2

The Binomial Method

2.1 INTRODUCTION

THE binomial option pricing model was first introduced by Sharpe (1978) and in detail by Cox, Ross and Rubinstein (1979) (CRR). The primary practical use for the binomial model was and still is for pricing American-style options. Since it is never optimal to exercise early an American call option on an asset that does not pay dividends[1] they can be valued using the Black–Scholes formula. But, for American put options and American call options on assets that pay dividends, early exercise can be optimal depending on the level of the underlying asset. The price of these options has no closed-form solution, and so numerical procedures must be used to solve the Black–Scholes partial differential equation. The binomial model provides a simple and intuitive numerical method for valuing American style options and it can also be extended to price more complex options. We begin our discussion of the binomial model by looking at non-dividend paying assets.

2.2 A BINOMIAL MODEL FOR A NON-DIVIDEND PAYING ASSET

The binomial model assumes that the underlying asset price follows a binomial process, that is at any time the asset price can only change to one of two possible values. Under this assumption the asset price has a binomial distribution.

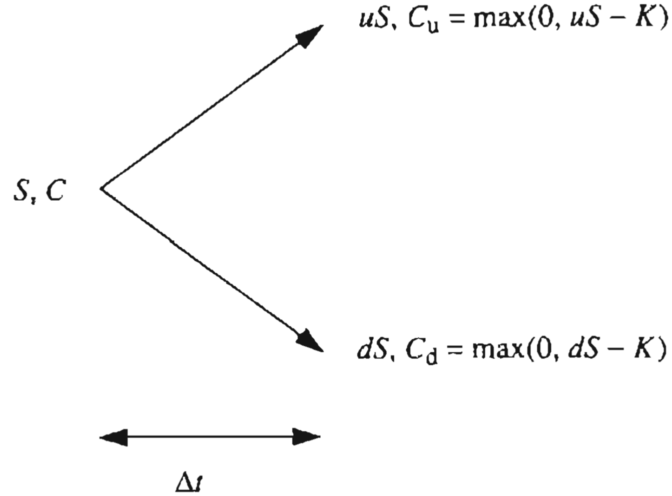
Consider an asset with a current price of S which follows a binomial process. That is, during a time period Δt the asset price can go up to uS or down to dS, we refer to this as the multiplicative binomial process. The parameters u and d determine the average behaviour and the volatility of the asset. Consider also a call option on this asset which matures at the end of the time period Δt, Figure 2.1 illustrates the asset prices and call option prices. These are the first two branches of a binomial tree starting from its root node which represents today and evolving out in time by one time step. In the same way as with the Black and Scholes model in Chapter 1, we can set up a riskless portfolio consisting of the underlying asset and the call option. Consider a long position of Δ units of the asset and a short position of one call option. We want the value of the portfolio to be the same regardless of whether the asset price goes up or down over the period Δt:

$$-C_{\mathrm{u}} + \Delta uS = -C_{\mathrm{d}} + \Delta dS \tag{2.1}$$

Rearranging we obtain

$$\Delta = \frac{C_{\mathrm{u}} - C_{\mathrm{d}}}{(u-d)S} \tag{2.2}$$

FIGURE 2.1 Binomial Model of an Asset Price and Call Option



Since this portfolio is riskless it must earn the riskless rate of interest r (continuously compounded)

$$(-C_u + \Delta uS) = e^{r\Delta t}(-C + \Delta S) \tag{2.3}$$

Substituting into equation (2.3) for ΔS, using equation (2.2) and rearranging for the call price at the start of the period C, we obtain

$$C = e^{-r\Delta t}\left(\frac{e^{r\Delta t} - d}{u - d}C_u + \frac{u - e^{r\Delta t}}{u - d}C_d\right) \tag{2.4}$$

Defining

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

we can rewrite the above equation in the simpler form:

$$C = e^{-r\Delta t}(pC_u + (1 - p)C_d) \tag{2.5}$$

This is the price of a call option with one period to maturity. In order to value a put we simply have to change the pay-off condition, that is, the values of C_u and C_d, to those for a put

$$C_u = \max(0, K - uS)$$
$$C_d = \max(0, K - dS)$$

Note that in the same way as for the Black–Scholes model the actual probabilities of the stock moving up or down are never used in deriving the option price — the option price is independent of the expected return of the stock. The option price is therefore independent of the risk preferences of investors which allows us to interpret p and $(1 - p)$ as risk-neutral probabilities. Equation (2.4) can therefore be interpreted as taking discounted expectations of future pay-offs under the risk-neutral probabilities. This gives us a very simple way of calculating the risk-neutral probabilities directly from the asset price, the return of which we can now assume is the riskless rate

$$uSp + dS(1 - p) = S e^{r\Delta t} \tag{2.6}$$

Rearranging leads, as before, to

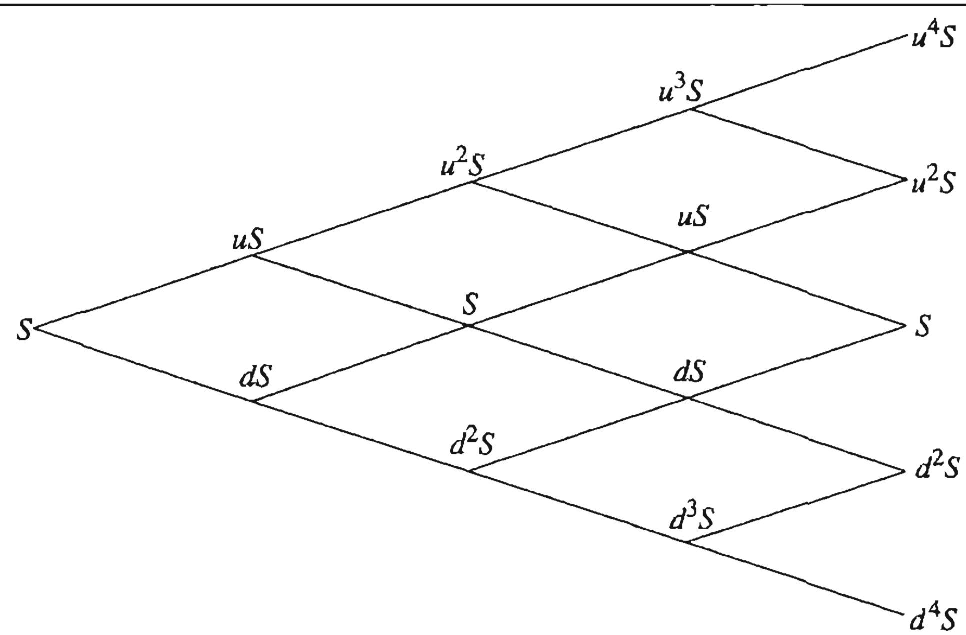
$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{2.7}$$

To price options with more than one period to go to maturity we extend the binomial tree outwards for the required number of periods to the maturity date of the option. For example, for an option which matures in four periods of time, Figure 2.2 illustrates the binomial tree we obtain.

We will refer to a state in the tree as a node and label the nodes (i, j), where i indicates the number of time steps from time zero and j indicates the number of upward movements the asset price has made since time zero. Therefore the level of the asset price at node (i, j) is $S_{i,j} = Su^j d^{i-j}$ and the option price will be $C_{i,j}$. Note in particular that $j = 0$ at the lowest node at every time step and in going from one node to another via a downward branch j remains the same since the number of upward moves which have occurred has not changed. In general we will assume we have N time steps in total, where the Nth time step corresponds to the maturity date of the option. As with the one period example, we note that the value of the option at the maturity date is known, it is simply the pay-off, for example for a call option

$$C_{N,j} = \max(0, S_{N,j} - K) \tag{2.8}$$

FIGURE 2.2 A Four-step Binomial Tree for an Asset



We have shown above that the value of the option at any node in the tree is its discounted expected future value, therefore at every node in the tree before maturity we have

$$C_{i,j} = e^{-r\Delta t}(pC_{i+1,j+1} + (1-p)C_{i+1,j}) \tag{2.9}$$

Using equations (2.8) and (2.9) we can compute the value of the option at every node at time step $N-1$. We can then reapply equation (2.9) at every node at every time step, working backwards through the tree, to compute the value of the option at every node in the tree. This procedure computes the value of the European option at every node in the tree. In order to compute the value of an American option we simply compare, at every node, the value of the option if exercised with the value if not exercised and set the option value at that node equal to the greater of the two. For example for an American put option we have

$$C_{i,j} = \max(e^{-r\Delta t}(pC_{i+1,j+1} + (1-p)C_{i+1,j}), K - S_{i,j}) \tag{2.10}$$

To illustrate the procedures, Figure 2.3 gives a pseudo-code implementation for the valuation of a European call in a multiplicative binomial tree.

Note firstly that we can precompute the one step discount factor (disc). Secondly, the asset prices at maturity can be efficiently computed because every node differs from the one below by a factor u/d. Also we do not have to store the entire binomial tree either for the asset price or the option price. We only need the asset prices at the final (maturity) time in order to evaluate the maturity condition. In stepping back through the tree we

FIGURE 2.3 Pseudo-code for Multiplicative Binomial Tree Valuation of a European Call

```
initialise_parameters { K, T, S, r, N, u, d }

{ precompute constants }

dt = T/N
p = (exp(r*dt)-d)/(u-d)
disc = exp(-r*dt)

{ initialise asset prices at maturity time step N }

St[0] = S*d^N
for j = 1 to N do St[j] = St[j-1]*u/d

{ initialise option values at maturity }

for j = 0 to N do C[j] = max( 0.0 , St[j] - K )

{ step back through the tree }

for i = (N-1) downto 0 do
  for j = 0 to i do
    C[j] = disc * ( p*C[j+1] + (1-p)*C[j] )

European_call = C[0]
```

can overwrite the previous time steps option values with the current values as they are computed.

Example : Multiplicative Binomial Tree Valuation of a European Call

We price a one-year maturity, at-the-money European call option with the current asset price at 100. The binomial tree has three time steps and up and down proportional jumps of 1.1 and 0.9091 respectively. The continuously compounded interest rate is assumed to be 6 per cent per annum, i.e. $K = 100$, $T = 1$, $S = 100$, $r = 0.06$, $N = 3$, $u = 1.1$, and $d = 1/u = 0.9091$. Figure 2.4 illustrates the numerical results, where nodes in the tree are represented by the boxes in which the upper value is the asset price and the lower value is the option price.

Firstly the constants; $\Delta t(dt)$, p, and $disc$ are precomputed:

$$\Delta t = \frac{T}{N} = \frac{1}{3} = 0.3333$$

$$p = \frac{e^{r\times\Delta t} - d}{u - d} = \frac{e^{0.06\times 0.3333} - 0.9091}{1.1 - 0.9091} = 0.5820$$

$$disc = e^{-r\times\Delta t} = e^{-0.06\times 0.3333} = 0.9802$$

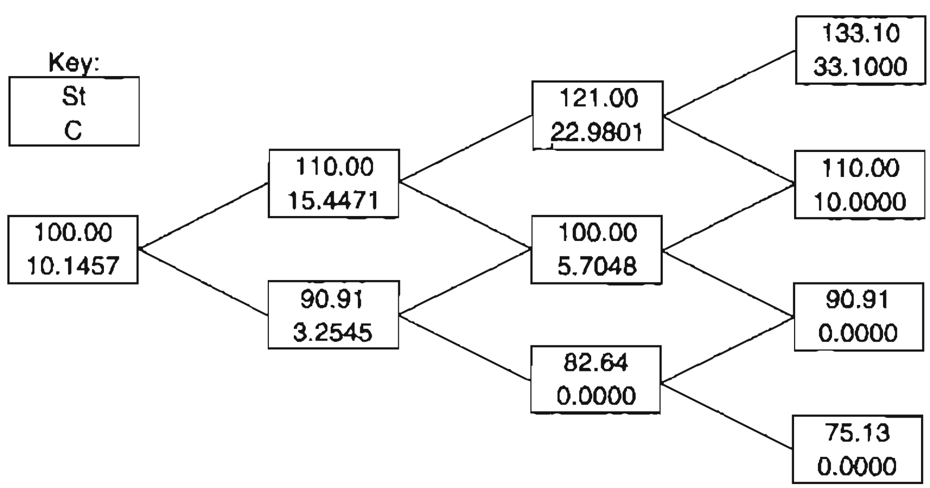
Then the asset prices at maturity are computed, for example the asset price at node (3, 0) is computed as

FIGURE 2.4 Multiplicative Binomial Tree Valuation of a European Call

K	T	S	r	N	u	d
100	1	100	0.06	3	1.1000	0.9091

dt	p	disc
0.3333	0.5820	0.9802

i	0	1	2	3
t	0	0.3333	0.6667	1



$$S_{3,0} = S \times d^N = 100 \times 0.9091^3 = 75.13$$

The other asset prices are computed from this, for example at node (3, 2) the asset is computed as

$$S_{3,2} = S_{3,1} \times \frac{u}{d} = 90.91 \times \frac{1.1}{0.9091} = 110.00$$

Next the option values at maturity are computed, for node (3, 2) we have

$$C_{3,2} = \max(0, S_{3,2} - K) = \max(0, 110.00 - 100.00) = 10.00$$

Finally we perform discounted expectations back through the tree. For node (2, 2) we have

$$\begin{aligned} C_{2,2} &= disc \times (p \times C_{3,3} + (1 - p) \times C_{3,2}) \\ &= 0.9802 \times (0.5820 \times 33.1000 + (1 - 0.5820) \times 10.0000) = 22.9801 \end{aligned}$$

Figure 2.5 presents the pseudo-code for the valuation of an American put by the multiplicative binomial model. The changes made from the code for the European call are in bold.

The first change from Figure 2.3 is to the pay-off, which becomes that for a put. Secondly, in order to apply the early exercise condition we must adjust the asset prices

FIGURE 2.5 Pseudo-code for Multiplicative Binomial Tree Valuation of an American Put

```
initialise_parameters { K, T, S, r, N, u, d }

{ precompute constants }

dt = T/N
p = (exp(r*dt)-d)/(u-d)
disc = exp(-r*dt)

{ initialise asset prices at maturity time step N }

St[0] = S*d^N
for j = 1 to N do St[j] = St[j-1]*u/d

{ initialise option values at maturity }

for j = 0 to N do C[j] = max( 0.0 , K - St[j] )

{ step back through the tree applying the
  early exercise condition }

for i = (N-1) downto 0 do
  for j = 0 to i do
    C[j] = disc * ( p*C[j+1] + (1-p)*C[j] )
    St[j] = St[j]/d
    C[j] = max( C[j] , K - St[j] )
  next j
next i

American_put = C[0]
```

so they apply to the current time step. Since the asset prices are indexed by the number of up jumps, as we step backwards, for each index level, the asset price must have had one less down jump. Finally, the early exercise condition is added.

Example : Multiplicative Binomial Tree Valuation of an American Put

We price a one-year maturity, at-the-money American put option with the current asset price at 100. The binomial tree has three time steps and up and down proportional jumps of 1.1 and 0.9091 respectively. The continuously compounded interest rate is assumed to be 6 per cent per annum, i.e. $K = 100$, $T = 1$, $S = 100$, $r = 0.06$, $N = 3$, $u = 1.1$ and $d = 1/u = 0.9091$. Figure 2.6 illustrates the numerical results, where nodes in the tree are represented by the boxes in which the upper value is the asset price and the lower value is the option price.

Firstly the constants; $\Delta t\,(dt)$, p, and $disc$ are precomputed:

$$\Delta t = \frac{T}{N} = \frac{1}{3} = 0.3333$$

$$p = \frac{e^{r\times\Delta t} - d}{u - d} = \frac{e^{0.06\times0.3333} - 0.9091}{1.1 - 0.9091} = 0.5820$$

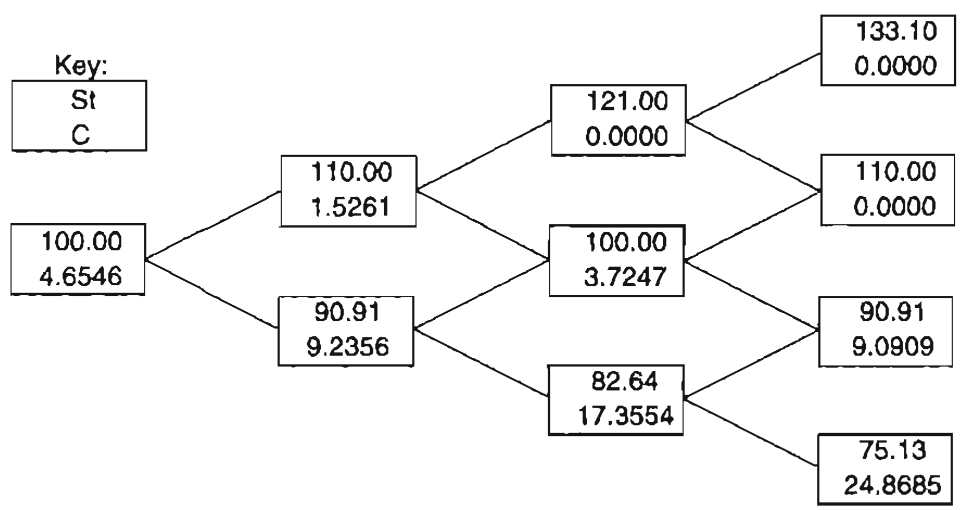
$$disc = e^{-r\times\Delta t} = e^{-0.06\times0.3333} = 0.9802$$

FIGURE 2.6 Multiplicative Binomial Tree Valuation of an American Put

K	T	S	r	N	u	d
100	1	100	0.06	3	1.1	0.9091

dt	p	disc
0.3333	0.5820	0.9802

i	0	1	2	3
t	0	0.3333	0.6667	1



Then the asset prices at maturity are computed, for example the asset price at node (3, 0) is computed as

$$S_{3,0} = S \times d^N = 100 \times 0.9091^3 = 75.13$$

The other asset prices are computed from this, for example at node (3, 2) the asset is computed as

$$S_{3,2} = S_{3,1} \times \frac{u}{d} = 90.91 \times \frac{11}{0.9091} = 110.00$$

Next the option values at maturity are computed, for node (3, 1) we have

$$C_{3,1} = \max(0, K - S_{3,1}) = \max(0, 100.00 - 90.91) = 9.0909$$

Finally we perform discounted expectations back through the tree. For node (2, 0) we have:

$$\begin{aligned} C_{2,0} &= disc \times (p \times C_{3,1} + (1 - p) \times C_{3,0}) \\ &= 0.9802 \times (0.5820 \times 9.0909 + (1 - 0.5820) \times 24.8685) = 15.3754 \end{aligned}$$

We then compute the asset price as

$$S_{2,0} = \frac{S_{3,0}}{d} = \frac{75.13}{0.9091} = 82.64$$

and apply the early exercise test:

$$C_{2,0} = \max(C_{2,0}, K - S_{2,0}) = \max(15.3754, 100 - 82.64) = 17.3554$$

We have not yet said anything about how to choose u and d to capture the volatility of the asset price and how to choose the time period Δt. In order to do this we will describe in the following section a more general approach to constructing binomial trees which will allow us to create more sophisticated trees later in this and subsequent chapters.

2.3 A GENERAL FORMULATION OF THE BINOMIAL MODEL

The binomial model is an approximation to the true behaviour of a real asset price. In Chapter 1 we described the two key properties of an asset price — its average behaviour or drift and its randomness or volatility. These properties can be captured by the mean and variance of changes in the asset price over a particular interval of time. Therefore, a reasonable way to proceed, in computing the parameters u and d and the probability p, is so that the mean and variance of the discrete binomial process match those of the risk-neutral process of the asset price over the time step of the binomial tree. As we stated in the introduction (section 2.1), the binomial model was originally constructed to price American-style options. It seems natural, therefore, to construct the binomial tree so as to be consistent with the Black–Scholes model for European options, and therefore to choose u, d and p to match the risk-neutral mean and variance of the GBM process:

$$dS = rS\,dt + \sigma S\,dz \tag{2.11}$$

This is the standard approach for choosing the binomial parameters. However, since there are three parameters and we are only trying to match two values we have a free choice for one of the parameters. Cox, Ross and Rubinstein (1979) (CRR) chose the probabilities to be one-half which leads to

$$
\begin{aligned}
u &= \exp\left(\left(r - \tfrac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\right) \\
d &= \exp\left(\left(r - \tfrac{1}{2}\sigma^2\right)\Delta t - \sigma\sqrt{\Delta t}\right) \qquad (2.12) \\
p &= \tfrac{1}{2}
\end{aligned}
$$

Jarrow and Rudd (1983) (JR) set the jump sizes to be equal, which leads to the probabilities being unequal:

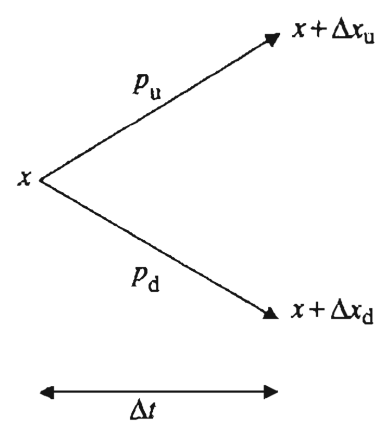
$$
\begin{aligned}
u &= \exp\left(\sigma\sqrt{\Delta t}\right) \\
d &= \exp\left(-\sigma\sqrt{\Delta t}\right) \qquad (2.13) \\
p &= \frac{1}{2} + \frac{r - \frac{1}{2}\sigma^2}{2\sigma}\sqrt{\Delta t}
\end{aligned}
$$

A problem with these formulations is that the approximation is only good over a small time interval, we cannot freely choose arbitrarily large time steps. In order to solve this problem and to obtain a more general and flexible formulation which is consistent with later chapters we reformulate the model in terms of the natural logarithm of the asset price ($x = \ln(S)$). The natural logarithm of the asset price under GBM is normally distributed with a constant mean and variance. Applying Itô's lemma the continuous time risk-neutral process for x can be shown to be

$$
\begin{aligned}
dx &= \nu\, dt + \sigma\, dz \\
\nu &= r - \tfrac{1}{2}\sigma^2
\end{aligned} \qquad (2.14)
$$

The discrete time binomial model for x is illustrated in Figure 2.7. The variable x can either go up to a level of $x + \Delta x_u$ with a probability of p_u or down to a level of

FIGURE 2.7 Binomial Model of the Natural Logarithm of an Asset



$x + \Delta x_d$ with a probability of $p_d = 1 - p_u$. We describe this as the additive binomial process. We now equate the mean and variance of the binomial process for x with the mean and variance of the continuous time process over the time interval Δt. This leads to the following equations:

$$\begin{aligned} E[\Delta x] &= p_u \Delta x_u + p_d \Delta x_d = \nu \Delta t \\ E[\Delta x^2] &= p_u \Delta x_u^2 + p_d \Delta x_d^2 = \sigma^2 \Delta t + \nu^2 \Delta t^2 \end{aligned} \tag{2.15}$$

We also have that $p_u + p_d = 1$ and therefore have three equations in four unknowns, or equivalently we can trivially substitute $p_d = 1 - p_u$ and obtain two equations in three unknowns. So, as we have already mentioned, we have a "free" choice for one of the parameters. The two obvious choices, analogous to those made by CRR and JR, are to set the probabilities to be equal to one-half or to set the jump sizes to be equal. Equal probabilities of one-half leads to the following:

$$\begin{aligned} \tfrac{1}{2}\Delta x_u + \tfrac{1}{2}\Delta x_d &= \nu \Delta t \\ \tfrac{1}{2}\Delta x_u^2 + \tfrac{1}{2}\Delta x_d^2 &= \sigma^2 \Delta t + \nu^2 \Delta t^2 \end{aligned} \tag{2.16}$$

which gives

$$\begin{aligned} \Delta x_u &= \tfrac{1}{2}\nu \Delta t + \tfrac{1}{2}\sqrt{4\sigma^2 \Delta t - 3\nu^2 \Delta t^2} \\ \Delta x_d &= \tfrac{3}{2}\nu \Delta t - \tfrac{1}{2}\sqrt{4\sigma^2 \Delta t - 3\nu^2 \Delta t^2} \end{aligned} \tag{2.17}$$

Equal jump sizes lead to

$$\begin{aligned} p_u(\Delta x) + p_d(-\Delta x) &= \nu \Delta t \\ p_u \Delta x^2 + p_d \Delta x^2 &= \sigma^2 \Delta t + \nu^2 \Delta t^2 \end{aligned} \tag{2.18}$$

which gives

$$\begin{aligned} \Delta x &= \sqrt{\sigma^2 \Delta t + \nu^2 \Delta t^2} \\ p_u &= \frac{1}{2} + \frac{1}{2}\frac{\nu \Delta t}{\Delta x} \end{aligned} \tag{2.19}$$

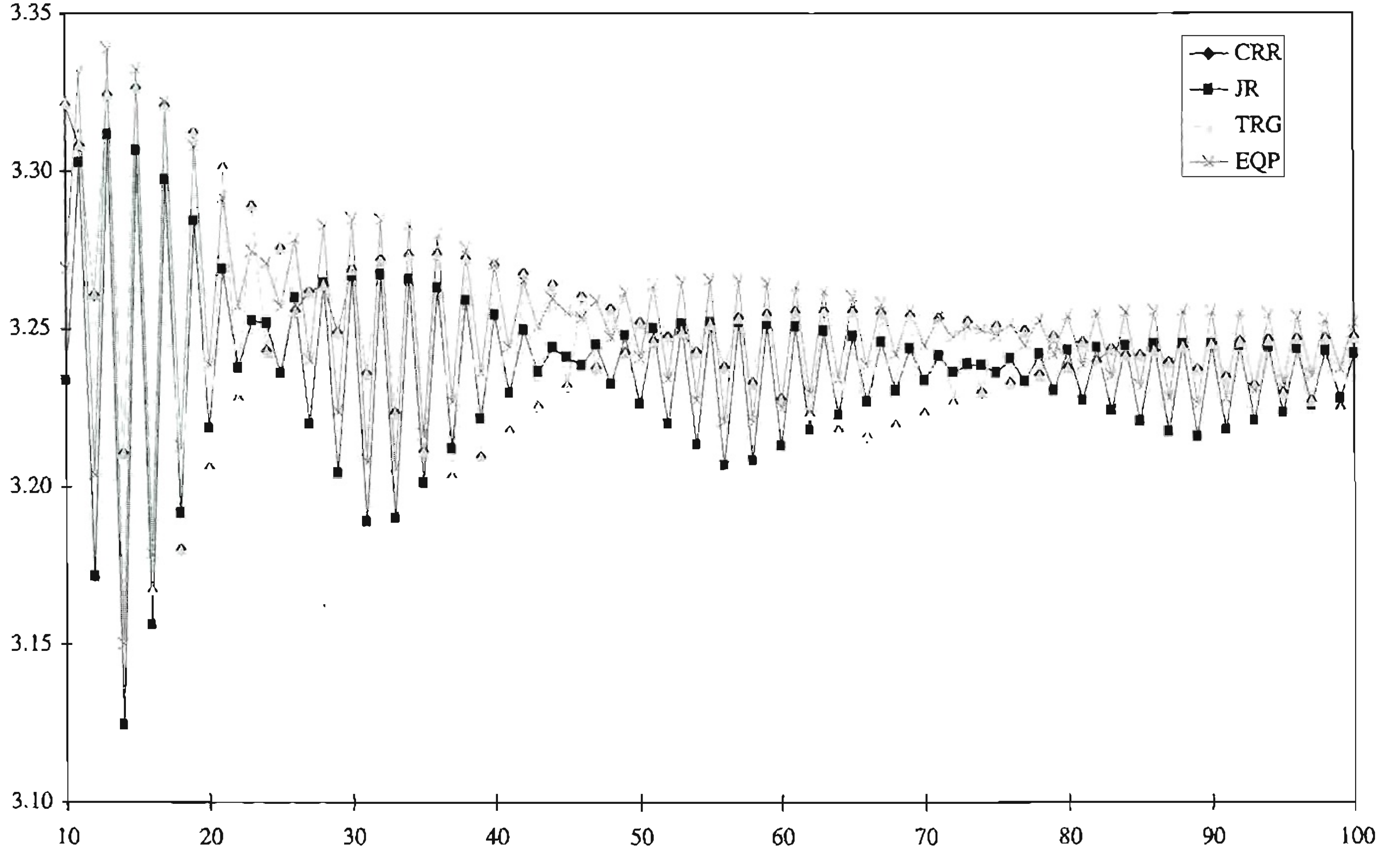
This latter solution was proposed by Trigeorgis (1992) (TRG) and on average has slightly better accuracy than the CRR, JR and the general formulation with equal probabilities (EQP). Figure 2.8 shows a comparison of the convergence of the four methods for typical choices of the parameters.

Figure 2.8 shows that the convergence behaviour of all the methods is quite complicated. It is also unsatisfactory in the sense that the error can actually increase with an increase in the number of time steps. The finite difference methods we describe in Chapter 3 solve this problem.

2.4 IMPLEMENTATION OF THE GENERAL BINOMIAL MODEL

The general additive binomial model has a similar structure to that of the multiplicative model in section 2.2. The nodes in the tree will be identified by a pair of indices (i, j),

FIGURE 2.8 The Price of an American Put Option Computed by the Binomial Method as a Function of the Number of Time Steps



$K = 90$, $T = 0.5$, $S = 100$, $\sigma = 0.2$, $r = 0.06$

$j = 0, 1, \ldots, i$ such that the node is i periods in the future and the asset has made j upwards moves to reach that node. Therefore the level of the asset price at node (i, j) is given by

$$S_{i,j} = \exp(x_{i,j}) = \exp(x + j\Delta x_{\mathrm{u}} + (i - j)\Delta x_{\mathrm{d}}) \tag{2.20}$$

The option price, as before, will be $C_{i,j}$ and again we will assume we have N time steps in total, where the Nth time step corresponds to the maturity date of the option. Figure 2.9 illustrates the structure of the general additive binomial tree.

Firstly, we illustrate the basic algorithm for valuation of a European call by the general additive binomial model. Figure 2.10 gives the pseudo-code implementation which is identical to the pseudo-code for the multiplicative model except for the initialisation of the parameters and asset price array.

Example : General Additive Binomial Tree Valuation of an European Call

We price a one-year maturity, at-the-money European call option with the current asset price at 100 and volatility of 20 per cent. The continuously compounded interest rate is assumed to be 6 per cent per annum and the binomial tree has three time steps, i.e. $K = 100$, $T = 1$, $S = 100$, $\sigma = 0.20$, $r = 0.06$, $N = 3$. Figure 2.11 illustrates the results of the calculations, where nodes in the tree are represented by the boxes in which the upper value is the asset price and the lower value is the option price.

FIGURE 2.9 The General Additive Binomial Tree

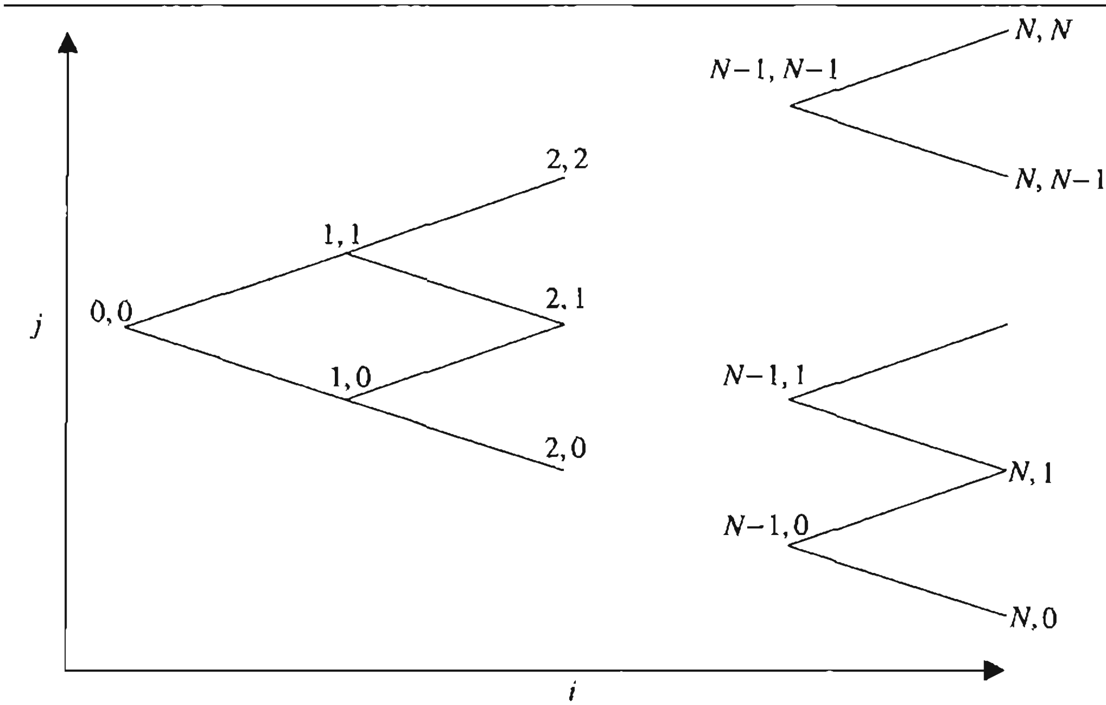


FIGURE 2.10 Pseudo-code for General Additive Binomial Valuation of a European Call

```
initialise_parameters { K, T, S, sig, r, N }

{ set coefficients - Trigeorgis }

dt = T/N
nu = r - 0.5*sig^2
dxu = sqrt( sig^2*dt + (nu*dt)^2 )
dxd = -dxu
pu = 1/2 + 1/2*( nu*dt/dxu )
pd = 1 - pu

{ precompute constants }

disc = exp(-r*dt)

{ initialise asset prices at maturity N }

St[0] = S*exp(N*dxd)
for j = 1 to N do St[j] = St[j-1]*exp( dxu-dxd )

{ initialise option values at maturity }

for j = 0 to N do C[j] = max( 0.0 , St[j] - K )

{ step back through the tree }

for i = (N-1) downto 0 do
  for j = 0 to i do
    C[j] = disc * ( pu*C[j+1] + pd*C[j] )

European_call = C[0]
```

Firstly the constants are precomputed (symbols in parentheses denote pseudo code equivalent): $\Delta t(dt)$, $\nu(nu)$, $\Delta x_u(dxu)$, $\Delta x_d(dxd)$, $p_u(pu)$, $p_d(pd)$, and *disc*:

$$\Delta t = \frac{T}{N} = \frac{1}{3} = 0.3333$$

$$\nu = r - \tfrac{1}{2}\sigma^2 = 0.06 - \tfrac{1}{2}0.20^2 = 0.0400$$

$$\Delta x_u = \sqrt{\sigma^2 \times \Delta t + (\nu \times \Delta t)^2} = \sqrt{0.20^2 \times 0.3333 + (0.0400 \times 0.3333)^2} = 0.1162$$

$$\Delta x_d = -\Delta x_u = -0.1162$$

$$p_u = \frac{1}{2} + \frac{1}{2}\left(\frac{\nu \times \Delta t}{\Delta x_u}\right) = \frac{1}{2} + \frac{1}{2}\left(\frac{0.040 \times 0.3333}{0.1162}\right) = 0.5574$$

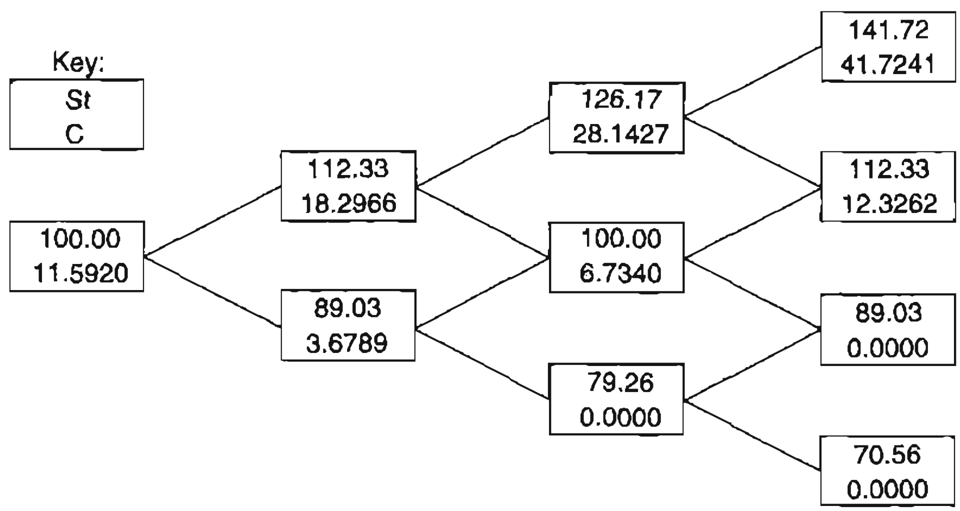
$$p_d = 1 - p_u = 1 - 0.5574 = 0.4426$$

$$disc = \exp(-r \times \Delta t) = \exp(-0.06 \times 0.3333) = 0.9802$$

FIGURE 2.11 General Additive Binomial Tree Valuation of an European Call

K	T	S	sig	rate	N	
100	1	100	0.2	0.06	3	
dt	**nu**	**dxu**	**dxd**	**pu**	**pd**	**disc**
0.3333	0.0400	0.1162	−0.1162	0.5574	0.4426	0.9802

i	0	1	2	3
t	0	0.3333	0.6667	1



Then the asset prices at maturity are computed, for example the asset price at node (3, 0) is computed as

$$S_{3,0} = S \times \exp(N \times \Delta x_d) = 100 \times e^{(3\times(-0.1162))} = 70.56$$

The other asset prices are computed from this, for example the asset price at node (3, 2) is computed as

$$S_{3,2} = S_{3,1} \times e^{(\Delta x_u - \Delta x_d)} = 89.03 \times e^{(0.1162-(-0.1162))} = 112.33$$

Next the option values at maturity are computed. For node (3, 2) we have

$$C_{3,2} = \max(0, S_{3,2} - K) = \max(0, 112.33 - 100) = 12.326$$

Finally we perform discounted expectations back through the tree. For node (2, 2) we have

$$\begin{aligned} C_{2,2} &= disc \times (p_u \times C_{3,3} + p_d \times C_{3,2}) \\ &= 0.9802 \times (0.5574 \times 41.7241 + 0.4426 \times 12.3262) = 28.1427 \end{aligned}$$

Figure 2.12 gives the pseudo-code algorithm for the valuation of an American put in the general additive binomial tree. Here we also introduce some simple optimisations

FIGURE 2.12 Pseudo-code for General Additive Binomial Valuation of an American Put

```
initialise_parameters { K, T, S, sig, r, N }

{ set coeffients - Trigeorgis }

dt = T/N
nu = r - 0.5*sig^2
dxu = sqrt( sig^2*dt + (nu*dt)^2 )
dxd = -dxu
pu = 1/2 + 1/2*( nu*dt/dxu )
pd = 1 - pu

{ precompute constants }

disc = exp(-r*dt)
dpu = disc*pu
dpd = disc*pd
edxud = exp( dxu - dxd )
edxd = exp( dxd )

{ initialise asset prices at maturity N }

St[0] = S*exp( N*dxd )
for j = 1 to N do St[j] = St[j-1]*edxud

{ initialise option values at maturity }

for j = 0 to N do C[j] = max( 0.0 , K - St[j] )

{ step back through the tree applying the
  early exercise condition }

for i = (N-1) downto 0 do
  for j = 0 to i do

    C[j] = dpd*C[j] + dpu*C[j+1]

    { adjust asset price to current time step }
    St[j] = St[j]/edxd

    { Apply the early exercise condition }
    C[j] = max( C[j] , K - St[j] )

  next j
next i

American_put = C[0]
```

to improve the efficiency. Firstly, the probabilities can be premultiplied by the discount factor (disc) saving one multiplication at every node. Secondly, by precomputing the proportional difference between the asset price levels (edxud) the asset price array can be computed using only one call to exp().

The changes from the corresponding European code are highlighted in bold. The calculation of the asset price array is more efficient, the pay-off condition is changed to that for a put, the asset price array is adjusted for the current time step, and the early exercise condition is applied.

Example : General Additive Binomial Tree Valuation of an American Put

We price a one-year maturity, at-the-money American put option with the current asset price at 100 and volatility of 20 per cent. The continuously compounded interest rate is assumed to be 6 per cent per annum and the binomial tree has three time steps, i.e. $K = 100$, $T = 1$, $S = 100$, $\sigma = 0.20$, $r = 0.06$, $N = 3$. Figure 2.13 illustrates the results of the calculations, where nodes in the tree are represented by the boxes in which the upper value is the asset price and the lower value is the option price.

Firstly the constants are precomputed: $\Delta t(dt)$, $\nu(nu)$, $\Delta x_{\mathrm{u}}(dxu)$, $\Delta x_{\mathrm{d}}(dxd)$, $p_{\mathrm{u}}(pu)$, $p_{\mathrm{d}}(pd)$, *disc*, *dpu*, *dpd*, *edxud*, *edxd*:

$$\Delta t = \frac{T}{N} = \frac{1}{3} = 0.3333$$

$$\nu = r - \tfrac{1}{2}\sigma^2 = 0.06 - \tfrac{1}{2}0.20^2 = 0.0400$$

$$\Delta x_{\mathrm{u}} = \sqrt{\sigma^2 \times \Delta t + (\nu \times \Delta t)^2)} = \sqrt{0.20^2 \times 0.3333 + (0.0400 \times 0.3333)^2} = 0.1162$$

$$\Delta x_{\mathrm{d}} = -\Delta x_{\mathrm{u}} = -0.1162$$

$$p_{\mathrm{u}} = \frac{1}{2} + \frac{1}{2}\left(\frac{\nu \times \Delta t}{\Delta x_{\mathrm{u}}}\right) = \frac{1}{2} + \frac{1}{2}\left(\frac{0.040 \times 0.3333}{0.1162}\right) = 0.5574$$

$$p_{\mathrm{d}} = 1 - p_{\mathrm{u}} = 1 - 0.5574 = 0.4426$$

$$dpu = disc \times p_{\mathrm{u}} = 0.9802 \times 0.5574 = 0.5463$$

$$dpd = disc \times p_{\mathrm{d}} = 0.9802 \times 0.4426 = 0.4339$$

$$edxud = e^{(\Delta x_{\mathrm{u}} - \Delta x_{\mathrm{d}})} = e^{(0.1162-(-0.1162))} = 1.2617$$

$$edxd = e^{\Delta x_{\mathrm{d}}} = e^{-0.1162} = 0.8903$$

Then the asset prices at maturity are computed, for example the asset price at node (3, 0) is computed as

$$S_{3,0} = S \times \exp(N \times \Delta x_{\mathrm{d}}) = 100 \times \exp(3 \times (-0.1162)) = 70.56$$

The other asset prices are computed from this, for example the asset price at node (3, 2) is computed as

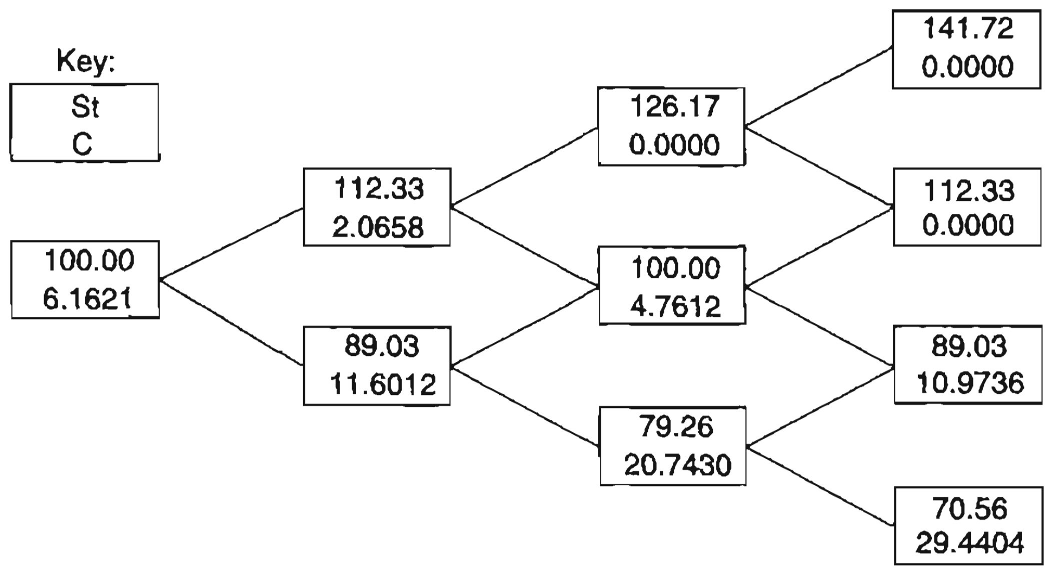
$$S_{3,2} = S_{3,1} \times e^{(\Delta x_{\mathrm{u}} - \Delta x_{\mathrm{d}})} = 89.03 \times e^{(0.1162-(-0.1162))} = 112.33$$

FIGURE 2.13 **General Additive Binomial Tree Valuation of an American Put**

K	T	S	sig	r	N
100	1	100	0.2	0.06	3

dt	nu	dxu	dxd	pu	pd	disc	dpu	dpd	edxud	edxd
0.3333	0.0400	0.1162	−0.1162	0.5574	0.4426	0.9802	0.5463	0.4339	1.2617	0.8903

i	0	1	2	3
t	0	0.3333	0.6667	1



Next the option values at maturity are computed. For node (3, 1) we have

$$C_{3,1} = \max(0, K - S_{3,1}) = \max(0, 100 - 89.03) = 10.974$$

Finally we perform discounted expectations back through the tree. For node (2, 0) we have:

$$\begin{aligned} C_{2,0} &= dpu \times C_{3,1} + dpd \times C_{3,0} \\ &= 0.5463 \times 10.9736 + 0.4339 \times 29.4404 = 18.7691 \end{aligned}$$

We then compute the asset price as

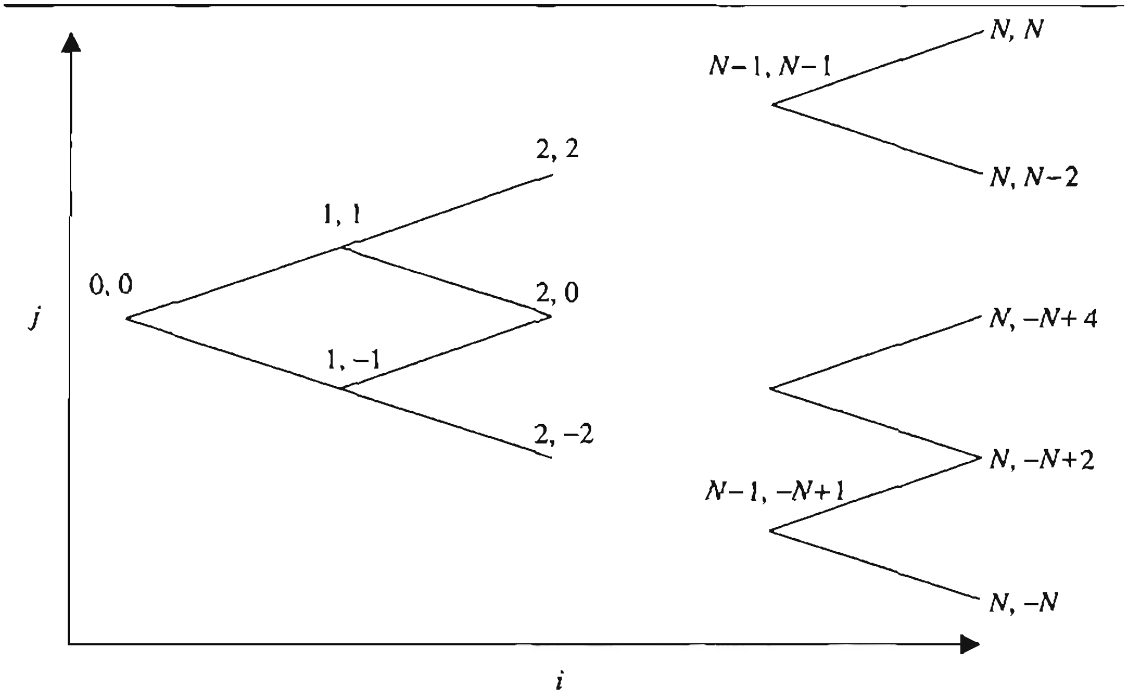
$$S_{2,0} = \frac{S_{3,0}}{edxd} = \frac{70.56}{0.8902} = 79.26$$

and apply the early exercise test:

$$C_{2,0} = \max(C_{2,0}, K - S_{2,0}) = \max(18.7691, 100 - 79.26) = 20.7430$$

For this general case where the up and down jumps may not be equal, we must adjust the asset price vector as we step backwards through the tree. This is necessary because we need the asset prices at each time step in order to apply the early exercise condition for the American option. If we restrict ourselves to equal up and down jumps in the asset price $\Delta x = \Delta x_u = \Delta x_d$ we can obtain a much more efficient algorithm. In order to achieve the more efficient implementation we change the structure of the binomial tree slightly; the new structure is illustrated in Figure 2.14.

FIGURE 2.14 The Equal Jumps General Additive Binomial Tree



The only change from Figure 2.9 is the way the levels of the asset price are indexed. The index j no longer refers to the number of up jumps to reach a particular asset price, instead it indicates the level of the asset price

$$S_{i,j} = \exp(x_{i,j}) = \exp(x + j\Delta x) \tag{2.21}$$

The pseudo-code algorithm for an American put is given in Figure 2.15 with the changes highlighted in bold.

The important point to note in Figure 2.15 is that the j index now steps by two at each time step. This is because j now indicates the level of the asset price rather than the number of upward branches which have occurred and between each pair of nodes at a given time step there is an asset level which is not considered (see Figure 2.14).

FIGURE 2.15 Pseudo-code for the Equal Jump General Additive Binomial Valuation of an American Put

```
initialise_parameters { K, T, S, sig, r, N }

{ set coeffients - Trigeorgis }

dt = T/N
nu = r - 0.5*sig^2
dx = sqrt( sig^2*dt + (nu*dt)^2 )
pu = 1/2 + 1/2*( nu*dt/dx )
pd = 1 - pu

{ precompute constants }

disc = exp(-r*dt)
dpu = disc*pu
dpd = disc*pd
edx = exp( dx )

{ initialise asset prices at maturity N }

St[-N] = S*exp( -N*dx )
for j = -N+1 to N do St[j] = St[j-1]*edx

{ initialise option values at maturity }

for j = -N to N step 2 do C[j] = max( 0.0 , K - St[j] )

{ step back through the tree applying early exercise }

for i = N-1 downto 0 do
  for j = -i to +i step 2 do
    C[j] = dpd*C[j] + dpu*C[j+1]
    C[j] = max( C[j] , K - St[j] )
  next j
next i

American_put = C[0]
```

Finally, the accuracy of the binomial method can be improved by using the Black-Scholes formula to compute the prices of the option at time step $N-1$, rather than using discounted expectations back from time step N with the pay-off values. However, this technique can make the binomial method significantly slower.

2.5 COMPUTING HEDGE SENSITIVITIES

As important as computing price of an option is computing the standard hedge sensitivities; *delta*, *gamma*, *vega*, *theta* and *rho*. The calculation of *delta*, *gamma* and *theta* is straightforward since they can be approximated by finite difference ratios in a binomial tree. One estimate of the *delta* of an option is given by[2]

$$delta = \frac{\partial C}{\partial S} \approx \frac{\Delta C}{\Delta S} = \frac{C_{1,1} - C_{1,0}}{S_{1,1} - S_{1,0}} \tag{2.22}$$

For example, using the American put numerical example illustrated in Figure 2.13, we have

$$delta = \frac{2.066 - 11.601}{112.33 - 89.03} = -0.40923$$

Notice that with this approximation the asset and the option prices are one time step into the future. This may be regarded as the *delta* one time step in the future. Similarly, an estimate of the *gamma* is

$$gamma = \frac{\partial^2 C}{\partial S^2} \approx \frac{[(C_{2,2} - C_{2,1})/(S_{2,2} - S_{2,1})] - [(C_{2,1} - C_{2,0})/(S_{2,1} - S_{2,0})]}{\frac{1}{2}(S_{2,2} - S_{2,0})} \tag{2.23}$$

For example, again using the numerical example illustrated in Figure 2.13, we have

$$\begin{aligned} gamma &= \frac{[(0.000 - 4.761)/(126.17 - 100.00)] - [(4.761 - 20.743)/(100.00 - 79.26)]}{\frac{1}{2}(126.17 - 79.26)} \\ &= 0.0250975 \end{aligned}$$

This can be regarded as an estimate two time steps into the future. To obtain estimates of *delta* and *gamma* at the current time we build the tree starting from two time steps back from today. Figure 2.16 illustrates the start of the tree, where the current value of the option is $C_{0,1}$.

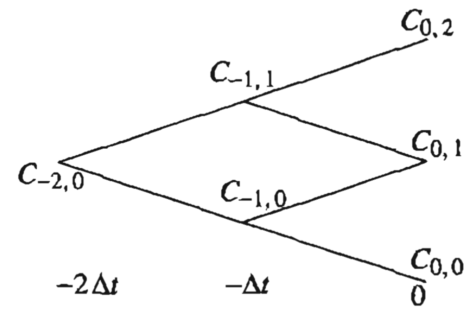
We do not actually have to build the tree starting two time steps back from today as this is inefficient. All we do is extend the tree at the upper and lower edges by one node. So in Figure 2.16 at time zero we add the nodes (0, 2) and (0, 0). An estimate of *delta* can now be obtained from

$$delta = \frac{C_{0,2} - C_{0,0}}{S_{0,2} - S_{0,0}} \tag{2.24}$$

and *gamma* by

$$gamma = \frac{\partial^2 C}{\partial S^2} \approx \frac{[(C_{0,2} - C_{0,1})/(S_{0,2} - S_{0,1})] - [(C_{0,1} - C_{0,0})/(S_{0,1} - S_{0,0})]}{\frac{1}{2}(S_{0,2} - S_{0,0})} \tag{2.25}$$

FIGURE 2.16 A Binomial Tree Starting Two Time Steps Back From Today



Vega and *rho* can be computed by re-evaluation of the price for small changes in the volatility and the interest rate respectively:

$$vega = \frac{\partial C}{\partial \sigma} \approx \frac{C(\sigma + \Delta\sigma) - C(\sigma - \Delta\sigma)}{2\Delta\sigma} \tag{2.26}$$

$$rho = \frac{\partial C}{\partial r} \approx \frac{C(r + \Delta r) - C(r - \Delta r)}{2\Delta r} \tag{2.27}$$

where, for example, $C(\sigma + \Delta\sigma)$ is the value computed using an initial volatility of $\sigma + \Delta\sigma$, where $\Delta\sigma$ is a small fraction of σ, e.g. $\Delta\sigma = 0.001\sigma$.

2.6 THE BINOMIAL MODEL FOR ASSETS PAYING A CONTINUOUS DIVIDEND YIELD

If the underlying asset pays a continuous dividend yield at a rate of δ per unit time then, as we saw in Chapter 1, in a Black–Scholes world the stochastic differential equation governing its evolution is

$$dS = (r - \delta)S\,dt + \sigma S\,dz \tag{2.28}$$

To a reasonable approximation assets that fit into this category include: options on (broad-based) stock indices, where δ represents the dividend yield on the index; options on foreign exchange rates, where δ represents the foreign exchange rate; options on futures contracts, where δ is equal to the risk-free rate ensuring that the contract has a zero risk-neutral drift; and options on commodities, where δ is interpreted as the convenience yield on the commodity.

It is straightforward to take into account a continuous dividend yield in the binomial model. We simply replace r by $r - \delta$ wherever it appears in the formulae for the probabilities and jump sizes. For example the general additive formulation equal jump size formulae become

$$\begin{aligned} \Delta x &= \sqrt{\sigma^2 \Delta t + \nu^2 \Delta t^2} \\ p_u &= \frac{1}{2} + \frac{1}{2}\frac{\nu \Delta t}{\Delta x} \\ \nu &= r - \delta - \tfrac{1}{2}\sigma^2 \end{aligned} \tag{2.29}$$

2.7 THE BINOMIAL MODEL WITH A KNOWN DISCRETE PROPORTIONAL DIVIDEND

If the asset pays a known discrete proportional dividend at a known time in the future then adjusting the binomial tree to take account of the dividend is again straightforward. Let $\hat{\delta}$ now be the known proportional dividend or the proportional amount by which the asset price decreases on the dividend date τ. Assume also that the dividend date corresponds to one of the dates in the binomial tree[3]. The binomial tree structure is shown in terms of the asset price in Figure 2.17.

If the time $i\Delta t$ is prior to the ex-dividend date then the nodes remain unchanged. If the time $i\Delta t$ is on or after the date on which the asset pays the dividend, then the value of the asset at node (i, j) becomes $S(1 - \hat{\delta})u^j d^{i-j}$, where

$$u = \exp(\Delta x_u),\ d = \exp(\Delta x_d)$$

Example : Additive Binomial Tree Valuation of an American Put with a Known Discrete Proportional Dividend

We price a one-year maturity, at-the-money American put option with the current asset price at 100 and volatility of 20 per cent. The continuously compounded interest rate is assumed to be 6 per cent per annum and the binomial tree has three time steps. The asset is assumed to pay a discrete proportional dividend of 3 per cent after eight months, i.e. $K = 100$, $T = 1$, $S = 100$, $\sigma = 0.20$, $r = 0.06$, $\hat{\delta}(\text{dvh}) = 0.03$, $\tau = 0.66667$, $N = 3$. Figure 2.18 illustrates the results of the calculations, where nodes in the tree are represented by the boxes in which the upper value is the asset price and the lower value is the option price.

FIGURE 2.17 Binomial Tree Adjusted for a Known Discrete Proportional Dividend

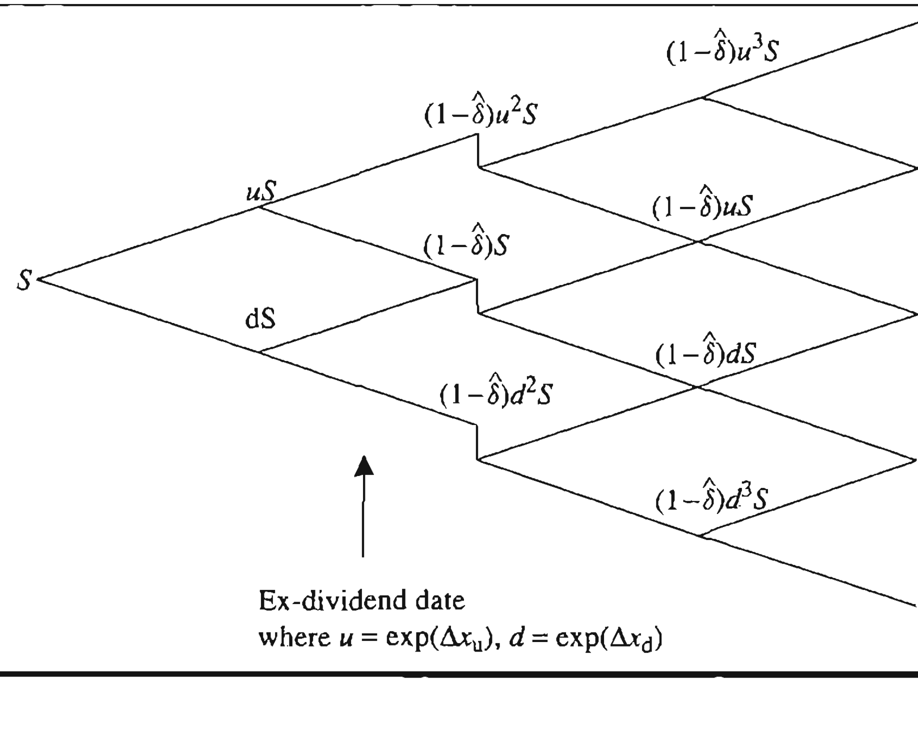
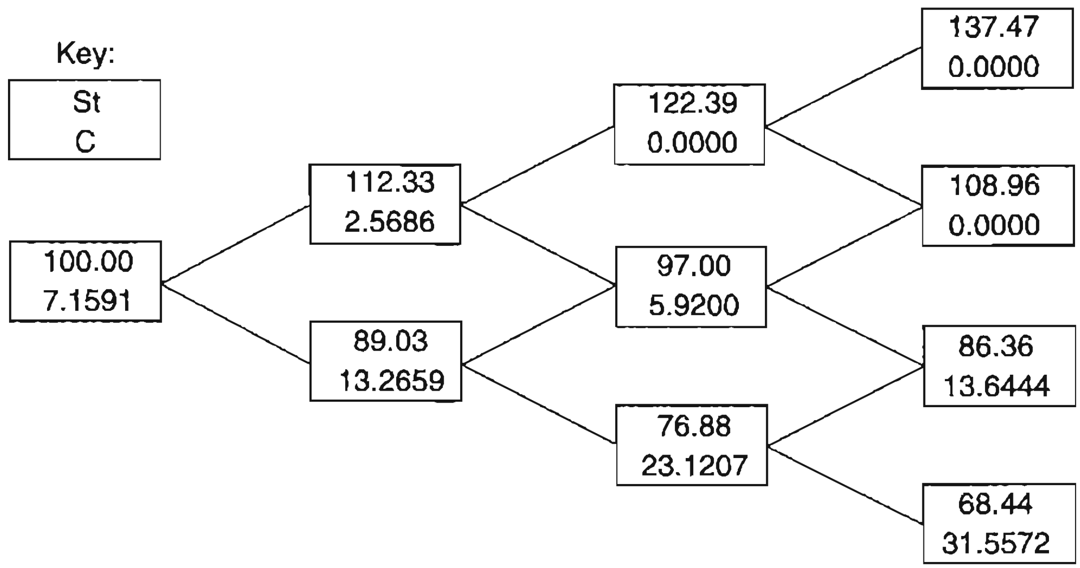


FIGURE 2.18 Additive Binomial Tree Valuation of an American Put with a Known Discrete Proportional Dividend

K	T	S	sig	r	dvh	tau	N
100	1	100	0.2	0.06	0.03	0.6667	3

dt	nu	dxu	dxd	pu	pd	disc	dpu	dpd	edxud	edxd
0.3333	0.0400	0.1162	–0.1162	0.5574	0.4426	0.9802	0.5463	0.4339	1.2617	0.8903

i	0	1	2	3
t	0	0.3333	0.6667	1



Firstly the constants are precomputed; $\Delta t(dt)$, $\nu(nu)$, $\Delta x_u(dxu)$, $\Delta x_d(dxd)$, $p_u(pu)$, $p_d(pd)$, *disc*, *dpu*, *dpd*, *edxud*, *edxd*:

$$\Delta t = \frac{T}{N} = \frac{1}{3} = 0.3333$$

$$\nu = r - \tfrac{1}{2}\sigma^2 = 0.06 - \tfrac{1}{2}0.20^2 = 0.0400$$

$$\Delta x_u = \sqrt{\sigma^2 \times \Delta t + (\nu \times \Delta t)^2)} = \sqrt{0.20^2 \times 0.3333 + (0.0400 \times 0.3333)^2} = 0.1162$$

$$\Delta x_d = -\Delta x_u = -0.1162$$

$$p_u = \frac{1}{2} + \frac{1}{2}\left(\frac{\nu \times \Delta t}{\Delta x_u}\right) = \frac{1}{2} + \frac{1}{2}\left(\frac{0.040 \times 0.3333}{0.1162}\right) = 0.5574$$

$$p_d = 1 - p_u = 1 - 0.5574 = 0.4426$$

$$dpu = disc \times p_u = 0.9802 \times 0.5574 = 0.5463$$

$$dpd = disc \times p_d = 0.9802 \times 0.4426 = 0.4339$$

$$edxud = e^{(\Delta x_u - \Delta x_d)} = e^{(0.1162-(-0.1162))} = 1.2617$$

$$edxd = e^{\Delta x_d} = e^{-0.1162} = 0.8903$$

Then the asset prices at maturity are computed. The asset price at node (3, 0) is computed as

$$S_{3,0} = S \times (1 - \hat{\delta})e^{N \times \Delta x_d} = 100 \times (1 - 0.03)e^{3\times(-0.1162)} = 68.44$$

The other asset prices are computed from this value. Consider node (3, 2) the asset price is computed as

$$S_{3,2} = S_{3,1} \times edxud = 86.36 \times 1.2617 = 108.96$$

Next the option values at maturity are computed. For node (3, 1) we have

$$C_{3,1} = \max(0, K - S_{3,1}) = \max(0, 100 - 86.36) = 13.6444$$

Finally, perform discounted expectations back through the tree applying the early exercise test. For node (2, 0) we have

$$C_{2,0} = dpu \times C_{3,1} + dpd \times C_{3,0} = 0.5463 \times 13.6444 + 0.4339 \times 31.5572 = 21.1466$$

We then compute the asset price as

$$S_{2,0} = \frac{S_{3,0}}{edxd} = \frac{68.44}{0.8903} = 76.88$$

and apply the early exercise test:

$$C_{2,0} = \max(C_{2,0}, K - S_{2,0}) = \max(21.1466, 100 - 76.88) = 23.1207$$

For node (1, 0) we have

$$C_{1,0} = dpu \times C_{2,1} + dpd \times C_{2,0} = 0.5463 \times 5.9200 + 0.4339 \times 23.1207 = 13.2659$$

We then compute the asset price as

$$S_{1,0} = \frac{S_{2,0}}{edxd \times (1-\hat{\delta})} = \frac{76.88}{0.8903 \times (1-0.03)} = 89.03$$

and apply the early exercise test:

$$C_{1,0} = \max(C_{1,0}, K - S_{1,0}) = \max(13.2659, 100 - 89.03) = 13.2659$$

2.8 THE BINOMIAL MODEL WITH A KNOWN DISCRETE CASH DIVIDEND

If the asset pays a known cash dividend then the situation is more difficult as the binomial tree becomes non-recombining for nodes after the ex-dividend date. Suppose that the asset pays a dividend as a cash amount D at a time τ such that $k\Delta t < \tau < (k+1)\Delta t$. Figure 2.19 illustrates the structure of the binomial tree.

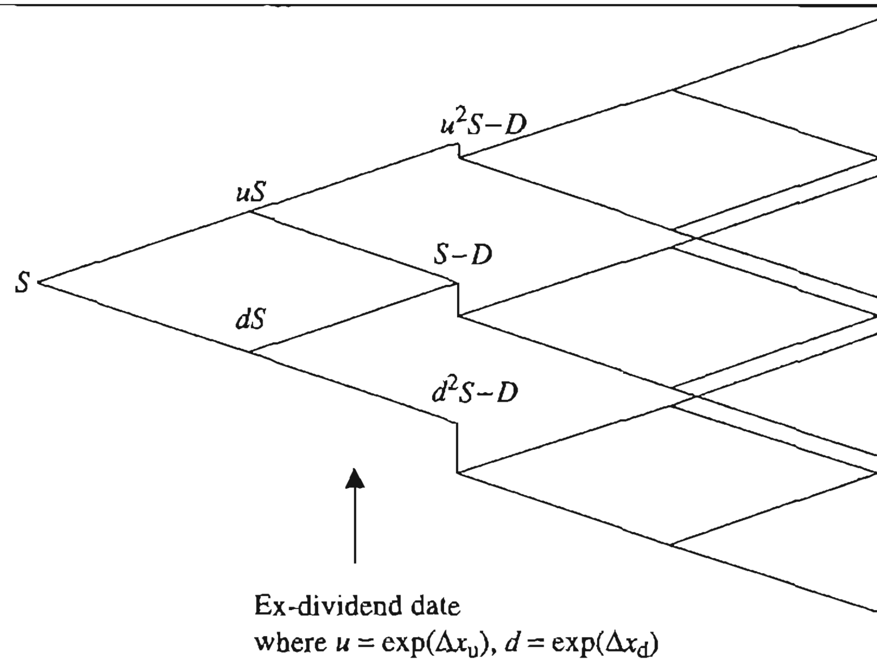
If the time $i\Delta t$ is prior to the dividend date then the nodes remain unchanged. If the time $i\Delta t$ is on or after the date on which the asset pays the dividend then the value of the asset at node (i, j) becomes

$$S\exp(\Delta x_{\mathrm{u}})^{j}\exp(\Delta x_{\mathrm{d}})^{i-j} - D$$

Therefore at time $(k+m)\Delta t$ there are $m(k+1)$ nodes rather than $k+m+1$.

We can overcome this problem and obtain a recombining tree by making a particular assumption about the volatility of the asset price. Suppose that the asset price, S_t, has two components, a part that is uncertain, $\tilde{S}_t$, and a certain part that is the present value

FIGURE 2.19 Binomial Tree Adjusted for a Known Discrete Cash Dividend



of the future dividend stream. The value of $\tilde{S}_t$ is given by

$$\tilde{S}_t = S_t \quad \text{when } t > \tau$$

and

$$\tilde{S}_t = S_t - D\,e^{-r(\tau-t)} \quad \text{when } t \leq \tau$$

Define $\tilde{\sigma}$ as the volatility of $\tilde{S}_t$ and assume it is constant. The binomial tree parameters p_u, p_d, Δx_u, Δx_d are calculated in the usual way but with σ replaced by $\tilde{\sigma}$. The binomial tree is then constructed in the same way as before. The value of the asset is given by

$$\tilde{S}_t \exp(\Delta x_u)^j \exp(\Delta x_d)^{i-j} + D\,e^{-r(\tau-t)}$$

when $t = i\Delta t < \tau$ and

$$\tilde{S}_t \exp(\Delta x_u)^j \exp(\Delta x_d)^{i-j}$$

when $t = i\Delta t \geq \tau$.

Example : Additive Binomial Tree Valuation of an American Put with a Known Discrete Cash Dividend

We price a one-year maturity, at-the-money American put option with the current asset price at 100 and volatility of 20 per cent. The continuously compounded interest rate is assumed to be 6 per cent per annum and the binomial tree has three time steps. The asset is assumed to pay a discrete cash dividend of 3 after six months i.e. $K = 100$, $T = 1$, $S = 100$, $\sigma = 0.20$, $r = 0.06$, $N = 3$, $D(\text{Div}) = 3$, $\tau = 0.5$. Figure 2.20 illustrates the results of the calculations, where nodes in the tree are represented by the boxes in which the upper value is the asset price and the lower value is the option price.

Firstly the constants are precomputed: $\Delta t(dt)$, $\nu(nu)$, $\Delta x_u(dxu)$, $\Delta x_d(dxd)$, $p_u(pu)$, $p_d(pd)$, *disc*, *dpu*, *dpd*, *edxud*, *edxd*:

$$\Delta t = \frac{T}{N} = \frac{1}{3} = 0.3333$$

$$\nu = r - \tfrac{1}{2}\sigma^2 = 0.06 - \tfrac{1}{2}0.20^2 = 0.0400$$

$$\Delta x_u = \sqrt{\sigma^2 \times \Delta t + (\nu \times \Delta t)^2} = \sqrt{0.20^2 \times 0.3333 + (0.0400 \times 0.3333)^2} = 0.1162$$

$$\Delta x_d = -\Delta x_u = -0.1162$$

$$p_u = \frac{1}{2} + \frac{1}{2}\left(\frac{\nu \times \Delta t}{\Delta x_u}\right) = \frac{1}{2} + \frac{1}{2}\left(\frac{0.040 \times 0.3333}{0.1162}\right) = 0.5574$$

$$p_d = 1 - p_u = 1 - 0.5574 = 0.4426$$

$$dpu = disc \times p_u = 0.9802 \times 0.5574 = 0.5463$$

$$dpd = disc \times p_d = 0.9802 \times 0.4426 = 0.4339$$

$$edxud = e^{(\Delta x_u - \Delta x_d)} = e^{(0.1162-(-0.1162))} = 1.2617$$

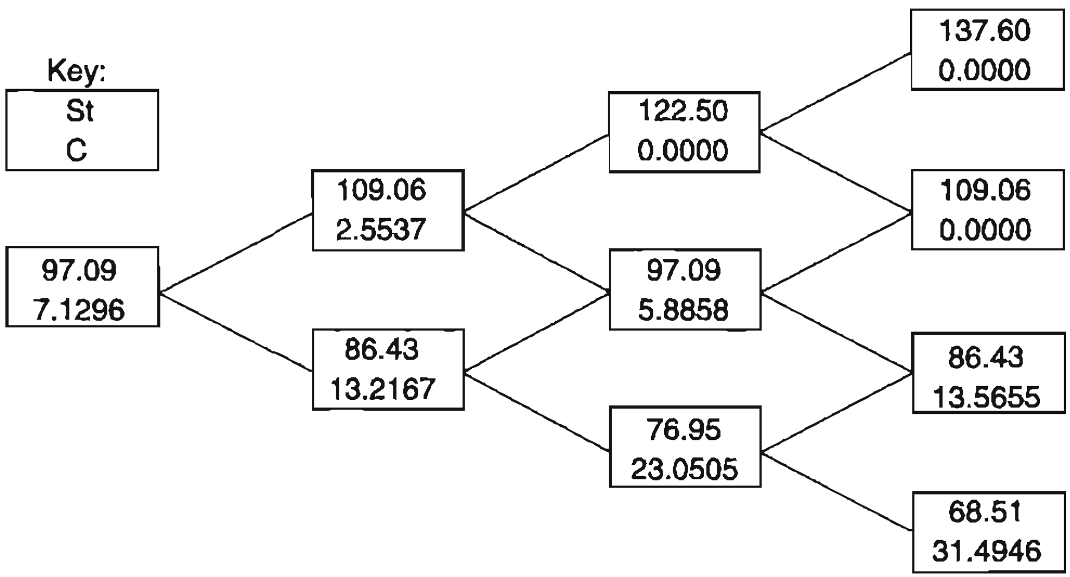
$$edxd = e^{\Delta x_d} = e^{-0.1162} = 0.8903$$

FIGURE 2.20 Additive Binomial Tree Valuation of an American Put with a Known Discrete Cash Dividend

K	T	S	sig	r	Div	tau	N
100	1	100	0.2	0.06	3.00	0.5	3

dt	nu	dxu	dxd	pu	pd	disc	dpu	dpd	edxud	edxd
0.3333	0.0400	0.1162	−0.1162	0.5574	0.4426	0.9802	0.5463	0.4339	1.2617	0.8903

i	0	1	2	3
t	0	0.3333	0.6667	1



Then the asset prices at maturity are computed. The asset price at node (3, 0) is computed as

$$S_{3,0} = (S - D \times e^{-r \times \tau})e^{N \times \Delta x_d} = (100 - 3.0 \times e^{-0.06 \times 0.5})e^{3 \times (-0.1162)} = 68.51$$

where $(S - D \times e^{-r \times \tau})$ is the current value of $\tilde{S}$. The other asset prices are computed from this value, the node (3,2) asset price is computed as

$$S_{3,2} = S_{3,1} \times edxud = 86.43 \times 1.2617 = 109.06$$

Next the option values at maturity are computed. For node (3, 1) we have:

$$C_{3,1} = \max(0, K - S_{3,1}) = \max(0, 100 - 86.43) = 13.5655$$

Finally we perform discounted expectations back through the tree applying the early exercise test. For node (2, 0) we have

$$C_{2,0} = dpu \times C_{3,1} + dpd \times C_{3,0} = 0.5463 \times 13.5655 + 0.4339 \times 31.4946 = 21.0763$$

We then compute the asset price as

$$S_{2,0} = \frac{S_{3,0}}{edxd} = \frac{68.51}{0.8903} = 76.95$$

and apply the early exercise test:

$$C_{2,0} = \max(C_{2,0}, K - S_{2,0}) = \max(21.0763, 100 - 76.95) = 23.0505$$

For node (1, 0) we have

$$C_{1,0} = dpu \times C_{2,1} + dpd \times C_{2,0} = 0.5463 \times 5.8858 + 0.4339 \times 23.0505 = 13.2167$$

We then compute the asset price as

$$S_{1,0} = \frac{S_{2,0}}{edxd} + D \times e^{-r(\tau - t)} = \frac{76.95}{0.8903} + 3.00 \times e^{-0.06(0.5-0.3333)} = 89.40$$

and apply the early exercise test:

$$C_{1,0} = \max(C_{1,0}, K - S_{1,0}) = \max(13.2167, 100 - 89.40) = 13.2167$$

2.9 ADAPTING THE BINOMIAL MODEL TO TIME VARYING VOLATILITY

A common requirement in practice is the incorporation of time varying volatility (and to a lesser extent interest rates) into a model. This arises for example when implied volatilities are obtained from the market prices of options. In order to price other options consistently with the market the model must be consistent with these observed volatilities. This can be achieved with the binomial model (although we will see in Chapter 3 that it is simpler to achieve with trinomial trees).

The simplest and most robust way to adapt the binomial model to time-varying volatility is to fix the space step and vary the probabilities and time step. This ensures that the binomial tree recombines. If we fix the space step at Δx and have a time-varying volatility σ_i and interest rate r_i for time step i, which leads to time varying probabilities p_i, time step Δt_i, and risk-neutral drift ν_i, equations (2.15) become

$$\begin{aligned} p_i \Delta x - (1 - p_i)\Delta x &= \nu_i \Delta t_i \\ p_i \Delta x^2 + (1 - p_i)\Delta x^2 &= \sigma_i^2 \Delta t_i + \nu_i^2 \Delta t_i^2 \end{aligned} \tag{2.30}$$

which leads to

$$\begin{aligned} \Delta t_i &= \frac{1}{2\nu_i^2}\left(-\sigma_i^2 \pm \sqrt{\sigma_i^4 + 4\nu_i^2 \Delta x^2}\right) \\ p_i &= \frac{1}{2} + \frac{\nu_i \Delta t_i}{2\Delta x} \end{aligned} \tag{2.31}$$

If Δx is set according to

$$\Delta x = \sqrt{\overline{\sigma}^2 \overline{\Delta t} + \overline{\nu}^2 \overline{\Delta t}^2} \tag{2.32}$$

where

$$\overline{\sigma} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i \quad \text{and} \quad \overline{\nu} = \frac{1}{N}\sum_{i=1}^{N} \nu_i$$

then $\overline{\Delta t}$ will be approximately the average time step which is obtained when the tree is built. Figure 2.21 illustrates the typical parameters obtained and Figure 2.22 illustrates the binomial tree obtained using this method.

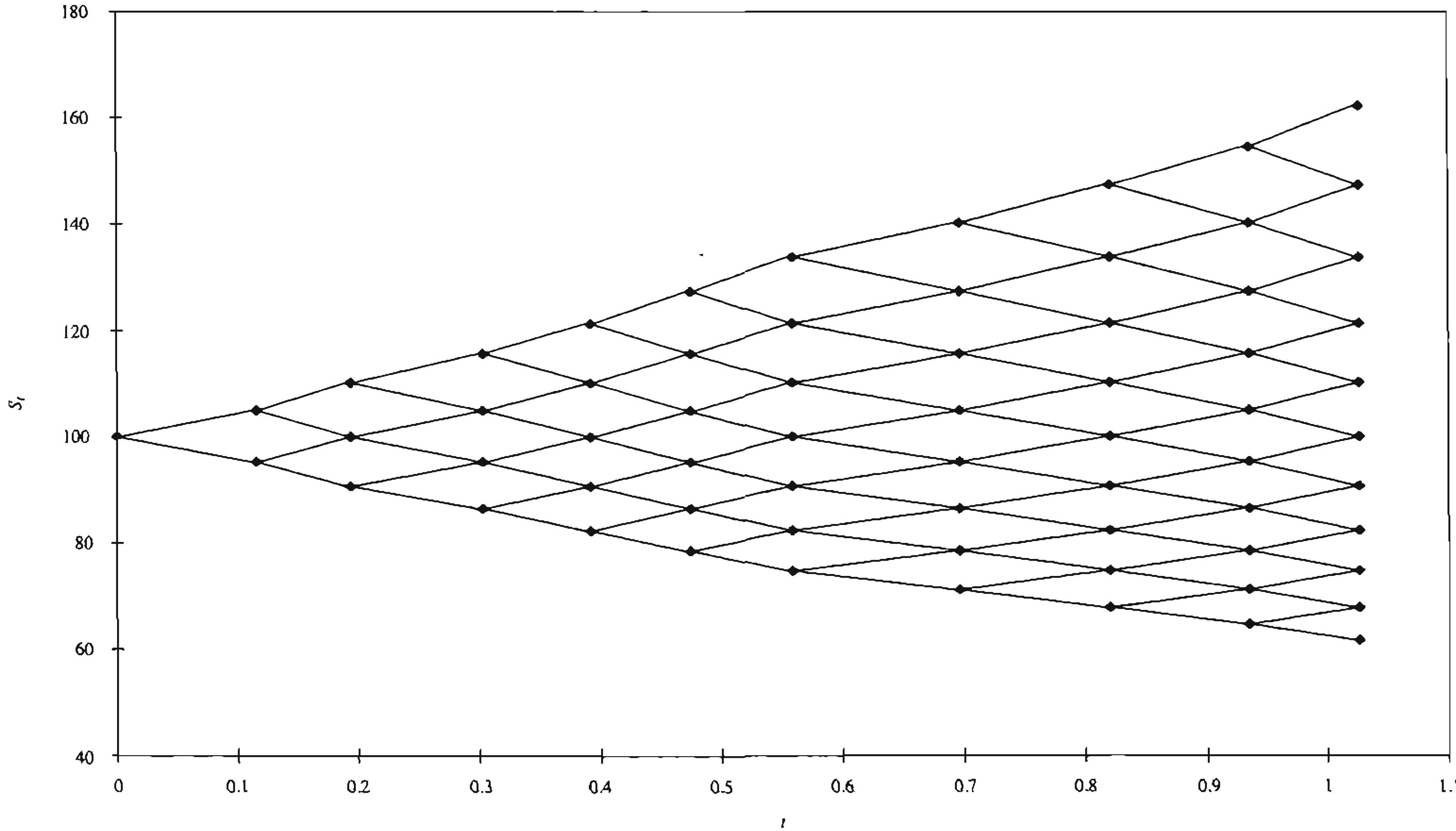
The input parameters are the interest rate r_i which varies between approximately 4 and 6 per cent and the volatility σ_i which varies between approximately 14 and 17 per cent over the total time period which is approximately one year.

The problem with this method is that the time steps are now constrained by the interest rates and volatilities and we are no longer free to choose them to correspond, for example, to option exercise dates or cashflow dates. In Figure 2.22 the time step varies between

FIGURE 2.21 Typical Time-varying Binomial Tree Parameters

i	r_i	σ_i	ν_i	Δt_i	p_i
0	0.059292	0.14103	0.049347	0.116065	0.559184
1	0.040042	0.173789	0.024941	0.077396	0.519947
2	0.040604	0.146126	0.029927	0.109148	0.533754
3	0.051304	0.162250	0.038141	0.088505	0.534882
4	0.058875	0.166929	0.044943	0.083516	0.538786
5	0.058571	0.166925	0.044639	0.083527	0.538528
6	0.046959	0.129180	0.038616	0.138586	0.555300
7	0.052179	0.136914	0.042806	0.123411	0.554588
8	0.050334	0.142216	0.040221	0.114708	0.547675
9	0.054212	0.159661	0.041466	0.091283	0.539114

FIGURE 2.22 Typical Binomial Tree With Time Varying Interest Rate and Volatility



approximately 0.08 and 0.14 of a year. This inconvenient variation in the time step can clearly be seen in Figure 2.22. The trinomial tree methods we describe in Chapter 3 solve this problem.

2.10 PRICING PATH-DEPENDENT OPTIONS

Trees (or lattices) can be used to price path-dependent American style exotic options. Binomial trees are not ideal because of the problems with accuracy, convergence and the simplicity of the binomial tree structure which we have discussed in the previous sections. However, the simplicity of the structure does make the method clear so we will introduce the ideas here and refer the reader to Chapter 5 for more detailed examples.

As a simple first example of the method and the problems which can be encountered we describe the pricing of an American down-and-out call option. This is a standard American call option except that if the asset price falls below a predetermined level H, the barrier level, then the option ceases to exist, or knocks out, and pays off nothing[4]. Figure 2.23 illustrates three example paths of the asset price.

Path 1 does not go below the barrier level and finishes above the strike price and therefore pays off, path 2 does not go below the barrier level, but finishes below the strike price and therefore pays off zero and finally path 3 goes below the barrier and therefore pays off nothing even though it finishes above the strike price. The valuation of this option in a binomial tree is similar to that for the American put option in section 2.4, the only difference is that the value of the option at every node below the barrier is set to zero. These sections are highlighted in bold in Figure 2.24 which gives the pseudo-code implementation.

FIGURE 2.23 Example Asset Paths for a Down-and-Out Call Option

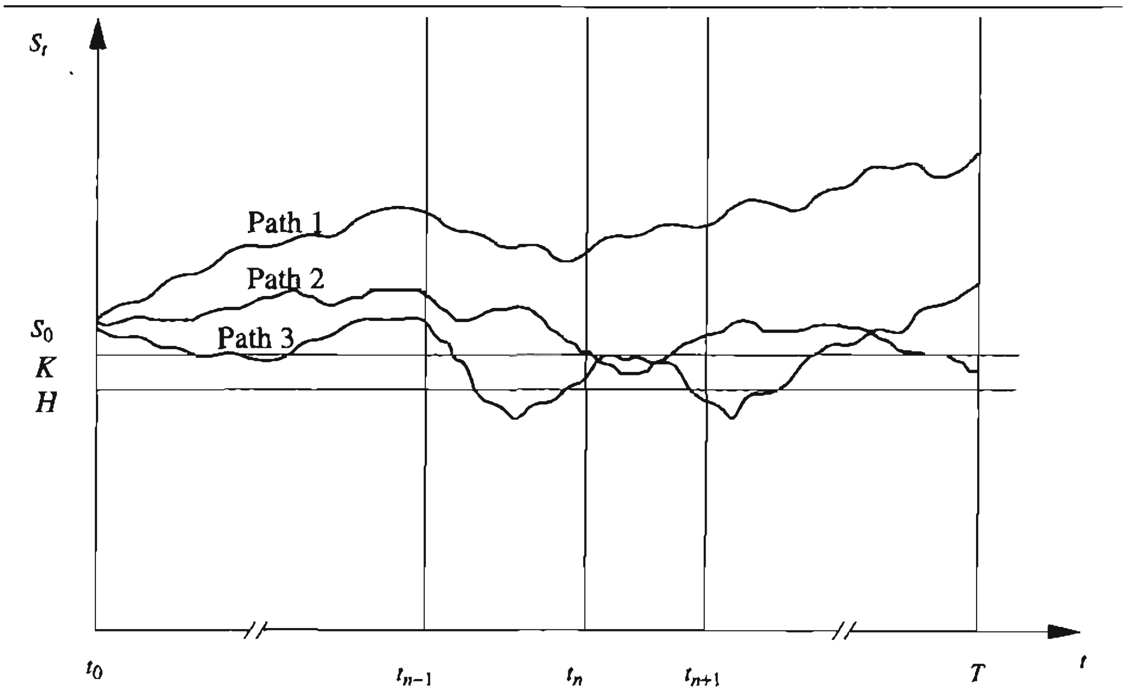


FIGURE 2.24 Pseudo-code for General Additive Binomial Valuation of an American Down-and-Out Call

```
initialise_parameters { K, T, S, sig, r, H, N }

{ set coefficients - Trigeorgis }

dt = T/N
nu = r - 0.5*sig^2
dxu = sqrt( sig^2*dt + (nu*dt)^2 )
dxd = -dxu
pu = 1/2 + 1/2*( nu*dt/dxu )
pd = 1 - pu

{ precompute constants }

disc = exp(-r*dt)
dpu = disc*pu
dpd = disc*pd
edxud = exp( dxu - dxd )
edxd = exp( dxd )

{ initialise asset prices at maturity N }

St[0] = S*exp( N*dxd )
for j = 1 to N do St[j] = St[j-1]*edxud

{ initialise option values at maturity }

for j = 0 to N do
  if ( St[j] > H ) then
    C[j] = max( 0.0 , St[j] - K )
  else
    C[j] = 0.0
next j

{ step back through the tree applying the barrier and early
  exercise condition }

for i = (N-1) downto 0 do
  for j = 0 to i do

  { adjust asset price to current time step }
  St[j] = St[j]/edxd

  if ( St[j] > H ) then

    C[j] = dpd*C[j] + dpu*C[j+1]

    { Apply the early exercise condition }
    C[j] = max( C[j] , St[j] - K )
```

(continues)

FIGURE 2.24 *(continued)*

```
    else

      C[j] = 0.0

    next j
  next i

  American_Down_and_Out_Call = C[0]
```

Example : Additive Binomial Tree Valuation of an American Down-and-Out Call

We price a one year maturity, at-the-money American down-and-out call option with the current asset price at 100 and volatility of 20 per cent. The barrier is set at 95, the continuously compounded interest rate is 6 per cent per annum and the binomial tree has three time steps, i.e. $K = 100$, $T = 1$, $S = 100$, $\sigma = 0.20$, $r = 0.06$, $H = 95$, $N = 3$. Figure 2.25 illustrates the results of the calculations, where nodes in the tree are represented by the boxes in which the upper value is the asset price and the lower value is the option price.

Firstly the constants are precomputed: $\Delta t(dt)$, $\nu(nu)$, $\Delta x_u(dxu)$, $\Delta x_d(dxd)$, $p_u(pu)$, $p_d(pd)$, *disc, dpu, dpd, edxud, edxd*:

$$\Delta t = \frac{T}{N} = \frac{1}{3} = 0.3333$$

$$\nu = r - \tfrac{1}{2}\sigma^2 = 0.06 - \tfrac{1}{2}0.20^2 = 0.0400$$

$$\Delta x_u = \sqrt{\sigma^2 \times \Delta t + (\nu \times \Delta t)^2)} = \sqrt{0.20^2 \times 0.3333 + (0.0400 \times 0.3333)^2} = 0.1162$$

$$\Delta x_d = -\Delta x_u = -0.1162$$

$$p_u = \frac{1}{2} + \frac{1}{2}\left(\frac{\nu \times \Delta t}{\Delta x_u}\right) = \frac{1}{2} + \frac{1}{2}\left(\frac{0.040 \times 0.3333}{0.1162}\right) = 0.5574$$

$$p_d = 1 - p_u = 1 - 0.5574 = 0.4426$$

$$dpu = disc \times p_u = 0.9802 \times 0.5574 = 0.5463$$

$$dpd = disc \times p_d = 0.9802 \times 0.4426 = 0.4339$$

$$edxud = e^{(\Delta x_u - \Delta x_d)} = e^{(0.1162-(-0.1162))} = 1.2617$$

$$edxd = e^{\Delta x_d} = e^{-0.1162} = 0.8903$$

Then the asset prices at maturity are computed, for example the asset price at node (3, 0) is computed as

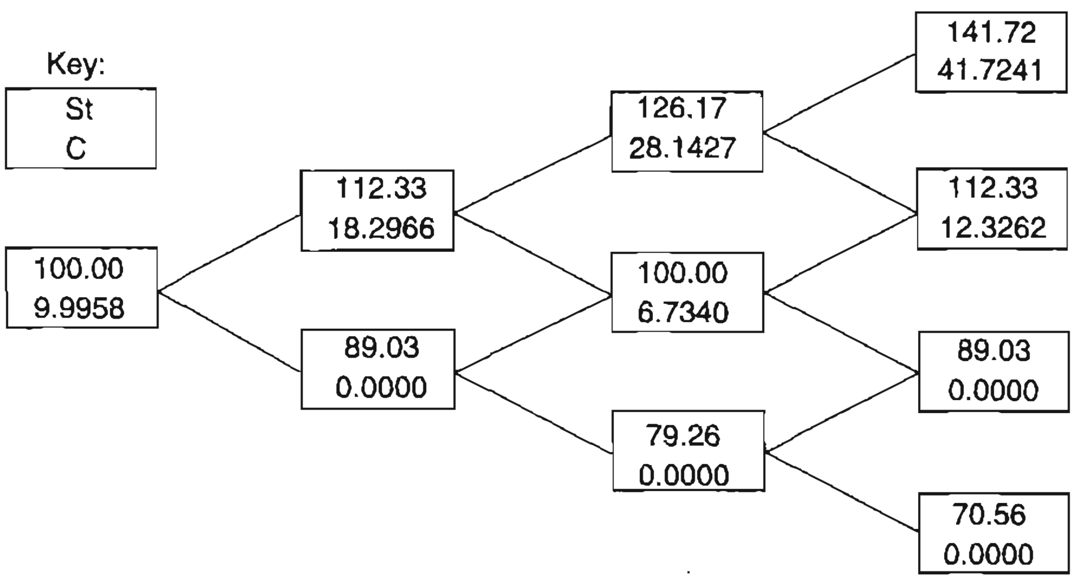
$$S_{3,0} = S \times e^{N \times \Delta x_d} = 100 \times e^{3\times(-0.1162)} = 70.56$$

FIGURE 2.25 Additive Binomial Tree Valuation of an American Down-and-Out Call

K	T	S	sig	r	H	N
100	1	100	0.2	0.06	95	3

dt	nu	dxu	dxd	pu	pd	disc	dpu	dpd	edxud	edxd
0.3333	0.0400	0.1162	–0.1162	0.5574	0.4426	0.9802	0.5463	0.4339	1.2617	0.8903

i	0	1	2	3
t	0	0.3333	0.6667	1



The other asset prices are computed from this, for example the asset price at node (3, 2) is computed as

$$S_{3,2} = S_{3,1} \times e^{(\Delta x_u - \Delta x_d)} = 89.03 \times e^{0.1162-(-0.1162))} = 112.33$$

Next the option values at maturity are computed. For node (3, 1) we have $S_{3,1} < H$, i.e. $89.03 < 95$ and therefore $C_{3,1} = 0.0$. For node (3, 2) we have $S_{3,2} > H$, i.e. $112.33 > 95$ and therefore

$$C_{3,2} = \max(0, S_{3,2} - K) = \max(0, 112.33 - 100) = 12.3262$$

Finally we perform discounted expectations back through the tree, applying the barrier condition. For example at node (1, 0) we have $S_{1,0} < H$, i.e. $89.03 < 95$, therefore $C_{1,0} = 0.0$.

2.11 THE MULTIDIMENSIONAL BINOMIAL METHOD

If we want to price options whose pay-off depends on more than one asset, for example options on the maximum or minimum of a basket of equity indices, then we must model all the assets simultaneously. This can be done by modelling the assets in a multidimensional binomial tree[5]. Consider the case of an option which pays off based on the values of two assets, S_1 and S_2 , which follow correlated GBMs

$$dS_1 = (r - \delta_1)S_1 dt + \sigma_1 S_1 dz_1 \tag{2.33}$$

$$dS_2 = (r - \delta_2)S_2 dt + \sigma_2 S_2 dz_2 \tag{2.34}$$

where the two assets have correlation ρ, i.e. $dz_1 dz_2 = \rho\, dt$ and the other symbols have their usual meaning.

We can model the joint evolution of S_1 and S_2 with a two-variable binomial lattice. The first step of a simple multiplicative two variable binomial tree is illustrated in Figure 2.26.

We now have four branches instead of two at every node, corresponding to the four possible combinations of the two assets going up or down. This tree can be extended in exactly the same way as the single asset tree in section 2.2. In order to specify the jump sizes and probabilities we equate the means, variances and correlations for the binomial process with those of the continuous time process. As with the single asset binomial tree it is easier to work in terms of the processes for the natural logarithms of the asset prices:

FIGURE 2.26 Multiplicative Two-Variable Binomial Process

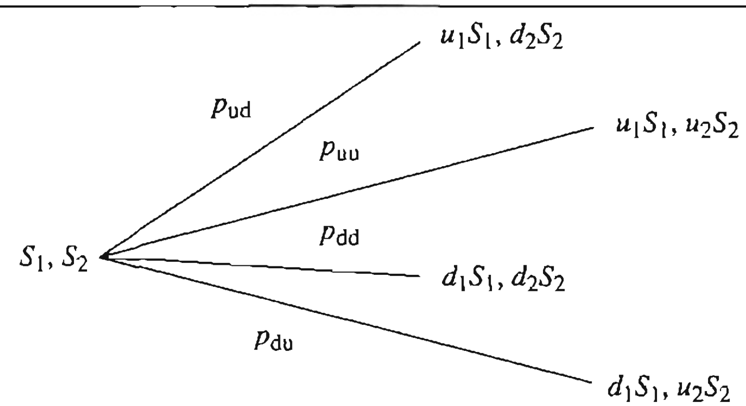
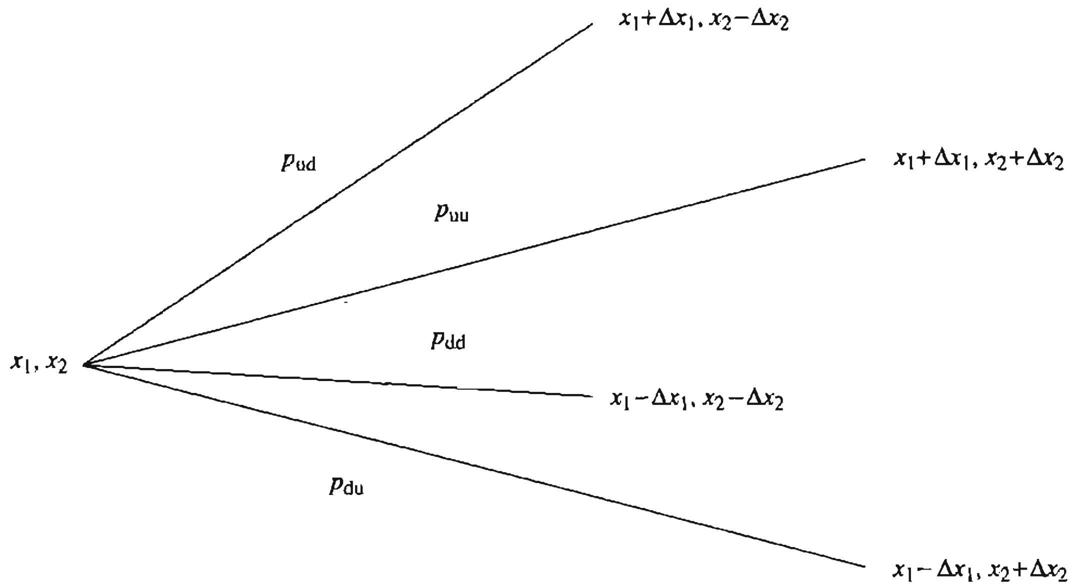


FIGURE 2.27 Additive Two-variable Binomial Process



$$dx_1 = \nu_1 dt + \sigma_1 dz_1 \tag{2.35}$$

$$dx_2 = \nu_2 dt + \sigma_2 dz_2 \tag{2.36}$$

where $\nu_1 = r - \delta_1 - \frac{1}{2}\sigma_1^2$ and $\nu_2 = r - \delta_2 - \frac{1}{2}\sigma_2^2$. Figure 2.27 shows the first step of the additive two-asset binomial tree, where we have chosen equal up and down jump sizes for each asset. The probabilities $(p_{uu}, p_{ud}, p_{du}, p_{dd})$ and the jump sizes (equal up and down jumps) $(\Delta x_1, \Delta x_2)$ are chosen to match the means and variances of the risk-neutral process:

$$E[\Delta x_1] = (p_{uu} + p_{ud})\Delta x_1 - (p_{du} + p_{dd})\Delta x_1 = \nu_1 \Delta t \tag{2.37}$$

$$E[\Delta x_1^2] = (p_{uu} + p_{ud})\Delta x_1^2 - (p_{du} + p_{dd})\Delta x_1^2 = \sigma_1^2 \Delta t \tag{2.38}$$

$$E[\Delta x_2] = (p_{uu} + p_{du})\Delta x_2 - (p_{ud} + p_{dd})\Delta x_2 = \nu_2 \Delta t \tag{2.39}$$

$$E[\Delta x_2^2] = (p_{uu} + p_{du})\Delta x_2^2 - (p_{ud} + p_{dd})\Delta x_2^2 = \sigma_2^2 \Delta t \tag{2.40}$$

$$E[\Delta x_1 \Delta x_2] = (p_{uu} - p_{ud} - p_{du} - p_{dd})\Delta x_1 \Delta x_2 = \rho\sigma_1\sigma_2 \Delta t \tag{2.41}$$

and we have that the probabilities must sum to one

$$p_{uu} + p_{ud} + p_{du} + p_{dd} = 1 \tag{2.42}$$

The solution[6] to this system of equations is

$$\Delta x_1 = \sigma_1 \sqrt{\Delta t} \tag{2.43}$$

$$\Delta x_2 = \sigma_2 \sqrt{\Delta t} \tag{2.44}$$

$$p_{uu} = \frac{1}{4}\frac{(\Delta x_1 \Delta x_2 + \Delta x_2 \nu_1 \Delta t + \Delta x_1 \nu_2 \Delta t + \rho\sigma_1\sigma_2\Delta t)}{\Delta x_1 \Delta x_2} \tag{2.45}$$

$$p_{ud} = \frac{1}{4}\frac{(\Delta x_1 \Delta x_2 + \Delta x_2 \nu_1 \Delta t - \Delta x_1 \nu_2 \Delta t - \rho\sigma_1\sigma_2\Delta t)}{\Delta x_1 \Delta x_2} \tag{2.46}$$

$$p_{\mathrm{du}} = \frac{1}{4}\frac{(\Delta x_1 \Delta x_2 - \Delta x_2 \nu_1 \Delta t + \Delta x_1 \nu_2 \Delta t - \rho\sigma_1\sigma_2\Delta t)}{\Delta x_1 \Delta x_2} \tag{2.47}$$

$$p_{\mathrm{dd}} = \frac{1}{4}\frac{(\Delta x_1 \Delta x_2 - \Delta x_2 \nu_1 \Delta t - \Delta x_1 \nu_2 \Delta t + \rho\sigma_1\sigma_2\Delta t)}{\Delta x_1 \Delta x_2} \tag{2.48}$$

Nodes in this tree are referred to by (i, j, k) which indicates the node at time step i, level j in asset 1 and level k in asset 2, i.e.

$$S_{1,i,j,k} = S_1 \exp(j\Delta x_1) \quad \text{and} \quad S_{2,i,j,k} = S_2 \exp(k\Delta x_2)$$

This tree therefore has the same structure as the equal jump size tree in section 2.4, but with two asset prices so at each time step nodes are separated by two space steps and the space indices will step by two.

As an example of using the two-variable binomial tree we consider an American spread option on the difference between the two assets S_1 and S_2 with strike price K which has the pay-off

$$\max(0, S_{1,T} - S_{2,T} - K)$$

Figure 2.28 gives a pseudo-implementation of this example. Note that the probabilities have been multiplied by the one step discount factor for efficiency.

FIGURE 2.28 Pseudo-code for American Spread Call Option by Two-variable Binomial

```
initialise_parameters { K, T, S1, S2, sig1, sig2, div1, div2, rho, r, N }

{ precompute constants }

dt = T/N
nu1 = r - div1 - 0.5*sig1^2
nu2 = r - div2 - 0.5*sig2^2
dx1 = sig1*sqrt(dt)
dx2 = sig2*sqrt(dt)
disc = exp(-r*dt)

puu = ( dx1*dx2 + ( dx2*nu1 + dx1*nu2 + rho*sig1*sig2 )*dt )/
      ( 4*dx1*dx2 ) * disc
pud = ( dx1*dx2 + ( dx2*nu1 - dx1*nu2 - rho*sig1*sig2 )*dt )/
      ( 4*dx1*dx2 ) * disc
pdu = ( dx1*dx2 + ( -dx2*nu1 + dx1*nu2 - rho*sig1*sig2 )*dt )/
      ( 4*dx1*dx2 ) * disc
pdd = ( dx1*dx2 + ( -dx2*nu1 - dx1*nu2 + rho*sig1*sig2 )*dt )/
      ( 4*dx1*dx2 ) * disc

edx1 = exp( dx1 )
edx2 = exp( dx2)

{ initialise asset prices at time step N }

S1t[-N] = S1*exp( -N*dx1 )
S2t[-N] = S2*exp( -N*dx2 )
for j = -N+1 to N do
```

FIGURE 2.28 ***(continued)***

```
  S1t[j] = S1t[j-1]*edx1
  S2t[j] = S2t[j-1]*edx2
next j

{ initialise option values at maturity }

for j = -N to N step 2 do
  for k = -N to N step 2 do
    C[j, k] = max( 0.0 , S1t[j] - S2t[k] - K )

{ step back through the tree applying early exercise }

for i = N-1 downto 0 do
  for j = -i to i step 2 do
    for k = -i to i step 2 do

      C[j, k] = pdd*C[j-1, k-1] + pud*C[j+1, k-1] +
                pdu*C[j-1, k+1] + puu*C[j+1, k+1]

    C[j, k] = max( C[j, k] , S1t[j] - S2t[k] - K )

    next k
  next j
next i

American_spread_option = C[0, 0]
```

Example : American Spread Call Option by Two-variable Binomial

We price a one-year maturity, American spread call option with a strike price of 1, current asset prices of 100, volatilities of 20 and 30 per cent, continuous dividend yields of 3 and 4 per cent and a correlation of 50 per cent. The continuously compounded interest rate is assumed to be 6 per cent per annum and the binomial tree has three time steps, i.e. $K = 1$, $T = 1$, $S_1 = 100$, $S_2 = 100$, $\sigma_1 = 0.20$, $\sigma_2 = 0.30$, $d_1 = 0.03$, $d_2 = 0.04$, $\rho = 0.50$, $r = 0.06$, $N = 3$. Figure 2.29 illustrates the results of the calculations, where nodes in the tree are represented by the grids of boxes and the asset prices by the separate rows and columns of boxes.

Firstly the constants are precomputed: $\Delta t(dt)$, $\nu_1(nu1)$, $\nu_2(nu2)$, $\Delta x_1(dx1)$, $\Delta x_2(dx2)$, *disc*, $p_{uu}(puu)$, $p_{ud}(pud)$, $p_{du}(pdu)$, $p_{dd}(pdd)$, *edx1, and edx2*:

$$\Delta t = \frac{T}{N} = \frac{1}{3} = 0.3333$$

$$\nu_1 = r - \delta_1 - \tfrac{1}{2}\sigma_1^2 = 0.06 - 0.03 - \tfrac{1}{2}0.20^2 = 0.0100$$

$$\nu_2 = r - \delta_2 - \tfrac{1}{2}\sigma_2^2 = 0.06 - 0.04 - \tfrac{1}{2}0.30^2 = -0.0250$$

$$\Delta x_1 = \sigma_1\sqrt{\Delta t} = 0.2\sqrt{0.3333} = 0.1155$$

FIGURE 2.29 American Spread Call Option by Two-variable Binomial

K	T	S_1	S_2	sig1	sig2	div1	div2	rho	r	N
1	1	100	100	0.2	0.3	0.03	0.04	0.50	0.06	3

dt	nu1	nu2	dx_1	dx_2	disc	puu	pud	pdu	pdd	edx1	edx2
0.3333	0.0100	-0.0250	0.1155	0.1732	0.9802	0.3629	0.1414	0.1037	0.3723	1.1224	1.1891

i 0
t 0

S2t							
168.14							
141.40							
118.91							
100.00				10.04479			
84.10							
70.72							
59.47							
S1t	70.72	79.38	89.09	100.00	112.24	125.98	141.40

i 1
t 0.333333

S2t							
168.14							
141.40							
118.91			0.9635		6.7420		
100.00							
84.10			9.4563		28.1353		
70.72							
59.47							
S1t	70.72	79.38	89.09	100.00	112.24	125.98	141.40

i 2
t 0.666667

S2t							
168.14							
141.40		0.0000		0.0000		3.0381	
118.91							
100.00		0.5653		5.3263		25.8626	
84.10							
70.72		9.3123		28.2778		54.2561	
59.47							
S1t	70.72	79.38	89.09	100.00	112.24	125.98	141.40

i 3
t 1

S2t							
168.14	0.0000		0.0000		0.0000		0.0000
141.40							
118.91	0.0000		0.0000		0.0000		21.4873
100.00							
84.10	0.0000		3.9982		27.1436		56.3017
70.72							
59.47	10.2473		28.6198		51.7652		80.9233
S1t	70.72	79.38	89.09	100.00	112.24	125.98	141.40

$$\Delta x_2 = \sigma_2 \sqrt{\Delta t} = 0.3\sqrt{0.3333} = 0.1732$$

$$disc = \exp(-r \times \Delta t) = 0.9802$$

The probabilities multiplied by *disc* are:

$$p_{uu} = \frac{(\Delta x_1 \times \Delta x_2 + (\Delta x_2 \times \nu_1 + \Delta x_1 \times \nu_2 + \rho \times \sigma_1 \times \sigma_2) \times \Delta t)}{(4 \times \Delta x_1 \times \Delta x_2)} \times disc$$

$$= \frac{\left\{\begin{matrix}(0.1154 \times 0.1732 + (0.1732 \times 0.01 + 0.1154 \times (-0.0250) \\ + 0.5 \times 0.2 \times 0.3) \times 0.3333)\end{matrix}\right\}}{(4 \times 0.1154 \times 0.1732)} \times 0.9802$$

$$= 0.3629$$

$$p_{\mathrm{ud}} = \frac{(\Delta x_1 \times \Delta x_2 + (\Delta x_2 \times v_1 - \Delta x_1 \times v_2 - \rho \times \sigma_1 \times \sigma_2) \times \Delta t)}{(4 \times \Delta x_1 \times \Delta x_2)} \times disc$$

$$= \frac{\left\{\begin{matrix}(0.1154 \times 0.1732 + (0.1732 \times 0.01 - 0.1154 \times (-0.0250) \\ - 0.5 \times 0.2 \times 0.3) \times 0.3333)\end{matrix}\right\}}{(4 \times 0.1154 \times 0.1732)} \times 0.9802$$

$$= 0.1414$$

$$p_{\mathrm{du}} = \frac{(\Delta x_1 \times \Delta x_2 + (-\Delta x_2 \times v_1 + \Delta x_1 \times v_2 - \rho \times \sigma_1 \times \sigma_2) \times \Delta t)}{(4 \times \Delta x_1 \times \Delta x_2)} \times disc$$

$$= \frac{\left\{\begin{matrix}(0.1154 \times 0.1732 + (-0.1732 \times 0.01 + 0.1154 \times (-0.0250) \\ - 0.5 \times 0.2 \times 0.3) \times 0.3333)\end{matrix}\right\}}{(4 \times 0.1154 \times 0.1732)} \times 0.9802$$

$$= 0.1037$$

$$p_{\mathrm{dd}} = \frac{(\Delta x_1 \times \Delta x_2 + (-\Delta x_2 \times v_1 - \Delta x_1 \times v_2 + \rho \times \sigma_1 \times \sigma_2) \times \Delta t)}{(4 \times \Delta x_1 \times \Delta x_2)} \times disc$$

$$= \frac{\left\{\begin{matrix}(0.1154 \times 0.1732 + (-0.1732 \times 0.01 \\ - 0.1154 \times (-0.0250) + 0.5 \times 0.2 \times 0.3) \times 0.333)\end{matrix}\right\}}{(4 \times 0.1154 \times 0.1732)} \times 0.9802$$

$$= 0.3723$$

$$edx1 = e^{\Delta x_1} = e^{0.1154} = 1.1224$$

$$edx2 = e^{\Delta x_2} = e^{0.1732} = 1.1891$$

Then the asset prices at maturity are computed (these are the same for every time step). The asset price 1 at node $(3, -3, -3)$ is computed as

$$S_{1,3,-3,-3} = S_1 \times e^{-N \times \Delta x_1} = 100 \times e^{-3 \times 0.1154} = 70.72$$

and asset price 2 at node $(3, -3, -3)$ is computed as

$$S_{2,3,-3,-3} = S_2 \times e^{-N \times \Delta x_2} = 100 \times e^{-3 \times 0.1732} = 59.47$$

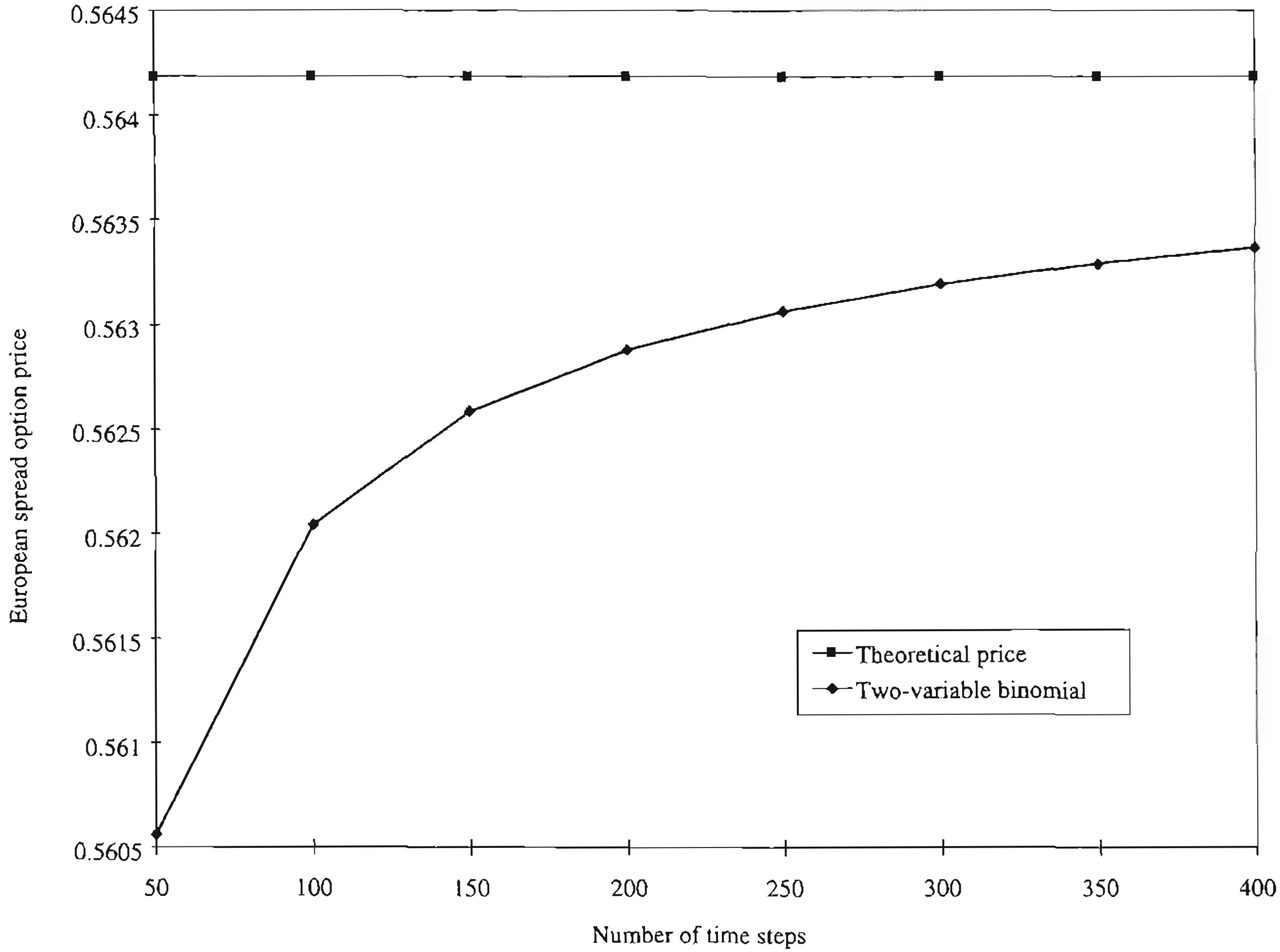
The other asset prices are computed from this, e.g. node $(3, -2, -2)$ asset price 1 is computed as

$$S_{1,3,-2,-2} = S_{1,3,-3,-2} \times edx1 = 70.72 \times 1.1224 = 79.38$$

Next the option values at maturity are computed. For node $(3, 1, -1)$ we have

$$C_{3,1,-1} = \max(0, S_{1,3,1,-1} - S_{2,3,1,-1} - K) = \max(0, 112.24 - 84.10 - 1) = 27.144$$

FIGURE 2.30 Convergence of the Two-variable Binomial Method



Finally we perform discounted expectations back through the tree applying the early exercise test. For node (2, 0, 0) we have

$$C_{2,0,0} = p_{dd} \times C_{3,-1,-1} + p_{ud} \times C_{3,1,-1} + p_{du} \times C_{3,-1,1} + p_{uu} \times C_{3,1,1}$$
$$= 0.3723 \times 3.998 + 0.1414 \times 27.144 + 0.1037 \times 0.000 + 0.3629 \times 0.000 = 5.3269$$

then applying the early exercise test we have

$$C_{2,0,0} = \max(C_{2,0,0}, S_{1,2,0,0} - S_{2,2,0,0} - K) = \max(5.3269, 100 - 100 - 1) = 5.3269$$

Unfortunately the binomial model is even less efficient for two variables than for one variable. Figure 2.30 illustrates the convergence of the price of an example European spread option (for which we can compute an accurate value by numerical integration) as a function of the number of time steps.

It can be seen that convergence cannot be achieved with a reasonable number of time steps. To achieve efficient pricing for more than one variable problems implicit methods must be used. These are discussed in Chapter 3.

2.12 SUMMARY

In this chapter we have introduced the binomial model of asset prices and shown how it can be generalised and used for pricing American-style options. We then described in detail how the binomial model can be efficiently implemented and how hedge ratios can be computed. An important consideration in option pricing is whether the underlying asset has an associated income or dividend stream, we therefore discussed methods for dealing with asset paying various forms of dividend streams. These methods are quite general and can be applied to the trinomial trees and finite difference methods in Chapters 3 and 5. Finally, we showed how the binomial method can be extended to deal with time-varying volatility, path-dependent options and options whose payoff depends on more than one asset. However, these types of problems can be far more efficiently dealt with using trinomial tree and finite difference methods.

ENDNOTES

1. Strictly, American call options on assets paying a dividend less than or equal to zero can be valued using the Black–Scholes formula.
2. The indexing for the nodes in this section refer to the general additive tree — see Figure 2.9.
3. If this is not the case then we can apply an appropriately inflated amount to the nearest date in the tree after the dividend date. With trinomial trees it is straightforward to arrange for the tree dates to match important dates such as dividends (see Chapter 3).
4. Sometimes the option will pay a predetermined cash rebate, X_{rebate}, when the option knocks out, we deal with this case in Chapter 5.
5. These methods were introduced by Boyle (1988) and Boyle, Evnine and Gibbs (1989), see also Kamrad and Ritchken (1991).
6. Computer algebra packages are useful for solving systems of equation such as these.