Structure-preserving FEM for MHD systems

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Magnetohydrodynamics (MHD) is a coupled multi-physics system with wide applications in plasma physics. The differential structures and nonlinearity in MHD raise challenges for numerical discretization and solvers. In this work, we construct and analyze finite element methods that preserve key structures of the MHD system, and investigate robust preconditioners for solving the algebraic system.

Main Objectives

1. finite element methods (FEM) for MHD that preserve discrete energy law, magnetic Gauss law, magnetic and cross helicity,
2. preconditioners and solvers that are robust with physical and discretization parameters.

PDEs and key structures

$$
\begin{cases}
\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - R_e^{-1}\Delta\boldsymbol{u} - s\boldsymbol{j}\times\boldsymbol{B} + \nabla p = \boldsymbol{f}, & \text{(fluid momentum)}\\
\nabla\cdot\boldsymbol{u} = 0, & \text{(incompressible flow)}\\
\boldsymbol{j} - R_m^{-1}\nabla\times\boldsymbol{B} = \boldsymbol{0}, & \text{(reduced Maxwell, eddy current model)}\\
\boldsymbol{B}_t + \nabla\times\boldsymbol{E} = \boldsymbol{0}, & \text{(Faraday's law)}\\
\boldsymbol{j} = \boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}, & \text{(Ohm's law, constitutive law in MHD models)}\\
\nabla\cdot\boldsymbol{B} = 0. & \text{(magnetic Gauss law)}
\end{cases}
$$

with boundary conditions

$$\boldsymbol{u} = 0, \quad \boldsymbol{E}\times\boldsymbol{n} = 0, \quad \boldsymbol{B}\cdot\boldsymbol{n} = 0, \quad \text{on } \partial\Omega.$$

Variables: $\boldsymbol{u}$: fluid velocity, p: fluid pressure, $\boldsymbol{B}$: magnetic field, $\boldsymbol{E}$: electric field, $\boldsymbol{j}$: current density

Parameters: R_e: fluid Reynolds number, R_m: magnetic Reynolds number, s: coupling number $(\alpha := sR_m^{-1})$

- energy law:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^2 + \frac{1}{2}\alpha\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{B}\|^2 + R_e^{-1}\|\nabla\boldsymbol{u}\|^2 + s\|\boldsymbol{j}\|^2 = (\boldsymbol{f},\boldsymbol{u}).$$

- constraint automatically preserved:

$$\boldsymbol{B}_t + \nabla\times\boldsymbol{E} = 0 \ \Rightarrow \partial_t(\nabla\cdot\boldsymbol{B}) = 0.$$

important for MHD computation, see, e.g., Brackbill, Barnes, 1980.

- helicity conservation ($R_e^{-1} = R_m^{-1} = 0$):
  - magnetic helicity $\mathcal{H}_m := \int\boldsymbol{A}\cdot\boldsymbol{B}$, with any $\nabla\times\boldsymbol{A} = \boldsymbol{B}$,
  - cross helicity $\mathcal{H}_c := \int\boldsymbol{u}\cdot\boldsymbol{B}$,
  - describe knot of fields, provide local lower bound for energy (V.I. Arnold 1973), important for turbulence.

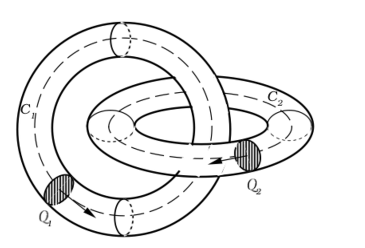


Figure 1: Example of a divergence-free field ξ consisting of two linking rings, $\mathcal{H}_\xi := \int \mathrm{curl}^{-1}\xi\cdot\xi = 2l(C_1, C_2)Q_1\cdot Q_2$, where $l(C_1, C_2)$ is the linking number. Arnold, Khesin, *Topological methods in hydrodynamics*, 1999

Variational form and finite elements

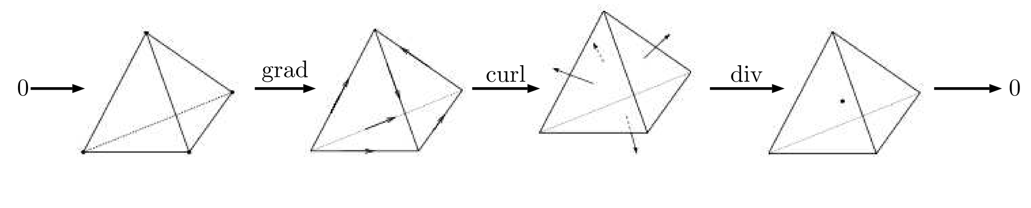
find $(\boldsymbol{u},\boldsymbol{E},\boldsymbol{B},p)\in[H_0^1]^3\times H_0(\mathrm{curl})\times H_0(\mathrm{div})\times L_0^2$, such that for any $(\boldsymbol{v},\boldsymbol{F},\boldsymbol{C},q)\in[H_0^1]^3\times H_0(\mathrm{curl})\times H_0(\mathrm{div})\times L_0^2$,

$$
\begin{cases}
(\boldsymbol{u}_t,\boldsymbol{v}) + 1/2\left[((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{v}) - ((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{u})\right] + R_e^{-1}(\nabla\boldsymbol{u},\nabla\boldsymbol{v})\\
\qquad -s((\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{B},\boldsymbol{v}) - (p,\nabla\cdot\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}),\\
(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B},\boldsymbol{F}) - R_m^{-1}(\boldsymbol{B},\nabla\times\boldsymbol{F}) = 0,\\
(\boldsymbol{B}_t,\boldsymbol{C}) + (\nabla\times\boldsymbol{E},\boldsymbol{C}) = 0,\\
(\nabla\cdot\boldsymbol{u},q) = 0.
\end{cases}
$$

- continuous and discrete de Rham complexes:

$$
\begin{array}{ccccccc}
 & & \boldsymbol{E} & & \boldsymbol{B} & & \\
H_0(\mathrm{grad}) & \xrightarrow{\mathrm{grad}} & H_0(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H_0(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L_0^2\\
\downarrow \Pi_h^{\mathrm{grad}} & & \downarrow \Pi_h^{\mathrm{curl}} & & \downarrow \Pi_h^{\mathrm{div}} & & \downarrow \Pi_h^0\\
H_0^h(\mathrm{grad}) & \xrightarrow{\mathrm{grad}} & H_0^h(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H_0^h(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L_0^{2,h}\\
 & & \boldsymbol{E}_h & & \boldsymbol{B}_h & &
\end{array}
$$

lowest order FEs (other spaces, e.g., splines, are possible):



(classical Stokes pairs or Stokes complexes for $\boldsymbol{u}$ and p).

- features of discrete de Rham complexes:
  - respect physics, e.g., electric fields discretized on edges of tetrahedron, magnetic fields/fluxes on faces,

> Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area.
>
> – James Clerk Maxwell, 1873

  - preserve differential structures: e.g., $\nabla\times H_0^h(\mathrm{curl})\subset H_0^h(\mathrm{div})$,
  - preserve topology (de Rham theorem),
  - a framework for constructing algorithms and rigorous analysis,
  - arbitrary order and dimension, implemented in finite element packages, e.g., FEniCS.

- discrete variational form: $(\boldsymbol{u},\boldsymbol{E},\boldsymbol{B},p)\to(\boldsymbol{u}_h,\boldsymbol{E}_h,\boldsymbol{B}_h,p_h)$,
- discrete energy law:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_h\|^2 + \frac{1}{2}\alpha\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{B}_h\|^2 + R_e^{-1}\|\nabla\boldsymbol{u}_h\|^2 + s\|\boldsymbol{E}_h+\boldsymbol{u}_h\times\boldsymbol{B}_h\|^2 = (\boldsymbol{f},\boldsymbol{u}_h),$$

- Gauss law: $\partial_t\boldsymbol{B}_h + \nabla\times\boldsymbol{E}_h = 0 \ \Rightarrow \partial_t(\nabla\cdot\boldsymbol{B}_h) = 0$, divergence-free initial data yields divergence-free solutions.
- well-posedness: Babuška and Brezzi theory with weighted norms (with time step size k):

$$\|(\boldsymbol{u},p,\boldsymbol{E},\boldsymbol{B})\|_{\mathscr{H}}^2 := \|\boldsymbol{u}\|_{\mathcal{H}_1}^2 + \|p\|_{\mathcal{H}_2}^2 + \|\boldsymbol{B}\|_{\mathcal{H}_3}^2 + \|\boldsymbol{E}\|_{\mathcal{H}_4}^2,$$

$$\|\boldsymbol{u}\|_{\mathcal{H}_1}^2 := k^{-1}\|\boldsymbol{u}\|^2 + R_e^{-1}\|\nabla\boldsymbol{u}\|^2 + k^{-1}\|\mathbb{P}\nabla\cdot\boldsymbol{u}\|^2 + s\left\|\boldsymbol{u}\times\boldsymbol{B}^-\right\|^2,$$

$$\|p\|_{\mathcal{H}_2}^2 := k\|p\|^2,$$

$$\|\boldsymbol{B}\|_{\mathcal{H}_3}^2 := k^{-1}\alpha\|\boldsymbol{B}\|^2 + \alpha\|\nabla\cdot\boldsymbol{B}\|^2,$$

$$\|\boldsymbol{E}\|_{\mathcal{H}_4}^2 := s\|\boldsymbol{E}\|^2 + k\alpha\|\nabla\times\boldsymbol{E}\|^2.$$

Preconditioners

Operator form of MHD system $\mathcal{A}\boldsymbol{x} = \boldsymbol{f}$:

$$
\begin{pmatrix}
\mathcal{A}_1 & -\mathrm{div}^* & 0 & \mathcal{F}^*\\
-\mathrm{div} & 0 & 0 & 0\\
0 & 0 & -\alpha k^{-1}\mathcal{I}_3 & -\alpha\mathrm{curl}\\
\mathcal{F} & 0 & -\alpha\mathrm{curl}^* & s\mathcal{I}_4
\end{pmatrix}
\begin{pmatrix}\boldsymbol{u}\\ p\\ \boldsymbol{B}\\ \boldsymbol{E}\end{pmatrix}
=
\begin{pmatrix}\boldsymbol{h}_1\\ g\\ \boldsymbol{h}_2\\ \boldsymbol{h}_3\end{pmatrix},
$$

$$\mathcal{A}_1\boldsymbol{u} = k^{-1}\boldsymbol{u} - Re^{-1}\Delta\boldsymbol{u} + k^{-1}\mathrm{div}^*\mathrm{div}\boldsymbol{u} - s(\boldsymbol{u}\times\boldsymbol{B}^-)\times\boldsymbol{B}^-,$$

$$\mathcal{F}\boldsymbol{u} = s\boldsymbol{u}\times\boldsymbol{B}^-.$$

- idea: well-posedness analysis provides a natural block diagonal operator induced by the $\|\cdot\|_{\mathscr{H}}$ norm:

$$\mathcal{D} = \mathrm{diag}\left[\mathcal{H}_1, \mathcal{H}_2, \alpha k^{-1}\mathcal{I}_3, \mathcal{H}_4\right],$$

which is spectrally equivalent to $\mathcal{A}$. It is easy to invert $\mathcal{D}$, as a *norm equivalent preconditioner* (Mardal, Winther 2011).

- similar idea for block triangular preconditioners:

$$
\mathcal{M}_{\mathcal{L}} =
\begin{pmatrix}
\mathcal{A}_1 & 0 & 0 & 0\\
\mathrm{div} & k\mathcal{I}_2 & 0 & 0\\
0 & 0 & \alpha k^{-1}\mathcal{I}_3 & 0\\
\mathcal{F} & 0 & -\alpha\mathrm{curl}^* & \mathcal{H}_4
\end{pmatrix}^{-1},
$$

- rigorous analysis, robust with the h (mesh size), k, R_m and s,

> *(Reminder of the Field-of-Value (FOV) analysis for non-symmetric systems). Let $\mathcal{M}$ be an SPD matrix. If*
>
> $$\gamma \le \frac{(x,\mathcal{M}_{\mathcal{L}}\mathcal{A}x)_{\mathcal{M}^{-1}}}{(x,x)_{\mathcal{M}^{-1}}}, \quad \frac{\|\mathcal{M}_{\mathcal{L}}\mathcal{A}x\|_{\mathcal{M}^{-1}}}{\|x\|_{\mathcal{M}^{-1}}} \le \Gamma,$$
>
> *then the m^{th}-iteration solution x^m of the GMRES satisfies*
>
> $$\frac{\left\|\mathcal{M}_{\mathcal{L}}\mathcal{A}\left(x-x^m\right)\right\|_{\mathcal{M}^{-1}}}{\left\|\mathcal{M}_{\mathcal{L}}\mathcal{A}\left(x-x^0\right)\right\|_{\mathcal{M}^{-1}}} \le \left(1-\frac{\gamma^2}{\Gamma^2}\right)^{m/2}.$$

- decoupling at the preconditioner level, solving uniphysics problem for each block:
  - $\boldsymbol{u}$: GMG/AMG, DDM etc.
  - p: Gauss-Seidel, Jacobi etc.
  - $\boldsymbol{B}$ and $\boldsymbol{E}$: HX preconditioners, AFW smoothers etc.
- robust convergence also for inexact uniphysics solvers,
- $\nabla\cdot\boldsymbol{B}_h = 0$ preserved in preconditioned MINRES/GMRES solvers.

Numerical tests

2D lid-driven cavity on $\Omega = [0,1]\times[0,1]$, $\boldsymbol{u} = \boldsymbol{u}_0$ on the boundary $y = 1$. The constant background magnetic field is $\boldsymbol{B}_0 = (0,1)^T$. The velocity field perturbs the magnetic fields, and the Lorentz force acts on the fluid and changes the motion of the fluid flow.

- number of iterations:

(a) $R_e = 1$, $R_m = 1$

Δt \ h	1/32	1/64	1/128
0.02	23	23	22
0.01	24	24	22
0.005	24	23	22
0.0025	23	23	22

(b) $R_e = 1$, $R_m = 400$

Δt \ h	1/32	1/64	1/128
0.02	19	23	23
0.01	17	19	21
0.005	16	17	19
0.0025	14	16	17

(c) $R_e = 400$, $R_m = 1$

Δt \ h	1/32	1/64	1/128
0.02	15	15	15
0.01	13	15	15
0.005	13	13	15
0.0025	13	13	13

(d) $R_e = 400$, $R_m = 400$

Δt \ h	1/32	1/64	1/128
0.02	14	16	17
0.01	10	14	16
0.005	10	12	15
0.0025	9	10	13

Table 1: Block diagonal preconditioner $\mathcal{D}$ for MINRES method (diagonal blocks are solved exactly)

(a) $R_e = 1$, $R_m = 1$

Δt \ h	1/32	1/64	1/128
0.02	6	6	6
0.01	6	6	6
0.005	7	5	5
0.0025	6	6	6

(b) $R_e = 1$, $R_m = 400$

Δt \ h	1/32	1/64	1/128
0.02	5	5	5
0.01	5	5	5
0.005	6	5	5
0.0025	6	6	6

(c) $R_e = 400$, $R_m = 1$

Δt \ h	1/32	1/64	1/128
0.02	5	5	5
0.01	4	4	4
0.005	4	4	4
0.0025	5	4	4

(d) $R_e = 400$, $R_m = 400$

Δt \ h	1/32	1/64	1/128
0.02	5	4	4
0.01	4	4	4
0.005	4	4	4
0.0025	5	4	4

Table 2: Block lower triangular preconditioner $\mathcal{M}_{\mathcal{L}}$ for FGMRES method (diagonal blocks are solved exactly)

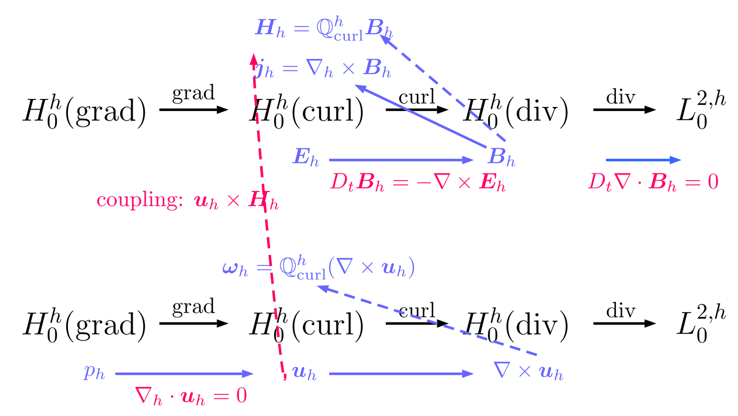
- More numerical tests (including 3D cases and inexact uniphysics solvers) in [4, 3].

Helicity preserving algorithms

In addition to preserving the discrete energy law and the Gauss law, we modify the algorithms to preserve helicity in the ideal limit as well.

- idea: use projections of $\nabla\times\boldsymbol{u}$ and $\boldsymbol{B}$ as additional variables.
  - $\nabla\times\boldsymbol{u}\in H_0^h(\mathrm{div})$, $\quad\boldsymbol{\omega} := \mathbb{Q}_{\mathrm{curl}}^h(\nabla\times\boldsymbol{u})\in H_0^h(\mathrm{curl})$,
  - $\boldsymbol{B}\in H_0^h(\mathrm{div})$, $\quad\boldsymbol{H} := \mathbb{Q}_{\mathrm{curl}}^h\boldsymbol{B}\in H_0^h(\mathrm{curl})$,

  where $\mathbb{Q}_{\mathrm{curl}}^h : L^2\to H_0^h(\mathrm{curl})$ is the L^2 projection.



- differential form point of view: introducing Hodge dual
  - $\boldsymbol{B}$: 2-form, $\quad\boldsymbol{H} := *\boldsymbol{B}$: 1-form,
  - $d\boldsymbol{u}$: 2-form, $\quad\boldsymbol{\omega} := *d\boldsymbol{u}$: 1-form.
- algorithm (with Crank-Nicolson stepping):

$$D_t\boldsymbol{u} := k^{-1}(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n), \quad \boldsymbol{u} := 1/2(\boldsymbol{u}^{n+1} + \boldsymbol{u}^n).$$

$$\boldsymbol{X}_h := [H_0^h(\mathrm{curl},\Omega)]^5\times H_0^h(\mathrm{div},\Omega)\times H_0^h(\mathrm{grad}).$$

Find $(\boldsymbol{u},\boldsymbol{\omega},\boldsymbol{j},\boldsymbol{E},\boldsymbol{H},\boldsymbol{B},p)\in\boldsymbol{X}_h$, s.t. $\forall(\boldsymbol{v},\boldsymbol{\mu},\boldsymbol{k},\boldsymbol{F},\boldsymbol{G},\boldsymbol{C},q)\in\boldsymbol{X}_h$:

$$
\begin{aligned}
(D_t\boldsymbol{u},\boldsymbol{v}) - (\boldsymbol{u}\times\boldsymbol{\omega},\boldsymbol{v})&\\
+(\nabla p,\boldsymbol{v}) - \mathsf{c}(\boldsymbol{j}\times\boldsymbol{H},\boldsymbol{v}) &= (\boldsymbol{f},\boldsymbol{v}),\\
(D_t\boldsymbol{B},\boldsymbol{C}) + (\nabla\times\boldsymbol{E},\boldsymbol{C}) &= 0,\\
(\boldsymbol{u},\nabla q) &= 0,\\
(\boldsymbol{\omega},\boldsymbol{\mu}) &= (\nabla\times\boldsymbol{u},\boldsymbol{\mu}),\\
(\boldsymbol{B},\boldsymbol{F}) &= (\boldsymbol{H},\boldsymbol{F}),\\
(\boldsymbol{E},\boldsymbol{G}) &= -(\boldsymbol{u}\times\boldsymbol{H},\boldsymbol{G}),\\
(\boldsymbol{j},\boldsymbol{k}) &= (\boldsymbol{B},\nabla\times\boldsymbol{k}).
\end{aligned}
$$

several local projections, easy to solve.

- discrete energy law, precise magnetic Gauss law,
- helicity conservation: with suitable boundary conditions,

$$D_t\int_\Omega\boldsymbol{B}\cdot\boldsymbol{A} = 0, \quad D_t\int_\Omega\boldsymbol{u}\cdot\boldsymbol{B} = 0,$$

for any $\boldsymbol{A} = \mathrm{curl}^{-1}\boldsymbol{B}$.

Summary and future directions

- key: discretize variables on discrete de Rham complexes and preserve structures,
  - differential structures: differential forms and de Rham complex,
  - nonlinear structures and symmetry: *Lorentz force* and *magnetic convection* cancel each other in the energy law, *fluid convection* and *magnetic convection* cancel in the helicity conservation,
- future directions:
  - stabilization for convection-dominated problems (large R_e, R_m),
  - temporal discretization, Hamiltonian and Lagrangian mechanics,
  - multiscale,
  - applications in plasma physics and collaborations.

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