D^\bullet)$, $(\tilde{Z}^\bullet, \tilde{D}^\bullet)$, connecting maps $S^i : \tilde{Z}^i \to Z^{i+1}$, satisfying
  - (anti-)commutativity: $S^{i+1}\tilde{D}^i = -D^{i+1}S^i$,
  - injectivity/surjectivity condition: S^i injective for $i \le J$, surjective for $i \ge J$.

$$\begin{tikzcd}
0 \arrow[r] & Z^0 \arrow[r, "D^0"] & Z^1 \arrow[r, "D^1"] & \cdots \arrow[r, "D^{n-1}"] & Z^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{Z}^0 \arrow[r, "\tilde{D}^0"] \arrow[ur, "S^0"] & \tilde{Z}^1 \arrow[r, "\tilde{D}^1"] \arrow[ur, "S^1"] & \cdots \arrow[r, "\tilde{D}^{n-1}"] \arrow[ur, "S^{n-1}"] & \tilde{Z}^n \arrow[r] & 0
\end{tikzcd}$$

- output:

$$\cdots \longrightarrow \operatorname{coker}(S^{J-2}) \xrightarrow{D^{J-1}} \operatorname{coker}(S^{J-1}) \xrightarrow{D^J} \xrightarrow{(S^J)^{-1}} \xrightarrow{\tilde{D}^J} \mathcal{N}(S^{J+1}) \xrightarrow{\tilde{D}^{J+1}} \mathcal{N}(S^{J+2}) \xrightarrow{\tilde{D}^{J+2}} \cdots$$

- conclusion:

$$\dim \mathscr{H}^i(\Upsilon^\bullet, \mathscr{D}^\bullet) \le \dim \mathscr{H}^i(Z^\bullet, D^\bullet) + \dim \mathscr{H}^i\left(\tilde{Z}^\bullet, \tilde{D}^\bullet\right), \quad \forall i = 0, 1, \cdots, n$$

  Equality holds if and only if S^i induces the zero maps on cohomology, i.e., $S^i\mathcal{N}(\tilde{D}^i) \subset \mathcal{R}(D^i)$.

Example: de Rham complexes

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathrm{Alt}^{0,J-1} \arrow[r, "d"] & H^{q-1} \otimes \mathrm{Alt}^{1,J-1} \arrow[r, "d"] & \cdots \arrow[r, "d"] & H^{q-n} \otimes \mathrm{Alt}^{n,J-1} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathrm{Alt}^{0,J} \arrow[r, "d"] \arrow[ur, "s^{0,J}"] & H^{q-2} \otimes \mathrm{Alt}^{1,J} \arrow[r, "d"] \arrow[ur, "s^{1,J}"] & \cdots \arrow[r, "d"] \arrow[ur, "s^{n-1,J}"] & H^{q-n-1} \otimes \mathrm{Alt}^{n,J} \arrow[r] & 0
\end{tikzcd}
$$

where $\mathrm{Alt}^{i,J} := \mathrm{Alt}^i \otimes \mathrm{Alt}^J$

$$
s^{i,J}\mu(v_0, \cdots, v_i)(w_1, \cdots, w_{J-1}) := \sum_{l=0}^{i} (-1)^l \mu(v_0, \cdots, \widehat{v_l}, \cdots, v_i)(v^l, w_1, \cdots, w_{J-1}),
$$

$$
\forall v_0, \cdots, v_i, w_1, \cdots, w_{J-1} \in \mathbb{R}^n.
$$

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "I"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "I"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

Hessian complex:

$$
0 \longrightarrow H^q \otimes \mathbb{R} \xrightarrow{\mathrm{hess}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\mathrm{curl}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\mathrm{div}} H^{q-4} \otimes \mathbb{V} \longrightarrow 0.
$$

biharmonic equations, plate theory, Einstein-Bianchi system of general relativity

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "I"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "I"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

elasticity complex:

$$
0 \longrightarrow H^{q-1} \otimes \mathbb{V} \xrightarrow{\mathrm{def}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\mathrm{inc}} H^{q-4} \otimes \mathbb{S} \xrightarrow{\mathrm{div}} H^{q-5} \otimes \mathbb{V} \longrightarrow 0.
$$

elasticity, defects, metric, curvature

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "I"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "I"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

divdiv complex:

$$
0 \longrightarrow H^{q-2} \otimes \mathbb{V} \xrightarrow{\mathrm{dev\,grad}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\mathrm{sym\,curl}} H^{q-4} \otimes \mathbb{S} \xrightarrow{\mathrm{div\,div}} H^{q-6} \otimes \mathbb{R} \longrightarrow 0.
$$

plate theory, elasticity

Definition and motivation

Poincaré operators: $P^k : V^k \mapsto V^{k-1}$, satisfying null-homotopy property

$$d^{k-1}P^k + P^{k+1}d^k = I_{V^k},$$

Motivation 1: constructing exact sequences and finite elements

$$du = 0 \quad \Longrightarrow \quad u = (dP + Pd)u = d(Pu).$$

e.g., local exactness of de-Rham complexes.

If $V^\bullet$ is a complex with both $d^\bullet$ and $P^\bullet$, then both $(V^\bullet, d^\bullet)$ and $(V^\bullet, P^\bullet)$ are exact.

$$\cdots \rightleftarrows V^{i-1} \underset{P^i}{\overset{d^{i-1}}{\rightleftarrows}} V^i \underset{P^{i+1}}{\overset{d^i}{\rightleftarrows}} V^{i+1} \rightleftarrows \cdots$$

Examples:

$$\cdots \longrightarrow \mathcal{P}_{r-(k-1)}\Lambda^{k-1} \xrightarrow{d^{k-1}} \mathcal{P}_{r-k}\Lambda^k \xrightarrow{d^k} \mathcal{P}_{r-(k+1)}\Lambda^{k+1} \longrightarrow \cdots,$$

$$\cdots \longrightarrow \mathcal{P}_r\Lambda^{k-1} + P^k\mathcal{P}_r\Lambda^k \xrightarrow{d^{k-1}} \mathcal{P}_r\Lambda^k + P^{k+1}\mathcal{P}_r\Lambda^{k+1} \xrightarrow{d^k} \mathcal{P}_r\Lambda^k + P^{k+1}\mathcal{P}_r\Lambda^{k+1} \longrightarrow$$

Motivation 2: p-robustness of finite element methods

bounded, *polynomial-preserving* Poincaré operators imply results that are uniform with the polynomial degree.

Motivation 3: well-posedness of Stokes problem

given $f \in L^2$, find $u \in [H_0^1]^n$ and $p \in L^2/\mathbb{R}$, such that

$$\begin{aligned} -\Delta u + \nabla p &= f, \\ \operatorname{div} u &= 0. \end{aligned}$$

inf-sup condition: for any $q \in L^2/\mathbb{R}$, $\exists\ u = P(q) \in [H_0^1]^n$, s.t. $\operatorname{div} u = q$, $\|u\|_1 \leq C\|q\|$.

Motivation 4: analytic results, e.g., regular decomposition, compactness.

How to construct

Smooth de-Rham complex (see books on manifolds)

Let $F_t : \Omega \to \Omega, t \in [0,1]$ be a continuous family of operators indexed by t.
If u is a k-form:

$$(\mathfrak{p}[F]u)_x(\xi_2, \ldots, \xi_k) = \int_0^1 u_{F_t(x)}(\partial_t F_t(x), DF_t(x)\xi_2, \cdots, DF_t(x)\xi_k)\, dt.$$

Suppose that F_1 is identity and F_0 is constant x_0, then we have $d\mathfrak{p} + \mathfrak{p}d = I$ for $k \geq 1$ and $\mathfrak{p}du(x) = u(x) - u(x_0)$ for $k = 0$.

Simplification

- 1D with *base point* x_0

$$\mathfrak{p}(u) := \int_{x_0}^x u(y)\, dy, \quad \partial\mathfrak{p}(u) = u, \ \mathfrak{p}(\partial v) = v(x) - v(x_0).$$

- 3D vector proxy (choose a curve $\gamma(t) = tx$ connecting 0 and x):

$$\mathfrak{p}^1 u = \int_0^1 u_{tx} \cdot x dt, \quad \mathfrak{p}^2 v = \int_0^1 t v_{tx} \wedge x dt, \quad \mathfrak{p}^3 w = \int_0^1 t^2 w_{tx} x dt.$$

Sobolev de-Rham complex: Costabel-McIntosh 2010

$$0 \longrightarrow H^q\Lambda^0 \xrightarrow{d^0} H^{q-1}\Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H^{q-n}\Lambda^n \longrightarrow 0,$$

where q is any real number.

- regularized Poincaré operators: averaging the base points in the smooth de-Rham version, mapping $H^q\Lambda^k$ to $H^{q+1}\Lambda^{k-1}$, polynomial-preserving
- generalized Bogolvskiĭ operators: "dual" of Poincaré, , mapping $H_0^q\Lambda^k$ to $H_0^{q+1}\Lambda^{k-1}$

pseudo-differential operators of order -1, which implies boundedness between various spaces (Sobolev, Besov...)

General Lipschitz domain:

$$dP + Pd = I - L,$$

where $L(u) \in C^\infty$ for any u. The smoothing operator L comes from partition of unity (into a union of star-shaped patches).

BGG complexes: overview

More details of the BGG machinery (explicit way of doing homological algebra) are needed.

$$
\begin{tikzcd}
Y^i \arrow[r, shift left, "d^i"'] \arrow[d, "F"'] & Y^{i+1} \arrow[l, shift left, "P^{i+1}"'] \arrow[d, "F"'] \\
Y^i \arrow[r, shift left, "d_V^i"'] \arrow[d, shift right, "B^i"'] & Y^{i+1} \arrow[l, shift left, "P_V^{i+1}"'] \arrow[d, shift right, "B^{i+1}"'] \\
\Upsilon^i \arrow[r, shift left, "\mathscr{D}^i"'] \arrow[u, shift right, "A^i"'] & \Upsilon^{i+1} \arrow[l, shift left, "\mathscr{P}^{i+1}"'] \arrow[u, shift right, "A^{i+1}"']
\end{tikzcd}
$$

de-Rham complex $\longrightarrow$ twisted de-Rham complex $\longrightarrow$ BGG complex

If we know Poincaré operators for the de-Rham complex (Costabel-McIntosh), then the rest is derived using commuting diagram.

F: isomorphism (bijective). For BGG complexes:

$$\mathscr{P} := B \circ F \circ P \circ F^{-1} \circ A.$$

If $P^\bullet$ satisfies $dP + Pd = I$, then $\mathscr{P}^\bullet$ satisfies $\mathscr{D}\mathscr{P} + \mathscr{P}\mathscr{D} = I$.

Physics interpretation

- twisted complexes: Timoshenko beam, Reissner-Mindlin plate, Cosserat elasticity
- BGG complexes: Euler-Bernoulli beam, Kirchhoff-Love plate, standard elasticity.

Recall the two-row BGG diagram

$$\begin{tikzcd}
0 \arrow[r] & Z^0 \arrow[r, "d^0"] & Z^1 \arrow[r, "d^1"] & \cdots \arrow[r, "d^{n-1}"] & Z^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{Z}^0 \arrow[r, "d^0"] \arrow[ur, "S^0"] & \tilde{Z}^1 \arrow[r, "d^1"] \arrow[ur, "S^1"] & \cdots \arrow[r, "d^{n-1}"] \arrow[ur, "S^{n-1}"] & \tilde{Z}^n \arrow[r] & 0
\end{tikzcd}$$

Define $Y^j := Z^j \times \tilde{Z}^j$ and define the twisted complex

$$\cdots \longrightarrow Y^{i-1} \xrightarrow{d_V^{i-1}} Y^i \xrightarrow{d_V^i} Y^{i+1} \longrightarrow \cdots,$$

where

$$d_V^k = \begin{pmatrix} d^k & -S^k \\ 0 & d^k \end{pmatrix}.$$

Observation: $d(PS) - (PS)d = S$, and thus $F \circ d = d_V \circ F$, where

$$F^k = \begin{pmatrix} I & P^{k+1} \circ S^k \\ 0 & I \end{pmatrix}.$$

Define P_V by the commuting diagram: $P_V := B \circ P \circ A$. Then $d_V P_V + P_V d_V = I$.

1D example

BGG diagram

$$
\begin{tikzcd}
H^q \arrow[r, "\partial"] & H^{q-1} \\
H^{q-1} \arrow[r, "\partial"] \arrow[ur, "I"] & H^{q-2},
\end{tikzcd}
$$

where $\partial := \frac{d}{dx}$ with $P_\sharp : H^{q-1} \to H^q$ and $P_\flat : H^{q-2} \to H^{q-1}$.

We want to derive Poincaré operators for $H^q \xrightarrow{\partial^2} H^{q-2}$.

$$
\begin{tikzcd}
\begin{pmatrix} H^q \\ H^{q-1} \end{pmatrix} \arrow[r, "\begin{pmatrix} \partial & \\ & \partial \end{pmatrix}"] \arrow[d, "F^0"] & \begin{pmatrix} H^{q-1} \\ H^{q-2} \end{pmatrix} \arrow[d, "F^1"] \\
\begin{pmatrix} H^q \\ H^{q-1} \end{pmatrix} \arrow[r, "\begin{pmatrix} \partial & -I \\ 0 & \partial \end{pmatrix}"] & \begin{pmatrix} H^{q-1} \\ H^{q-2} \end{pmatrix},
\end{tikzcd}
$$

where

$$
F^0 := \begin{pmatrix} I & P_\sharp \\ 0 & I \end{pmatrix}, \quad F^1 := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.
$$

$$
P_V := F^0 \circ P \circ (F^1)^{-1} = \begin{pmatrix} I & P_\sharp \\ 0 & I \end{pmatrix} \begin{pmatrix} P_\sharp & 0 \\ 0 & P_\flat \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} P_\sharp & P_\sharp P_\flat \\ 0 & P_\flat \end{pmatrix}.
$$

From twisted complex to BGG complex: $\mathscr{P} = BP_V A$, where

$$A = \begin{pmatrix} I & 0 \\ Td & P_{\mathcal{N}} \end{pmatrix}, \quad B = \begin{pmatrix} P_{\Upsilon} & 0 \\ P_{\Upsilon} dT & P_{\Upsilon} \end{pmatrix}$$

1D example continued

$$\begin{array}{ccc} \begin{pmatrix} H^q \\ H^{q-1} \end{pmatrix} & \xrightarrow{\begin{pmatrix} \partial & -I \\ & \partial \end{pmatrix}} & \begin{pmatrix} H^{q-1} \\ H^{q-2} \end{pmatrix} \\ \Big\downarrow B^0 & & \Big\downarrow B^1 \\ \begin{pmatrix} H^q \\ 0 \end{pmatrix} & \xrightarrow{\begin{pmatrix} 0 & 0 \\ \partial^2 & 0 \end{pmatrix}} & \begin{pmatrix} 0 \\ H^{q-2} \end{pmatrix} \end{array},$$

$$B^0 := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad B^1 := \begin{pmatrix} 0 & 0 \\ \partial & I \end{pmatrix},$$

and

$$A^0 := \begin{pmatrix} I & 0 \\ \partial & 0 \end{pmatrix}, \quad A^1 := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Then we obtain Poincaré operators:

$$\tilde{\mathscr{P}} = B^0 \circ P_V \circ A^1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_\sharp & P_\sharp P_\flat \\ 0 & P_\flat \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} P_\sharp & P_\sharp P_\flat \\ 0 & 0 \end{pmatrix}.$$

This readily gives the Poincaré operator $\mathscr{P} := P_\sharp P_\flat$.

3D elasticity complex

$$\mathscr{P}^1(e) = \int_0^x e(y)\,dy - \int_0^x dy \wedge \int_0^y \nabla \times e(z)dz,$$

for smooth e. Generalization of Cesàro-Volterra formula, satisfying

$$\mathscr{P}^1 \operatorname{def}(w) = w(x) - w(0) + \frac{1}{2}\int_0^x dy \wedge \nabla w(y).$$

For general Sobolev functions, $\mathscr{P}^1 = P(PS - SP)Td$ (all ingredients given in BGG diagram).

Compared to

- '*A Cesàro-Volterra formula with little regularity*', Ciarlet, Gratie, Mardare, 2010 JMPA,

the new formulas are explicit, polynomial-preserving, work for a broad class of functions, for the entire complex.

Complex property $P \circ P = 0$?

Let $P^\bullet$ be Poincaré operators satisfying $dP + Pd = I$, but not necessarily $P \circ P = 0$. We can generally modify $P^\bullet$ to $\tilde{P}^\bullet$, defined by $\tilde{P} := P - DP^2$. Then straightforward algebra implies $d\tilde{P} + \tilde{P}d = I$ and $\tilde{P} \circ \tilde{P} = 0$.

Nontrivial cohomology?

Standard procedure: cover $\Omega = \cup_j \Omega_j$, partition of unity, $1 = \sum_j \xi_j$ with ξ_j supported on Ω_j. Use $dP + Pd = I$ on each Ω_j, leading to $dP + Pd = I - L$ globally, where L is a smoothing operator.

Some applications

- for any $v \in L_0^2 \otimes \mathbb{V}$, $\exists \sigma = \mathscr{P}(v) \in H_0^1 \otimes \mathbb{S}$, s.t., $\operatorname{div} \sigma = v$, and $\mathscr{P}$ is polynomial-preserving.
  $\Longrightarrow$ p-robustness of finite element methods for elasticity (Aznaran, KH, Parker, in preparation)

- exactness of polynomial BGG complexes, e.g.,

$$\mathrm{RM} \longrightarrow \mathcal{P}_r \otimes \mathbb{V} \xrightarrow{\operatorname{def}} \mathcal{P}_{r-1} \otimes \mathbb{S} \xrightarrow{\operatorname{inc}} \mathcal{P}_{r-3} \otimes \mathbb{S} \xrightarrow{\operatorname{div}} \mathcal{P}_{r-4} \otimes \mathbb{V} \longrightarrow 0.$$

- intrinsic elasticity and intrinsic defect model

$$\begin{cases} \operatorname{inc}(\mathbb{A} \operatorname{inc} E) = K, & \text{in } \Omega, \\ \operatorname{div} E = 0, & \text{in } \Omega, \\ E \cdot n = 0, & \text{on } \partial\Omega, \\ \mathcal{T}_0(\mathbb{B} \operatorname{inc} E) = \mathcal{T}_1(\mathbb{B} \operatorname{inc} E) = 0, & \text{on } \partial\Omega, \end{cases}$$

  where $\mathcal{T}_0$ and $\mathcal{T}_1$ are boundary terms (Amstutz, Van Goethem 2019).
  Equivalent intrinsic formulation (with $B = \operatorname{inc} E$):

$$\inf_{B \in H(\operatorname{div}, \mathbb{S}), \operatorname{div} B = 0} \frac{1}{2}(B, B)_{\mathbb{A}} - (K, \mathscr{P} B),$$

  where $\mathscr{P}$ is the Poincaré operator of degree 2 in the elasticity complex.
  (intrinsic elasticity: Ciarlet, Gratie, Mardare 2019)

References:

- *Complexes from complexes*, Douglas Arnold, KH; *Foundations of Computational Mathematics* (2021). framework, analytic results from homological algebraic structures
- *BGG sequences with weak regularity and applications*, Andreas Čap, KH; *arXiv:2203.01300* (2022) more general framework
- *Poincaré path integrals for elasticity*, Christiansen, KH, Sande, *Journal de Mathématiques Pures et Appliquées*, (2019) smooth elasticity complex in 3D
- *Bounded Poincaré operators for BGG complexes*, Andreas Čap, KH; *in preparation* (2022) main results