Complexes from complexes

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Motivation

decomposition of flows: $\Omega \subset \mathbb{R}^3$, for any $u \in [L^2(\Omega)]^3$:

$$u = \operatorname{grad} \phi + \operatorname{curl} \psi + w$$

- any flow $=$ potential part (rotation-free) $\oplus_\perp$ rotational part (source-free) $\oplus_\perp$ harmonic,
- w: both rotation-free and source-free (*harmonic forms*),
  dim of such functions $=$ Betti number, reflecting topology of Ω,
- rough idea of cohomology theories: using w to study topology.

Key of these results: algebraic structures, complexes!

de Rham complex

de Rham complex (3D version)

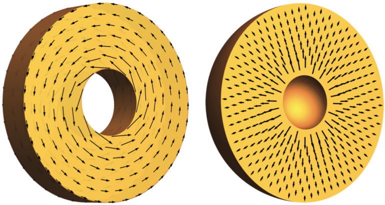
$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0.$$

$$d^0 := \text{grad}, \quad d^1 := \text{curl}, \quad d^2 := \text{div}\,.$$

- complex property: $d^k \circ d^{k-1} = 0, \ \Rightarrow \ \mathcal{R}(d^{k-1}) \subset \mathcal{N}(d^k)$,
  $\text{curl} \circ \text{grad} = 0 \Rightarrow \mathcal{R}(\text{grad}) \subset \mathcal{N}(\text{curl}), \quad \text{div} \circ \text{curl} = 0 \Rightarrow \mathcal{R}(\text{curl}) \subset \mathcal{N}(\text{div})$
- cohomology: $\mathscr{H}^k := \mathcal{N}(d^k)/\mathcal{R}(d^{k-1})$,
  $\mathscr{H}^0 := \mathcal{N}(\text{grad}), \quad \mathscr{H}^1 := \mathcal{N}(\text{curl})/\mathcal{R}(\text{grad}), \quad \mathscr{H}^2 := \mathcal{N}(\text{div})/\mathcal{R}(\text{curl})$
- exactness (contractible domains): $\mathcal{N}(d^k) = \mathcal{R}(d^{k-1})$, i.e., $d^k u = 0 \ \Rightarrow u = d^{k-1} v$
  $\text{curl}\, u = 0 \Rightarrow u = \text{grad}\,\phi, \quad \text{div}\, v = 0 \Rightarrow v = \text{curl}\,\psi.$

de Rham complex and topology:

dimension of $\mathscr{H}^k$ = number of "k-dimensional holes" (c.f. de Rham theorem)



Examples where $\dim \mathscr{H}^1 = 1$ and $\dim \mathscr{H}^2 = 1$, respectively.
Left: curl-free field which is not grad, Right: div-free field with is not curl.

(figure from *Finite element exterior calculus*, D.N.Arnold, SIAM 2008.)

From complexes to PDEs

Formal adjoint of operators:

$$\operatorname{grad}^* = -\operatorname{div}, \quad \operatorname{curl}^* = \operatorname{curl}, \quad \operatorname{div}^* = -\operatorname{grad}.$$

$$\int_\Omega \operatorname{grad} u \cdot v = -\int_\Omega u \operatorname{div} v + \text{bound. term}, \quad \int_\Omega \operatorname{curl} u \cdot v = \int_\Omega u \cdot \operatorname{curl} v + \text{bound. term}$$

$$(\operatorname{grad} u, v) = (u, -\operatorname{div} v), \qquad (\operatorname{curl} u, v) = (u, \operatorname{curl} v)$$

Formal adjoint of de Rham complex:

$$0 \longleftarrow C^\infty(\Omega) \xleftarrow{-\operatorname{div}} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{\operatorname{curl}} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{-\operatorname{grad}} C^\infty(\Omega) \longleftarrow 0.$$

$$d_2^* := -\operatorname{div}, \quad d_1^* := \operatorname{curl}, \quad d_0^* := -\operatorname{grad}.$$

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d_{k-1}^* + d_k^* d^k)u = f.$$

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d^*_{k-1} + d^*_k d^k)u = f.$$

$$0 \rightleftarrows {\color{red}C^\infty(\Omega)} \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$-\operatorname{div}\operatorname{grad} u = f.$$

Poisson equation.

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d^*_{k-1} + d^*_k d^k)u = f.$$

$$0 \qquad C^\infty(\Omega) \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \underset{\operatorname{curl}}{\overset{\operatorname{curl}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$-\operatorname{grad}\operatorname{div} v + \operatorname{curl}\operatorname{curl} v = f.$$

Maxwell equations.

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d_{k-1}^{*} + d_k^{*}d^k)u = f.$$

$$0 \qquad C^\infty(\Omega) \qquad C^\infty(\Omega;\mathbb{R}^3) \underset{\text{curl}}{\overset{\text{curl}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \underset{-\,\text{grad}}{\overset{\text{div}}{\rightleftarrows}} C^\infty(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$\operatorname{curl}\operatorname{curl} v - \operatorname{grad}\operatorname{div} v = f.$$

Maxwell equations.

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d_{k-1}^* + d_k^* d^k)u = f.$$

$$0 \qquad C^\infty(\Omega) \qquad C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega;\mathbb{R}^3) \underset{-\,\mathrm{grad}}{\overset{\mathrm{div}}{\rightleftarrows}} C^\infty(\Omega) \rightleftarrows 0.$$

Hodge-Laplacian problem:

$$-\operatorname{div}\operatorname{grad} u = f.$$

Poisson equation.

Analytic properties

$$0 \longrightarrow H(\mathrm{grad}) \xrightarrow{\mathrm{grad}} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \longrightarrow 0$$

$H(d) := \{u \in L^2 : du \in L^2\}$.

- Hodge decomposition: $L^2 = \mathcal{R}(d^{k-1}) \oplus \mathcal{R}(d_k^*) \oplus \mathscr{H}^k$,

$$L^2 = \mathcal{R}(\mathrm{div}) \oplus \mathscr{G},$$
$$[L^2]^3 = \mathcal{R}(\mathrm{grad}) \oplus_\perp \mathcal{R}(\mathrm{div}) \oplus \mathscr{H}.$$

- Poincaré inequalities: $\|u\| \leq C\|d^k u\|$, $\forall u \perp \mathcal{N}(d^k)$,

$$\|u\| \leq \|\,\mathrm{grad}\, u\|, \quad u \perp \mathcal{N}(\mathrm{grad}),$$
$$\|u\| \leq \|\,\mathrm{curl}\, u\|, \quad u \perp \mathcal{N}(\mathrm{curl}),$$
$$\|u\| \leq \|\,\mathrm{div}\, u\|, \quad u \perp \mathcal{N}(\mathrm{div}).$$

- other properties: regular decomposition, existence of regular potentials, compactness properties, div-curl lemma, etc.

Why are complexes useful for PDEs?

- well-posedness: analytic properties + standard variational argument,
- discretization: structure-preservation!

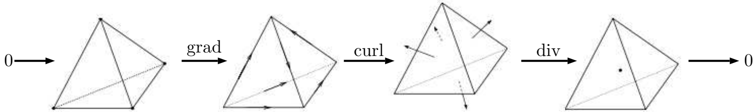
> Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area.
>
> – James Clerk Maxwell, 1873

discrete differential forms (Bossavit 1988, Hiptmair 1999, ...),
Finite Element Exterior Calculus (Arnold, Falk, Winther 2006, ...)

discrete spaces fit into de Rham complexes.

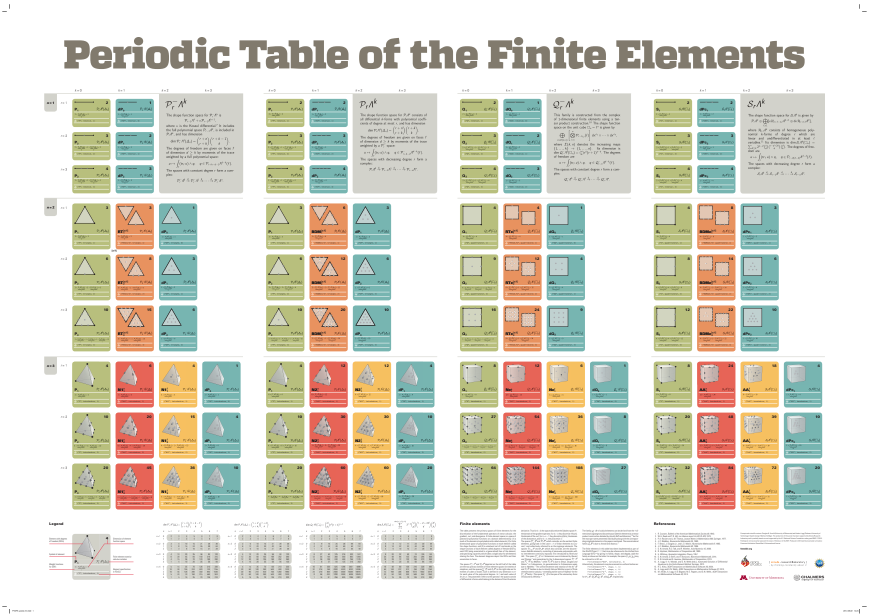
Discretization

finite element de Rham complex in 3D



$$0 \longrightarrow \mathcal{P}_1 \xrightarrow{\text{grad}} [\mathcal{P}_0]^3 + [\mathcal{P}_0]^3 \times x \xrightarrow{\text{curl}} [\mathcal{P}_0]^3 + \mathcal{P}_0 \otimes x \xrightarrow{\text{div}} \mathcal{P}_0 \longrightarrow 0.$$

Raviart-Thomas (1977), Nédélec (1980) in numerical analysis, Bossavit (1988), Hiptmair (1999) for differential forms, Whitney (1957) for studying topology.



Arnold, Logg 2014, SIAM news

Elasticity: deformation and mechanics of solids

elasticity equation:

$$-\operatorname{div}(A \operatorname{def} u) = f.$$

u displacement (vector),

$e := \operatorname{def} u := 1/2(\nabla u + \nabla u^T)$ strain (linearized deformation),

$\sigma := A \operatorname{def} u$ stress.

analogy to Poisson equation:

$$-\operatorname{div}(A \operatorname{grad} v) = g.$$

Elasticity-electromagnetism analogue

KRÖNER [13] has developed a most useful analogy between the theory of internal stresses and strains as described in sections 2 to 6 and the theory of the magnetic field of distributions of stationary electric currents. Table 1 contains a list of the corresponding physical quantities, differential operators, and equations. We hope that this table is understandable without any further comments (see also the review article by DE WIT [10]).

Table 1

Correspondences in elasticity and magnetism

Elasticity	Magnetism
vector quantity	scalar quantity
tensor rank two	vector
tensor rank four	tensor rank two
Div	div
Ink	curl
Div Ink $\equiv 0$	div curl $\equiv 0$
Def	grad
Ink Def $\equiv 0$	curl grad $\equiv 0$
Burgers vector $\boldsymbol{b}$	current I
incompatibility tensor $\boldsymbol{\eta}$	current density $\boldsymbol{J}$
strain tensor $\boldsymbol{\epsilon}$	magnetic intensity $\boldsymbol{H}$
stress tensor $\boldsymbol{\sigma}$	magnetic induction $\boldsymbol{B}$
stress function tensor $\boldsymbol{\chi}$, $\boldsymbol{\chi}'$	vector potential $\boldsymbol{A}$
elastic constants $\boldsymbol{C}$ (or G, K)	permeability μ
displacement $\boldsymbol{s}$	scalar potential ψ
equation (3)	$\boldsymbol{H} = \operatorname{grad} \psi$
equation (5)	$\operatorname{curl} \boldsymbol{H} = \boldsymbol{J}$
equation (17)	$\operatorname{div} \boldsymbol{B} = 0$
equation (18)	$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}$
equations (19), (19a)	$\nabla^2 \boldsymbol{A} = -\mu \boldsymbol{J}$
equation (20)	$\operatorname{div} \boldsymbol{A} = 0$
equation (22)	$\boldsymbol{A} = \frac{\mu}{4\pi} \iiint \frac{\boldsymbol{J}(\boldsymbol{r}')}{\lvert\boldsymbol{r} - \boldsymbol{r}'\rvert} d\tau_{\boldsymbol{r}'}$

Seeger, 1961, *Recent Advances in the Theory of Defects in Crystals*.

A cohomological approach: elasticity complex

Kröner's continuum description of dislocations/defects, internal stress

displacement formulation

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{def}} \overbrace{C^\infty \otimes \mathbb{S} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{S}} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0$$

displacement *strain (metric)* *stress (curvature)* *force*

intrinsic elasticity (Ciarlet et al.)

Hellinger-Reissner principle of elasticity

$\mathbb{V} := \mathbb{R}^3$ vectors, $\mathbb{S} := \mathbb{R}^{3\times 3}_{\text{sym}}$ symmetric matrices

$$\text{def}\, u := 1/2(\nabla u + \nabla u^T), \quad (\text{def}\, u)_{ij} = 1/2(\partial_i u_j + \partial_j u_i).$$

$$\text{inc}\, g := \nabla \times g \times \nabla, \quad (\text{inc}\, g)^{ij} = \epsilon^{ikl}\epsilon^{jst}\partial_k\partial_s g_{lt}.$$

$$\text{div}\, v := \nabla \times v, \quad (\text{div}\, v)_i = \partial^j u_{ij}.$$

g metric $\Rightarrow$ $\text{inc}\, g$ linearized Einstein tensor ($\backsimeq$ Riem $\backsimeq$ Ric in 3D)

$\text{inc} \circ \text{def} = 0$: Saint-Venant compatibility

$\text{div} \circ \text{inc} = 0$: Bianchi identity

systematic study still missing, until
Complexes from complexes, Arnold, Hu 2020.

Algebraic and analytic construction (Arnold, Hu 2020): derive elasticity from deRham

$$
\begin{tikzcd}
0 \arrow[r] & H^s\otimes\mathbb{V} \arrow[r, "\mathrm{grad}"] & H^{s-1}\otimes\cancelto{\mathbb{S}}{\mathbb{M}} \arrow[r, "\mathrm{curl}"] & H^{s-2}\otimes\mathbb{M} \arrow[r, "\mathrm{div}"] & \cancel{H^{s-3}\otimes\mathbb{V}} \arrow[r] & 0 \\
0 \arrow[r] & \cancel{H^{s-1}\otimes\mathbb{V}} \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^0:=\mathrm{mskw}"] & H^{s-2}\otimes\mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S^1"] & H^{s-3}\otimes\cancelto{\mathbb{S}}{\mathbb{M}} \arrow[r, "\mathrm{div}"] \arrow[ur, "S^2:=\mathrm{vskw}"] & H^{s-4}\otimes\mathbb{V} \arrow[r] & 0.
\end{tikzcd}
$$

$S^1 u := u^T - \mathrm{tr}(u)I.$

key: Sobolev complexes ($\forall s \in \mathbb{R}$), match indices, commuting diagrams, injectivity & surjective.

output: elasticity complex

$$
\begin{tikzcd}
0 \arrow[r] & H^s\otimes\mathbb{V} \arrow[r, "\mathrm{def}"] & H^{s-1}\otimes\mathbb{S} \arrow[dl, "\mathrm{curl}\ \mathrm{T}\ \mathrm{curl}"] & & \\
& H^{s-3}\otimes\mathbb{S} \arrow[r, "\mathrm{div}"] & H^{s-4}\otimes\mathbb{V} \arrow[r] & 0.
\end{tikzcd}
\tag{1}
$$

Theorem

Cohomology of (1) *is isomorphic to the smooth de Rham cohomology:*

$$\mathcal{N}(\mathscr{D}^i) = \mathcal{R}(\mathscr{D}^{i-1}) \oplus \mathscr{H}^i_\infty, \quad \mathscr{H}^i_\infty \backsimeq \mathcal{H}^i_{\mathrm{deRham}} \otimes (\mathbb{V} \times \mathbb{V})$$

Proof: Homological algebra + results for de Rham by Costabel & McIntosh.

Corollary: finite dimensional cohomology $\Longrightarrow$ operators have closed range.

inspired by Bernstein-Gelfand-Gelfand (BGG) resolution
(c.f., Eastwood 2000, Čap, Slovák, Souček 2001, Arnold, Falk, Winther 2006).

Consequences:

- analytic results (Poincaré inequality, Hodge decomposition, etc.)
  e.g., Korn inequality $\|u\|_1 \leq C\|\operatorname{def} u\|, \;\; u \perp \mathcal{N}(\operatorname{def})$.
- explicit representatives of elasticity cohomology.

Systematic construction: generating new complexes from existing ones

- input: $(Z^\bullet, D^\bullet)$, $(\tilde{Z}^\bullet, \tilde{D}^\bullet)$, connecting maps $S^i : \tilde{Z}^i \to Z^{i+1}$, satisfying
    - (anti-)commutativity: $S^{i+1}\tilde{D}^i = -D^{i+1}S^i$,
    - injectivity/surjectivity condition: S^i injective for $i \leq J$, surjective for $i \geq J$.

$$\begin{tikzcd}
0 \arrow[r] & Z^0 \arrow[r, "D^0"] & Z^1 \arrow[r, "D^1"] & \cdots \arrow[r, "D^{n-1}"] & Z^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{Z}^0 \arrow[r, "\tilde{D}^0"] \arrow[ur, "S^0"] & \tilde{Z}^1 \arrow[r, "\tilde{D}^1"] \arrow[ur, "S^1"] & \cdots \arrow[r, "\tilde{D}^{n-1}"] \arrow[ur, "S^{n-1}"] & \tilde{Z}^n \arrow[r] & 0
\end{tikzcd}$$

- output:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{coker}(S^{J-2}) \arrow[r, "D^{J-1}"] & \mathrm{coker}(S^{J-1}) \arrow[r, "D^J"] & {} \arrow[dll, "(S^J)^{-1}"'] & & \\
& & {} \arrow[r, "\tilde{D}^J"] & \mathcal{N}(S^{J+1}) \arrow[r, "\tilde{D}^{J+1}"] & \mathcal{N}(S^{J+2}) \arrow[r, "\tilde{D}^{J+2}"] & \cdots
\end{tikzcd}$$

- conclusion:

$$\dim \mathscr{H}^i(\Upsilon^\bullet, \mathscr{D}^\bullet) \leq \dim \mathscr{H}^i(Z^\bullet, D^\bullet) + \dim \mathscr{H}^i\left(\tilde{Z}^\bullet, \tilde{D}^\bullet\right), \quad \forall i = 0, 1, \cdots, n$$

Equality holds if and only if S^i induces the zero maps on cohomology, i.e., $S^i\mathcal{N}(\tilde{D}^i) \subset \mathcal{R}(D^i)$.

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}$$

Hessian complex:

$$0 \longrightarrow H^q \otimes \mathbb{R} \xrightarrow{\mathrm{hess}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\mathrm{curl}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\mathrm{div}} H^{q-4} \otimes \mathbb{V} \longrightarrow 0.$$

biharmonic equations, plate theory, Einstein-Bianchi system of general relativity

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\,\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

elasticity complex:

$$
0 \longrightarrow H^{q-1} \otimes \mathbb{V} \xrightarrow{\mathrm{def}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\mathrm{inc}} H^{q-4} \otimes \mathbb{S} \xrightarrow{\mathrm{div}} H^{q-5} \otimes \mathbb{V} \longrightarrow 0.
$$

elasticity, defects, metric, curvature

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V}; \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\,\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

divdiv complex:

$$0 \longrightarrow H^{q-2} \otimes \mathbb{V} \xrightarrow{\mathrm{dev\,grad}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\mathrm{sym\,curl}} H^{q-4} \otimes \mathbb{S} \xrightarrow{\mathrm{div\,div}} H^{q-6} \otimes \mathbb{R} \longrightarrow 0.$$

plate theory, elasticity

Iterating the construction

$$
\begin{tikzcd}
0 \arrow[r] & H^{q+1}\otimes\mathbb{R} \arrow[r, "\operatorname{grad}"] & H^{q}\otimes\mathbb{V} \arrow[r, "\operatorname{curl}"] & H^{q-1}\otimes\mathbb{V} \arrow[r, "\operatorname{div}"] & H^{q-2}\otimes\mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q}\otimes\mathbb{V} \arrow[r, "\operatorname{dev}\operatorname{grad}"] \arrow[ur, "-\mathrm{id}"] & H^{q-1}\otimes\mathbb{T} \arrow[r, "\operatorname{sym}\operatorname{curl}"] \arrow[ur, "2\operatorname{vskw}"] & H^{q-2}\otimes\mathbb{S} \arrow[r, "\operatorname{div}\operatorname{div}"] \arrow[ur, "\operatorname{tr}"] & H^{q-4}\otimes\mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1}\otimes\mathbb{V} \arrow[r, "\operatorname{def}"] \arrow[ur, "-\operatorname{mskw}"] & H^{q-2}\otimes\mathbb{S} \arrow[r, "\operatorname{inc}"] \arrow[ur, "S"] & H^{q-4}\otimes\mathbb{S} \arrow[r, "\operatorname{div}"] \arrow[ur, "\operatorname{tr}"] & H^{q-5}\otimes\mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \arrow[r, "\operatorname{hess}"] \arrow[ur, "\iota"] & H^{q-4}\otimes\mathbb{S} \arrow[r, "\operatorname{curl}"] \arrow[ur, "S"] & H^{q-5}\otimes\mathbb{T} \arrow[r, "\operatorname{div}"] \arrow[ur, "2\operatorname{vskw}"] & H^{q-6}\otimes\mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-4}\otimes\mathbb{R} \arrow[r, "\operatorname{grad}"] \arrow[ur, "\iota"] & H^{q-5}\otimes\mathbb{V} \arrow[r, "\operatorname{curl}"] \arrow[ur, "\operatorname{mskw}"] & H^{q-6}\otimes\mathbb{V} \arrow[r, "\operatorname{div}"] \arrow[ur, "-\mathrm{id}"] & H^{q-7}\otimes\mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

Example from iterative constructions

"conformal complex"

ker of dev def: conformal Killing v.f. Cotton-York: flatness in conformal geometry

$$0 \longrightarrow H^q(\Omega)\otimes\mathbb{V} \xrightarrow{\operatorname{dev}\operatorname{def}} \underbrace{H^{q-1}(\Omega)\otimes(\mathbb{S}\cap\mathbb{T}) \xrightarrow{\operatorname{cott}} H^{q-4}(\Omega)}_{}\otimes \underbrace{(\mathbb{S}\cap\mathbb{T}) \xrightarrow{\operatorname{div}} H^{q-5}(\Omega)}_{}\otimes\mathbb{V} \longrightarrow 0$$

gravitational wave variable: transverse-traceless (TT) gauge
(= symmetric, trace-free, div-free)

stress like variable def$u$ in NS
(Gopalakrishnan, Lederer, Schöberl, 2019)

Hodge decomp.: York split, Einstein constraint eqns.

trace-free Korn inequality:

$$\|u\|_1 \le C\|\operatorname{dev}\operatorname{def} u\|, \quad \forall u \in \mathcal{N}(\operatorname{dev}\operatorname{def}).$$

$\mathcal{N}(\operatorname{dev}\operatorname{def})$: conformal Killing fields

open problem (Chipot 2020): minimal number of linear functionals l_i, s.t. generalized Korn inequality holds

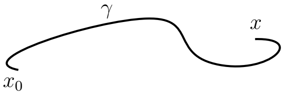
$$\|u\|_1 \le C\Big(\sum_{i=1}^{N}\|l_i(\nabla u)\|_{L^2} + \|u\|_{L^2}\Big).$$

e.g., 3D Poincaré: N=9; Korn: N= 6; trace-free Korn: N=5.

Poincaré integral operators

Question: $Du = 0$, find *explicit* potential ϕ, s.t. $u = \tilde{D}\phi$.

Example: $u = \operatorname{grad}\phi$, $\phi = \phi(x_0) + \int_{\gamma(y)} u\, dy$



Applications: prove exactness, construct finite elements etc.

Idea (Christiansen, Hu, Sande 2019): using de Rham results and homological algebra.

Result: systematic construction, Cesàro-Volterra integral (1906, 1907) in classical elasticity as a special case.

Finite elements: mimic continuous construction

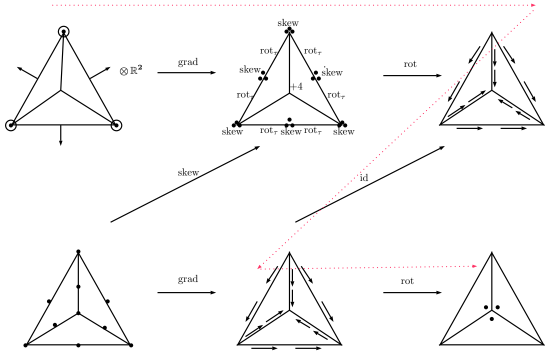
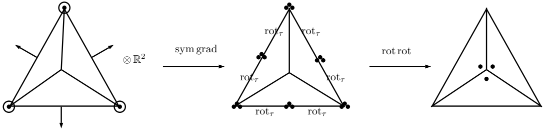


diagram chase or using the new Poincaré operators (Christiansen, Hu 2019):



$$0 \longrightarrow \mathrm{RM} \xrightarrow{\subset} H^1 \otimes \mathbb{V} \xrightarrow{\mathrm{def}} H(\operatorname{rot}\operatorname{rot}; \mathbb{S}) \xrightarrow{\operatorname{rot}\operatorname{rot}} L^2 \longrightarrow 0.$$

first conforming discretization for metric and linearized curvature

Applications

continuum mechanics and geometry: modeling, analysis, numerics

Cosserat (micropolar) continuum v.s. elasticity $\Leftrightarrow$ BGG complexes v.s. elasticity complex

Cosserat continuum: incorporating size effects by introducing rotational degrees of freedom.
(c.f. *E. Cartan's attempt at bridge-building between Einstein and the Cosserat - or how translational curvature became to be known as torsion*, Scholz 2019.)

structure-preserving numerical relativity: solving the Einstein equations.

Take home messages:

- cohomological structures play a key role in modeling, analysis, and numerics,
- (elasticity, continuum mechanics, geometry, relativity) complexes from (de Rham) complexes.

References

- *Complexes from complexes*, Douglas N. Arnold and Kaibo Hu; 2020, arXiv:2005.12437.
- *Poincaré path integrals for elasticity*; Snorre H. Christiansen, Kaibo Hu and Espen Sande; *Journal de Mathématiques Pures et Appliquées*, (2019).
- *Finite element systems for vector bundles: elasticity and curvature*, Snorre H. Christiansen and Kaibo Hu; 2019, Available online as arXiv:1906.09128.