

Conformal Korn inequality in 2D

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On a bounded Lipschitz domain:

- Poincaré inequalities:

$$\|u\|_{H^1} \leq \|\operatorname{grad} u\|_{L^2}, \quad u \perp \mathcal{N}(\operatorname{grad}).$$

Trivial generalization: vector-valued functions (9 components in $\operatorname{grad} u$).

- Korn inequality:

$$\|u\|_{H^1} \leq C \|\operatorname{sym} \operatorname{grad} u\|_{L^2}, \quad u \perp \mathcal{N}(\operatorname{sym} \operatorname{grad}).$$

- Conformal Korn inequality:

$$\|u\|_{H^1} \leq C \|\operatorname{dev} \operatorname{sym} \operatorname{grad} u\|_{L^2}, \quad u \perp \mathcal{N}(\operatorname{dev} \operatorname{sym} \operatorname{grad}),$$

holds in nD for $n \geq 3$! ($\operatorname{dev} w := w - \frac{1}{n} \operatorname{tr}(w)I$)

Fails in 2D:

$$\operatorname{dev} \operatorname{sym} \operatorname{grad} u = \begin{pmatrix} \frac{1}{2}(\partial_x u_1 - \partial_y u_2) & \frac{1}{2}(\partial_y u_1 + \partial_x u_2) \\ \frac{1}{2}(\partial_y u_1 + \partial_x u_2) & -\frac{1}{2}(\partial_x u_1 - \partial_y u_2) \end{pmatrix}$$

Cauchy-Riemann operator

Applications in GR, failure in 2D: Dain, S. (2006). Generalized Korn's inequality and conformal Killing vectors. Calculus of variations and partial differential equations, 25(4), 535-540.

An algebraic proof: closed range implies inequalities (Banach theorem), $\text{range} = \text{kernel}$.

- Poincaré inequalities: $\text{curl} \circ \text{grad} = 0$, $\mathcal{R}(\text{grad}) = \mathcal{N}(\text{curl})$
de Rham complex, topology
- Korn inequality: $(\text{curl} \circ T \circ \text{curl}) \circ (\text{sym grad}) = 0$, $\mathcal{R}(\text{sym grad}) = \mathcal{N}(\text{curl} \circ T \circ \text{curl})$
Einstein (Ricci/Riemann) tensor, Riemannian geometry
- Conformal Korn inequality (3D):
 $(\text{curl} \circ S^{-1} \circ \text{curl} \circ S^{-1} \circ \text{curl}) \circ (\text{dev sym grad}) = 0$, $\mathcal{R}(\text{sym grad}) = \mathcal{N}(\text{curl} \circ T \circ \text{curl})$
Cotton-York tensor, conformal geometry

(assume contractible domain. generalization to finite dimensional cohomology is similar)

Failure of the 2D conformal Korn inequality means that the Cauchy-Riemann operator does not have closed range. How to fix?

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^q \otimes \mathbb{V} & \xrightarrow{\text{grad}} & H^{q-1} \otimes \mathbb{M} & \xrightarrow{\text{rot}} & H^{q-2} \otimes \mathbb{V} \longrightarrow 0 \\
& & & \searrow S^{0,1} & & \searrow S^{1,1} & \\
0 & \longrightarrow & H^{q-1} \otimes (\mathbb{R} \times \mathbb{R}) & \xrightarrow{\text{grad}} & H^{q-2} \otimes (\mathbb{V} \times \mathbb{V}) & \xrightarrow{\text{rot}} & H^{q-3} \otimes (\mathbb{R} \times \mathbb{R}) \longrightarrow 0 \\
& & & \searrow S^{0,2} & & \searrow S^{1,2} & \\
0 & \longrightarrow & H^{q-2} \otimes \mathbb{V} & \xrightarrow{\text{grad}} & H^{q-3} \otimes \mathbb{M} & \xrightarrow{\text{rot}} & H^{q-4} \otimes \mathbb{V} \longrightarrow 0.
\end{array}$$

Here $S^{0,1} = -\perp$, $S^{1,1} = \text{tr}$, $S^{0,2} = -\iota$, and $S^{1,2} = I$.

\mathbb{V} : vector, \mathbb{M} : matrix, \mathbb{S} : symmetric matrix, \mathbb{T} : tracefree matrix

Horizontal: de Rham complex. Diagonal: Lie algebra (co)homology.

Output:

$$0 \longrightarrow H^q \otimes \mathbb{V} \xrightarrow{D^0} \begin{pmatrix} H^{q-1} \otimes (\mathbb{S} \cap \mathbb{T}) \\ H^{q-3} \otimes (\mathbb{S} \cap \mathbb{T}) \end{pmatrix} \xrightarrow{D^1} H^{q-3} \longrightarrow 0,$$

where

$$D^0 := \begin{pmatrix} \text{dev def grad} \\ \frac{1}{2}(\text{hess div} - \text{grad curl rot})u \end{pmatrix}, \quad D^1 = \begin{pmatrix} \frac{1}{2}(-\text{grad div} + \text{curl rot}) \text{rot} \\ \text{rot} \end{pmatrix}^T.$$

2D conformal Korn inequality: $\|u\|_{H^1} \leq C(\|\text{dev def grad } u\|_{L^2} + \|(\text{hess div} - \text{grad curl rot})u\|_{L^2})$.
Möbius structure.

Open problem: minimal number of linear functionals l_i , s.t. generalized Korn inequality holds

$$\|u\|_{H^1} \leq C \left(\sum_{i=1}^N \|l_i(\nabla u)\|_{L^2} + \|u\|_{L^2} \right).$$

e.g., 3D Poincaré: $N=9$; Korn: $N=6$; trace-free Korn: $N=5$.

Chipot, M. (2021). *On inequalities of Korn's type*. Journal de Mathématiques Pures et Appliquées, 148, 199-220.

Todo: nonlinear Korn inequality? (Pfaff system, Poincaré lemma, diagram chasing, rigidity and compatibility of Riemannian geometry...)

References:

- *Complexes from complexes*, Douglas Arnold, Kaibo Hu; *Foundations of Computational Mathematics* (2021). [framework, analytic results from homological algebraic structures](#)
- *Some Bernstein-Gelfand-Gelfand sequences with weak regularity*, Andreas Čap, Kaibo Hu; *in preparation* (2022)
[more general framework that implies the 2D conformal Korn inequality, more examples.](#)