Fourier Transforms!

PHYS 250 (Autumn 2019) – Lecture 13

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Outline

Reminders from last time

We discussed both Poisson's equation and Fourier Series in the last lecture.

PDEs and Fourier Series

- **PDEs: Poisson's Equation**
  - We discussed and then wrote a function for computing the solution to Poisson's equation
  - Specifically, we elaborated on how we could structure the (relatively simple) function in order to take various constraints (i.e. sources) for the solution into account more easliy
- **Fourier Series:**
  - Started discussing the basics of Fourier Series
  - Evaluated, computationally, the coefficients of a simple series for both a square and sawtooth wave

Today we will go much more in depth with Fourier Transforms and Analysis!
This will be a mixture of Python Notebooks and Lecture Slides

Outline

Square Wave

First, I want to do a little follow-up with the Fourier Series for the **square wave** function that we discussed last time.
Recall that we determined the coefficients of a sine and cosine expansion to be:

$$a_n = \frac{2}{n\pi} \sin\left(n\omega_0 \frac{\pi}{2}\right) \tag{1}$$

which yields a discrete function:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \tag{2}$$

Now open up the `Fourier-Transforms-Analysis.ipynb` jupyter notebook so that we can look at this in more detail!

Outline

Euler's Formula

We can generalize this by making use of Euler's Formula.

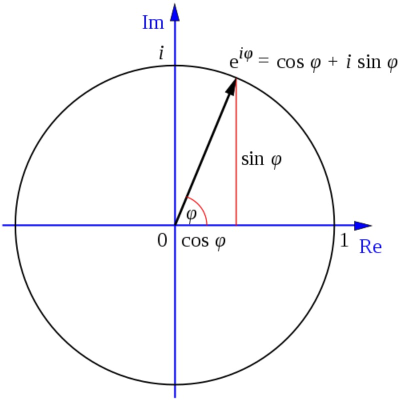
Euler's formula states that for any real number ϕ:

$$e^{i\phi} = \cos(\phi) + i\sin(\phi) \tag{3}$$

When $\phi = \pi$, Euler's formula evaluates to

$$e^{i\pi} + 1 = 0, \tag{4}$$

which is known as Euler's identity.



The implication is that it is possible to recover the amplitude of each wave in a Fourier series using an integral, which has many useful properties (in particular, that it's then continuous).

Fourier Transforms (I)

I will use the following definitions for the Fourier transform $\hat{f}(\xi)$ of a function $f(x)$, where x typically represents either a **spatial or time domain**, and ξ typically represents a corresponding inverse notion of **spatial or time frequency**.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx \tag{5}$$

The **inverse transform** is then obtained via

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi \tag{6}$$

In the case of spatial coordinates, x denotes length and ξ denotes inverse wavelength: $\xi = \frac{1}{\lambda}$. In the time domain, x denotes time and ξ denotes frequency. In the case that $x = t$ is in seconds, but ξ is **angular** frequency ω then a factor of 2π appears to get the normalization correct.

Fourier Transforms (II)

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$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \quad (7)$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega \quad (8)$$

Since $\omega = 2\pi\xi = \frac{2\pi}{\lambda}$.

The $\frac{1}{\sqrt{2\pi}}$ factor in both these integrals is a common normalization in quantum mechanics but maybe not in engineering where only a single $\frac{1}{2\pi}$ factor is often used.

Discrete Fourier Transforms (I)

If $\hat{f}(\omega)$ or $f(t)$ are known analytically or numerically, the Fourier transform integrals can be evaluated using the integration techniques studied earlier.

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In this way the **DFT can be thought of as a technique for interpolating, compressing, and extrapolating data**.

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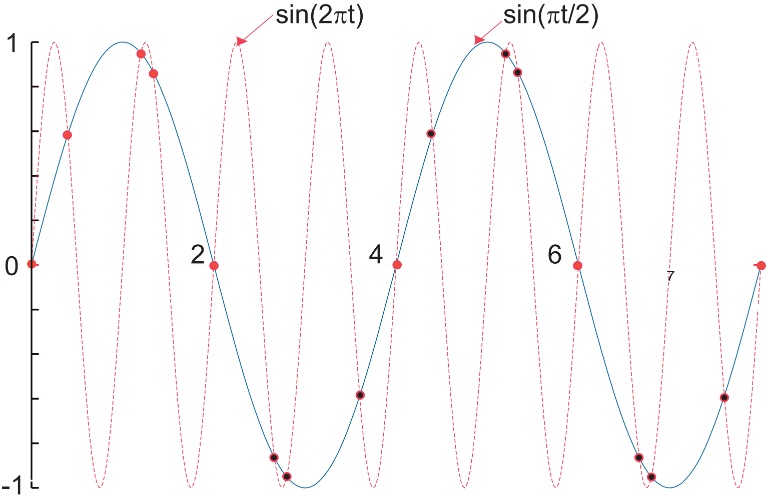

Discrete Fourier Transforms (II)

The DFT algorithm results from evaluating the integral not from 1 to $+1$ but rather from time 0 to time T over which the signal is measured, and from approximating the integration of the integral by computing a discrete sum:

$$\begin{aligned}
\hat{f}(\omega_n) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega_n t} dt && (9) \\
&\simeq \frac{1}{\sqrt{2\pi}} \int_{0}^{T} f(t) e^{-i\omega_n t} dt && (10) \\
&\simeq \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{N} h f(t_k) e^{-i\omega_n t_k} \qquad (h \equiv \text{stepsize}) && (11) \\
&\simeq \frac{h}{\sqrt{2\pi}} \sum_{k=1}^{N} f_k e^{-2\pi i k n/N} && (12) \\
\hat{f}_n \equiv \frac{\hat{f}(\omega_n)}{h} &= \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{N} f_k e^{-2\pi i k n/N} && (13)
\end{aligned}$$

Discrete Fourier Transforms (III)

We then need the inverse as well, which we can obtain with $d\omega \to 2\pi/Nh$ we invert the $\hat{f}_n$

$$f_k = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{N} \frac{2\pi}{Nh} \hat{f}_n e^{i\omega_n t} \tag{14}$$

Once we know the N values of the transform $\hat{f}_n$, we can use this expression to evaluate $f(t)$ for any time t. The frequencies ωn are determined by the number of samples taken and by the total sampling time $T = Nh$ as

$$\omega_n = n\frac{2\pi}{Nh} \tag{15}$$

Clearly, the larger we make the time $T = Nh$ over which we sample the function, the smaller will be the frequency steps or resolution. Accordingly,if you want a smooth frequency spectrum, you need to have a small frequency step $2\pi/T$.

Discrete Fourier Transforms (IV)

Lastly, we can simplify this expression to yield a clear computational approach:

$$f_k = \frac{\sqrt{2\pi}}{N} \sum_{n=1}^{N} Z^{-nk} \hat{f}_n \qquad (Z = e^{-2\pi i/N}) \tag{16}$$

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{N} Z^{nk} f_k \qquad (n = 0, 1, \cdots, N) \tag{17}$$

With this formulation, the computer needs to compute only powers of Z.