Inference in High-Dimensional Panel Models: Two-Way Dependence and Unobserved Heterogeneity

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Abstract

Panel data allows for the modeling of unobserved heterogeneity, which significantly increases the number of nuisance parameters, making high dimensionality a practical issue rather than just a theoretical concern. However, unobserved heterogeneity, along with potential temporal and cross-sectional dependence in panel data, further complicates estimation and inference for high-dimensional models. This paper proposes a toolkit for robust estimation and inference in high-dimensional panel models with large cross-sectional and time sample sizes. To reduce the dimensionality, I propose a weighted LASSO using two-way cluster-robust penalty weights. Due to the cluster dependence driven by the underlying components, the rate of convergence is slow even in an oracle case. Nevertheless, by leveraging a clustered-panel cross-fitting approach for bias-correction, the asymptotic normality on low-dimensional parameters can be established using the weighted LASSO for nuisance estimation. As a special case, in a partial linear model with non-additive unobserved time and unit effects, inferential results are also established using the full sample. In a panel estimation of the government spending multiplier, I demonstrate how high dimensionality can be hidden and how the proposed toolkit enables flexible modeling and robust inference.

Keywords: high-dimensional regression, two-way cluster dependence, correlated time effects, unobservable heterogeneity, LASSO, Post-LASSO, double/debiased machine learning, cross-fitting.
JEL Classification: C01, C14, C23, C33

1. Introduction

In economic research, high dimensionality typically refers to the large number of unknown parameters relative to the sample size, under which traditional estimations are either infeasible or tend to yield estimates too noisy to be informative. The issue of high dimensionality becomes more relevant as data availability grows and economic modeling involves more flexibility. Commonly, the problem of high dimensionality appears in at least the following three scenarios:

- The dimension of observable and potentially relevant variables can be large relative to the sample. In trade literature, preferential trade agreements (PTAs) usually involve a large number of provisions even

though most policy analysis only focuses on the effect of a small subset of the provisions [1]. In demand analysis, even if the focus is on the own-price elasticity, the prices of relevant goods should also be included, unless strong assumptions for aggregation are assumed (see Chernozhukov et al., 2019).

- With nonparametric or semiparametric modeling, the unknown functions are viewed as infinite-dimensional parameters regardless of the dimension of observable variables. If the unknown function $g(X)$ is approximately sparse and can be well-approximated by a linear combination of the 3rd-order polynomial transformation of X, then it would involves 285 transformed regressors when the dimension of X is 10 and 1770 when we start with a dimension of 20. [2]

- The modeling of heterogeneity can raise the number of nuisance parameters drastically. In demand analysis, income effects are specific to products if the homothetic preference assumption fails. For difference-in-difference analysis, allowing unit-specific trends and heterogeneous trends across the covariates can relax/test the parallel trend assumption. For models with unobserved heterogeneity that appears in a nonlinear way, either treating them as parameters to be estimated (fixed effects) or modeling them in a flexible way (correlated random effects) contributes to high dimensionality. [3].

Particularly, the modeling of heterogeneity in panel models makes high dimensionality more of a practical issue rather than just a theoretical concern. As a concrete example, let's consider a panel model where all three sources of high dimensionality are involved:

$$Y_{it} = D_{it}\theta_0 + g_0(X_{it}, c_i, d_t) + U_{it}, \tag{1.1}$$

where D_{it} is a vector of low-dimensional treatment or policy variables. X_{it} is a vector of potentially high-dimensional control variables. D_{it} can also contain some higher order effects and interactive effects with a subset of the controls to allow for nonlinear and heterogeneous effects in a parametric way. When D_{it} is not conditionally uncounfounded, instrumental variables would be used for identification of θ_0. ; $g(.)$ is an unknown function, e.g. an infinite dimensional parameter; c_i and d_t are unobserved heterogeneous effects, either as fixed-effect parameters or correlated random variables. The interest lies in the estimation and inference on the low-dimensional parameters of interest θ_0.

Without considering the features of panel data and the unobserved heterogeneity, it is a classic partial linear model that has been well-studied in previous semiparametric literature. o address the high-dimensional issues in the model, regularization approaches, also known as machine learning, have been employed for

[1] Based on data from Mattoo et al. (2020), 282 PTAs were signed and notified to the WTO between 1958 and 2017, encompassing 937 provisions across 17 policy areas. See Breinlich et al. (2022).

[2] For a vector X with dimension k, it is easy to show that the 2nd-order polynomial transformation generates $\frac{k^2}{2} + \frac{3}{2}k$ terms and the 3rd-order polynomial transformation generates $k + \frac{1}{2}k(k+1) + \frac{1}{2}\sum_{l=1}^{k} l(l+1) = \frac{1}{6}k^3 + k^2 + \frac{11}{6}k$ terms.

[3] This is particularly relevant in trade literature where the unobserved heterogeneity derived from the gravity model takes a pairwise form among the importers, exporters, and the time. As each of these three dimensions expands, the number of nuisance parameters explodes quickly. See Correia et al. (2020), Chiang et al. (2021), and Chiang et al. (2023b), for example.

estimation, which trades off bias for smaller variance. However, due to the bias introduced by regularization and overfitting, inference is challenging. Typically, some types of bias-correction approaches are involved to obtain desirable statistical properties of high-dimensional estimation and inference approaches.

In a panel data setting, it is soon realized that three challenges would appear if researchers attempt to apply the existing high-dimensional approaches directly. First of all, the statistical properties of many high-dimensional estimators remain unknown with panel data, which is potentially dependent across space and time. Secondly, some bias-correction procedures for inference such as sample-splitting/cross-fitting are very specific to the dependence structure of the data and existing approaches are not valid under two-way dependence in panels. Thirdly, panel data models often consider unobserved individual and time effects, which may lead to another source of high dimensionality and further complicate estimation and inference.

To reduce the dimensionality, as the first challenge, I proposed a variant of LASSO that uses regressor-specific penalty weights robust to two-way cluster dependence and weak temporal dependence across clusters. Such a LASSO approach is named a two-way cluster-LASSO, corresponding to the heteroskedasticity-robust LASSO in Belloni et al. (2012) and the cluster-LASSO in Belloni et al. (2016). This approach identifies the common penalty level λ up to a constant and a small-order sequence that do not vary across different data generating processes. Therefore, data-driven tuning, such as cross-validation, is not needed, which makes it more computationally efficient and to avoid non-trivial theories that take data-driven tuning into account.

A common and important condition for obtaining the desirable statistical properties of LASSO selection/estimation is the so-called "regularization event", which states that the (overall) penalty level is sufficient large to dominate the "noise" in the high-dimensional estimation (but not too large at the same time to avoid under-selection and slow rate of convergence). However, existing approaches for guaranteeing such an event to happen with probability approaching one does not extend to this case due to the two-way cluster dependence. Instead, by considering the component structure characterization of the two-way dependence and decomposing the correlated error terms using Hajek projections, I am able to leveraging the moderate deviation theorems by Peña et al., 2009 and Gao et al., 2022 and the concentration inequality by Fuk and Nagaev (1971) for bounding the tail probability of the "noise" term. Combining with existing non-asymptotic bounds for the LASSO approach in Belloni et al. (2012), I derive the rate of convergence for the (post) two-way cluster LASSO.

According to the rate of convergence results, the proposed (post) LASSO is consistent under certain sparsity conditions. However, it is also revealed that the convergence rate is not as fast as the common rates for LASSO estimation due to the two-way cluster dependence. The problem lies in the underlying component structure. To illustrate, consider the simplest multivariate mean model through a component structure representation:

$$Y_{it} = \theta_0 + f(\alpha_i, \gamma_t, \epsilon_{it}) \tag{1.2}$$

where Y_{it} is a high-dimensional vector with dimension $s = o(NT)$ and $\theta_0 = \mathrm{E}[Y_{it}]$; α_i, γ_t, and ε_{it} are unobserved random elements. This is a common characterization of cluster dependence in the literature of cluster-robust inference: we notice that α_i introduce cluster/temporal dependence within group i and γ_t introduce cluster/cross-sectional dependence within group t. To estimate the high-dimensional vector θ_0, we consider the sample mean estimator $\widehat{\theta} = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} Y_{it}$. We can rewrite the estimator through a Hajek projection:

$$\widehat{\theta} - \theta_0 = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(a_i + g_t + e_{it}) = \frac{1}{N}\sum_{i=1}^{N} a_i + \frac{1}{T}\sum_{t=1}^{T} g_t + \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} e_{it}, \tag{1.3}$$

where $a_i := \mathrm{E}[Y_{it} - \theta_0|\alpha_i], g_t := \mathrm{E}[Y_{it} - \gamma_t]$, and $e_{it} := Y_{it} - \theta_0 - a_i - g_t$. For illustration purpose, suppose those components are i.i.d sequences and independent of each other. Then it can be shown that, under some regularity conditions, for each $j = 1, ..., s$, $\widehat{\theta}_j - \theta_0 = O_P\left(\frac{1}{\sqrt{N\wedge T}}\right)$ and $\|\widehat{\theta} - \theta_0\|_2 = \left(\sum_{j=1}^{s}\left(\widehat{\theta}_j - \theta_{0j}\right)^2\right)^{1/2} = O_P\left(\sqrt{\frac{s}{N\wedge T}}\right)$. We see that for $\widehat{\theta} - \theta_0$ to have a well-behaved asymptotic distribution (e.g., equipped with finite second moment while not degenerate), we need to scale it by $\frac{1}{\sqrt{N\wedge T}}$ while also requiring both N, T diverging to infinity. While $\widehat{\theta}$ is still consistent when N, T diverge at the same rate, $\|\widehat{\theta} - \theta_0\|_2$ converges slower than $o_P\left(\sqrt{\frac{1}{N\wedge T}}\right) = o_P\left((NT)^{-1/4}\right)$, which is a common rate requirement for inferential theory.

This is where the second challenge arises: if a faster rate of convergence is not achievable due to the two-way cluster-dependence, some bias-correction approaches are needed to relax the rate requirement for valid inference. One common approach in semiparametric literature is to add a correction term to the original identifying moment function which can result in an orthogonalized moment condition. The orthogonal moments often feature multiplicative error terms, closely related to the doubly robust estimators, and therefore the nuisance estimations are allowed to be relatively more noisy. By the orthogonality itself, however, in general it does not ensure valid inference in a high-dimensional setting. An extra bias-correction approach, sample splitting or its generalization cross-fitting, has been proposed for inference in high-dimensional regression models. The idea of sample splitting in a two-step estimation is to split the sample in a proper way and use the sub-samples separately for each step. If the sub-samples are independent of each other, then the first-step estimates will be independent of the sample used for the second-step estimation. With this property, the error term that causes the bias can vanish with a less stringent rate requirement on the first step. Intuitively, the dependence between the two steps is eliminated so that a potentially over-fitted nuisance estimate from the first step does not pollute the second-step estimator as much as it would otherwise do. However, sample-splitting as well as cross-fitting is very sensitive to the sampling assumption. Building upon recent development of cross-fitting approaches for dependent data (Chiang et al. (2022); Semenova et al. (2023a)), I propose a clustered-panel cross-fitting scheme and I show the constructed main and auxiliary samples are "approximately" and independent of each other. Effectively, this inferential procedure

extends the double/debiased machine learning (DML, hereafter) approach by Chernozhukov et al. (2018a) to panel data models, and it is labeled as panel DML. Asymptotic normality for the panel DML estimator is established given high-level assumptions on the convergence rates regarding the first-step estimator. It is shown that the crude requirement on the rate of convergence can be relaxed to $o\left((N \wedge T)^{-1/4}\right)$ in L^2 norm, which admits the first-step estimation through the two-way cluster LASSO.

For the third challenge caused by the unobserved heterogeneity, existing literature has proposed to use fixed-effect or common correlated effect approaches by assuming the unknown function g_0 in 1.1 is linear in (c_i, d_t) (Belloni et al., 2016; Kock and Tang, 2019) or linear in the interactive fixed effects (Vogt et al., 2022). To allow for flexible function forms while remaining tractable, I propose to model (c_i, d_t) as correlated random effects through a generalized Mundlak device while assuming the unknown function to be approximately sparse. In that way, a very rich form of heterogeneity is permitted. While not all of those are relevant and the identity of the truly relevant heterogeneous effects is unknown to the researcher, we can use suitable machine learning approaches, e.g. the two-way cluster-LASSO, to select the relevant effects. However, there is one more subtle issue: common approaches including Mundlak device that deal with the unobserved heterogeneous effects introduce cross-sectional and temporal sample averages which in turn bring dependence across cross-fitting sub-samples. Furthermore, even if it remains valid under extra conditions, cross-fitting often causes the loss of efficiency due to the exclusion of observations. On the other hand, without cross-fitting, valid inference remain challenging for high-dimensional panel models in general. Nevertheless, in the case of partial linear panel model, I show that inferential theory can be established using the full sample.

In the empirical application, I re-examine the effects of government spending on the output of an open economy following the framework of Nakamura and Steinsson (2014), a well-cited empirical-macro paper. While they study it using a panel data approach considering unobserved heterogeneous effects that raise the nuisance parameters as the sample size grows, it is not considered as a high-dimensional problem in the baseline setting: a linear panel model with only a few covariates and additive unobserved heterogeneous effects; the identification is through the instrumental variable. However, even in a conventionally low-dimensional setting, high dimensionality can be hidden in the sense that the true model can be highly nonlinear in the covariates and that the unobserved heterogeneous effects can enter the model in a more flexible way. To avoid the endogeneity caused by potential misspecification in the function form, I consider extending the baseline model in a more flexible way as in 1.1, which introduces high-dimensional nuisance parameters. Due to potential two-way cluster dependence, existing high-dimensional methods designed for independent or weakly dependent data may not be valid. This is where the proposed dependence-robust estimation and inference for high-dimensional models can be leveraged and the results can be used for a robustness check. It is shown that the estimates are consistent with the baseline results, which indicates the nonlinear and interactive effects may not be very relevant in this model; other estimation approaches that are not robust to either high-dimensionality or two-way cluster dependence tend to over-fit and result in more noisy estimates (with larger standard errors).

The rest of the paper is outlined as follows: The next sub-section reviews relevant literature and sum-

marizes the differences and contributions of this paper relative to the existing ones. Section 2 presents the two-way cluster-LASSO estimator and the investigation of its asymptotic properties under two-way cluster dependence. Section 3 introduces a sub-sampling scheme designed for cross-fitting that allows within-cluster dependence and weak dependence across clusters. It is then used as a bias-correction approach for valid inference on the low-dimensional parameter considering the effect of high-dimensional nuisance estimates. In Section 4, the partial linear model with unobserved heterogeneity is studied in detail as a leading example. Simulation evidence is given in Section 5 where the proposed approaches are competed with existing ones. In Section 6, the empirical estimation of the government spending multiplier is used as an illustration of hidden high dimensionality and the application of the proposed toolkit. Section 7 concludes the paper with a discussion of limitations and detailed empirical recommendations.

1.1. ***Relation to the Literature***

This paper builds upon literature on $l1$ regularization methods in high-dimensional regression. Bickel et al. (2009) derive the convergence rate of the prediction risk in terms of the empirical norm under homogeneous Gaussian error, restricted eigenvalue, and sparsity assumption. Bühlmann and Van De Geer (2011) instead assumes a sub-Gaussian tail property to derive similar results of convergence rates. See Section 29.11 of Hansen (2022) for an illustration and extension of Bickel et al. (2009)'s analysis under heteroskedasticity. Under Gaussian or sub-Gaussian errors, Basu and Michailidis (2015); Kock and Callot (2015); Lin and Michailidis (2017) study LASSO-based approaches for dependent data. To allow for both non-Gaussian errors and dependent data, Wu and Wu (2016), Chernozhukov et al. (2021a), Babii et al. (2022, 2023) Gao et al. (2024) derive Nagaev-type concentration inequalities to bound the tail probability assuming a proper order of the penalty level. However, all aforementioned LASSO-based approaches require delicate tuning of the penalty level to ensure a desirable finite sample performance. The common cross-validation approaches and bootstrap in Chernozhukov et al. (2021a) for choosing the penalty level are computationally costly and are very sensitive to the sampling assumption. Plus, the statistical analysis accounting for the data-driven penalty level is highly non-trivial (see Chetverikov et al., 2021 for validity on cross-validation LASSO under random sampling). As another strand, Belloni et al. (2011, 2012, 2016) propose other variants of LASSO approaches and leverage (self-normalized) moderate deviation theorems to derive theoretically-driven penalty levels. However, their methodologies cannot be easily extended to the the settings with two-way dependence. The proposed variant of LASSO is built upon aforementioned literature and employ both Nagaev-type inequalities (Fuk and Nagaev, 1971) and moderate deviation theorem for self-normalized sums (Peña et al., 2009; Gao et al., 2022). Up to my knowledge, it is the first LASSO and high-dimensional estimator that is robust to the two-way cluster dependence and weak dependence across clusters and the common penalty level is also theoretically driven.

The inferential theory in high-dimensional regression models typically relies on some bias-correction methods and they are particularly important here due to the two-way cluster dependence that results in a slow rate of convergence. Bias-correction approaches for inference purpose take various forms in the literature:

for example, the low-dimensional projection adjustment in Zhang and Zhang (2014), the de-sparsification procedure in Van de Geer et al. (2014), the decorrelating matrix adjustment in Javanmard and Montanari (2014), the double selection approach in Belloni et al. (2014), the decorrelated score construction in Ning and Liu (2017), the Neyman orthogonal moment construction in Chernozhukov et al. (2018a, 2022a). The last strand of the literature is often labeled as the debiased machine learning (DML) approach, which is closely related to previous semiparametric literature including Ichimura (1987), Robinson (1988), Powell et al. (1989), Newey (1994), and Andrews (1994). The idea of the orthogonalization is to add a correction term to the original identifying moment function so that the second-step estimator is less sensitive to the plug-in of noisy first steps. Due to the resulting multiplicative error term in the orthogonal moment condition, it is also related to the doubly-robust literature. Newey (1994) provides a general construction of the orthogonal moment condition through the influence functions. It is further facilitated by Ichimura and Newey (2022) for identifying moment conditions satisfying certain restrictions. See Chernozhukov et al. (2018a) and Chernozhukov et al. (2022a) for a summary of such constructions and known orthogonal moment functions. More recently, Chernozhukov et al. (2018b, 2021b, 2022b,c); Jordan et al. (2023) provide an alternative approach by estimating the correction term without knowing its analytical form. For the inferential theory in high-dimensional panel models, this paper takes the orthogonalization step as given and focuses on nuisance estimation and cross-fitting.

Sample-splitting and cross-fitting serves as another bias-correction approach in this paper has been widely used in many two-step estimations. The role of cross-fitting in high-dimensional inferential theory is to remove the dependence between the nuisance estimation and the second-step estimation so that the over-fitting bias from the first step has less impact on the second step. Technically, it allows for a slower rate of convergence in the first step and it in turn relaxes the sparsity condition (e.g., Belloni et al., 2014). Chernozhukov et al. (2018a) generalize the sample-splitting procedure as a cross-fitting scheme which further improves finite sample performance by reducing the noise due to arbitrary splitting of the sample. Chiang et al. (2021, 2022) propose a cross-fitting scheme robust to separately and jointly exchangeable arrays. Semenova et al. (2023a) propose a leave-one-neighborhood-out cross-fitting and introduce a coupling approach (due to Strassen, 1965 and Berbee, 1987) to prove the validity of cross-fitting under temporal dependence. The idea of leave-one-neighborhood-out sub-sampling scheme is also shared by $h-$block cross-validation (Burman et al., 1994; Racine, 2000) and big-block-small-block technique in time series literature (e.g., Gao et al., 2022). Built upon previous literature, I propose a more robust cross-fitting scheme that is valid under not only cluster dependence but also weak temporal dependence across clusters.

This paper also belongs to the cluster-robust inference literature. The characterization of the two-way cluster dependence is based on the Aldous-Huber-Kallenberg (AHK) type representation, which is common is this literature (e.g., Djogbenou et al., 2019, Roodman et al., 2019, Davezies et al., 2019, and Menzel, 2021). This original representation only works for exchangeable arrays, which is violated in panel data settings with autocorrelation over time. Chiang et al. (2024) generalizes this representation by allowing the time factor to be correlated over time and Chen and Vogelsang (2024) also considers this representation

when deriving fixed-b asymptotic results for inference. Differing from the original representation theorem, strictly speaking, it is not a representation anymore but more of an assumption and a general characterization of cluster dependence. Such characterization of the dependence structure is common in economics studies (e.g., Rajan and Zingales, 1998, Fama and French, 2000, Li et al., 2004, Larrain, 2006, Thompson, 2011, Nakamura and Steinsson, 2014, Guvenen et al., 2017, Ellison et al., 2024, and Nakamura and Steinsson, 2014 among many others). In this paper, AHK representation introduces dependence both within and across clusters, and the asymptotic variance of the DML estimator has two components: one is due to within-unit dependence and one is due to both within-time dependence and across-time dependence. Therefore, the usual one-way or two-way cluster variance estimator is not valid. I propose variance estimators similar to Chiang et al. (2024) and Chen and Vogelsang (2024), with careful adjustment due to cross-fitting procedures.

1.2. Notation.

Here is a collection of the most frequently used notations in this paper. Some extra notations are defined along with the context. I use E and P as generic expectation and probability operators. I denote $\mathcal{P}_{NT}$ as an expanding collection of all data-generating processes P that satisfy certain conditions. I denote P_{NT} as a sequence of probability laws such that $P_{NT} \in \mathcal{P}_{NT}$. for each (N, T) We will suppress the dependence on (N, T) and P_{NT} whenever clear in its setting. We will use the following vector and matrix norms: we denote $\|.\|$ as the Euclidean (Frobenius) norm for a matrix. Let $\mathbf{x}$ be a generic $k \times 1$ real vector, then the l^q norm is denoted as $\|\mathbf{x}\|_q := \left(\sum_{j=1}^k x_j^q\right)^{1/q}$ for $1 \leq q < \infty$, and $\|\mathbf{x}\|_\infty := \max_{1\leq j\leq k} |x_j|$. The $L^q(P)$ norm is denoted as $\|f\|_{P,q} := \left(\int \|f(\omega)\|^q dP(\omega)\right)^{1/q}$ where f is a random element with probability law P. I denote the empirical average of f_{it} over $i = 1, ..., N$ and $t = 1, ..., T$ as $\mathbb{E}_{NT}[f_{it}] = \frac{1}{NT}\sum_{i=1}^N \sum_{t=1}^T f_{it}$ and the empirical L^2 norm as $\|f_{it}\|_{NT,2} = \left(\frac{1}{NT}\sum_{i=1}^N \sum_{t=1}^T \|f_{it}\|^2\right)^{1/2}$. Correspondingly, I denote the empirical average of f_{it} over the sub-sample $i \in I_k$ and $t \in S_l$ as $\mathbb{E}_{kl}[f_{it}] = \frac{1}{N_k T_l}\sum_{i\in I_k, t\in S_l} f_{it}$ and the empirical L^2 norm over the subsample as $\|f_{it}\|_{kl,2} = \left(\frac{1}{N_k T_l}\sum_{i\in N_l} \sum_{t\in T_l} \|f_{it}\|^2\right)^{1/2}$, where I_k, S_l are sub-sample index sets and N_k, T_l are sub-sample sizes that will be introduced next section.

2. Two-Way Cluster LASSO

In the literature, not much is known in terms of statistical properties for high-dimensional methods under cluster dependence in both space and time. In this section, a variant of the $l1$-regularization methods, also known as the LASSO, will be constructed and examined. Besides a common penalty level that is theoretically driven, regressor-specific penalty weights robust to two-way cluster dependence and temporal weak dependence across clusters will be utilized. Such a LASSO approach is named two-way cluster LASSO, corresponding to the existing heteroskedasticity-robust LASSO in Belloni et al. (2012) and the cluster-LASSO in Belloni et al. (2016). The intuition is to replace the regressor-specific penalty weights employed in the aforementioned papers with some alternative choices that are robust to two-way cluster dependence in the

panel. However, the valid choices are not immediately clear and the theory is highly non-trivial due to the dependence in both cross-sectional and temporal dimensions.

To focus on the LASSO approach under two-way dependence, I consider a simple conditional expectation model of a scalar outcome given a potentially high-dimensional vector of covariates, and the regression model throughout this section is free from the endogeneity issue caused by unobserved heterogeneous effects. Let (Y_{it}, X_{it}) be a sample with $i = 1, ..., N$ and $t = 1, ..., T$. The conditional expectation model can be expressed as follows:

$$Y_{it} = f(X_{it}) + V_{it},\ \mathrm{E}[V_{it}|X_{it}] = 0 \tag{2.1}$$

where $f(X) := E[Y|X]$ is an unknown conditional expectation function of potentially high-dimensional covariates X and V_{it} is the associated stochastic error.

To characterize the two-way cluster dependence in the panel, I assume the random elements $W_{it} := (Y_{it}, X_{it}, V_{it})$ are generated from the following process:

Assumption AHK (Aldous-Hoover-Kallenberg Component Structure Characterization)**.**

$$W_{it} = \mu + f\left(\alpha_i, \gamma_t, \varepsilon_{it}\right),\ \ \forall i \geq 1, t \geq 1, \tag{2.2}$$

where $\mu = \mathrm{E}_P[W_{it}]$, f is some unknown measurable function; $(\alpha_i)_{i\geq 1}$, $(\gamma_t)_{t\geq 1}$, and $(\varepsilon_{it})_{i\geq 1,t\geq 1}$ are mutually independent sequences, α_i is i.i.d across i, ε_{it} is i.i.d across i and t, and γ_t is strictly stationary.

Assumption AHK is motivated by a representation theorem for an exchangeable array, named after Aldous-Hoover-Kallenberg (AHK, hereafter), which states that if an array of random variables $(X_{ij})_{i\geq 1,j\geq 1}$ is separately or jointly exchangeable[4], then $X_{ij} = f(\nu, \xi_i, \iota_j, \zeta_{ij})$ where $\nu, (\xi_i)_{i\geq 1}, (\iota_j)_{j\geq 1}, (\zeta_{ij})_{i\geq 1,j\geq 1}$ are mutually independent, uniformly distributed i.i.d. random variables[5]. However, the exchangeability is not likely to hold for arrays with the presence of a temporal dimension since it is naturally ordered. In macroeconomics, for instance, we can interpret the time components $(\gamma_t)_{t\geq 1}$ as unobserved common time shocks, which are naturally correlated over time, implying the exchangeability violated. Therefore, by allowing γ_t to be correlated, it introduces temporal dependence across all clusters, making the characterization more sensible. The relaxation of the independence condition on $(\gamma_t)_{t\geq 1}$ can be viewed as a generalization of the component structure representation, as argued by Chiang et al. (2024). It is clear that under Assumption AHK, W_{it} and W_{is} are correlated for any i, t, s due to sharing the same cross-sectional cluster. Similarly, due to sharing the same temporal cluster, W_{it} and W_{jt} are dependent for any t, i, j. Furthermore, even if sharing neither the

[4]An array $(X_{ij})_{i\geq 1,j\geq 1}$ is separately exchangeable if $\left(X_{\pi(i),\pi'(j)}\right) \overset{d}{=} \left(X_{ij}\right)$, and jointly exchangeable if the same condition holds with $\pi = \pi'$.

[5]This is first proved in Aldous (1981) and independently proved and generalized to higher dimensional arrays in Hoover (1979). It is then further studied in Kallenberg (1989). For a formal statement of the theorem, see, for example, Theorem 7.22 in Kallenberg (2005).

cross-sectional or temporal dimensions, observations can still be correlated due to correlated time effects γ_t. It is important to notice that the components in 2.2 simply characterize the dependence in panel data in a fairly general. Differing from factor models or models with unobserved heterogeneity, they do not affect the identification of the regression model in any way.

Throughout the paper, I consider the time effects γ_t to be weakly dependent, and some regularity condition is introduced for tractability. Before that, a few more concepts and notations need to be introduced. Let (X, Y) be random elements taking values in Euclidean space $S = (S_1 \times S_2)$ with laws P_X and P_Y, respectively. Let $\|\nu\|_{TV}$ denote the total variation norm of a signed measure ν on a measurable space (S, Σ) where Σ is a σ-algebra on S:

$$\|\nu\|_{TV} = \sup_{A \in \Sigma} \nu(A) - \nu(A^c).$$

Define the dependence coefficient of X and Y as:

$$\beta(X, Y) = \frac{1}{2} \|P_{X,Y} - P_X \times P_Y\|_{TV}.$$

The next assumption regulates the dependence of γ_t using the beta-mixing coefficient:

Assumption AR (Absolute Regularity)**.** *The sequence $\{\gamma_t\}_{t \geq 1}$ is beta-mixing at a geometric rate:*

$$\beta_\gamma(m) = \sup_{s \leq T} \beta\left(\{\gamma_t\}_{t \leq s}, \{\gamma_t\}_{t \geq s+m}\right) \leq c_\kappa exp(-\kappa m), \forall m \in \mathbb{Z}^+, \tag{2.3}$$

for some constants $\kappa > 0$ and $c_\kappa \geq 0$.

Condition AR, also known as the beta-mixing condition, restricts the temporal dependence of the common time effects to decay at a certain rate that is common in literature (for example, see Hahn and Kuersteiner (2011); Fernández-Val and Lee (2013), and can be generated by common autoregressive models as in Baraud et al. (2001).

Due to the potential high dimensionality in X, traditional nonparametric methods are not feasible for estimating the unknown function f. To reduce the dimensionality, a common approach is to assume sparsity and reduce the dimension through regularization. However, the unknown function f is an infinite dimensional parameter, which is not exactly sparse. Therefore, I take a sparse approximation approach following Belloni et al. (2012):

Assumption ASM (Approximate Sparse Model)**.** *The unknown function f can be well-approximated by a dictionary of transformations $f_{it} = F(X_{it})$ where f_{it} is a $p \times 1$ vector and F is a measurable map, such that*

$$f(X_{it}) = f_{it}\zeta_0 + r_{it}$$

where the coefficients ζ_0 and the approximation error r_{it} satisfy

$$\|\zeta_0\|_0 \leq s = o(N \wedge T),\ \|r_{it}\|_{NT,2} = O_P\left(\sqrt{\frac{s}{N \wedge T}}\right).$$

Assumption ASM views the high-dimensional linear regression as an approximation. It requires a subset of the parameters ζ_0 to be zero while controlling the size of the approximation error. Compared to the sparsity assumption in previous literature in high-dimensional regression, it requires a relatively slow rate of growth restriction on the non-zero slope coefficients. For example, $s = o(NT)$ corresponds to the case of heteroskedasticity-robust LASSO under i.i.d data in Belloni et al. (2012); $s = (Nl_T)$ corresponds to the cluster-robust LASSO under temporal dependence panel data in Belloni et al. (2016) where $l_T \in [1, T]$ is an information index that equals T when there is no temporal dependence and equals 1 when there is cross-sectional independence and perfect temporal dependence. In other words, the underlying factor structure restricts the growth of nonzero slope coefficients of the model in a similar way to the perfect temporal dependence case in Belloni et al. (2016).

Under Assumption ASM, we can rewrite the model 2.1 as

$$Y_{it} = f_{it}\zeta_0 + r_{it} + V_{it},\ \mathrm{E}[V_{it}|X_{it}] = 0. \tag{2.4}$$

Using 2.4, we can estimate ζ_0 allowing its dimension to be greater than the sample size by applying $l1$ regularization in the least squared error problem. Let λ be some non-negative common penalty level and ω be some non-negative $p \times p$ diagonal matrix of regressor-specific penalty weights. Consider the following generic weighted LASSO estimator:

$$\widehat{\zeta} = \arg\min_{\zeta} \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(Y_{it} - f_{it}\zeta)^2 + \frac{\lambda}{NT}\|\omega^{1/2}\zeta\|_1. \tag{2.5}$$

To obtain the desirable property of LASSO estimation or variable selection, one needs to choose λ and ω in an optimal way that the penalty level is large enough to avoid noisy estimation due to overfitting but also the smallest possible since the size of penalty determines the performance bound of LASSO estimation and too large a penalty level can cause missing variable bias. In other words, the overall penalty level given by both λ and ω decides the trade-off between overfitting variance and regularization bias. For example, let $\dot{f}_{it}$ be the demeaned f_{it} by the sample averages[6] and one common choice of ω is the empirical Gram matrix $\mathrm{E}[\dot{f}_{it}'\dot{f}_{it}]$ that is used to standardize the regressors and so the model selection is not affected by the scale of the regressors. The common penalty level λ is often chosen by some cross-validation algorithms. If the chosen λ satisfies a certain asymptotic order, then the key condition that regularizes the tail behavior of the

[6]The demeaning is done because of the inclusion of the intercept term. To avoid it to be penalized, it is usually projected out first.

error term can be established under conditional Gaussian error or sub-Gaussian error conditions (see Bickel et al., 2009, Bühlmann and Van De Geer, 2011, and Theorem 29.3 of Hansen, 2022):

$$\max_{j=1,...,p} \left| \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \omega_j^{-1/2} f_{it,j} V_{it} \right| \leq \frac{\lambda}{2c_1 NT}. \tag{2.6}$$

Condition 2.6 is referred to as the "regularization event" in the literature. Combining with some algebraic results, the rate of convergence for LASSO estimation can be derived. This approach, however, is not applicable when the error terms are considered to exhibit heavy tails. Alternatively, Fuk-Nagaev types of concentration inequality are established to verify Condition 2.6 without relying on the Gaussian or sub-Gaussian assumption (e.g. Babii et al. (2024, 2023); Gao et al. (2024)). These alternative approaches, again, rely on cross-validation for choosing penalty levels and are computationally costly and sensitive to the tuning of the cross-validation. It is further complicated when the cross-validation needs to be adjusted for dependent data and the validity of cross-validation is highly non-trivial.

Belloni et al. (2012) propose to self-normalize the regressors through regressor-specific penalty weights and leverage moderate deviation theorems (see Jing et al., 2003 and Peña et al., 2009) for the self-normalized sums to verify Condition 2.6. This common penalty level λ though this approach is theoretically derived and only determined by the sample size, the number of regressors, a small-order sequence and some constants that do not vary across data generate processes. When the dependence is considered only in the temporal dimension, then the existing approach for independent data can be extended by clustering over the cross-sectional dimension (see Belloni et al. (2016)). However, there is no simple extension if the dependence is present in both temporal and cross-sectional dimensions. Instead, I utilize the component structure characterization of the dependence and decompose the high-dimensional error term $f_{it}V_{it}$ using Hajek projection into three components: $a_i = \mathrm{E}[f_{it}V_{it}|\alpha_i]$, $g_t = \mathrm{E}[f_{it}V_{it}|\gamma_t]$, $e_{it} = f_{it}V_{it} - a_i - g_t$, where a_i are i.i.d over i, g_t are weakly dependent over t, and the remainder can be shown as small order and is well-behaved too. With this observation, I can construct the regressor-specific penalty weights in a way that existing moderate deviation theorems for i.i.d and weakly dependent sums can be leveraged for the first two Hajek components separately while some concentration inequality can be established to the third component.

For that purpose, I propose the following common penalty level λ and (infeasible) penalty weights:

$$\lambda = \frac{C_\lambda NT}{(N \wedge T)^{1/2}} \Phi^{-1}\left(1 - \frac{\gamma}{2p}\right), \tag{2.7}$$

$$\omega_j = \frac{N \wedge T}{N^2} \sum_{i=1}^{N} a_{i,j}^2 + \frac{N \wedge T}{T^2} \sum_{b=1}^{B} \left(\sum_{t \in H_b} g_{t,j} \right)^2. \tag{2.8}$$

where $a_{i,j} = \mathrm{E}[f_{it,j}V_{it}|\alpha_i]$, $g_{t,j} = \mathrm{E}[f_{it,j}V_{it}|\gamma_t]$ for $j = 1, ..., p$. C_λ is some sufficiently large constant and γ is a small order sequence. The convergence rate of γ affects the convergence rate of the LASSO estimator: as is revealed later, γ should be $o(1)$ for LASSO to be consistent while a larger γ guarantees a faster convergence

rate of LASSO. Again, both C_λ and γ do not vary across different DGPs. While there is some guidance about choosing C_λ and γ through the asymptotic theory, the choice is not given exactly. In practice, it is found that $C_\lambda = 2$ and $\lambda = 0.1/\log(p \vee N \vee T)$ delivers desirable finite sample performance. Looking at the definition of ω in 2.8, we notice that the first term in 2.8 is a variance estimator for i.i.d random variables and the second term is a cluster variance estimator as in Bester et al., 2008 where B is the number of clusters/blocks, h is the block length and H_b is the associated index set. Technically, they are chosen as $B = \text{round}(T/h)$, $h = \text{round}(T^{1/5}) + 1$, and, for $b = 1, .., B$, $H_b = \{t : h(b-1) + 1 \leq t \leq hb\}$.

To implement the penalty weights in 2.8, however, we need to estimate $a_{i,j} = \mathrm{E}[f_{it,j}V_{it}|\alpha_i]$ and $g_{t,j} = \mathrm{E}[f_{it,j}V_{it}|\gamma_t]$ with two challenges. With some initial estimation, we can replace V_{it} with some initial residual $\tilde{V}_{it}$ and then $\tilde{V}_{it}$ can be updated iteratively by the residuals from the estimation in 2.5 until it converges, meaning that $\tilde{V}_{it}$ does not update anymore up to a small difference. A common estimator for $a_{i,j}$ is then $\hat{a}_{i,j} = \frac{1}{N}\sum_{t=1}^{T} f_{it,j}\tilde{V}_{it}$. Similarly, we use $\hat{g}_{t,j} = \frac{1}{N}\sum_{i=1}^{N} f_{it,j}\tilde{V}_{it}$ for estimating $g_{t,j}$. Observe that this choice of implementing $\sum_{i=1}^{N} a_{i,j}^2$ is equivalent to the (unscaled) cluster variance estimator for $\mathrm{Var}(\mathrm{E}_{NT}[f_{it,j}V_{it}])$, which is also used as the regressor-specific penalty weights for cluster-LASSO in Belloni et al. (2016). It shows that the first term of ω clusters over time so it adjusts for the temporal dependence within each unit cluster, while the second term clusters over individual first and then clusters over time so it adjusts for cross-sectional dependence within each time cluster and temporal dependence across time clusters. The validity of estimating those components through cross-sectional and temporal averages is given in Menzel (2021) for exchangeable arrays. Extending the consistency results of those sample averages for non-exchangeable arrays is not trivial and establishing the uniform convergence result, required due to the high dimensionality, is rather challenging and not the focus of this paper. Following the practice in Belloni et al. (2012) and Belloni et al. (2016), the asymptotic analysis in this paper is based on high-level assumptions on the feasible penalty weights: Let $\widehat{\omega}$ be the feasible diagonal weights and suppose there exists $0 < 1/c_1 < l \leq 1$ and $1 \leq u < \infty$ such that $l \to 1$ and

$$l\omega_j^{1/2} \leq \widehat{\omega}_j^{1/2} \leq u\omega_j^{1/2}, \text{ uniformly over j} = 1, ..., \text{p}, \tag{2.9}$$

where $\{\omega_j\}$ and $\{\widehat{\omega}_j\}$ are diagonal entries of ω and $\widehat{\omega}$, respectively.

Algorithm: Implementation of the Two-Way Cluster-LASSO

i Obtain the initial residuals $\tilde{V}$: estimate the model without penalization if feasible; if not feasible, include a certain (user-specified) number of the most correlated regressors. [7]

ii Set λ according to 2.7 with $C_\lambda = 2$ and $\gamma = 0.1/\log(p \vee N \vee T)$. Calculate $\tilde{\omega}$ according to 2.8 using $\tilde{V}$.

[7] This step is for better convergence of the iterative estimation of the penalty weights. A small number of initially included regressors can cause failure to converge.

iii Using $\tilde{\omega}$ for LASSO estimation as in 2.5 and update the residual $\tilde{V}$ using the (post) LASSO estimation.[8]

iv Repeat steps ii-iii until it converges or up to a pre-set number of iterations. Obtains the (post) LASSO estimates from the last iteration.

Before delivering the main results of the weighted LASSO estimator above, two more sets of regularity conditions are needed. In the low dimensional case, a key identifying condition is that the population Gram matrix $\mathrm{E}_P[f_{it}' f_{it}]$ is non-singular so that the empirical Gram matrix is also non-singular with high probability. However, as we allow the dimension of f_{it} to be larger than the sample size, the empirical Gram matrix $\mathbb{E}_{NT} f_{it}' f_{it}$ is singular. Fortunately, it turns out that we only need certain sub-matrices to be well-behaved for identification. Define

$$\phi_{\min}(m)(M_f) := \min_{\delta \in \Delta(m)} \delta' M_f \delta \text{ and } \phi_{\max}(Cs)(M_f) := \max_{\delta \in \Delta(m)} \delta' M_f \delta,$$

where $\Delta(m) = \{\delta : \|\delta\|_0 = m, \|\delta\|_2 = 1\}$ and $M_f = \mathbb{E}_{NT}[f_{it}' f_{it}]$.

Assumption SE (Sparse Eigenvalues)**.** *For any $C > 0$, there exists constants $0 < \kappa_1 < \kappa_2 < \infty$ such that with probability approaching one, as $(N, T) \to \infty$ jointly, $\kappa_1 \leq \phi_{\min}(Cs)(M_f) < \phi_{\max}(Cs)(M_f) \leq \kappa_2$.*

The sparse eigenvalue assumption follows from Belloni et al. (2012). It implies a restricted eigenvalue condition, which represents a modulus of continuity between the prediction norm and the norm of δ within a restricted set. More primitive conditions for both types of assumptions are given in Belloni et al. (2012).

Assumption REG (Regularity Conditions)**.** *For $j = 1, ..., p$: (i) $\left[\mathrm{E}(a_{i,j})^2\right]^{1/2} / \left[\mathrm{E}(a_{i,j})^3\right]^{1/3} = O(1)$. (ii) $\log(p/\gamma) = o\left(T^{1/6}/(\log T)^2\right)$ and $p = o\left(T^{7/6}/(\log T)^2\right)$. (iii) For some $\mu > 1, \delta > 0$, $\mathrm{E}[|f_{it,j}|^{8(\mu+\delta)}] < \infty$, $\mathrm{E}[|V_{it}|^{8\mu+\delta)}] < \infty$. (iv) Either $\Sigma_{a,j} := \left[\mathrm{E}(a_{i,j}^2)\right]^{1/2} > c_\sigma$ or $\Sigma_{g,j} := \left[\sum_{\ell=-\infty}^{\infty} \mathrm{E}[g_{t,j} g_{t+\ell,j}]\right]^{1/2} > c_\sigma$ for some $c_\sigma > 0$.*

Assumption REG(i) is needed for applying the moderate deviation theorem from Peña et al. (2009). Assumption REG(ii) restricts the dimension of f_{it} relative to the sample size. Although the number of regressors is constrained to be of a small order relative to the overall sample size NT as $N, T \to \infty$ jointly, it is still allowed to be greater than the sample size in finite sample. Note that this requirement is more of a technical constraint and may be further relaxed with refined concentration inequalities for two-way dependent arrays. The moment conditions in Assumption REG(iii) are common in the literature. REG(iv) is a non-degeneracy condition, which is the main case of interest.

A common way to mitigate the shrinkage bias of LASSO is to apply least square estimation based on the selected model by LASSO, which is named Post-LASSO. The next theorem delivers a similar result. Let $\widehat{\Gamma} = j \in 1, ..., p : |\widehat{\zeta}_j| > 0$ where $\widehat{\zeta}_j$ are two-way LASSO estimates. In general, $\widehat{\Gamma}$ also allows the inclusion of additional variables chosen by the researcher. However, such generalization is not considered in this paper

[8] While they are asymptotically equivalent, post-LASSO suffers from less shrinkage bias in the finite sample.

to avoid further complications. As is shown in Belloni et al. (2012), the Post-LASSO is able to achieve rates of convergence no worse than LASSO, and under certain conditions, it improves upon LASSO. This finding is also supported by our simulation. The next theorem gives convergence rates for both two-way cluster-LASSO and its associated Post-LASSO.

Theorem 2.1. *Suppose Assumptions AHK, ASM, AR, REG hold for model 2.1 as $N, T \to \infty$ jointly with $N/T \to c$. Then, by setting λ as 2.7 with some sufficiently large C_λ, we have (i) the event 2.6 happens with probability approaching one. Additionally, suppose that Assumption SE holds and $\widehat{\omega}$ satisfies condition 2.9. Let $\widehat{\zeta}$ be the two-way cluster-LASSO estimator or the post-LASSO estimator based on the two-way cluster-LASSO selection. Then, (ii) $\|\widehat{\zeta}\|_0 = O_P(s)$, and (iii)*

$$\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(f_{it}\widehat{\zeta} - f_{it}\zeta_0\right)^2 &= O_P\left(\frac{s\log(p/\gamma)}{N\wedge T}\right),\\
\left\|\widehat{\zeta}-\zeta_0\right\|_1 &= O_P\left(s\sqrt{\frac{\log(p/\gamma)}{N\wedge T}}\right),\\
\left\|\widehat{\zeta}-\zeta_0\right\|_2 &= O_P\left(\sqrt{\frac{s\log(p/\gamma)}{N\wedge T}}\right).
\end{aligned}$$

Theorem 2.1 establishes convergence rates in terms of the prediction, $l1$, and $l2$ norms for the two-way cluster-LASSO estimator in an approximately sparse model. These results are the first that give convergence rates for a LASSO-based estimator allowing for two-way cluster dependence. It is shown that under the two-way cluster dependence, driven by an underlying factor structure, the two-way cluster-LASSO is consistent but, unfortunately, has a convergence rate slower than those of LASSO-based methods under the random sampling condition or the cross-section independence. Without loss of generality, let $N = N \wedge T$, then by choosing γ according to $\log(1/\gamma) \simeq \log(p \vee NT)$, we have $\left\|\widehat{\zeta}-\zeta_0\right\|_2 = O_P\left(\sqrt{\frac{s\log(p\vee NT)}{N}}\right)$. As a comparison, the rate of convergence in terms of the $l2$ norm is $O_P\left(\sqrt{\frac{s\log p}{NT}}\right)$ under the random sampling and the homoskedasticity Gaussian error assumptions in Bickel et al. (2009) or the heteroskedasticity Gaussian error in Theorem 19.3 of Hansen (2022), $O_P\left(\sqrt{\frac{s\log(p\vee NT)}{NT}}\right)$ under random sampling in Belloni et al. (2012), and $O_P\left(\sqrt{\frac{s\log(p\vee NT)}{Nl_T}}\right)$ under cross-sectional independence in Belloni et al. (2016) where the information index $l_T = 1$ when there is perfect dependence within the cross-sectional cluster.

As is revealed in the proof of Theorem 2.1 in the Appendix and briefly illustrated in the Introduction, the slow rate of convergence is due to the underlying factor structure. It is unclear if valid inference is still possible under the rate of convergence results in Theorem 2.1 and if it is possible to relax the requirement through a cross-fitting procedure? These questions are addressed in the next section.

3. Clustered-Panel Cross-Fitting and Inference

In this section, I will first propose a sub-sampling scheme for cross-fitting in a two-way clustered panel and then propose a general inference procedure using cross-fitting for a high-dimensional panel model. The idea of the sub-sampling scheme is to split the sample in a proper way so that two resulting sub-samples are independent or, at least, "approximately" independent. With such properties, the sub-sampling scheme can be used for various purposes. For example, it can be used for cross-fitting in a two-step estimation since it effectively eliminates the dependence between the two steps, which in turn relaxes the rate of convergence requirement for the first step for valid inference. It can also be used for cross-validation when choosing tuning parameters in panel data models. In this paper, we will focus on its application in cross-fitting. Building upon the cross-fitting algorithm for exchangeable arrays in Chiang et al. (2022) and for weakly dependent panels in Semenova et al. (2023a), I propose a cross-fitting scheme robust to two-way cluster dependence and weak dependence over time.

Let $\{W_{it} : i = 1, ..., N \text{ and } t = 1, ..., T\}$ denote a sample of sizes (N, T) from a sequence of random elements $(W_{it})_{i\geq 1, t\geq 1}$ defined on a common measurable space $(\Omega, \mathcal{F})$ and taking values in Euclidean spaces. To allow the dimension of W_{it} to grow with N, T, we denote $(\mathcal{P}_{NT})_{N\geq 1, T\geq 1}$ as an expanding class of probability laws of $\{W_{it} : i = 1, ..., N \text{ and } t = 1, ..., T\}$ and denote $P \in \mathcal{P}_{NT}$ as a generic probability law for the sample with sizes (N, T).

Under the AHK characterization in Assumption AHK, W_{it} are cluster-dependent with both W_{is} and W_{jt}. Importantly, these types of cluster dependence do not vanish as the distance between observations (if there is any ordering) increases. If γ_t is weakly dependent, which is the focus of this paper, then the dependence between observations that don't share the same cluster in either dimension dies out as the temporal distance grows. In that case, intuitively, one can split the sample so that the sub-samples do not share the same cluster and are away from each other in temporal distance. This is exactly how this scheme works:

Definition 3.1 (Two-Way Clustered-Panel Cross-Fitting)**.**

(i) Select some positive integers (K, L)*. Randomly partition the cross-sectional index set* $\{1, 2, ..., N\}$ *into* K *folds* $\{I_1, I_2, ..., I_K\}$ *and partition the temporal index set* $\{1, 2, ..., T\}$ *into* L *adjacent folds* $\{S_1, S_2, ..., S_L\}$ *so that* $\bigcup_{k=1}^{K} I_k = \{1, ..., N\}$, $\bigcup_{l=1}^{L} S_l = \{1, ..., T\}$[9].

(ii) For each $k = 1, ..., K$ *and* $l = 1, .., L$*, construct the main sample*

$$W(k, l) = \{W_{it} : i \in I_k, t \in S_l\},$$

[9]For simplicity, I assume N and T are divisible by K and L, respectively. In practice, if N is not divisible by K, the size for each cross-sectional block can be chosen differently with some length equal to $\text{floor}(N/K)$ and others equal to $\text{ceil}(N/K)$. and the same applies to the temporal dimension.

and the auxiliary sample (typically larger)

$$W(-k,-l) = \left\{ W_{it} : i \in \bigcup_{k' \neq k} I_{k'}, t \in \bigcup_{l' \neq l, l \pm 1} S_{l'} \right\},$$

Later on, we also use I_{-k} and S_{-l} to denote the index sets for the auxiliary sample $W(-k,-l)$. Similarly, we denote N_{-k} and T_{-l} as the cross-sectional and temporal sample sizes for the auxiliary sample $W(-k,-l)$. Figure 1 illustrates the cross-fitting with $K = 4$ and $L = 8$.

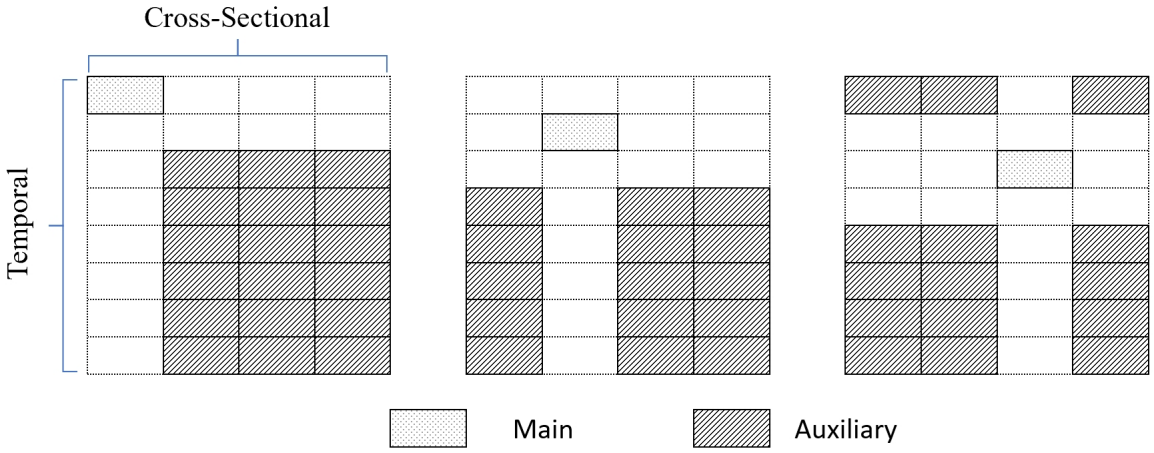

Figure 1: Clustered-Panel cross-fitting with $K = 4$ and $L = 8$. Three graphs from left to right correspond to the main and auxiliary sample constructions with $(k,l) = (1,1)$, $(k,l) = (2,2)$, $(k,l) = (3,3)$. For a simple illustration, observations in the main sample are all adjacent in the cross-sectional dimension but it is not necessary in practice; the same applies to the auxiliary sample.

Since the sub-samples $W(k,l)$ and $W(-k,-l)$ do not share any cluster, they are free from cluster dependence and what's left is the weak dependence over time. Unless imposing $m-$dependence, the sub-samples above will not be independent. However, under certain regularity conditions regarding the weak dependence, it can be shown through the coupling technique that as long as the temporal distance between the sub-samples diverges at a certain rate, there exists coupling sub-samples that are independent of each other while having the same marginal distributions as the our constructed sub-samples with probability coverging to 1. Such coupling tecnique is common in time series context and the idea in cross-fitting in time dimension follows from Semenova et al. (2023a). The following lemma delievers such a result

Lemma 3.1 (Independent Coupling)**.** *Consider the sub-samples $W(k,l)$ and $W(-k,-l)$ for $k = 1,...,K$ and $l = 1,...,L$. Suppose Assumptions AHK, AR hold and $log(N)/T = o(1)$ as $T \to \infty$. Then, we can construct $\tilde{W}(k,l)$ and $\tilde{W}(-k,-l)$ such that: (i) they are independent of each other; (ii) have the same marginal distribution as $W(k,l)$ and $W(-k,-l)$, respectively; (iii)*

$$\mathrm{P}\left\{(W(k,l), W(-k,-l)) \neq \left(\tilde{W}(k,l), \tilde{W}(-k,-l)\right), \text{ for some } (k,l)\right\} = o(1).$$

Lemma 3.1 shows that the main and auxiliary samples from the proposed clustered-panel cross-fitting scheme are approximately independent as N, T diverge. Note that the hypothetical sample $\tilde{W}(k,l)$ and $\tilde{W}(-k,-l)$ do not matter in practice, but they allow us to treat $W(k,l)$ and $W(-k,-l)$ as $\tilde{W}(k,l)$ and $\tilde{W}(-k,-l)$ with probability approaching 1. The proof of Lemma 3.1 is based on independence coupling results (Strassen, 1965, Dudley and Philipp, 1983, and Berbee, 1987) introduced in Semenova et al. (2023a).

As mentioned at the beginning of the section, one of the primary uses of the sub-sampling scheme is cross-fitting in a two-step estimation. To be concrete, I will define a two-step estimator using the cross-fitting algorithm in the context of a semi-parametric moment restriction model. The theoretical properties of the estimator will be studied in Section 3.1.

Let $\varphi(W_{it};\theta,\eta)$ denote some identifying moment functions where θ is a low-dimensional vector of parameters of interest and η are nuisance functions. For example, $\eta = g_0$ in 1.1. Let $\psi(W_{it};\theta,\eta)$ denote some orthogonalized moment function based on $\varphi(W_{it};\theta,\eta)$. The formal definition of the orthogonality will be delivered in the next subsection. For now, it suffices to be aware that both functions are mean zero but $\psi(W_{it};\theta,\eta)$ is adjusted for the fact that η_0 needs to be estimated. In model 1.1, $\varphi(W_{it};\theta,\eta) = D_{it}U_{it}$ and $\psi(W_{it};\theta,\eta) = \left(D_{it} - E[D_{it}|X_{it},c_i,d_t]\right)\left(Y_{it} - D_{it}\theta - g(X_{it},c_i,d_t)\right)$. In the treatment effect model with unconfoundedness conditional on covariates and unobserved heterogeneous effects, $\varphi(W_{it};\theta,\eta) = \mathrm{E}[Y_{it}|D_{it} = 1, X_{it}, c_i, d_t] - \mathrm{E}[Y_{it}|D_{it} = 0, X_{it}, c_i, d_t] - \theta^{\mathrm{ATE}}$ and $\psi(W_{it};\theta,\eta)$ is the moment function corresponding to the well-known augmented inverse probability weighting estimator, which is doubly robust.

The panel cross-fitting procedure goes as follows. For each k and l, we use the sub-sample $W(-k,-l)$ to estimate η with the estimator denoted as $\widehat{\eta}_{kl}$. For each $i \in I_k$ and $t \in S_l$, we plug-in $\widehat{\eta}_{kl}$ to the orthogonal moment function, $\psi(W_{it};\theta,\widehat{\eta}_{kl})$. By averaging $\psi(W_{it};\theta,\widehat{\eta}_{kl})$ across all $k = 1,...,K$ and $l = 1,...,L$, we obtain

$$\overline{\psi}_{kl} := \mathbb{E}_{kl}\left[\psi(W_{it};\theta,\widehat{\eta}_{kl})\right],$$

which is a sample analogue of the population orthogonal moment condition used for estimation. Note that the larger sub-sample $W(-k,-l)$, instead of the smaller sub-sample $W(k,l)$, is used for first-step nuisance estimation because it usually involves high-dimensional unknown parameters. For reference, $W(k,l)$ is referred to as the main sample and $W(-k,-l)$ is referred to as the auxiliary sample. The next definition summarizes the panel DML estimation and inference procedures for a semiparametric moment restriction model:

Definition 3.2 (Panel DML Algorithm)**.**

(i) Given the identifying moment functions $\varphi(W;\theta,\eta)$ such that $\mathrm{E}_P[\varphi(W;\theta_0,\eta_0)] = 0$, find the orthogonalized moment function $\psi(W,\theta,\eta)$.

(ii) Select (K,L) and then randomly partition $\{1,2,...,N\}$ into K folds $\{I_1,I_2,...,I_K\}$ and partition $\{1,2,...,T\}$ into L adjacent folds $\{S_1,S_2,...,S_L\}$. For each $k = 1,...,K$ and $l = 1,..,L$, construct the

main sample

$$W(k,l) = \{W_{it} : i \in I_k, t \in S_l\},$$

and the auxiliary sample

$$W(-k,-l) = \left\{ W_{it} : i \in \bigcup_{k' \neq k} I_{k'}, t \in \bigcup_{l' \neq l, l \pm 1} S_{l'} \right\}.$$

(iii) For each k and l, use the sample $W(-k,-l)$ for the first-step estimation and obtain $\widehat{\eta}_{kl}$, then construct $\overline{\psi}_{kl}(\theta) = \mathbb{E}_{kl}[\psi(W_{it};\theta,\widehat{\eta}_{kl})]$ for each (k,l). Finally, obtain the DML estimator $\widehat{\theta}$ as the solution to

$$\frac{1}{KL}\sum_{k=1}^{K}\sum_{l=1}^{L}\overline{\psi}_{kl}(\theta) = 0. \tag{3.1}$$

Remark 3.1 (The Choice of K and L)**.** *Notice there is a trade-off in setting (K, L) between the first step and second step accuracy: the bigger values of (K, L), the bigger sample size of the auxiliary sample $W(-k,-l)$, which is beneficial for high-dimensional first-steps but at the cost of a noisier parametric second step. In our case, it necessitates an $L \geq 4$ for feasible implementation (if $L = 3$, for example, any main sample $W(k,l)$ with $l = 2$ does not have a well-defined auxiliary sample). On the other hand, it is computationally costly to set the values of (K, L) too large. In practice, $K = 2$ to 4 and $L = 4$ to 8 work well in simulations.*

3.1. ***Panel DML: Inferential Theory***

To investigate the required convergence rate of a high-dimensional estimator for valid inference, I will study a general inference procedure for a high-dimensional panel model characterized by a semiparametric moment restriction. Such an inference procedure is based on the panel cross-fitting approach proposed in Section 3 and the prototypical DML approach proposed in Chernozhukov et al. (2018a).

With the same notation from Section 3, the model is characterized by a semiparametric moment condition $\mathrm{E}[\varphi(W_{it};\theta_0,\eta_0)] = 0$ where W_{it} are again characterized by an underlying component structure as in Assumption AHK. Let $\psi(W;\theta,\eta)$ be the orthogonalized moment function. Formally, the orthogonality means that it is mean zero and its pathwise or Gateaux derivative with respect to the nuisance parameter is 0 when evaluated at the true values:

$$\mathrm{E}_P[\psi(W_{it};\theta_0,\eta_0)] = 0, \tag{3.2}$$

$$\partial_r \mathrm{E}_P\left[\psi\left(W_{it};\theta_0,\eta_0 + r(\eta-\eta_0)\right)\right]|_{r=0} = 0. \tag{3.3}$$

In other words, the nuisance functions have no first-order effect locally on the orthogonalized moment conditions, based on which the estimation of θ_0 is therefore robust to the plug-in of noisy estimates of γ_0. In contrast, the original identifying moment conditions do not possess such a property.

Again, the orthogonal moment construction is taken as given. The panel DML procedure is defined in Definition 3.2. Differing from the existing literature, the approach in this paper focuses on estimation and inference robust to two-way cluster dependence and weak dependence across clusters, characterized by Assumption AHK. Note that Assumption AHK also includes i.i.d data as a special case. Although the panel DML procedure is also robust to the i.i.d case or, more generally, the case of the degeneracy in components, the theoretical properties are not formally given in this paper. The rates of convergence for both the nuisance estimator and the second-step estimator are different and faster for the i.i.d case but that's not surprising and is not the focus of this paper. To restrict the focus, I will assume a non-degeneracy condition in terms of Hajek projection components. First, I define the Hajek components and their corresponding (long-run) variance-covariance matrices as follows:

$$
\begin{aligned}
a_i &:= \mathrm{E}_P\left[\psi\left(W_{it};\theta_0,\eta_0\right)|\alpha_i\right], \quad \Sigma_a := \mathrm{E}_P[a_i a_i'], \\
g_t &:= \mathrm{E}_P\left[\psi\left(W_{it};\theta_0,\eta_0\right)|\gamma_t\right], \quad \Sigma_g := \sum_{l=-\infty}^{\infty} \mathrm{E}_P[g_t g_{t+l}'], \\
e_{it} &:= \psi\left(W_{it};\theta_0,\eta_0\right) - a_i - g_t, \; \Sigma_e := \sum_{l=-\infty}^{\infty} \mathrm{E}_P[e_{it} e_{i,t+l}'].
\end{aligned}
$$

Let $\lambda_{min}[.]$ denote the smallest eigenvalue of a square matrix. The next assumption specifies the non-degenerate condition.

Assumption ND (Non-Degeneracy)**.** *There exists some constant $c_\Sigma > 0$ such that either $\lambda_{min}[\Sigma_a] > c_\Sigma$ or $\lambda_{min}[\Sigma_g] > c_\Sigma$.*

Assumption ND implies that at least one of the components drives the cluster dependence.

The next two assumptions follow the same format as Chernozhukov et al. (2018a) but, importantly, they characterize some different rates of convergence required for inferential theory. Let (δ_{NT}) and (Δ_{NT}) be some sequence of positive constants converging to 0 as $N, T \to \infty$. Let $\mathcal{T}_{NT}$ be a nuisance realization set such that it contains η_0 and that $\widehat{\eta}_{kl}$ belongs to $\mathcal{T}_{NT}$ with probability $1 - \Delta_{NT}$ for each (k, l).

Assumption DML1 (Linear Moment Conditions, Smoothness, and Identification)**.**

(i) $\psi(W;\theta,\eta)$ *is linear in* θ*:*

$$\psi(w;\theta,\eta) = \psi^a(W,\eta)\theta + \psi^b(W,\eta), \; \forall\, w \in \mathcal{W}, \theta \in \Theta, \eta \in \mathcal{T}.$$

(ii) $\psi(W;\theta,\eta)$ *satisfy the Neyman orthogonality conditions 3.2 and 3.3 with respect to the probability measure P, or, more generally, 3.3 can be replaced by a* λ_{NT} *near-orthogonality condition*

$$\lambda_{NT} := \sup_{\eta \in \mathcal{T}_{NT}} \left\| \partial_r \mathrm{E}_P[\psi(W;\theta_0,\eta_0 + r(\eta - \eta_0))]|_{r=0} \right\| \leq \delta_{NT}/\sqrt{N}.$$

(iii) The map $\eta \to \mathrm{E}_P[\psi(W_{it};\theta,\eta)]$ *is twice continuously Gateaux-differentiable on* $\mathcal{T}$*.*

(iv) The singular values of the matrix $A_0 := \mathrm{E}_P[\psi^a(W_{it};\eta_0)]$ *are bounded below by* $c_a > 0$.

Assumption DML1(i) restricts the focus of this paper to models with linear orthogonal moment conditions, which covers many applications and the model in Section 4. For nonlinear orthogonal moment conditions, Chernozhukov et al. (2018a) has shown that the DML estimator has the same desirable properties under more complicated regularity conditions. Focusing on the linear cases allows us to pay more attention to issues specifically attributed to panel data. Assumption DML1(ii) slightly relaxes the orthogonality condition 3.3 by a near-orthogonality condition, which is useful for the approximate sparse model considered in Section 4 because the corresponding orthogonal moment condition does not satisfy 3.3 exactly due to approximation errors. Assumption DML1(iii) imposes a mild smoothness assumption on the orthogonal moment condition and Assumption DML1(iv) is a common condition for identification.

Assumption DML2 (Moment Regularity and First-Steps)**.**

(i) For all $i \geq 1$, $t \geq 1$, *and some* $q > 2$, $c_m < \infty$, *the following moment conditions hold:*

$$m_{NT} := \sup_{\eta \in \mathcal{T}_{NT}} \left(\mathrm{E}_P \left\|\psi(W_{it};\theta_0,\eta)\right\|^q\right)^{1/q} \leq c_m,$$
$$m'_{NT} := \sup_{\eta \in \mathcal{T}_{NT}} \left(\mathrm{E}_P \left\|\psi^a(W_{it};\eta)\right\|^q\right)^{1/q} \leq c_m.$$

(ii) The following conditions on the statistical rates r_{NT}, r'_{NT}, λ'_{NT} *hold for all* $i \geq 1$, $t \geq 1$*:*

$$r_{NT} := \sup_{\eta \in \mathcal{T}_{NT}} \left\|\mathrm{E}_P[\psi^a(W_{it};\eta) - \psi^a(W_{it};\eta_0)]\right\| \leq \delta_{NT},$$
$$r'_{NT} := \sup_{\eta \in \mathcal{T}_{NT}} \left(\mathrm{E}_P \left\|\psi(W_{it};\theta_0,\eta) - \psi(W_{it};\theta_0,\eta_0)\right\|^2\right)^{1/2} \leq \delta_{NT},$$
$$\lambda'_{NT} := \sup_{r \in (0,1), \eta \in \mathcal{T}_{NT}} \left\|\partial_r^2 \mathrm{E}_P[\psi(W_{it};\theta_0,\eta_0 + r(\eta - \eta_0))]\right\| \leq \delta_{NT}/\sqrt{N}.$$

Assumption DML2 regulates the quality of the first-step nuisance estimators. It follows from Chernozhukov et al. (2018a) and it can be verified under primitive conditions in the next section. Observe that, if the orthogonal moment function $\psi(W;\theta,\eta)$ is smooth in η, then λ'_{NT} is the dominant rate and it imposes a crude rate requirement of order $\varepsilon_{NT} = o(N^{-1/4})$ on the first-step nuisance parameter in $L^2(P)$ norm, which is possible for the two-way cluster LASSO estimator to achieve under proper sparsity assumption. Furthermore, in some models including the partial linear model, λ'_{NT} can be exactly 0, then it is possible to obtain the weakest possible rate requirement for the first-step estimator, i.e. $\varepsilon_{NT} = o(1)$.

Theorem 3.1 (Asymptotic Normality and Variance)**.** *Suppose Assumptions AHK, AR, ND, DML1, DML2 hold for any* $P \in \mathcal{P}_{NT}$, *then for some* $\delta_{NT} \geq N^{-1/2}$, *as* $(N,T) \to \infty$ *jointly,*

$$\sqrt{N}\left(\widehat{\theta} - \theta_0\right) = -\sqrt{N} A_0^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \psi(W_{it};\theta_0,\eta_0) + o_P(1) \Rightarrow \mathcal{N}(0,V),$$

where

$$V \coloneqq A_0^{-1}\Omega A_0^{-1'},$$
$$\Omega \coloneqq \Sigma_a + c\Sigma_g.$$

We observe that the convergence rate of the two-step estimator $\widehat{\theta}$ resulting from the panel DML procedure is non-standard. It is $\sqrt{N}$-consistent instead of $\sqrt{NT}$-consistent. This is because the cluster dependence introduced by the unit and time components does not decay over time or space. Intuitively, with more persistence, the information carried by data is accumulated more slowly. It is a common feature in the literature of robust inference with cluster dependence[10] and it is also related to inferential theory under strong cross-sectional dependence as in Gonçalves (2011).

Due to the presence of unit and time components, the asymptotic variance is made of (long-run) variance-covariance matrices of both factors. I consider a two-way cluster robust variance estimator similar to Chiang et al. (2024) (CHS estimator) with adjustment due to cross-fitting. The variance estimator is motivated under arbitrary dependence in panel data and is shown to be robust to two-way clustering with correlated time effects in linear panel models. As is shown in Chen and Vogelsang (2024), such variance estimator can be written as an affine combination of three well-known robust variance estimators: Liang-Zeger-Arellano estimator, Driscoll-Kraay estimator, and the "average of HACs" estimator. Applying this result, we can define the CHS-type variance estimator as follows:

$$\widehat{V}_{\text{CHS}} = \widehat{\bar{A}}^{-1}\widehat{\Omega}_{\text{CHS}}\widehat{\bar{A}}^{-1'},$$
$$\widehat{\Omega}_{\text{CHS}} = \widehat{\Omega}_{\text{A}} + \widehat{\Omega}_{\text{DK}} - \widehat{\Omega}_{\text{NW}},$$

where $\widehat{\bar{A}} := \frac{1}{KL}\sum_{k=1}^{K}\sum_{l=1}^{L}\frac{1}{N_k T_l}\sum_{i\in I_k, s\in S_l}\psi^a(W_{it};\widehat{\eta}_{kl})$ and, with $k\left(\frac{m}{M}\right) := 1 - \frac{m}{M}$ for $m = 0, 1, ..., M-1$ and 0 otherwise (i.e., Bartlett kernel) and the bandwidth parameter M chosen from 1 to T_l,

$$\widehat{\Omega}_{\text{A}} := \frac{1}{KL}\sum_{k=1}^{K}\sum_{l=1}^{L}\frac{1}{N_k T_l^2}\sum_{i\in I_k, t\in S_l, r\in S_l}\psi(W_{it};\widehat{\theta},\widehat{\eta}_{kl})\psi(W_{ir};\widehat{\theta},\widehat{\eta}_{kl})',$$
$$\widehat{\Omega}_{\text{DK}} := \frac{1}{KL}\sum_{k=1}^{K}\sum_{l=1}^{L}\frac{K/L}{N_k T_l^2}\sum_{t\in S_l, r\in S_l}k\left(\frac{|t-r|}{M}\right)\sum_{i\in I_k, j\in I_k}\psi(W_{it};\widehat{\theta},\widehat{\eta}_{kl})\psi_{jr}(\widehat{\theta},\widehat{\eta}_{kl})',$$
$$\widehat{\Omega}_{\text{NW}} := \frac{1}{KL}\sum_{k=1}^{K}\sum_{l=1}^{L}\frac{K/L}{N_k T_l^2}\sum_{i\in I_k, t\in S_l, r\in S_l}k\left(\frac{|t-r|}{M}\right)\psi(W_{it};\widehat{\theta},\widehat{\eta}_{kl})\psi(W_{ir};\widehat{\theta},\widehat{\eta}_{kl})'.$$

It is noted that the variance estimator under the cross-fitting is equivalent to estimating the variance in

[10]For example, see Hansen, 2007, MacKinnon et al., 2021,Menzel, 2021, Chiang et al., 2022,Chiang et al., 2023a,Chiang et al., 2024,Chen and Vogelsang, 2024 among many others.

each sub-sample and then averaging across all sub-samples. Since K, L are fixed, the asymptotic analysis is done at the sub-sample level. The next theorem establishes the consistency of this variance estimator under the conventional small-bandwidth assumption.

Theorem 3.2 (Consistent Variance Estimator)**.** *Assumptions AHK, AR, ND, DML1, DML2 hold for any $P \in \mathcal{P}_{NT}$, and some $q > 4$ (defined in Assumption DML2), and $M/T^{1/2} = o(1)$. Then, as $N, T \to \infty$ and $N/T \to c$ where $0 < c < \infty$,*

$$\widehat{V}_{\text{CHS}} = V + o_P(1).$$

Theorem 3.2 can be seen as a generalization of the consistency result for the CHS variance estimator in Chiang et al. (2024) by allowing for the estimated nuisance parameters in the moment functions.

A remaining practical issue is that $\widehat{V}$ is not ensured to be positive semi-definite. It has been shown in Chen and Vogelsang (2024) that negative variance estimates happen with a non-trivial number of times under certain data-generating processes. Accordingly, an alternative two-term variance estimator was proposed in Chen and Vogelsang (2024). Following the same idea, I propose an alternative variance estimator by dropping the double-counting term $\widehat{\Omega}_{\text{NW}}$:

$$\begin{aligned}\widehat{V}_{\text{DKA}} &= \widehat{A}^{-1}\widehat{\Omega}_{\text{DKA}}\widehat{A}^{-1'},\\ \widehat{\Omega}_{\text{DKA}} &= \widehat{\Omega}_{\text{A}} + \widehat{\Omega}_{\text{DK}}.\end{aligned}$$

The estimator is referred to as the DKA variance estimator because it is a sum of Driscoll-Krray and Arellano variance estimators.[11] Similar approaches can be found in MacKinnon et al. (2021). It relies on the fact that the double-counting term is of small order asymptotically when the panel is two-way clustering. Similar to other two-term cluster-robust variance estimators, it has the computational advantage of guaranteeing positive semi-definiteness but at the cost of inconsistency in the case of no clustering or clustering at the intersection. For theoretical results and more detailed discussions on the trade-off between the ensured positive-definiteness and the risk of being too conservative/losing power, readers are referred to MacKinnon et al. (2021) and Chen and Vogelsang (2024).

Theorem 3.3 (Alternative Consistent Variance Estimator)**.** *Under the same conditions as Theorem 3.2, we have, as $N, T \to \infty$ and $N/T \to c$ where $0 < c < \infty$,*

$$\widehat{V}_{\text{DKA}} = \widehat{V}_{\text{CHS}} + o_P(1).$$

Theorem 3.3 formally shows that the double-counting term is of small order under two-way clustering

[11] Note that, the DKA estimator defined in Chen and Vogelsang (2024) differs from the DKA estimator here by a constant term based on fixed-b asymptotic analysis. Such bias correction is not considered here since the fixed-b properties are not directly applicable in this setting. The conjecture is that the same form of bias correction can be applied here but formally establishing the fixed-b asymptotic results with the presence of estimated nuisance parameters is challenging and out of the scope of this paper, and so is left for future research.

and it implies that the $\widehat{V}_{\text{DKA}}$ is also consistent for Ω under two-way clustering.

To conclude, in this section, the inferential theory is established for the panel DML estimator, under high-level assumptions on the first-step estimator. Even though the rate of convergence can be slow for the nuisance estimations due to the two-way cluster dependence, the cross-fitting approach for panel models allows for valid inference in a general semiparametric moment restriction model with growing dimension in the nuisance parameters. In the next section, I will study a special case of the semiparametric restriction model but consider the complication due to unobserved heterogeneity. I will demonstrate how to apply the results from previous sections and discuss an extra subtle issue.

4. Partial Linear Model with Unobserved Heterogeneity

In this section, a partial linear model with non-additive unobserved heterogeneous effects is considered and I will illustrate how the proposed high-dimensional methods proposed in the previous two sections can be leveraged. Additionally, with an extra subtle issue for cross-fitting due to the unobserved heterogeneous effects, I also propose an valid inferential procedure using the full sample. The proposed toolkit is flexible enough to allow for models with instrumental variables used for identification, so I consider the following model: for $i = 1, ..., N$ and $t = 1, ..., T$,

$$Y_{it} = D_{it}\theta_0 + g(X_{it}, c_i, d_t) + U_{it}, \;\; \mathrm{E}[U_{it}|X_{it}, c_i, d_t] = 0, \tag{4.1}$$

where D_{it} is a low-dimensional vector of endogenous variables; g is an unknown function of potentially high-dimensional control variables X_{it} and unobserved heterogeneous effects (c_i, d_t). For clearer presentation, D_{it} is treated as a scalar variable. In practice, D_{it} can contain some high-order terms and interactions with a low-dimensional vector of controls. If the lags or leads of D_{it} are considered to be exogenous, they can also included in X_{it}. Doing so would not change the theory for estimation and inference but doing so would change the interpretation of θ_0. Consider an excludable instrumental variable Z_{it} such that $\mathrm{E}[Z_{it}U_{it}] = 0$, which gives the identifying moment condition.

To apply the estimation and inference methods proposed in previous sections, g is again assumed to be approximately sparse. However, it does not suffice since (c_i, d_t) are not observed. To deal with the unobserved heterogeneous effects that cause the endogeneity, I take a correlated random-effects approach through the generalized Mundlak device:

Assumption GMD (Generalized Mundlak Device)**.** *For each $i = 1, ..., N$ and $t = 1, ..., T$,*

$$c_i = h_c(\bar{F}_i, \epsilon_i^c), \tag{4.2}$$

$$d_t = h_d(\bar{F}_t, \epsilon_t^d), \tag{4.3}$$

where $\bar{F}_i = \frac{1}{T}\sum_{t=1}^{T} F_{it}$, $\bar{F}_t = \frac{1}{N}\sum_{i=1}^{N} F_{it}$, $F_{it} := (D_{it}, X_{it}')'$; h_c and h_d are some unknown measurable functions; the stochastic errors $(\epsilon_i^c, \epsilon_t^d)$ are independent of $(\bar{F}_i, \bar{F}_t, X_{it}, Z_{it}, U_{it}^D, U_{it}^Y)$; and $(c_i, d_t, \epsilon_i, \epsilon_t)$ are

independent of U_{it}.

To justify its use, we shall recall the idea of the conventional Mundlak device. Due to the correlation between (c_i, d_t) and the covariates, the endogeneity issue arises if we don't control for the unobserved heterogeneity. To explicitly model the correlation between the random effects and the covariates, Mundlak (1978) proposes an auxiliary regression between the random effects and the cross-sectional sample average and shows that if the random effects enter the model linearly then the resulting estimator GLS estimator is equivalent to the common within-estimator. Wooldridge (2021) further shows that the equivalence relations exist among the POLS estimators resulting from the Mundlak device, within-transformation, and the fixed-effects dummies. Therefore, if the within-transformation and including fixed-effects dummies are sensible and commonly accepted ways of dealing with unobserved heterogeneity, then allowing the Mundlak regression to have a more flexible function form should also be sensible and more robust. A similar assumption is considered in Wooldridge and Zhu (2020).

Under model 4.1, $g(X_{it}, c_i, d_t) = \mathrm{E}[Y_{it} - D_{it}\theta_0 | X_{it}, c_i, d_t]$. We can rewrite 4.1 as follows:

$$Y_{it} = \left(D_{it} - g_D(X_{it}, c_i, d_t)\right)\theta_0 + g_Y(X_{it}, c_i, d_t) + U_{it}.$$

where $g_D(X_{it}, c_i, d_t) := \mathrm{E}[D_{it}|X_{it}, c_i, d_t]$ and $g_Y(X_{it}, c_i, d_t) := \mathrm{E}[Y_{it}|X_{it}, c_i, d_t]$. Under Assumption GMD, $g_D(X_{it}, c_i, d_t)$ and $g_Y(X_{it}, c_i, d_t)$ can be rewritten as compound functions, which are assumed to be well-approximated by a linear combination of a $\tau-$th order polynomial transformation L^τ as follows:

$$g_D^*(X_{it}, \bar{F}_i, \epsilon_i^c, \bar{F}_t, \epsilon_t^d) := g_D(X_{it}, h_c(\bar{F}_i, \epsilon_i^c), h_d(\bar{F}_t, \epsilon_t^d)) = L^\tau\left(X_{it}, \bar{F}_i, \bar{F}_t, \epsilon_i^c, \epsilon_t^d\right)\eta_D + r_{it}^D \tag{4.4}$$

$$g_Y^*(X_{it}, \bar{F}_i, \epsilon_i^c, \bar{F}_t, \epsilon_t^d) := g_Y(X_{it}, h_c(\bar{F}_i, \epsilon_i^c), h_d(\bar{F}_t, \epsilon_t^d)) = L^\tau\left(X_{it}, \bar{F}_i, \bar{F}_t, \epsilon_i^c, \epsilon_t^d\right)\eta_Y + r_{it}^Y \tag{4.5}$$

where (η_D, η_Y) are slope coefficients and (r_{it}^D, r_{it}^Y) are the approximation errors. Furthermore, we can define a vector of transformed regressors as $L_{1,it} = L^\tau(X_{it}, \bar{F}_i, \bar{F}_t)$ and a vector of unobserved regressors as $L_{2,it} = L^\tau(X_{it}, \bar{F}_i, \bar{F}_t, \epsilon_i^c, \epsilon_t^d) \backslash L^\tau(X_{it}, \bar{F}_i, \bar{F}_t)$. Let $(\eta_{D,1}, \eta_{D,2})$ be such that $\eta_D = \eta_{D,1} \bigcup \eta_{D,2}$ and $L^\tau\left(X_{it}, \bar{F}_i, \bar{F}_t, \epsilon_i^c, \epsilon_t^d\right)\eta_D = L_{1,it}\eta_{D,1} + L_{2,it}\eta_{D,2}$; and $(\eta_{Y,1}, \eta_{Y,2})$ are defined in the same way. Under the sparse approximation and Assumption GMD, we can rewrite model 4.1 as follows:

$$Y_{it} = \left(D_{it} - L_{1,it}\eta_{D,1} - L_{2,it}\eta_{D,2} - r_{it}^D\right)\theta_0 + L_{1,it}\eta_{Y,1} + L_{2,it}\eta_{Y,2} + r_{it}^Y + U_{it},$$

By defining a new error term $V_{it}^g := \left(L_{2,it} - \mathrm{E}[L_{2,it}]\right)\left(\eta_{Y,2} - \eta_{D,2}\theta_0\right) + U_{it}$, a new approximation error $r_{it} = r_{it}^Y + r_{it}^D\theta_0$, the vector of observables $f_{it} := \left(L_{1,it}, 1\right)$ with dimension denoted by p, and the nuisance vectors $\beta_0 := \left(\eta_{Y,1}, \mathrm{E}[L_{2,it}]\eta_{Y,2}\right)$, $\pi_0 := \left(\eta_{D,1}, \mathrm{E}[L_{2,it}]\eta_{D,2}\right)$, we can rewrite the model above as

$$Y_{it} = \left(D_{it} - f_{it}\pi_0\right)\theta_0 + f_{it}\beta_0 + r_{it} + V_{it}^g. \tag{4.6}$$

Noticeably, in this case, the parameters associated with the unobservables $L_{2,it}$ can be arbitrarily non-sparse.

Given $\mathrm{E}[Z_{it}U_{it}]$ and the independence between Z_{it} and $(\epsilon_i^c, \epsilon_t^d)$, we have the identifying moment condition $\mathrm{E}[Z_{it}V_{it}^g] = 0$. Let ζ_0 be the linear projection parameter of Z_{it} onto f_{it} and let V_{it}^Z be the corresponding linear projection errors. By Chernozhukov et al., 2018a, (2.18), the near-Neyman orthogonal moment function is given by:

$$\psi_{it}(\theta_0, \eta_0) := \left(Z_{it} - f_{it}\zeta_0\right)\left(Y_{it} - f_{it}\beta_0 - \left(D_{it} - f_{it}\pi_0\right)\theta_0\right). \tag{4.7}$$

where we denote $\eta_0 = (\zeta_0, \beta_0, \pi_0)$. Under the sparse approximation, we can also rewrite the conditional expectation models for Y and D as

$$\begin{aligned} Y_{it} &= E[Y_{it}|X_{it}, c_i, d_t] + U_{it}^Y = f_{it}\beta_0 + r_{it}^Y + V_{it}^Y \\ D_{it} &= E[Y_{it}|X_{it}, c_i, d_t] + U_{it}^D = f_{it}\pi_0 + r_{it}^D + V_{it}^D. \end{aligned}$$

where $V_{it}^Y = \left(L_{2,it} - \mathrm{E}[L_{2,it}]\right)\eta_{Y,2} + U_{it}^Y$ and $V_{it}^D = \left(L_{2,it} - \mathrm{E}[L_{2,it}]\right)\eta_{D,2} + U_{it}^D$. For $l = Z, Y, D$, let ω_l, as defined in 2.8 with V_{it} replaced by V_{it}^l be the infeasible penalty weights for the two-way cluster LASSO estimations of $(\zeta_0, \beta_0, \pi_0)$. Correspondingly, let $\widehat{V}^l$ be the residuals and let $\widehat{\omega}_l$ be the feasible penalty weights. The two-step debiased estimator $\widehat{\theta}$ for θ_0 using the full-sample is defined as the solution of $\mathbb{E}_{NT}[\psi_{it}(\theta, \widehat{\eta})] = 0$ where $\widehat{\eta}$ are the (post) two-way cluster LASSO estimators for η_0 obtained in the first step using the full-sample.

Before presenting the regularity conditions, we further introduce the following notations:

$$\begin{aligned} &a_i = E[V_{it}^Z V_{it}^g|\alpha_i], \quad g_t = E[V_{it}^Z V_{it}^g|\gamma_t], \quad \Sigma_a = E[a_i a_i'], \quad \Sigma_g = \sum_{l=-\infty}^{\infty} E[g_t \tilde{g}_{t+l}'] \\ &A_0 = E_P[V_{it}^Z V_{it}^D], \quad \Omega_0 = \Sigma_a + c\Sigma_g, \end{aligned}$$

and, for $l = Z, Y, D$,

$$a_{i,j,l} = E[f_{it,j}V_{it}^l|\alpha_i], \quad g_{t,j,l} = E[f_{it,j}V_{it}^l|\gamma_t], \quad \Sigma_{a,j}^l = E[a_{i,j,l}^2], \quad \Sigma_{g,j}^l = \sum_{m=-\infty}^{\infty} E[g_{t,j,l}g_{t+m,j,l}'].$$

Assumption REG-P (Regularity Conditions for the Partial Linear Model)**.**

(i) A_0 *is non-singular.*

(ii) *For any* ϵ, $h_c(F, \epsilon)$ *and* $h_d(F, \epsilon)$ *are invertible in* F.

(iii) *For some* $\mu > 1, \delta > 0$, $\mathrm{E}[|f_{it,j}|^{8(\mu+\delta)}] < \infty$ *for* $j = 1, ..., p$, $\mathrm{E}[|V_{it}^l|^{8(\mu+\delta)}] < \infty$ *for* $l = g, D, Y, Z$.

(iv) *Either* $\lambda_{min}[\Sigma_a] > 0$ *or* $\lambda_{min}[\Sigma_g] > 0$, *and either* $\Sigma_{a,j}^l > c_\sigma$ *or* $\Sigma_{g,j}^l > c_\sigma$ *for some* $c_\sigma > 0$, $j = 1, ..., p$, $l = D, Y, Z$.

(v) $\log(p/\gamma) = o\left(T^{1/6}/(\log T)^2\right)$ *and* $p = o(T^{7/6}/(\log T)^2)$.

(vi) $\left[\mathrm{E}(a_{i,j}^l)^2\right]^{1/2} / \left[\mathrm{E}(a_{i,j}^l)^3\right]^{1/3} = O(1)$ *for* $j = 1, ..., p$, $l = D, Y, Z$.

(vii) The feasible penalty weights $\widehat{\omega}_l$ *satisfy condition 2.9 for* $l = D, Y, Z$.

This set of regularity conditions follows from the assumptions for two-way cluster-LASSO and the panel-DML inference. The only extra condition is Assumption REG-P(ii) which is a smoothness condition that ensures the exogeneity properties of (c_i, ϵ_i) and (d_t, ϵ_t) are inherited by $\bar{F}_i$ and $\bar{F}_t$.

Theorem 4.1. *Suppose, for* $P = P_{NT}$ *for each* (N, T)*, the following conditions hold for model 4.1 and* $W_{it} = \left(Y_{it}, D_{it}, X_{it}, Z_{it}, U_{it}, c_i, d_t, \epsilon_i, \epsilon_t\right)$*: (i) Assumptions 2, 2, 2, 4, 4; (ii) sparse approximation in 4.4 and 4.5 with* $s = o\left(\frac{\sqrt{N \wedge T}}{\log(p/\gamma)}\right)$, $\|r_{it}^l\|_{NT,2} = o_P\left(\sqrt{\frac{1}{N \wedge T}}\right)$ *for* $l = Y, D$*. Then, as* $N, T \to \infty$ *and* $N/T \to c$ *where* $0 < c < \infty$*,*

$$\sqrt{N}(\widehat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0, V)$$

where $V := A_0^{-1} \Omega_0 A_0^{-1}$.

Theorem 4.1 establishes the validity of the proposed inference procedure using the full sample. Note that the sparsity condition and the condition of the approximation errors are stronger than the ones needed for two-way LASSO estimation itself.

We define the variance estimators adapted from Chiang et al. (2024) and Chen and Vogelsang (2024) using the full sample as follows:

$$\widehat{V}_{\mathrm{CHS}} = \widehat{A}_{NT}^{-1} \widehat{\Omega}_{\mathrm{CHS}} \widehat{A}_{NT}^{-1\prime}, \quad \widehat{\Omega}_{\mathrm{CHS}} = \widehat{\Omega}_{\mathrm{A}} + \widehat{\Omega}_{\mathrm{DK}} - \widehat{\Omega}_{\mathrm{NW}}, \tag{4.8}$$

$$\widehat{V}_{\mathrm{DKA}} = \widehat{A}_{NT}^{-1} \widehat{\Omega}_{\mathrm{DKA}} \widehat{A}_{NT}^{-1\prime}, \quad \widehat{\Omega}_{\mathrm{DKA}} = \widehat{\Omega}_{\mathrm{A}} + \widehat{\Omega}_{\mathrm{DK}}, \tag{4.9}$$

where

$$\begin{aligned}
\widehat{A}_{NT} :=& \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (Z_{it} - f_{it}\widehat{\zeta})(D_{it} - f_{it}\widehat{\pi}), \\
\widehat{\Omega}_{\mathrm{A}} :=& \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{r=1}^{T} \psi_{it}(\widehat{\theta}, \widehat{\eta}) \psi_{ir}(\widehat{\theta}, \widehat{\eta})', \\
\widehat{\Omega}_{\mathrm{DK}} :=& \frac{1}{NT^2} \sum_{t=1}^{T} \sum_{r=1}^{T} k\left(\frac{|t-r|}{M}\right) \sum_{i=1}^{N} \sum_{j=1}^{N} \psi_{it}(\widehat{\theta}, \widehat{\eta}) \psi_{jr}(\widehat{\theta}, \widehat{\eta})', \\
\widehat{\Omega}_{\mathrm{NW}} :=& \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{r=1}^{T} k\left(\frac{|t-r|}{M}\right) \psi_{it}(\widehat{\theta}, \widehat{\eta}) \psi_{ir}(\widehat{\theta}, \widehat{\eta})'.
\end{aligned}$$

For simplicity, we deliver the consistency results of variance estimators assuming the approximation is exact. Allowing for approximation errors does not change the main idea but only requires more regularity conditions on the approximation error and more cumbersome derivations in the proof.

Theorem 4.2. *Suppose assumptions for Theorem 4.1 holds for $P = P_{NT}$ for each (N,T) with $r_{it}^D = r_{it}^Y = 0$ a.s., and $M/T^{1/2} = o(1)$. Then, $(N,T) \to \infty$ and $N/T \to c$ where $0 < c < \infty$,*

$$\begin{aligned}\widehat{V}_{\text{CHS}} =& V + o_P(1),\\ \widehat{V}_{\text{DKA}} =& \widehat{V}_{\text{CHS}} + o_P(1).\end{aligned}$$

5. Monte Carlo Simulation

In this section, we examine the performance of the panel DML estimation and inference procedure in a Monte Carlo simulation study. To focus on the performance of the panel DML procedure and the LASSO estimator, the DGPs considered in this section are free of correlated random effects. Firstly, I consider the following triangular model without approximation error, simplified from the partial linear model in Section 4:

$$\begin{aligned}\textbf{Linear Model}: \; Y_{it} =& \; D_{it}\theta_0 + X_{it}\beta_0 + U_{it},\\ D_{it} =& \; X_{it}\pi_0 + V_{it},\end{aligned}$$

where $\theta_0 = 1/2$ and $\beta_0 = \pi_0 = a\,(1, 1, ..., 1, 0, ..., 0)'$ are p-dimensional parameter vectors where the first s entries are 1 and the rest of the elements are 0; a is a constant that controls the relevance of the covariates.

Secondly, I also consider the case where the true nuisance function form is unknown but it is smooth enough so it can be well-approximated by a linear combination of polynomially transformed variables:

$$\begin{aligned}\textbf{Nonlinear Model}: \; Y_{it} =& \; D_{it}\theta_0 + \frac{X_{it}\beta_0}{1 + \left(X_{it}\beta_0\right)^2} + U_{it},\\ D_{it} =& \; \frac{X_{it}\pi_0}{1 + [\exp(X_{it}\pi_0)]^{-1}} + V_{it},\end{aligned}$$

where the parametrization is the same as DGP(i). For DGP(ii), we pretend the nuisance functions of X_{it} are unknown, but, since they are sufficiently smooth functions of X_{it} with bounded derivatives, it can be shown by Taylor expansion that they can be well-approximated (in the sense of Assumption ASM) by a polynomial series of sufficiently large order.

To feature in the two-way dependence in $V_{it}U_{it}$ as well as $X_{it}U_{it}$ and $X_{it}V_{it}$, (X_{it}, U_{it}, V_{it}) are generated by the underlying components as follows: for each $j = 1, ..., p$,

$$\begin{aligned}\textbf{Additive Components}: \; X_{it,j} =& \; w_1\alpha_{i,j} + w_2\gamma_{t,j} + w_3\varepsilon_{it,j},\\ U_{it} =& \; w_1\alpha_i^u + w_2\gamma_t^u + w_3\varepsilon_{it}^u,\\ V_{it} =& \; w_1\alpha_i^v + w_2\gamma_t^v + w_3\varepsilon_{it}^v,\end{aligned}$$

where the components $\alpha_i^u, \alpha_i^v, \varepsilon_{it}^u, \varepsilon_{it}^v, \alpha_{i,j}, \gamma_{t,j}$ are each random draws from a uniform distribution with support

$(-\sqrt{3}, \sqrt{3})$ for each j; $\varepsilon_{it} = (\varepsilon_{it,1}, ..., \varepsilon_{it,p})'$ is a random draw from a joint normal distribution with mean 1 and variance-covariance matrix equal to $\iota^{|j-k|}$, $\iota \in [0, 1)$, in the (j, k)'s entry; The components γ_t^u, γ_t^v each follow a AR(1) process with the coefficient equal to ρ and the initial values randomly drawn from the normal distribution with mean 0 and variance $1 - \rho^2$ for some $\rho \in [0, 1)$. The weights (w_1, w_2, w_3) are non-negative with $w_1 + w_2 + w_3 = 1$.

In practice, it is common to apply a within-transformation for a linear model with additive unobserved heterogeneity. In that case, if the component structure is exactly as described above, then the within-transformation not only deals with the endogeneity due to the unobserved heterogeneity but also eliminates the two-way dependence introduced by the linear components. To illustrate it is not necessarily the case in general and verify the theory under a different scenario, I also consider a multiplicative component structure DGP that features two-way dependence in the moment function even after the within-transformation:

$$\begin{aligned}
\textbf{Multiplicative Components}: \quad X_{it,j} &= w_1\alpha_{i,j} + w_2\gamma_{t,j} + w_3\varepsilon_{it,j}, \\
U_{it} &= \frac{w_4}{c_p}\sum_{j=1}^{p}\left[\alpha_i^u\gamma_{t,j} + \alpha_{i,j}\gamma_t^u\right] + w_5\varepsilon_{it}^u, \\
V_{it} &= \frac{w_4}{c_p}\sum_{j=1}^{p}\left[\alpha_i^v\gamma_{t,j} + \alpha_{i,j}\gamma_t^v\right] + w_5\varepsilon_{it}^v,
\end{aligned}$$

where the components are generated the same way as the Linear Components. The weights $(w_1, w_2, w_3, w_4, w_5)$ are non-negative with $w_1^2 + w_2^2 + w_3^2 = 1$ and $w_4^2 + w_5^2 = 1$. c_p is a scaling factor that ensures the sums of multiplicative components in both U_{it} and V_{it} are variance 1.

The multiplicative components construction here is a generalization of the example in Chiang et al. (2024) where it is used to illustrate that the two-way within-transformation of the original linear panel model may not eliminate the underlying components. To see why $U_{it}V_{it}$ features a component structure, we can expand the product and observe that it includes terms like $\alpha_i^u\alpha_i^v\gamma_{t,j}^2$ for $j = 1, ..., p$ whose conditional expectations given $\alpha = (\alpha_i^u, \alpha_i^v, \alpha_{i,1}, ..., \alpha_{i,p})$ are $\alpha_i^u\alpha_i^v$ since $\gamma_{t,j}$ has variance 1 and is independent of α. Likewise, the product also includes terms like $\gamma_t^u\gamma_t^v\alpha_{i,j}^2$ whose conditional expectations given $\gamma = (\gamma_t^u, \gamma_t^v, \gamma_{t,1}, ..., \gamma_{t,p})$ are $\gamma_t^u\gamma_t^v$. We can also show that $X_{it,j}U_{it}$ and $X_{it,j}V_{it}$ possess a components structure in a similar way. Importantly, these underlying common factors do not introduce endogeneity as they may seem to.

The simulation study examines the Monte Carlo bias(Bias), standard deviation (SD), mean square error (MSE), and coverage probability of estimators for θ_0. All estimations are based on the orthogonal moment condition. The comparison will be among procedures with and without cross-fitting. The first-step estimations will be based on the POLS estimator (if feasible), the heteroskedasticity-robust LASSO from Belloni et al. (2012), the square-root LASSO from Belloni et al. (2011), the cluster-robust LASSO from Belloni et al. (2016), and the two-way cluster-LASSO. The CHS-type and DKA-type variance estimators (different formulas for estimations with and without cross-fitting) will be used to obtain sample coverage probabilities. In some unreported simulations, I also compare CHS/DKA type variance estimators with Eicker-Huber-White

type estimators in Chernozhukov et al. (2018a) for random sampling data and Cameron-Galbach-Miller type estimator from Chiang et al. (2022) for multiway clustered data. Since it is well-known that inference based on variance estimators not sufficiently accounting for the dependence would cause over-rejection, it is omitted here.

Table 5.1: Linear Model with Additive Components
with $N = T = 30$, $s = 5$, $p = 200$, $\iota = 0.5$, $\rho = 0.75$, $a = 0.5$

Cross	First-Step	First-Step Ave.		Second-Step			Coverage (%)	
Fitting	Estimator	Sel. Y	Sel. D	Bias	SD	RMSE	CHS	DKA
No	POLS	200	200	0.001	0.047	0.047	78.6	94.4
	H LASSO	24.3	24.7	0.056	0.058	0.080	63.4	82.0
	R LASSO	20.1	20.6	0.057	0.060	0.082	68.0	83.0
	C LASSO	8.8	8.7	0.042	0.085	0.095	80.0	86.0
	TW LASSO	5.0	4.9	0.005	0.107	0.108	82.8	87.4
Yes	POLS	200	200	0.001	0.128	0.128	97.6	98.4
	H LASSO	15.5	15.7	0.057	0.139	0.150	93.6	95.8
	R LASSO	11.7	11.9	0.054	0.141	0.151	94.6	96.4
	C LASSO	5.5	5.7	0.008	0.152	0.152	93.4	95.0
	TW LASSO	5.0	4.7	0.012	0.159	0.159	90.4	93.6

Note: Simulation results are based on 500 replications. Tuning parameters: $(K, L) = (4, 8)$, $C_\lambda = 2.1$, $m = 2$, and $\gamma = 0.1/\log(p \vee N \vee T)$. H: heteroskedastic-LASSO; R: square-root-LASSO; C: cluster-LASSO; TW: two-way cluster-LASSO. Post-LASSO POLS is performed in all first steps. Based on the sample correlation with the outcome, the 5 most relevant regressors are used to obtain the initial feasible penalty weights for all three LASSO approaches. Nominal coverage probability: 0.95.

The simulation results are based on 500 Monte Carlo replications. It is a relatively small number of replications but it is necessitated by the high computational cost of multiple high-dimensional estimation and inference procedures. The estimation is based on the orthogonal moment condition given as follows by 4.7 with $Z_{it} = D_{it}$, $f_{it} = X_{it}$, $\zeta_0 = \pi_0$. Results are obtained across DGPs varied by the sample sizes (N, T), the dimensions of covariates p, the number of non-zero slope coefficients s, the other sparsity parameter b, the common coefficient a, the multicollinearity parameter ι and the temporal correlation parameter ρ. For the panel DML inferential procedure with cross-fitting, the tuning parameters (K, L), the number of cross-fitting blocks, needs to be chosen. For variance estimation, bandwidth parameters M of the Bartlett kernel are required. I use the min-MSE rule from Andrews (1991) for both purposes. For a generic scalar score v_{it}, the formula is given as follows:

$$\hat{M} = 1.8171 \left(\frac{\widehat{\rho}^2}{\left(1 - \widehat{\rho}^2\right)^2} \right)^{1/3} T^{1/3} + 1,$$

where $\widehat{\rho}$ is the OLS estimator from the regression $\bar{\hat{v}}_t = \rho\bar{\hat{v}}_{t-1} + \eta_t$ where $\bar{\hat{v}}_t = \frac{1}{N}\sum_{i=1}^{N}\hat{v}_{it}$. For variance estimation, $\hat{v}_{it} = \hat{U}_{it}\hat{V}_{it}$; For feasible weights estimation, $\hat{v}_{it} = x_{it}\hat{U}_{it}$ or $\hat{v}_{it} = x_{it}\hat{V}_{it}$ where x_{it} is a generic scalar regressor.

Table 5.2: Linear Model with Additive Components
with $N = T = 30$, $s = 5$, $p = 800$, $\iota = 0.5$, $\rho = 0.75$, $a = 0.5$

Cross	First-Step	First-Step Ave.		Second-Step			Coverage (%)	
Fitting	Estimator	Sel. Y	Sel. D	Bias	SD	RMSE	CHS	DKA
No	POLS	800	800	0.004	0.097	0.097	47.6	65.8
	H LASSO	39.1	39.6	0.066	0.043	0.078	47.8	74.4
	R LASSO	31.9	32.2	0.068	0.048	0.083	49.2	70.2
	C LASSO	13.4	13.8	0.058	0.078	0.097	66.6	76.2
	TW LASSO	5.0	4.8	0.011	0.108	0.108	79.8	86.6
Yes	H LASSO	24.6	25.7	0.029	0.151	0.154	91.8	94.2
	R LASSO	18.1	18.3	0.060	0.133	0.146	94.0	97.0
	C LASSO	6.5	6.5	0.011	0.151	0.151	93.2	95.6
	TW LASSO	5.1	4.7	0.021	0.157	0.158	89.0	94.0

Note: Simulation results are based on 500 replications. Tuning parameters: $(K, L) = (4, 8)$, $C_\lambda = 2.1$, $m = 2$, and $\gamma = 0.1/\log(p \vee N \vee T)$. H: heteroskedastic-LASSO; R: square-root-LASSO; C: cluster-LASSO; TW: two-way cluster-LASSO. Post-LASSO POLS is performed in all first steps. Based on the sample correlation with the outcome, the 5 most relevant regressors are used to obtain the initial feasible penalty weights for all three LASSO approaches. Nominal coverage probability: 0.95.

Table 5.1 presents a set of baseline results that are obtained for a reasonably large number of regressors ($p = 200$) among which 5 are associated with non-zero slope coefficients. The number of covariates is much larger than either cross-sectional or temporal dimensions. Still, the number of non-zero coefficients can be regarded as a small order of the sample sizes, satisfying the sparsity condition. In the first step, model selections are done using different LASSO approaches reported in the second column. The number of selected regressors for both Y and D are reported in the third and fourth columns. First, we take a look at the rows without using cross-fitting. It is shown that the proposed two-way cluster-LASS, either performed with CHS or DKA feasible weights, selects almost perfectly while other LASSO approaches all over-select to a different degree. Among the LASSO-based approaches, the proposed methods suffer from the least finite sample bias. However, the proposed methods have slightly larger variability in terms of Monte Carlo SD and RMSE. For sample coverage, both CHS- and DKA-type variance estimators are used for all methods. In terms of variance estimation and inference, all high-dimensional methods suffer from under-coverage slightly while the proposed methods have the least size distortion. This can result from a strong serial correlation in the time effects ($\rho = 0.75$) and can also be explained by the fact that under high-dimensionality and two-way cluster dependence, the asymptotic normality is not guaranteed without cross-fitting, as discussed in

Section 2. Surprisingly, POLS performs slightly better than high-dimensional methods for a reasonably high-dimensional model. However, as we will see later, it is not true anymore when the dimension of covariates grows further.

Table 5.3: Nonlinear Model with Additive Components
with $N = T = 30$, $s = 5$, $p = 10$, $\iota = 0.5$, $\rho = 0.75$, $a = 0.33$

Cross	First-Step	First-Step Ave.		Second-Step			Coverage (%)	
Fitting	Estimator	Sel. Y	Sel. D	Bias	SD	RMSE	CHS	DKA
	POLS	494	494	0.004	0.091	0.091	78.2	85.0
	H LASSO	9.8	18.1	0.007	0.091	0.092	85.2	89.4
	R LASSO	8.3	16.0	0.007	0.092	0.092	86.0	89.4
No	C LASSO	8.0	15.4	0.002	0.093	0.093	86.6	89.4
	TW LASSO	7.0	14.2	0.008	0.093	0.093	85.4	91.2
	POLS	494	494	0.011	0.166	0.166	99.2	99.8
	H LASSO	7.3	13.5	0.037	0.159	0.163	88.4	93.0
	R LASSO	5.2	10.4	0.037	0.157	0.162	87.6	92.8
Yes	C LASSO	4.9	9.2	0.027	0.148	0.150	91.4	93.6
	TW LASSO	4.8	9.8	0.027	0.139	0.142	92.0	94.6

Note: Simulation results are based on 500 replications. Tuning parameters: $(K, L) = (4, 8)$, $C_\lambda = 2.1$, $m = 5$, and $\gamma = 0.1/\log(p \vee N \vee T)$. H: heteroskedastic-LASSO; R: square-root-LASSO; C: cluster-LASSO; TW: two-way cluster-LASSO. Post-LASSO POLS is performed in all first steps. Based on the sample correlation with the outcome, the 5 most relevant regressors are used to obtain the initial feasible penalty weights for all three LASSO approaches. Nominal coverage probability: 0.95.

When cross-fitting is employed, all methods have witnessed a significant improvement in terms of sample coverage. This is particularly true for LASSO-based methods that are not designed for dependent data, which is not predicted by the theory. As a cost, we can see from the increased Monte Carlo SD that there is a loss of efficiency by doing cross-fitting, as expected. It is also worth emphasizing that the CHS- and DKA-type variance estimators designed for cross-fitting approaches play an important role in the desirable sample coverage. In some unreported simulations, it is shown that inference based on the cross-fitting variance estimators proposed in Chernozhukov et al. (2018a) and Chiang et al. (2022) suffer from severe under-coverage. This is not surprising but the implication is more subtle: while two-way dependence potentially affects both estimation and inference, its negative impact on the inference is more salient.

As the dimension of the covariates is as large as the overall sample size, a different pattern is revealed. Table 5.2 considers the same DGP as Table 5.1 except that the dimension p now increases to 800, slightly smaller than the sample size 900. In principle, we can compare a case with the dimension of covariates larger than the sample size, but then it is not possible to compare the results based on the POLS first-steps. First, we compare the results without cross-fitting. The simulation results demonstrate that the methods based on the

POLS first-steps with no selection and those based on the existing LASSO approaches with over-selection all suffer from severe under-coverage. The proposed methods, in contrast, continue to select almost perfectly regardless of the increased number of irrelevant regressors. Again, when cross-fitting is performed, there is a significant improvement across all approaches in terms of the sample coverage. This is expected: as shown in the proof of Theorem 4.2, the error term R_{NT}^2 contains terms such as $V_{it}^D(\beta_0 - \tilde{\beta})$ that are not mean 0 and may not vanish fast enough; this is more severe when no selection or inconsistent selection is performed in the first stage. Through the proper cross-fitting scheme, such terms are conditionally mean-zero even with over-selection. Such property of cross-fitting methods is thus translated into better coverages across all approaches considered here.

Table 5.4: Linear Model with Multiplicative Components
with $N = T = 30$, $s = 5$, $p = 200$, $\iota = 0.5$, $\rho = 0.75$, $a = 0.5$

Cross	First-Step	First-Step Ave.		Second-Step			Coverage (%)	
Fitting	Estimator	Sel. Y	Sel. D	Bias	SD	RMSE	CHS	DKA
	POLS	200	200	0.001	0.139	0.139	77.0	81.6
	H LASSO	5.8	5.9	0.002	0.135	0.135	83.8	89.4
No	R LASSO	5.6	5.6	0.001	0.135	0.135	84.2	89.2
	C LASSO	5.9	6.1	0.001	0.137	0.137	84.2	89.6
	TW LASSO	12.2	12.6	0.001	0.137	0.137	83.6	89.2
	POLS	200	200	0.004	0.148	0.148	91.8	94.6
	H LASSO	6.0	6.0	0.003	0.149	0.149	95.0	97.2
Yes	R LASSO	5.4	5.4	0.002	0.149	0.149	94.8	97.6
	C LASSO	6.2	6.1	0.013	0.152	0.153	94.0	96.6
	TW LASSO	12.7	12.2	0.024	0.152	0.153	92.6	96.2

Note: Simulation results are based on 500 replications. Tuning parameters: $(K, L) = (4, 8)$, $C_\lambda = 2.1$, $m = 2$, and $\gamma = 0.1/\log(p \vee N \vee T)$. H: heteroskedastic-LASSO; R: square-root-LASSO; C: cluster-LASSO; TW: two-way cluster-LASSO. Post-LASSO POLS is performed in all first steps. Based on the sample correlation with the outcome, the 5 most relevant regressors are used to obtain the initial feasible penalty weights for all three LASSO approaches. Nominal coverage probability: 0.95.

We have seen the case with exact sparsity in Tables 5.1 and 5.2. As argued in the theory, the proposed estimation and inference procedures are also valid under approximate sparsity. In Table 5.3, we consider a case where 10 observable control variables are considered by the researcher while only 5 of those are relevant. Those 5 relevant controls enter the model through unknown but smooth nuisance functions as described above. The researchers are not aware of the irrelevance of the other five controls and neither do researchers have knowledge of the nuisance function forms. However, the smoothness allows for the approximation of the nuisance function using polynomials series. In particular, we use the 3rd-order polynomial transformation of the 10 observable controls for approximation. Due to bounded derivatives of the smooth nuisance functions

and the irrelevance of some of the controls, the sparse approximation is valid. As is shown in Table 5.3, compared to previous sets of results, the advantage of the proposed methods is not obvious in this case. From the number of selected regressors and the Monte Carlo SD across all methods, it seems the dependence is not as strong as in previous DGPs. Given the small number of Monte Carlo replications, the differences in sample coverage probabilities across different approaches are not really distinguishable. On the other hand, the improvement made by the use of the cross-fitting remains significant.

Lastly, I use the multiplicative components to generate two-way cluster dependence and compare the results in Table 5.4. There are two main differences compared to previous results. First, the two-way dependence is present but relatively weak. In some unreported simulation results using variance estimators not accounting for two-way dependence, all methods suffer from under-coverage issues, implying the presence of two-way dependence. Additionally, we can see the over-selection problem is almost not present for heteroskedasticity-LASSO, square-root-LASSO, and cluster-LASSO, also indicating the dependence is quite weak. Secondly, notice that the two-way cluster LASSO with CHS-type penalty weights over-selects, in a more severe way than existing approaches not accounting for two-way dependence. Further examination of the issue reveals that it is due to the finite-sample bias of the HAC-type formula: it is well known that HAC-type variance estimators are not guaranteed to be positive-semi definite. Under certain DGPs, the HAC-type estimator is more likely to be negative or close to 0, causing large finite sample bias in variance estimation. This is likely to be the case here. Indeed, some penalty weights estimated using the CHS formula are negative and replaced by 0. There are other finite sample adjustments available but in general, it is a disadvantage for the CHS-type penalty weights. The DKA-type feasible weights are guaranteed to be positive and it has been shown in the literature that the two-term formulas for variance estimation tend to have better finite sample performance (Chen and Vogelsang (2024)). However, it is also well-known that the two-term formulas for variance estimation suffer from an overestimation problem when the true DGP is i.i.d over both clusters or is only clustered at the intersection of the two clusters (MacKinnon et al. (2021)). In practice, it is very rare to encounter panel data not featuring any dependence across space or time. As a result, it is in general recommended to use the DKA-type formula for both penalty weights and variance estimation.

6. Empirical Application

In this section, I re-examine the effects of government spending on the output of an open economy following the framework of Nakamura and Steinsson (2014). It is one of the most cited empirical-macro papers on the American Economic Review and it investigates one classic quantity of interest in economics: the government spending multiplier. The question is can we improve on the estimation and inference through more robust and flexible methods? As I will show, it is made possible by the proposed toolkit in this paper.

This framework utilizes the regional variation in military spending in the US to estimate the percentage increase in output that results from the increase of government spending by 1 percent of GDP, i.e. government spending multiplier. It is referred to as the "open economy relative multiplier" because this framework takes advantage of uniform monetary and tax policies across the regions in the US to difference-out their effects on

government spending and output. The parameter of interest is a scalar and the baseline model does not even need a control for identification, so why is the high-dimensionality relevant here? As it will be revealed very soon, indeed, the high dimensionality from heterogeneity and flexible modeling can be hidden in settings to which researchers don't usually relate high dimensionality.

Due to the endogeneity in the variation of the regional military procurement, Nakamura and Steinsson (2014) achieves identification through an instrumental variable (IV) approach. As argued by the authors, the national military spending is largely determined by geopolitical events so it is likely exogenous to the unobserved factors of regional military spending and it affects the regional military spending disproportionally. In other words, the identifying assumption is that the buildups and drawdowns in national military spending are not due to unbalanced military development across regions. Based on this observation, a share-shift type IV is considered and the share is estimated by regressing the regional military spending on the national military spending allowing for region-specific constant slope coefficients.[12] To focus on the main idea, the shares are taken as given and the resulting instrument variable is treated as observable instead of generated regressors to avoid further complication.

In this paper, I extend the linear model with additive unobserved heterogeneous effects to a partial linear model with non-additive unobserved heterogeneous effects. Let D_{it} be the percentage change in per capita regional military spending in state i and time t and Z_{it} be the IV. Specifically, the baseline model from the original study and the one from this paper differ as follows:

$$\begin{aligned}
&\text{Baseline model}: \\
&Y_{it} = \theta_0 D_{it} + \pi_i W_t + c_i + d_t + U_{it}. \\
&\text{Partial linear model}: \\
&Y_{it} = \theta_0 D_{it} + g(X_{it}, W_t, c_i, d_t) + U_{it}.
\end{aligned}$$

where θ_0 is the parameter of interest, i.e. the true multiplier; X_{it} and W_t are exogenous control variables with the latter being only time-varying; π_i are non-random unit specific slope coefficients of W_t; (c_i, d_t) are unobserved heterogeneous effects. In the original study, the linear model is estimated by the two-stage least square (2SLS) with two-way fixed effects. In the extended model, I model the unobserved heterogeneous effects as correlated random effects and take a sparse approximation approach for the infinite-dimensional nuisance parameters as in Section 4. Specifically, c_i is assumed to be a function of $(\bar{D}_i, \bar{X}_i)$ and d_t is assumed to be a function of $(\bar{D}_t, \bar{X}_t, W_t)$. Then, through sparse approximation, the feasible (near) Neyman-orthogonal

[12] All quantities, unless specifically defined, are in terms of two-year growth rate of the real per capita values. Per capita is in terms of total population. Nakamura and Steinsson (2014) also presents results when per capita is calculated using the working age population as a robustness check.

moment function is given by

$$\left(Z_{it} - f_{it}\zeta_0\right)\left(Y_{it} - f_{it}\beta_0 - \theta_0\left(D_{it} - f_{it}\pi_0\right)\right)$$

where $f_{it} = (L^r(X_{it}, W_t, \bar{D}_i, \bar{D}_t, \bar{X}_i, \bar{X}_t), 1)$ and (β, π, ζ) are associated slope coefficients defined the same way as in Section 4.

In the original study, W_t are not included in the baseline model. In the alternative specifications, W_t is chosen as the real interest rate or the change in national oil price. These two variables are never included together in the original study. Note that allowing the unit-specific slope coefficients for controls generates many nuisance parameters: with 51 state groups[13], one control would increase 51 parameters and two controls would generate 102 parameters, given no interactions or higher order terms. With a sample size less than 2000, the high dimensionality in nuisance parameters could result in a noisy estimate of θ_0. In this paper, to obtain a more precise estimate and make the excludability assumption of the IV to be more plausible, besides the controls from the original study, I also consider additional controls. For X_{it}, I include the change in state population. As is shown in Table 3 of Nakamura and Steinsson (2014), the state population is likely not affected by the treatment (the regional military spending), so it is immune to the "bad control" problem[14]; But it could affect the treatment and the outcome. By considering more flexible function forms and additional exogenous control variables, the excludability condition of the instruments is more plausible. On the other hand, the high-dimensionality arose from the flexible function form and the unobserved heterogeneity necessitates the use of high-dimensional selection methods. Moreover, state-level yearly variables of those macroeconomic characteristics are often considered to be cluster-dependent in both space and time groups due to correlated time shocks and state-unobserved factors. These concerns justify the use of robust estimation and inference methods proposed in this paper.

The data is available through Nakamura and Steinsson (2014). It is a balanced (after trimming) state-level yearly panel data with 51 states from 1971-2005 years. The military spending data is collected from the electronic database of DD-350 military procurement forms of the US Department of Defense. The state output is measured by state DGP collected from the US Bureau of Economics Analysis (BEA). The state population data is from the Census Bureau. Data on oil prices is from West Texas Intermediate. The Federal Funds rate is from the FRED database of the St. Louis Federal Reserve. The state inflation measures are constructed from several sources. For more details on data construction, readers are referred to Nakamura and Steinsson (2014).

Table 6.1 provides benchmark results for the original model with different choices of control variables. All estimates (columns 6 and 9) of are given by 2SLS with two-way fixed effects and the standard errors

[13]The regions in this analysis are defined by the states. Nakamura and Steinsson (2014) also presents results on regions as clusters of states.

[14]Angrist and Pischke, 2009, Frölich, 2008, and Chen and Kim, 2024 provide detailed discussions on when endogenous control pollute the identification/estimation and when they are innocuous.

Table 6.1: Multiplier estimates from the original model

(1) Unobs. Heterog.	(2) Oil Price	(3) Real Int.	(4) Pop.	(5) First Stage	(6) IV 1 $\widehat{\theta}$	(7) CHS s.e.	(8) DKA s.e.
Fixed Effects	No	No	No	POLS	1.43	0.68	0.81
	Yes	No	No	POLS	1.30	0.56	0.72
	No	Yes	No	POLS	1.40	0.57	0.70
	Yes	Yes	No	POLS	1.27	0.45	0.71
	Yes	Yes	Yes	POLS	1.36	0.43	0.56

Note: Standard errors are calculated with the truncation parameter M chosen by the min-MSE rule given in Section 5.

(s.e.) are calculated using CHS and DKA formulas given in Section 4. The estimates of the multiplier are matched with those given in Nakamura and Steinsson (2014) with significant differences in the standard errors. It is because the variance estimates here account for the potential two-way dependence while the variance estimator used in Nakamura and Steinsson (2014) assumes cross-sectional independence.

Table 6.2: Estimates of the open economy relative multiplier from the extended model.

(1) Cross-Fitting	(2) Unobs. Heterog.	(3) Poly. Trans.	(4) Param. Gen.	(5) First Stage	(6) Z: Param. Sel.	(7) $\widehat{\theta}$	(8) CHS s.e.	(9) DKA s.e.
No	Mundlak	None	7	POLS	7	1.51	0.66	0.82
				H LASSO	2	1.43	0.70	0.84
				C LASSO	4	1.43	0.66	0.81
				TW LASSO	1	1.47	0.61	0.77
No	Mundlak	2nd	35	POLS	35	1.73	0.99	1.15
				H LASSO	6	1.73	1.03	1.19
				CR LASSO	5	1.40	0.70	0.86
				TW LASSO	3	1.47	0.61	0.77
No	Mundlak	3rd	119	POLS	119	2.20	1.19	1.37
				H LASSO	9	2.00	1.18	1.39
				CR LASSO	6	0.97	0.65	0.80
				TW LASSO	5	1.47	0.61	0.77

Note: The tuning parameters are chosen as $m = 10$, $C_\lambda = 2.1$, $\gamma = 0.1/\log(p \vee N)$. The control variables and the Mundlak sample averages $(X_{it}, W_t, \bar{D}_i, \bar{D}_t, \bar{X}_i, \bar{X}_t)$ are used to obtain the initial feasible penalty weights for all three LASSO approaches. Number of predictors generated by the polynomial transformation and the number of selected predictors for Z are reported in columns (4) and (6). Standard errors are calculated with the truncation parameter M chosen by the min-MSE rule given in Section 5.

The main comparisons are done in Tables 6.2 and 6.3. In Table 6.2, no cross-fitting is performed in the first stage. The number of parameters associated with regressors generated by polynomials transformations are reported in column (4) and the number of selected parameters associated with Z are reported in column

(6)[15]. Overall, with more controls and the polynomial transformation of the observables, the standard errors are generally larger than those in 6.1. With no transformations of the original regressors, the estimates obtained by four different methods are similar and they are consistent with the baseline results. It is noticeable that the proposed approach TW LASSO using the DKA-type penalty weights achieves an estimate that is consistent with the baseline results and has the least variability. As the flexibility and number of nuisance parameters increase with the higher-order polynomial transformations, the number of selected regressors increases across all methods. While the standard errors of most approaches climb become larger and the estimates deviate from the baseline results, the proposed approach remains less noisy. This indicates that many higher-order polynomials included in the extended model for robustness in the function form may not matter that much but sorely contribute to the noise; while the existing approaches tend to over-select those terms under potential two-way dependence, the proposed method is robust against over-selection.

Table 6.3: Estimates of the open economy relative multiplier from the extended model.

(1) Cross-Fitting	(2) Unobs. Heterog.	(3) Poly. Trans.	(4) Param. Gen.	(5) First Stage	(6) Z: Param. Ave. Sel.	(7) $\widehat{\theta}$	(8) CHS s.e.	(9) DKA s.e.
Yes	Mundlak	None	7	H LASSO	2.0	1.45	1.66	1.93
				C LASSO	2.7	1.32	1.75	2.03
				TW LASSO	2.0	1.51	1.29	1.56
Yes	Mundlak	2nd	35	H LASSO	5.1	0.98	2.10	2.43
				C LASSO	5.5	1.12	1.90	2.18
				TW LASSO	4.6	1.01	1.22	1.47
Yes	Mundlak	3rd	119	H LASSO	8.2	2.05	2.92	3.46
				C LASSO	6.5	1.23	1.57	1.88
				TW LASSO	6.0	1.09	1.18	1.46

Note: The tuning parameters are chosen as $(K, L) = (4, 8)$, $m = 10$, $C_\lambda = 2.1$, $\gamma = 0.1/\log(p \vee N \vee T)$. The control variables and the Mundlak sample averages $(X_{it}, W_t, \bar{D}_i, \bar{D}_t, \bar{X}_i, \bar{X}_t)$ are used to obtain the initial feasible penalty weights for all three LASSO approaches. Number of predictors generated by the polynomial transformation and the average (over cross-fitting sub-samples) number of selected predictors for Z are reported in columns (4) and (6). Standard errors are calculated with the truncation parameter M chosen by the min-MSE rule given in Section 5.

For more robustness in inference, as is shown in both theoretical and simulation results, I further consider different methods implemented with cross-fitting in Table 6.3[16]. It shows a similar pattern as in Table 6.2: The variability of different methods increases as the model approximated by higher-order polynomial series, except for the proposed approach which witnesses more accuracy as the approximation is made more flexible. What's going on here could be potential nonlinearity in the true model unknown to the researcher. When no

[15] Across all first-step LASSO approaches, more parameters associated with Z are selected compared to those associated with Y and X. The difference in the LASSO selection is less evident for Y and X while the pattern is similar.

[16] Due to smaller samples in the first step and multicollinearity among the polynomial terms, methods based on the POLS first-step is too noisy and so they are omitted for comparison here.

transformation or the 2nd-order polynomial transformation is used, a limited amount of nonlinearity is captured in the misspecified model. With the 3rd-order polynomial approximation, while the nonlinearity might be captured by all three methods in comparison, as revealed by the increased number of selected regressors, the proposed two-way cluster LASSO achieves that with fewer and likely more accurate selections. Unfortunately, although the cross-fitting approach is asymptotically valid and has better size control as shown in the simulation, it is in the cost of efficient loss, as told by the larger standard errors.

7. Conclusion and Discussion

The inferential theory for high-dimensional models is particularly relevant in panel data settings where the modeling of unobserved heterogeneity commonly leads to high-dimensional nuisance parameters. This paper enriches the toolbox of researchers in dealing with high-dimensional panel models. Particularly, I propose a package of tools that deal with the estimation and inference in high-dimensional panel models that feature in two-way cluster dependence and unobserved heterogeneity. I first develop a weighted LASSO approach that is robust to two-way cluster dependence in the panel data. As is shown in the asymptotic analysis of the two-way cluster LASSO, the convergence rates are slow due to the cluster dependence, making it challenging for inference purposes. However, by utilizing a cross-fitting method designed for a two-way clustered panel, the rate requirement for the first step can be substantially relaxed, making the proposed two-way cluster-LASSO a valid first-step estimator for inference purposes in a high-dimensional semiparametric model. Individually, both the two-way cluster-LASSO and the cross-fitting can be of independent interest; Together, they extend the DML approach to panel data settings. I further consider the unobserved heterogeneity in panel models. Due to the potential non-compatibility of cross-fitting with common fixed-effect and random-effect methods, I study the statistical properties of the proposed estimation and inference procedures using the full sample and establish the validity under slightly stronger sparsity assumptions in a partial linear panel model, as a special case.

The estimation and inferential theory are empirically relevant. I illustrate the proposed approaches in an empirical example and exemplify that high-dimensionality can be hidden in questions not traditionally considered high-dimensional. In practice, when the question is naturally high-dimensional and answered by panel data, then there is no reason not to apply the approaches in this paper. When the questions are originally not high-dimensional, it is reasonable to start with a simple model as a baseline and then extend it to a more general and flexible model for a robustness check.

While both theoretical and simulation results support the proposed approaches, some limitations remain in certain scenarios. First, the Mundlak device and, in general, many other approaches for dealing with unobserved heterogeneity are not compatible with the cross-fitting schemes due to the dependence introduced by the full history of the data that is used for modeling the unobserved heterogeneous effects. On the other hand, it is generally not feasible to establish the inferential theory without cross-fitting in high-dimensional semi-parametric models (except the special cases such as the partial linear models). In that sense, the cross-fitting approach is naturally limited in use for panel data models. Secondly, the feasible penalty weight

estimation is highly non-trivial due to two-way cluster dependence and high dimensionality. The asymptotic analysis of the two-way cluster LASSO relies on high-level assumptions on the feasible penalty weights. Even though the iterative feasible weights estimation possesses desirable finite sample properties among the scenarios considered in the Monte Carlo simulation, there are many subtle issues that are lack of theoretical guarantee. A devoted exploration of such issues requires a more comprehensive treatment and is another important direction of future research.

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Appendix A

We will first introduce two lemmas regarding the law of large number (LLN) and the central limit theorem (CLT) for two-way clustered arrays with correlated time effects. They are restated and generalized from Theorems 1 and 2 in Chiang et al. (2024). The following notations will also be used frequently throughout the appendices: Let $\{W_{it} : i = 1, .., N; t = 1, ..., T\}$ be an array of random vectors taking values in $\mathbb{R}^p$. Let $F : \mathbb{R}^p \to \mathbb{R}^k$ be a measurable function where k is a constant. We define the Hajek projection terms $a_i = E[F(W_{it}) - E[F(W_{it})]|\alpha_i]$, $g_t = E[F(W_{it}) - E[F(W_{it})]|\gamma_t]$, and $e_{it} = W_{it} - E[F(W_{it})] - a_i - g_t$ and their corresponding (long-run) variance covariance matrices:

$$\Sigma_a = E[a_i a_i'], \quad \Sigma_g = \sum_{l=-\infty}^{\infty} E[g_t g_{t+l}'], \quad \Sigma_e = \sum_{l=-\infty}^{\infty} E[e_{it} e_{i,t+l}'].$$

We can rewrite $F(W_{it}) = a_i + g_t + e_{it}$. Suppose that W_{it} satisfy Assumptions AHK and AR, then the decomposition has the following properties:

(i) $\{a_i\}_{i\geq 1}$ is a sequence of i.i.d random vectors, $\{g_t\}_{t\geq 1}$ are strictly stationary and β-mixing with the the mixing coefficient $\beta_g(m) \leq \beta_\gamma(m)$ for all $m \geq 1$; for each i, $\{e_{it}\}_{t\geq 1}$ is also strictly stationary; and a_i is independent of g_t.

(ii) a_i, b_t, e_{it} are mean zero.

(iii) Conditional on (γ_t, γ_r), e_{it} and e_{jr} are independent for $j \neq i$.

(iv) The sequences $\{a_i\}$, $\{g_t\}$, $\{e_{it}\}$ are mutually uncorrelated.

Properties (i) and (ii) are straightforward. Property (iii) is due to the assumption that $\{\alpha_i\}$ and $\{\varepsilon_{it}\}$ are each i.i.d sequence and independent of each other. Property (iv) is less obvious. One can show $\mathrm{E}_P[e_{it}|\gamma_r] = 0$ and $\mathrm{E}_P[e_{it}|\alpha_j]$ for any i, t, j, r. It is less obvious to see $\mathrm{E}_P[e_{it}|\gamma_r] = 0$ for some $r \neq t$:

$$\begin{aligned}
\mathrm{E}_P[e_{it}|\gamma_r] &= \mathrm{E}_P[\psi\left(W_{it}; \theta_0, \eta_0\right)|\gamma_r] - \mathrm{E}_P[a_i|\gamma_r] - \mathrm{E}_P[g_s|\gamma_t] \\
&= \mathrm{E}_P[\mathrm{E}_P[\psi\left(f(\alpha_i, \gamma_t, \varepsilon_{it}); \theta_0, \eta_0\right)|\gamma_t, \gamma_r]|\gamma_r] - \mathrm{E}_P[a_i] - \mathrm{E}_P[g_t|\gamma_r] \\
&= \mathrm{E}_P[\mathrm{E}_P[\psi\left(f(\alpha_i, \gamma_t, \varepsilon_{it}); \theta_0, \eta_0\right)|\gamma_t]|\gamma_r] - \mathrm{E}_P[a_i] - \mathrm{E}_P[g_t|\gamma_r] \\
&= \mathrm{E}_P[g_t|\gamma_r] - \mathrm{E}_P[g_t|\gamma_r] = 0
\end{aligned}$$

where the second equality follows from the iterated expectation and the independence of α_i and γ_r and the third equality follows from that given γ_t, γ_r is independent of $(\alpha_i, \gamma_t, \varepsilon_{it})$.

Using the properties above, one can derive the LLN and CLT for two-way clustered panel data. The following lemma is regarding the LLN.

Lemma A.1 *Suppose that W_{it} satisfy Assumptions AHK and AR and $E\left[\|F(W_{it})\|^{4(r+\delta)}\right] < \infty$. Then,*

i $\|\Sigma_a\| < \infty$, $\|\Sigma_g\| < \infty$, *and* $\|\Sigma_e\| < \infty$ *where*

ii $\mathrm{Var}\left(\mathrm{E}_{NT}[F(W_{it})]\right) = \frac{1}{N}\Sigma_a + \frac{1}{T}\Sigma_g(1 + o(1)) + \frac{1}{NT}\Sigma_e(1 + o(1))$ *as* $N, T \to \infty$.

iii $\mathrm{E}_{NT}[F(W_{it})] \xrightarrow{p} E[F(W_{it})]$ *as* $N, T \to \infty$.

Lemma A.2 *With the same setting as in Lemma A.1, further assume that either* $\lambda_{min}\left[\Sigma_a\right] > 0$ *or* $\lambda_{min}\left[\Sigma_g\right] > 0$. *Then, as* $N, T \to \infty$ *and* $N/T \to c$, $\sqrt{N}\left(\mathrm{E}_{NT}[F(W_{it})] - E[F(W_{it})]\right) \xrightarrow{d} \mathcal{N}(0, \Sigma_a + c\Sigma_g)$

Lemmas A.1 and A.2 are the same as those for Theorems 1 and 2 in Chiang et al. (2024) except that W_{it} are replaced by $F(W_{it})$ and we don't consider the i.i.d case here. The proofs with W_{it} replaced by $F(W_{it})$ still go through so they are not repeated here.

The following lemma provides a probability limit of the infeasible penalty weights.

Lemma A.3 *Let* ω_j *be as defined in 2.8 with the bandwidth* M *such that* $M/T^{0.5} = o(1)$. *With the same setting as in Lemma A.2 for* $F(W_{it}) = f_{it,j}V_{it}$, $\omega_j \xrightarrow{p} \frac{N\wedge T}{N}\Sigma_a + \frac{N\wedge T}{T}\Sigma_g$ *as* $N, T \to \infty$ *and* $N/T \to c$.

Proof of Lemma A.3. Since $a_{i,j}$ is independent over i, we can apply the weak law of large number and obtain

$$\frac{N \wedge T}{N^2}\sum_{i=1}^{N} a_{i,j}^2 = \frac{N \wedge T}{N}\Sigma_a + o_P(1)$$

To show the convergence of the second term, we can apply Proposition 2 of Bester et al. (2008) by verifying its Assumption 7. Since the block size here $h = round(T^{1/5}) + 1$, it diverges with the time sample size and $h/T \to 0$ as $T \to \infty$. and Assumption 7(i) follows. Note that the β-mixing property of $g_{t,j}$ implies that it is also α-mixing with the mixing coefficient $\alpha_g(q) \le \beta_g(q) \le \beta_\gamma(q) = c_\kappa \exp(-\kappa q)$ for all $q \ge 1$. Let ζ be some positive constant, then we have

$$\sum_{q=1}^{\infty} q^2 \alpha_g(q)^{\zeta/(4+\zeta)} \le c_\kappa^{\zeta/(4+\zeta)} \sum_{q=1}^{\infty} q^2 \exp(-\kappa\zeta q/(4+\zeta)) = c_\kappa^{\zeta/(4+\zeta)} \sum_{q=1}^{\infty} q^2 \exp(-aq)$$

where $a := \frac{\kappa\zeta}{4+\zeta}$. We can use the ratio test to examine the convergence of sum:

$$\lim_{q\to\infty} \frac{(q+1)^2 \exp(-a(q+1))}{q^2 \exp(-aq)} = \lim_{q\to\infty} \left(\frac{q+1}{q}\right) \exp(-a) = \exp(-a)$$

Since $\kappa > 0$ and $\zeta > 0$, we have $a > 0$ and so $\exp(-a) < 1$. Thus we conclude the infinite sum does not diverge to infinity. The third condition is ensured by our assumptions directly. Thus, by Proposition 2 of Bester et al. (2008), we have

$$\frac{N \wedge T}{T^2}\sum_{b=1}^{B}\left(\sum_{t\in H_b} g_{t,j}\right)^2 = \frac{N \wedge T}{T}\Sigma_g + o_P(1).$$

The conclusion follows. □

The following notations and the lemma is used for deriving the performance bounds for post-LASSO.

Corresponding to $\widehat{\Gamma}$ defined above Theorem 2.1, here we define Γ_0 as the support of ζ_0. Define $\widehat{m} = \|\widehat{\Gamma}\backslash\Gamma_0\|_0$. Define $\mathcal{P}_\Gamma$ as the projection matrix such that it projects an $NT \times 1$ vector onto the linear span of $NT \times 1$ vector f_j with $j \in \Gamma$. The post-LASSO estimator $\widehat{\xi}_{PL}$ is defined as the OLS estimator of the linear projection of Y_{it} onto $\{f_{it,j} : j \in \widehat{\Gamma}\}$.

Lemma A.4 *Under Assumption ASM, if* $S_{\max} := \max_{1\le j\le p} |\mathbb{E}_{NT}[\omega^{-1/2} f_{it,j} V_{it}]| \le \frac{\lambda}{2c_1 NT}$, $0 < a = \min_j \omega^{1/2} \le \max_j \omega^{1/2} = b < \infty$, *and* $u \ge 1 \ge l \ge 1/c_1$, *then*

$$\|f(X_{it}) - f_{it}\widehat{\xi}_{PL}\|_{NT,2} = \left(\sqrt{\frac{s}{\phi_{min}(s)(M_f)}} + \sqrt{\frac{\widehat{m}}{\phi_{min}(\widehat{m})(M_f)}}\right) O_P\left(\frac{\lambda}{NT}\right) + O_P\left(\|f(X_{it}) - \left(\mathcal{P}_{\widehat{\Gamma}} f\right)_{it}\|_{NT,2}\right).$$

Proof of Lemma A.4. We can decompose $f(X_{it}) - f_{it}\widehat{\xi}_{PL}$ as follows:

$$\begin{aligned} f(X_{it}) - f_{it}\widehat{\xi}_{PL} =& f(X_{it}) - (\mathcal{P}_{\widehat{\Gamma}} Y)_{it} = \left((I_{NT} - \mathcal{P}_{\widehat{\Gamma}}) f(X) - \mathcal{P}_{\widehat{\Gamma}} V\right)_{it} = \left((I_{NT} - \mathcal{P}_{\widehat{\Gamma}}) f - (\mathcal{P}_{\widehat{\Gamma}\backslash\Gamma_0} + \mathcal{P}_{\Gamma_0}) V\right)_{it}, \\ \le& \left\|\left(I_{NT} - \mathcal{P}_{\widehat{\Gamma}}) f\right)_{it}\right\|_{NT,2} + \left\|\left(\mathcal{P}_{\Gamma_0} V\right)_{it}\right\|_{NT,2} + \left\|\left(\mathcal{P}_{\widehat{\Gamma}\backslash\Gamma_0} V\right)_{it}\right\|_{NT,2}. \end{aligned}$$

where the last equality follows from the property of the linear projection and the inequality follows from Minkowski's inequality. By Hölder's inequality and the property of spectral norm, we have

$$\begin{aligned} &\left\|\left(\mathcal{P}_{\widehat{\Gamma}\backslash\Gamma_0} V\right)_{it}\right\|_{NT,2} = \frac{1}{\sqrt{NT}} \left\|\mathcal{P}_{\widehat{\Gamma}\backslash\Gamma_0} V\right\|_2 \le \frac{1}{\sqrt{NT}} \left\|f_{\widehat{\Gamma}\backslash\Gamma_0} (f'_{\widehat{\Gamma}\backslash\Gamma_0} f_{\widehat{\Gamma}\backslash\Gamma_0})^{-1}\right\|_\infty \left\|f'_{\widehat{\Gamma}\backslash\Gamma_0} V\right\|_2 \\ \le& \frac{1}{\sqrt{NT}} \sqrt{\frac{1}{NT\phi_{min}(\widehat{m})(M_f)}} \left(\sum_{j\in\widehat{\Gamma}\backslash\Gamma_0} \left(\sum_{i=1}^N \sum_{t=1}^T f_{it,j} V_{it}\right)^2\right)^{1/2} \le \sqrt{\frac{\widehat{m}}{\phi_{min}(\widehat{m})(M_f)}} S_{\max} = \sqrt{\frac{\widehat{m}}{\phi_{min}(\widehat{m})(M_f)}} O_P\left(\frac{\lambda}{NT}\right) \end{aligned}$$

where the last line follows from $\min_j \omega_j^{1/2} = a > 0$ and $S_{\max} \le \frac{\lambda}{2c_1 NT}$. By similar arguments, we have

$$\begin{aligned} &\left\|\left(\mathcal{P}_{\Gamma_0} V\right)_{it}\right\|_{NT,2} = \frac{1}{\sqrt{NT}} \left\|\mathcal{P}_{\Gamma_0} V\right\|_2 \le \frac{1}{\sqrt{NT}} \left\|f_{\Gamma_0} (f'_{\Gamma_0} f_{\Gamma_0})^{-1}\right\|_\infty \left\|f'_{\Gamma_0} V\right\|_2 \\ \le& \frac{1}{\sqrt{NT}} \sqrt{\frac{1}{NT\phi_{min}(s)(M_f)}} \left(\sum_{j\in\Gamma_0} \left(\sum_{i=1}^N \sum_{t=1}^T f_{it,j} V_{it}\right)^2\right)^{1/2} \le \sqrt{\frac{s}{\phi_{min}(s)(M_f)}} O_P\left(\frac{\lambda}{NT}\right). \end{aligned}$$

□

Proof of Theorem 2.1. In the proof, we will show L1 and L2 convergence rates for $\widehat{\zeta}$. We will first show the regularization event in terms of the infeasible penalty weights ω as defined in 2.8. Due to the AHK representation as in Assumption AHK, we can decompose $f_{it,j} V_{it}$ as $f_{it,j} V_{it} = a_{i,j} + g_{t,j} + e_{it,j}$ where $a_{i,j} := \mathrm{E}[f_{it,j} V_{it} | \alpha_i]$, $g_{t,j} = \mathrm{E}[f_{it,j} V_{it} | \gamma_t]$, and $e_{it,j} = f_{it,j} V_{it} - a_{i,j} - g_{t,j}$, for $j = 1, .., p$.

To show the regularization event holds with probability approaching one, we bound the probability of the following event for each $j = 1, ..., p$:

$$\begin{aligned}
&\mathrm{P}\left(\frac{1}{NT}\left|\sum_{i=1}^{N}\sum_{t=1}^{T}\omega_j^{-1/2}f_{it,j}V_{it}\right| > \frac{\lambda}{2c_1NT}\right) = \mathrm{P}\left(\omega_j^{-1/2}\left|\frac{1}{N}\sum_{i=1}^{N}a_{i,j} + \frac{1}{T}\sum_{t=1}^{T}g_{t,j} + \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}e_{it,j}\right| > \frac{\lambda}{2c_1NT}\right) \\
\leq &\mathrm{P}\left(\left|\frac{1}{N}\sum_{i=1}^{N}\omega_{a,j}^{-1/2}a_{i,j}\right| + \left|\frac{1}{T}\sum_{t=1}^{T}\omega_{g,j}^{-1/2}g_{t,j}\right| + \left|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\omega_j^{-1/2}e_{it,j}\right| > \frac{\lambda}{2c_1NT}\right) \\
\leq &\mathrm{P}\left(\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\omega_{a,j}^{-1/2}a_{i,j}\right| > \frac{\sqrt{N}\lambda}{6c_1NT}\right) + \mathrm{P}\left(\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\omega_{g,j}^{-1/2}g_{t,j}\right| > \frac{\sqrt{T}\lambda}{6c_1NT}\right) + \mathrm{P}\left(\left|\sum_{i=1}^{N}\sum_{t=1}^{T}\omega_j^{-1/2}e_{it,j}\right| > \frac{\lambda}{6c_1}\right) \\
:= &p_{1,j}(\lambda) + p_{2,j}(\lambda) + p_{3,j}(\lambda)
\end{aligned}$$

where $\omega_{a,j} := \frac{N\wedge T}{N^2}\sum_{i=1}^{N}a_{i,j}^2$ and $\omega_{g,j} := \frac{N\wedge T}{T^2}\sum_{b=1}^{B}\left(\sum_{t\in H_b}g_{t,j}\right)^2$. The first inequality follows from the triangle inequality and the fact that $\omega_j^{1/2} = \left(\omega_{a,j} + \omega_{g,j}\right)^{1/2} \geq \max\{\omega_{a,j}^{1/2}, \omega_{g,j}^{1/2}\}$. The third inequality follows from a union-bound inequality. Applying union-bound inequality again, we obtain

$$\mathrm{P}\left(\max_{j=1,...,p}\left|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\omega_j^{-1/2}f_{it,j}V_{it}\right| > \frac{\lambda}{2c_1NT}\right) \leq \sum_{j=1}^{p}\left[p_{1,j}(\lambda) + p_{2,j}(\lambda) + p_{3,j}(\lambda)\right]$$

To bound $p_{1,j}(\lambda)$, we will apply a moderate deviation theorem for self-normalized sums of independent random variables. For $j = 1, ..., p$, define $\Xi_{a,j} = \frac{\left[\mathrm{E}(a_{i,j})^2\right]^{1/2}}{\left[\mathrm{E}(a_{i,j})^3\right]^{1/3}}$. Let $l_{a,N}$ be some positive increasing sequence. By Theorem 7.4 of Peña et al. (2009) with $\delta = 1$, we have for any $x \in [0, N^{1/6}\Xi_{a,j}/l_{a,N}]$ that

$$\mathrm{P}\left(\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\omega_{a,j}^{-1/2}a_{i,j}\right| > x\right) \leq 2\left(1 - \Phi(x)\right)\left[1 + O(1)\left(\frac{1}{l_{a,N}}\right)^3\right]$$

Then, setting $\lambda = 6c_1\frac{NT}{\sqrt{N}}\Phi^{-1}\left(1 - \frac{\gamma}{2p}\right)$ gives

$$\sum_{j=1}^{p}p_{1,j}(\lambda) \leq 2p(1 - \Phi(\Phi^{-1}(1 - \gamma/2p))) \leq \gamma[1 + O(1)(1/l_{a,N})^3]$$

given that $\Phi^{-1}\left(1 - \frac{\gamma}{2p}\right) \in [0, N^{1/6}\Xi_{a,j}/l_{a,N}]$. To show the right-hand-side converges to 0 as $\gamma \to 0$ and $(N, T) \to \infty$, there should exist an increasing sequence $l_{a,N}$ such that $\Phi^{-1}\left(1 - \frac{\gamma}{2p}\right) \in [0, N^{1/6}\min_j\{\Xi_{a,j}\}/l_{a,N}]$. Under Assumption REG(i), $\min_j\{\Xi_{a,j}\} = O(1)$. Note that $\Phi^{-1}\left(1 - \frac{\gamma}{2p}\right) \lesssim \sqrt{\log(p/\gamma)} = o\left(N^{1/12}/\log N\right)$ under Assumption REG(ii) and $N/T \to c$ as $N, T \to \infty$. Therefore, by taking some $l_{a,N} = O(\log N)$, it follows that $\sum_{j=1}^{p}p_{1,kl}(\lambda) \to 0$ as $\gamma \to 0$ and $(N, T) \to \infty$.

To bound $p_{2,j}(\lambda)$, we utilize a moderate deviation theorem for self-normalized sums of weakly dependent random variables. Observe that $g_{t,j} = \mathrm{E}[f_{it,j}V_{it}|\gamma_t]$ is beta-mixing with coefficient $\beta_g(q)$ satisfying

$$\beta_g(q) \leq \beta_\gamma(q) \leq c_\kappa exp(-\kappa q)\ \forall q \in \mathbb{Z}^+$$

Furthermore, by the strict stationarity, non-degeneracy condition, and the properties of the Hajek projection components listed in the beginning of Appendix A, we can verify that for some $\nu > 0$, $\mathrm{E}\left[\sum_{t=r}^{r+m} f_{it,j}V_{it}\right]^2 \geq \nu^2 m$ for all $r \geq 0, m \geq 1$. By Assumption REG(iii) and Holder's inequality, we have $\mathrm{E}|f_{it,j}V_{it}|^{4(\mu+\delta)} < \infty$ for some $\mu > 1,\ \delta > 0$. Then, by Theorem 3.2 of Gao et al. (2022) with $\tau = 1$ and $\alpha = \frac{1}{1+2\tau}$, we have

$$\sum_{j=1}^{p} \mathrm{P}\left(\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\omega_{g,j}^{-1/2}g_{t,j}\right| > x\right) \leq 2p\,(1-\Phi(x))\left[1 + O(1)\left(\frac{1}{l_{g,T}}\right)^2\right]$$

uniformly for $x \in \left(0, d_0(\log T)^{-1/2}T^{1/12}/l_{g,T}\right)$ where d_0 is some positive constant and $l_{g,T}$ is some positive increasing sequence. Then, setting $\lambda = 6c_1\frac{NT}{\sqrt{T}}\Phi^{-1}(1-\frac{\gamma}{2p})$ gives, for all $j = 1, ..., p$,

$$\sum_{j=1}^{p} p_{2,j}(\lambda) \leq \gamma\left[1 + O(1)\left(\frac{1}{l_{g,T}}\right)^2\right]$$

given that $\Phi^{-1}(1-\frac{\gamma}{2p}) \in \left(0, d_0(\log T)^{-1/2}T^{1/12}/l_{g,T}\right)$. To show the right-hand-side converge to 0 as $\gamma \to 0$ and $(N,T) \to 0$, there should exists an increasing sequence $l_{g,T}$ such that $\Phi^{-1}\left(1-\frac{\gamma}{2p}\right) \in (0, d_0(\log T)^{-1/2}T^{1/12}/l_{g,T})$. Under Assumption REG(ii), $\log(p/\gamma) = o\left(T^{1/6}/(\log T)^2\right)$ and so $\Phi^{-1}\left(1-\frac{\gamma}{2p}\right) \lesssim \sqrt{\log(p/\gamma)} = o\left(T^{1/12}/(\log T\right)$. Therefore, by taking $l_{g,T} = O\left((\log T)^{1/2}\right)$, it follows that $\sum_{j=1}^{p} p_{2,j}(\lambda) \to 0$ as $\gamma \to 0$ and $(N,T) \to \infty$.

Consider $p_{3,j}(\lambda)$. Define $\bar{e}_{i,j} := \frac{1}{T}\sum_{t=1}^{T} e_{it,j}$. Observe that $\mathrm{E}[\bar{e}_{i,j}] = 0$ by iterated expectation and conditional on $\{\gamma_t\}_{t=1}^{T}$, $\bar{e}_{i,j}$ are independent over i . We have shown previously that $\mathrm{E}|f_{it,j}V_{it}|^{4(\mu+\delta)} < \infty$ for some $\mu > 1,\ \delta > 0$. Given that $e_{it,j} = f_{it,j}V_{it} - a_{i,j} - g_{t,j}$ and $\mathrm{E}|a_{i,j}|^{4(\mu+\delta)} < \infty$, $\mathrm{E}|g_{t,j}|^{4(\mu+\delta)} < \infty$ due to Jansen's inequality and iterated expectation, we have $\mathrm{E}|e_{it,j}|^{4(\mu+\delta)} < \infty$ and so $\mathrm{E}|\bar{e}_{i,j}|^{4(\mu+\delta)} < \infty$ due to Minkowski's inequality. Note that

$$\mathrm{Var}\left(\bar{e}_{i,j}\right) = \frac{1}{T}\sum_{l=-(T-1)}^{T-1}\left(1 - \frac{|l|}{T}\right)\mathrm{E}(e_{it,j}e_{i,t+l,j}) = \frac{1}{T}\Sigma_e(1+o(1)).$$

where Σ_e is defined in the beginning Appendix A with $k = 1$ in this case. By Lemma A.1, $|\Sigma_{e,j}| < \infty$. Furthermore, as is shown below, $\omega_j^{1/2}$ is bounded from below by some constant $a > 0$. Now, by the conditional

version of Corollary 4 from Fuk and Nagaev (1971), there exists some constant a_1 and a_2 such that

$$\begin{aligned}
&\mathrm{P}\left(\left|\sum_{i=1}^{N}\omega_j^{-1/2}\bar{e}_{i,j}\right| > \frac{\lambda}{6c_1T}|\{\gamma_t\}_{t=1}^T\right) \leq \mathrm{P}\left(\left|\sum_{i=1}^{N}\bar{e}_{i,j}\right| > \frac{a\lambda}{6c_1T}|\{\gamma_t\}_{t=1}^T\right) \\
&\leq a_1(\lambda/T)^{-4}\sum_{i=1}^{N}E\left(|\bar{e}_{i,j}|^4|\{\gamma_t\}_{t=1}^T\right) + \exp\left(-\frac{a_2(\lambda/T)^2}{\sum_{i=1}^{N}\mathrm{Var}\left(\bar{e}_{i,j}|\{\gamma_t\}_{t=1}^T\right)}\right)
\end{aligned}$$

Note that $\exp(-1/z)$ is not globally concave but it is concave for $z > 1/2$ and is bounded by z/e^2 for $z \in (0, 1/2)$ where e is the Euler's number. Denote $z = \frac{(T/\lambda)^2}{a_2}\sum_{i=1}^{N}\mathrm{Var}(\bar{e}_{i,j}|\{\gamma_t\}_{t=1}^T)$. Then, we have

$$\exp\left(-\frac{a_2(\lambda/T)^2}{\sum_{i=1}^{N}\mathrm{Var}(\bar{e}_{i,j}|\{\gamma_t\}_{t=1}^T)}\right) = \exp(-1/z) \leq z/e^2 1\{z \in (0,1/2)\} + \exp(-1/z)1\{z > 1/2\}.$$

By Fubini theorem, Jensen's inequality, and the bounded moments, we have

$$\begin{aligned}
p_{3,j}(\lambda) =& \mathrm{P}\left(\left|\sum_{i=1}^{N}\omega_j^{-1/2}\bar{e}_{i,j}\right| > \frac{\lambda}{6c_1T}\right) \\
\leq& a_1(\lambda/T)^{-4}\sum_{i=1}^{N}E\left(|\bar{e}_{i,j}|^4\right) + \frac{(T/\lambda)^2}{a_2}\sum_{i=1}^{N}\mathrm{Var}(\bar{e}_{i,j})/e^2 + \exp\left(-\frac{a_2(\lambda/T)^2}{\sum_{i=1}^{N}\mathrm{Var}(\bar{e}_{i,j})}\right) \\
=& O\left((\lambda/T)^{-4}N\right) + O\left(\frac{TN}{\lambda^2}\right) + \exp\left(-\frac{a_2(\lambda/T)^2}{N/T\Sigma_e}\right).
\end{aligned}$$

Therefore, we have $\sum_{j=1}^{p}p_{3,j}(\lambda) = O\left(p(\lambda/T)^{-4}N\right) + O\left(\frac{pTN}{\lambda^2}\right) + p\exp\left(-\frac{a_2(\lambda/T)^2}{N/T\Sigma_e}\right)$. Given that $N/T \to c$ where $0 < c < \infty$, by taking $\lambda_e = \frac{p^{1/4}TN^{1/4}}{\varepsilon^{1/4}} \vee \frac{\sqrt{pNT}}{\varepsilon^{1/2}} \vee \sqrt{\frac{NT\log(p/\gamma)}{a_2/\Sigma_e}}$ for some $\varepsilon = o(1)$, $\sum_{j=1}^{p}p_{3,j}(\lambda_e) = o(1)$. Under REG(ii), $\log(p/\gamma) = o\left(T^{1/6}/(\log T)^2\right)$ and $p = o(T^{7/6}/(\log T)^2)$, then $\lambda_e = O(\lambda)$ where $\lambda = 6c_1\frac{NT}{\sqrt{N\wedge T}}\Phi^{-1}(1-\frac{\gamma}{2p})$. Therefore, we have shown $\sum_{j=1}^{p}p_{3,j}(\lambda) \to 0$ for $\lambda = 6c_1\frac{NT}{\sqrt{N\wedge T}}\Phi^{-1}(1-\frac{\gamma}{2p})$.

Put together, we have shown

$$\mathrm{P}\left(\max_{j=1,\ldots,p}\left|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\omega_j^{-1/2}f_{it,j}V_{it}\right| \leq \frac{\lambda}{2c_1NT}\right) \to 1. \tag{7.1}$$

Now, we can apply Lemma 6 of Belloni et al. (2012) to obtain the finite sample bounds on $\left\|f_{it}\left(\widehat{\zeta}-\zeta_0\right)\right\|_{NT,2}$ and $\left|\omega^{1/2}\left(\widehat{\zeta}-\zeta_0\right)\right|_1$. Let δ be some generic vector of nuisance parameters and let J_p^1 be a subset of an index set $J_p = 1, ..., p$ and $J_p^0 = J_p \backslash J_p^1$. Let δ^1 be a copy of δ with its j-th element replaced by 0 for all $j \in J_p^0$ and similarly let δ^0 be a copy of δ with its j-th element replaced by 0 for all $j \in J_p^1$. Define the restricted

eigenvalues and Gram matrix as follows:

$$K_C(M_f) = \min_{\delta:\|\delta^0\|_1 \le C\|\delta^1\|_1,\ \|\delta\|\neq 0,\ |J_p^1|\le s} \frac{\sqrt{s\delta' M_f \delta}}{\left\|\delta^1\right\|_1}, \quad M_f = \mathrm{E}_{NT}[f_{it}' f_{it}].$$

Define the weighted restricted eigenvalues as follows:

$$K_C^{\omega}(M_f) = \min_{\delta:\|\omega^{1/2}\delta^0\|_1 \le C\|\omega^{1/2}\delta^1\|_1,\ \|\delta\|\neq 0,\ |J_p^1|\le s} \frac{\sqrt{s\delta' M_f \delta}}{\left\|\omega^{1/2}\delta^1\right\|_1}.$$

Let $a := \min_{j=1,...,p} \omega_j^{1/2}$, $b := \max_{j=1,...,p} \omega_j^{1/2}$. As is shown in Belloni et al. (2016),

$$K_C^{\omega}(M_f) \ge \frac{1}{b} K_{bC/a}(M_f). \tag{7.2}$$

By Lemma A.2 above, we have $\omega_j \xrightarrow{p} \frac{N\wedge T}{N}\Sigma_{a,j} + \frac{N\wedge T}{T}\Sigma_{g,j}$. By Lemma A.1, $|\Sigma_{a,j}| < \infty$ and $|\Sigma_{g,j}| < \infty$. The non-degeneracy condition implies that either $\Sigma_{a,j}^2 > c_\sigma > 0$ or $\Sigma_{g,j}^2 > c_\sigma > 0$. Therefore, we have ω_j bounded below by zero and bounded above for each $j = 1, .., p$ with probability approaching one as $N, T \to \infty$. Assumption (ASM), the condition 2.9, and 7.1, Lemma 6 of Belloni et al. (2012) implies that

$$\begin{aligned}
\left\| f_{it}\left(\widehat{\zeta} - \zeta_0\right)\right\|_{NT,2} \le& \left(u + \frac{1}{c_1}\right)\frac{\sqrt{s}\lambda}{NTK_{c_0}^{\omega}(M_f)} + 2\left\|r\right\|_{NT,2} = O_P\left(\frac{1}{K_{c_0}^{\omega}(M_f)}\sqrt{\frac{s\log(p/\gamma)}{N\wedge T}} + \sqrt{\frac{s}{N\wedge T}}\right),\\
\left\|\omega^{1/2}\left(\widehat{\zeta} - \zeta_0\right)\right\|_1 \le& \frac{3c_0\sqrt{s}}{K_{2c_0}^{\omega}(M_f)}\left[\left(u + \frac{1}{c_1}\right)\frac{\sqrt{s}\lambda}{NTK_{c_0}^{\omega}(M_f)} + 2\left\|r\right\|_{NT,2}\right] + 3c_0\frac{NT}{\lambda}\left\|r\right\|_{NT,2}^2,\\
=& O_P\left(\frac{s}{K_{2c_0}^{\omega}(M_f)K_{c_0}^{\omega}(M_f)}\sqrt{\frac{\log(p/\gamma)}{N\wedge T}} + \sqrt{\frac{s}{N\wedge T}} + \frac{s/\sqrt{N\wedge T}}{\log(p/\gamma)}\right)
\end{aligned}$$

where $c_0 := \frac{uc+1}{lc-1} > 1$. By 7.2, we have $1/K_{c_0}^{\omega}(M_f) \le b/K_{\bar{C}}(M_f)$ where $\bar{C} := bc_0/a$. By arguments given in Bickel et al. (2009), Assumption SE implies that $1/K_C(M_f) = O_P(1)$ for any $C > 0$. Therefore,

$$\left\| f_{it}\left(\widehat{\zeta} - \zeta_0\right)\right\|_{NT,2} = O_P\left(\sqrt{\frac{s\log(p/\gamma)}{N\wedge T}}\right), \ \left|\omega^{1/2}\left(\widehat{\zeta} - \zeta_0\right)\right|_1 = O_P\left(s\sqrt{\frac{\log(p/\gamma)}{N\wedge T}}\right).$$

By Holder's inequality and that $\min_j \omega_j^{1/2} \ge a > 0$

$$\|\widehat{\zeta} - \zeta_0\|_1 \le \|\omega^{-1/2}\|_\infty \left|\omega^{1/2}\left(\widehat{\zeta} - \zeta_0\right)\right|_1 = O_P\left(s\sqrt{\frac{\log(p/\gamma)}{N\wedge T}}\right) = O_P\left(s\sqrt{\frac{\log(p\vee NT)}{N\wedge T}}\right)$$

where the first inequality follows from the .

The $l2$ rate of convergence will be derived after the sparsity bounds. We now switch the focus to the Post-LASSO. By the finite sample bounds of Lemma A.4, we have

$$\begin{aligned}\|f(X_{it}) - f_{it}\widehat{\zeta}_{PL}\|_{NT,2} = &\left(\sqrt{\frac{s}{\phi_{min}(s)(M_f)}} + \sqrt{\frac{\widehat{m}}{\phi_{min}(\widehat{m})(M_f)}}\right) O_P\left(\frac{\lambda}{NT}\right) \\ &+ O_P\left(\|f(X_{it}) - \left(\mathcal{P}_{\widehat{\Gamma}}f\right)_{it}\|_{NT,2}\right),\end{aligned} \tag{7.3}$$

By the finite sample bounds of Lemma 7 from Belloni et al. (2012), we have

$$\|f_{it}(\widehat{\zeta}_{PL} - \zeta_0)\|_{NT,2} \leq \|f_{it}(X_{it}) - f_{it}\widehat{\zeta}_{PL}\|_{NT,2} + \|r_{it}\|_{NT,2}, \tag{7.4}$$

$$\|\omega^{1/2}(\widehat{\zeta}_{PL} - \zeta_0)\|_1 \leq \frac{b\sqrt{\widehat{m}+s}}{\sqrt{\phi_{min}(\widehat{m}+s)(M_f)}} \times \|f_{it}(\widehat{\zeta}_{PL} - \zeta_0)\|_{NT,2} \tag{7.5}$$

$$\|f(X_{it}) - \mathcal{P}_{\widehat{\Gamma}}f(X_{it})\|_{NT,2} \leq \left(u + \frac{1}{c_1}\right)\frac{\lambda\sqrt{s}}{NTK^{\omega}_{c_0}(M_f)} + 3\|r_{it}\|_{NT,2}. \tag{7.6}$$

The finite sample bound of Lemma 8 from Belloni et al. (2012) gives

$$\widehat{m} \leq \phi_{max}(\widehat{m})(M_f)a^{-2}\left(\frac{2c_0\sqrt{s}}{K^{\omega}_{c_0}(M_f)} + \frac{6c_0NT\|r_{it}\|_{NT,2}}{\lambda}\right)^2.$$

where $a > 0$ has been shown previously. Let $\mathcal{M} = \left\{m \in \mathbb{N} : m > 2\phi_{max}(m)(M_f)a^{-2}\left(\frac{2c_0\sqrt{s}}{K^{\omega}_{c_0}(M_f)} + \frac{6c_0NT\|r_{it}\|_{NT,2}}{\lambda}\right)^2\right\}$. Lemma 10 of Belloni et al. (2012) gives

$$\widehat{m} \leq \min_{m\in\mathcal{M}} \phi_{max}(m \wedge NT)(M_f)a^{-2}\left(\frac{2c_0\sqrt{s}}{K^{\omega}_{c_0}(M_f)} + \frac{6c_0NT\|r_{it}\|_{NT,2}}{\lambda}\right)^2. \tag{7.7}$$

Note that $\frac{6c_0NT\|r_{it}\|_{NT,2}}{\lambda\sqrt{s}} = O_P(1/\log(p \wedge NT)) \xrightarrow{p} 0$. Recall that $1/K^{\omega}_{c_0}(M_f) \leq b/K_{\bar{C}}(M_f) < \infty$. Let $\mu := \min_m\left\{\sqrt{\phi_{max}(m)(M_f)/\phi_{min}(m)(M_f)} : m > 18\bar{C}^2s\phi_{max}(m)(M_f)/K^2_{\bar{C}}(M_f)\right\}$, and let $\bar{m}$ be the integer associated with μ. By the definition of $\mathcal{M}$, it implies that $\bar{m} \in \mathcal{M}$ with probability approaching one, which implies $\bar{m} > \widehat{m}$ due to 7.7. By Lemma 9 (the sub-linearity of sparse eigenvalues) from Belloni et al. (2012) and 7.7, we have

$$\widehat{m} \lesssim_P s\mu^2\phi_{min}(\bar{m}+s)/K^2_{\bar{C}} \lesssim s\mu^2\phi_{min}(\hat{m}+s)/K^2_{\bar{C}}.$$

Combining the results above with 7.3 and 7.6 to gives

$$\|f(X_{it}) - f_{it}\widehat{\zeta}_{PL})\|_{NT,2} = O_P\left(\sqrt{\frac{s\mu^2\log(p/\gamma)}{(N\wedge T)K_{\bar{C}}^2}} + \|r_{it}\|_{NT,2} + \frac{\lambda\sqrt{s}}{NTK_{c_0}^{\omega}(M_f)}\right).$$

Recall that $b < \infty$ and Condition SE imply $1/K_{c_0}^{\omega}(M_f) \leq 1/K_{\bar{C}}(M_f) < \infty$. Then, Condition SE, Condition ASM and the choice of λ together imply

$$\|f(X_{it}) - f_{it}\widehat{\zeta}_{PL})\|_{NT,2} = O_P\left(\sqrt{\frac{s\log(p/\gamma)}{N\wedge T}}\right).$$

For the $l1$ convergence rate, note that $\|\widehat{\zeta}_{PL} - \zeta_0\|_0 \leq \widehat{m} + s$. Then, applying Cauchy-Schwarz inequality to $\|\widehat{\zeta}_{PL} - \zeta_0\|_1 = \sum_{j=1}^{p}|\widehat{\zeta}_{PL} - \zeta_0| = \sum_{j\in\{\widehat{\Gamma}\bigcup\Gamma_0\}}|\widehat{\zeta}_{PL} - \zeta_0|$ gives

$$\|\widehat{\zeta}_{PL} - \zeta_0\|_1 \leq \sqrt{\widehat{m}+s}\|\widehat{\zeta}_{PL} - \zeta_0\|_2$$

To derive the convergence rates in $l2$-norm of the Post-LASSO estimator (the $l2$ rate for the LASSO estimator is obtained similarly), we will utilize the sparse eigenvalue condition and the prediction norm. If $\widehat{\zeta}_{PL} - \zeta_0 = 0$, then the conclusion holds trivially. Otherwise, define $b = \left(\widehat{\zeta}_{PL} - \zeta_0\right)/\|\widehat{\zeta}_{PL} - \zeta_0\|_2^{-1}$. Then, we have $\|b\|_2 = 1$ and so $b \in \Delta(\widehat{m}+s) = \{\delta : \|\delta\|_0 = \widehat{m}+s, \|\delta\|_2 = 1\}$. By Assumption SE, we have

$$0 < \kappa_1 \leq \phi_{min}(\widehat{m}+s)(M_f) \leq \frac{(b'M_f b)^{1/2}}{\|b\|_2} = \frac{\left\|f_{it}\left(\widehat{\zeta}_{PL} - \zeta_0\right)\right\|_{NT,2}}{\|\widehat{\zeta}_{PL} - \zeta_0\|_2},$$

Therefore, using the bound on the prediction norm above, we conclude that

$$\|\widehat{\zeta}_{PL} - \zeta_0\|_2 \leq \frac{\left\|f_{it}\left(\widehat{\zeta}_{PL} - \zeta_0\right)\right\|_{NT,2}}{\kappa_1} = O_P\left(\sqrt{\frac{s\log(p/\gamma)}{N\wedge T}}\right).$$

It implies that $\|\widehat{\zeta}_{PL} - \zeta_0\|_1 = \sqrt{\widehat{m}+s}O_P\left(\sqrt{\frac{s\log(p/\gamma)}{N\wedge T}}\right) = O_P\left(\sqrt{\frac{s^2\log(p/\gamma)}{N\wedge T}}\right)$.

□

Appendix B

The following lemma, quoted from Semenova et al. (2023a)(Lemma A.3), is a result follows from the weak form of Strassen's coupling Strassen (1965) and the strong form of Strassen's coupling via Lemma 2.11 of Dudley and Philipp (1983):

Lemma B.1 *Let (X, Y) be random element taking values in Polish space $S = (S_1 \times S_2)$ with laws P_X and P_Y, respectively. Then, we can construct $(\tilde{X}, \tilde{Y})$ taking values in (S_1, S_2) such that (i) they are independent of each other; (ii) their laws $\mathcal{L}(\tilde{X}) = P_X$ and $\mathcal{L}(\tilde{Y}) = P_Y$; (iii)*

$$\mathrm{P}\left\{(X, Y) \neq (\tilde{X}, \tilde{Y})\right\} = \frac{1}{2}\|P_{X,Y} - P_X \times P_Y\|_{TV}$$

The proof is provided in Semenova et al. (2023b). To apply the independence coupling result for cross-fitting in the panel data, we need to introduce another lemma:

Lemma B.2 *Let $X_1, ..., X_q$* and Y be random elements taking values in Polish space $S = (S_1 \times ... \times S_m \times S_y)$.

$$\beta((X_1, ..., X_m), Y) \leq \sum_{i=1}^{q} \beta(X_i, Y).$$

Proof of Lemma B.2. By Lemma B.1, we have

$$\begin{aligned}\beta((X_1, ..., X_m), Y) &= \frac{1}{2}\left\|P_{(X_1,...,X_q),Y} - P_{(X_1,...,X_m)} \times P_Y\right\|_{TV} \\ &= \mathrm{P}((X_1, ..., X_m, Y) \neq (\tilde{X}_1, ..., \tilde{X}_m, \tilde{Y})) \leq \sum_{i=1}^{m} \mathrm{P}((X_i, Y) \neq (\tilde{X}_i, \tilde{Y})) = \sum_{i=1}^{m} \beta(X_i, Y),\end{aligned}$$

where the inequality follows from the union bound. □

Now we can prove Lemma 3.1 from the main body of the paper:

Proof of Lemma 3.1. By Lemma B.1, for each (k, l) we have

$$\begin{aligned}&\mathrm{P}\{(W(k,l), W(-k,-l)) \neq (\tilde{W}(k,l), \tilde{W}(-k,-l))\} \\ =&\beta\left(W(k,l), W(-k,-l)\right) = \beta\left(\{W_{it}\}_{i\in I_k, t\in S_l}, \bigcup_{k'\neq k, l'\neq l, l\pm 1} \{W_{it}\}_{i\in I_{k'}, t\in S_{l'}}\right) \\ \leq &\sum_{i\in I_k} \beta\left(\{W_{it}\}_{t\in S_l}, \bigcup_{k'\neq k, l'\neq l, l\pm 1} \{W_{it}\}_{i\in I_{k'}, t\in S_{l'}}\right) \leq \sum_{k'\neq k, l'\neq l, l\pm 1} \sum_{j\in I_{k'}} \sum_{i\in I_k} \beta\left(\{W_{it}\}_{t\in S_l}, \{W_{jt}\}_{t\in S_{l'}}\right)\end{aligned}$$

where the last two inequalities follow from Lemma B.2. Note that for $s, m \geq 1$, we have

$$\begin{aligned}&\beta(\{W_{it}\}_{t\leq s}, \{W_{jt}\}_{t\geq s+m}) = \left\|P_{\{W_{it}\}_{t\leq s}, \{W_{jt}\}_{t\geq s+m}} - P_{\{W_{it}\}_{t\leq s}} \times P_{\{W_{jt}\}_{t\geq s+m}}\right\|_{TV} \\ \leq &\sup_{A\in\sigma(\{W_{jt}\}_{t\geq s+m})} \mathrm{E}_P\left|\mathrm{P}(A|\sigma(\{W_{it}\}_{t\leq s})) - \mathrm{P}(A)\right| = \sup_{A\in\sigma(\{W_{jt}\}_{t\geq s+m})} \mathrm{E}_P\left|\mathrm{P}(\mathrm{P}(A|\sigma(\alpha_i, \{\gamma_t\}_{t\leq s}, \{\varepsilon_{it}\}_{t\leq s}))|\sigma(\{W_{it}\}_{t\leq s})) - \mathrm{P}(A)\right| \\ = &\sup_{A\in\sigma(\{W_{jt}\}_{t\geq s+m})} \mathrm{E}_P\left|\mathrm{P}(A|\sigma(\{\gamma_t\}_{t\leq s}) - \mathrm{P}(A)\right| = \sup_{A\in\sigma(\{\gamma_t\}_{t\geq s+m})} \mathrm{E}_P\left|\mathrm{P}(A|\sigma(\{\gamma_t\}_{t\leq s}) - \mathrm{P}(A)\right| \leq c_\kappa exp(-\kappa m),\end{aligned}$$

where the last inequality follows from Assumption 2. Therefore,

$$\mathrm{P}\{(W(k,l),W(-k,-l)) \neq (\tilde{W}(k,l),\tilde{W}(-k,-l))\} \leq KLN^2 c_\kappa exp(-\kappa T_l),$$

which in turn gives

$$\mathrm{P}\{(W(k,l),W(-k,-l)) \neq (\tilde{W}(k,l),\tilde{W}(-k,-l)),\ for\ some\ (k,l)\} \leq K^2L^2N^2 c_\kappa exp(-\kappa T_l),$$

where $T_l = T/L$. Given that $\log(N)/T = o(1)$ and (K, L) are finite, it follows that

$$\mathrm{P}\{(W(k,l),W(-k,-l)) \neq (\tilde{W}(k,l),\tilde{W}(-k,-l)),\ for\ some\ (k,l)\} = o(1)$$

□

Proof of Theorem 3.1. By Assumption DML2(i), with probability $1 - \Delta_{NT}$, $\widehat{\eta}_{kl} \in \mathcal{T}_{NT}$. So, $\mathrm{P}(\widehat{\eta}_{kl} \in \mathcal{T}_{NT},\ \forall(k,l)) \geq 1 - KL\Delta_{NT} = 1 - o(1)$. Let's denote the event $\mathrm{P}(\widehat{\eta}_{kl} \in \mathcal{T}_\eta,\ \forall(k,l))$ as $\mathcal{E}_\eta$ and the event $\{(W(k,l),W(-k,-l)) = (\tilde{W}(k,l),\tilde{W}(-k,-l)),\ for\ some\ (k,l)\}$ as $\mathcal{E}_{cp}$. By Lemma 3.1, we have $\mathrm{P}(\mathcal{E}_{cp}) = 1 - o(1)$. By union bound inequality, we have $\mathrm{P}(\mathcal{E}_\eta^c \cup \mathcal{E}_{cp}^c) \leq \mathrm{P}(\mathcal{E}_\eta^c) + \mathrm{P}(\mathcal{E}_{cp}^c) = o(1)$. So, $\mathrm{P}(\mathcal{E}_\eta \cap \mathcal{E}_{cp}) = 1 - \mathrm{P}(\mathcal{E}_\eta^c \cup \mathcal{E}_{cp}^c) \geq 1 - o(1)$.

Let $\widehat{\theta}$ be a solution from equation 3.1. To simplify the notation, we denote

$$\widehat{A}_{kl} = \mathbb{E}_{kl}[\psi^a(W_{it},\widehat{\eta}_{kl})],\ \widehat{\bar{A}} = \frac{1}{KL}\sum_{k=1}^{K}\sum_{l=1}^{L}\widehat{A}_{kl},\ A_0 = \mathrm{E}_P[\psi^a(W_{it};\eta_0)],$$
$$\widehat{B}_{kl} = \mathbb{E}_{kl}[\psi^b(W_{it},\widehat{\eta}_{kl})],\ \widehat{\bar{B}} = \frac{1}{KL}\sum_{k=1}^{K}\sum_{l=1}^{L}\widehat{B}_{kl},\ B_0 = \mathrm{E}_P[\psi^b(W_{it};\eta_0)],$$
$$\widehat{\bar{\psi}}(\theta) = \widehat{\bar{A}}\theta + \widehat{\bar{B}},\ \bar{\psi}(\theta,\eta) = \mathbb{E}_{NT}\psi(W_{it};\theta,\eta).$$

Claim B.1. *On event* $\{\mathcal{E}_\eta \cap \mathcal{E}_{cp}\}$, $\|\widehat{\bar{A}} - A_0\| = O_P(N^{-1/2} + r_{NT})$.

By Claim 1 and Assumption 3(iii) that all singular values of A_0 are bounded below by zero, it follows that all singular values of $\widehat{\bar{A}}$ are also bounded below from zero, on event $\mathcal{E}_\eta$. Then, by the linearity in Assumption 3(i), we can write $\widehat{\theta} = -\widehat{\bar{A}}^{-1}\widehat{\bar{B}},\ \theta_0 = -A_0^{-1}B_0$. Then, by basic algebra, we have

$$\begin{aligned}\sqrt{N}(\widehat{\theta} - \theta_0) =& \sqrt{N}(-\widehat{\bar{A}}^{-1}\widehat{\bar{B}} - \theta_0) = -\sqrt{N}\widehat{\bar{A}}^{-1}(\widehat{\bar{B}} + \widehat{\bar{A}}\theta_0) = -\sqrt{N}\widehat{\bar{A}}^{-1}\widehat{\bar{\psi}}(\theta_0)\\ =& \sqrt{N}A_0^{-1}\bar{\psi}(\theta_0,\eta_0) + \sqrt{N}A_0^{-1}\left(\widehat{\bar{\psi}}(\theta_0) - \bar{\psi}(\theta_0,\eta_0)\right)\\ &+ \sqrt{N}\left[\left(A_0 + \widehat{\bar{A}} - A_0\right)^{-1} - A_0^{-1}\right]\left(\bar{\psi}(\theta_0,\eta_0) + \widehat{\bar{\psi}}(\theta_0) - \bar{\psi}(\theta_0,\eta_0)\right)\end{aligned}$$

Claim B.2. *On event* $\{\mathcal{E}_\eta \cap \mathcal{E}_{cp}\}$, $\|\widehat{\bar{\psi}}(\theta_0) - \bar{\psi}(\theta_0, \eta_0)\| = O_P(r'_{NT}/\sqrt{N} + \lambda_{NT} + \lambda'_{NT})$.

By Assumption DML2(i) and Jensen's inequality, we have $\|A_0\| \leq m'_{NT} \leq c_m$. Then, Claim B.2 implies that

$$\begin{aligned}\|\sqrt{N}A_0^{-1}\left(\widehat{\bar{\psi}}(\theta_0) - \bar{\psi}(\theta_0, \eta_0)\right)\| =& O_P(1)O_P(\sqrt{N}r'_{NT} + \sqrt{N}\lambda_{NT} + \sqrt{N}\lambda'_{NT}) \\ =& O_P(r'_{NT} + \sqrt{N}\lambda_{NT} + \sqrt{N}\lambda'_{NT}),\end{aligned}$$

Since $\mathrm{E}[\bar{\psi}(\theta_0, \eta_0)] = 0$, by Lemma A.2, we have $\sqrt{N}\bar{\psi}(\theta_0, \eta_0) \xrightarrow{d} \mathcal{N}(0, \Omega)$ where $\Omega = \Sigma_a + c\Sigma_g$ and $\|\Omega\| < \infty$. By Claims B.1, B.2, and the asymptotic normality of $\sqrt{N}\bar{\psi}(\theta_0, \eta_0)$, we have

$$\begin{aligned}&\left\|\sqrt{N}\left[\left(A_0 + \widehat{\dot{A}} - A_0\right)^{-1} - A_0^{-1}\right]\left(\bar{\psi}(\theta_0, \eta_0) + \widehat{\bar{\psi}}(\theta_0) - \bar{\psi}(\theta_0, \eta_0)\right)\right\| \\ \leq& \left\|\widehat{\dot{A}}^{-1}\right\|\left\|\widehat{\dot{A}} - A_0\right\|\left\|A_0^{-1}\right\|\left\|\sqrt{N}\left(\bar{\psi}(\theta_0, \eta_0) + \widehat{\bar{\psi}}(\theta_0) - \bar{\psi}(\theta_0, \eta_0)\right)\right\| \\ =& O_P(1)O_P\left(N^{-1/2} + r_{NT}\right)O_P(1)\left(O_P(1) + O_P\left(r'_{NT} + \sqrt{N}\lambda_{NT} + \sqrt{N}\lambda'_{NT}\right)\right) \quad = O_P\left(N^{-1/2} + r_{NT}\right),\end{aligned}$$

and $\sqrt{N}\left(\widehat{\theta} - \theta_0\right) = A_0^{-1}\mathcal{N}(0, \Omega) + O_P\left(N^{-1/2} + r_{NT} + r'_{NT} + \sqrt{N}\lambda_{NT} + \sqrt{N}\lambda'_{NT}\right) \xrightarrow{d} A_0^{-1}\mathcal{N}(0, \Omega)$.

Proof of Claim B.1. Fix any (k, l), we have

$$\left\|\widehat{A}_{kl} - A_0\right\| \leq \left\|\widehat{A}_{kl} - \mathrm{E}_P[\widehat{A}_{kl}|W(-k, -l)]\right\| + \left\|\mathrm{E}_P[\widehat{A}_{kl}|W(-k, -l)] - A_0\right\| =: \|\Delta_{A,1}\| + \|\Delta_{A,2}\|.$$

On the event $\{\mathcal{E}_\eta \cap \mathcal{E}_{cp}\}$, we have $\widehat{\eta}_{kl} \in \mathcal{T}_{NT}$ and the independence between $W(-k, -l)$ and $W(k, l)$. So, due to Assumption DML2, we have $\left\|\Delta_{A,2}\right\| \leq r_{NT}$. By iterated expectation,- $\mathrm{E}_P[\Delta_{A,1}] = 0$. To simplify the notation, we denote $\ddot{\psi}_{it}^{a,kl} := \psi^a(W_{it}, \widehat{\eta}_{kl}) - \mathrm{E}_P[\psi^a(W_{it}, \widehat{\eta}_{kl})|W(-k, -l)]$. Consider $\left\|\Delta_{A,1}\right\|$:

$$\begin{aligned}&\mathrm{E}\left(\left\|\Delta_{A,1}\right\|^2|W(-k, -l)\right) = \left(\frac{1}{N_k T_l}\right)^2 \mathrm{E}_P\left[\left\|\sum_{i\in I_k, t\in S_l}\ddot{\psi}_{it}^{a,kl}\right\|^2|W(-k, -l)\right] \\ \leq& \left(\frac{1}{N_k T_l}\right)^2 \sum_{i\in I_k, t\in S_l, r\in S_l}\left|\mathrm{E}_P\left[\langle\ddot{\psi}_{it}^{a,kl}, \ddot{\psi}_{is}^{a,kl}\rangle|W(-k, -l)\right]\right| + \sum_{t\in S_l, i\in I_k, j\in I_k}\left|\mathrm{E}_P\left[\langle\ddot{\psi}_{it}^{a,kl}, \ddot{\psi}_{jt}^{a,kl}\rangle|W(-k, -l)\right]\right| \\ &+ \sum_{t\in S_l, i\in I_k}\left|\mathrm{E}_P\left[\langle\ddot{\psi}_{it}^{a,kl}, \ddot{\psi}_{it}^{a,kl}\rangle|W(-k, -l)\right]\right| + 2\sum_{m=1}^{T_l-1}\sum_{t=\min(S_l)}^{\max(S_l)-m}\sum_{i,j\in I_k}\left|\mathrm{E}_P\left[\langle\ddot{\psi}_{it}^{a,kl}, \ddot{\psi}_{j,t+m}^{a}\rangle|W(-k, -l)\right]\right| \\ &+ 2\sum_{m=1}^{T_l-1}\sum_{t=\min(S_l)}^{\max(S_l)-m}\sum_{i\in I_k}\left|\mathrm{E}_P\left[\langle\ddot{\psi}_{it}^{a,kl}, \ddot{\psi}_{i,t+m}^{a}\rangle|W(-k, -l)\right]\right| =: \left(\frac{1}{N_k T_l}\right)^2 (a(1) + a(2) + a(3) + 2a(4) + 2a(5)).\end{aligned}$$

By conditional Cauchy-Schwarz inequality, for any i, t, j, s, we have

$$\left|\mathrm{E}_P\left[\langle\ddot{\psi}_{it}^{a,kl}, \ddot{\psi}_{js}^{a,kl}\rangle | W(-k,-l)\right]\right| \leq \left(\mathrm{E}_P\left[\|\ddot{\psi}_{it}^{a,kl}\|^2 | W(-k,-l)\right]\mathrm{E}_P\left[\|\ddot{\psi}_{js}^{a,kl}\|^2 | W(-k,-l)\right]\right)^{1/2}$$
$$= \mathrm{E}_P\left[\|\ddot{\psi}_{it}^{a,kl}\|^2 | W(-k,-l)\right].$$

Therefore, we have

$$a(1) \leq N_k T_l^2 \mathrm{E}_P\left[\|\ddot{\psi}_{it}^{a,kl}\|^2 | W(-k,-l)\right], \quad a(2) \leq N_k^2 T_l \mathrm{E}_P\left[\|\ddot{\psi}_{it}^{a,kl}\|^2 | W(-k,-l)\right],$$
$$a(3) \leq N_k T_l \mathrm{E}_P\left[\|\ddot{\psi}_{it}^{a,kl}\|^2 | W(-k,-l)\right], \quad a(5) \leq N_k T_l^2 \mathrm{E}_P\left[\|\ddot{\psi}_{it}^{a,kl}\|^2 | W(-k,-l)\right].$$

On the event $\mathcal{E}_\eta \cap \mathcal{E}_{cp}$, we have, for $i \in I_k, t \in S_l$,

$$\left(\mathrm{E}_P\left[\|\ddot{\psi}_{it}^{a,kl}\|^2 | W(-k,-l)\right]\right)^{1/2} \lesssim \left(\mathrm{E}_P\left[\|\psi^a(W_{it}, \widehat{\eta}_{kl})\|^2 | W(-k,-l)\right]\right)^{1/2} < \infty,$$

where the first inequality follows from expanding the term and applying Jensen's inequality and the second inequality follows from Assumption DML2(i). Let D denote the dimension of $\psi^a(W, \eta)$, then we have

$$a(4) = a(5) + \sum_{m=1}^{T_l - 1} \sum_{t=\min(S_l)}^{\max(S_l)-m} \sum_{i,j \in I_k, i \neq j} \sum_{d=1}^{D} \mathrm{E}_P\left[\ddot{\psi}_{d,i,t}^{a,kl} \ddot{\psi}_{d,j,t+m}^{a,kl} | W(-k,-l)\right]$$

For each $i \in I_k, t \in S_l$, we can decompose $\ddot{\psi}_{d,i,t}^{a,kl} = a_i^{kl} + g_t^{kl} + e_{it}^{kl}$ where $a_i = \mathrm{E}[\ddot{\psi}_{d,i,t}^{a,kl} | \alpha_i]$, $g_i = \mathrm{E}[\ddot{\psi}_{d,i,t}^{a,kl} | \gamma_t]$, and $e_{it} = \ddot{\psi}_{d,i,t}^{a,kl} - a_i - g_t$. Conditional on $W(-k,-l)$, $(a_i^{kl}, g_t^{kl}, e_{it}^{kl})$ are mutually uncorrelated, $a_i \perp a_j$ for $i \neq j$, and g_t^{kl} is also beta-mixing with $\beta_g(m) \leq \beta_\gamma(m)$. Therefore, we have

$$\mathrm{E}_P\left[\ddot{\psi}_{d,i,t}^{a,kl} \ddot{\psi}_{d,j,t+m}^{a,kl} | W(-k,-l)\right] = \mathrm{E}_P\left[g_t^{kl} g_{t+m}^{kl} + e_{it}^{kl} e_{j,t+m}^{kl} | W(-k,-l)\right]$$
$$= \mathrm{E}_P\left[g_t^{kl} g_{t+m}^{kl} | W(-k,-l)\right] + \mathrm{E}_P\left[\mathrm{E}_P\left[e_{it}^{kl} e_{j,t+m}^{kl} | \alpha_i, \alpha_j, W(-k,-l)\right] | W(-k,-l)\right]$$

Note that β-mixing of γ_t implies α-mixing with the mixing coefficient $\alpha_\gamma(m) \leq \beta_\gamma(m)$ for all $m \in \mathbb{Z}^+$, and conditional on $W(-k,-l)$ and α_i, e_{it}^{kl} is also α-mixing with the mixing coefficient not larger than $\alpha_\gamma(m)$ by Theorem 14.12 of Hansen (2022). Then, we have

$$\left|\mathrm{E}_P\left[\mathrm{E}_P\left[e_{it}^{kl} e_{j,t+m}^{kl} | \alpha_i, \alpha_j, W(-k,-l)\right] | W(-k,-l)\right]\right| \leq \mathrm{E}_P\left[\left|\mathrm{E}_P\left[e_{it}^{kl} e_{j,t+m}^{kl} | \alpha_i, \alpha_j, W(-k,-l)\right]\right| | W(-k,-l)\right]$$
$$\lesssim 8\alpha_\gamma(m)^{1-2/q}\left(\mathrm{E}_P[|\ddot{\psi}_{d,i,t}^{a,kl}|^q | W(-k,-l)]\right)^{1/q}\left(\mathrm{E}_P[|\ddot{\psi}_{d,j,t+m}^{a,kl}|^q | W(-k,-l)]\right)^{1/q} \lesssim 32\alpha_\gamma(m)^{1-2/q} c_m^2,$$

where the first inequality follows from the Jensen's inequality; the second inequality follows from the fact that $\mathrm{E}[e_{it}^{kl} | \alpha_i, W(-k,-l)] = 0$, and Theorem 14.13(ii) of Hansen (2022); the last inequality follows from the

moment conditions in Assumption DML2 and that $W(-k,-l)$ is independent of $W(k,l)$ on $\mathfrak{E}_{cp}$. Similarly,

$$\left|\mathrm{E}_P\left[g_t^{kl} g_{t+m}^{kl} | W(-k,-l)\right]\right| \lesssim \alpha_\gamma(m)^{1-2/q} c_m^2,$$

Then, we have

$$\frac{1}{N_k^2 T_l} \sum_{m=1}^{T_l-1} \sum_{t=\min(S_l)}^{\max(S_l)-m} \sum_{i,j\in I_k, i\neq j} \sum_{d=1}^{D} \mathrm{E}_P\left[\ddot{\psi}_{d,i,t}^{a,kl} \ddot{\psi}_{d,j,t+m}^{a,kl} | W(-k,-l)\right]$$
$$\lesssim c_m^2 \frac{1}{N_k^2 T_l} \sum_{m=1}^{T_l-1} \sum_{t=\min(S_l)}^{\max(S_l)-m} \sum_{i,j\in I_k, i\neq j} \sum_{d=1}^{D} \alpha_\gamma(m)^{1-2/q} \leq c_m^2 D \sum_{m=1}^{\infty} c_\kappa exp(-\kappa m)^{1-2/q} \leq \frac{c_m^2 D c_\kappa}{exp(\kappa(1-2/q))-1} < \infty,$$

where the last inequality follows from the geometric sum. Thus, as $(N_k, T_l) \to \infty$ we have

$$\mathrm{E}\left(\left\|\Delta_{A,1}\right\|^2 | W(-k,-l)\right) = \left(\frac{1}{N_k T_l}\right)^2 [a(1) + a(2) + (3) + 2a(4) + 2a(5)] = O_P(1/T_l) = O_P(1/N).$$

where the last step follows from that L is constant and $N/T \to c$ as $N, T \to \infty$. By Markov's inequality, we conclude that conditional on $W(-k,-l)$, $\left\|\Delta_{A,1}\right\| = O_P(1/\sqrt{N})$. By Lemma 6.1 that conditional convergence implies unconditional convergence, we have $\left\|\Delta_{A,1}\right\| = O_P(1/\sqrt{N})$. To summarize, we have $\left\|\widehat{A}_{kl} - A_0\right\| = O_P(N^{-1/2} + \delta_{NT})$, which implies $\left\|\widehat{\bar{A}} - A_0\right\| = O_P(N^{-1/2} + r_{NT})$.

Proof of Claim B.2: Since K and L are finite, it suffices to show for any k, l,

$$\left\|\mathbb{E}_{kl}\left[\psi(W_{it}; \theta_0, \widehat{\eta}_{kl}) - \psi(W_{it}; \theta_0, \eta_0)\right]\right\| = O_P(r'_{NT}/\sqrt{N_k} + \lambda_{NT} + \lambda'_{NT}).$$

To simplify the notation, we denote

$$\begin{aligned}
\ddot{\psi}_{it}^{kl} &= \psi(W_{it}; \theta_0, \widehat{\eta}_{kl}) - \psi(W_{it}; \theta_0, \eta_0), \\
\tilde{\ddot{\psi}}_{it}^{kl} &= \ddot{\psi}_{it}^{kl} - \mathrm{E}_P[\ddot{\psi}_{it}^{kl} | W(-k,-l)], \\
b(1) &= \left\| \frac{\sqrt{N_k}}{N_k T_l} \sum_{i\in I_k, t\in S_l} \left[\ddot{\psi}_{it}^{kl} - \mathrm{E}_P[\ddot{\psi}_{it}^{kl} | W(-k,-l)]\right] \right\| \\
b(2) &= \left\| \frac{1}{N_k T_l} \mathrm{E}_P\left[\psi(W_{it}; \theta_0, \widehat{\eta}_{kl}) | W(-k,-l)\right] - \mathrm{E}_P\left[\psi(W_{it}; \theta_0, \eta_0)\right] \right\|.
\end{aligned}$$

We also denote $\tilde{\ddot{\psi}}_{d,it}$ as each element in the vector $\tilde{\ddot{\psi}}_{it}^{kl}$ for $d = 1, ..., D$, while suppressing the subscripts k, l for convenience. By triangle inequality, we have

$$\left\|\mathbb{E}_{kl}\left[\psi(W_{it}; \theta_0, \widehat{\eta}_{kl}) - \psi(W_{it}; \theta_0, \eta_0)\right]\right\| \leq b(1)/\sqrt{N_k} + b(2).$$

To bound $b(1)$, first note that it is mean zero by the iterated expectation argument. On the event $\mathcal{E}_\eta \cap \mathcal{E}_{cp}$, we have

$$
\begin{aligned}
&\mathrm{E}_P[b(1)^2|W(-k,-l)] \leq \frac{1}{N_k T_l^2} \sum_{i\in I_k, t\in S_l, r\in S_l} \left|\mathrm{E}_P\left[\langle\tilde{\ddot{\psi}}_{it}^{kl}, \tilde{\ddot{\psi}}_{is}^{kl}\rangle|W(-k,-l)\right]\right| \\
&+ \sum_{t\in S_l, i\in I_k, j\in I_k} \left|\mathrm{E}_P\left[\langle\tilde{\ddot{\psi}}_{it}^{kl}, \tilde{\ddot{\psi}}_{jt}^{kl}\rangle|W(-k,-l)\right]\right| + \sum_{t\in S_l, i\in I_k} \left|\mathrm{E}_P\left[\langle\tilde{\ddot{\psi}}_{it}^{kl}, \tilde{\ddot{\psi}}_{it}^{kl}\rangle|W(-k,-l)\right]\right| \\
&+ 2\sum_{m=1}^{T_l-1\,\max(S_l)-m} \sum_{t=\min(S_l)} \sum_{i,j\in I_k} \left|\mathrm{E}_P\left[\langle\tilde{\ddot{\psi}}_{it}^{kl}, \tilde{\ddot{\psi}}_{j,t+m}^{kl}\rangle|W(-k,-l)\right]\right| + 2\sum_{m=1}^{T_l-1\,\max(S_l)-m} \sum_{t=\min(S_l)} \sum_{i\in I_k} \left|\mathrm{E}_P\left[\langle\tilde{\ddot{\psi}}_{it}^{kl}, \tilde{\ddot{\psi}}_{i,t+m}^{kl}\rangle|W(-k,-l)\right]\right| \\
&=: c(1) + c(2) + c(3) + 2c(4) + 2c(5).
\end{aligned}
$$

By conditional Cauchy-Schwarz inequality, for any i, t, j, s, we have

$$
\left|\mathrm{E}_P\left[\langle\tilde{\ddot{\psi}}_{it}^{kl}, \tilde{\ddot{\psi}}_{js}^{kl}\rangle|W(-k,-l)\right]\right| \leq \left(\mathrm{E}_P\left[\|\tilde{\ddot{\psi}}_{it}^{kl}\|^2|W(-k,-l)\right]\mathrm{E}_P\left[\|\tilde{\ddot{\psi}}_{js}^{kl}\|^2|W(-k,-l)\right]\right)^{1/2}.
$$

Applying Minkowski's inequality, Jensen's inequality on the event $\mathcal{E}_\eta \cap \mathcal{E}_{cp}$, we have, for $i \in I_k, t \in S_l$,

$$
\begin{aligned}
\left(\mathrm{E}_P\left[\|\tilde{\ddot{\psi}}_{it}^{kl}\|^2|W(-k,-l)\right]\right)^{1/2} &\leq \left(\mathrm{E}_P\left[\|\ddot{\psi}_{it}^{kl}\|^2|W(-k,-l)\right]\right)^{1/2} + \left(\mathrm{E}_P\left[\|\mathrm{E}_P[\ddot{\psi}_{it}^{kl}|W(-k,-l)]\|^2|W(-k,-l)\right]\right)^{1/2} \\
&\leq 2\left(\mathrm{E}_P\left[\|\ddot{\psi}_{it}^{kl}\|^2|W(-k,-l)\right]\right)^{1/2} \\
&\leq 2r'_{NT}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&c(1) \leq \mathrm{E}_P\left[\|\tilde{\ddot{\psi}}_{it}^{kl}\|^2|W(-k,-l)\right] = O({r'_{NT}}^2), && c(2) \leq c\mathrm{E}_P\left[\|\tilde{\ddot{\psi}}_{it}^{kl}\|^2|W(-k,-l)\right] = O({r'_{NT}}^2), \\
&c(3) \leq \frac{1}{N_k}\mathrm{E}_P\left[\|\tilde{\ddot{\psi}}_{it}^{kl}\|^2|W(-k,-l)\right] = O({r'_{NT}}^2/N), && c(5) \leq \mathrm{E}_P\left[\|\tilde{\ddot{\psi}}_{it}^{kl}\|^2|W(-k,-l)\right]] = O({r'_{NT}}^2).
\end{aligned}
$$

Following similar arguments as for bounding $a(4)$, $c(4)$ is of order $O({r'_{NT}}^2)$. So, we have shown

$$
\mathrm{E}_P[b(1)^2|W(-k,-l)] = O_P\left({r'_{NT}}^2\right),
$$

which implies $b(1) = O_P(r'_{NT})$ by Markov inequality and Lemma 6.1 of Chernozhukov et al. (2018a).

To bound $b(2)$, we first define

$$
f_{kl}(r) := \mathrm{E}_P\left[\psi(W_{it}, \theta_0, \eta_0 + r(\widehat{\eta}_{kl} - \eta_0)|W(-k,-l)\right] - \mathrm{E}_P\left[\psi(W_{it}; \theta_0, \eta_0)\right],\ r \in [0,1],
$$

for some $i \in I_k, t \in S_l$. So, $b(2) = \|f_{kl}(1)\|$. By expanding $f_{kl}(r)$ around 0 using mean value theorem and

evaluating at $r = 1$, we have

$$f_{kl}(r) = f_{kl}(0) + f'_{kl}(0) + f''_{kl}(\tilde{r})/2,$$

where $\tilde{r} \in (0, 1)$. We note that $f_{kl}(0) = 0$ on the event $\mathcal{E}_{cp}$. On the event $\mathcal{E}_\eta \cap \mathcal{E}cp$ and under Assumption DML1(ii)(near-orthogonality), we have $\left\|f'_{kl}(0)\right\| \leq \lambda_{NT}$ and $\left\|f''_{kl(0)}\right\| \leq \lambda'_{NT}$. Therefore, we have shown that $b(2) = O_P(\lambda_{NT}) + O_P(\lambda'_{NT})$. Combining the bounds for $b(1)$ and $b(2)$ completes the proof of Claim B.2. □

Proof of Theorem 3.2. By the same arguments for Theorem 3.1, we have $\mathrm{P}(\mathcal{E}_\eta \cap \mathcal{E}_{cp}) = 1 - \mathrm{P}(\mathcal{E}^c_\eta \cup \mathcal{E}^c_{cp}) \geq 1 - o(1)$. By Claim B.1, we have $\|\widehat{\widehat{A}} - A_0\| = O_P(N^{-1/2} + r_{NT})$ on event $\{\mathcal{E}_\eta \cap \mathcal{E}_{cp}\}$. Therefore, due to $\|A_0^{-1}\| \leq a_0^{-1}$ ensured by Assumption DML1(iv) and $\Omega < \infty$ as shown in Claim B.2, it suffices to show $\|\widehat{\Omega}_{\mathrm{CHS}} - \Omega\| = o_P(1)$. Furthermore, since K, L are fixed constants, it suffices to show for each (k, l) that $\|\widehat{\Omega}_{\mathrm{CHS},kl} - \Omega\| = o_P(1)$ where

$$\begin{aligned}
\widehat{\Omega}_{\mathrm{CHS},kl} :=& \widehat{\Omega}_{a,kl} + \widehat{\Omega}_{b,kl} - \widehat{\Omega}_{c,kl} + \widehat{\Omega}_{d,kl} + \widehat{\Omega}'_{d,kl}, \\
\widehat{\Omega}_{a,kl} :=& \frac{1}{N_k T_l^2} \sum_{i\in I_k, t\in S_l, r\in S_l} \psi(W_{it}; \widehat{\theta}, \widehat{\eta}_{kl})\psi(W_{ir}; \widehat{\theta}, \widehat{\eta}_{kl})', \\
\widehat{\Omega}_{b,kl} :=& \frac{K/L}{N_k T_l^2} \sum_{t\in S_l, i\in I_k, j\in I_k} \psi(W_{it}; \widehat{\theta}, \widehat{\eta}_{kl})\psi(W_{jt}; \widehat{\theta}, \widehat{\eta}_{kl})', \\
\widehat{\Omega}_{c,kl} :=& \frac{K/L}{N_k T_l^2} \sum_{i\in I_k, t\in S_l} \psi(W_{it}; \widehat{\theta}, \widehat{\eta}_{kl})\psi(W_{it}; \widehat{\theta}, \widehat{\eta}_{kl})', \\
\widehat{\Omega}_{d,kl} :=& \frac{K/L}{N_k T_l^2} \sum_{m=1}^{M-1} k\left(\frac{m}{M}\right) \sum_{t=\lfloor S_l \rfloor}^{\lceil S_l \rceil - m} \sum_{i\in I_k, j\in I_k, j\neq i} \psi(W_{it}; \widehat{\theta}, \widehat{\eta}_{kl})\psi(W_{j,t+m}; \widehat{\theta}, \widehat{\eta}_{kl})'.
\end{aligned}$$

Since a sequence of symmetric matrices Ω_n converges to a symmetric matrix Ω_0 if and only if $e'\Omega_n e \to e'\Omega_0 e$ for all comfortable e, it suffices to assume without loss of generality that the dimension of ψ to be 1. To simplify the expression, we denote

$$\psi^{(0)}_{it} = \psi(W_{it}; \theta_0, \eta_0), \quad \widehat{\psi}^{(kl)}_{it} = \psi(W_{it}; \widehat{\theta}, \widehat{\eta}_{kl})$$

Claim B.3. *On event* $\{\mathcal{E}_\eta \cap \mathcal{E}_{cp}\}$, $\left|\widehat{\Omega}_{a,kl} - \Sigma_a\right| = O_P\left(N^{-1/2} + r'_{NT}\right)$.
Claim B.4. *On event* $\{\mathcal{E}_\eta \cap \mathcal{E}_{cp}\}$, $\left|\widehat{\Omega}_{b,kl} - c\mathrm{E}_P[g_t g_t]\right| = O_P\left(N^{-1/2} + r'_{NT}\right)$.
Claim B.5. *On event* $\{\mathcal{E}_\eta \cap \mathcal{E}_{cp}\}$, $\left|\widehat{\Omega}_{c,kl}\right| = O_P\left(T^{-1}\right)$.
Claim B.6. *On event* $\{\mathcal{E}_\eta \cap \mathcal{E}_{cp}\}$, $\left|\widehat{\Omega}_{d,kl} - c\sum_{m=1}^{\infty} \mathrm{E}_P[g_t g_{t+m}]\right| = o_P(1)$.

The decomposition techniques used in the proofs of Claims A.4, A.5, and A.7 follow the proofs of Lemma 1 and Lemma 2 in Appendix E of Chiang et al. (2024). Combining the Claims A.4-A.7 completes the proof

of Theorem 3.2.

Proof of Claim B.3. By triangle inequality, we have

$$\left|\widehat{\Omega}_{a,kl} - \Sigma_a\right| \leq \left|I_{a,1}^{(kl)}\right| + \left|I_{a,2}^{(kl)}\right| + \left|I_{a,2}^{(kl)}\right|,$$

where

$$\begin{aligned}
I_{a,1}^{(kl)} &:= \frac{1}{N_k T_l^2} \sum_{i\in I_k, t\in S_l, r\in S_l} \left\{\widehat{\psi}_{it}^{(kl)}\widehat{\psi}_{ir}^{(kl)} - \psi_{it}^{(0)}\psi_{ir}^{(0)}\right\}, \\
I_{a,2}^{(kl)} &:= \frac{1}{N_k T_l^2} \sum_{i\in I_k, t\in S_l, r\in S_l} \left\{\psi_{it}^{(0)}\psi_{ir}^{(0)} - \mathrm{E}_P[\psi_{it}^{(0)}\psi_{ir}^{(0)}]\right\}, \\
I_{a,2}^{(kl)} &:= \frac{1}{N_k T_l^2} \sum_{i\in I_k, t\in S_l, r\in S_l} \mathrm{E}_P[\psi_{it}^{(0)}\psi_{ir}^{(0)}] - \mathrm{E}_P[a_i a_i].
\end{aligned}$$

By law of total covariance and mean-zero property of $\psi_{it}^{(0)}$, we have

$$\mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{ir}^{(0)}\right] = \mathrm{E}_P[\mathrm{E}_P(\psi_{it}^{(0)}, \psi_{ir}^{(0)}|\alpha_i)] + \mathrm{E}_P\left(\mathrm{E}_P[\psi_{it}^{(0)}|\alpha_i]\mathrm{E}_P[\psi_{ir}^{(0)}|\alpha_i]\right)$$

Due to the identical distribution of α_i and mean zero, we have

$$\frac{1}{N_k T_l^2} \sum_{i\in I_k, t\in S_l, r\in S_l} \mathrm{E}_P[\psi_{it}^{(0)}\psi_{ir}^{(0)}] = \frac{1}{T_l^2} \sum_{t\in S_l, r\in S_l} \left\{\mathrm{E}_P[\mathrm{E}_P(\psi_{it}^{(0)}\psi_{ir}^{(0)}|\alpha_i)] + \mathrm{E}_P(\mathrm{E}_P[\psi_{it}^{(0)}|\alpha_i]\mathrm{E}_P[\psi_{ir}^{(0)}|\alpha_i])\right\}$$

Conditional on α_i, $\{\psi_{it}^{(0)}\}_{t\geq 1}$ is β-mixing with the mixing coefficient same as γ_t. Therefore, we can apply Theorem 14.13(ii) in Hansen (2022) and Jensen's inequality:

$$\mathrm{E}_P\left|\mathrm{E}_P\left[\psi_{it}^{(0)}, \psi_{ir}^{(0)}|\alpha_i\right]\right| \leq 8\left(\mathrm{E}_P|\psi_{it}^{(0)}|^q\right)^{2/q} \beta_\gamma(|t-r|)^{1-2/q}$$

Note that $\sum_{t\in S_l, r\in S_l} \beta_\gamma(|t-r|)^{1-2/q} \leq \infty$ under Assumption 2. So, $I_{a,2}^{(kl)} = O(1/T_l^2) = O(T^{-2})$.

To bound $I_{a,2}^{(kl)}$, we can rewrite it by triangle inequality as follows:

$$\begin{aligned}
\left|I_{a,2}^{(kl)}\right| &\leq \left|\frac{1}{N_k}\sum_{i\in I_k} I_{a,2,i}^{(kl)}\right| + \left|\frac{1}{N_k}\sum_{i\in I_k} \tilde{I}_{a,2,i}^{(kl)}\right|, \\
I_{a,2,i}^{(kl)} &:= \frac{1}{T_l^2}\sum_{t,r\in S_l}\left\{\psi_{it}^{(0)}\psi_{ir}^{(0)} - \mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{ir}^{(0)}|\{\gamma_t\}_{t\in S_l}\right]\right\}, \\
\tilde{I}_{a,2,i}^{(kl)} &:= \frac{1}{T_l^2}\sum_{t,r\in S_l}\left\{\mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{ir}^{(0)}|\{\gamma_t\}_{t\in S_l}\right] - \mathrm{E}_P[\psi_{it}^{(0)}\psi_{ir}^{(0)}]\right\}.
\end{aligned}$$

Due to identical distribution of α_i, $\tilde{I}^{(kl)}_{a,2,i}$ does not vary over i so that $\mathrm{E}_P\left|\frac{1}{N_k}\sum_{i\in I_k}\tilde{I}^{(kl)}_{a,2,i}\right|^2 = \mathrm{E}_P\left|\tilde{I}^{(kl)}_{a,2,i}\right|^2$. Denote $h_i(\gamma_t,\gamma_r) = \mathrm{E}_P[\psi^{(0)}_{it}\psi^{(0)}_{ir}|\gamma_t,\gamma_r] - \mathrm{E}_P[\psi^{(0)}_{it}\psi^{(0)}_{ir}]$. By direct calculation, we have

$$\mathrm{E}_P\left|\tilde{I}^{(kl)}_{a,2,i}\right|^2 = \frac{1}{T_l^4}\sum_{t,r,t',r'\in S_l}\mathrm{E}_P\left[h_i(\gamma_t,\gamma_r)h_i(\gamma_{t'},\gamma_{r'})\right].$$

To bound the RHS above, we can apply Lemma 3.4 in Dehling and Wendler (2010) by verifying the following two conditions:

$$\mathrm{E}_P\left|h_i(\gamma_t,\gamma_r)\right|^{2+\delta} < \infty, \tag{7.8}$$

$$\int\int\left|h_i(u,v)\right|^{2+\delta} dF(u)dF(v) < \infty, \tag{7.9}$$

for some $\delta > 0$ and $F(.)$ is the common CDF of γ_t. Consider condition 7.8. By Minkowski's inequality, Jensen's inequality, and the law of iterated expectation, we have

$$\left(\mathrm{E}_P\left|h_i(\gamma_t,\gamma_r)\right|^{2+\delta}\right)^{\frac{1}{2+\delta}} \le \left(\mathrm{E}_P\left|\psi^{(0)}_{it}\psi^{(0)}_{ir}\right|^{2+\delta}\right)^{\frac{1}{2+\delta}} + \mathrm{E}_P\left|\psi^{(0)}_{it}\psi^{(0)}_{ir}\right| \le \left(\mathrm{E}_P\left|\psi^{(0)}_{it}\right|^{4+2\delta}\right)^{\frac{1}{2+\delta}} + \mathrm{E}_P\left|\psi^{(0)}_{it}\right|^2$$

where the second inequality follows from Hölder's inequality and the identical distribution of γ_t. Let $\delta = \frac{p-4}{2}$, then $\left(\mathrm{E}_P\left|\psi^{(0)}_{it}\right|^{4+2\delta}\right)^{\frac{1}{2+\delta}} < c_m$ and $\mathrm{E}_P\left|\psi^{(0)}_{it}\right|^2 \le c_m^2$ follows from Assumption DML2(i). Therefore, condition 7.8 is satisfied.

Consider condition 7.9. By Minkowski's inequality and Jensen's inequality, we have

$$\begin{aligned}
&\left(\int\int\left|\mathrm{E}_P[\psi^{(0)}_{it}\psi^{(0)}_{ir}|\gamma_t=u,\gamma_r=v] - \mathrm{E}_P[\psi^{(0)}_{it}\psi^{(0)}_{ir}]\right|^{2+\delta} dF(u)dF(v)\right)^{\frac{1}{2+\delta}} \\
\le&\left(\int\int\left|\mathrm{E}_P[\psi^{(0)}_{it}\psi^{(0)}_{ir}|\gamma_t=u,\gamma_r=v]\right|^{2+\delta} dF(u)dF(v)\right)^{\frac{1}{2+\delta}} + \mathrm{E}_P\left|\psi^{(0)}_{it}\psi^{(0)}_{ir}\right| \\
\le&\left(\int\int\left(\mathrm{E}_P\left[\left|\psi^{(0)}_{it}\right|^2|\gamma_t=u\right]\right)^{\frac{2+\delta}{2}}\left(\mathrm{E}_P\left[\left|\psi^{(0)}_{ir}\right|^2|\gamma_r=v\right]\right)^{\frac{2+\delta}{2}} dF(u)dF(v)\right)^{\frac{1}{2+\delta}} + \mathrm{E}_P\left|\psi^{(0)}_{it}\right|^2 \\
\le&\left(\int\int\mathrm{E}_P\left[\left|\psi^{(0)}_{it}\right|^{2+\delta}|\gamma_t=u\right]\mathrm{E}_P\left[\left|\psi^{(0)}_{ir}\right|^{2+\delta}|\gamma_r=v\right] dF(u)dF(v)\right)^{\frac{1}{2+\delta}} + \mathrm{E}_P\left|\psi^{(0)}_{it}\right|^2 \\
=&\left(\mathrm{E}_P\left|\psi^{(0)}_{it}\right|^{4+2\delta}\right)^{\frac{1}{2+\delta}} + \mathrm{E}_P\left|\psi^{(0)}_{it}\right|^2
\end{aligned}$$

where the second inequality follows from (conditional) Hölder's inequality and identical distribution of γ_t; the third inequality follows from Jensen's inequality; the last equality follows from the law of iterated expectation

and the identical distribution of γ_t. Therefore, condition 7.9 is also satisfied with $\delta = \frac{p-4}{2}$. By Lemma 3.4 in Dehling and Wendler (2010), we conclude

$$\mathrm{E}_P\left|\tilde{I}_{a,2,i}^{(kl)}\right|^2 = \frac{1}{T_l^4}\sum_{t,r,t',r'\in S_l}\mathrm{E}_P\left[h_i(\gamma_t,\gamma_r)h_i(\gamma_{t'},\gamma_{r'})\right] = o(T_l^{-1}) = o(T^{-1}).$$

Therefore, by Markov inequality, we have $\tilde{I}_{a,2,i}^{(kl)} = o_P(T^{-1/2})$. Next. consider $\left|\frac{1}{N_k}\sum_{i\in I_k} I_{a,2,i}^{(kl)}\right|$. Note that conditional on $\{\gamma_t\}_{t\in S_l}$, $I_{a,2,i}^{(kl)}$ is i.i.d over i. So, we have

$$\mathrm{E}_P\left[\left|\frac{1}{N_k}\sum_{i\in I_k} I_{a,2,i}^{(kl)}\right|^2 |\{\gamma_t\}_{t\in S_l}\right] = \frac{1}{N_k^2}\sum_{i\in I_k}\mathrm{E}_P\left[\left|I_{a,2,i}^{(kl)}\right|^2 |\{\gamma_t\}_{t\in S_l}\right] = \frac{1}{N_k}\mathrm{E}_P\left[\left|I_{a,2,i}^{(kl)}\right|^2 |\{\gamma_t\}_{t\in S_l}\right]$$

By conditional Markov inequality, we have

$$\mathrm{P}\left(\left|\frac{1}{N_k}\sum_{i\in I_k} I_{a,2,i}^{(kl)}\right| > \varepsilon |\{\gamma_t\}_{t\in S_l}\right) = O\left(\frac{1}{N_k}\mathrm{E}_P\left[\left|I_{a,2,i}^{(kl)}\right|^2 |\{\gamma_t\}_{t\in S_l}\right]\right)$$

By Minkowski's inequality for infinite sums, Jensen's inequality, and Hölder's inequality, we have

$$\left(\mathrm{E}_P\left[\left|I_{a,2,i}^{(kl)}\right|^2\right]\right)^{1/2} \lesssim \frac{1}{T_l^2}\sum_{t,r\in S_l}\left(\mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{ir}^{(0)}\right]^2\right)^{1/2} \le \frac{1}{T_l^2}\sum_{t,r\in S_l}\left(\mathrm{E}_P\left[\psi_{it}^{(0)}\right]^4\right)^{1/2} \le c_m^2,$$

where the last inequality follows from Assumption DML2(i) . Then, by law of iterated expectation, we have

$$\mathrm{P}\left(\left|\frac{1}{N_k}\sum_{i\in I_k} I_{a,2,i}^{(kl)}\right| > \varepsilon\right) = O\left(N_k^{-1}\right),$$

and $\left|\frac{1}{N_k}\sum_{i\in I_k} I_{a,2,i}^{(kl)}\right| = O_P\left(N_k^{-1/2}\right) = O_P\left(N^{-1/2}\right)$. Therefore, we have shown $\left|I_{a,2}^{kl}\right| = O_P\left(N^{-1/2}\right) + o_P\left(T^{-1/2}\right)$.

Next, consider $I_{a,1}^{kl}$. By product decomposition, triangle inequality, and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\left|I_{a,1}^{kl}\right| \le& \frac{1}{N_k T_l^2}\sum_{i\in I_k, t\in S_l, r\in S_l}\left|\widehat{\psi}_{it}^{(kl)}\widehat{\psi}_{ir}^{(kl)\prime} - \psi_{it}^{(0)}\psi_{ir}^{(0)}\right| \\
\le& \frac{1}{N_k T_l^2}\sum_{i\in I_k, t\in S_l, r\in S_l}\left\{\left|\widehat{\psi}_{it}^{(kl)} - \psi_{it}^{(0)}\right|\left|\widehat{\psi}_{ir}^{(kl)} - \psi_{ir}^{(0)}\right| + \left|\psi_{it}^{(0)}\right|\left|\widehat{\psi}_{ir}^{(kl)} - \psi_{ir}^{(0)}\right| + \left|\widehat{\psi}_{it}^{(kl)} - \psi_{it}^{(0)}\right|\left|\widehat{\psi}_{ir}^{(kl)\prime}\right|\right\} \\
\lesssim& R_{kl}\left\{\left\|\psi_{it}^{(0)}\right\|_{kl,2} + R_{kl}\right\},
\end{aligned}$$

where $R_{kl} = \left\| \widehat{\psi}_{it}^{(kl)} - \psi_{it}^{(0)} \right\|_{kl,2}$. By Markov inequality and under Assumption DML2(i), we have

$$\mathrm{E}_P \left[\frac{1}{N_k T_l} \sum_{i \in I_k, t \in S_l} \left(\psi_{it}^{(0)} \right)^2 \right] = \mathrm{E}_P \left| \psi(W_{it}; \theta_0, \eta_0) \right|^2 \leq c_m^2.$$

Therefore, $\frac{1}{N_k T_l} \sum_{i \in I_k, t \in S_l} \left(\psi_{it}^{(0)} \right)^2 = O_P(1)$. To bound R_{kl}, note that by Assumption DML1(i) (linearity) we have

$$\begin{aligned} R_{kl}^2 =& \frac{1}{N_k T_l} \sum_{i \in I_k, t \in S_l} \left(\psi^a(W_{it}; \widehat{\eta}_{kl})(\widehat{\theta} - \theta_0) + \psi(W_{it}; \theta_0, \widehat{\eta}_{kl}) - \psi(W_{it}; \theta_0, \eta_0) \right)^2 \\ \lesssim& \frac{1}{N_k T_l} \sum_{i \in I_k, t \in S_l} \left| \psi^a(W_{it}; \widehat{\eta}_{kl}) \right|^2 \left| \widehat{\theta} - \theta_0 \right|^2 + \frac{1}{N_k T_l} \sum_{i \in I_k, t \in S_l} \left| \widehat{\psi}_{it}^{(kl)} - \psi_{it}^{(0)} \right|^2 \end{aligned}$$

By Markov inequality and Assumption DML2(i), we have $\frac{1}{N_k T_l} \sum_{i \in I_k, t \in S_l} \left| \psi^a(W_{it}; \widehat{\eta}_{kl}) \right|^2 = O_P(1)$. By Theorem 3.1, $\left| \widehat{\theta} - \theta_0 \right|^2 = O_p(N^{-1})$. Therefore, the first term on RHS is $O_P(N^{-1})$. For the second term on RHS, consider its conditional expectation given the auxiliary sample $W(-k, -l)$. On the event $\mathcal{E}_\eta \cap \mathcal{E}_{cp}$, we have

$$\mathrm{E}_P \left[\frac{1}{N_k T_l} \sum_{i \in I_k, t \in S_l} \left| \widehat{\psi}_{it}^{(kl)} - \psi_{it}^{(0)} \right|^2 | W(-k, -l) \right] = \mathrm{E}_P \left[\left| \psi(W_{it}; \theta_0, \widehat{\eta}_{kl}) - \psi(W_{it}; \theta_0, \eta_0) \right|^2 | W(-k, -l) \right] \leq \delta_{NT}^2,$$

where the last inequality follows from Assumption DML2(ii). Then, by Markov inequality and Lemma 6.1 from Chernozhukov et al. (2018a), we have $R_{kl}^2 = O_P\left(N^{-1} + (r'_{NT})^2 \right)$ and so $\left| I_{a,1}^{kl} \right| = O_P\left(N^{-1/2} + r'_{NT} \right)$. To summarize, we have shown

$$\left| \widehat{\Omega}_{a,kl} - \Sigma_a \right| = O_P\left(N^{-1/2} + r'_{NT} \right) + O_P(N^{-1/2}) + o_P(T^{-1/2}) + O(T^{-2}) = O_P\left(N^{-1/2} + r'_{NT} \right)$$

Proof of Claim B.4. By triangle inequality, we have

$$\begin{aligned} \left| \widehat{\Omega}_{b,kl} - c\mathrm{E}_P[g_t g_t'] \right| \leq& \left| I_{b,1}^{(kl)} \right| + \left| I_{b,2}^{(kl)} \right| + \left| I_{b,3}^{(kl)} \right|, \\ I_{b,1}^{(kl)} :=& \frac{K/L}{N_k T_l^2} \sum_{t \in S_l, i \in I_k, j \in I_k} \left\{ \widehat{\psi}_{it}^{(kl)} \widehat{\psi}_{jt}^{(kl)} - \psi_{it}^{(0)} \psi_{jt}^{(0)} \right\}, \\ I_{b,2}^{(kl)} :=& \frac{K/L}{N_k T_l^2} \sum_{t \in S_l, i \in I_k, j \in I_k} \left\{ \psi_{it}^{(0)} \psi_{jt}^{(0)} - \mathrm{E}_P[\psi_{it}^{(0)} \psi_{jt}^{(0)}] \right\}, \\ I_{b,3}^{(kl)} :=& \frac{K/L}{N_k T_l^2} \sum_{t \in S_l, i \in I_k, j \in I_k} \mathrm{E}_P[\psi_{it}^{(0)} \psi_{jt}^{(0)}] - c\mathrm{E}_P[g_t g_t'], \end{aligned}$$

and $\frac{K/L}{N_k T_l^2} = \frac{c}{N_k^2 T_l}$.

Consider $I_{b,3}^{(kl)}$. By the the law of total covariance, we have

$$\mathrm{E}_P[\psi_{it}^{(0)}\psi_{jt}^{(0)}] = cov(\psi_{it}^{(0)}, \psi_{jt}^{(0)}) = \mathrm{E}_P[cov(\psi_{it}^{(0)}, \psi_{jt}^{(0)}|\gamma_t)] + cov(\mathrm{E}_P[\psi_{it}^{(0)}|\gamma_t], \mathrm{E}_P[\psi_{jt}^{(0)}|\gamma_t]) = 0 + \mathrm{E}_P[g_t g_t'],$$

Due to identical distribution of γ_t, $\mathrm{E}_P[g_t g_t']$ does not vary over t and so $I_{b,3}^{(kl)} = 0$.

To bound $I_{b,2}^{(kl)}$, we can rewrite it by triangle inequality as follows

$$\begin{aligned}
&\frac{1}{c}\left|I_{b,2}^{kl}\right| \leq \left|\frac{1}{T_l}\sum_{t\in S_l} I_{b,2,t}^{(kl)}\right| + \left|\frac{1}{T_l}\sum_{t\in S_l} \tilde{I}_{b,2,t}^{(kl)}\right|, \\
&I_{b,2,t}^{(kl)} := \frac{1}{N_k^2}\sum_{i,j\in I_k}\left\{\psi_{it}^{(0)}\psi_{jt}^{(0)} - \mathrm{E}_P[\psi_{it}^{(0)}\psi_{jt}^{(0)}|\{\alpha_i\}_{i\in I_k}]\right\} \\
&\tilde{I}_{b,2,t}^{(kl)} := \frac{1}{N_k^2}\sum_{i,j\in I_k}\left\{\mathrm{E}_P[\psi_{it}^{(0)}\psi_{jt}^{(0)}|\{\alpha_i\}_{i\in I_k}] - \mathrm{E}_P[\psi_{it}^{(0)}\psi_{jt}^{(0)}]\right\}
\end{aligned}$$

Due to identical distribution of γ_t, $\tilde{I}_{b,2,t}^{(kl)}$ does not vary over t so that $\mathrm{E}_P\left|\frac{1}{T_l}\sum_{t\in S_l}\tilde{I}_{b,2,t}^{(kl)}\right|^2 = \mathrm{E}_P\left|\tilde{I}_{b,2,t}^{(kl)}\right|^2$. Denote $\zeta_{ij,t} = \psi_{it}^{(0)}\psi_{jt}^{(0)}$. By direct calculation, we have

$$\begin{aligned}
\mathrm{E}_P\left|\tilde{I}_{b,2,t}^{(kl)}\right|^2 =& \frac{1}{N_k^4}\sum_{i,j\in I_k}\sum_{i',j'\in I_k}\mathrm{E}_P\left[\left(\mathrm{E}_P[\zeta_{ij,t}|\alpha_i,\alpha_j] - \mathrm{E}_P[\zeta_{ij,t}]\right)\left(\mathrm{E}_P[\zeta_{i'j'}|\alpha_{i'},\alpha_{j'}] - \mathrm{E}_P[\zeta_{i'j'}]\right)\right] \\
\lesssim& \frac{1}{N_k}\mathrm{E}_P[\zeta_{ij,t}]^2 < \frac{1}{N_k}\mathrm{E}_P\left[\psi_{it}^{(0)}\right]^4 = O(1/N_k).
\end{aligned}$$

where the first inequality follows from the assumption that α_i is independent over i and an application of Hölder's inequality and Jensen's inequality. The second inequality follows from Hölder's inequality and the last equality follows from Assumption DML2(i) with some $q > 4$. Therefore, by Markov inequality, we have $\left|\frac{1}{T_l}\sum_{t\in S_l}\tilde{I}_{b,2,t}^{(kl)}\right| = O_P(N_k^{-1/2}) = O_P(N^{-1/2})$.

Now consider $\left|\frac{1}{T_l}\sum_{t\in S_l} I_{b,2,t}^{(kl)}\right|$. Note that conditional on $\{\alpha_i\}$, $I_{b,2,t}^{(kl)}$ is also β-mixing with the mixing coefficient same as γ_t. Then, with an application of the conditional version of Theorem 14.2 from Davidson (1994), we have

$$\left(\mathrm{E}_P\left[\left|\mathrm{E}_P[I_{b,2,t}^{(kl)}|\{\alpha_i\}_{i\in I_k}, \mathcal{F}_{-\infty}^{t-l}]\right|^2|\{\alpha_i\}_{i\in I_k}\right]\right)^{1/2}] \leq 2(2^{1/2}+1)\beta(l)^{1/2-\frac{2}{q}}\left(\mathrm{E}_P\left[|I_{b,2,t}^{(kl)}|^{\frac{q}{2}}|\{\alpha_i\}_{i\in I_k}\right]\right)^{\frac{2}{q}}.$$

Then, we can apply the conditional version of Lemma A from Hansen (1992) to show that

$$\left(\mathrm{E}_P\left[\left|\frac{1}{T_l}\sum_{t\in S_l} I_{b,2,t}^{(kl)}\right|^2 |\{\alpha_i\}_{i\in I_k}\right]\right)^{1/2} \lesssim \frac{1}{T_l}\sum_{l=1}^{\infty}\beta(l)^{1/2-\frac{2}{q}}\left(\sum_{t\in S_l}\left(\mathrm{E}_P\left[|I_{b,2,t}^{(kl)}|^{\frac{q}{2}}|\{\alpha_i\}_{i\in I_k}\right]\right)^{\frac{4}{q}}\right)^{1/2}$$
$$\lesssim \frac{1}{\sqrt{T_l}}\left(\mathrm{E}_P\left[|I_{b,2,t}^{(kl)}|^{\frac{q}{2}}|\{\alpha_i\}_{i\in I_k}\right]\right)^{\frac{2}{q}}$$

By conditional Markov inequality, we have

$$\mathrm{P}\left(\left|\frac{1}{T_l}\sum_{t\in S_l} I_{b,2,t}^{(kl)}\right| > \varepsilon|\{\alpha_i\}_{i\in I_k}\right) = O\left(T_l^{-1}\mathrm{E}_P\left[\left|I_{b,2,t}^{(kl)}\right|^{\frac{q}{2}}|\{\alpha_i\}_{i\in I_k}\right]\right)$$

By Minkowski's inequality for infinite sums, Jensen's inequality, and Hölder's inequality, we have

$$\left(\mathrm{E}_P\left[\left|I_{b,2,t}^{(kl)}\right|^{\frac{q}{2}}\right]\right)^{\frac{2}{q}} \lesssim \frac{1}{N_k^2}\sum_{i,j\in I_k}\left(\mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{jt}^{(0)}\right]^{\frac{q}{2}}\right)^{\frac{2}{q}} \le \frac{1}{N_k^2}\sum_{i,j\in I_k}\left(\mathrm{E}_P\left[\psi_{it}^{(0)}\right]^q\right)^{\frac{2}{q}} \le c_m^2,$$

where the last inequality follows from Assumption DML2(i) . Then, by the law of iterated expectation, we have

$$\mathrm{P}\left(\left|\frac{1}{T_l}\sum_{t\in S_l} I_{b,2,t}^{(kl)}\right| > \varepsilon\right) = O\left(T_l^{-1/2}\right).$$

Therefore, we have shown $\left|I_{b,2}^{kl}\right| = O_P(N_k^{-1}) + O_P(T_l^{-1/2}) = O_P(T^{-1/2})$.

Consider $I_{b,1}^{kl}$. By the similar inequality for $\left|I_{a,1}^{kl}\right|$, we have

$$\frac{1}{c}\left|I_{b,1}^{kl}\right| \lesssim R_{kl}\left\{\left(\frac{1}{N_k T_l}\sum_{i\in I_k, t\in S_l}\left(\psi_{it}^{(0)}\right)^2\right)^{1/2} + R_{kl}\right\},$$

where $R_{kl} = \left\|\widehat{\psi}_{it}^{(kl)} - \psi_{it}^{(0)}\right\|_{kl,2}$. We have shown in the proof of Claim B.3 that $\left\|\psi_{it}^{(0)}\right\|_{kl,2} = O_P(1)$ and $R_{kl}^2 = O_P\left(N^{-1} + (r'_{NT})^2\right)$. So $\left|I_{b,1}^{kl}\right| = O_P\left(N^{-1/2} + r'_{NT}\right)$. To summarize

$$\left|\widehat{\Omega}_{b,kl} - c\mathrm{E}_P[g_t g_t']\right| = O_P\left(N^{-1/2}\right) + O_P\left(T^{-1/2}\right) + O_P\left(N^{-1/2} + r'_{NT}\right) = O_P\left(N^{-1/2} + r'_{NT}\right),$$

which completes the proof of Claim B.4.

Proof of Claim B.5. By triangle inequality, we have

$$\left|\widehat{\Omega}_{c,kl}\right| \leq \left|I_{c,1}^{(kl)}\right| + \left|I_{c,2}^{(kl)}\right| + \left|I_{c,3}^{(kl)}\right|$$

where

$$\begin{aligned}
I_{c,1}^{(kl)} &:= \frac{K/L}{N_k T_l^2} \sum_{i\in I_k, t\in S_l} \left\{ \widehat{\psi}_{it}^{(kl)} \widehat{\psi}_{it}^{(kl)} - \psi_{it}^{(0)} \psi_{it}^{(0)} \right\}, \\
I_{c,2}^{(kl)} &:= \frac{K/L}{N_k T_l^2} \sum_{i\in I_k, t\in S_l} \left\{ \psi_{it}^{(0)} \psi_{it}^{(0)} - \mathrm{E}_P[\psi_{it}^{(0)} \psi_{it}^{(0)}] \right\}, \\
I_{c,3}^{(kl)} &:= \frac{K/L}{N_k T_l^2} \sum_{i\in I_k, t\in S_l} \mathrm{E}_P[\psi_{it}^{(0)} \psi_{it}^{(0)}],
\end{aligned}$$

Consider $I_{c,3}^{(kl)}$. Note that under Assumption DML2(i), we have

$$\mathrm{E}_P[\psi_{it}^{(0)} \psi_{it}^{(0)}] \leq c_m^2.$$

Thus, $I_{c,3}^{(kl)} = O_P(1/T_l) = O_P(T^{-1})$.

Consider $I_{c,2}^{kl}$. We denote $\xi_{it} = \psi_{it}^{(0)} \psi_{it}^{(0)} - \mathrm{E}_P[\psi_{it}^{(0)} \psi_{it}^{(0)}]$. By expanding $\mathrm{E}\left|I_{c,2}^{kl}\right|^2$ and applying Hölder's inequality, we have

$$\begin{aligned}
\mathrm{E}\left|I_{c,2}^{kl}\right|^2 \leq \left(\frac{K/L}{N_k T_l^2}\right)^2 \Bigg\{ & \sum_{i\in I_k, t\in S_l, r\in S_l} \mathrm{E}_P\left|\xi_{it}\right|^2 + \sum_{t\in S_l, i\in I_k, j\in I_k} \mathrm{E}_P\left|\xi_{it}\right|^2 + \sum_{t\in S_l, i\in I_k} \mathrm{E}_P\left|\xi_{it}\right|^2 \\
& + 2\sum_{m=1}^{T_l-1} \sum_{t=\min(S_l)}^{\max(S_l)-m} \sum_{i,j\in I_k} \mathrm{E}_P\left|\xi_{it}\right|^2 + 2\sum_{m=1}^{T_l-1} \sum_{t=\min(S_l)}^{\max(S_l)-m} \sum_{i\in I_k} \mathrm{E}_P\left|\xi_{it}\right|^2 \Bigg\}.
\end{aligned}$$

where the last inequality follows from Note that for each i, t, by Hölder's inequality and Assumption DML2(i), we have

$$\mathrm{E}_P\left|\xi_{it}\right|^2 \lesssim \mathrm{E}_P\left[\psi(W_{it}; \theta_0, \eta_0)^4\right] \leq c_m^4.$$

Thus, $\mathrm{E}\left|I_{c,2}^{(kl)}\right|^2 = O(T^{-2})$ and so $I_{c,2}^{(kl)} = O_P(T^{-1})$.

Now consider $I_{c,1}^{(kl)}$. Following the same steps for $I_{b,1}^{(kl)}$, we have

$$\left|I_{c,1}^{(kl)}\right| \lesssim \frac{K/L}{T_l} R_{kl} \left\{ \left\|\psi_{it}^{(0)}\right\|_{kl,2} + R_{kl} \right\},$$

where $R_{kl} = \left\|\widehat{\psi}_{it}^{(kl)} - \psi_{it}^{(0)}\right\|_{kl,2}$. We have shown in the proof of Claim B.3 that $\left\|\psi_{it}^{(0)}\right\|_{kl,2} = O_P(1)$ and

$R_{kl}^2 = O_P\left(N^{-1} + (r'_{NT})^2\right)$. So, $\left|I_{c,1}^{(kl)}\right| = O_P\left(N^{-1/2}/T + r'_{NT}/T\right)$. To summarize

$$\left|\widehat{\Omega}_{c,kl}\right| = O_P\left(T^{-1}\right) + O_P\left(N^{-1/2}/T + r'_{NT}/T\right) = O_P\left(T^{-1}\right),$$

which completes the proof of Claim B.5.

Proof of Claim B.6. By triangle inequality, we have

$$\left|\widehat{\Omega}_{d,kl} - c\sum_{m=1}^{\infty}\mathrm{E}_P[g_t g_t']\right| \leq \left|I_{d,1}^{(kl)}\right| + \left|I_{d,2}^{(kl)}\right| + \left|I_{d,3}^{(kl)}\right| + \left|I_{d,4}^{(kl)}\right| + \left|I_{d,5}^{(kl)}\right| + \left|I_{d,6}^{(kl)}\right|$$

where

$$\begin{aligned}
I_{d,1}^{(kl)} &:= \frac{K/L}{N_k T_l^2}\sum_{m=1}^{M-1} k\left(\frac{m}{M}\right)\sum_{t=\lfloor S_l\rfloor}^{\lceil S_l\rceil - m}\sum_{i\in I_k, j\in I_k, j\neq i}\left\{\widehat{\psi}_{it}^{(kl)}\widehat{\psi}_{j,t+m}^{(kl)} - \psi_{it}^{(0)}\psi_{j,t+m}^{(0)}\right\},\\
I_{d,2}^{(kl)} &:= \frac{K/L}{N_k T_l^2}\sum_{m=1}^{M-1} k\left(\frac{m}{M}\right)\sum_{t=\lfloor S_l\rfloor}^{\lceil S_l\rceil - m}\sum_{i\in I_k, j\in I_k, j\neq i}\left\{\psi_{it}^{(0)}\psi_{j,t+m}^{(0)} - \mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{j,t+m}^{(0)}\right]\right\},\\
I_{d,3}^{(kl)} &:= \frac{K/L}{N_k T_l^2}\sum_{m=1}^{M-1}\left(k\left(\frac{m}{M}\right) - 1\right)\sum_{t=\lfloor S_l\rfloor}^{\lceil S_l\rceil - m}\sum_{i\in I_k, j\in I_k, j\neq i}\mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{j,t+m}^{(0)}\right],\\
I_{d,4}^{(kl)} &:= \frac{K/L}{N_k T_l^2}\sum_{m=M}^{\infty}\sum_{t=\lfloor S_l\rfloor}^{\lceil S_l\rceil - m}\sum_{i\in I_k, j\in I_k, j\neq i}\mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{j,t+m}^{(0)}\right],\\
I_{d,5}^{(kl)} &:= \frac{K/L}{N_k T_l^2}\sum_{m=1}^{M-1}\sum_{t=\lfloor S_l\rfloor}^{\lceil S_l\rceil - m}\sum_{i\in I_k, j\in I_k, j\neq i}\mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{j,t+m}^{(0)}\right] - c\sum_{m=1}^{\infty}\mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{j,t+m}^{(0)}\right],\\
I_{d,6}^{(kl)} &:= c\sum_{m=1}^{\infty}\mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{j,t+m}^{(0)}\right] - c\sum_{m=1}^{\infty}\mathrm{E}_P\left[g_t g_{t+m}\right]
\end{aligned}$$

and $\frac{K/L}{N_k T_l^2} = \frac{c}{N_k^2 T_l}$.

Consider $I_{d,6}^{(kl)}$. By the law of total covariance, we have

$$\begin{aligned}
\mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{j,t+m}^{(0)}\right] &= cov(\psi_{it}^{(0)}, \psi_{j,t+m}^{(0)})\\
&= \mathrm{E}_P[cov(\psi_{it}^{(0)}, \psi_{j,t+m}^{(0)})|\gamma_t, \gamma_{t+m}] + cov(\mathrm{E}_P[\psi_{it}^{(0)}|\gamma_t], \mathrm{E}_P[\psi_{j,t+m}^{(0)}|\gamma_{t+m}])\\
&= 0 + \mathrm{E}_P[g_t g_{t+m}'],
\end{aligned}$$

where the last equality follows from the properties of Hajek projection components, as discussed in the beginning of Appendix A. Therefore, $I_{d,6}^{(kl)} = 0$.

Consider $I_{d,5}^{(kl)}$. The strict stationarity of γ_t implies that $\psi_{it}^{(0)}$ is also strictly stationary over t. And under Assumption 2, there is no heterogeneity across i. Then, as $M, T \to \infty$, we have $I_{d,5}^{(kl)} = o(1)$.

Consider $I_{d,4}^{(kl)}$. Under Assumption DML2(i), $\left(\mathrm{E}_P|\psi_{it}^{(0)}|^q\right)^{1/q} \leq c_m$ for $q > 4$. And conditional on α_i, $\psi_{it}^{(0)}$ is β-mixing with the mixing coefficient not larger than that of γ_t. Then by Theorem 14.13(ii) in Hansen (2022), we have

$$\left|\mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{j,t+m}^{(0)}|\{\alpha_i\}_{i\in I_k}\right]\right| \leq 8\left(\mathrm{E}_P\left[|\psi_{it}^{(0)}|^q|\alpha_i\right]\right)^{1/q}\left(\mathrm{E}_P\left[|\psi_{j,t+m}^{(0)}|^q|\alpha_j\right]\right)^{1/q}\alpha_\gamma(m)^{1-2/q}$$

By iterated expectation and Jensen's inequality, we have

$$\begin{aligned}
\left|\mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{j,t+m}^{(0)}\right]\right| \leq & \mathrm{E}_P\left[\left|\mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{j,t+m}^{(0)}|\{\alpha_i\}_{i\in I_k}\right]\right|\right] \\
\leq & 8\mathrm{E}_P\left[\left(\mathrm{E}_P\left[|\psi_{it}^{(0)}|^q|\alpha_i\right]\right)^{1/q}\left(\mathrm{E}_P\left[|\psi_{j,t+m}^{(0)}|^q|\alpha_j\right]\right)^{1/q}\alpha_\gamma(m)^{1-2/q}\right] \\
\leq & 8\mathrm{E}_P\left[\left(\mathrm{E}_P\left[|\psi_{it}^{(0)}|^q|\alpha_i\right]\right)^{1/q}\right]\mathrm{E}_P\left[\left(\mathrm{E}_P\left[|\psi_{j,t+m}^{(0)}|^q|\alpha_j\right]\right)^{1/q}\right]\alpha_\gamma(m)^{1-2/q} \\
\lesssim & c_m^2\alpha_\gamma(m)^{1-2/q}
\end{aligned}$$

where the third inequality follows from that α_i are independent over i. Then, as $M \to \infty$,

$$\begin{aligned}
\left|I_{d,4}^{(kl)}\right| \leq & \frac{K/L}{N_k T_l^2}\sum_{m=M}^{\infty}\sum_{t=\lfloor S_l\rfloor}^{\lceil S_l\rceil-m}\sum_{i\in I_k, j\in I_k, j\neq i}\left|\mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{j,t+m}^{(0)}\right]\right| \lesssim \sum_{m=M}^{\infty}\alpha_\gamma(m)^{1-2/q} \leq \sum_{m=M}^{\infty}\beta_\gamma(m)^{1-2/q} \\
\leq & c_\kappa\sum_{m=M}^{\infty}exp(-\kappa m) = c_\kappa\left(\frac{1}{1-e^{-k}} - \frac{1-e^{-kM}}{1-e^{-\kappa}}\right) = O\left(e^{-\kappa M}\right).
\end{aligned}$$

Consider $I_{d,3}^{(kl)}$.

$$\begin{aligned}
\left|I_{d,3}^{(kl)}\right| \leq & \frac{K/L}{N_k T_l^2}\sum_{m=1}^{M-1}\left|k\left(\frac{m}{M}\right)-1\right|\sum_{t=\lfloor S_l\rfloor}^{\lceil S_l\rceil-m}\sum_{i\in I_k, j\in I_k, j\neq i}\left|\mathrm{E}_P\left[\psi_{it}^{(0)}\psi_{j,t+m}^{(0)}\right]\right| \\
\leq & cc_m^2\sum_{m=1}^{M-1}\left|k\left(\frac{m}{M}\right)-1\right|\alpha_\gamma(m)^{1-2/q}.
\end{aligned}$$

Note that for each m, $\left|k\left(\frac{m}{M}\right)-1\right| \to 0$ as $M \to \infty$. Since $\left|k\left(\frac{m}{M}\right)-1\right|\alpha_\gamma(m)^{1-2/q} \leq 1$, we can apply dominated convergence theorem to conclude that $I_{d,3}^{(kl)} = o(1)$.

To bound $I_{d,2}^{(kl)}$, we can rewrite it by triangle inequality as follows

$$\frac{1}{c}\left|I_{d,2}^{(kl)}\right| \leq \left|\sum_{m=1}^{M-1} \frac{k\left(\frac{m}{M}\right)}{T_l} \sum_{t=\lfloor S_l \rfloor}^{\lceil S_l \rceil - m} I_{d,2,tm}^{(kl)}\right| + \left|\sum_{m=1}^{M-1} \frac{k\left(\frac{m}{M}\right)}{T_l} \sum_{t=\lfloor S_l \rfloor}^{\lceil S_l \rceil - m} \tilde{I}_{d,2,tm}^{(kl)}\right|,$$

where

$$\begin{aligned} I_{d,2,tm}^{(kl)} :=& \frac{1}{N_k^2} \sum_{i,j \in I_k, i \neq j} \left\{ \psi_{it}^{(0)} \psi_{j,t+m}^{(0)} - \mathrm{E}_P[\psi_{it}^{(0)} \psi_{j,t+m}^{(0)} | \{\alpha_i\}_{i \in I_k}] \right\} \\ \tilde{I}_{d,2,tm}^{(kl)} :=& \frac{1}{N_k^2} \sum_{i,j \in I_k, i \neq j} \left\{ \mathrm{E}_P[\psi_{it}^{(0)} \psi_{j,t+m}^{(0)} | \{\alpha_i\}_{i \in I_k}] - \mathrm{E}_P[\psi_{it}^{(0)} \psi_{j,t+m}^{(0)}] \right\} \end{aligned}$$

Due to identical distribution of γ_t, $\tilde{I}_{d,2,tm}^{(kl)}$ does not vary over t so that $\mathrm{E}_P \left|\sum_{m=1}^{M-1} \frac{k\left(\frac{m}{M}\right)}{T_l} \sum_{t=\lfloor S_l \rfloor}^{\lceil S_l \rceil - m} \tilde{I}_{d,2,tm}^{(kl)}\right|^2 \leq \mathrm{E}_P \left|\sum_{m=1}^{M-1} k\left(\frac{m}{M}\right) \tilde{I}_{d,2,tm}^{(kl)}\right|^2$. And by Minkowski's inequality, we have

$$\left(\mathrm{E}_P \left|\sum_{m=1}^{M-1} k\left(\frac{m}{M}\right) \tilde{I}_{d,2,tm}^{(kl)}\right|^2\right)^{1/2} \leq \sum_{m=1}^{M-1} k\left(\frac{m}{M}\right) \left(\mathrm{E}_P\left[\tilde{I}_{d,2,tm}^{(kl)}\right]^2\right)^{1/2}$$

Denote $\zeta_{ijm} = \psi_{it}^{(0)} \psi_{j,t+m}^{(0)}$. By direct calculation, we have

$$\begin{aligned} \mathrm{E}_P \left|\tilde{I}_{d,2,tm}^{(kl)}\right|^2 =& \frac{1}{N_k^4} \sum_{i,j \in I_k} \sum_{i',j' \in I_k} \mathrm{E}_P \left[\left(\mathrm{E}_P[\zeta_{ijm} | \alpha_i, \alpha_j] - \mathrm{E}_P[\zeta_{ij,t}]\right) \left(\mathrm{E}_P[\zeta_{i'j'} | \alpha_{i'}, \alpha_{j'}] - \mathrm{E}_P[\zeta_{i'j'}]\right)\right] \\ \lesssim& \frac{1}{N_k} \mathrm{E}_P[\zeta_{ijm}]^2 < \frac{1}{N_k} \mathrm{E}_P\left[\psi_{it}^{(0)}\right]^4 = O(1/N_k). \end{aligned}$$

where the first inequality follows from the assumption that α_i is independent over i and an application of Hölder's inequality and Jensen's inequality. The second inequality follows from Hölder's inequality and the last equality follows from Assumption DML2(i) with some $q > 4$. Therefore, we have $\left(\mathrm{E}_P \left|\sum_{m=1}^{M-1} k\left(\frac{m}{M}\right) \tilde{I}_{d,2,tm}^{(kl)}\right|^2\right)^{1/2} \leq O_P\left(\frac{M}{N^{1/2}}\right) = O_P\left(\frac{M}{T^{1/2}}\right)$. By Markov inequality, we have $\left|\sum_{m=1}^{M-1} \frac{k\left(\frac{m}{M}\right)}{T_l} \sum_{t=\lfloor S_l \rfloor}^{\lceil S_l \rceil - m} \tilde{I}_{d,2,tm}^{(kl)}\right| = O_P\left(\frac{M}{T^{1/2}}\right)$.

Now consider $\left|\sum_{m=1}^{M-1} \frac{k\left(\frac{m}{M}\right)}{T_l} \sum_{t=\lfloor S_l \rfloor}^{\lceil S_l \rceil - m} I_{d,2,tm}^{(kl)}\right|$. By Minkowski's inequality, we have

$$\left(\mathrm{E}_P \left| \sum_{m=1}^{M-1} \frac{k\left(\frac{m}{M}\right)}{T_l} \sum_{t=\lfloor S_l \rfloor}^{\lceil S_l \rceil - m} I_{d,2,tm}^{(kl)} \right|^2\right)^{1/2} \leq \sum_{m=1}^{M-1} k\left(\frac{m}{M}\right) \left(\mathrm{E}_P \left| \frac{1}{T_l} \sum_{t=\lfloor S_l \rfloor}^{\lceil S_l \rceil - m} I_{d,2,tm}^{(kl)} \right|^2\right)^{1/2}$$

Following the same steps as for $I_{b,2,tm}^{(kl)}$, we can show

$$\mathrm{E}_P \left| \frac{1}{T_l} \sum_{t=\lfloor S_l \rfloor}^{\lceil S_l \rceil - m} I_{d,2,tm}^{(kl)} \right|^2 = O\left(T_l^{-1}\right).$$

Therefore, $\left|\sum_{m=1}^{M-1} \frac{k\left(\frac{m}{M}\right)}{T_l} \sum_{t=\lfloor S_l \rfloor}^{\lceil S_l \rceil - m} I_{d,2,tm}^{(kl)}\right| = O_P\left(\frac{M}{T_l^{-1/2}}\right) = O_P\left(\frac{M}{T^{-1/2}}\right)$. We have shown $\left|I_{b,2}^{(kl)}\right| = O_P(1/N_k) + O_P\left(\frac{M}{T^{-1/2}}\right) = O_P\left(\frac{M}{T^{-1/2}}\right)$.

Consider $I_{d,1}^{(kl)}$. Denote

$$I_{d,1,m}^{(kl)} = \frac{K/L}{N_k T_l^2} \sum_{t=\lfloor S_l \rfloor}^{\lceil S_l \rceil - m} \sum_{i \in I_k, j \in I_k, j \neq i} \left\{ \widehat{\psi}_{it}^{(kl)} \widehat{\psi}_{j,t+m}^{(kl)} - \psi_{it}^{(0)} \psi_{j,t+m}^{(0)} \right\},$$

for each m. Then, $I_{d,1}^{(kl)} = \sum_{m=1}^{M-1} k\left(\frac{m}{M}\right) I_{d,1,m}^{(kl)}$. Following the same steps as for $I_{a,1}^{(kl)}$, we can show

$$\left|I_{d,1,m}^{(kl)}\right| = O_P(T^{-1/2} + r'_{NT}),$$

for each m. Therefore, $\left|I_{d,1}^{(kl)}\right| = O_P\left(\frac{M}{T^{-1/2}} + M r'_{NT}\right)$. Note that $M r'_{NT} \leq M \delta_{NT} N^{-1/2} = \frac{M}{T^{1/2}} \frac{T^{1/2}}{N^{1/2}} \delta_{NT} = o(1)$.

To summarize

$$\begin{aligned}\left|\widehat{\Omega}_{d,kl} - c \sum_{m=1}^{\infty} \mathrm{E}_P[g_t g'_t]\right| &= O_P\left(\frac{M}{T^{-1/2}} + M r'_{NT}\right) + O_P\left(\frac{M}{T^{1/2}}\right) + o(1) + O(e^{-\kappa M}) + o(1) + 0 \\ &= o_P(1).\end{aligned}$$

which completes the proof of Claim B.6.

□

Proof of Theorem 3.3. Since (K, L) are fixed constants, it suffices to show for each (k, l) that $\widehat{\Omega}_{\mathrm{NW},kl} :=$

$\frac{K/L}{N_k T_l^2} \sum_{i\in I_k, t\in S_l, r\in S_l} k\left(\frac{|t-r|}{M}\right) \psi(W_{it};\widehat{\theta},\widehat{\eta}_{kl})\psi(W_{ir};\widehat{\theta},\widehat{\eta}_{kl})' = o_P(1)$. Note that we can rewrite $\widehat{\Omega}_{\mathrm{NW},kl}$ as

$$\widehat{\Omega}_{\mathrm{NW},kl} = \widehat{\Omega}_{c,kl} + \widehat{\Omega}_{e,kl} - \widehat{\Omega}_{d,kl}$$

where $\widehat{\Omega}_{c,kl}$ and $\widehat{\Omega}_{d,kl}$ are defined in the beginning of the proof of Theorem 3.2, and $\widehat{\Omega}_{e,kl}$ is defined as follows:

$$\widehat{\Omega}_{e,kl} := \frac{K/L}{N_k T_l^2} \sum_{m=1}^{M-1} k\left(\frac{m}{M}\right) \sum_{t=\lfloor S_l \rfloor}^{\lceil S_l \rceil - m} \sum_{i\in I_k, j\in I_k} \psi(W_{it};\widehat{\theta},\widehat{\eta}_{kl})\psi(W_{j,t+m};\widehat{\theta},\widehat{\eta}_{kl})'.$$

Observe that by replacing $\widehat{\Omega}_{d,kl}$ by $\widehat{\Omega}_{e,kl}$, each step in the proof of Claim B.6 also follows. It implies that $\widehat{\Omega}_{e,kl} = \widehat{\Omega}_{d,kl} + o_P(1)$. By Lemma A.6, we have $\widehat{\Omega}_{c,kl} = O_P(T^{-1})$. Therefore, we conclude that $\widehat{\Omega}_{\mathrm{NW},kl} = o_P(1)$.

□

Appendix C

Proof of Theorem 4.1. Let $P \in \mathcal{P}_{NT}$ for each (N,T). We denote

$$\begin{aligned} A_{NT} &= \frac{1}{NT}(V^Z)'V^D,\ \hat{A}_{NT} = \frac{1}{NT}(Z - f\hat{\zeta}_0)'(D - f\hat{\pi}_0), \\ \psi_{NT} &= \frac{1}{NT}(V^Z)'V^g,\ \hat{\psi}_{NT} = \frac{1}{NT}(Z - f\hat{\zeta}_0)'\left(Y - f\hat{\beta} - (D - f\hat{\zeta})'\theta_0\right). \end{aligned}$$

We can write $\widehat{\theta} - \theta_0 = \hat{A}_{NT}^{-1}\hat{\psi}_{NT}$. By product decomposition, we have

$$\widehat{\theta} - \theta_0 = A_{NT}^{-1}\psi_{NT} + A_{NT}^{-1}\left[\hat{\psi}_{NT} - \psi_{NT}\right] + \left[\hat{A}_{NT}^{-1} - A_{NT}^{-1}\right]\left[\hat{\psi}_{NT} - \psi_{NT}\right] + \left[\hat{A}_{NT}^{-1} - A_{NT}^{-1}\right]\psi_{NT}$$

For the asymptotic normality of $\sqrt{N\wedge T}\left(\widehat{\theta} - \theta_0\right)$, we need to show the following statements: (i) $A_{NT} \xrightarrow{p} A_0 = \mathrm{E}[V_{it}^Z V_{it}^D]$; (ii) $\sqrt{N\wedge T}\psi_{NT} \xrightarrow{d} \mathcal{N}(0,\Omega_0)$; (iii) $\sqrt{N\wedge T}\left[\hat{\psi}_{NT} - \psi_{NT}\right] = o(1)$; (iv) $\widehat{A}_{NT} - A_{NT} = o_P(1)$. With statements (i) - (iv) and the identification condition in Assumption REG-P(i) such that $\tilde{A}_0$ is non-singular, $\sqrt{N\wedge T}\left(\widehat{\theta} - \theta_0\right) \xrightarrow{d} \mathcal{N}\left(0, A_0^{-1}\Omega_0 A_0^{-1'}\right)$. Then, the conclusion of the theorem follows.

Before we show Statement (i) - (iv), we note that Assumptions REG-P(ii) and AHK imply that $(\bar{F}_i, \bar{F}_t)$ are functions of only $(\alpha_i, \gamma_t, \varepsilon_{it})$, and so are f_{it} and V_{it}^l for $l = g, D, Y, Z$. Therefore, the results based on Hajek projection are still applicable. Also, due to Assumptions REG-P(ii), $\bar{F}_i$ is a function of only (c_i, ϵ_i) and $\bar{F}_t$ is a function of only (d_t, ϵ_t), so f_{it} is a function of $(X_{it}, c_i, \epsilon_i, d_t, \epsilon_t)$ which are mean independent of U_{it}^D. Therefore, $\mathrm{E}_P[f_{it}V_{it}^D] = \mathrm{E}_P\left[f_{it}[(L_{2,it} - E[L_{2,it}])\eta_{D,2} + U_{it}^D]\right] = 0$ given that f_{it} is uncorrelated with $L_{2,it}$ as discussed in the main text. Similarly, we have $\mathrm{E}_P[f_{it}V_{it}^D] = 0$.

Statement (i) follows from Lemma A.1 under Assumptions AHK, AR, and REG-P(iii). For Statement (ii), we first observe that $V_{it}^Z = Z_{it}(1-\zeta_0)$ where $\zeta_0 = \left(E[f_{it}' f_{it}]\right)^{-1} E[f_{it}' Z_{it}]$. Due to the exogeneity condition $\mathrm{E}_P[Z_{it}U^g] = 0$ and the independence between $(\bar{F}_i, \bar{F}_t, Z_{it}, X_{it})$ and (ϵ_i, ϵ_t), we have $\mathrm{E}_P[V_{it}^Z V_{it}^g] = 0$. With the additional Assumption REG-P(iv), Statement (ii) follows from Lemma A.2.

Consider Statement (iii). By product decomposition and triangle inequality, we have

$$\begin{aligned}
NT|\hat{\psi}_{NT} - \psi_{NT}| \leq & |(f(\zeta_0 - \widehat{\zeta}))'(f(\beta_0 - \widehat{\beta}) + V^Y + r^Y - \theta_0(f(\pi_0 - \widehat{\pi}) + V^D + r^D))| \\
& + |(Z - f\zeta_0)'(\theta_0(f(\widehat{\pi} - \pi_0)) - f(\beta_0 - \widehat{\beta}) + r^g)| \\
\lesssim & |(f(\zeta_0 - \widehat{\zeta}))' f(\beta_0 - \widehat{\beta})| + |(f(\zeta_0 - \widehat{\zeta}))' V^Y| + |(f(\zeta_0 - \widehat{\zeta}))' r^Y| \\
& + |(f(\zeta_0 - \widehat{\zeta}))' f(\pi_0 - \widehat{\pi})| + |(f(\zeta_0 - \widehat{\zeta}))' V^D| + |(f(\zeta_0 - \widehat{\zeta}))' r^D| \\
& + |(V^Z)' f(\widehat{\pi} - \pi_0)| + |(V^Z)' f(\beta_0 - \widehat{\beta})| + + |(V^Z)' r^g| \qquad (7.10)
\end{aligned}$$

Under Assumptions AHK, AR, the sparse approximation conditions as well as Assumption REG-P(ii) - (vii), we can apply Theorem 2.1 to obtain that $\|f_{it}(\eta_0 - \widehat{\eta})\|_{NT,2} = O_P\left(\sqrt{\frac{s\log(p/\gamma)}{N\wedge T}}\right)$, $\|\eta_0 - \widehat{\eta}\|_1 = O_P\left(s\sqrt{\frac{\log(p/\gamma)}{N\wedge T}}\right)$ for $\eta = \zeta, \pi, \beta$, and $P\left(\max_{j=1,\ldots,p}\left|\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T \omega_{j,l}^{-1/2} f_{it,j} V_{it}^l\right| \geq \frac{\lambda}{2c_1 NT}\right) \to 0$ for $l = Z, D, Y$ where $\lambda = \frac{6c_1 NT}{\sqrt{N\wedge T}}\Phi^{-1}(1-\gamma/2p)$. By Lemma A.2, $\omega_{j,l} \xrightarrow{p} \frac{A\wedge T}{N}\Sigma_{a,j,l} + \frac{N\wedge T}{T}\Sigma_{g,j,l}$ where $\Sigma_{a,j}^l > c_\sigma > 0$ and $\Sigma_{g,j}^l > c_\sigma > 0$ by Lemma A.1. Therefore, $\min_j \omega_{j,l}^{-1/2} > \sqrt{c_\sigma}$, which implies $\|f' V^l\|_\infty = O_P(\Phi^{-1}(1-\gamma/2p)/\sqrt{N\wedge T}) = O_P\left(\sqrt{\frac{\log(p/\gamma)}{N\wedge T}}\right)$ for $l = D, Y, Z$.

Consider the first term in 7.10. By Cauchy-Swartz inequality, we have $\frac{\sqrt{N\wedge T}}{NT}|(f(\zeta_0 - \widehat{\zeta}))' f(\beta_0 - \widehat{\beta})| \leq \sqrt{N\wedge T}\|f_{it}(\zeta_0 - \widehat{\zeta})\|_{NT,2}\|f_{it}(\beta_0 - \widehat{\beta})\|_{NT,2} = O_P\left(\frac{s\log(p/\gamma)}{\sqrt{N\wedge T}}\right)$. Consider the second term in 7.10. By Holder's inequality, we have $\frac{\sqrt{N\wedge T}}{NT}|(f(\zeta_0 - \widehat{\zeta}))' V^Y| \leq \frac{\sqrt{N\wedge T}}{NT}\|\zeta_0 - \widehat{\zeta}\|_1 \|f' V^Y\|_\infty = O_P\left(s\frac{\log(p/\gamma)}{\sqrt{N\wedge T}}\right)$. Consider the third term in 7.10. By Cauchy-Swartz inequality, we have $\frac{\sqrt{N\wedge T}}{NT}|(f(\zeta_0 - \widehat{\zeta}))' r^Y| \leq \sqrt{N\wedge T}\|f_{it}(\zeta_0 - \widehat{\zeta})\|_{NT,2}\|r_{it}^Y\|_{NT,2} = O_P\left(\sqrt{\frac{s\log(p/\gamma)}{N\wedge T}}\right)$. For the last term of 7.10, Cauchy-Swartz inequality implies that $\frac{\sqrt{N\wedge T}}{NT}|(V^Z)' r| \leq \sqrt{N\wedge T}\|V_{it}^Y\|_{NT,2}\|r_{it}^Y\|_{NT,2}$. By Assumption REG-P(ii), we have $|\mathrm{E}[(V_{it}^Y)^2]^{4(\mu+\delta)}| < \infty$. Then we can apply Lemma A.1 and obtain that $\|V_{it}^Y\|_{NT,2} \to (\mathrm{E}[(V_{it}^Y)^2])^{1/2}$. Therefore, we have $\frac{\sqrt{N\wedge T}}{NT}|(V^Z)' r| = o_P(1)$. The arguments for the rest of the terms in 7.10 are similar. Under the sparsity condition $s = \frac{\sqrt{N\wedge T}}{\log(p/\gamma)}$, we conclude that $\sqrt{NT}|\hat{\psi}_{NT} - \psi_{NT}| = o_P(1)$.

Consider Statement (vi). By product decomposition, we have

$$\begin{aligned} NT\left\|\widehat{A}_{NT}-A_{NT}\right\|_1 &= \left\|\left(f(\zeta_0-\widehat{\zeta})\right)' f(\pi_0-\widehat{\pi})+\left(f(\zeta_0-\widehat{\zeta})\right)'(D-f\pi_0)+(Z-f\zeta_0)'f(\pi_0-\widehat{\pi})\right\|_1 \\ &\leq \left\|\left(f(\zeta_0-\widehat{\zeta})\right)' f(\pi_0-\widehat{\pi})\right\|_1+\left\|\left(f(\zeta_0-\widehat{\zeta})\right)'(r^D+V^D)\right\|_1+\left\|(V^Z)'f(\pi_0-\widehat{\pi})\right\|_1 \end{aligned}$$

We observe that, by similar arguments for Statement (v), $\left\|\widehat{A}_{NT}-A_{NT}\right\|_1 = o_P(1)$. We have shown Statement (i)-(iv), completing the proof.

□

Proof of Theorem 4.2. We have shown in the proof of Theorem 4.1 that $\widehat{A}_{NT}-A_{NT}=o_P(1)$ and $A_{NT}-A_0=o_P(1)$. By triangle inequality, we have $\widehat{A}_{NT}-A_0=o_P(1)$. Then, it suffices to show $\widehat{\Omega}_{\text{CHS}}-\Omega=o_P(1)$. We decompose $\widehat{\Omega}_{\text{CHS}}$ as follows:

$$\begin{aligned} \widehat{\Omega}_{\text{CHS}} :=& \widehat{\Omega}_a+\widehat{\Omega}_b-\widehat{\Omega}_c+\widehat{\Omega}_d+\widehat{\Omega}_d', \\ \widehat{\Omega}_a :=& \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{r=1}^{T}\psi_{it}(\widehat{\theta},\widehat{\eta})\psi_{ir}(\widehat{\theta},\widehat{\eta})', \;\; \widehat{\Omega}_b := \frac{1}{NT^2}\sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{j=1}^{N}\psi_{it}(\widehat{\theta},\widehat{\eta})\psi_{jt}(\widehat{\theta},\widehat{\eta})', \\ \widehat{\Omega}_c :=& \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\psi_{it}(\widehat{\theta},\widehat{\eta})\psi_{it}(\widehat{\theta},\widehat{\eta})', \;\; \widehat{\Omega}_d := \frac{1}{NT^2}\sum_{m=1}^{M-1}k\left(\frac{m}{M}\right)\sum_{t=1}^{T-m}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\psi_{it}(\widehat{\theta},\widehat{\eta})\psi_{j,t+m}(\widehat{\theta},\widehat{\eta})'. \end{aligned}$$

where $\psi_{it}(\theta,\eta)=(Z_{it}-f_{it}\zeta)(Y_{it}-f_{it}\beta-\theta(D_{it}-f_{it}\pi))$ and $\eta=(\zeta,\beta,\pi)$. We need to show $\widehat{\Omega}_a \xrightarrow{p} \Sigma_a = \mathrm{E}_P[a_i^2]$, $\widehat{\Omega}_b \xrightarrow{p} c\mathrm{E}[g_t^2]$, $\widehat{\Omega}_c=o_P(1)$, and $\widehat{\Omega}_d \xrightarrow{p} c\sum_{m=1}^{\infty}\mathrm{E}_P[g_t g_{t+m}]$.

First, consider $\widehat{\Omega}_a-\mathrm{E}_P[a_i^2]$. By triangle inequality, we have

$$\left|\widehat{\Omega}_a-\mathrm{E}_P[a_i^2]\right| \leq \left|I_{a,1}\right|+\left|I_{a,2}\right|+\left|I_{a,2}\right|,$$

where

$$\begin{aligned} I_{a,1} :=& \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{r=1}^{T}\left\{\psi_{it}(\widehat{\theta},\widehat{\eta})\psi_{ir}(\widehat{\theta},\widehat{\eta})-\psi_{it}(\theta_0,\eta_0)\psi_{ir}(\theta_0,\eta_0)\right\}, \\ I_{a,2} :=& \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{r=1}^{T}\left\{\psi_{it}(\theta_0,\eta_0)\psi_{ir}(\theta_0,\eta_0)-\mathrm{E}[\psi_{it}(\theta_0,\eta_0)\psi_{ir}(\theta_0,\eta_0)]\right\}, \\ I_{a,2} :=& \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{r=1}^{T}\left\{\mathrm{E}[\psi_{it}(\theta_0,\eta_0)\psi_{ir}(\theta_0,\eta_0)]-\mathrm{E}[a_i^2]\right\}. \end{aligned}$$

Note that in proving Claim B.3, the cross-fitting device is only used to show that $I_{a,1}$ is of small order. Since the arguments for showing $I_{a,2}$ and $I_{a,3}$ to be of small order are basically the same as those in the proof of Claim B.3, they are not repeated here.

Consider $I_{a,1}$. By product decomposition, triangle inequality, and Cauchy-Schwarz inequality, we have

$$\begin{aligned}|I_{a,1}| \lesssim & R_{NT}\left\{|\psi_{it}(\theta_0,\eta_0)|_{NT,2}+R_{NT}\right\}\\ R_{NT} := & \left\|\psi_{it}(\widehat{\theta},\widehat{\eta})-\psi_{it}(\theta_0,\eta_0)\right\|_{NT,2}\end{aligned}$$

By Minkowski's inequality, we have

$$\begin{aligned}R_{NT} &= \left\|\psi^a_{it}(\eta_0)(\widehat{\theta}-\theta_0)+(\psi^a_{it}(\eta_0)-\psi^a_{it}(\widehat{\eta}))(\widehat{\theta}-\theta_0)+\psi_{it}(\theta_0,\widehat{\eta})-\psi_{it}(\theta_0,\eta_0)\right\|_{NT,2}\\ &\le \left\|\psi^a_{it}(\eta_0)(\widehat{\theta}-\theta_0)\right\|_{NT,2}+\left\|(\psi^a_{it}(\eta_0)-\psi^a_{it}(\widehat{\eta}))(\widehat{\theta}-\theta_0)\right\|_{NT,2}+\left\|\psi_{it}(\theta_0,\widehat{\eta})-\psi_{it}(\theta_0,\eta_0)\right\|_{NT,2} : R_{a,1}+R_{a,2}+R_{a,3},\end{aligned}$$

where $\psi^a_{it}(\eta) := \left(Z_{it}-f_{it}\zeta\right)(D_{it}-f_{it}\pi)$. Under Assumption REG-P(ii), we have $\mathrm{E}_P[\psi^a_{it}(\eta_0)]^2 = \mathrm{E}_P[V^Z_{it}(V^D_{it}+r^D_{it})]^2 = O_P(1)$, and Markov inequality implies that $\left\|\psi^a_{it}(\eta_0)\right\|_{NT,2} = O_P(1)$. By Theorem 4.1, we have $\widehat{\theta}-\theta_0 = O_P\left(\frac{1}{\sqrt{N\wedge T}}\right)$. Therefore, $R_{a,1} \le \left\|\psi^a_{it}(\eta_0)\right\|_{NT,2}|\widehat{\theta}-\theta_0| = O_P\left(\frac{1}{\sqrt{N\wedge T}}\right)$. To bound $R_{a,2}$, we note

$$\left\|\psi^a_{it}(\eta_0)-\psi^a_{it}(\eta_0)\right\|_{NT,2} = \left\|f_{it}(\widehat{\zeta}-\zeta_0)(D_{it}-f_{it}\pi_0)+f_{it}(\widehat{\zeta}-\zeta_0)f_{it}(\widehat{\pi}-\pi_0)+(Z_{it}-f_{it}\zeta_0)f_{it}(\widehat{\pi}-\pi_0)\right\|_{NT,2}$$

Under Assumption REG-P(iii), we have $\mathrm{E}_P|V^D_{it}|^{8(\mu+\delta)} < \infty$, which implies $\mathrm{E}_P\left[\max_{i\le N,t\le T}|V^D_{it}|^2\right] \lesssim (NT)^{\frac{1}{4(\mu+\delta)}}$. By Markov inequality, we have $\max_{i\le N,t\le T}|V^D_{it}|^2 = O_P((NT)^{\frac{1}{4(\mu+\delta)}})$. As in the proof of Theorem 4.1, Theorem 2.1 can be applied to obtain $\left\|f_{it}(\widehat{\zeta}-\zeta_0)\right\|_{NT,2} = O_P\left(\sqrt{\frac{s\log(p/\gamma)}{N\wedge T}}\right)$. Then, we have

$$\begin{aligned}R_{a,2} &= \left\|f_{it}(\widehat{\zeta}-\zeta_0)V^D_{it}\right\|_{NT,2} \le \left(\max_{i\le N,t\le T}|V^D_{it}|^2\right)^{1/2}\left\|f_{it}(\widehat{\zeta}-\zeta_0)\right\|_{NT,2}\\ &= O_P((NT)^{\frac{1}{8(\mu+\delta)}})O_P\left(\sqrt{\frac{s\log(p/\gamma)}{N\wedge T}}\right) = O_P((NT)^{\frac{1}{8(\mu+\delta)}})o_P\left(\frac{1}{(N\wedge T)^{1/4}}\right) = o_P(1).\end{aligned}$$

Similar arguments can be made to show $R_{a,3}$. Therefore, we have $R_{NT} = o_P(1)$ and so $\widehat{\Omega}_a \xrightarrow{p} \Sigma_a$

It is left to show that $\widehat{\Omega}_b \xrightarrow{p} c\mathrm{E}[g_t^2]$, $\widehat{\Omega}_c = o_P(1)$, and $\widehat{\Omega}_d \xrightarrow{p} c\sum_{m=1}^{\infty}\mathrm{E}_P[g_tg_{t+m}]$. As is shown in the proof of Theorem 3.2 (Lemmas A.5-A.7), the only step in showing these claims that involve cross-fitting technique is to show the same term R_{NT} to converge to 0 in probability. Otherwise, the arguments are basically the same and not repeated here. Combining these results, we obtain $\widehat{\Omega} \xrightarrow{p} \mathrm{E}_P(a_i^2)+c\mathrm{E}_P(g_t^2)+c\sum_{m=1}^{\infty}\mathrm{E}_P(g_tg_{t+m}) = \Sigma_a+c\Sigma_g$.

To show $\widehat{V}_{\mathrm{DKA}} = \widehat{V}_{\mathrm{CHS}}+o_P(1)$, it suffices to show $\widehat{\Omega}_{\mathrm{NW}} = o_P(1)$. We decompose Ω_{NW} as follows:

$$\widehat{\Omega}_{\mathrm{NW}} = \widehat{\Omega}_c+\widehat{\Omega}_e-\widehat{\Omega}_d,$$

where $\widehat{\Omega}_c$ and $\widehat{\Omega}_d$ are defined as above and $\widehat{\Omega}_e$ is defined as follows:

$$\widehat{\Omega}_e := \frac{1}{NT^2} \sum_{m=1}^{M-1} k\left(\frac{m}{M}\right) \sum_{t=1}^{T-m} \sum_{i=1}^{N} \sum_{j=1}^{N} \psi(W_{it}; \widehat{\theta}, \tilde{\eta}) \psi(W_{j,t+m}; \widehat{\theta}, \tilde{\eta}).$$

Following the same arguments as in the proof of Claim B.6, we have $\widehat{\Omega}_e = \widehat{\Omega}_d + o_P(1)$. We have shown $\widehat{\Omega}_c = o_P(1)$. Therefore, we conclude that $\widehat{\Omega}_{NW} = o_P(1)$. So it is proved. □