Well-conditioned frames for high order finite elements

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What is a good basis? conditioning, sparsity, fast evaluation, rotational symmetry, etc.

> *... In this sense, preconditioning will always be an art rather than a science.*
>
> – Andy Wathen, *Preconditioning*, Acta Numer. 2015

same remains true for bases.

This talk: conditioning for high order finite element bases (dependence on degree).

1. Condition number: a revisit

2. High order finite elements

- finite element methods: with some bases $\{\phi_i\}_{i=1}^N$,

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx, \quad (u, v) := \int_\Omega u \cdot v \, dx,$$

stiffness matrix $(\mathbb{A}_h)_{ij} := a(\phi_i, \phi_j)$, mass matrix $(\mathbb{M}_h)_{ij} := (\phi_i, \phi_j)$.

- condition number $\kappa(\mathbb{A}_h)$, $\kappa(\mathbb{M}_h)$: depend on the basis
- design bases: a lot of efforts

  Fuentes, Keith, Demkowicz, Nagaraj, 2015; Beuchler, Pillwein, Schöberl, Zaglmayr 2012; Szabo, Babuška 1991; Karniadakis, Sherwin 2013; Bonazzoli, Gaburro, Dolean, Rapetti 2014; Ainsworth, Coyle, 2004; Dubiner 1991 etc.

- Question: Which condition number to optimize? How?

  e.g., convection-diffusion equation

$$u_t + (\beta(x) \cdot \nabla)u - \epsilon \Delta u = f.$$

mass varying coefficients stiffness

Our choice: L^2 (mass) condition number

Reasons:

- “basis condition number”.

  representation $\tau_h : \mathbb{R}^n \mapsto V_h$, $\tau_h(c) = \sum_j c_j \phi_j$,
  operator $\mathcal{A}_h : V_h \to V_h^*$, matrix representation $\mathbb{A}_h := \tau_h^* \mathcal{A}_h \tau_h : \mathbb{R}^n \to \mathbb{R}^n$

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\mathbb{A}_h} & \mathbb{R}^n \\ \downarrow \tau_h & & \tau_h^* \uparrow \\ V_h & \xrightarrow{\mathcal{A}_h} & V_h^* . \end{array}$$

  Riesz operator $\mathcal{I}_h : V_h^* \mapsto V_h$, mass matrix $\mathbb{M}_h = \tau_h^* \mathcal{I}_h^{-1} \tau_h$,

$$\mathbb{A}_h = \mathbb{M}_h(\tau_h^{-1} \mathcal{I}_h \mathcal{A}_h \tau_h), \quad \Rightarrow \quad \kappa(\mathbb{A}_h) \leq \kappa(\mathcal{I}_h \mathcal{A}_h) \cdot \kappa(\mathbb{M}_h),$$

- L^2 condition number controls others: $\quad \kappa(\mathcal{I}_h \mathcal{A}_h) = \kappa(\mathbb{M}_h^{-1} \mathbb{A}_h) \leq C r^4$

  Poincaré + inverse inequalities
  $\|\nabla u_h\| \leq C h^{-1} \|u_h\|, \quad \|\nabla u_h\| \leq C r^2 \|u_h\|$. (sharp bound : Bernardi, Maday 1997)

- resemblance of h-method: in a position to handle stillffness matrix by preconditioners

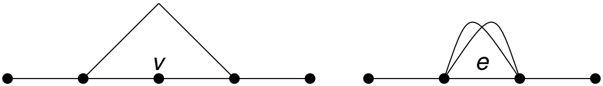
High order finite elements: continuous piecewise polynomial

$$V_r := \{u_h \in C^0(\Omega) : u_h|_T \in \mathscr{P}_r(T),\ \forall T \in \mathcal{T}_h\}.$$

- natural bases (1D): vertex modes + interior modes/bubbles

$$V_r = \sum_{v \in \mathcal{V}} V_v \oplus \sum_{e \in \mathcal{E}} V_e$$

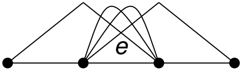
Example: $r = 3$.



Local orthogonality is not enough for well conditioning

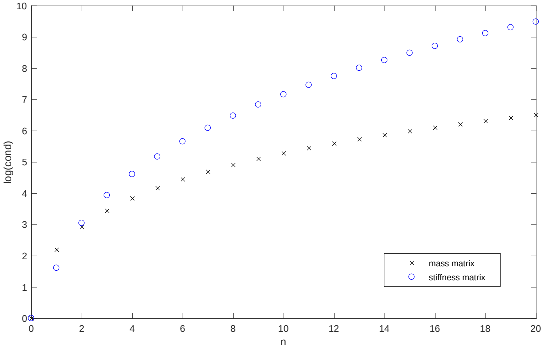
$$V_r = \sum_{v \in \mathcal{V}} V_v \oplus \sum_{e \in \mathcal{E}} V_e$$

hat function $\oplus$ orthogonal bubbles



condition numbers grow with the polynomial degree r.

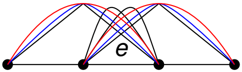
Reason: vertex hat functions interfere with bubbles



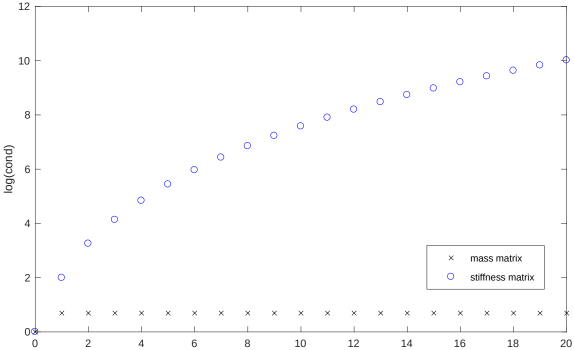
Quick overview of our solution: stable decomposition of V_r by introducing redundancy

$$V_r = \sum_{v\in\mathcal{V}} \tilde{V}_v + \sum_{e\in\mathcal{E}} V_e$$

more hat functions + orthogonal bubbles

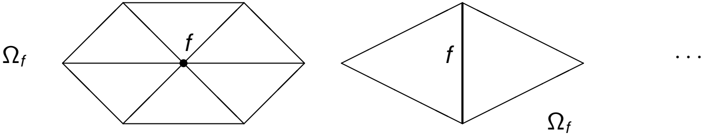


L^2 frame condition numbers ($\lambda_{\max}/\lambda_{\min,+}$) remains constant for all r:
(reminder: frame = bases with redundancy)



More details in *n*D: well conditioning = stable decomposition + local orthogonality

- notation: $f \in \Delta$: vertex, edge, face etc., Ω_f: *patch* associated with f.



- tool: bubble transform (Falk, Winther, 2016)
  stable decomposition of H^1 into local patches, $u = \sum_{f\in\Delta} B_f u$,

$$a\|u\|^2 \le \sum_{f\in\Delta} \|B_f u\|^2 \le b\|u\|^2, \quad \forall u \in H^1(\Omega).$$

- bubble transform on finite element spaces:

$$V_r = \sum_{f\in\Delta} B_f V_r =: \sum_{f\in\Delta} V_f,$$

  where $V_f \subset \mathring{\mathscr{P}}_r(\Omega_f)$ is a pull back of polynomial spaces defined on reference elements to Ω_f. not a direct sum

Theorem

Assume that there exists $B_f : V_r \mapsto V_f$, and (stable decomposition)

$$a\|u\|^2 \leq \sum_{f\in\Delta} \|B_f u\|^2 \leq b\|u\|^2,$$

and (local basis $\phi_{f,k}$ in V_f)

$$\alpha_f \sum_k c_{f,k}^2 \leq \left\| \sum_k c_{f,k}\phi_{f,k} \right\|^2 \leq \beta_f \sum_k c_{f,k}^2, \quad \forall c_{f,k}.$$

Define the local condition number $\kappa_f = \beta_f/\alpha_f$. We have

$$\kappa \leq (a^{-1}b)\left(\max_{f\in\Delta} \kappa_f\right) \cdot \max_{f,g\in\Delta} \frac{\alpha_f}{\alpha_g},$$

where κ is the (global) frame condition number.

frame condition number $\kappa := \lambda_{\max}/\lambda_{\min,+}$.

Scaling of local blocks

$\max_{f,g\in\Delta} \frac{\alpha_f}{\alpha_g}$: scaling of local blocks

Example:

$$A = \begin{pmatrix} I & 0 \\ 0 & \epsilon I \end{pmatrix},$$

condition number of each block is 1, but $\max \frac{\alpha_k}{\alpha_j} = \epsilon^{-1}$.

- mutually orthogonal (Jacobi) polynomials on $(m+1)$-simplexes with weight $\boldsymbol{w}_m$:

$$J_{\boldsymbol{s}}(\lambda) := c_{\boldsymbol{s}}^{-1} \prod_{j=0}^{m} \left(1 - \sum_{i=0}^{j-1} \lambda_i\right)^{s_j} J_{s_j}^{a_j,2}\left(\frac{2\lambda_j}{1-\sum_{i=0}^{j-1}\lambda_i} - 1\right), \quad |\boldsymbol{s}| \le s,$$

  where $\boldsymbol{s} = (s_0, \ldots, s_m)$ is a multi-index,

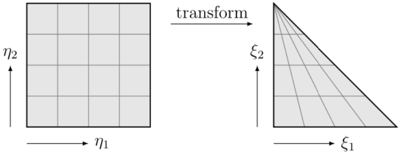
$$a_j = 2\sum_{i=j+1}^{m} s_i + d + 2m - 3j - 1,$$

  and

$$c_{\boldsymbol{s}}^{-2} = \Pi_{j=0}^{m} 2^{a_j+3} = \Pi_{j=0}^{m} 2^{2\sum_{i=j+1}^{m} s_i + d + 2m - 3j + 2}.$$

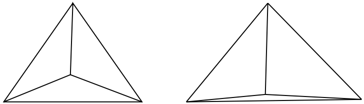
  (c.f. Dunkl, Xu 2014.)

- Duffy transform:



singularity and non-symmetry arise (figure from Mengaldo, Grazia, Moxey, Vincent, Sherwin 2015)

2D tests : robust with mesh distorsion



r	3	5	7	9
λ_{max}	6.623	6.819	6.893	6.930
$\lambda_{min,+}$	0.427	0.457	0.472	0.481
cond. num.	15.503	14.922	14.600	14.414
dim. of frame	33	98	199	336
rank of frame	19	46	85	136

Table: Results for Test 1.

r	3	5	7	9
λ_{max}	6.624	6.819	6.893	6.930
$\lambda_{min,+}$	0.333	0.383	0.414	0.435
cond. num.	19.869	17.809	16.647	15.929
dim. of frame	33	98	199	336
rank of frame	19	46	85	136

Table: Results for Test 2.

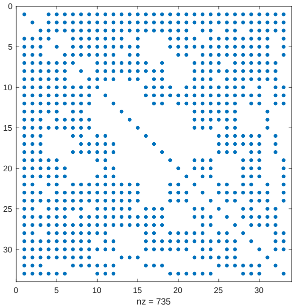


Figure: Pattern of mass matrix, Test 1, $r = 3$.

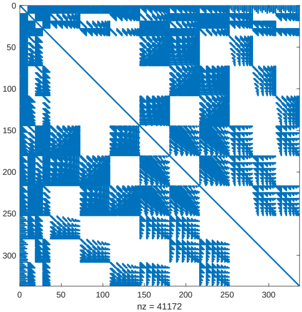


Figure: Pattern of mass matrix, Test 1, $r = 9$.

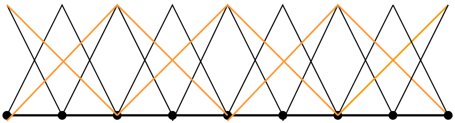
Solving semi-definite systems

- most iterative methods work for semi-definite systems
- connections to iterative methods:

  Jacobi/Gauss-Seidel on frames $\Longleftrightarrow$ subspace correction iterations (MG, DDM etc.),

  c.f., Xu 1992, Griebel 1994; Griebel, Oswald 1995, Chen 2010.

  Example of multigrid:



  p- preconditioning with a similar decomposition:

  Additive Schwarz preconditioning for p-version triangular and tetrahedral finite elements, Schöberl, Melenk, Pechstein, Zaglmayr 2008.

Conclusion

- take-home message:
  - consider L^2 (mass) condition number,
  - redundancy + local orthogonality $\Rightarrow$ well-conditioning.
- further directions:
  - more general local shape functions (other than polynomials) (bubble transform still works)
  - electromagnetism, fluid, elasticity... : curl/div problems
  - *p*- preconditioning for the local problem (spectral methods).
- Reference:

Kaibo Hu and Ragnar Winther, *Well-conditioned frames for high order finite element methods*. Journal of Computational Mathematics, 2021.