CS 517: Problem Set 1

1. Consider the following function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$:

$$f(x) = \{i \mid \text{the } 2^i\text{'s place in the binary expansion of } x \text{ is } 1\}$$

Why does this f not contradict Cantor's theorem? Give an *explicit* counterexample to the claim that this f is a bijection.

2. A Turing printer (TP) is a Turing machine with a work tape, a special "print" tape, and a special "print" state. The TM starts with an empty work tape. Every time it enters the "print" state, we consider the current contents of the print-tape to be "printed." If P is a TP, then we write $L(P)$ to denote the set of strings that P eventually prints.

   (a) Prove that a language L is Turing-recognizable **if and only if** $L = L(P)$ for some TP P.

   (b) Prove that a language L is Turing-decidable **if and only if** $L = L(P)$ for some TP P that prints strings in lexicographic order.

   (c) Prove that every *infinite* Turing-recognizable language L contains an *infinite* subset that is Turing-decidable.

3. A language C is said to **separate** A and B if $A \subseteq C$ and $B \subseteq \overline{C}$. Construct two Turing-recognizable languages A and B so that no Turing-decidable language separates them.