Differential complexes for linearized geometry

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de Rham complex

$$0 \longrightarrow \Lambda^0 \xrightarrow{d^0} \Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \Lambda^n \longrightarrow 0.$$

- Λ^k: differential k-forms, d^k: exterior derivatives, complex: $d^{k+1} \circ d^k = 0$.
- connections to PDEs:
  Hodge-Laplacian problems: $(d^{k-1} d^*_{k-1} + d^*_k d^k)u = f$,
  $d^*_k : \Lambda^{k+1} \to \Lambda^k$, adjoint of d^k.
- 3D vector proxies:

  $$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega; \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0.$$

  e.g. Hodge-Laplacian problem with $k = 1$:

  $$\operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u = f.$$

  Maxwell equations.
- Finite Element Exterior Calculus (FEEC, Arnold-Falk-Winther 2006, Arnold 2018): cohomological approach of finite element methods, framework for discretizing complexes and PDEs.

Elasticity complex

kernel of def: Killing fields — inc ∘ def = 0: imbedding — div ∘ inc = 0: Bianchi identity

$$0 \longrightarrow H^q \otimes \mathbb{V} \xrightarrow{\text{def}} H^{q-1} \otimes \mathbb{S} \xrightarrow{\text{inc}} H^{q-3} \otimes \mathbb{S} \xrightarrow{\text{div}} H^{q-4} \otimes \mathbb{V} \longrightarrow 0$$

displacement — metric — Einstein tensor (≃Ric, Riem in 3D)

Conformal complex

kernel of dev def:
conformal Killing fields

cott: Cotton-York tensor
flatness in conformal geometry

$$0 \to \underbrace{H^q \otimes \mathbb{V} \xrightarrow{\text{dev def}} H^{q-1} \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{cott}} H^{q-4}}_{\substack{\text{Hodge decomposition:}\\ \text{York split, Einstein constraint eqn.}}} \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{div}} H^{q-5} \otimes \mathbb{V} \to 0$$

gravitational wave variables:
Traceless Transverse (TT) gauge
(div-free, trace-free, symmetric)

Hessian complex

$$0 \longrightarrow H^q \otimes \mathbb{R} \xrightarrow{\text{hess}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\text{curl}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\text{div}} H^{q-4} \otimes \mathbb{V} \longrightarrow 0$$

linearized Einstein-Bianchi system:

$$E_t + \operatorname{curl} B = 0$$
$$B_t - \operatorname{curl} E = 0$$

B, E: symmetric, traceless ($\mathbb{S} \cap \mathbb{T}$)

Anderson, Choquet-Bruhat, York 1997, Quenneville-Bélair 2015.

$\mathbb{V}$: vectors, $\mathbb{S}$: symmetric matrices, $\mathbb{T}$: traceless matrices

Arnold, Hu, *Complexes from complexes*, arXiv:2005.12437.