Tensor product finite element BGG complexes

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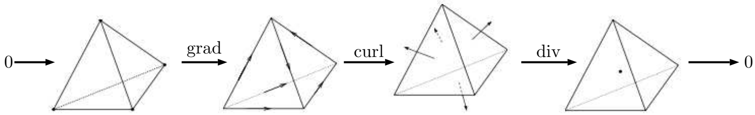
Basic homological algebra

$$\cdots \longrightarrow V^{i-1} \xrightarrow{d^{i-1}} V^{i} \xrightarrow{d^{i}} V^{i+1} \longrightarrow \cdots$$

V^i: vector spaces, d^i: linear (or nonlinear) operators

- complex: $d^i V^i \subset V^{i+1}$, $d^{i+1} \circ d^i = 0$, $\forall i$, (implies $\mathcal{R}(d^{i-1}) \subset \mathcal{N}(d^i)$)
- exact: $\mathcal{N}(d^i) = \mathcal{R}(d^{i-1})$,
- cohomology (when d is linear): $\mathscr{H}^i := \mathcal{N}(d^i)/\mathcal{R}(d^{i-1})$.

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0.$$



Raviart-Thomas, Nédélec in numerical analysis, Whitney forms for topology.

Finite element exterior calculus (FEEC): cohomological framework for studying numerical methods. (c.f., Arnold, Falk, Winther 2006, Arnold 2018)

Elasticity: deformation and mechanics of solids

elasticity equation:

$$-\operatorname{div}(A \operatorname{def} u) = f.$$

u	displacement (vector),
$e := \operatorname{def} u := 1/2(\nabla u + \nabla u^T)$	strain (linearized deformation),
$\sigma := A \operatorname{def} u$	stress.

analogy to Poisson equation:

$$-\operatorname{div}(A \operatorname{grad} v) = g.$$

A cohomological approach: elasticity complex

displacement formulation

Kröner's continuum description of dislocations/defects, internal stress

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0$$

displacement *strain (metric)* *stress (curvature)* *force*

intrinsic elasticity (Ciarlet et al.)

Hellinger-Reissner principle of elasticity

$$\mathbb{V} := \mathbb{R}^3 \text{ vectors}, \quad \mathbb{S} := \mathbb{R}^{3\times 3}_{\mathrm{sym}} \text{ symmetric matrices}$$

$$\operatorname{def} u := 1/2(\nabla u + \nabla u^T), \quad (\operatorname{def} u)_{ij} = 1/2(\partial_i u_j + \partial_j u_i).$$

$$\operatorname{inc} g := \nabla \times g \times \nabla, \quad (\operatorname{inc} g)^{ij} = \epsilon^{ikl}\epsilon^{jst}\partial_k \partial_s g_{lt}.$$

$$\operatorname{div} v := \nabla \cdot v, \quad (\operatorname{div} v)_i = \partial^j u_{ij}.$$

g metric $\Rightarrow$ $\operatorname{inc} g$ linearized Einstein tensor ($\backsimeq \mathrm{Riem} \backsimeq \mathrm{Ric}$ in 3D)

$\operatorname{inc} \circ \operatorname{def} = 0$: Saint-Venant compatibility

$\operatorname{div} \circ \operatorname{inc} = 0$: Bianchi identity

Bernstein-Gelfand-Gelfand (BGG) construction:

Eastwood 1999, Čap, Slovák, Souček 2001, Arnold, Falk, Winther 2006, Arnold, Hu 2021, Čap, Hu 2022.

Continuous level

$$0 \longrightarrow H^2 \xrightarrow{\partial_x^2} L^2 \longrightarrow 0.$$

$$\begin{tikzcd}
0 \arrow[r] & H^2 \arrow[r, "\partial_x"] & H^1 \arrow[r] & 0 \\
0 \arrow[r] & H^1 \arrow[r, "\partial_x"] \arrow[ur, "I"] & L^2 \arrow[r] & 0.
\end{tikzcd}$$

- two de-Rham complexes with different continuity,
- cohomology: $\mathcal{N}(\partial_x^2) \cong \mathcal{N}(\partial_x) \oplus \mathcal{N}(\partial_x)$, ∂_x^2 is onto.

Algebraic and analytic construction (Arnold, KH 2021): derive elasticity from deRham

$$
\begin{tikzcd}
0 \arrow[r] & H^s\otimes\mathbb{V} \arrow[r, "\mathrm{grad}"] & H^{s-1}\otimes\cancel{\mathbb{M}}\,\mathbb{S} \arrow[r, "\mathrm{curl}"] & H^{s-2}\otimes\mathbb{M} \arrow[r, "\mathrm{div}"] & \cancel{H^{s-3}\otimes\mathbb{V}} \arrow[r] & 0 \\
0 \arrow[r] & \cancel{H^{s-1}\otimes\mathbb{V}} \arrow[r, "\mathrm{grad}"'] \arrow[ur, "S^0:=\mathrm{mskw}"] & H^{s-2}\otimes\mathbb{M} \arrow[r, "\mathrm{curl}"'] \arrow[ur, "S^1"] & H^{s-3}\otimes\cancel{\mathbb{M}}\,\mathbb{S} \arrow[r, "\mathrm{div}"'] \arrow[ur, "S^2:=\mathrm{vskw}"] & H^{s-4}\otimes\mathbb{V} \arrow[r] & 0.
\end{tikzcd}
$$

$S^1 u := u^T - \mathrm{tr}(u)I.$

output: elasticity complex

$$
\begin{tikzcd}
0 \arrow[r] & H^s\otimes\mathbb{V} \arrow[r, "\mathrm{def}"] & H^{s-1}\otimes\mathbb{S} \arrow[r, "\mathrm{curl}"] & {} \arrow[dl, "\mathrm{T}"] & & & \\
& & {} \arrow[r, "\mathrm{curl}"] & H^{s-3}\otimes\mathbb{S} \arrow[r, "\mathrm{div}"] & H^{s-4}\otimes\mathbb{V} \arrow[r] & 0.
\end{tikzcd}
$$

de Rham results + homological algebra $\quad\Rightarrow\quad$ elasticity/geometry results

Consequence: cohomology of derived complex is isomorphic to de Rham
Info of cohomology implies analytic results: Poincaré-Korn inequalities, compactness...

Inspired by the Bernstein-Gelfand-Gelfand (BGG) resolution
c.f., Eastwood 2000, Čap, Slovák, Souček 2001, Arnold, Falk, Winther 2006.

A general picture

- input: $(Z^\bullet, D^\bullet)$, $(\tilde{Z}^\bullet, \tilde{D}^\bullet)$, connecting maps $S^i : \tilde{Z}^i \to Z^{i+1}$, satisfying
  - (anti-)commutativity: $S^{i+1}\tilde{D}^i = -D^{i+1}S^i$,
  - injectivity/surjectivity condition: S^i injective for $i \leq J$, surjective for $i \geq J$.

$$
\begin{tikzcd}
0 \arrow[r] & Z^0 \arrow[r, "D^0"] & Z^1 \arrow[r, "D^1"] & \cdots \arrow[r, "D^{n-1}"] & Z^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{Z}^0 \arrow[r, "\tilde{D}^0"] \arrow[ur, "S^0"] & \tilde{Z}^1 \arrow[r, "\tilde{D}^1"] \arrow[ur, "S^1"] & \cdots \arrow[r, "\tilde{D}^{n-1}"] \arrow[ur, "S^{n-1}"] & \tilde{Z}^n \arrow[r] & 0
\end{tikzcd}
$$

- output:

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathrm{coker}(S^{J-2}) \arrow[r, "D^{J-1}"] & \mathrm{coker}(S^{J-1}) \arrow[dr, "D^J", "(S^J)^{-1}" description, "\tilde{D}^J"'] & & \\
& & & \mathcal{N}(S^{J+1}) \arrow[r, "\tilde{D}^{J+1}"] & \mathcal{N}(S^{J+2}) \arrow[r, "\tilde{D}^{J+2}"] & \cdots
\end{tikzcd}
$$

- conclusion:

$$
\dim \mathscr{H}^i\left(\Upsilon^\bullet, \mathscr{D}^\bullet\right) \leq \dim \mathscr{H}^i\left(Z^\bullet, D^\bullet\right) + \dim \mathscr{H}^i\left(\tilde{Z}^\bullet, \tilde{D}^\bullet\right), \quad \forall i = 0, 1, \cdots, n
$$

  Equality holds if and only if S^i induces the zero maps on cohomology, i.e., $S^i\mathcal{N}(\tilde{D}^i) \subset \mathcal{R}(D^i)$.

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "I"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "I"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

Hessian complex:

$$
0 \longrightarrow H^q \otimes \mathbb{R} \xrightarrow{\mathrm{hess}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\mathrm{curl}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\mathrm{div}} H^{q-4} \otimes \mathbb{V} \longrightarrow 0.
$$

biharmonic equations, plate theory, Einstein-Bianchi system of general relativity

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "I"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "I"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

elasticity complex:

$$
0 \longrightarrow H^{q-1} \otimes \mathbb{V} \xrightarrow{\mathrm{def}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\mathrm{inc}} H^{q-4} \otimes \mathbb{S} \xrightarrow{\mathrm{div}} H^{q-5} \otimes \mathbb{V} \longrightarrow 0.
$$

elasticity, defects, metric, curvature

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "I"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\,\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "I"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

divdiv complex:

$$0 \longrightarrow H^{q-2} \otimes \mathbb{V} \xrightarrow{\mathrm{dev\,grad}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\mathrm{sym\,curl}} H^{q-4} \otimes \mathbb{S} \xrightarrow{\mathrm{div\,div}} H^{q-6} \otimes \mathbb{R} \longrightarrow 0.$$

plate theory, elasticity

Additional motivation for discretizing the entire BGG diagram: twisted complexes

BGG diagram

$$\begin{tikzcd}
 \Lambda^0\otimes\mathbb{R}^3 \arrow[r, "\mathrm{grad}"] & \Lambda^1\otimes\mathbb{R}^3 \arrow[r, "\mathrm{curl}"] & \Lambda^2\otimes\mathbb{R}^3 \arrow[r, "\mathrm{div}"] & \Lambda^3\otimes\mathbb{R}^3 \\
 \Lambda^0\otimes\mathbb{R}^3 \arrow[r, "\mathrm{grad}"] \arrow[ur, "\text{-mskw}"] & \Lambda^1\otimes\mathbb{R}^3 \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & \Lambda^2\otimes\mathbb{R}^3 \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{vskw}"] & \Lambda^3\otimes\mathbb{R}^3
\end{tikzcd}$$

twisted complex

$$\begin{array}{c} \text{displacement} \\ \begin{bmatrix}\Lambda^0\otimes\mathbb{R}^3\\ \\ \Lambda^0\otimes\mathbb{R}^3\end{bmatrix} \\ \text{rotation}\end{array}
\xrightarrow{\begin{bmatrix}\mathrm{grad} & \mathrm{mskw}\\ & \mathrm{grad}\end{bmatrix}}
\begin{array}{c} \text{coframe} \\ \begin{bmatrix}\Lambda^1\otimes\mathbb{R}^3\\ \\ \Lambda^1\otimes\mathbb{R}^3\end{bmatrix} \\ \text{connection 1-form}\end{array}
\xrightarrow{\begin{bmatrix}\mathrm{curl} & \text{-S}\\ & \mathrm{curl}\end{bmatrix}}
\begin{array}{c} \text{torsion} \\ \begin{bmatrix}\Lambda^2\otimes\mathbb{R}^3\\ \\ \Lambda^2\otimes\mathbb{R}^3\end{bmatrix} \\ \text{(Riemann-Cartan) curvature}\end{array}
\xrightarrow{\begin{bmatrix}\mathrm{div} & \text{-vskw}\\ & \mathrm{div}\end{bmatrix}}
\begin{bmatrix}\Lambda^3\otimes\mathbb{R}^3\\ \\ \Lambda^3\otimes\mathbb{R}^3\end{bmatrix}$$

Cosserat elasticity

Cosserat with defects

BGG complex

$$\underbrace{\Lambda^0\otimes\mathbb{R}^3 \xrightarrow{\mathrm{deff}} \overset{\text{metric}}{(\Lambda^1\otimes\mathbb{R}^3)\cap\mathbb{S}}}_{\text{elasticity}} \xrightarrow{\mathrm{inc}} \underbrace{\overset{\text{(Riemann) curvature}}{(\Lambda^2\otimes\mathbb{R}^3)\cap\mathbb{S}}}_{\text{elasticity with defects}} \xrightarrow{\mathrm{div}} \Lambda^3\otimes\mathbb{R}^3$$

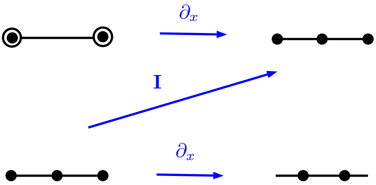
In 1D, 2D, 3D (Čap-KH 2022):

- twisted complexes: Timoshenko beam, Reissner-Mindlin plate, Cosserat elasticity
- BGG complexes: Euler-Bernoulli beam, Kirchhoff-Love plate, standard elasticity.

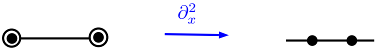
Discretization of complexes:

- 2D stress: Arnold-Winther 2002, J.Hu-S.Zhang 2014, Christiansen-KH 2018,
- 2D strain: Chen-J.Hu-Huang 2014 (Regge/HHJ), Christiansen-KH 2018 (conforming), Chen-Huang 2020, DiPietro-Droniou 2021 (polygonal meshes)
- 3D elasticity: various results on last part of complex, Hauret-Kuhl-Ortiz 2007 (discrete geometry/mechanics), Arnold-Awanou-Winther 2008, Christiansen 2011 (Regge), Christiansen-Gopalakrishnan-Guzmán-Hu 2020, Chen-Huang 2021
- 3D Hessian: Chen-Huang 2020, J.Hu-Liang 2021, Arf-Simeon 2021 (splines)
- 3D divdiv: Chen-Huang 2021, J.Hu-Liang-Ma 2021, Sander 2021 ($H(\mathrm{sym\,curl})$, $H(\mathrm{dev\,sym\,curl})$), J.Hu-Liang-Ma-Zhang 2022
- nD: Chen-Huang 2021 (last two spaces), 2D arbitrary regularity: Chen-Huang 2022
- conformal complexes: open.

Discrete level



implies



Two dimensions: continuous level

$\mathbb{R}$: scalars, $\mathbb{V}$: vectors, $\mathbb{M}$: matrices, $\mathbb{S}$: symmetric matrices

$$\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{rot}"] & H^{q-2} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "I"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{rot}"] \arrow[ur, "-2\,\mathrm{sskw}"] & H^{q-3} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{V} \arrow[r, "\mathrm{rot}"] \arrow[ur, "\mathrm{id}"] & H^{q-4} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}$$

The output complexes using two consecutive rows:

$$0 \longrightarrow H^q \xrightarrow{\mathrm{hess}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\mathrm{rot}} H^{q-3} \otimes \mathbb{V} \longrightarrow 0,$$

$$0 \longrightarrow H^{q-1} \otimes \mathbb{V} \xrightarrow{\mathrm{def}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\mathrm{rot\,rot}} H^{q-4} \longrightarrow 0.$$

Input: two 1D diagrams, $i = 1, 2$, $\mathcal{S}^q_r$: spline space with regularity index r, polynomial degree q.

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{S}^{p_i}_{r_i}(\mathcal{I}) \arrow[r, "\partial"] & \mathcal{S}^{p_i-1}_{r_i-1}(\mathcal{I}) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{S}^{p_i-1}_{r_i-1}(\mathcal{I}) \arrow[r, "\partial"] \arrow[ur, "I"] & \mathcal{S}^{p_i-2}_{r_i-2}(\mathcal{I}) \arrow[r] & 0,
\end{tikzcd}
$$

General patterns: $\mathcal{S}^{p_1,p_2}_{r_1,r_2} := \mathcal{S}^{p_1}_{r_1} \otimes \mathcal{S}^{p_2}_{r_2}$, spline de Rham: Buffa-Rivas-Sangalli-Vazquez 2011

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{S}^{p_1,p_2}_{r_1,r_2} \arrow[r, "\mathrm{grad}"] & \begin{pmatrix} \mathcal{S}^{p_1-1,p_2}_{r_1-1,r_2} \\ \mathcal{S}^{p_1,p_2-1}_{r_1,r_2-1} \end{pmatrix} \arrow[r, "\mathrm{rot}"] & \mathcal{S}^{p_1-1,p_2-1}_{r_1-1,r_2-1} \arrow[r] & 0 \\
0 \arrow[r] & \begin{pmatrix} \mathcal{S}^{p_1-1,p_2}_{r_1-1,r_2} \\ \mathcal{S}^{p_1,p_2-1}_{r_1,r_2-1} \end{pmatrix} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{I}"] & \begin{pmatrix} \mathcal{S}^{p_1-2,p_2}_{r_1-2,r_2} & \mathcal{S}^{p_1-1,p_2-1}_{r_1-1,r_2-1} \\ \mathcal{S}^{p_1-1,p_2-1}_{r_1-1,r_2-1} & \mathcal{S}^{p_1,p_2-2}_{r_1,r_2-2} \end{pmatrix} \arrow[r, "\mathrm{rot}"] \arrow[ur, "\mathrm{sskw}"] & \begin{pmatrix} \mathcal{S}^{p_1-2,p_2-1}_{r_1-2,r_2-1} \\ \mathcal{S}^{p_1-1,p_2-2}_{r_1-1,r_2-2} \end{pmatrix} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{S}^{p_1-1,p_2-1}_{r_1-1,r_2-1} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"] & \begin{pmatrix} \mathcal{S}^{p_1-2,p_2-1}_{r_1-2,r_2-1} \\ \mathcal{S}^{p_1-1,p_2-2}_{r_1-1,r_2-2} \end{pmatrix} \arrow[r, "\mathrm{rot}"] \arrow[ur, "S"] & \mathcal{S}^{p_1-2,p_2-2}_{r_1-2,r_2-2} \arrow[r] & 0.
\end{tikzcd}
$$

- first row, first column: de Rham complexes,
- regularity decreases in each row and in each column, (larger form degrees correspond to lower regularity and polynomial degree)
- then diagonal S operators match spaces well.

General tensor product construction: any dimension and degree

$$\mathscr{S}^{\boldsymbol{p}}_{\boldsymbol{r}}\Lambda^{I,J} := \oplus_{(i_i,\cdots,i_n)\in\chi_I,(j_1,\cdots,j_n)\in\chi_J}(\mathcal{S}^{p_1-i_1-j_1}_{r_1-i_1-j_1}\Lambda^{i_1,j_1}\otimes\cdots\otimes\mathcal{S}^{p_n-i_n-j_n}_{r_n-i_n-j_n}\Lambda^{i_n,j_n}),$$

where $\mathcal{I} := [0,1]$ (1D domain), $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$,

$$\mathcal{S}^{p_l-i_l-j_l}_{r_l-i_l-j_l}\Lambda^{i,j}(\mathcal{I}) := \mathcal{S}^{p_l-i_l-j_l}_{r_l-i_l-j_l}\otimes\mathrm{Alt}^i\otimes\mathrm{Alt}^j.$$

BGG diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathscr{S}^{\boldsymbol{p}}_{\boldsymbol{r}}\Lambda^{I-1,J-1} \arrow[r, "d"] & \mathscr{S}^{\boldsymbol{p}}_{\boldsymbol{r}}\Lambda^{I,J-1} \arrow[r, "d"] & \mathscr{S}^{\boldsymbol{p}}_{\boldsymbol{r}}\Lambda^{I+1,J-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathscr{S}^{\boldsymbol{p}}_{\boldsymbol{r}}\Lambda^{I-1,J} \arrow[r, "d"] \arrow[ur, "S^{I-1,J}"] & \mathscr{S}^{\boldsymbol{p}}_{\boldsymbol{r}}\Lambda^{I,J} \arrow[r, "d"] \arrow[ur, "S^{I,J}"] & \mathscr{S}^{\boldsymbol{p}}_{\boldsymbol{r}}\Lambda^{I+1,J} \arrow[r] & \cdots .
\end{tikzcd}$$

Derivation of BGG complexes: same as continuous level, consisting of kernels and cokernels of S.

The construction of degrees of freedom also comes from tensor products.

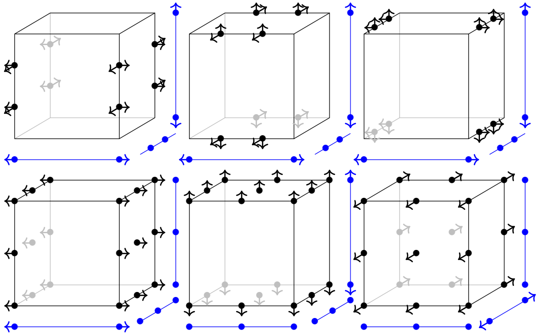


Figure: Degrees of freedom for the lowest order strain element $V^{1,1}$ of the elasticity complex. Diagonal entries $\sigma_{11}, \sigma_{22}, \sigma_{33}$ in the top row. Off-diagonal entries $\sigma_{23} = \sigma_{32}$, $\sigma_{13} = \sigma_{31}$, and $\sigma_{12} = \sigma_{21}$ in the bottom row. Pairs of arrows indicate first order and mixed second order derivatives.

Special cases (div div complex) coincide with J.Hu, Y.Liang, R.Ma, M.Zhang 2022.

References:

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- *BGG sequences with weak regularity and applications*, Andreas Čap, KH; *arXiv:2203.01300* (2022) more general framework, conformal complexes, applications
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- *Spline and tensor product finite element BGG complexes*, Francesca Bonizzoni, KH, Guido Kanschat, Duygu Sap; *in preparation*. any dimension, any degree