Finite elements for curvature

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Outline

1. Elasticity complex and motivation

2. Finite element sequences

De Rham complexes

De Rham complex in 3D:

$$0 \longrightarrow L^2(\Omega) \xrightarrow{\text{grad}} L^2(\Omega)\otimes\mathbb{V} \xrightarrow{\text{curl}} L^2(\Omega)\otimes\mathbb{V} \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow 0,$$

with the domain complex:

$$0 \longrightarrow H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl},\Omega) \xrightarrow{\text{curl}} H(\text{div},\Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow 0.$$

- $\mathbb{V}$: vectors, $\mathbb{M}$: matrices, $\mathbb{S}$: symmetric matrices,
- $H(\mathscr{D},\Omega) := \left\{u \in L^2 : \mathscr{D}u \in L^2\right\}$,
- complex: $\text{curl}\,\text{grad} = 0$, $\text{div}\,\text{curl} = 0$,
- cohomology: $\mathcal{N}(\text{curl})/\mathcal{R}(\text{grad})$, $\mathcal{N}(\text{div})/\mathcal{R}(\text{curl})$.

The de Rham complex, as an example of more general Hilbert complexes, plays a vital role in the Finite Element Exterior Calculus (FEEC).

Elasticity complex (linearized Calabi complex)

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0$$

displacement *strain (metric)* *stress (curvature)* *force*

- $\mathbb{V}$: vectors, $\mathbb{S}$: symmetric matrices,
- linearized deformation $\quad \operatorname{def} u := 1/2(\nabla u + u\nabla)$,
- linearized curvature $\quad \operatorname{inc} v := \nabla \times v \times \nabla$,

  Saint-Venant compatibility condition: $\quad e = \operatorname{def} u \;\Rightarrow \operatorname{inc} e = 0$
- $\operatorname{inc} \circ \operatorname{def} = 0, \quad \operatorname{div} \circ \operatorname{inc} = 0$,
- classical elasticity
  - displacement formulation: displacement,
  - Hellinger-Reissner principle: stress+force,
  - intrinsic formulation: strain,

    Ciarlet, Gratie, Mardare, 2009; Ciarlet, Ciarlet, 2008 etc.
- continuum description of defects (incompatibility theory)
  - elasto-plastic decomposition: $e = e^e + e^p$, $\operatorname{inc} e^e = 0$,
  - Beltrami decomposition: $e = \operatorname{def} w + \operatorname{inc} v$.

(A little bit) more on the continuous level

Continuous level has not been clarified yet...

Basically, we have all the analogous properties as the de Rham version. (Arnold, H., *Construction of Hilbert complexes*, in preparation.)

$$\mathrm{RM} \xrightarrow{\subset} H^1(\mathbb{V}) \xrightarrow{\mathrm{def}} H(\mathrm{inc};\mathbb{S}) \xrightarrow{\mathrm{inc}} H(\mathrm{div};\mathbb{S}) \xrightarrow{\mathrm{div}} L^2(\mathbb{V}) \longrightarrow 0.$$

- cohomology is isomorphic to the de Rham cohomology $\mathscr{H}^{\bullet}_{\mathrm{dR}} \otimes (\mathbb{V} \otimes \mathbb{V})$,
- operators have closed range,
- Poincaré type inequalities (-> Korn's inequality),
- Hodge decomposition and well-posed Hodge Laplacian boundary value problems, (-> Beltrami type decomposition)
- regular decomposition,
- compactness property,
- Poincaré/Koszul operators (-> Cesàro-Volterra path integral),

Elasticity-electromagnetism analogue

Seeger, 1961, *Recent Advances in the Theory of Defects in Crystals*.

KRÖNER [13] has developed a most useful analogy between the theory of internal stresses and strains as described in sections 2 to 6 and the theory of the magnetic field of distributions of stationary electric currents. Table 1 contains a list of the corresponding physical quantities, differential operators, and equations. We hope that this table is understandable without any further comments (see also the review article by DE WIT [10]).

Table 1

Correspondences in elasticity and magnetism

Elasticity	Magnetism
vector quantity	scalar quantity
tensor rank two	vector
tensor rank four	tensor rank two
Div	div
Ink	curl
Div Ink $\equiv 0$	div curl $\equiv 0$
Def	grad
Ink Def $\equiv 0$	curl grad $\equiv 0$
Burgers vector $\boldsymbol{b}$	current I
incompatibility tensor $\boldsymbol{\eta}$	current density $\boldsymbol{J}$
strain tensor $\boldsymbol{\epsilon}$	magnetic intensity $\boldsymbol{H}$
stress tensor $\boldsymbol{\sigma}$	magnetic induction $\boldsymbol{B}$
stress function tensor $\boldsymbol{\chi}$, $\boldsymbol{\chi}'$	vector potential $\boldsymbol{A}$
elastic constants $\boldsymbol{C}$ (or G, K)	permeability μ
displacement $\boldsymbol{s}$	scalar potential ψ
equation (3)	$\boldsymbol{H} = \operatorname{grad} \psi$
equation (5)	$\operatorname{curl} \boldsymbol{H} = \boldsymbol{J}$
equation (17)	$\operatorname{div} \boldsymbol{B} = 0$
equation (18)	$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}$
equations (19), (19a)	$\nabla^2 \boldsymbol{A} = -\mu \boldsymbol{J}$
equation (20)	$\operatorname{div} \boldsymbol{A} = 0$
equation (22)	$\boldsymbol{A} = \frac{\mu}{4\pi} \iiint \frac{\boldsymbol{J}(\boldsymbol{r}')}{\lvert \boldsymbol{r} - \boldsymbol{r}' \rvert} d\tau_{r'}$

- a concrete model describing dislocation and defects (Amstutz, Van Goethem 2016):

$$\operatorname{inc}\operatorname{inc} e = f,$$
$$\operatorname{div} e = 0,$$

elasticity analogue of the Maxwell equations

$$\operatorname{curl}\operatorname{curl} E = g,$$
$$\operatorname{div} E = 0.$$

- Question: how to discretize, even in 2D?
 3D elasticity complex:

$$\mathrm{RM} \xrightarrow{\subset} H^1 \otimes \mathbb{V} \xrightarrow{\operatorname{def}} H(\operatorname{inc};\mathbb{S}) \xrightarrow{\operatorname{inc}} H(\operatorname{div};\mathbb{S}) \xrightarrow{\operatorname{div}} L^2 \otimes \mathbb{V} \longrightarrow 0,$$

 2D stress complex:

$$\mathscr{P}_1 \xrightarrow{\subset} H^2 \xrightarrow{\operatorname{airy}} H(\operatorname{div};\mathbb{S}) \xrightarrow{\operatorname{div}} L^2 \otimes \mathbb{V} \longrightarrow 0,$$

 2D strain complex:

$$\mathrm{RM} \xrightarrow{\subset} H^1 \otimes \mathbb{V} \xrightarrow{\operatorname{def}} H(\operatorname{rot}\operatorname{rot};\mathbb{S}) \xrightarrow{\operatorname{rot}\operatorname{rot}} L^2 \longrightarrow 0.$$

Existing work

$H(\mathrm{div}, \mathbb{S})$-$L^2(\mathbb{V})$ pair:

- 2D macroelements: Johnson, Mercier 1978; Arnold, Douglas, Gupta 1984
- 2D and 3D : Arnold, Winther 2002; Arnold, Awanou, Winther 2008
- nD canonical construction: Hu, Zhang 2015
- other nonconforming methods, e.g., Pechstein, Schöberl 2012, TDNNS, $H(\mathrm{div}\,\mathrm{div}; \mathbb{S})$-$H(\mathrm{curl})$

$H(\mathrm{inc})$ element fitting into a sequence:

- Regge calculus from the finite element point of view (Christiansen 2011) piecewise $\mathcal{P}^0(\mathbb{S})$, tangential-tangential continuity, highly nonconforming,
- Li 2018 thesis: higher order Regge.

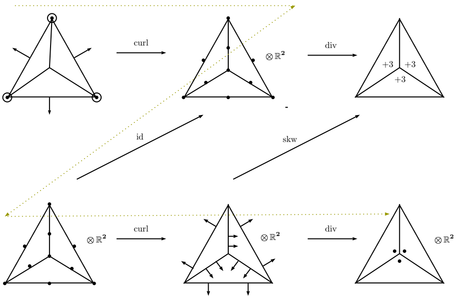
Goal of this work: first conforming 2D $H(\mathrm{inc})$ element fitting in a sequence (conforming Regge type element)

Bernstein-Gelfand-Gelfand construction: Eastwood 1999; Arnold, Falk, Winther 2006.

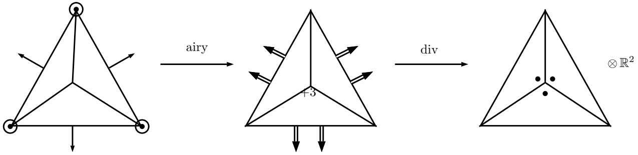
$$\mathscr{P}_1 \xrightarrow{\subset} H^2 \xrightarrow{\text{airy}} H(\mathrm{div};\mathbb{S}) \xrightarrow{\text{div}} L^2 \otimes \mathbb{V} \longrightarrow 0,$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\subset"] & H^2 \arrow[r, "\mathrm{curl}"] & H^1 \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & L^2 \arrow[r] & 0 \\
\mathbb{V} \arrow[r, "\subset"] & H^1 \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{id}"] & H(\mathrm{div}) \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{skw}"] & L^2 \otimes \mathbb{V} \arrow[r] & 0
\end{tikzcd}$$

(anti-)commuting diagram, $dS = -Sd$, bijectivity, surjectivity



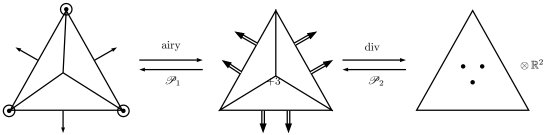
Output: Johnson-Mercier type elements.



Applying this BGG type diagram chase to other Stokes type complexes:

- Arnold-Winther pair: Arnold, Falk, Winther 2006
- Hu-Zhang pair: Christiansen, Hu, H. 2016
- other macroelements

Discrete stress complex: Poincaré operator approach



- Poincaré operators:
  - explicit potential, Poincaré lemma, canonical construction of FEs,
  - null-homotopy identity: $\mathscr{D}^{i-1}\mathscr{P}^{i} + \mathscr{P}^{i+1}\mathscr{D}^{i} = \mathrm{id}$,
  - complex property: $\mathscr{P}^{i-1}\mathscr{P}^{i} = 0$,
  - polynomial-preserving property.
- Koszul operators: Poincaré acting on homogeneous polynomials, similar properties.
- for the stress complex (Christiansen, Hu, Sande 2019):

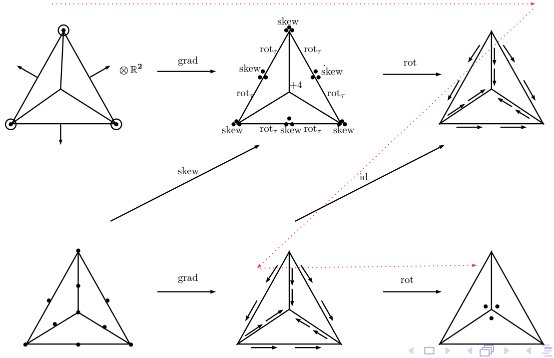
$$\mathscr{K}_r^1 u = x^\perp \cdot u \cdot x^\perp,$$

$$(\mathscr{K}_r^2 u)(x) = \mathrm{sym}\left(\frac{1}{r+2} u \otimes x + \frac{1}{(r+2)(r+3)} \mathrm{curl}\left(x^\perp \cdot ux\right)\right).$$

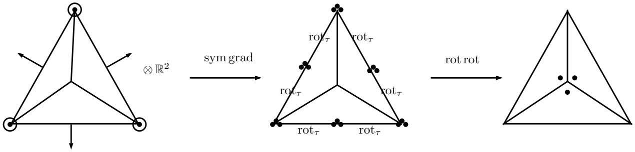
Discrete strain complex: BGG approach

$$\mathrm{RM} \xrightarrow{\subset} H^2 \otimes \mathbb{V} \xrightarrow{\mathrm{def}} H^1(\mathrm{rot}\,\mathrm{rot}; \mathbb{S}) \xrightarrow{\mathrm{rot}\,\mathrm{rot}} L^2 \longrightarrow 0.$$

$$\begin{tikzcd}
\mathbb{V} \arrow[r, "\subset"] & H^2 \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] & H^1(\mathrm{rot}\,\mathrm{rot}; \mathbb{M}) \arrow[r, "\mathrm{rot}"] & H(\mathrm{rot}) \arrow[r] & 0 \\
\mathbb{R} \arrow[r, "\subset"] & H^1 \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{skw}"] & H(\mathrm{rot}) \arrow[r, "\mathrm{rot}"] \arrow[ur, "id"] & L^2 \arrow[r] & 0
\end{tikzcd}$$



Output:



Equivalent edge DOFs: $\int_e \operatorname{rot} u \cdot \tau \Longleftrightarrow \int_e \partial_n(\tau \cdot u \cdot \tau)$ for $u \in C^0\mathcal{P}^2$.

Discrete strain complex: lower regularity

- Koszul operators for the strain complex:

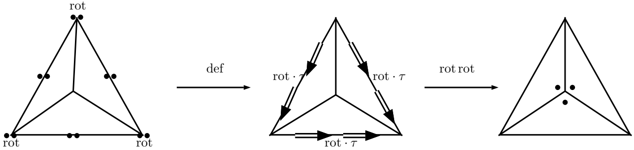
$$\mathscr{K}_1^r(E) = \frac{1}{r+1} E \cdot x + \frac{1}{(r+1)(r+2)} x \wedge (\mathrm{rot}\, E) \cdot x : C^\infty(\mathbb{S}) \mapsto C^\infty(\mathbb{V}),$$

$$\mathscr{K}_2^r(V) = \frac{1}{(r+2)(r+3)} x^\perp \otimes V \otimes x^\perp : C^\infty(\mathbb{S}) \mapsto C^\infty(\mathbb{S}).$$

- stain complexes with lower regularity and fewer DOFs are possible:

$$\mathbb{V} \xrightarrow{\subset} H^1(\mathrm{rot}) \xrightarrow{\mathrm{def}} H(\mathrm{rot};\mathbb{S}) \cap H(\mathrm{rot}\,\mathrm{rot};\mathbb{S}) \xrightarrow{\mathrm{rot}\,\mathrm{rot}} L^2 \longrightarrow 0$$

- can also be obtained by a diagram chase.



Conclusions

- Reference:
    - Finite element systems for vector bundles: elasticity and curvature; Christiansen, H., *arXiv:1906.09128*.
    - Poincaré path integrals for elasticity; Christiansen, H., Sande, *Journal de Mathématiques Pures et Appliquées*, accepted, 2019
    - Construction of Hilbert complexes; Arnold, H., in preparation.
- discrete BGG diagram chase is based on several Stokes type complexes with various regularity, which is an important topic by itself,
- 3D is ongoing,
- comparison to discrete differential geometry and potential applications in defect theory, geometric problems (Ricci flows, Einstein equations etc.).