Construction of Hilbert complexes

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Review: de Rham complexes

- Sobolev de Rham complex:

$$0 \longrightarrow H^s\Lambda^0 \xrightarrow{d^0} H^{s-1}\Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H^{s-n}\Lambda^n \longrightarrow 0,$$

Theorem 1 (Costabel, McIntosh, 2010)**.**

$$\mathcal{N}\left(d^k, H^{s-k}\Lambda^k\right) = d^{k-1}H^{s-(k-1)}\Lambda^{k-1} \oplus \mathcal{H}^k_\infty,$$

– *Lipschitz domain, any real number s,*
– *cohomology representation $\mathcal{H}^k_\infty \subset C^\infty$, not depending on s.*

Idea of proof: estimating Bogovskiĭ and regularized Poincaré path integral operators as pseudodifferential operators of order -1.

- L^2 Hilbert complex

$$0 \longrightarrow L^2\Lambda^0 \xrightarrow{d^0} L^2\Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} L^2\Lambda^n \longrightarrow 0,$$

with the domain complex ($H(\mathscr{D})$ type de Rham complex):

$$0 \longrightarrow H\Lambda^0 \xrightarrow{d^0} H\Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H\Lambda^n \longrightarrow 0,$$

where $H\Lambda^k := \{u \in L^2\Lambda^k \ : \ d^k u \in L^2\Lambda^{k+1}\}$. ($H(\mathrm{curl})/H(\mathrm{div})$ type spaces)

Analytic results hold: closed range, Hodge decomposition, Poincaré inequalities etc., which lay the foundation of the Finite Element Exterior Calculus (Arnold, Falk, Winther 2006) and applications.

Question: elasticity complex (Kröner/linearized Calabi complex)

displacement formulation — Kröner's continuum description of defects, internal stress

$$\mathrm{RM} \xrightarrow{\subseteq} C^\infty(\Omega;\mathbb{V}) \xrightarrow{\mathrm{def}} C^\infty(\Omega;\mathbb{S}) \xrightarrow{\mathrm{inc}} C^\infty(\Omega;\mathbb{S}) \xrightarrow{\mathrm{div}} C^\infty(\Omega;\mathbb{V}) \longrightarrow 0$$

displacement *strain* *stress* *load*

intrinsic elasticity (Ciarlet et al.) — Hellinger-Reissner principle of elasticity

- $\mathbb{V}$: vectors, $\mathbb{S}$: symmetric matrices,
- linearized deformation $\mathrm{def}\, u := 1/2(\nabla u + u\nabla)$, linearized curvature $\mathrm{inc}\, v := \nabla \times v \times \nabla$,
- $\mathrm{inc} \circ \mathrm{def} = 0$, $\mathrm{div} \circ \mathrm{inc} = 0$,
- Saint-Venant compatibility condition: $e = \mathrm{def}\, u \Rightarrow \mathrm{inc}\, e = 0$,

$\boxed{\text{Functional analysis foundation for } L^2 \text{ and Sobolev complexes?}}$

Solution: algebraic construction of elasticity complex from de Rham complexes

Bernstein-Gelfand-Gelfand type construction:
(Eastwood 1999; Arnold, Falk, Winther 2006, for C^∞ functions)

$$\begin{tikzcd}
0 \arrow[r] & H^s\otimes\mathbb{V} \arrow[r, "\mathrm{grad}"] & H^{s-1}\otimes\mathbb{M} \arrow[r, "\mathrm{curl}"] & H^{s-2}\otimes\mathbb{M} \arrow[r, "\mathrm{div}"] & H^{s-3}\otimes\mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{s-1}\otimes\mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^0:=\mathrm{mskw}"] & H^{s-2}\otimes\mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S^1"] & H^{s-3}\otimes\mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "S^2:=\mathrm{vskw}"] & H^{s-4}\otimes\mathbb{V} \arrow[r] & 0.
\end{tikzcd}$$

- $\mathbb{M}$: matrices, $\mathrm{skw} : \mathbb{M} \mapsto \mathbb{K}$: skew symmetrization, $\mathrm{mskw} : \mathbb{V} \mapsto \mathbb{K}$: axial vector to skew symmetric matrix, $\mathrm{vskw} := \mathrm{mskw}^{-1} \circ \mathrm{skw} : \mathbb{M} \mapsto \mathbb{V}$, $S^1 u := u^T - \mathrm{tr}(u)I$,
- key structures:
  – $DS = -SD$: $\mathrm{curl}\,\mathrm{mskw} = -S^1\,\mathrm{grad}$, $\mathrm{div}\, S^1 = -\mathrm{vskw}\,\mathrm{curl}$,
  – S^i operators, $i = 0, 1, 2$: injective, bijective, surjective,
- homological algebra: eliminate the skew part, connect the curl operators

$$\begin{tikzcd}
0 \arrow[r] & H^s\otimes\mathbb{V} \arrow[r, "\mathrm{grad}"] & H^{s-1}\otimes\mathbb{M} \arrow[r, "\mathrm{curl}"] & H^{s-2}\otimes\mathbb{M} \arrow[r, "\mathrm{div}"] & \cancel{H^{s-3}\otimes\mathbb{V}} \arrow[r] & 0 \\
0 \arrow[r] & \cancel{H^{s-1}\otimes\mathbb{V}} \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^0:=\mathrm{mskw}"] & H^{s-2}\otimes\mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S^1"] & H^{s-3}\otimes\mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "S^2:=\mathrm{vskw}"] & H^{s-4}\otimes\mathbb{V} \arrow[r] & 0.
\end{tikzcd}$$

def $\quad \mathbb{S} \quad \mathbb{S}$

Derive the elasticity complex ($\mathrm{inc} = \mathrm{curl} \circ \mathrm{T} \circ \mathrm{curl}$):

$$0 \longrightarrow H^s\otimes\mathbb{V} \xrightarrow{\mathrm{def}} H^{s-1}\otimes\mathbb{S} \xrightarrow{\mathrm{curl}} \xrightarrow{\mathrm{T}} \xrightarrow{\mathrm{curl}} H^{s-3}\otimes\mathbb{S} \xrightarrow{\mathrm{div}} H^{s-4}\otimes\mathbb{V} \longrightarrow 0. \tag{1}$$

Theorem 2. *Cohomology of* (1) *is isomorphic to the de Rham cohomology $\mathcal{H}^\bullet_\infty \otimes (\mathbb{V}\times\mathbb{V})$. Explicit formulas of isomorphism exist.*

Proof. Homological algebra + Theorem 1 (Costabel, McIntosh). □

General framework

Input: two complexes:

$$\begin{tikzcd}
0 \arrow[r] & Z^0 \arrow[r, "D^0"] & Z^1 \arrow[r, "D^1"] & \cdots \arrow[r, "D^{n-1}"] & Z^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{Z}^0 \arrow[r, "\tilde{D}^0"] \arrow[ur, "S^0"] & \tilde{Z}^1 \arrow[r, "\tilde{D}^1"] \arrow[ur, "S^1"] & \cdots \arrow[r, "\tilde{D}^{n-1}"] \arrow[ur, "S^{n-1}"] & \tilde{Z}^n \arrow[r] & 0,
\end{tikzcd}$$

- Hilbert spaces $Z^i := V^i \otimes \mathbb{E}^i$, and $\tilde{Z}^i := V^{i+1} \otimes \tilde{\mathbb{E}}^i$, with given Hilbert spaces $V^i, i = 0, 1, \cdots, n+1$, and finite dimensional inner product spaces $\mathbb{E}^i, \tilde{\mathbb{E}}^i, i = 0, 1, \cdots, n$,
- $D^i, \tilde{D}^i, i = 0, 1, \cdots, n-1$, are bounded linear operators,
- assumptions on the S operators: $S^i := \mathrm{id} \otimes s^i$,
  – anti-commutativity: $S^{i+1}\tilde{D}^i = -D^{i+1}S^i$, $i = 0, 1, \cdots, n-2$.
  – for some j, $0 \le j \le n-1$, s^i is injective for $0 \le i \le j$, and s^i is surjective for $j \le i \le n-1$ (consequently, s^j is bijective).

Output: derived complex

$$0 \longrightarrow \Upsilon^0 \xrightarrow{\mathscr{D}^0} \Upsilon^1 \xrightarrow{\mathscr{D}^1} \cdots \xrightarrow{\mathscr{D}^{n-1}} \Upsilon^n \longrightarrow 0,$$

where (with $\mathcal{R}\left(s^{-1}\right)^\perp := \mathbb{E}^0$ and $\mathcal{N}\left(s^n\right) := \tilde{\mathbb{E}}^n$):

$$\Upsilon^i := \begin{cases} V^i \otimes \mathbb{W}^i, & 0 \le i \le j; \\ V^{i+1} \otimes \mathbb{W}^i, & j < i \le n, \end{cases} \qquad \mathbb{W}^i := \begin{cases} \mathcal{R}\left(s^{i-1}\right)^\perp \subset \mathbb{E}^i, & 0 \le i \le j; \\ \mathcal{N}\left(s^i\right) \subset \tilde{\mathbb{E}}^i, & j < i \le n, \end{cases}$$

$$\mathscr{D}^i = \begin{cases} (\mathrm{id} \otimes P_{\mathcal{R}^\perp})D^i, & i < j; \\ \tilde{D}^j(S^j)^{-1}D^j, & i = j; \\ \tilde{D}^i, & i > j. \end{cases}$$

Properties

- The derived complex has finite dimensional cohomology $\Rightarrow$ closed range property,
- Hodge-Beltrami type decomposition,
  Proof: closed range + general results on Hilbert complexes.
- Poincaré-Korn type inequalities: (thus leads to a homological proof for the Korn inequality)

$$\|u\|_{L^2} \le C\|\mathscr{D}u\|_{L^2}, \quad \forall u \in \mathcal{N}\left(\mathscr{D}^i\right)^\perp \subset H\left(\mathscr{D}^i, \Omega; \mathbb{W}^i\right).$$

  Proof: Banach theorem.
- well-posed Hodge Laplacian boundary value problems,
- regular decomposition: let $D^\bullet$ and $\tilde{D}^\bullet$ be first order operators,

$$H\left(\mathscr{D}, \mathbb{W}^i\right) = \mathscr{D}^{i-1}\left(H^1 \otimes \mathbb{W}^{i-1}\right) + H^1 \otimes \mathbb{W}^i.$$

  Proof: use the derived Sobolev complexes with various s.

- compactness property: the following imbedding is compact:

$$H\left(\mathscr{D}^i; \mathbb{W}^i\right) \cap D\left(\mathscr{D}^*_i\right) \hookrightarrow L^2 \otimes \mathbb{W}^i,$$

where $D\left(\mathscr{D}^*_i\right)$ is the domain of the adjoint operator in the sense of unbounded operators.
Proof: existence of regular potential (Sobolev complex) and standard Rellich compactness of H^1.

More examples

$\mathbb{T}$: trace-free matrices, tr: trace, $\iota : \mathbb{R} \mapsto \mathbb{M}, w \mapsto wI$, $S^1 u := u^T - \mathrm{tr}(u)I$
$\mathrm{dev}\, u := u - 1/n\,\mathrm{tr}(u)I$: deviator, $\mathrm{hess} := \mathrm{grad}\,\mathrm{grad}$: Hessian.

Input: Arnold, Lectures at Peking University, 2015 (with C^∞ functions)

$$\begin{tikzcd}
0 \arrow[r] & H^s\otimes\mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{s-1}\otimes\mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{s-2}\otimes\mathbb{V} \arrow[r, "\mathrm{div}"] & H^{s-3}\otimes\mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{s-1}\otimes\mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & H^{s-2}\otimes\mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{vskw}"] & H^{s-3}\otimes\mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{s-4}\otimes\mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{s-2}\otimes\mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{mskw}"] & H^{s-3}\otimes\mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S^1"] & H^{s-4}\otimes\mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{vskw}"] & H^{s-5}\otimes\mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{s-3}\otimes\mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\iota"] & H^{s-4}\otimes\mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{s-5}\otimes\mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & H^{s-6}\otimes\mathbb{R} \arrow[r] & 0.
\end{tikzcd}$$

Diagonal maps are isomorphism, subdiagonal maps are injective, superdiagonal maps are surjective.

Output:
- Hessian complex (1st + 2nd rows): $\mathrm{hess} = \mathrm{grad} \circ \mathrm{id} \circ \mathrm{grad}$, Einstein eqn. (Quenneville-Bélair, 2015)

$$0 \longrightarrow H^s\otimes\mathbb{R} \xrightarrow{\mathrm{grad}} \xrightarrow{\mathrm{id}} \xrightarrow{\mathrm{grad}} H^{s-2}\otimes\mathbb{S} \xrightarrow{\mathrm{curl}} H^{s-3}\otimes\mathbb{T} \xrightarrow{\mathrm{div}} H^{s-4}\otimes\mathbb{V} \longrightarrow 0,$$

- elasticity/Kröner/linearized Calabi complex (2nd + 3rd rows),
- div div complex (3rd + 4th rows):

$$0 \longrightarrow H^{s-2}\otimes\mathbb{V} \xrightarrow{\mathrm{dev\,grad}} H^{s-3}\otimes\mathbb{T} \xrightarrow{\mathrm{sym\,curl}} H^{s-4}\otimes\mathbb{S} \xrightarrow{\mathrm{div}} \xrightarrow{\mathrm{id}} \xrightarrow{\mathrm{div}} H^{s-6}\otimes\mathbb{R} \longrightarrow 0.$$

Stokes (Gopalakrishnan,Lederer,Schöberl, 2018&2019), biharmonic eqn (Pauly,Zulerhner, 2018).

- 1st + 3rd: $\mathrm{grad}\,\mathrm{curl}$ (with connections to $\mathrm{curl}\,\mathrm{curl}$); 2nd + 4th: $\mathrm{curl}\,\mathrm{div}$; 1st + 4th: $\mathrm{grad}\,\mathrm{div}$.

Example of iterative construction

$$\begin{tikzcd}
0 \arrow[r] & H^s(\Omega)\otimes\mathbb{V} \arrow[r, "\mathrm{def}"] & H^{s-1}(\Omega)\otimes\cancel{\mathbb{S}} \arrow[r, "\mathrm{inc}"] & H^{s-3}(\Omega)\otimes\mathbb{S} \arrow[r, "\mathrm{div}"] & \cancel{H^{s-4}(\Omega)\otimes\mathbb{V}} \arrow[r] & 0 \\
0 \arrow[r] & \cancel{H^{s-1}(\Omega)} \arrow[r, "\mathrm{hess}"] \arrow[ur, "\iota"] & H^{s-3}(\Omega)\otimes\mathbb{S} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S^1"] & H^{s-4}(\Omega)\otimes\cancel{\mathbb{T}} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{vskw}"] & H^{s-5}(\Omega)\otimes\mathbb{V} \arrow[r] & 0.
\end{tikzcd}$$

dev def $\quad \mathbb{S}\cap\mathbb{T} \quad \mathbb{S}\cap\mathbb{T}$

ker of dev def: conformal Killing v.f. — Cotton-York: flatness in conformal geometry

$$0 \longrightarrow H^s(\Omega)\otimes\mathbb{V} \xrightarrow{\mathrm{dev\,def}} \underbrace{H^{s-1}(\Omega)\otimes(\mathbb{S}\cap\mathbb{T}) \xrightarrow{\mathrm{Cott}} H^{s-4}(\Omega)}_{\substack{\text{gravitational wave variable: transverse-traceless (TT) gauge} \\ \text{(= symmetric, trace-free, div-free)}}} \otimes \underbrace{(\mathbb{S}\cap\mathbb{T}) \xrightarrow{\mathrm{div}} H^{s-5}(\Omega)}_{\substack{\text{stress like variable } \mathrm{def}\,u \text{ in NS} \\ \text{(Gopalakrishnan, Lederer, Schöberl, 2019)}}} \otimes\mathbb{V} \longrightarrow 0$$

- $\mathrm{dev\,def}$: conformal invariant, trace-free Korn inequality; $\mathrm{Cott} := \mathrm{curl}\left(S^1\right)^{-1}\mathrm{curl}\left(S^1\right)^{-1}\mathrm{curl}$,

Concluding remarks

- from algebra to analysis: cohomology is obtained from homological algebra, which in turn proves closed range property and analytic properties of derived complexes,
- a systematic construction rooted in algebra and geometry, established analytic foundation of FEEC for these derived complexes,
- future work: Hodge-Laplacian and modeling (geometry, relativity, continuum mechanics with microstructures, defects), discrete level.