Conformal Korn inequality in 2D

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On a bounded Lipschitz domain:

- Poincaré inequalities:
$$\|u\|_{H^1} \leq \|\operatorname{grad} u\|_{L^2}, \quad u \perp \mathcal{N}(\operatorname{grad}).$$
Trivial generalization: vector-valued functions (9 components in grad u).
- Korn inequality:
$$\|u\|_{H^1} \leq C\|\operatorname{sym}\operatorname{grad} u\|_{L^2}, \quad u \perp \mathcal{N}(\operatorname{sym}\operatorname{grad}).$$
- Conformal Korn inequality:
$$\|u\|_{H^1} \leq C\|\operatorname{dev}\operatorname{sym}\operatorname{grad} u\|_{L^2}, \quad u \perp \mathcal{N}(\operatorname{dev}\operatorname{sym}\operatorname{grad}),$$
holds in nD for $n \geq 3$! ($\operatorname{dev} w := w - \frac{1}{n}\operatorname{tr}(w)I$)

Fails in 2D:
$$\operatorname{dev}\operatorname{sym}\operatorname{grad} u = \begin{pmatrix} \frac{1}{2}(\partial_x u_1 - \partial_y u_2) & \frac{1}{2}(\partial_y u_1 + \partial_x u_2) \\ \frac{1}{2}(\partial_y u_1 + \partial_x u_2) & -\frac{1}{2}(\partial_x u_1 - \partial_y u_2) \end{pmatrix}$$
Cauchy-Riemann operator

Applications in GR, failure in 2D: Dain, S. (2006). Generalized Korn's inequality and conformal Killing vectors. Calculus of variations and partial differential equations, 25(4), 535-540.

An algebraic proof: closed range implies inequalities (Banach theorem), range=kernel.

- Poincaré inequalities: $\mathrm{curl} \circ \mathrm{grad} = 0$, $\mathcal{R}(\mathrm{grad}) = \mathcal{N}(\mathrm{curl})$
  de Rham complex, topology

- Korn inequality: $(\mathrm{curl} \circ \mathrm{T} \circ \mathrm{curl}) \circ (\mathrm{sym}\,\mathrm{grad}) = 0$, $\mathcal{R}(\mathrm{sym}\,\mathrm{grad}) = \mathcal{N}(\mathrm{curl} \circ \mathrm{T} \circ \mathrm{curl})$
  Einstein (Ricci/Riemann) tensor, Riemannian geometry

- Conformal Korn inequality (3D):
  $(\mathrm{curl} \circ S^{-1} \circ \mathrm{curl} \circ S^{-1} \circ \mathrm{curl}) \circ (\mathrm{dev}\,\mathrm{sym}\,\mathrm{grad}) = 0$, $\mathcal{R}(\mathrm{sym}\,\mathrm{grad}) = \mathcal{N}(\mathrm{curl} \circ \mathrm{T} \circ \mathrm{curl})$
  Cotton-York tensor, conformal geometry

  (assume contractible domain. generalization to finite dimensional cohomology is similar)

  Failure of the 2D conformal Korn inequality means that the Cauchy-Riemann operator does not have closed range. How to fix?

$$\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{M} \arrow[r, "\mathrm{rot}"] & H^{q-2} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes (\mathbb{R} \times \mathbb{R}) \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^{0,1}"] & H^{q-2} \otimes (\mathbb{V} \times \mathbb{V}) \arrow[r, "\mathrm{rot}"] \arrow[ur, "S^{1,1}"] & H^{q-3} \otimes (\mathbb{R} \times \mathbb{R}) \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^{0,2}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{rot}"] \arrow[ur, "S^{1,2}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0.
\end{tikzcd}$$

Here $S^{0,1} = -\perp$, $S^{1,1} = \mathrm{tr}$, $S^{0,2} = -\iota$, and $S^{1,2} = I$.
$\mathbb{V}$: vector, $\mathbb{M}$: matrix, $\mathbb{S}$: symmetric matrix, $\mathbb{T}$: tracefree matrix

Horizontal: de Rham complex. Diagonal: Lie algebra (co)homology.

Output:

$$0 \longrightarrow H^q \otimes \mathbb{V} \xrightarrow{D^0} \begin{pmatrix} H^{q-1} \otimes (\mathbb{S} \cap \mathbb{T}) \\ H^{q-3} \otimes (\mathbb{S} \cap \mathbb{T}) \end{pmatrix} \xrightarrow{D^1} H^{q-3} \longrightarrow 0,$$

where

$$D^0 := \begin{pmatrix} \mathrm{dev}\,\mathrm{def}\,\mathrm{grad} \\ \mathrm{grad}\, T^{1,1}\, \mathrm{grad}\, T^{1,0}\, \mathrm{grad} \end{pmatrix}, \quad D^1 = \begin{pmatrix} \mathrm{rot}\, T^{2,1}\, \mathrm{rot}\, T^{2,0}\, \mathrm{rot} \\ \mathrm{rot} \end{pmatrix}^T.$$

$T^{1,0} = (\frac{1}{2}\mathrm{tr}, -\mathrm{sskw})$, $T^{2,0} = \frac{1}{2}(\perp, I)$, $T^{1,1} = \frac{1}{2}(I, -\perp)^T$, and $T^{2,1} = (-\mathrm{mskw}, -\frac{1}{2}\iota)$

2D conformal Korn inequality:

$$\|u\|_3 \le C(\|\mathrm{dev}\,\mathrm{def}\,\mathrm{grad}\, u\|_2 + \|\mathrm{grad}\, T^{1,1}\, \mathrm{grad}\, T^{1,0}\, \mathrm{grad}\, u\|),$$
$$\forall u \perp \mathcal{N}(\mathrm{dev}\,\mathrm{def}\,\mathrm{grad}) \cap \mathcal{N}(\mathrm{grad}\, T^{1,1}\, \mathrm{grad}\, T^{1,0}\, \mathrm{grad}).$$

Möbius structure.

Open problem: minimal number of linear functionals l_i, s.t. generalized Korn inequality holds

$$\|u\|_{H^1} \leq C\left(\sum_{i=1}^{N} \|l_i(\nabla u)\|_{L^2} + \|u\|_{L^2}\right).$$

e.g., 3D Poincaré: N=9; Korn: N= 6; trace-free Korn: N=5.

Chipot, M. (2021). *On inequalities of Korn's type.* Journal de Mathématiques Pures et Appliquées, 148, 199-220.

References:

- *Complexes from complexes*, Douglas Arnold, Kaibo Hu; *Foundations of Computational Mathematics* (2021). framework, analytic results from homological algebraic structures
- *BGG sequences with weak regularity and applications*, Andreas Čap, Kaibo Hu; *arXiv:2203.01300* (2022)
  more general framework that implies the 2D conformal Korn inequality, more examples.