A cohomological perspective for high order problems (II)

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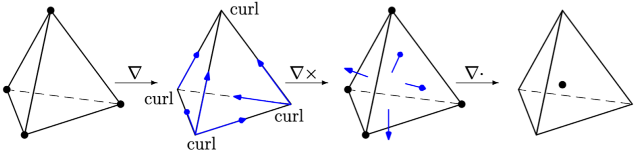
From Part I: Hu-Q.Zhang-Z.Zhang 2020 arXiv

construct finite element subcomplexes for

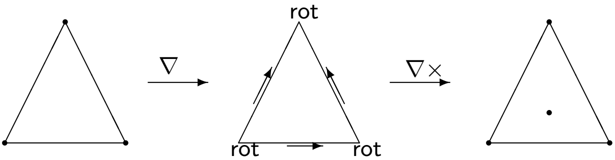
$$0 \longrightarrow H^1 \xrightarrow{\text{grad}} H(\text{grad}\,\text{curl}) \xrightarrow{\text{curl}} [H^1]^3 \xrightarrow{\text{div}} L^2 \longrightarrow 0,$$



- relate FEs for high order problems ($H(\text{grad}\,\text{curl})$) to Stokes ($[H^1]^3$-$L^2$),
- new elements inspired by the complex property,
- why $H(\text{grad}\,\text{curl})$ FEs important?

2D gradcurl-conforming elements: Hu-Q.Zhang-Z.Zhang, 2020 SISC

$$0 \longrightarrow H^1 \xrightarrow{\text{grad}} H(\text{curl}\,\text{rot}) \xrightarrow{\text{rot}} H^1 \longrightarrow 0.$$



primal variational formulation

Find $(\lambda, \boldsymbol{u}) \in \mathbb{R} \times H(\operatorname{grad}\operatorname{rot};\Omega) \cap H_0(\operatorname{div};\Omega)$, s.t.,

$$(\operatorname{grad}\operatorname{rot}\boldsymbol{u}, \operatorname{grad}\operatorname{rot}\boldsymbol{v}) + (\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{v}) = \lambda(\boldsymbol{u},\boldsymbol{v}), \forall \boldsymbol{v} \in H(\operatorname{grad}\operatorname{rot};\Omega) \cap H_0(\operatorname{div};\Omega)$$

mixed variational formulation ($\sigma = -\operatorname{div}\boldsymbol{u}$)

Find $(\lambda, \boldsymbol{u}, \sigma) \in \mathbb{R} \times H(\operatorname{grad}\operatorname{rot};\Omega) \times H^1(\Omega)$, s.t.,

$$(\operatorname{grad}\operatorname{rot}\boldsymbol{u}, \operatorname{grad}\operatorname{rot}\boldsymbol{v}) + (\nabla\sigma, \boldsymbol{v}) = \lambda(\boldsymbol{u},\boldsymbol{v}), \ \forall \boldsymbol{v} \in H(\operatorname{grad}\operatorname{rot};\Omega),$$

$$(\boldsymbol{u}, \nabla\tau) - (\sigma,\tau) = 0, \ \forall \tau \in H^1(\Omega).$$

PDE problem

$$-\operatorname{curl}\Delta\operatorname{rot}\boldsymbol{u} - \nabla\operatorname{div}\boldsymbol{u} = \lambda\boldsymbol{u}, \quad \text{in } \Omega,$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \text{and natural b.c.,} \quad \text{on } \partial\Omega.$$

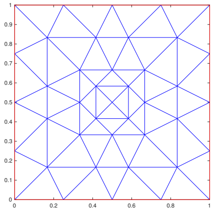


Table: Argyris FEs, primal formulation

n	λ_1	λ_2	λ_3	λ_4	λ_5
0	1.0000000001	1.0000000001	2.0000000015	4.00000016958	4.0000002587
1	1.0000000000	1.0000000000	2.0000000000	3.73248049670	4.0000000005
2	1.0000000001	1.0000000001	2.0000000001	3.26637184925	4.0000000003
3	0.9999999981	1.0000000001	2.0000000005	2.90370225476	3.9999999996
4	1.0000000000	1.0000000000	1.9999999960	2.61348572818	3.9999999430

Table: grad rot-conforming FEs, mixed formulation

n	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
0	0	1.0290	1.0290	2.0871	4.3970	4.3970
1	0	1.0072	1.0072	2.0220	4.1016	4.1038
2	0	1.0018	1.0018	2.0055	4.0255	4.0263
3	0	1.0005	1.0005	2.0014	4.0064	4.0066
4	0	1.0001	1.0001	2.0003	4.0016	4.0016

K.Hu, Q.Zhang, J. Han, L. Wang, and Z. Zhang, *Spurious solutions for high order curl problems*, in preparation.

Question: how to understand zero eigenvalues? Why Argyris element failed?

- nontrivial topology?
- higher dimensions?
- ...

Overview of our answer: look at complexes!

e.g., 3D grad curl complex ($\mathbb{V} := \mathbb{R}^3$ vectors; $\mathbb{T}$ trace-free matrices, $\mathbb{M}$: full matrices):

$$0 \longrightarrow H^q \xrightarrow{\text{grad}} H^{q-1} \otimes \mathbb{V} \xrightarrow{\text{grad curl}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\text{curl}} H^{q-4} \otimes \mathbb{M} \xrightarrow{\text{div}} H^{q-5} \otimes \mathbb{V} \longrightarrow 0.$$

$\boxed{\mathbb{R} \times 0}$ $\boxed{0 \times \mathbb{R}^3}$

- complex: $d \circ d = 0$,
- Hodge-Laplacian at index 1:
  $(\text{grad curl})^*(\text{grad curl}) + \text{grad}(\text{grad}^*) = -\text{curl}\,\Delta\,\text{curl} - \text{grad div}$,
- Hodge decomposition:

$$[L^2]^3 := \text{grad}\, H^1 \oplus_\perp (\text{grad curl})^* H_0((\text{grad curl})^*) \oplus_\perp \mathscr{H},$$

$$\mathscr{H} := \mathcal{N}(\text{div}) \cap \mathcal{N}(\text{grad curl}), \quad \#_{\lambda=0} = \dim \mathscr{H}$$

$$\dim \mathscr{H} = \mathrm{Betti}_1 + 3!$$

  deRham: $[L^2]^3 := \text{grad}\, H^1 \oplus_\perp \text{curl}\, H_0(\text{curl}) \oplus_\perp \mathscr{G}$, $\dim \mathscr{G} = \mathrm{Betti}_1$.
- On squares: 3 (1 for 2D) dimensional "rigid body motion". (grad curl-free, not something grad). Argyris element fails to incorporate the complex structure.

Motivation: generating new complexes and properties.

Elasticity: deformation and mechanics of solids

elasticity equation:

$$-\operatorname{div}(A \operatorname{def} u) = f.$$

u	displacement (vector),
$e := \operatorname{def} u := 1/2(\nabla u + \nabla u^T)$	strain (linearized deformation),
$\sigma := A \operatorname{def} u$	stress.

analogy to Poisson equation:

$$-\operatorname{div}(A \operatorname{grad} v) = g.$$

Elasticity-electromagnetism analogue

Kröner [13] has developed a most useful analogy between the theory of internal stresses and strains as described in sections 2 to 6 and the theory of the magnetic field of distributions of stationary electric currents. Table 1 contains a list of the corresponding physical quantities, differential operators, and equations. We hope that this table is understandable without any further comments (see also the review article by de Wit [10]).

Table 1

Correspondences in elasticity and magnetism

Elasticity	Magnetism
vector quantity	scalar quantity
tensor rank two	vector
tensor rank four	tensor rank two
Div	div
Ink	curl
Div Ink $\equiv 0$	div curl $\equiv 0$
Def	grad
Ink Def $\equiv 0$	curl grad $\equiv 0$
Burgers vector $\boldsymbol{b}$	current I
incompatibility tensor $\boldsymbol{\eta}$	current density $\boldsymbol{J}$
strain tensor $\boldsymbol{\epsilon}$	magnetic intensity $\boldsymbol{H}$
stress tensor $\boldsymbol{\sigma}$	magnetic induction $\boldsymbol{B}$
stress function tensor $\boldsymbol{\chi}, \boldsymbol{\chi}'$	vector potential $\boldsymbol{A}$
elastic constants $\boldsymbol{C}$ (or G, K)	permeability μ
displacement $\boldsymbol{s}$	scalar potential ψ
equation (3)	$\boldsymbol{H} = \text{grad}\, \psi$
equation (5)	curl $\boldsymbol{H} = \boldsymbol{J}$
equation (17)	div $\boldsymbol{B} = 0$
equation (18)	$\boldsymbol{B} = \text{curl}\, \boldsymbol{A}$
equations (19), (19a)	$\nabla^2 \boldsymbol{A} = -\mu \boldsymbol{J}$
equation (20)	div $\boldsymbol{A} = 0$
equation (22)	$\boldsymbol{A} = \frac{\mu}{4\pi} \int\int\int \frac{\boldsymbol{J}(\boldsymbol{r}')}{\lvert \boldsymbol{r} - \boldsymbol{r}' \rvert} d\tau_{\boldsymbol{r}'}$

Seeger, 1961, *Recent Advances in the Theory of Defects in Crystals*.

A cohomological approach: elasticity complex

displacement formulation

Kröner's continuum description of dislocations/defects, internal stress

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0$$

displacement *strain (metric)* *stress (curvature)* *force*

intrinsic elasticity (Ciarlet et al.)

Hellinger-Reissner principle of elasticity

$\mathbb{V} := \mathbb{R}^3$ vectors, $\mathbb{S} := \mathbb{R}^{3\times 3}_{\text{sym}}$ symmetric matrices

$$\operatorname{def} u := 1/2(\nabla u + \nabla u^T), \quad (\operatorname{def} u)_{ij} = 1/2(\partial_i u_j + \partial_j u_i).$$

$$\operatorname{inc} g := \nabla \times g \times \nabla, \quad (\operatorname{inc} g)^{ij} = \epsilon^{ikl}\epsilon^{jst}\partial_k\partial_s g_{lt}.$$

$$\operatorname{div} v := \nabla \times v, \quad (\operatorname{div} v)_i = \partial^j u_{ij}.$$

g metric $\Rightarrow$ $\operatorname{inc} g$ linearized Einstein tensor ($\backsimeq \mathrm{Riem} \backsimeq \mathrm{Ric}$ in 3D)

$\operatorname{inc} \circ \operatorname{def} = 0$: Saint-Venant compatibility

$\operatorname{div} \circ \operatorname{inc} = 0$: Bianchi identity

systematic study still missing, until

Complexes from complexes, Arnold, Hu 2020.

Algebraic and analytic construction (Arnold, Hu 2020): derive elasticity from deRham

$$
\begin{tikzcd}
0 \arrow[r] & H^s \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] & H^{s-1} \otimes \cancel{\mathbb{M}}\ \mathbb{S} \arrow[r, "\mathrm{curl}"] & H^{s-2} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] & \cancel{H^{s-3} \otimes \mathbb{V}} \arrow[r] & 0 \\
0 \arrow[r] & \cancel{H^{s-1} \otimes \mathbb{V}} \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^0:=\mathrm{mskw}"] & H^{s-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S^1"] & H^{s-3} \otimes \cancel{\mathbb{M}}\ \mathbb{S} \arrow[r, "\mathrm{div}"] \arrow[ur, "S^2:=\mathrm{vskw}"] & H^{s-4} \otimes \mathbb{V} \arrow[r] & 0.
\end{tikzcd}
$$

$S^1 u := u^T - 1/2\,\mathrm{tr}(u) I.$

key: Sobolev complexes ($\forall s \in \mathbb{R}$), match indices, commuting diagrams, injectivity & surjective.

output: elasticity complex

$$
\begin{tikzcd}
0 \arrow[r] & H^s \otimes \mathbb{V} \arrow[r, "\mathrm{def}"] & H^{s-1} \otimes \mathbb{S} \arrow[r, "\mathrm{curl}\,\mathrm{T}\,\mathrm{curl}"] & H^{s-3} \otimes \mathbb{S} \arrow[r, "\mathrm{div}"] & H^{s-4} \otimes \mathbb{V} \arrow[r] & 0.
\end{tikzcd}
$$

Theorem

Cohomology of the derived complex is isomorphic to the smooth de Rham cohomology:

$$\mathcal{N}(\mathscr{D}^i) = \mathcal{R}(\mathscr{D}^{i-1}) \oplus \mathscr{H}^i_\infty, \quad \mathscr{H}^i_\infty \backsimeq \mathcal{H}^i_{\mathrm{deRham}} \otimes (\mathbb{V} \times \mathbb{V})$$

Proof: Homological algebra + results for de Rham by Costabel & McIntosh.

Corollary: finite dimensional cohomology $\Longrightarrow$ operators have closed range.

inspired by Bernstein-Gelfand-Gelfand (BGG) resolution
(c.f., Eastwood 2000, Čap, Slovák, Souček 2001, Arnold, Falk, Winther 2006).

Consequences:

- analytic results (Poincaré inequality, Hodge decomposition, etc.)
  e.g., Korn inequality $\|u\|_1 \leq C\| \operatorname{def} u\|, \quad u \perp \mathcal{N}(\operatorname{def})$.
- explicit representatives of elasticity cohomology.

Systematic construction: generating new complexes from existing ones

- input: $(Z^\bullet, D^\bullet)$, $(\tilde{Z}^\bullet, \tilde{D}^\bullet)$, connecting maps $S^i : \tilde{Z}^i \to Z^{i+1}$, satisfying
  - (anti-)commutativity: $S^{i+1}\tilde{D}^i = -D^{i+1}S^i$,
  - injectivity/surjectivity condition: S^i injective for $i \leq J$, surjective for $i \geq J$.

$$\begin{tikzcd}
0 \arrow[r] & Z^0 \arrow[r, "D^0"] & Z^1 \arrow[r, "D^1"] & \cdots \arrow[r, "D^{n-1}"] & Z^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{Z}^0 \arrow[r, "\tilde{D}^0"] \arrow[ur, "S^0"] & \tilde{Z}^1 \arrow[r, "\tilde{D}^1"] \arrow[ur, "S^1"] & \cdots \arrow[r, "\tilde{D}^{n-1}"] \arrow[ur, "S^{n-1}"] & \tilde{Z}^n \arrow[r] & 0
\end{tikzcd}$$

- output:

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{coker}(S^{J-2}) \arrow[r, "D^{J-1}"] & \operatorname{coker}(S^{J-1}) \arrow[r, "D^J"] & {} \arrow[dl, "(S^J)^{-1}"'] & & \\
& & {} \arrow[r, "\tilde{D}^J"] & \mathcal{N}(S^{J+1}) \arrow[r, "\tilde{D}^{J+1}"] & \mathcal{N}(S^{J+2}) \arrow[r, "\tilde{D}^{J+2}"] & \cdots
\end{tikzcd}$$

- conclusion:

$$\dim \mathscr{H}^i(\Upsilon^\bullet, \mathscr{D}^\bullet) \leq \dim \mathscr{H}^i(Z^\bullet, D^\bullet) + \dim \mathscr{H}^i\left(\tilde{Z}^\bullet, \tilde{D}^\bullet\right), \quad \forall i = 0, 1, \cdots, n$$

Equality holds if and only if S^i induces the zero maps on cohomology, i.e., $S^i\mathcal{N}(\tilde{D}^i) \subset \mathcal{R}(D^i)$.

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\,\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

Hessian complex:

$$0 \longrightarrow H^q \otimes \mathbb{R} \xrightarrow{\mathrm{hess}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\mathrm{curl}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\mathrm{div}} H^{q-4} \otimes \mathbb{V} \longrightarrow 0.$$

biharmonic equations, plate theory, Einstein-Bianchi system of general relativity

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\text{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\text{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\text{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\text{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\text{curl}"] \arrow[ur, "2\,\text{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\text{div}"] \arrow[ur, "\text{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\text{grad}"] \arrow[ur, "-\,\text{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\text{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\text{div}"] \arrow[ur, "2\,\text{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\text{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\text{curl}"] \arrow[ur, "\text{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\text{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

elasticity complex:

$$
0 \longrightarrow H^{q-1} \otimes \mathbb{V} \xrightarrow{\text{def}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\text{inc}} H^{q-4} \otimes \mathbb{S} \xrightarrow{\text{div}} H^{q-5} \otimes \mathbb{V} \longrightarrow 0.
$$

elasticity, defects, metric, curvature

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V}; \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\,\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

divdiv complex:

$$
0 \longrightarrow H^{q-2} \otimes \mathbb{V} \xrightarrow{\text{dev grad}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\text{sym curl}} H^{q-4} \otimes \mathbb{S} \xrightarrow{\text{div div}} H^{q-6} \otimes \mathbb{R} \longrightarrow 0.
$$

plate theory, elasticity

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q\otimes\mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1}\otimes\mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2}\otimes\mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3}\otimes\mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1}\otimes\mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2}\otimes\mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3}\otimes\mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4}\otimes\mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2}\otimes\mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\,\mathrm{mskw}"] & H^{q-3}\otimes\mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4}\otimes\mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5}\otimes\mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3}\otimes\mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4}\otimes\mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5}\otimes\mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6}\otimes\mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

grad curl complex:

$$0 \longrightarrow H^q \xrightarrow{\mathrm{grad}} H^{q-1}\otimes\mathbb{V} \xrightarrow{\mathrm{grad\,curl}} H^{q-3}\otimes\mathbb{T} \xrightarrow{\mathrm{curl}} H^{q-4}\otimes\mathbb{M} \xrightarrow{\mathrm{div}} H^{q-5}\otimes\mathbb{V} \longrightarrow 0.$$

couple stress, size effects

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\,\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

curl div complex:

$$
0 \to H^q \otimes \mathbb{V} \xrightarrow{\mathrm{grad}} H^{q-1} \otimes \mathbb{M} \xrightarrow{\mathrm{dev\,curl}} H^{q-2} \otimes \mathbb{T} \xrightarrow{\mathrm{curl\,div}} H^{q-4} \otimes \mathbb{V} \xrightarrow{\mathrm{div}} H^{q-5} \to 0.
$$

couple stress, size effects

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

grad div complex:

$$
0 \to H^q \xrightarrow{\mathrm{grad}} H^{q-1} \otimes \mathbb{V} \xrightarrow{\mathrm{curl}} H^{q-2} \otimes \mathbb{V}
$$

$$
\xrightarrow{\mathrm{grad\,div}} H^{q-4} \otimes \mathbb{V} \xrightarrow{\mathrm{curl}} H^{q-5} \otimes \mathbb{V} \xrightarrow{\mathrm{div}} H^{q-6} \to 0.
$$

Iterating the construction

$$
\begin{tikzcd}
0 \arrow[r] & H^{q+1}\otimes\mathbb{R} \arrow[r, "\mathsf{grad}"] & H^{q}\otimes\mathbb{V} \arrow[r, "\mathsf{curl}"] & H^{q-1}\otimes\mathbb{V} \arrow[r, "\mathsf{div}"] & H^{q-2}\otimes\mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q}\otimes\mathbb{V} \arrow[r, "\mathsf{dev\,grad}"] \arrow[ur, "-\mathrm{id}"] & H^{q-1}\otimes\mathbb{T} \arrow[r, "\mathsf{sym\,curl}"] \arrow[ur, "2\,\mathsf{vskw}"] & H^{q-2}\otimes\mathbb{S} \arrow[r, "\mathsf{div\,div}"] \arrow[ur, "\mathsf{tr}"] & H^{q-4}\otimes\mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1}\otimes\mathbb{V} \arrow[r, "\mathsf{def}"] \arrow[ur, "-\,\mathsf{mskw}"] & H^{q-2}\otimes\mathbb{S} \arrow[r, "\mathsf{inc}"] \arrow[ur, "S"] & H^{q-4}\otimes\mathbb{S} \arrow[r, "\mathsf{div}"] \arrow[ur, "\mathsf{tr}"] & H^{q-5}\otimes\mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \arrow[r, "\mathsf{hess}"] \arrow[ur, "\iota"] & H^{q-4}\otimes\mathbb{S} \arrow[r, "\mathsf{curl}"] \arrow[ur, "S"] & H^{q-5}\otimes\mathbb{T} \arrow[r, "\mathsf{div}"] \arrow[ur, "2\,\mathsf{vskw}"] & H^{q-6}\otimes\mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-4}\otimes\mathbb{R} \arrow[r, "\mathsf{grad}"] \arrow[ur, "\iota"] & H^{q-5}\otimes\mathbb{V} \arrow[r, "\mathsf{curl}"] \arrow[ur, "\mathsf{mskw}"] & H^{q-6}\otimes\mathbb{V} \arrow[r, "\mathsf{div}"] \arrow[ur, "-\mathrm{id}"] & H^{q-7}\otimes\mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

Example from iterative constructions

"conformal complex"

ker of dev def: conformal Killing v.f. Cotton-York: flatness in conformal geometry

$$0 \longrightarrow H^q(\Omega)\otimes \mathbb{V} \xrightarrow{\operatorname{dev}\operatorname{def}} H^{q-1}(\Omega)\otimes(\mathbb{S}\cap\mathbb{T}) \xrightarrow{\operatorname{cott}} H^{q-4}(\Omega)\otimes(\mathbb{S}\cap\mathbb{T}) \xrightarrow{\operatorname{div}} H^{q-5}(\Omega)\otimes\mathbb{V} \longrightarrow 0$$

gravitational wave variable: transverse-traceless (TT) gauge
(= symmetric, trace-free, div-free)

stress like variable def$u$ in NS
(Gopalakrishnan, Lederer, Schöberl, 2019)

Hodge decomp.: York split, Einstein constraint eqns.

trace-free Korn inequality:

$$\|u\|_1 \le C\|\operatorname{dev}\operatorname{def} u\|, \quad \forall u \in \mathcal{N}(\operatorname{dev}\operatorname{def}).$$

$\mathcal{N}(\operatorname{dev}\operatorname{def})$: conformal Killing fields

open problem (Chipot 2020): minimal number of linear functionals l_i, s.t. generalized Korn inequality holds

$$\|u\|_1 \le C\Big(\sum_{i=1}^{N}\|l_i(\nabla u)\|_{L^2} + \|u\|_{L^2}\Big).$$

e.g., 3D Poincaré: N=9; Korn: N= 6; trace-free Korn: N=5.

Back to the eigenvalue problem: a closer look at the grad curl complex

$$
\begin{tikzcd}
 & \boxed{\mathbb{R}} & \boxed{0} & & & & \\
0 \arrow[r] & H^q \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[ur, "0"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "-\mathrm{tr}"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "0"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0. \\
 & \boxed{0} & \boxed{\mathbb{R}^3} & & & &
\end{tikzcd}
$$

grad curl complex:

$$
\begin{tikzcd}
0 \arrow[r] & H^q \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad\,curl}"] & H^{q-3} \otimes \mathbb{T} \arrow[r, "\mathrm{curl}"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0. \\
 & \boxed{\mathbb{R} \times 0} & \boxed{0 \times \mathbb{R}^3} & & & &
\end{tikzcd}
$$

explicit representation of cohomology (tracking the algebraic construction): $a \wedge x$. (2D: $cx^{\perp}$), "rigid body motion"

Hodge decomposition, regularity decomposition, compact imbedding, ..., follow.

$$[L^2]^3 := \mathrm{grad}\, H^1 \oplus_{\perp} (\mathrm{grad\,curl})^* H_0((\mathrm{grad\,curl})^*) \oplus_{\perp} \mathscr{H},$$

$$\mathscr{H} := \mathcal{N}(\mathrm{div}) \cap \mathcal{N}(\mathrm{grad\,curl}), \quad \#_{\lambda=0} = \dim \mathscr{H} = \mathrm{Betti}_1 + 3!$$

Take home messages:

- reasons for spurious solutions:
  1. regularity (domains with corners),
  2. nontrivial topology,
  3. *intrinsic algebraic structures of differential operators*.
- potential of "complexes from complexes": a strong tie between analysis, topology, algebra, computation, (even) modeling.

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