

*Fourier Transforms: Discrete, Fast, and Practical*  
*PHYS 250 (Autumn 2024) – Lecture 12*

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# Outline

1

## *Reminders*

- Reminders from Lecture 13

2

## *DFT to FFT*

- Reminders of the DFT
- Cooley-Tukey algorithm
- Butterfly calculations
- Danielson-Lanczos Lemma

## *Reminders from last time*

We left off discussing details of our discrete Fourier transform and how we might speed it up.

### PDEs and Fourier Series

- **Fourier Series** → **Fourier Transforms**

- We discussed how we can move to a continuous function definition of the expansion over a basis of functions
- We then broke this down into discrete steps and obtained the **Discrete Fourier Transform**

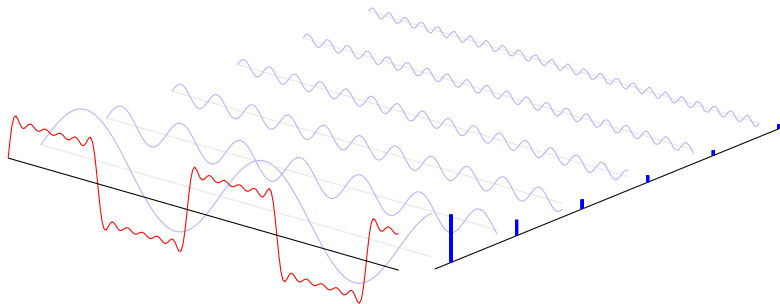
- **Issues encountered:**

- We realized that there is an issue related to the finite sampling of a function: **aliasing**
- Began to break down the Fourier transform even further for a **fast** implementation

Today we will discuss the evolution towards the **FFT**, some of the practical limitations, and specific real-world (scientific and otherwise!) examples of using FFT's!

# Square wave Fourier series

We already saw how we can break down a “simple” function into its components:



So let's figure out how to use this to its full capacity!

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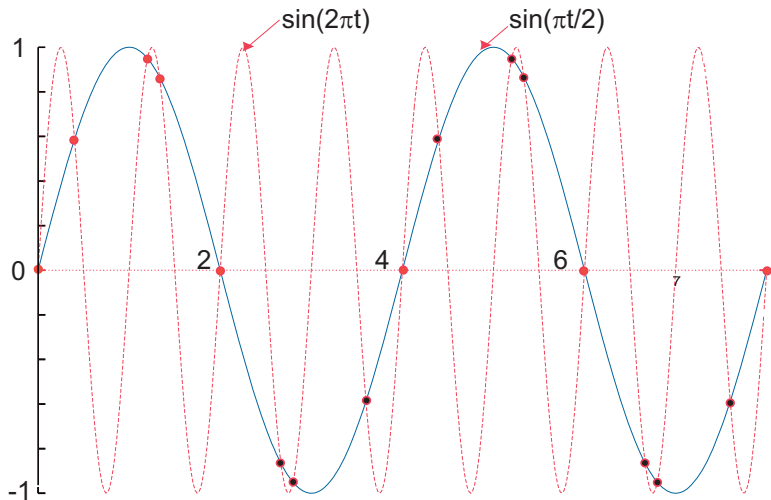
In this way the **DFT can be thought of as a technique for interpolating, compressing, and extrapolating data.**

## *Discussion*

**Do you see any issues with this “sampling”?**

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## Discrete Fourier Transforms (II)

The DFT algorithm results from evaluating the integral not from  $-\infty$  to  $+\infty$  but rather from time 0 to time  $T$  over which the signal is measured, and from approximating the integration of the integral by computing a discrete sum:

$$\hat{f}(\omega_n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega_n t} dt \quad (1)$$

$$\simeq \frac{1}{\sqrt{2\pi}} \int_0^T f(t) e^{-i\omega_n t} dt \quad (2)$$

$$\simeq \frac{1}{\sqrt{2\pi}} \sum_{k=1}^N h f(t_k) e^{-i\omega_n t_k} \quad (h \equiv \text{stepsize}) \quad (3)$$

$$\simeq \frac{h}{\sqrt{2\pi}} \sum_{k=1}^N f_k e^{-2\pi i k n / N} \quad (4)$$

$$\hat{f}_n \equiv \frac{\hat{f}(\omega_n)}{h} = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^N f_k e^{-2\pi i k n / N} \quad (5)$$

## Discrete Fourier Transforms (III)

We then need the inverse as well, which we can obtain with  $d\omega \rightarrow 2\pi/Nh$  we invert the  $\hat{f}_n$

$$f_k = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^N \frac{2\pi}{Nh} \hat{f}_n e^{i\omega_n t} \quad (6)$$

Once we know the  $N$  values of the transform  $\hat{f}_n$ , we can use this expression to evaluate  $f(t)$  for any time  $t$ . The frequencies  $\omega_n$  are determined by the number of samples taken and by the total sampling time  $T = Nh$  as

$$\omega_n = n \frac{2\pi}{Nh} \quad (7)$$

Clearly, the larger we make the time  $T = Nh$  over which we sample the function, the smaller will be the frequency steps or resolution. Accordingly, if you want a smooth frequency spectrum, you need to have a small frequency step  $2\pi/T$ .

## Discrete Fourier Transforms (IV)

Lastly, we can simplify this expression to yield a clear computational approach:

$$f_k = \frac{\sqrt{2\pi}}{N} \sum_{n=1}^N Z^{-nk} \hat{f}_n \quad (Z = e^{-2\pi i/N}) \quad (8)$$

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^N Z^{nk} f_k \quad (n = 0, 1, \dots, N) \quad (9)$$

With this formulation, the computer needs to compute only powers of  $Z$ .

## Recap of the Discrete Fourier Transform (DFT)

This is where we are at with the discretization of the Fourier Transform:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \quad \xrightarrow{\text{DFT}} \quad \hat{f}_n = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^N f_k e^{-2\pi i k n / N} \quad (10)$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega \quad \xrightarrow{\text{DFT}} \quad f_k = \frac{\sqrt{2\pi}}{N} \sum_{n=1}^N \hat{f}_n e^{-i\omega_n t} \quad (11)$$

This has certain drawbacks which we will discuss shortly, but it also has huge advantages. Namely, we can re-write this to see some amazing computational properties.

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^N Z_N^{nk} f_k \quad (Z_N = e^{-2\pi i / N}) \quad (12)$$

$$f_k = \frac{\sqrt{2\pi}}{N} \sum_{n=1}^N Z_N^{-nk} \hat{f}_n \quad (n = 0, 1, \dots, N) \quad (13)$$



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We're saying that with this formulation, the computer needs to compute only powers of  $Z \rightarrow Z_N^{nk}$ .

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**What if we can make this scale as  $N \ln N$ ???**

This may not seem like much of a difference, for  $N = 10^{2-3}$ , the difference of  $10^{3-5}$  is the difference between a minute and a week.

**This is what the FFT buys us!**

## Cooley-Tukey algorithm

Let's start with the simplified form of the DFT:

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^N f_k e^{-2\pi i k n / N} = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^N f_k Z_N^{nk} \quad (14)$$

First, **STOP AND LOOK**. What are some features of this expression?  
What operations are required?

- There are **imaginary components**

• However, if we define the signal elements  $f_k$  to be real (as we often find),  $Z_N$  is always a complex number, and therefore we must process both real and imaginary parts when we multiply them together.

- We have to add and/or multiply  $N^2$  times unless we break this down further

• For example, with  $N=4$  and  $N=8$  we have 16 integer values, and (22) is real (multiplications are real), but we require  $N^2$  multiplications and additions of complex numbers.

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• Each add and/or multiply costs a certain value, the cost of multiplication is typically more expensive than addition and/or subtraction.

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- There are **imaginary components**
  - Even if the signal elements  $f_k$  to be transformed are real,  $Z_N$  is always complex, and therefore we must process both real and imaginary parts when computing transforms.
- We have to add and/or multiply  $N^2$  times unless we break this down further
  - Both  $n$  and  $k$  range over  $N$  integer values, the  $(Z_N^n)^k f_k$  multiplications require  $N^2$  multiplications and additions of complex numbers.

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## Example Cooley-Tukey algorithm (I)

The time savings comes from taking advantage of **periodicity**.

The values of the **computationally expensive complex factor**  $Z^{nk} = ((Z)^n)^k$  **are repeated** as the integers  $n$  and  $k$  vary sequentially.

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## Example Cooley-Tukey algorithm (II)

**There are actually only 4 independent values!  $Z^0, Z^1, Z^2, Z^3$**

$$\begin{aligned} Z^0 &= \exp(0) = +1, & Z^1 &= \exp\left(-\frac{2\pi}{8}i\right) = +\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \\ Z^2 &= \exp\left(-\frac{2\pi}{8}2i\right) = -i, & Z^3 &= \exp\left(-\frac{2\pi}{8}3i\right) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \\ Z^4 &= \exp\left(-\frac{2\pi}{8}4i\right) = -Z^0, & Z^5 &= \exp\left(-\frac{2\pi}{8}5i\right) = -Z^1 \\ Z^6 &= \exp\left(-\frac{2\pi}{8}6i\right) = -Z^2, & Z^7 &= \exp\left(-\frac{2\pi}{8}7i\right) = -Z^3 \\ Z^8 &= \exp\left(-\frac{2\pi}{8}8i\right) = +Z^0, & Z^9 &= \exp\left(-\frac{2\pi}{8}9i\right) = +Z^1 \\ Z^{10} &= \exp\left(-\frac{2\pi}{8}10i\right) = +Z^2, & Z^{11} &= \exp\left(-\frac{2\pi}{8}11i\right) = +Z^3 \\ Z^{12} &= \exp\left(-\frac{2\pi}{8}11i\right) = -Z^0, & \dots & \end{aligned}$$



## Example Cooley-Tukey algorithm (III)

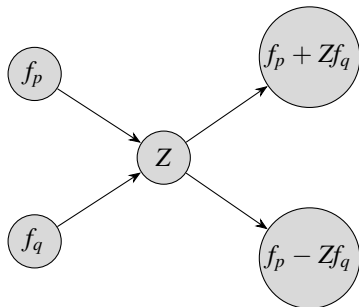
We can now put these equations in an appropriate form for computing by regrouping the terms into sums and differences of the  $f$ 's:

$$\begin{aligned}\hat{f}^0 &= Z^0(f_0 + f_4) + Z^0(f_1 + f_5) + Z^0(f_2 + f_6) + Z^0(f_3 + f_7) \\ \hat{f}^1 &= Z^0(f_0 - f_4) + Z^1(f_1 - f_5) + Z^2(f_2 - f_6) + Z^3(f_3 - f_7) \\ \hat{f}^2 &= Z^0(f_0 + f_4) + Z^2(f_1 + f_5) - Z^0(f_2 + f_6) - Z^2(f_3 + f_7) \\ \hat{f}^3 &= Z^0(f_0 - f_4) + Z^3(f_1 - f_5) - Z^2(f_2 - f_6) + Z^1(f_3 - f_7) \\ \hat{f}^4 &= Z^0(f_0 + f_4) - Z^0(f_1 + f_5) + Z^0(f_2 + f_6) - Z^0(f_3 + f_7) \\ \hat{f}^5 &= Z^0(f_0 - f_4) - Z^1(f_1 - f_5) + Z^2(f_2 - f_6) - Z^3(f_3 - f_7) \\ \hat{f}^6 &= Z^0(f_0 + f_4) - Z^2(f_1 + f_5) - Z^0(f_2 + f_6) + Z^2(f_3 + f_7) \\ \hat{f}^7 &= Z^0(f_0 - f_4) - Z^3(f_1 - f_5) - Z^2(f_2 - f_6) - Z^1(f_3 - f_7) \\ \hat{f}^8 &= \hat{f}^0.\end{aligned}\tag{15}$$

## Butterfly calculations (I)

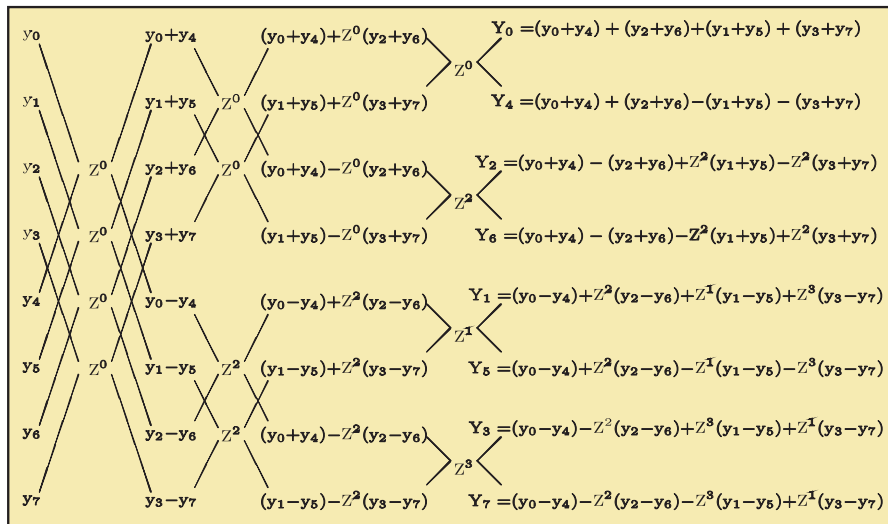
Now comes the real magic, and something that is used all over the place in fast, hardware-based calculations:

- notice the **repeating factors inside the parentheses**, they have the form  $f_p \pm f_q$ . These symmetries are systematized by introducing the **butterfly operation**.



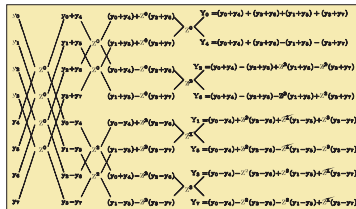
## Butterfly calculations (II)

With the mapping  $y \rightarrow f$ ,  $Y \rightarrow \hat{f}$ , this looks like a **network of complex additions and multiplications** for our  $N = 8$  FFT:



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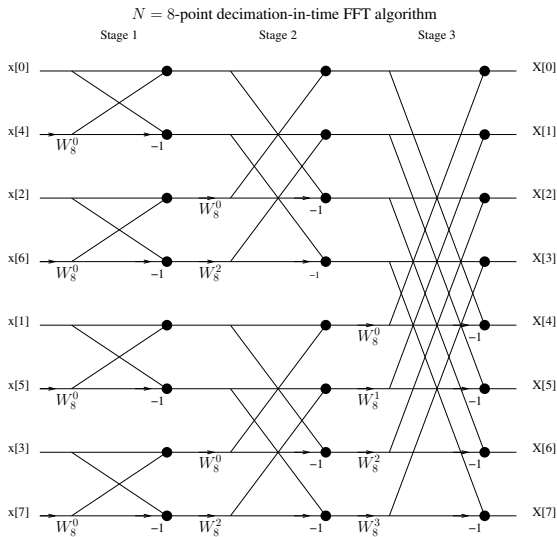


Notice how the number of multiplications of complex numbers has been reduced:

- For the first butterfly operation there are 8 multiplications by  $Z^0$
- For the second butterfly operation there are 8 multiplications
- A total of 24 multiplications is made in four butterfly operations

## Butterfly calculations (III)

This is often written in a slightly different form (**notice anything?**):



$$W_N = e^{-j\frac{2\pi}{N}}$$

## Danielson-Lanczos Lemma

The discrete Fourier transform of length  $N$  (where  $N$  is even) can be rewritten as the **sum of two discrete Fourier transforms**, each of length  $N/2$ , one for **even-numbered** points and the other for **odd-numbered** points.

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^N f_k Z_N^{nk} \quad (16)$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{N/2} f_{2k} Z_{N/2}^{nk} + Z_N^n \sum_{k=1}^{N/2} Z_{N/2}^{nk} f_{2k+1} \quad (17)$$

$$= \hat{f}_n^{\text{even}} + Z_N^n \hat{f}_n^{\text{odd}}, \quad (18)$$

In fact, this procedure can be **applied recursively** to break up the  $N/2$  even and odd points to their  $N/4$  even and odd points.

If  $N$  is a power of 2, this procedure breaks up the original transform into  $\ln N$  transforms of length 1.