Fourier Transforms: Discrete, Fast, and Practical

PHYS 250 (Autumn 2018) – Lecture 14

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Outline

Reminders from last time

We left off discussing details of our discrete Fourier transform and how we might speed it up.

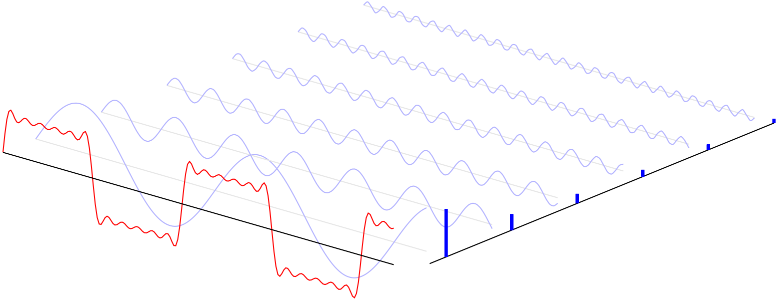
PDEs and Fourier Series

- **Fourier Series → Fourier Transforms**
  - We discussed how we can move to a continuous function definition of the expansion over a basis of functions
  - We then broke this down into discrete steps and obtained the **Discrete Fourier Transform**
- **Issues encountered:**
  - We realized that there is an issue related to the finite sampling of a function: **aliasing**
  - Began to break down the Fourier transform even further for a **fast** implementation

Today we will discuss the evolution towards the **FFT**, some of the practical limitations, and specific real-world (scientific and otherwise!) examples of using FFT's!

Square wave Fourier series

We already saw how we can break down a "simple" function into its components:



So let's figure out how to use this to its full capacity!

Outline

1. *Reminders*
   - Reminders from Lecture 13

2. *DFT to FFT*
   - Reminders of the DFT
   - Cooley-Tukey algorithm
   - Butterfly calculations
   - Danielson-Lanczos Lemma

Reminders of the Discrete Fourier Transform (DFT)

We determined the form of the **DFT** last time:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \quad \xrightarrow{DFT} \quad \hat{f}_n = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{N} f_k e^{-2\pi i k n / N} \tag{1}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega \quad \xrightarrow{DFT} \quad f_k = \frac{\sqrt{2\pi}}{N} \sum_{n=1}^{N} \hat{f}_n e^{-i\omega_n t} \tag{2}$$

This has certain drawbacks which we will discuss shortly, but it also has huge advantages. Namely, we can re-write this to see some amazing computational properties.

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{N} Z_N^{nk} f_k \qquad (Z_N = e^{-2\pi i/N}) \tag{3}$$

$$f_k = \frac{\sqrt{2\pi}}{N} \sum_{n=1}^{N} Z_N^{-nk} \hat{f}_n \qquad (n = 0, 1, \cdots, N) \tag{4}$$

*Moving to a **fast** DFT, aka Fast Fourier Transform*

As we mentioned last time, with this formulation, the computer needs to compute only powers of $Z \to Z_N^{nk}$.

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What if we can make this scales as $N \ln N$???

This may not seem like much of a difference, for $N = 10^{2-3}$, the difference of 10^{3-5} is the difference between a minute and a week.

This is what the FFT buys us!

Cooley-Tukey algorithm

Let's start with the simplified form of the DFT:

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{N} f_k e^{-2\pi i k n/N} = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{N} f_k Z_N^{nk} \tag{5}$$

First, **STOP AND LOOK**. **What are some features of this expression? What operations are required?**

- There are **imaginary components**
  - Even if the signal elements f_k to be transformed are real, Z_N is always complex, and therefore we must process both real and imaginary parts when computing transforms.
- We have to add and/or multiply N^2 times unless we break this down further
  - Both n and k range over N integer values, the $(Z_N^n)^k f_k$ multiplications require N^2 multiplications and additions of complex numbers.

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Example Cooley-Tukey algorithm (I)

The time savings comes from taking advantage of **periodicity**.

The values of the **computationally expensive complex factor** $Z^{nk} = ((Z)^n)^k$ **are repeated** as the integers n and k vary sequentially.

For instance, for $N = 8$ *(leaving out subscript and $1/\sqrt{2\pi}$, but note this is Z_8^{nk})*:

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$$\hat{f}^0 = Z^0 f_0 + Z^0 f_1 + Z^0 f_2 + Z^0 f_3 + Z^0 f_4 + Z^0 f_5 + Z^0 f_6 + Z^0 f_7$$

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\hat{f}^3 &= Z^0 f_0 + Z^3 f_1 + Z^6 f_2 + Z^9 f_3 + Z^{12} f_4 + Z^{15} f_5 + Z^{18} f_6 + Z^{21} f_7 \\
\hat{f}^4 &= Z^0 f_0 + Z^4 f_1 + Z^8 f_2 + Z^{12} f_3 + Z^{16} f_4 + Z^{20} f_5 + Z^{24} f_6 + Z^{28} f_7 \\
\hat{f}^5 &= Z^0 f_0 + Z^5 f_1 + Z^{10} f_2 + Z^{15} f_3 + Z^{20} f_4 + Z^{25} f_5 + Z^{30} f_6 + Z^{35} f_7 \\
\hat{f}^6 &= Z^0 f_0 + Z^6 f_1 + Z^{12} f_2 + Z^{18} f_3 + Z^{24} f_4 + Z^{30} f_5 + Z^{36} f_6 + Z^{42} f_7 \\
\hat{f}^7 &= Z^0 f_0 + Z^7 f_1 + Z^{14} f_2 + Z^{21} f_3 + Z^{28} f_4 + Z^{35} f_5 + Z^{42} f_6 + Z^{49} f_7
\end{aligned}$$

Example Cooley-Tukey algorithm (II)

There are actually only 4 independent values! Z^0, Z^1, Z^2, Z^3

$$Z^0 = \exp(0) = +1, \qquad Z^1 = \exp(-\frac{2\pi}{8}i) = +\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$Z^2 = \exp(-\frac{2\pi}{8}2i) = -i, \qquad Z^3 = \exp(-\frac{2\pi}{8}3i) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$Z^4 = \exp(-\frac{2\pi}{8}4i) = -Z^0, \qquad Z^5 = \exp(-\frac{2\pi}{8}5i) = -Z^1$$

$$Z^6 = \exp(-\frac{2\pi}{8}6i) = -Z^2, \qquad Z^7 = \exp(-\frac{2\pi}{8}7i) = -Z^3$$

$$Z^8 = \exp(-\frac{2\pi}{8}8i) = +Z^0, \qquad Z^9 = \exp(-\frac{2\pi}{8}9i) = +Z^1$$

$$Z^{10} = \exp(-\frac{2\pi}{8}10i) = +Z^2, \qquad Z^{11} = \exp(-\frac{2\pi}{8}11i) = +Z^3$$

$$Z^{12} = \exp(-\frac{2\pi}{8}11i) = -Z^0, \qquad \cdots$$

Example Cooley-Tukey algorithm (III)

We can now put these equations in an appropriate form for computing by regrouping the terms into sums and differences of the f's:

$$
\begin{aligned}
\hat{f}^0 &= Z^0(f_0+f_4)+Z^0(f_1+f_5)+Z^0(f_2+f_6)+Z^0(f_3+f_7)\\
\hat{f}^1 &= Z^0(f_0-f_4)+Z^1(f_1-f_5)+Z^2(f_2-f_6)+Z^3(f_3-f_7)\\
\hat{f}^2 &= Z^0(f_0+f_4)+Z^2(f_1+f_5)-Z^0(f_2+f_6)-Z^2(f_3+f_7)\\
\hat{f}^3 &= Z^0(f_0-f_4)+Z^3(f_1-f_5)-Z^2(f_2-f_6)+Z^1(f_3-f_7)\\
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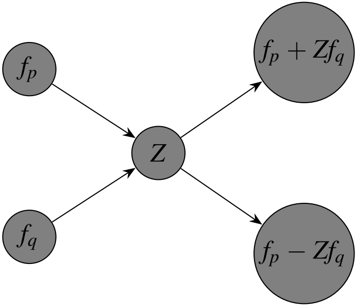
\hat{f}^7 &= Z^0(f_0-f_4)-Z^3(f_1-f_5)-Z^2(f_2-f_6)-Z^1(f_3-f_7)\\
\hat{f}^8 &= \hat{f}^0.
\end{aligned}
\tag{6}
$$

Butterfly calculations (I)

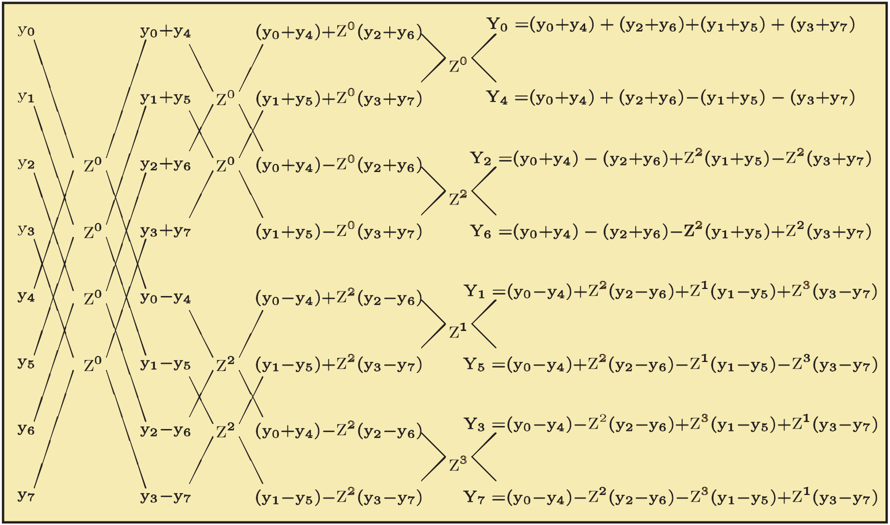
Now comes the real magic, and something that is used all over the place in fast, hardware-based calculations:

$\rightarrow$ notice the **repeating factors inside the parentheses**, they have the form $f_p \pm f_q$. These symmetries are systematized by introducing the **butterfly operation**.



Butterfly calculations (II)

With the mapping $y \to f$, $Y \to \hat{f}$, this looks like a **network of complex additions and multiplications** for our $N = 8$ FFT:



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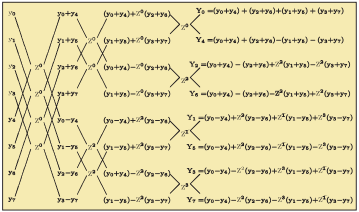
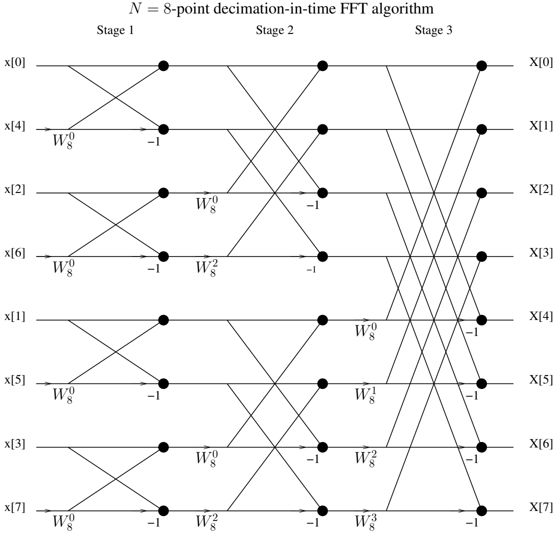


Notice how the number of multiplications of complex numbers has been reduced:

- For the first butterfly operation there are 8 multiplications by Z^0
- For the second butterfly operation there are 8 multiplications
- A total of 24 multiplications is made in four butterfly operations

Butterfly calculations (III)

This is often written in a slightly different form **(notice anything?)**:



$W_N = \mathrm{e}^{-\jmath\frac{2\pi}{N}}$

Danielson-Lanczos Lemma

The discrete Fourier transform of length N (where N is even) can be rewritten as the **sum of two discrete Fourier transforms**, each of length $N/2$, one for **even-numbered** points and the other for **odd-numbered** points.

$$\begin{aligned}
\hat{f}_n &= \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{N} f_k Z_N^{nk} && (7)\\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{N/2} f_{2k}\, Z_{N/2}^{nk} + Z_N^n \sum_{k=1}^{N/2} Z_{N/2}^{nk} f_{2k+1} && (8)\\
&= \hat{f}_n^{\text{even}} + Z_N^n \hat{f}_n^{\text{odd}}, && (9)
\end{aligned}$$

In fact, this procedure can be **applied recursively** to break up the $N/2$ even and odd points to their $N/4$ even and odd points.

If N is a power of 2, this procedure breaks up the original transform into $\ln N$ transforms of length 1.