Construction of Hilbert complexes

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Review: de Rham complexes

• Sobolev de Rham complex:

$$0 \longrightarrow H^s \Lambda^0 \xrightarrow{d^0} H^{s-1} \Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H^{s-n} \Lambda^n \longrightarrow 0,$$

Theorem 1 (Costabel, McIntosh, 2010).

$$\mathcal{N}\left(d^k, H^{s-k}\Lambda^k\right) = d^{k-1}H^{s-(k-1)}\Lambda^{k-1} \oplus \mathcal{H}_{\infty}^k,$$

- Lipschitz domain, any real number s,
- cohomology representation $\mathcal{H}_{\infty}^k \subset C^{\infty}$, not depending on s.

Idea of proof: estimating Bogovskiĭ and regularized Poincaré path integral operators as pseudodifferential operators of order -1.

• L^2 Hilbert complex

$$0 \longrightarrow L^2 \Lambda^0 \xrightarrow{d^0} L^2 \Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} L^2 \Lambda^n \longrightarrow 0.$$

with the domain complex $(H(\mathcal{D}))$ type de Rham complex):

$$0 \longrightarrow H\Lambda^0 \xrightarrow{d^0} H\Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H\Lambda^n \longrightarrow 0,$$

where $H\Lambda^k:=\{u\in L^2\Lambda^k: d^ku\in L^2\Lambda^{k+1}\}.$ $(H(\operatorname{curl})/H(\operatorname{div}) \text{ type spaces})$

Analytic results hold: closed range, Hodge decomposition, Poincaré inequalities etc., which lay the foundation of the Finite Element Exterior Calculus (Arnold, Falk, Winther 2006) and applications.

Question: elasticity complex (Kröner/linearized Calabi complex)

 $\overset{\text{displacement formulation}}{/} \overset{\text{Kröner's continuum description of defects,}}{} \overset{\text{internal stress}}{/} \\ RM \overset{\subseteq}{\longrightarrow} C^{\infty}(\Omega; \mathbb{V}) \overset{\text{def}}{\longrightarrow} C^{\infty}(\Omega; \mathbb{S}) \overset{\text{inc}}{\longrightarrow} C^{\infty}(\Omega; \mathbb{S}) \overset{\text{div}}{\longrightarrow} C^{\infty}(\Omega; \mathbb{V}) \longrightarrow 0$



intrinsic elasticity (Ciarlet et al.) Hellinger-Reissner principle of elasticity

- V: vectors, S: symmetric matrices,
- linearized deformation $\det u := 1/2(\nabla u + u\nabla)$, linearized curvature $\operatorname{inc} v := \nabla \times v \times \nabla$,
- $\operatorname{inc} \circ \operatorname{def} = 0$, $\operatorname{div} \circ \operatorname{inc} = 0$,
- Saint-Venant compatibility condition: $e = \operatorname{def} u \Rightarrow \operatorname{inc} e = 0$,

Functional analysis foundation for L^2 and Sobolev complexes?

Solution: algebraic construction of elasticity complex from de Rham complexes

Bernstein-Gelfand-Gelfand type construction:

(Eastwood 1999; Arnold, Falk, Winther 2006, for C^{∞} functions)

$$0 \longrightarrow H^{s} \otimes \mathbb{V} \xrightarrow{\operatorname{grad}} H^{s-1} \otimes \mathbb{M} \xrightarrow{\operatorname{curl}} H^{s-2} \otimes \mathbb{M} \xrightarrow{\operatorname{div}} H^{s-3} \otimes \mathbb{V} \longrightarrow 0$$

$$S^{0} := \operatorname{mskw} \longrightarrow S^{1} \longrightarrow S^{2} := \operatorname{vskw} \longrightarrow 0$$

$$0 \longrightarrow H^{s-1} \otimes \mathbb{V} \xrightarrow{\operatorname{grad}} H^{s-2} \otimes \mathbb{M} \xrightarrow{\operatorname{curl}} H^{s-3} \otimes \mathbb{M} \xrightarrow{\operatorname{div}} H^{s-4} \otimes \mathbb{V} \longrightarrow 0.$$

- M: matrices, skw: $\mathbb{M} \mapsto \mathbb{K}$: skew symmetrization, mskw: $\mathbb{V} \mapsto \mathbb{K}$: axial vector to skew symmetric matrix, vskw:= mskw⁻¹ \circ skw: $\mathbb{M} \mapsto \mathbb{V}$, $S^1 u := u^T \operatorname{tr}(u)I$,
- key structures:
- DS = -SD: curl mskw = $-S^1$ grad, div $S^1 = -$ vskw curl,
- S^i operators, i = 0, 1, 2: injective, bijective, surjective,

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• compactness property: the following imbedding is compact:

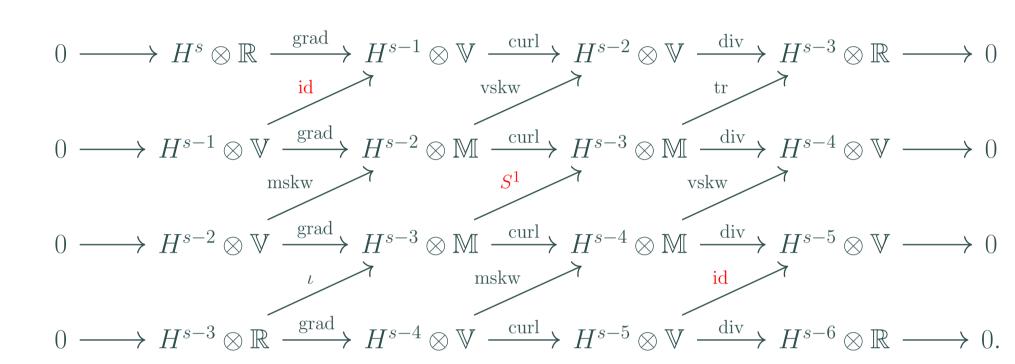
$$H\left(\mathscr{D}^i; \mathbb{W}^i\right) \cap D\left(\mathscr{D}_i^*\right) \hookrightarrow L^2 \otimes \mathbb{W}^i,$$

where $D\left(\mathcal{D}_{i}^{*}\right)$ is the domain of the adjoint operator in the sense of unbounded operators. Proof: existence of regular potential (Sobolev complex) and standard Rellich compactness of H^{1} .

More examples

 \mathbb{T} : trace-free matrices, $t: \text{trace}, \ \iota: \mathbb{R} \mapsto \mathbb{M}, w \mapsto wI, \ S^1u := u^T - \text{tr}(u)I$ $\text{dev}u := u - 1/n \, \text{tr}(u)I$: deviator, hess := grad grad: Hessian.

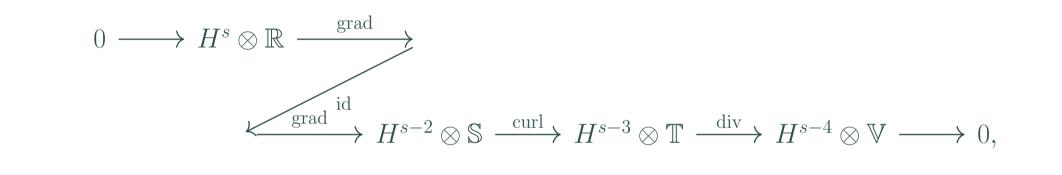
Input: Arnold, Lectures at Peking University, 2015 (with C^{∞} functions)



Diagonal maps are isomorphism, subdiagonal maps are injective, superdiagonal maps are surjective.

Output

• Hessian complex (1st + 2nd rows): hess = grad oid o grad, Einstein eqn. (Quenneville-Bélair, 2015)



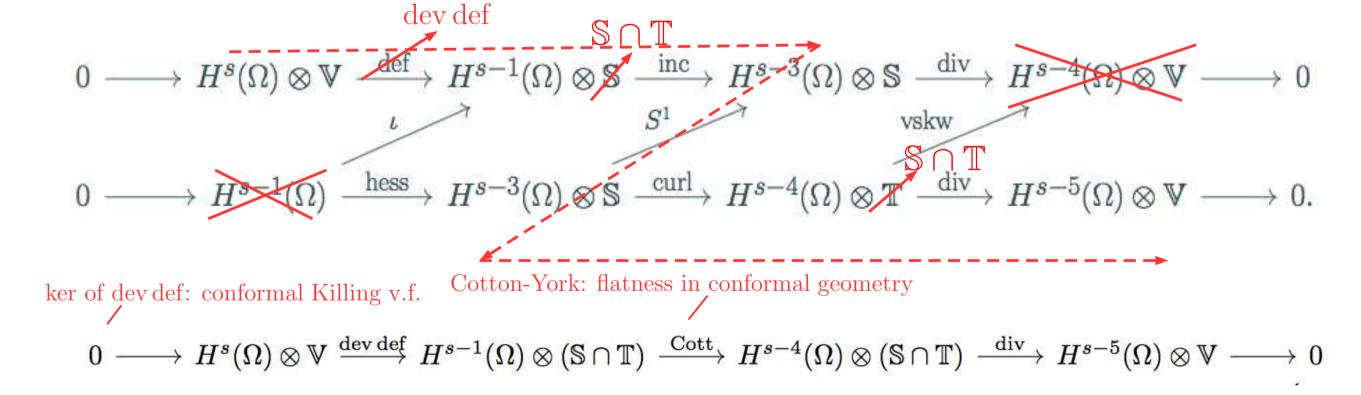
- elasticity/Kröner/linearized Calabi complex (2nd + 3rd rows),
- div div complex (3rd + 4th rows):

$$0 \longrightarrow H^{s-2} \otimes \mathbb{V} \xrightarrow{\operatorname{dev} \operatorname{grad}} H^{s-3} \otimes \mathbb{T} \xrightarrow{\operatorname{sym} \operatorname{curl}} H^{s-4} \otimes \mathbb{S} \xrightarrow{\operatorname{div}} H^{s-6} \otimes \mathbb{R} \longrightarrow 0.$$

Stokes (Gopalakrishnan, Lederer, Schöberl, 2018&2019), biharmonic eqn (Pauly, Zulerhner, 2018).

• 1st + 3rd: grad curl (with connections to curl curl); 2nd + 4th: curl div; 1st + 4th: grad div.

Example of iterative construction



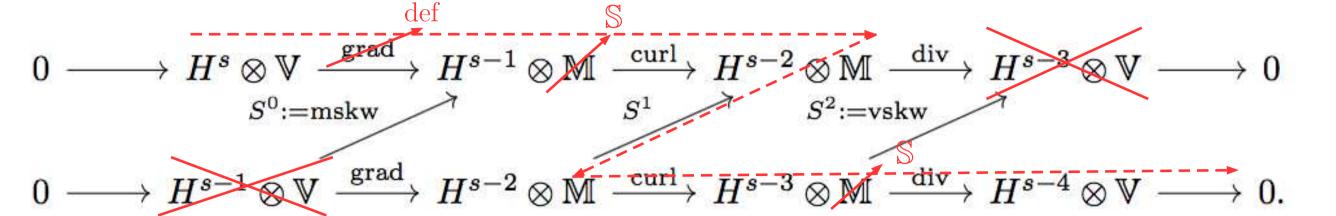
gravitational wave variable: transverse-traceless (TT) gauge $\$ stress like variable defu in NS (= symmetric, trace-free, div-free) (Gopalakrishnan, Lederer, Schöberl, 2019)

• dev def: conformal invariant, trace-free Korn inequality; Cott := curl $(S^1)^{-1}$ curl $(S^1)^{-1}$ curl,

Concluding remarks

- from algebra to analysis: cohomology is obtained from homological algebra, which in turn proves closed range property and analytic properties of derived complexes,
- a systematic construction rooted in algebra and geometry, established analytic foundation of FEEC for these derived complexes,
- future work: Hodge-Laplacian and modeling (geometry, relativity, continuum mechanics with microstructures, defects), discrete level.





Derive the elasticity complex (inc = $curl \circ T \circ curl$):

$$0 \longrightarrow H^{s} \otimes \mathbb{V} \xrightarrow{\operatorname{def}} H^{s-1} \otimes \mathbb{S} \xrightarrow{\operatorname{curl}} \tag{1}$$

$$\xrightarrow{\operatorname{Curl}} H^{s-3} \otimes \mathbb{S} \xrightarrow{\operatorname{div}} H^{s-4} \otimes \mathbb{V} \longrightarrow 0.$$

Theorem 2. Cohomology of (1) is isomorphic to the de Rham cohomology $\mathcal{H}_{\infty}^{\bullet} \otimes (\mathbb{V} \times \mathbb{V})$. Explicit formulas of isomorphism exist.

Proof. Homological algebra + Theorem 1 (Costabel, McIntosh).

General framework

Input: two complexes:

$$0 \longrightarrow Z^{0} \xrightarrow{D^{0}} Z^{1} \xrightarrow{D^{1}} \cdots \xrightarrow{D^{n-1}} Z^{n} \longrightarrow 0$$

$$0 \longrightarrow \tilde{Z}^{0} \xrightarrow{\tilde{D}^{0}} \tilde{Z}^{1} \xrightarrow{\tilde{D}^{1}} \cdots \xrightarrow{\tilde{D}^{n-1}} \tilde{Z}^{n} \longrightarrow 0,$$

- Hilbert spaces $Z^i := V^i \otimes \mathbb{E}^i$, and $\tilde{Z}^i := V^{i+1} \otimes \tilde{\mathbb{E}}^i$, with given Hilbert spaces $V^i, i = 0, 1, \dots, n+1$, and finite dimensional inner product spaces $\mathbb{E}^i, \tilde{\mathbb{E}}^i, i = 0, 1, \dots, n$,
- D^i , \tilde{D}^i , $i = 0, 1, \dots, n-1$, are bounded linear operators,
- assumptions on the S operators: $S^i := id \otimes s^i$,
- anti-commutativity: $S^{i+1}\tilde{D}^i = -D^{i+1}S^i$, $i = 0, 1, \dots, n-2$.
- for some $j, 0 \le j \le n-1$, s^i is injective for $0 \le i \le j$, and s^i is surjective for $j \le i \le n-1$ (consequently, s^j is bijective).

Output: derived complex

$$0 \longrightarrow \Upsilon^0 \xrightarrow{\mathscr{D}^0} \Upsilon^1 \xrightarrow{\mathscr{D}^1} \cdots \xrightarrow{\mathscr{D}^{n-1}} \Upsilon^n \longrightarrow 0.$$

where (with $\mathcal{R}\left(s^{-1}\right)^{\perp}:=\mathbb{E}^{0}$ and $\mathcal{N}\left(s^{n}\right):=\tilde{\mathbb{E}}^{n}$):

$$\Upsilon^{i} := \begin{cases} V^{i} \otimes \mathbb{W}^{i}, & 0 \leq i \leq j; \\ V^{i+1} \otimes \mathbb{W}^{i}, & j < i \leq n, \end{cases} \quad \mathbb{W}^{i} := \begin{cases} \mathcal{R}\left(s^{i-1}\right)^{\perp} \subset \mathbb{E}^{i}, & 0 \leq i \leq j; \\ \mathcal{N}\left(s^{i}\right) \subset \tilde{\mathbb{E}}^{i}, & j < i \leq n, \end{cases}$$

$$\mathcal{D}^{i} = \begin{cases} (\operatorname{id} \otimes P_{\mathcal{R}^{\perp}}) D^{i}, & i < j; \\ \tilde{D}^{j} (S^{j})^{-1} D^{j}, & i = j; \\ \tilde{D}^{i}, & i > j. \end{cases}$$

Properties

- The derived complex has finite dimensional cohomology \Rightarrow closed range property,
- Hodge-Beltrami type decomposition,
- Proof: closed range + general results on Hilbert complexes.
- Poincaré-Korn type inequalities: (thus leads to a homological proof for the Korn inequality)

$$||u||_{L^2} \le C||\mathscr{D}u||_{L^2}, \quad \forall u \in \mathcal{N}\left(\mathscr{D}^i\right)^{\perp} \subset H\left(\mathscr{D}^i, \Omega; \mathbb{W}^i\right).$$

Proof: Banach theorem.

- well-posed Hodge Laplacian boundary value problems,
- regular decomposition: let D^{\bullet} and \bar{D}^{\bullet} be first order operators,

$$H\left(\mathscr{D}, \mathbb{W}^i\right) = \mathscr{D}^{i-1}\left(H^1 \otimes \mathbb{W}^{i-1}\right) + H^1 \otimes \mathbb{W}^i.$$

Proof: use the derived Sobolev complexes with various s.