A cohomological perspective for high order problems (I)

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High order problems

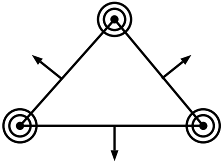
Examples of high order problems

- plate (thin structures):
$$\Delta^2 u = f,$$
- model problem for electromagnetic and generalized continuum:
$$\begin{aligned} -\operatorname{curl}\operatorname{div}(\operatorname{grad}\operatorname{curl})u &= f, \\ \operatorname{div} u &= 0. \end{aligned}$$

$\operatorname{div}\operatorname{grad} = \Delta = \operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div}$, formally $-\operatorname{curl}\operatorname{div}(\operatorname{grad}\operatorname{curl}) = \operatorname{curl}^4$.

Challenges for finite element discretization:

- conformity,
- inf-sup conditions (for vector/tensor fields and multi-fields).



Fluid mechanics

Stokes equations : strongly diffusive, stationary, incompressible

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p & = \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} & = 0, \end{cases}$$

Boundary condition: $\boldsymbol{u} = 0$ on $\partial\Omega$.

Variational form : find $\boldsymbol{u} \in [H_0^1(\Omega)]^n$, $p \in L_0^2(\Omega)$ s.t.,

$$\begin{cases} (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) & = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in [H_0^1(\Omega)]^n, \\ (\nabla \cdot \boldsymbol{u}, q) & = 0, \quad \forall q \in L_0^2(\Omega). \end{cases}$$

Discrete variational form : find $\boldsymbol{u}_h \in \boldsymbol{V}_h$, $p \in Q_h$ s.t.,

$$\begin{cases} (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (p_h, \nabla \cdot \boldsymbol{v}_h) & = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ (\nabla \cdot \boldsymbol{u}_h, q_h) & = 0, \quad \forall q_h \in Q_h. \end{cases}$$

How to choose $\boldsymbol{V}_h$ and Q_h?

- numerical stability (inf-sup condition): for any $q_h \in Q_h$, there exists $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $\|\boldsymbol{v}_h\| = 1$, s.t. $(\nabla \cdot \boldsymbol{v}_h, q_h) \geq \|q_h\|_{L^2}$; $\nabla \cdot \boldsymbol{V}_h$ should be "larger" than Q_h.
- precise divergence-free constraint:

  $(\nabla \cdot \boldsymbol{v}_h, q_h) = 0, \ \forall q_h \in Q_h \quad \Longrightarrow \quad \nabla \cdot \boldsymbol{v}_h = 0$ Q_h should be "larger" than $\nabla \cdot \boldsymbol{V}_h$.

  mass conservation, important for numerics, John et al. *SIAM Review 2017*
- balance: $\nabla \cdot \boldsymbol{V}_h = Q_h$,
- constructing finite elements satisfying $\nabla \cdot \boldsymbol{V}_h = Q_h$ turns out to be very challenging.

  Scott-Vogelius, stable Stokes pairs etc.

de Rham complex

de Rham complex (3D version)

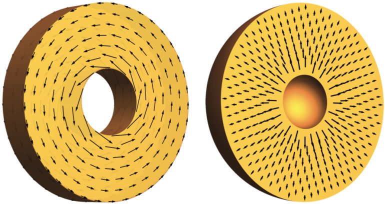
$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0.$$

$$d^0 := \text{grad}, \quad d^1 := \text{curl}, \quad d^2 := \text{div}\,.$$

- complex property: $d^k \circ d^{k-1} = 0, \;\Rightarrow\; \mathcal{R}(d^{k-1}) \subset \mathcal{N}(d^k)$,
  $\text{curl} \circ \text{grad} = 0 \Rightarrow \mathcal{R}(\text{grad}) \subset \mathcal{N}(\text{curl}), \quad \text{div} \circ \text{curl} = 0 \Rightarrow \mathcal{R}(\text{curl}) \subset \mathcal{N}(\text{div})$
- cohomology: $\mathscr{H}^k := \mathcal{N}(d^k)/\mathcal{R}(d^{k-1})$,
  $\mathscr{H}^0 := \mathcal{N}(\text{grad}), \quad \mathscr{H}^1 := \mathcal{N}(\text{curl})/\mathcal{R}(\text{grad}), \quad \mathscr{H}^2 := \mathcal{N}(\text{div})/\mathcal{R}(\text{curl})$
- exactness (contractible domains): $\mathcal{N}(d^k) = \mathcal{R}(d^{k-1})$, i.e., $d^k u = 0 \;\Rightarrow\; u = d^{k-1} v$
  $\text{curl}\, u = 0 \Rightarrow u = \text{grad}\,\phi, \quad \text{div}\, v = 0 \Rightarrow v = \text{curl}\,\psi.$

de Rham complex and topology:

dimension of $\mathscr{H}^k$ = number of "k-dimensional holes" (c.f. de Rham theorem)



Examples where $\dim \mathscr{H}^1 = 1$ and $\dim \mathscr{H}^2 = 1$, respectively.
Left: curl-free field which is not grad, Right: div-free field with is not curl.

(figure from *Finite element exterior calculus*, D.N.Arnold, SIAM 2008.)

From complexes to PDEs

Formal adjoint of operators:

$$\operatorname{grad}^* = -\operatorname{div}, \quad \operatorname{curl}^* = \operatorname{curl}, \quad \operatorname{div}^* = -\operatorname{grad}.$$

$$\int_\Omega \operatorname{grad} u \cdot v = -\int_\Omega u \operatorname{div} v + \text{bound. term}, \quad \int_\Omega \operatorname{curl} u \cdot v = \int_\Omega u \cdot \operatorname{curl} v + \text{bound. term}$$

$$(\operatorname{grad} u, v) = (u, -\operatorname{div} v), \qquad (\operatorname{curl} u, v) = (u, \operatorname{curl} v)$$

Formal adjoint of de Rham complex:

$$0 \longleftarrow C^\infty(\Omega) \xleftarrow{-\operatorname{div}} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{\operatorname{curl}} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{-\operatorname{grad}} C^\infty(\Omega) \longleftarrow 0.$$

$$d_2^* := -\operatorname{div}, \quad d_1^* := \operatorname{curl}, \quad d_0^* := -\operatorname{grad}.$$

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d_{k-1}^* + d_k^* d^k)u = f.$$

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d^*_{k-1} + d^*_k d^k)u = f.$$

$$0 \rightleftarrows \color{red}{C^\infty(\Omega)} \underset{-\,\mathrm{div}}{\overset{\mathrm{grad}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \qquad\qquad C^\infty(\Omega;\mathbb{R}^3) \qquad\qquad C^\infty(\Omega) \qquad\qquad 0.$$

Hodge-Laplacian problem:

$$-\,\mathrm{div}\,\mathrm{grad}\,u = f.$$

Poisson equation.

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d^*_{k-1} + d^*_k d^k)u = f.$$

$$0 \qquad C^\infty(\Omega) \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \underset{\operatorname{curl}}{\overset{\operatorname{curl}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$-\operatorname{grad}\operatorname{div} v + \operatorname{curl}\operatorname{curl} v = f.$$

Maxwell equations.

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d^*_{k-1} + d^*_k d^k)u = f.$$

$$0 \qquad C^\infty(\Omega) \qquad C^\infty(\Omega;\mathbb{R}^3) \underset{\text{curl}}{\overset{\text{curl}}{\rightleftarrows}} {\color{red}C^\infty(\Omega;\mathbb{R}^3)} \underset{-\,\text{grad}}{\overset{\text{div}}{\rightleftarrows}} C^\infty(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$\operatorname{curl}\operatorname{curl} v - \operatorname{grad}\operatorname{div} v = f.$$

Maxwell equations.

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d^*_{k-1} + d^*_k d^k)u = f.$$

$$0 \qquad C^\infty(\Omega) \qquad C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega;\mathbb{R}^3) \underset{-\,\mathrm{grad}}{\overset{\mathrm{div}}{\rightleftarrows}} {\color{red}C^\infty(\Omega)} \rightleftarrows 0.$$

Hodge-Laplacian problem:

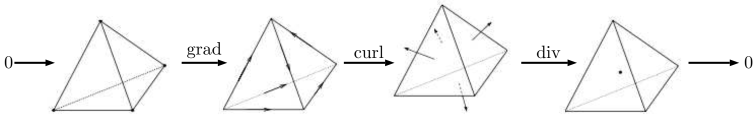
$$-\operatorname{div}\operatorname{grad} u = f.$$

Poisson equation.

Sobolev version and discretization

$$0 \longrightarrow H(\mathrm{grad}) \xrightarrow{\mathrm{grad}} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \longrightarrow 0$$

$H(d) := \{u \in L^2 : du \in L^2\}$.



$$0 \longrightarrow \mathcal{P}_1 \xrightarrow{\mathrm{grad}} [\mathcal{P}_0]^3 + [\mathcal{P}_0]^3 \times x \xrightarrow{\mathrm{curl}} [\mathcal{P}_0]^3 + \mathcal{P}_0 \otimes x \xrightarrow{\mathrm{div}} \mathcal{P}_0 \longrightarrow 0.$$

Raviart-Thomas (1977), Nédélec (1980) in numerical analysis, Bossavit (1988), Hiptmair (1999) for differential forms, Whitney (1957) for studying topology.

Smoother de Rham complexes

- connecting high order problems to Stokes problem

$$0 \longrightarrow H^1 \xrightarrow{\text{grad}} H(\text{grad}\,\text{curl}) \xrightarrow{\text{curl}} [H^1]^3 \xrightarrow{\text{div}} L^2 \longrightarrow 0,$$

$$0 \longrightarrow H^2 \xrightarrow{\text{grad}} H^1(\text{curl}) \xrightarrow{\text{curl}} [H^1]^3 \xrightarrow{\text{div}} L^2 \longrightarrow 0,$$

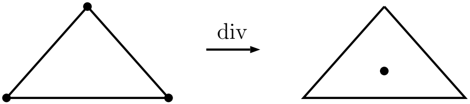
where $H^1(\text{curl}) := \{u \in [H^1]^3 : \text{curl}\, u \in [H^1]^3\}$.

- exactness (cohomology): c.f., Costabel, McIntosh 2010, for *any real number s*,

$$0 \longrightarrow H^s \xrightarrow{\text{grad}} H^{s-1} \otimes \mathbb{R}^3 \xrightarrow{\text{curl}} H^{s-2} \otimes \mathbb{R}^3 \xrightarrow{\text{div}} H^{s-3} \longrightarrow 0$$

Conservative Stokes discretization: $[H^1]^3$-L^2

- puzzle of Scott-Vogelius ($[C^0\mathcal{P}_r]^n$-$C^{-1}\mathcal{P}_{r-1}$) : 2D stable for $r \geq 4$, no "singular vertices"; 3D open.



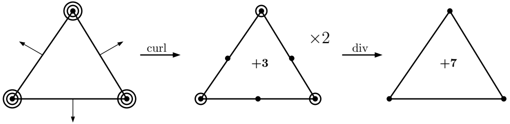
- a cohomological perspective

$$0 \longrightarrow H^1 \xrightarrow{\text{grad}} H(\text{grad}\,\text{curl}) \xrightarrow{\text{curl}} [H^1]^3 \xrightarrow{\text{div}} L^2 \longrightarrow 0.$$

$$0 \longrightarrow C^0 \text{ spline} \xrightarrow{\text{grad}} * \xrightarrow{\text{curl}} \boldsymbol{V}_h \xrightarrow{\text{div}} Q_h \longrightarrow 0.$$

(Incomplete) review on finite element Stokes complexes (on simplicial meshes)

- first FE Stokes complex: Falk-Neilan 2013,

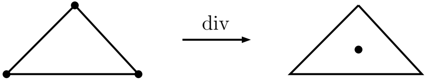


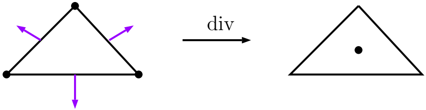
- 3D tetrahedral element: Neilan 2015 (starting with $\mathcal{P}_9$-C^1), Q.Zhang-Z.Zhang 2020 (H^1 complex, different 1-forms),
- construction on macroelements: Christiansen-Hu 2018 (low order complex), Fu-Guzmán-Neilan 2018 (Alfeld split, any dim), Guzmán-Lischke-Neilan, 2020 (Powell-Sabin, Worsey-Farin split),
- nonconforming elements: Mardal-Tai-Winther 2002, Tai-Winther 2006, Huang 2020 (fewer dofs),
- virtual elements: Zhao-Zhang-Mao-Chen 2019 (nonconforming), Beirão da Veiga-Dassi-Vacca 2020 (conforming 3D).

Stabilize Scott-Vogelius elements

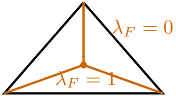
inf-sup condition: velocity space large enough, to "control" pressure

- piecewise constant pressure $\leftarrow$ face dofs of velocity, $\int_T \operatorname{div} u = \int_{\partial T} u \cdot n$,
- higher order pressure modes $\leftarrow$ interior dofs of velocity.



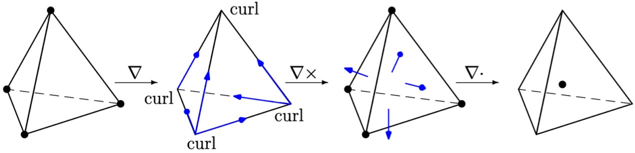


- Bernadi-Raugel bubble: $b_{BR} := \{b_F n_F\}$, scalar face bubble b_F, but $\operatorname{div} b_{BR} \not\subseteq \mathcal{P}_0$!
- modified Bernadi-Raugel bubble: $b_{mBR} := \{b_F n_F + \sum_{j=1}^{k} \lambda_F^j w_{k-j}\}$, $\operatorname{div} b_{mBR} \subset \mathcal{P}_0$.



lowest order : Guzmán-Neilan 2018

tetrahedra construction with canonical dofs: Hu-Q.Zhang-Z.Zhang 2020, arXiv:2008.03793



- construct 1-form by Poincaré integrals $\mathfrak{p}^2$:

$$V_h^1 := \mathrm{grad}\, V_h^1 \oplus \mathfrak{p}^2 V_h^2.$$

  basic idea of $\mathfrak{p}^2$: integral of functions

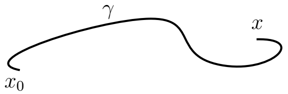
- canonical dofs: Whitney dofs + vertex evaluation.

Poincaré operators

Question:

- how to find out explicit potential? $u = D\phi$, knowing u, find out ϕ.
- how to prove the Poincaré lemma (local exactness of de Rham sequence)?

Example: $D = \text{grad}$, $\phi = \phi(x_0) + \int_{\gamma(y)} u \, dy$



General d^k (curl, div etc.):

- Poincaré operators (differential geometry books; Hiptmair 1999) $\mathfrak{p}^k : C^\infty\Lambda^k \mapsto C^\infty\Lambda^{k-1}$, satisfying

$$d^{k-1}\mathfrak{p}^k + \mathfrak{p}^{k+1}d^k = \mathrm{id}_{C^\infty\Lambda^k},$$

- exactness: $du = 0 \Rightarrow u = (d\mathfrak{p} + \mathfrak{p}d)u = d(\mathfrak{p}u)$.

$$\cdots \rightleftarrows V^{i-1} \underset{\mathfrak{p}^i}{\overset{d^{i-1}}{\rightleftarrows}} V^i \underset{\mathfrak{p}^{i+1}}{\overset{d^i}{\rightleftarrows}} V^{i+1} \rightleftarrows \cdots$$

Corollary: $V^i = dV^{i-1} \oplus \mathfrak{p}V^{i+1}$.

- explicit forms of $\mathfrak{p}^k$ for de Rham complexes: using Cartan's magic formula
- 3D vector proxy (with $W = 0$):

$$\mathfrak{p}_1 u = \int_0^1 u_{tx} \cdot x dt, \quad \mathfrak{p}_2 v = \int_0^1 t v_{tx} \times x dt, \quad \mathfrak{p}_3 w = \int_0^1 t^2 w_{tx} x dt.$$

applications in numerical analysis: constructing polynomial exact sequences. e.g.,

$$0 \longrightarrow \mathcal{P}_1 \xrightarrow{\text{grad}} [\mathcal{P}_0]^3 + [\mathcal{P}_0]^3 \times x \xrightarrow{\text{curl}} [\mathcal{P}_0]^3 + \mathcal{P}_0 \otimes x \xrightarrow{\text{div}} \mathcal{P}_0 \longrightarrow 0.$$

$$0 \longrightarrow \mathcal{P}_r \xrightarrow{\text{grad}} [\mathcal{P}_{r-1}]^3 \xrightarrow{\text{curl}} [\mathcal{P}_{r-2}]^3 \xrightarrow{\text{div}} \mathcal{P}_{r-3} \longrightarrow 0.$$

Finite element exterior calculus (FEEC): cohomological approach for numerical PDEs (Arnold, Falk, Winther 2006 Acta. Num., Arnold, Falk, Winther 2010 Bulletin of AMS, Arnold 2018 SIAM book).

Stokes case

$$V_h^1 := \text{grad}\, V_h^1 \oplus \mathfrak{p}^2 V_h^2.$$

interior point chosen as base point.

Take home messages:

- complex is important,
- enriching classical Scott-Vogelius pair with face and interior bubbles,
- application of Poincaré operators.

Overview of part 2: pseudo-solutions and grad curl-complexes grad curl complex:

$$0 \longrightarrow H^q \xrightarrow{\text{grad}} H^{q-1} \otimes \mathbb{V} \xrightarrow{\text{grad curl}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\text{curl}} H^{q-4} \otimes \mathbb{M} \xrightarrow{\text{div}} H^{q-5} \otimes \mathbb{V} \longrightarrow 0.$$

References

- *A family of finite element Stokes complexes in three dimensions*, Kaibo Hu, Qian Zhang, Zhimin Zhang; 2020, arXiv:2008.03793.
- *Simple curl-curl-conforming finite elements in two dimensions*; Kaibo Hu, Qian Zhang, Zhimin Zhang; SIAM Scientific Computing 2020 (accepted).
- *Generalized finite element systems for smooth differential forms and Stokes problem*, Snorre H. Christiansen and Kaibo Hu; Numerische Mathematik 2018.