Distributional finite element complexes

- What are BGG Whitney forms? -

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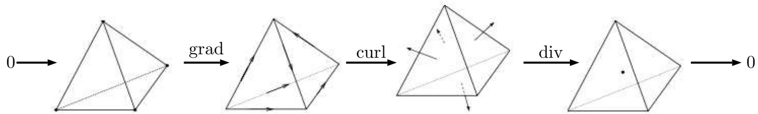
de Rham complexes
Discrete



$$0 \longrightarrow \mathcal{P}_1 \xrightarrow{\text{grad}} [\mathcal{P}_0]^3 + [\mathcal{P}_0]^3 \times x \xrightarrow{\text{curl}} [\mathcal{P}_0]^3 + \mathcal{P}_0 \otimes x \xrightarrow{\text{div}} \mathcal{P}_0 \longrightarrow 0.$$

Raviart-Thomas (1977), Nédélec (1980) in numerical analysis, Bossavit (1988), Hiptmair (1999) for differential forms, Whitney (1957) for studying topology.

Finite element exterior calculus (FEEC): structure-preserving FEM
Discrete exterior calculus (DEC): defining spaces and operators on primal and dual meshes
Topological data analysis (TDA): cohomology and Hodge-Laplacian on graphs

Lim, Lek-Heng. "Hodge Laplacians on graphs." SIAM Review 62.3 (2020).

BGG: GENERAL RECIPE

- input: $(Z^\bullet, D^\bullet)$, $(\tilde{Z}^\bullet, \tilde{D}^\bullet)$, connecting maps $S^i : \tilde{Z}^i \to Z^{i+1}$, satisfying
  - (anti-)commutativity: $S^{i+1}\tilde{D}^i = -D^{i+1}S^i$,
  - injectivity/surjectivity condition: S^i injective for $i \leq J$, surjective for $i \geq J$.

$$
\begin{tikzcd}
0 \arrow[r] & Z^0 \arrow[r, "D^0"] & Z^1 \arrow[r, "D^1"] & \cdots \arrow[r, "D^{n-1}"] & Z^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{Z}^0 \arrow[r, "\tilde{D}^0"] \arrow[ur, "S^0"] & \tilde{Z}^1 \arrow[r, "\tilde{D}^1"] \arrow[ur, "S^1"] & \cdots \arrow[r, "\tilde{D}^{n-1}"] \arrow[ur, "S^{n-1}"] & \tilde{Z}^n \arrow[r] & 0
\end{tikzcd}
$$

- output:

$$
\cdots \longrightarrow \mathrm{coker}(S^{J-2}) \xrightarrow{D^{J-1}} \mathrm{coker}(S^{J-1}) \xrightarrow{D^J} \xrightarrow{(S^J)^{-1}} \xrightarrow{\tilde{D}^J} \mathcal{N}(S^{J+1}) \xrightarrow{\tilde{D}^{J+1}} \mathcal{N}(S^{J+2}) \xrightarrow{\tilde{D}^{J+2}} \cdots
$$

- conclusion:

$$
\mathscr{H}^i(\Upsilon^\bullet, \mathscr{D}^\bullet) \cong \mathscr{H}^i(Z^\bullet, D^\bullet) \oplus \mathscr{H}^i\left(\tilde{Z}^\bullet, \tilde{D}^\bullet\right), \quad \forall i = 0, 1, \cdots, n.
$$

Inspired by Bernstein-Gelfand-Gelfand (BGG) resolution (Eastwood 2000, Čap,Slovák,Souček 2001, Arnold,Falk,Winther 2006)

BGG in 1D

BGG diagram:

$$
\begin{tikzcd}
0 \arrow[r] & H^2 \arrow[r, "\partial_x"] & H^1 \arrow[r] & 0 \\
0 \arrow[r] & H^1 \arrow[r, "\partial_x"] \arrow[ur, "I"] & L^2 \arrow[r] & 0.
\end{tikzcd}
$$

BGG complex:

$$
0 \longrightarrow H^2 \xrightarrow{\partial_x^2} L^2 \longrightarrow 0.
$$

nD: FORMS WITH DOUBLE INDICES

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathrm{Alt}^{0,J-1} \arrow[r, "d"] & H^{q-1} \otimes \mathrm{Alt}^{1,J-1} \arrow[r, "d"] & \cdots \arrow[r, "d"] & H^{q-n} \otimes \mathrm{Alt}^{n,J-1} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathrm{Alt}^{0,J} \arrow[r, "d"] \arrow[ur, "s^{0,J}"] & H^{q-2} \otimes \mathrm{Alt}^{1,J} \arrow[r, "d"] \arrow[ur, "s^{1,J}"] & \cdots \arrow[r, "d"] \arrow[ur, "s^{n-1,J}"] & H^{q-n-1} \otimes \mathrm{Alt}^{n,J} \arrow[r] & 0
\end{tikzcd}
$$

where $\mathrm{Alt}^{i,J} := \mathrm{Alt}^i \otimes \mathrm{Alt}^J$

$$
s^{i,J}\mu(v_0, \cdots, v_i)(w_1, \cdots, w_{J-1}) := \sum_{l=0}^{i} (-1)^l \mu(v_0, \cdots, \widehat{v_l}, \cdots, v_i)(v^l, w_1, \cdots, w_{J-1}),
$$

$$
\forall v_0, \cdots, v_i, w_1, \cdots, w_{J-1} \in \mathbb{R}^n.
$$

3D VECTOR/MATRIX PROXIES

$\mathbb{R}$: scalar $\quad$ $\mathbb{V}$: vector $\quad$ $\mathbb{M}$: matrix $\quad$ $\mathbb{S}$: symmetric matrix $\quad$ $\mathbb{T}$: trace-free matrix

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\,\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

Hessian complex:

$$
0 \longrightarrow H^q \otimes \mathbb{R} \xrightarrow{\mathrm{hess}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\mathrm{curl}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\mathrm{div}} H^{q-4} \otimes \mathbb{V} \longrightarrow 0.
$$

biharmonic equations, plate theory, Einstein-Bianchi system of general relativity

3D vector/matrix proxies

$\mathbb{R}$: scalar $\quad$ $\mathbb{V}$: vector $\quad$ $\mathbb{M}$: matrix $\quad$ $\mathbb{S}$: symmetric matrix $\quad$ $\mathbb{T}$: trace-free matrix

$$
\begin{tikzcd}
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0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

elasticity complex:

$$
0 \longrightarrow H^{q-1} \otimes \mathbb{V} \xrightarrow{\mathrm{def}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\mathrm{inc}} H^{q-4} \otimes \mathbb{S} \xrightarrow{\mathrm{div}} H^{q-5} \otimes \mathbb{V} \longrightarrow 0.
$$

elasticity, defects, metric, curvature

3D vector/matrix proxies

$\mathbb{R}$: scalar $\quad \mathbb{V}$: vector $\quad \mathbb{M}$: matrix $\quad \mathbb{S}$: symmetric matrix $\quad \mathbb{T}$: trace-free matrix

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\,\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

divdiv complex:

$$
0 \longrightarrow H^{q-2} \otimes \mathbb{V} \xrightarrow{\mathrm{dev\,grad}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\mathrm{sym\,curl}} H^{q-4} \otimes \mathbb{S} \xrightarrow{\mathrm{div\,div}} H^{q-6} \otimes \mathbb{R} \longrightarrow 0.
$$

plate theory, elasticity

DISCRETE LEVEL

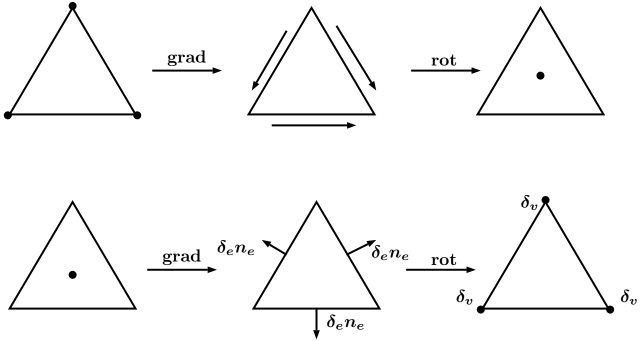
Goal: discrete spaces fitting in complexes.

- 2D stress: Arnold-Winther 2002, J.Hu-S.Zhang 2014, Christiansen-KH 2018,
- 2D strain: Chen-J.Hu-Huang 2014 (Regge/HHJ), Christiansen-KH 2018 (conforming), Chen-Huang 2020, DiPietro-Droniou 2021 (polygonal meshes), KH 2023
- 3D elasticity: various results on last part of complex, Hauret-Kuhl-Ortiz 2007 (discrete geometry/mechanics), Arnold-Awanou-Winther 2008, Christiansen 2011 (Regge), Christiansen-Gopalakrishnan-Guzmán-KH 2020, Chen-Huang 2021, J.Hu-Liang-Lin 2023, Gong-Gopalakrishnan-Guzmán-Neilan 2023
- 3D Hessian: Chen-Huang 2020, J.Hu-Liang 2021, Arf-Simeon 2021 (splines)
- 3D divdiv: Chen-Huang 2021, J.Hu-Liang-Ma 2021, Sander 2021 ($H(\mathrm{sym\,curl})$, $H(\mathrm{dev\,sym\,curl})$), J.Hu-Liang-Ma-Zhang 2022, J.Hu-Liang-Lin 2023
- nD: Chen-Huang 2021 (last two spaces), 2D arbitrary regularity: Chen-Huang 2022, Bonizzoni-KH-Kanschat-Sap 2023
- conformal complexes: open.

What is the analogue of Whitney forms (lowest order Lagrange, Nédélec, RT...)?

• encode topological/geometric information • canonical dofs (allowing generalizations)

Distributional complexes: 2D de Rham (Braess, Schöberl 2008)



grad of p.w. constants: for $\phi \in C_0^\infty$:

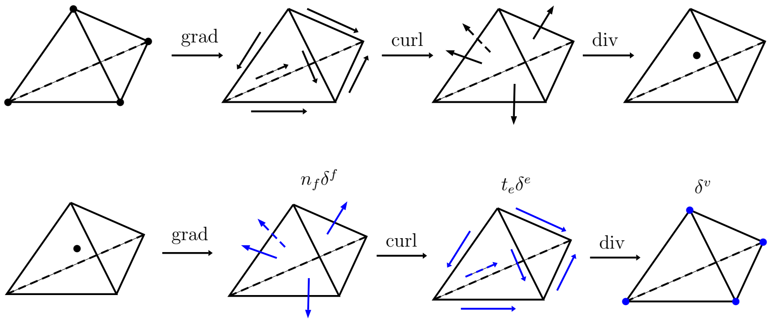
$$\langle \operatorname{grad} u, \phi\rangle := -(u, \operatorname{div}\phi) = -\sum_T \int_T u \operatorname{div}\phi = \sum_{\partial T}\int_{\partial T} u(\boldsymbol{n}\cdot\phi) = \sum_e \langle [u]_e \boldsymbol{n}\delta_e, \phi\rangle$$
$$\implies \operatorname{grad} u = [u]_e \boldsymbol{n}\delta_e.$$

rot of normal deltas $\boldsymbol{v} = \sum_e c_e \boldsymbol{n}\delta_e$: for $\psi \in C_0^\infty$:

$$\langle \operatorname{rot} \boldsymbol{v}, \psi\rangle := -\langle \boldsymbol{v}, \operatorname{curl}\psi\rangle = -\sum_e \int_e c_e \boldsymbol{n}\cdot \operatorname{curl}\psi = -\sum_e \int_e c_e \partial_{\boldsymbol{\tau}}\psi = \sum \text{vertex terms}$$
$$\implies \operatorname{rot} \boldsymbol{v} = [\boldsymbol{v}\cdot\boldsymbol{\tau}]_v \delta_v.$$

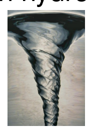
(recall DG, DEC.)

Distributional complexes: 3D de Rham



Perspectives:

- Finite element perspective: dual, complex of degrees of freedom
- DEC perspective: complex on dual meshes
- Fluid perspective: point vortex, vortex lines... (vorticity 2-form, delta on codim 2) (V.I.Arnold,B.Khesin, Topological methods in hydrodynamics)

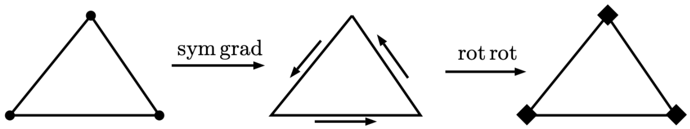
- Applications: equilibrated residual error estimators (Braess, Schöberl 2008)
- Cohomologies, analysis: Licht 2017 (double complex)

Images: Left: Science Behind the Vortex; Right: Wikipedia, “Vortex”.

Distributional complexes?

2D rot rot complex

$$0 \longrightarrow C^\infty \otimes \mathbb{R}^2 \xrightarrow{\text{sym grad}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{rot rot}} C^\infty \longrightarrow 0.$$

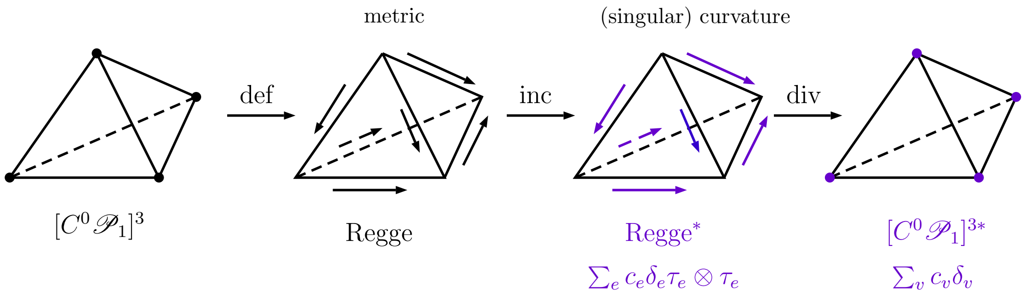


rot rot: linearized Gauss curvature.
Discrete curvature: angle deficit at vertices (discrete geometric approach), δ_V (finite element approach).

Cohomology can be reduced to de Rham with BGG diagrams.

3D ELASTICITY COMPLEX: ANALOGUE OF WHITNEY FORMS?

Christiansen 2011: Regge calculus = finite elements



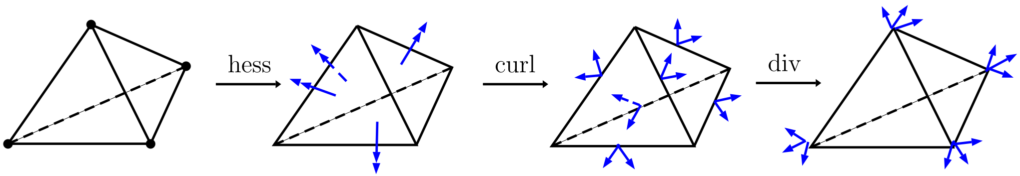
$$
\begin{array}{ccccccc}
\mathrm{Alt}^{0,1} & \longrightarrow & \mathrm{Alt}^{1,1} & \longrightarrow & & & \\
 & & & \swarrow & & & \\
 & & & \longrightarrow & \mathrm{Alt}^{2,2} & \longrightarrow & \mathrm{Alt}^{3,2}
\end{array}
$$

Regge calculus: quantum and numerical relativity, discrete geometry. Metric given by edge lengths; curvature as angle deficit.

Regge finite element: Metric: p.w. constant sym matrices, $\int_e t_e \cdot g \cdot t_e$ as dofs. Curvature: distributional (delta on codim 2).

nD: Lizao Li (2018 UMN thesis), nonlinear curvature with Regge elements (Christiansen 2013, Berchenko-Kogan,Gawlik 2022, Gopalakrishnan,Neunteufel,Schöberl,Wardetzky 2022, Gawlik,Neunteufel 2023)

3D HESSIAN



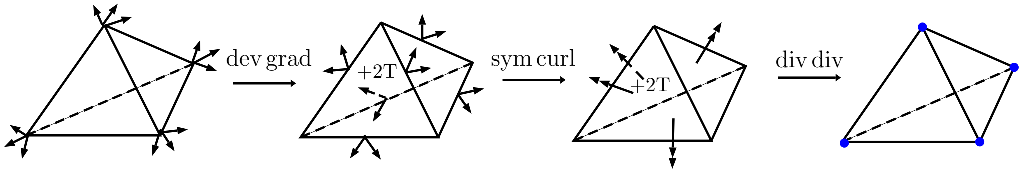
Lagrange $\longrightarrow$ face n-n delta $\longrightarrow$ edge t-n delta $\longrightarrow$ vertex delta

$$\begin{array}{ccccccc}
\mathrm{Alt}^{0,0} & \longrightarrow & & & & & \\
& \swarrow & & & & & \\
& \longrightarrow & \mathrm{Alt}^{1,1} & \longrightarrow & \mathrm{Alt}^{2,1} & \longrightarrow & \mathrm{Alt}^{3,1}
\end{array}$$

Cohomology: $\mathcal{P}_1 \otimes \mathcal{H}_{\mathrm{d}eRham}$

- Step 1: define an auxiliary sequence, cohomology = homology with $\mathcal{P}_1$ coefficients (resolution of $\mathcal{P}_1$)
- Step 2: cohomology of original complex = cohomology of auxiliary sequence (diagram chase, snake lemme)

3D DIVDIV



Lagrange vector $\longrightarrow$ $\mathbb{T} + x \times \mathbb{S}$, C^{tn} $\longrightarrow$ $\mathbb{S}$, C^{nn} $\longrightarrow$ vertex delta

$$\mathrm{Alt}^{0,2} \longrightarrow \mathrm{Alt}^{1,2} \longrightarrow \mathrm{Alt}^{2,2} \longrightarrow$$

$$\swarrow$$

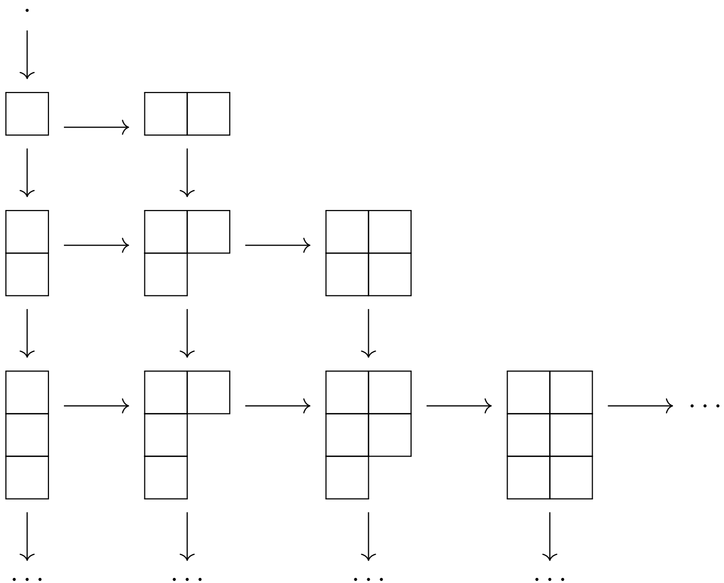
$$\longrightarrow \quad \mathrm{Alt}^{3,3}$$

Almost dual of Hessian complex , except for two interior dofs for unisolvency.
$\mathbb{T} + x \times \mathbb{S}$: analogy of Koszul (automatically trace-free, symbol version of $\mathrm{curl}\,\mathbb{S} \subset \mathbb{T}$).

A DIFFERENT PICTURE: HOW TO CHARACTERIZE HIGH-ORDER TENSORS?

Young tableaux

- number of boxes: order of tensors,
- shape: symmetry of tensors.



$$
\begin{tikzcd}
\mathbb{R} \arrow[d, "\mathrm{grad}"] & & & \\
\mathbb{V} \arrow[r, "\mathrm{sym\ grad}"] \arrow[d, "\mathrm{curl}"] & \mathbb{S} \arrow[d, "\mathrm{curl}"] & & \\
\mathbb{V} \arrow[r, "\mathrm{dev\ grad}"] \arrow[d, "\mathrm{div}"] & \mathbb{T} \arrow[r, "\mathrm{sym\ curl}"] \arrow[d, "\mathrm{div}"] & \mathbb{S} \arrow[d, "\mathrm{div}"] & \\
\mathbb{R} \arrow[r, "\mathrm{grad}"] & \mathbb{V} \arrow[r, "\mathrm{curl}"] & \mathbb{V} \arrow[r, "\mathrm{div}"] & \mathbb{R}
\end{tikzcd}
$$

Peter Olver, 'Differential hyperforms' 1982.

Distributional elements in 2D, 3D.

$$
\begin{tikzcd}
P_1(\mathcal{T}) \cap H^1 \arrow[d, "\mathrm{grad}'"] & & \\
\big(P_1^-(\mathcal{T}) \cap H(\mathrm{curl})\big)' \arrow[r, "\mathrm{def}"] \arrow[d, "\mathrm{curl}'"] & P_1(\mathcal{E})nn' \, \delta^{\mathcal{E}} \arrow[d, "\mathrm{curl}'"] & \\
P_0(\mathcal{T}) \arrow[r, "\mathrm{grad}"] & P_0(\mathcal{E})n \, \delta^{\mathcal{E}} + P(\mathcal{V})^2 \, \delta^{\mathcal{V}} \arrow[r, "\mathrm{curl}"] & P(\mathcal{V}) \, \delta^{\mathcal{V}} + \mathrm{curl}(P(\mathcal{V})^2 \delta^{\mathcal{V}}),
\end{tikzcd}
$$

Summary

- Natural as many structures are indeed singular (singular vortex, curvature of p.w. flat surfaces...)
- Solving PDEs (HHJ, Regge, TDNNS...).
- Examples and clues for ‘BGG Whitney forms’, not a complete answer yet; nD, (k, l)-form in progress.
- Prospects:
  - Discrete nD Riemann/Ricci/Einstein...?
    (nD Riemann etc. encoded in BGG complexes, to discretize the complexes)
  - Discrete Exterior Calculus as distributional ‘finite elements’?
    (dualizing dofs, rather than dualizing meshes) Can this shed a light on the definition and convergence theory for DEC type methods?
  - Nonlinear complexes?
    Exactness on the discrete level (rigidity, fundamental theorem of Riemannian geometry à la Ciarlet)

References:

- *Complexes from complexes*, Douglas Arnold, KH; FoCM (2021).
  framework, analytic results from homological algebraic structures
- *BGG sequences with weak regularity and applications*, Andreas Čap, KH; FoCM (2023)
  more general framework, conformal complexes, applications
- *Nonlinear elasticity complex and a finite element diagram chase*, KH; *arXiv* (2023).
  nonlinear complex, diagram chase
- ongoing projects: lowest order + cohomology (’Whitney forms’); Young tableaux, 2D & 3D & nD.