CHAPTER 10

Binomial Option Pricing: I

In earlier chapters we discussed how the price of one option is related to the price of another, but we did not explain how to determine the price of an option relative to the price of the underlying asset. In this chapter we discuss the binomial option pricing model, with which we can compute the price of an option, given the characteristics of the stock or other underlying asset.

The binomial option pricing model assumes that, over a period of time, the price of the underlying asset can move only up or down by a specified amount—that is, the asset price follows a binomial distribution. Given this assumption, it is possible to determine a no-arbitrage price for the option. Surprisingly, this approach, which appears at first glance to be overly simplistic, can be used to price options, and it conveys much of the intuition underlying more complex (and seemingly more realistic) option pricing models that we will encounter in later chapters. It is hard to overstate the value of thoroughly understanding the binomial approach to pricing options.

Because of its usefulness, we devote this and the next chapter to binomial option pricing. In this chapter, we will see how the binomial model works and use it to price both European and American call and put options on stocks, currencies, and futures contracts. As part of the pricing analysis, we will also see how market-makers can create options synthetically using the underlying asset and risk-free bonds. In the next chapter, we will explore the assumptions underlying the model.

10.1 A One-Period Binomial Tree

Binomial pricing achieves its simplicity by making a very strong assumption about the stock price: At any point in time, the stock price can change to either an up value or a down value. In-between, greater, or lesser values are not permitted. The restriction to two possible prices is why the method is called "binomial." The appeal of binomial pricing is that it displays the logic of option pricing in a simple setting, using only algebra to price options.

The binomial approach to pricing was first used by Sharpe (1978) as an intuitive way to explain option pricing. Binomial pricing was developed more formally by Cox et al. (1979) and Rendleman and Bartter (1979), who showed how to implement the model, demonstrated the link between the binomial model and the Black-Scholes model, and showed that the method provides a tractable way to price options for which early exercise may be optimal. The binomial model is often referred to as the "Cox-Ross-Rubinstein pricing model."

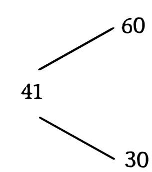
We begin with a simple example. Consider a European call option on the stock of XYZ, with a $40 strike and 1 year to expiration. XYZ does not pay dividends and its current price is $41. The continuously compounded risk-free interest rate is 8%. We wish to determine the option price.

Since the stock's return over the next year is uncertain, the option could expire either in-the-money or out-of-the-money, depending upon whether the stock price is more or less than $40. Intuitively, the valuation for the option should take into account both possibilities and assign values in each case. If the option expires out-of-the-money, its value is zero. If the option expires in-the-money, its value will depend upon how far in-the-money it is. To price the option, then, we need to characterize the uncertainty about the stock price at expiration.

Figure 10.1 represents the evolution of the stock price: Today the price is $41, and in 1 year the price can be either $60 or $30. This depiction of possible stock prices is called a **binomial tree.** For the moment we take the tree as given and price the option. Later we will learn how to construct such a tree.

FIGURE 10.1

Binomial tree depicting the movement of XYZ stock over 1 year. The current stock price is $41.



Computing the Option Price

Now we compute the price of our 40-strike 1-year call. Consider two portfolios:

Portfolio A. Buy one call option. The cost of this is the call premium, which is what we are trying to determine.

Portfolio B. Buy 2/3 of a share of XYZ and borrow $18.462 at the risk-free rate.[1] This position costs

$$2/3 \times \$41 - \$18.462 = \$8.871$$

[1] Of course, it is not possible to buy fractional shares of stock. As an exercise, you can redo this example, multiplying all quantities by 3. You would then compare three call options (Portfolio A) to buying two shares and borrowing $\$18.462 \times 3 = \$55.387$ (Portfolio B).

Now we compare the payoffs to the two portfolios 1 year from now. Since the stock can take on only two values, we can easily compute the value of each portfolio at each possible stock price.

For Portfolio A, the time 1 payoff is $\max[0, S_1 - \$40]$:

	Stock Price in 1 Year (S_1)	
	\$30	**\$60**
Payoff	0	\$20

In computing the payoff for Portfolio B, we assume that we sell the shares at the market price and that we repay the borrowed amount, plus interest ($\$18.462 \times e^{.08} = \$20$). Thus we have

	Stock Price in 1 Year (S_1)	
	\$30	**\$60**
2/3 purchased shares	\$20	\$40
Repay loan of \$18.462	−\$20	−\$20
Total payoff	0	\$20

Note that Portfolios A and B have the same payoff: Zero if the stock price goes down, in which case the option is out-of-the-money, and \$20 if the stock price goes up. Therefore, both portfolios should have the same cost. Since Portfolio B costs \$8.871, then given our assumptions, *the price of one option must be \$8.871*. Portfolio B is a *synthetic* call, mimicking the payoff to a call by buying shares and borrowing.

The idea that positions that have the same payoff should have the same cost is called the **law of one price**. This example uses the law of one price to determine the option price. We will see shortly that there is an arbitrage opportunity if the law of one price is violated.

The call option in the example is replicated by holding 2/3 shares, which implies that one option has the risk of 2/3 shares. The value 2/3 is the *delta* (Δ) of the option: the number of shares that replicates the option payoff. Delta is a key concept, and we will say much more about it later.

Finally, we can say something about the expected return on the option. Suppose XYZ has a positive risk premium (i.e., the expected return on XYZ is greater than the risk-free rate). Since we create the synthetic call by borrowing to buy the stock, the call is equivalent to a leveraged position in the stock, and therefore the call will have an expected return greater than that on the stock. The option elasticity, which we will discuss in Chapter 12, measures the amount of leverage implicit in the option.

The Binomial Solution

In the preceding example, how did we know that buying 2/3 of a share of stock and borrowing \$18.462 would replicate a call option?

We have two instruments to use in replicating a call option: shares of stock and a position in bonds (i.e., borrowing or lending). To find the replicating portfolio, we need to find a combination of stock and bonds such that the portfolio mimics the option.

To be specific, we wish to find a portfolio consisting of Δ shares of stock and a dollar amount B in lending, such that the portfolio imitates the option whether the stock rises or falls. We will suppose that the stock has a continuous dividend yield of δ, which we reinvest in the stock. Thus, as in Section 5.2, if you buy one share at time t, at time $t + h$ you will have $e^{\delta h}$ shares. The up and down movements of the stock price reflect the *ex-dividend* price.

We can write the stock price as uS_0 when the stock goes up and as dS_0 when the price goes down. We can represent the stock price tree as follows:

$$
\begin{tikzcd}
 & uS \\
S \arrow[ur, dash] \arrow[dr, dash] & \\
 & dS
\end{tikzcd}
$$

In this tree u is interpreted as one plus the rate of capital gain on the stock if it goes up, and d is one plus the rate of capital loss if it goes down. (If there are dividends, the total return is the capital gain or loss, plus the dividend.)

Let C_u and C_d represent the value of the option when the stock goes up or down, respectively. The tree for the stock implies a corresponding tree for the value of the option:

$$
\begin{tikzcd}
 & C_u \\
C_0 \arrow[ur, dash] \arrow[dr, dash] & \\
 & C_d
\end{tikzcd}
$$

If the length of a period is h, the interest factor per period is e^{rh}. The problem is to solve for Δ and B such that our portfolio of Δ shares and B in lending duplicates the option payoff. The value of the replicating portfolio at time h, with stock price S_h, is

$$\Delta S_h + e^{rh} B$$

At the prices $S_h = dS$ and $S_h = uS$, a successful replicating portfolio will satisfy[2]

$$
\begin{aligned}
(\Delta \times dS \times e^{\delta h}) + (B \times e^{rh}) &= C_d \\
(\Delta \times uS \times e^{\delta h}) + (B \times e^{rh}) &= C_u
\end{aligned}
$$

[2]The term e^{Sh} arises in the following equations because the owner of the stock receives a proportional dividend that we assume is reinvested in shares.

This is two equations in the two unknowns Δ and B. Solving for Δ and B gives

$$\boxed{\Delta = e^{-\delta h}\frac{C_u - C_d}{S(u-d)}} \tag{10.1}$$

$$\boxed{B = e^{-rh}\frac{uC_d - dC_u}{u-d}} \tag{10.2}$$

Note that when there are dividends, the formula adjusts the number of shares in the replicating portfolio, Δ, to offset the dividend income.

Given the expressions for Δ and B, we can derive a simple formula for the value of the option. The cost of creating the option is the net cash required to buy the shares and bonds. Thus, the cost of the option is $\Delta S + B$. Using equations (10.1) and (10.2), we have

$$\boxed{\Delta S + B = e^{-rh}\left(C_u \frac{e^{(r-\delta)h} - d}{u-d} + C_d \frac{u - e^{(r-\delta)h}}{u-d}\right)} \tag{10.3}$$

The assumed stock price movements, u and d, should not give rise to arbitrage opportunities. In particular, we require that

$$u > e^{(r-\delta)h} > d \tag{10.4}$$

To see why this condition must hold, suppose $\delta = 0$. If the condition were violated, we would short the stock to hold bonds (if $e^{rh} \geq u$), or we would borrow to buy the stock (if $d \geq e^{rh}$). Either way, we would earn an arbitrage profit. Therefore the assumed process could not be consistent with any possible equilibrium. Problem 10.21 asks you to verify that the condition must also hold when $\delta > 0$.

Note that because Δ is the number of shares in the replicating portfolio, it can also be interpreted as the sensitivity of the option to a change in the stock price. If the stock price changes by \$1, then the option price, $\Delta S + B$, changes by Δ. This interpretation will be quite important later.

Example 10.1 Here is the solution for Δ, B, and the option price using the stock price tree depicted in Figure 10.1. There we have $u = \$60/\$41 = 1.4634$, $d = \$30/\$41 = 0.7317$, and $\delta = 0$. In addition, the call option had a strike price of \$40 and 1 year to expiration—hence, $h = 1$. Thus $C_u = \$60 - \$40 = \$20$, and $C_d = 0$. Using equations (10.1) and (10.2), we have

$$\Delta = \frac{\$20 - 0}{\$41 \times (1.4634 - 0.7317)} = 2/3$$

$$B = e^{-0.08}\frac{1.4634 \times \$0 - 0.7317 \times \$20}{1.4634 - 0.7317} = -\$18.462$$

Hence, the option price is given by

$$\Delta S + B = 2/3 \times \$41 - \$18.462 = \$8.871$$

Note that *if we are interested only in the option price, it is not necessary to solve for* Δ *and* B; that is just an intermediate step. If we want to know only the option price, we can use equation (10.3) directly:

$$\begin{aligned}\Delta S + B &= e^{-0.08}\left(\$20 \times \frac{e^{0.08} - 0.7317}{1.4634 - 0.7317} + \$0 \times \frac{1.4634 - e^{0.08}}{1.4634 - 0.7317}\right)\\ &= \$8.871\end{aligned}$$

Throughout this chapter we will continue to report Δ and B, since we are interested not only in the price but also in the replicating portfolio.

Arbitraging a Mispriced Option

What if the observed option price differs from the theoretical price? Because we have a way to replicate the option using the stock, it is possible to take advantage of the mispricing and fulfill the dream of every trader—namely, to buy low and sell high.

The following examples illustrate that if the option price is anything other than the theoretical price, arbitrage is possible.

The option is overpriced Suppose that the market price for the option is \$9.00, instead of \$8.871. We can sell the option, but this leaves us with the risk that the stock price at expiration will be \$60 and we will be required to deliver the stock.

We can address this risk by buying a synthetic option at the same time we sell the actual option. We have already seen how to create the synthetic option by buying 2/3 shares and borrowing \$18.462. If we simultaneously sell the actual option and buy the synthetic, the initial cash flow is

$$\underbrace{\$9.00}_{\text{Receive option premium}} - \underbrace{2/3 \times \$41}_{\text{Cost of shares}} + \underbrace{\$18.462}_{\text{Borrowing}} = \$0.129$$

We earn \$0.129, the amount by which the option is mispriced.

Now we verify that there is no risk at expiration. We have

	Stock Price in 1 Year (S_1)	
	\$30	**\$60**
Written call	\$ 0	−\$20
2/3 Purchased shares	\$20	\$40
Repay loan of \$18.462	−\$20	−\$20
Total payoff	\$0	\$0

By hedging the written option, we eliminate risk.

The option is underpriced Now suppose that the market price of the option is \$8.25. We wish to buy the underpriced option. Of course, if we are unhedged and the stock price falls at expiration, we lose our investment. We can hedge by selling a synthetic option. We accomplish this by reversing the position for a synthetic purchased call: We short 2/3 shares and invest \$18.462 of the proceeds in Treasury bills. The cost of this is

$$\underbrace{-\$8.25}_{\text{Option premium}} + \underbrace{2/3 \times \$41}_{\text{Short-sale proceeds}} - \underbrace{\$18.462}_{\text{Invest in T-bills}} = \$0.621$$

At expiration we have

	Stock Price in 1 Year (S_1)	
	\$30	**\$60**
Purchased call	\$0	\$20
2/3 short-sold shares	−\$20	−\$40
Sell T-bill	\$20	\$20
Total payoff	\$0	\$0

We have earned the amount by which the option was mispriced and hedged the risk associated with buying the option.

A Graphical Interpretation of the Binomial Formula

The binomial solution for Δ and B, equations (10.1) and (10.2), is obtained by solving two equations in two unknowns. Letting C_h and S_h be the option and stock value after one binomial period, and supposing $\delta = 0$, the equations for the portfolio describe a line with the formula

$$C_h = \Delta \times S_h + e^{rh} B$$

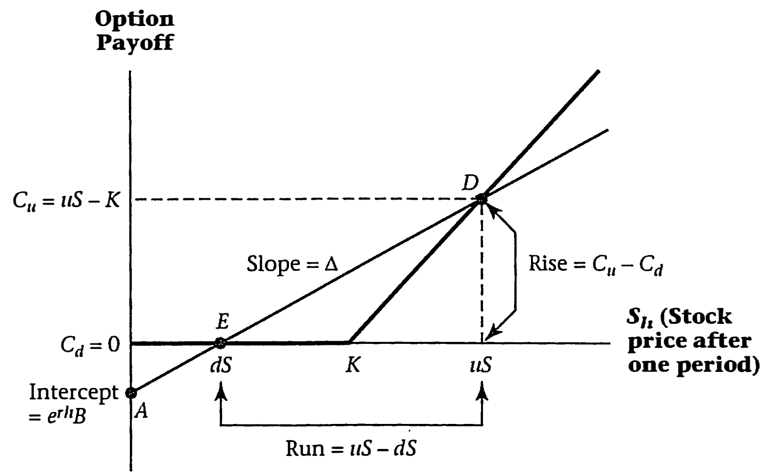
This is graphed as line AED in Figure 10.2, which shows the option payoff as a function of the stock price at expiration.

We choose Δ and B to yield a portfolio that pays C_d when $S_h = dS$ and C_u when $S_h = uS$. Hence, by construction this line runs through points E and D. We can control the slope of a payoff diagram by varying the number of shares, Δ, and its height by varying the number of bonds, B. It is apparent that a line that runs through both E and D must have slope $\Delta = (C_u - C_d)/(uS - dS)$. Also, the point A is the value of the portfolio when $S_h = 0$, which is the time-h value of the bond position, $e^{rh} B$. Hence, $e^{rh} B$ is the y-axis intercept of the line.

You can see by looking at Figure 10.2 that *any* line replicating a call will have a positive slope ($\Delta > 0$) and a negative intercept ($B < 0$). As an exercise, you can verify graphically that a portfolio replicating a put would have negative slope ($\Delta < 0$) and positive intercept ($B > 0$).

FIGURE 10.2

The payoff to an expiring call option is the dark heavy line. The payoff to the option at the points dS and uS are C_d and C_u (at point D). The portfolio consisting of Δ shares and B bonds has intercept $e^{rh}B$ and slope Δ, and by construction goes through both points E and D. The slope of the line is calculated as $\frac{\text{Rise}}{\text{Run}}$ between points E and D, which gives the formula for Δ.



Risk-Neutral Pricing

So far we have not specified the probabilities of the stock going up or down. In fact, probabilities were not used anywhere in the option price calculations. Since the strategy of holding Δ shares and B bonds replicates the option whichever way the stock moves, the probability of an up or down movement in the stock is irrelevant for pricing the option.

Although probabilities are not needed for pricing the option, there is a probabilistic interpretation of equation (10.3). Notice that in equation (10.3) the terms $(e^{(r-\delta)h} - d)/(u-d)$ and $(u - e^{(r-\delta)h})/(u-d)$ sum to 1 and are both positive (this follows from inequality 10.4). Thus, we can interpret these terms as probabilities. Let

$$\boxed{p^* = \frac{e^{(r-\delta)h} - d}{u - d}} \tag{10.5}$$

Equation (10.3) can then be written as

$$\boxed{C = e^{-rh}[p^* C_u + (1 - p^*)C_d]} \tag{10.6}$$

This expression has the appearance of a discounted expected value. It is peculiar, though, because we are discounting at the risk-free rate, even though the risk of the option is at least as great as the risk of the stock (a call option is a leveraged position in the stock since $B < 0$). In addition, there is no reason to think that p^* is the true probability that the stock will go up; in general it is not.

What happens if we use p^* to compute the expected *undiscounted* stock price? Doing this, we obtain

$$p^* uS + (1 - p^*)dS = e^{(r-\delta)h}S = F_{t,t+h} \tag{10.7}$$

When we use p^* as the probability of an up move, the expected stock price equals the forward price, $e^{(r-\delta)h}S$. (We derived this expression for the forward price of the stock in Chapter 5, equation 5.7.) Thus, *we can compute the forward price using the binomial tree*. In fact, one way to think about p^* is that it is the probability for which the expected stock price is the forward price.

We will call p^* the **risk-neutral probability** of an increase in the stock price. Equation (10.6) will prove very important and we will discuss risk-neutral pricing more in Chapter 11.

Constructing a Binomial Tree

We now explain the construction of the binomial tree.[3] Recall that the goal of the tree is to characterize future uncertainty about the stock price in an economically reasonable way.

As a starting point, we can ask: What if there were no uncertainty about the future stock price? Without uncertainty, the stock price next period must equal the forward price. Recall from Chapter 5 that the formula for the forward price is

$$F_{t,t+h} = S_t e^{(r-\delta)h} \tag{10.8}$$

Thus, without uncertainty we must have $S_{t+h} = F_{t,t+h}$. To interpret this, under certainty, the rate of return on the stock must be the risk-free rate. Thus, the stock price must rise at the risk-free rate less the dividend yield, $r - \delta$.

Now we incorporate uncertainty, but we first need to define what we mean by uncertainty. A natural measure of uncertainty about the stock return is the *annualized standard deviation of the continuously compounded stock return*, which we will denote by σ. The standard deviation measures how sure we are that the stock return will be close to the expected return. Stocks with a larger σ will have a greater chance of a return far from the expected return.

We incorporate uncertainty into the binomial tree by modeling the up and down moves of the stock price relative to the forward price, with the difference from the forward price being related to the standard deviation. We will see in Section 11.3 that if the annual standard deviation is σ, the standard deviation over a period of length h is $\sigma\sqrt{h}$. In other words, the standard deviation of the stock return is proportional to the *square root* of time.

We now model the stock price evolution as

$$\begin{aligned} uS_t &= F_{t,t+h}e^{+\sigma\sqrt{h}} \\ dS_t &= F_{t,t+h}e^{-\sigma\sqrt{h}} \end{aligned} \tag{10.9}$$

[3]This discussion is intended as a quick overview. Section 11.3 contains a more in-depth discussion.

Using equation (10.8), we can rewrite this as

$$\begin{aligned} u &= e^{(r-\delta)h+\sigma\sqrt{h}} \\ d &= e^{(r-\delta)h-\sigma\sqrt{h}} \end{aligned} \tag{10.10}$$

This is the formula we will use to construct binomial trees. Note that if we set volatility equal to zero (i.e., $\sigma = 0$), we will have $uS_t = dS_t = F_{t,t+h}$. Thus, with zero volatility, the price will still rise over time, just as with a Treasury bill. Zero volatility does not mean that prices are *fixed;* it means that prices are *known in advance.*

We will refer to a tree constructed using equation (10.10) as a "forward tree." In Section 11.3 we will discuss alternative ways to construct a tree, including the Cox-Ross-Rubinstein tree.

Another One-Period Example

We began this section by assuming that the stock price followed the binomial tree in Figure 10.1. The up and down stock prices of \$30 and \$60 were selected to make the example easy to follow. Now we present an example where everything is the same except that we use equation (10.10) to construct the up and down moves.

Suppose volatility is 30%. Since the period is 1 year, we have $h = 1$, so that $\sigma\sqrt{h} = 0.30$. We also have $S_0 = \$41$, $r = 0.08$, and $\delta = 0$. Using equation (10.10), we get

$$\begin{aligned} uS &= \$41e^{(0.08-0)\times 1+0.3\times\sqrt{1}} = \$59.954 \\ dS &= \$41e^{(0.08-0)\times 1-0.3\times\sqrt{1}} = \$32.903 \end{aligned} \tag{10.11}$$

Because the binomial tree is different than in Figure 10.1, the option price will be different as well.

Using the stock prices given in equation (10.11), we have $u = \$59.954/\$41 = 1.4623$ and $d = \$32.903/\$41 = 0.8025$. With $K = \$40$, we have $C_u = \$59.954 - \$40 = \$19.954$, and $C_d = 0$. Using equations (10.1) and (10.2), we obtain

$$\Delta = \frac{\$19.954 - 0}{\$41 \times (1.4623 - 0.8025)} = 0.7376$$

$$B = e^{-0.08}\frac{1.4623 \times \$0 - 0.8025 \times \$19.954}{1.4623 - 0.8025} = -\$22.405$$

Hence, the option price is given by

$$\Delta S + B = 0.7376 \times \$41 - \$22.405 = \$7.839$$

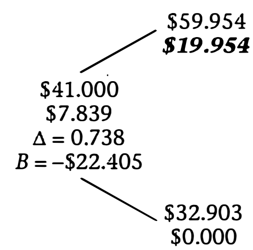
This example is summarized in Figure 10.3.

Summary

We have covered a great deal of ground in this section, so we pause for a moment to review the main points:

FIGURE 10.3

Binomial tree for pricing a European call option; assumes $S = \$41.00$, $K = \$40.00$, $\sigma = 0.30$, $r = 0.08$, $T = 1.00$ years, $\delta = 0.00$, and $h = 1.000$. At each node the stock price, option price, Δ, and B are given. Option prices in ***bold italic*** signify that exercise is optimal at that node.



- In order to price an option, we need to know the stock price, the strike price, the standard deviation of returns on the stock (in order to compute u and d), the dividend yield, and the risk-free rate.
- Using the risk-free rate, dividend yield, and σ, we can approximate the future distribution of the stock by creating a binomial tree using equation (10.10).
- Once we have the binomial tree, it is possible to price the option using equation (10.3). The solution also provides the recipe for synthetically creating the option: Buy Δ shares of stock (equation 10.1) and borrow B (equation 10.2).
- The formula for the option price, equation (10.3), can be written so that it has the appearance of a discounted expected value.

There are still many issues we have to deal with. The simple binomial tree seems too simple to provide an accurate option price. Unanswered questions include how to handle more than one binomial period, how to price put options, how to price American options, etc. With the basic binomial formula in hand, we can now turn to those questions.

10.2 Two or More Binomial Periods

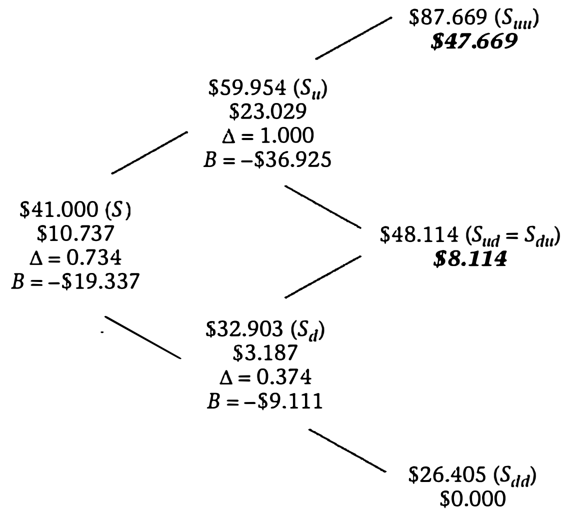
We now see how to extend the binomial tree to more than one period.

A Two-Period European Call

We begin first by adding a single period to the tree in Figure 10.3; the result is displayed in Figure 10.4. We can use that tree to price a 2-year option with a $40 strike when the current stock price is $41, assuming all inputs are the same as before.

Since we are increasing the time to maturity for a call option on a nondividend-paying stock, then based on the discussion in Section 9.3 we expect the option premium

FIGURE 10.4



Binomial tree for pricing a European call option; assumes $S = \$41.00$, $K = \$40.00$, $\sigma = 0.30$, $r = 0.08$, $T = 2.00$ years, $\delta = 0.00$, and $h = 1.000$. At each node the stock price, option price, Δ, and B are given. Option prices in ***bold italic*** signify that exercise is optimal at that node.

to increase. In this example the two-period tree will give us a price of \$10.737, compared to \$7.839 in Figure 10.3.

Constructing the tree To see how to construct the tree, suppose that we move up in year 1, to $S_u = \$59.954$. If we reach this price, then we can move further up or down according to equation (10.9). We get

$$S_{uu} = \$59.954e^{0.08+0.3} = \$87.669$$

and

$$S_{ud} = \$59.954e^{0.08-0.3} = \$48.114$$

The subscript *uu* means that the stock has gone up twice in a row and the subscript *ud* means that the stock has gone up once and then down.

Similarly if the price in one year is $S_d = \$32.903$, we have

$$S_{du} = \$32.903e^{0.08+0.3} = \$48.114$$

and

$$S_{dd} = \$32.903e^{0.08-0.3} = \$26.405$$

Note that an up move followed by a down move (S_{ud}) generates the same stock price as a down move followed by an up move (S_{du}). This is called a **recombining tree.** If an up move followed by a down move led to a different price than a down move followed by an up move, we would have a **nonrecombining tree.**[4] A recombining tree has fewer nodes, which means less computation is required to compute an option price. We will see examples of nonrecombining trees in Sections 11.5 and 24.4.

We also could have used equation (10.10) directly to compute the year-2 stock prices. Recall that $u = e^{0.08+0.3} = 1.462$ and $d = e^{0.08-0.3} = 0.803$. We have

$$S_{uu} = u^2 \times \$41 = e^{2\times(0.08+0.3)} \times \$41 = \$87.669$$

$$S_{ud} = S_{du} = u \times d \times \$41 = e^{(0.08+0.3)} \times e^{(0.08-0.3)} \times \$41 = \$48.114$$

$$S_{dd} = d^2 \times \$41 = e^{2\times(0.08-0.3)} \times \$41 = \$26.405$$

Pricing the call option How do we price the option when we have two binomial periods? The key insight is that we work *backward* through the binomial tree. In order to use equation (10.3), we need to know the option prices resulting from up and down moves in the subsequent period. At the outset, the only period where we know the option price is at expiration.

Knowing the price at expiration, we can determine the price in period 1. Having determined that price, we can work back to period 0.

Figure 10.4 exhibits the option price at each node as well as the details of the replicating portfolio at each node. Remember, however, when we use equation (10.3), it is not necessary to compute Δ and B in order to derive the option price.[5] Here are details of the solution:

Year 2, Stock Price = \$87.669 Since we are at expiration, the option value is $\max(0, S - K) = \$47.669$.

Year 2, Stock Price = \$48.114 Again we are at expiration, so the option value is \$8.114.

Year 2, Stock Price = \$26.405 Since the option is out of the money, the value is 0.

Year 1, Stock Price = \$59.954 At this node we use equation (10.3) to compute the option value. (Note that once we are at this node, the "up" stock price, uS, is \$87.669, and the "down" stock price, dS, is \$48.114.)

$$e^{-0.08}\left(\$47.669 \times \frac{e^{0.08} - 0.803}{1.462 - 0.803} + \$8.114 \times \frac{1.462 - e^{0.08}}{1.462 - 0.803}\right) = \$23.029$$

[4]In cases where the tree recombines, the representation of stock price movements is also (and, some argue, more properly) called a *lattice*. The term *tree* would then be reserved for nonrecombining stock movements.

[5]As an exercise, you can verify the Δ and B at each node.

Year 1, Stock Price = \$32.903 Again we use equation (10.3) to compute the option value:

$$e^{-0.08}\left(\$8.114 \times \frac{e^{0.08} - 0.803}{1.462 - 0.803} + \$0 \times \frac{1.462 - e^{0.08}}{1.462 - 0.803}\right) = \$3.187$$

Year 0, Stock Price = \$41 Again using equation (10.3):

$$e^{-0.08}\left(\$23.029 \times \frac{e^{0.08} - 0.803}{1.462 - 0.803} + \$3.187 \times \frac{1.462 - e^{0.08}}{1.462 - 0.803}\right) = \$10.737$$

Notice the following:

- The option price is greater for the 2-year than for the 1-year option, as we would expect.
- We priced the option by working backward through the tree, starting at the end and working back to the first period.
- The option's Δ and B are different at different nodes. In particular, at a given point in time, Δ increases to 1 as we go further into the money.
- We priced a European option, so early exercise was not permitted. However, permitting early exercise would have made no difference. At every node prior to expiration, the option price is greater than $S - K$; hence we would not have exercised even if the option had been American.
- Once we understand the two-period option it is straightforward to value an option using more than two binomial periods. The important principle is to work backward through the tree.

Many Binomial Periods

The generalization to many binomial periods is straightforward. We can represent only a small number of binomial periods here, but a spreadsheet or computer program can handle a very large number of binomial nodes.

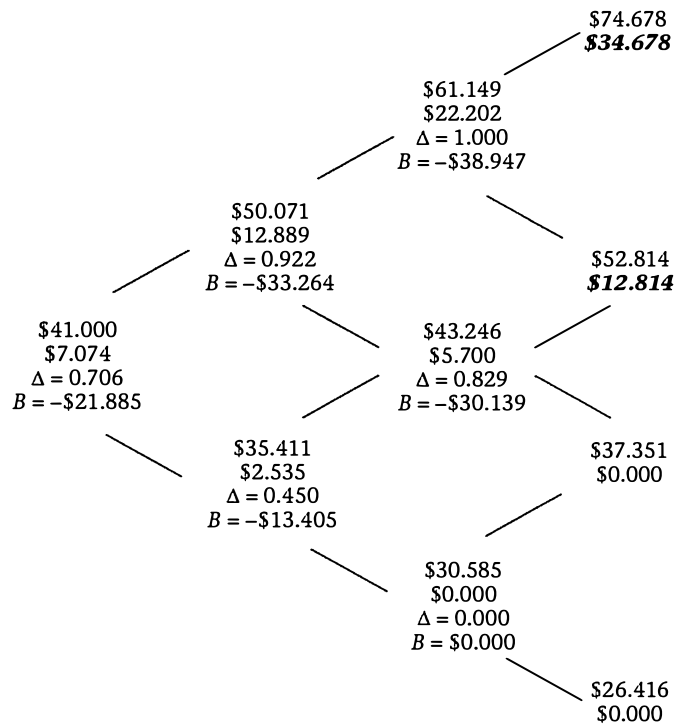
An obvious objection to the binomial calculations thus far is that the stock can only have two or three different values at expiration. It seems unlikely that the option price calculation will be accurate. The solution to this problem is to divide the time to expiration into more periods, generating a more realistic tree.

To illustrate how to do this, at the same time illustrating a tree with more than two periods, we will re-examine the 1-year European call option in Figure 10.3, which has a \$40 strike and initial stock price of \$41. Let there be three binomial periods. Since it is a 1-year call, this means that the length of a period is $h = \frac{1}{3}$. We will assume that other inputs stay the same, so $r = 0.08$ and $\sigma = 0.3$.

Figure 10.5 depicts the stock price and option price tree for this option. The option price is \$7.074, as opposed to \$7.839 in Figure 10.3. The difference occurs because

FIGURE 10.5

Binomial tree for pricing a European call option; assumes $S = \$41.00$, $K = \$40.00$, $\sigma = 0.30$, $r = 0.08$, $T = 1.00$ years, $\delta = 0.00$, and $h = 0.333$. At each node the stock price, option price, Δ, and B are given. Option prices in ***bold italic*** signify that exercise is optimal at that node.



the numerical approximation is different; it is quite common to see large changes in a binomial price when the number of periods, n, is changed, particularly when n is small.

Since the length of the binomial period is shorter, u and d are smaller than before (1.2212 and 0.8637 as opposed to 1.462 and 0.803 with $h = 1$). Just to be clear about the procedure, here is how the second-period nodes are computed:

$$S_u = \$41e^{0.08\times 1/3+0.3\sqrt{1/3}} = \$50.071$$

$$S_d = \$41e^{0.08\times 1/3-0.3\sqrt{1/3}} = \$35.411$$

The remaining nodes are computed similarly.

The option price is computed by working backward. The risk-neutral probability of the stock price going up in a period is

$$\frac{e^{0.08\times 1/3} - 0.8637}{1.2212 - 0.8637} = 0.4568$$

The option price at the node where $S = \$43.246$, for example, is then given by

$$e^{-0.08\times 1/3}\left([\$12.814 \times 0.4568] + [\$0 \times (1 - 0.4568)]\right) = \$5.700$$

Option prices at the remaining nodes are priced similarly.

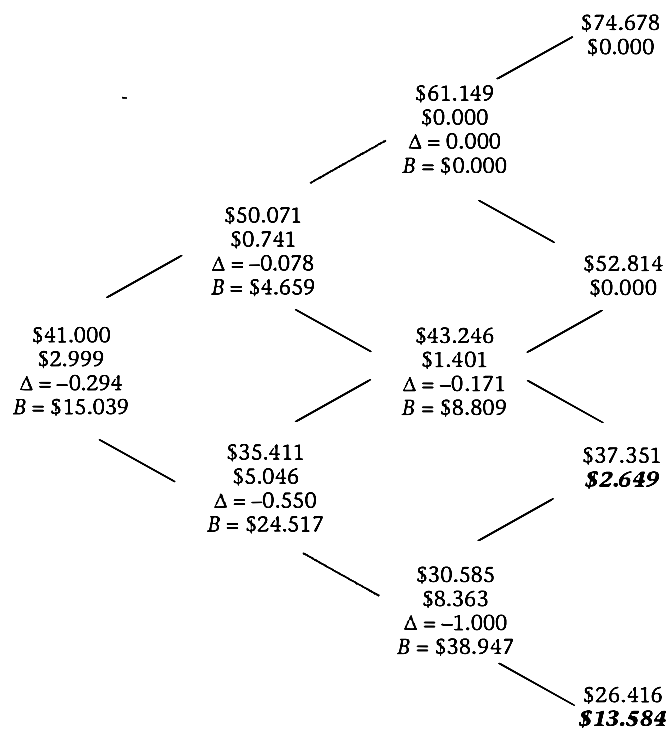
10.3 PUT OPTIONS

Thus far we have priced only call options. The binomial method easily accommodates put options also, as well as other derivatives. We compute put option prices using the same stock price tree and in almost the same way as call option prices; the only difference with a European put option occurs at expiration: Instead of computing the price as $\max(0, S - K)$, we use $\max(0, K - S)$.

Figure 10.6 shows the binomial tree for a European put option with 1 year to expiration and a strike of \$40 when the stock price is \$41. This is the same stock price tree as in Figure 10.5.

FIGURE 10.6

Binomial tree for pricing a European put option; assumes $S = \$41.00$, $K = \$40.00$, $\sigma = 0.30$, $r = 0.08$, $T = 1.00$ years, $\delta = 0.00$, and $h = 0.333$. At each node the stock price, option price, Δ, and B are given. Option prices in ***bold italic*** signify that exercise is optimal at that node.



To illustrate the calculations, consider the option price at the node where the stock price is \$35.411. The option price at that node is computed as

$$e^{-0.08\times 1/3}\left(\$1.401 \times \frac{e^{0.08\times 1/3}-0.8637}{1.2212-0.8637} + \$8.363 \times \frac{1.2212-e^{0.08\times 1/3}}{1.2212-0.8637}\right) = \$5.046$$

Figure 10.6 does raise one issue that we have not previously had to consider. Notice that at the node where the stock price is \$30.585, the option price is \$8.363. If this option were American, it would make sense to exercise at that node. The option is worth \$8.363 when held until expiration, but it would be worth \$40 − \$30.585 = \$9.415 if exercised at that node. Thus, in this case the American option should be more valuable than the otherwise equivalent European option. We will now see how to use the binomial approach to value American options.

10.4 AMERICAN OPTIONS

Since it is easy to check at each node whether early exercise is optimal, the binomial method is well-suited to valuing American options. The value of the option if it is left "alive" (i.e., unexercised) is given by the value of holding it for another period, equation (10.3). The value of the option if it is exercised is given by $\max(0, S - K)$ if it is a call and $\max(0, K - S)$ if it is a put.

Thus, for an American put, the value of the option at a node is given by

$$P(S, K, t) = \max\left(K - S, e^{-rh}\left[P(uS, K, t+h)p^* + P(dS, K, t+h)(1-p^*)\right]\right) \quad (10.12)$$

where, as in equation (10.5),

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

Figure 10.7 presents the binomial tree for the American version of the put option valued in Figure 10.6. The only difference in the trees occurs at the node where the stock price is \$30.585. The American option at that point is worth \$9.415, its early-exercise value. We have just seen in the previous section that the value of the option if unexercised is \$8.363.

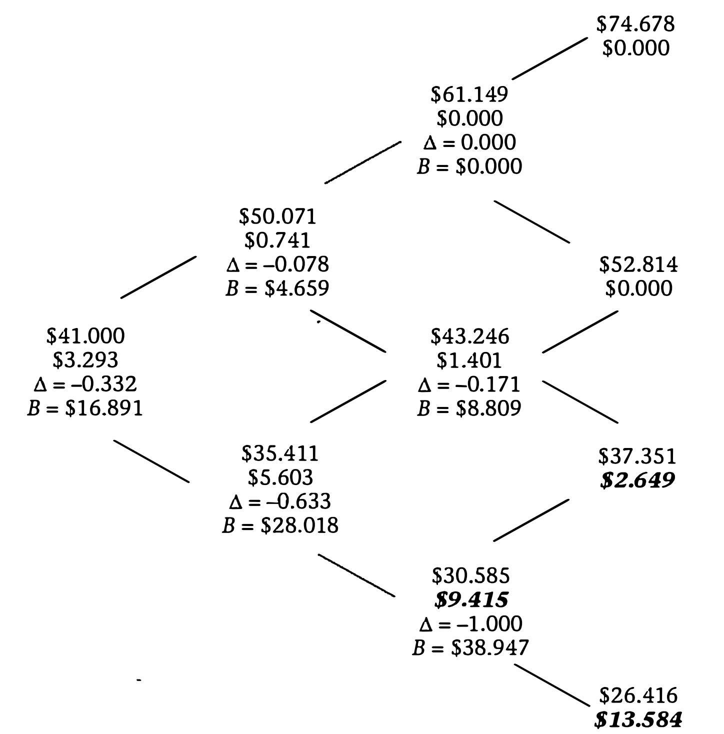
The greater value of the option at that node ripples back through the tree. When the option price is computed at the node where the stock price is \$35.411, the value is greater in Figure 10.7 than in Figure 10.6; the reason is that the price is greater at the subsequent node S_{dd} due to early exercise.

The initial option price is \$3.293, greater than the value of \$2.999 for the European option. This increase in value is due entirely to early exercise at the S_{dd} node.

In general the valuation of American options proceeds as in this example. At each node we check for early exercise. If the value of the option is greater when exercised, we assign that value to the node. Otherwise, we assign the value of the option unexercised. We work backward through the tree as usual.

FIGURE 10.7

Binomial tree for pricing an American put option; assumes $S = \$41.00$, $K = \$40.00$, $\sigma = 0.30$, $r = 0.08$, $T = 1.00$ years, $\delta = 0.00$, and $h = 0.333$. At each node the stock price, option price, Δ, and B are given. Option prices in ***bold italic*** signify that exercise is optimal at that node.



10.5 OPTIONS ON OTHER ASSETS

The model developed thus far can be modified easily to price options on underlying assets other than nondividend-paying stocks. In this section we present examples of how to do so. We examine options on stock indexes, currencies, and futures contracts. In every case the general procedure is the same: We compute the option price using equation (10.6). The difference for different underlying assets will be the construction of the binomial tree and the risk-neutral probability.

The valuation of an option on a stock that pays discrete dividends is more involved and is covered in Chapter 11.

Option on a Stock Index

Suppose a stock index pays continuous dividends at the rate δ. This type of option has in fact already been covered by our derivation in Section 10.1. The up and down index moves are given by equation (10.10), the replicating portfolio by equations (10.1) and

(10.2), and the option price by equation (10.3). The risk-neutral probability is given by equation (10.5).[6]

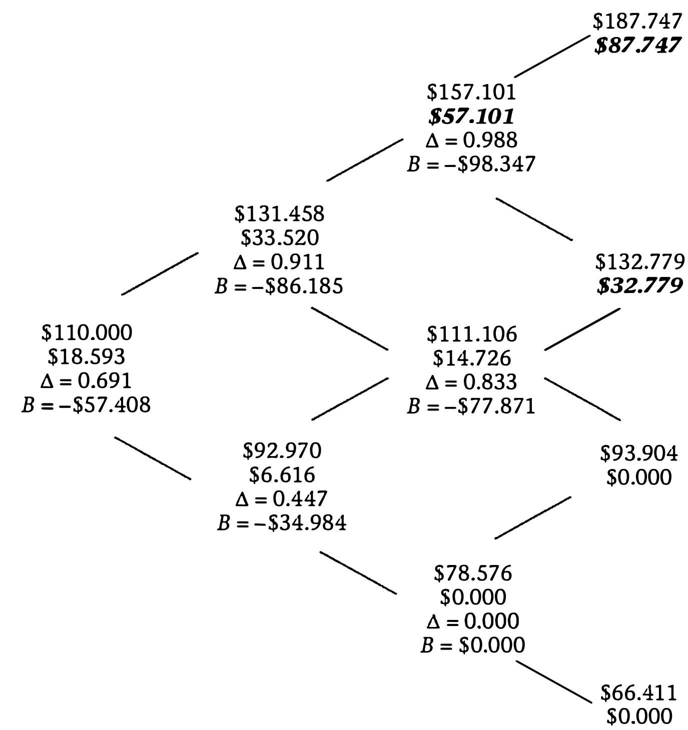
Figure 10.8 displays a binomial tree for an American call option on a stock index. Note that because of dividends, early exercise is optimal at the node where the stock price is \$157.101. Given these parameters, we have $p^* = 0.457$; hence, when $S = \$157.101$, the value of the option unexercised is

$$e^{-0.05\times 1/3}\left[0.457 \times \$87.747 + (1 - 0.457) \times \$32.779\right] = \$56.942$$

Since $57.101 > 56.942$, we exercise the option at that node.

FIGURE 10.8

Binomial tree for pricing an American call option on a stock index; assumes $S = \$110.00$, $K = \$100.00$, $\sigma = 0.30$, $r = 0.05$, $T = 1.00$ years, $\delta = 0.035$, and $h = 0.333$. At each node the stock price, option price, Δ, and B are given. Option prices in ***bold italic*** signify that exercise is optimal at that node.



[6] Intuitively, dividends can be taken into account either by (1) appropriately lowering the nodes on the tree and leaving risk-neutral probabilities unchanged, or (2) by reducing the risk-neutral probability and leaving the tree unchanged. The forward tree adopts the first approach.

Options on Currencies

With a currency with spot price x_0, the forward price is $F_{0,h} = x_0 e^{(r-r_f)h}$, where r_f is the foreign interest rate. Thus, we construct the binomial tree using

$$ux = xe^{(r-r_f)h+\sigma\sqrt{h}}$$
$$dx = xe^{(r-r_f)h-\sigma\sqrt{h}}$$

There is one subtlety in creating the replicating portfolio: Investing in a "currency" means investing in a money-market fund or fixed income obligation denominated in that currency. (We encountered this idea previously in Chapter 5.) Taking into account interest on the foreign-currency-denominated obligation, the two equations are

$$\Delta \times dxe^{r_f h} + e^{rh} \times B = C_d$$
$$\Delta \times uxe^{r_f h} + e^{rh} \times B = C_u$$

The risk-neutral probability of an up move in this case is given by

$$p^* = \frac{e^{(r-r_f)h} - d}{u - d} \tag{10.13}$$

Notice that if we think of r_f as the dividend yield on the foreign currency, these two equations look exactly like those for an index option. In fact the solution is the same as for an option on an index: Set the dividend yield equal to the foreign risk-free rate and the current value of the index equal to the spot exchange rate.

Figure 10.9 prices a dollar-denominated American put option on the euro. The current exchange rate is assumed to be \$1.05/€ and the strike is \$1.10/€. The euro-denominated interest rate is 3.1%, and the dollar-denominated rate is 5.5%.

Because volatility is low and the option is in-the-money, early exercise is optimal at three nodes prior to expiration.

Options on Futures Contracts

We now consider options on futures contracts. We assume the forward price is the same as the futures price. Since we build the tree based on the forward price, we simply add up and down movements around the current price. Thus, the nodes are constructed as

$$u = e^{\sigma\sqrt{h}}$$
$$d = e^{-\sigma\sqrt{h}}$$

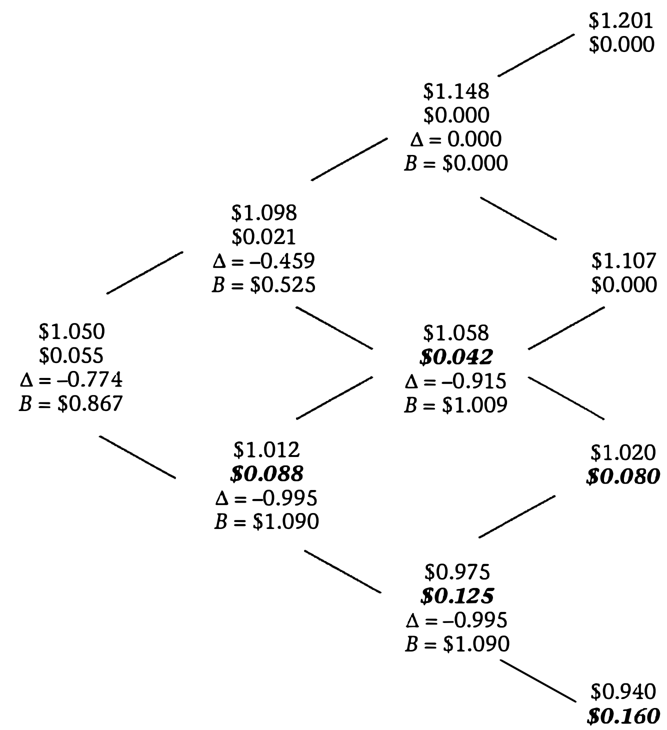
Note that this solution for u and d is exactly what we would get for an option on a stock index if δ, the dividend yield, were equal to the risk-free rate.

In constructing the replicating portfolio, recall that in each period a futures contract pays the change in the futures price, and there is no investment required to enter a futures contract. The problem is to find the number of futures contracts, Δ, and the lending, B, that replicates the option. We have

$$\Delta \times (dF - F) + e^{rh} \times B = C_d$$
$$\Delta \times (uF - F) + e^{rh} \times B = C_u$$

FIGURE 10.9

Binomial tree for pricing an American put option on a currency; assumes $S = \$1.05/€$, $K = \$1.10$, $\sigma = 0.10$, $r = 0.055$, $T = 0.50$ years, $\delta = 0.031$, and $h = 0.167$. At each node the stock price, option price, Δ, and B are given. Option prices in ***bold italic*** signify that exercise is optimal at that node.



Solving gives[7]

$$\Delta = \frac{C_u - C_d}{F(u-d)}$$

$$B = e^{-rh}\left(C_u \frac{1-d}{u-d} + C_d \frac{u-1}{u-d}\right)$$

While Δ tells us how many futures contracts to hold to hedge the option, the value of the option in this case is simply B. The reason is that the futures contract requires no

[7]The interpretation of Δ here is the number of futures contracts in the replicating portfolio. Another interpretation of Δ is the price sensitivity of the option when the price of the underlying asset changes. These two interpretations usually coincide, but not in the case of options on futures. The reason is that the futures price at time t reflects a price denominated in future dollars. The effect on the option price of a futures price change today is given by $e^{-rh}\Delta$. To see this, consider an option that is one binomial period (*continued*)

investment, so the only investment is that made in the bond. We can again price the option using equation (10.3).

The risk-neutral probability of an up move is given by

$$p^* = \frac{1-d}{u-d} \tag{10.14}$$

Figure 10.10 shows a tree for pricing an American call option on a gold futures contract. Early exercise is optimal when the price is \$336.720. The intuition for early exercise is that when an option on a futures contract is exercised, the option holder pays nothing, is entered into a futures contract, and receives mark-to-market proceeds of the difference between the strike price and the futures price. The motive for exercise is the ability to earn interest on the mark-to-market proceeds.

Options on Commodities

Many options exist on commodity futures contracts. However, it is also possible to have options on the physical commodity. If there is a market for lending and borrowing the commodity, then, in theory, pricing such an option is straightforward.

Recall from Chapter 6 that the *lease rate* for a commodity is conceptually similar to a dividend yield. If you borrow the commodity, you pay the lease rate. If you buy the commodity and lend it, you receive the lease rate. Thus, from the perspective of someone synthetically creating the option, the commodity is like a stock index, with the lease rate equal to the dividend yield.

Because this is conceptually the same as the pricing exercise in Figure 10.8 (imagine a commodity with a price of \$110, a lease rate of 3.5%, and a volatility of 30%), we do not present a pricing example.

In practice, pricing and hedging an option based on the physical commodity can be problematic. If an appropriate futures contract exists, a market-maker could use it to hedge a commodity option. Otherwise, transactions in physical commodities often have greater transaction costs than for financial assets. Short-selling a commodity may not be possible, for reasons discussed in Chapter 6. Market-making is then difficult.

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from expiration and for which $uF > dF > K$. Then

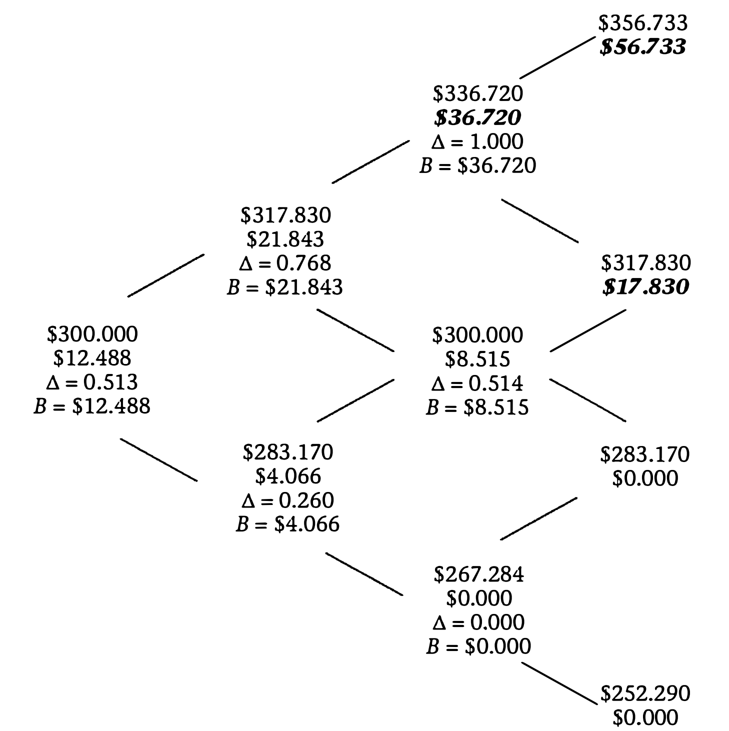
$$\Delta = \frac{uF - K - (dF - K)}{F(u-d)} = 1$$

But we also have

$$\begin{aligned} B &= e^{-rh}\left[(uF - K)\frac{1-d}{u-d} + (dF - K)\frac{u-1}{u-d}\right] \\ &= e^{-rh}(F - K) \end{aligned}$$

From the second expression, you can see that if the futures price changes by \$1, the option price changes by e^{-rh}.

FIGURE 10.10



Binomial tree for pricing an American call option on a futures contract; assumes $S = \$300.00$, $K = \$300.00$, $\sigma = 0.10$, $r = 0.05$, $T = 1.00$ years, $\delta = 0.05$, and $h = 0.333$. At each node the stock price, option price, Δ, and B are given. Option prices in ***bold italic*** signify that exercise is optimal at that node.

Options on Bonds

Finally, we will briefly discuss options on bonds. We devote a separate chapter later to discussing fixed-income derivatives, but it is useful to understand at this point some of the issues in pricing options on bonds. As a first approximation we could just say that bonds are like stocks that pay a discrete dividend (a coupon), and price bond options using the binomial model.

However, bonds differ from the assets we have been discussing in two important respects.

1. The volatility of a bond decreases over time as the bond approaches maturity. The prices of 30-day Treasury bills, for example, are much less volatile than the prices of 30-year Treasury bonds. The reason is that a given change in the interest rate, other things equal, changes the price of a shorter-lived bond by less.

2. We have been assuming in all our calculations that interest rates are the same for all maturities, do not change over time and are not random. While these assumptions may be good enough for pricing options on stocks, they are logically inconsistent for pricing options on bonds: If interest rates do not change unexpectedly, neither do bond prices.

In some cases, it may be reasonable to price bond options using the simple binomial model in this chapter. For example, consider a 6-month option on a 29-year bond. The underlying asset in this case is a 29.5-year bond. As a practical matter, the volatility difference between a 29.5- and a 29-year bond is likely to be very small. Also, because it is short-lived, this option will not be particularly sensitive to the short-term interest rate, so the correlation of the bond price and the 6-month interest rate will not matter much.

On the other hand, if we have a 3-year option to buy a 5-year bond, these issues might be quite important. Another issue is that bond coupon payments are discrete, so the assumption of a continuous dividend is an approximation.

In general, the conceptual and practical issues with bonds are different enough that bonds warrant a separate treatment. We will return to bonds in Chapter 24.

Summary

Here is the general procedure covering the other assets discussed in this section.

- Construct the binomial tree for the price of the underlying asset using

$$
\begin{aligned}
uS_t &= F_{t,t+h}e^{+\sigma\sqrt{h}} \quad \text{or} \quad u = \frac{F_{t,t+h}}{S_t}e^{+\sigma\sqrt{h}} \\
dS_t &= F_{t,t+h}e^{-\sigma\sqrt{h}} \quad \text{or} \quad d = \frac{F_{t,t+h}}{S_t}e^{-\sigma\sqrt{h}}
\end{aligned} \tag{10.15}
$$

  Since different underlying assets will have different forward price formulas, the tree will be different for different underlying assets.

- The option price at each node, if the option is unexercised, can then be computed as follows:

$$
\begin{aligned}
p^* &= \frac{F_{t,t+h}/S_t - d}{u - d} \\
&= \frac{e^{(r-\delta)h} - d}{u - d}
\end{aligned} \tag{10.16}
$$

  and, as before,

$$
C = e^{-rh}\left(p^*C_u + (1 - p^*)C_d\right) \tag{10.17}
$$

  where C_u and C_d are the up and down nodes relative to the current node. For an American option, at each node take the greater of this value and the value if exercised.

Pricing options with different underlying assets requires adjusting the risk-neutral probability for the borrowing cost or lease rate of the underlying asset. Mechanically, this means that we can use the formula for pricing an option on a stock index with an appropriate substitution for the dividend yield. Table 10.1 summarizes the substitutions.

TABLE 10.1 Substitutions for pricing options on assets other than a stock index.

Underlying Asset	Interest Rate	Dividend Yield
Stock index	Domestic risk-free rate	Dividend yield
Currency	Domestic risk-free rate	Foreign risk-free rate
Futures contract	Domestic risk-free rate	Domestic risk-free rate
Commodity	Domestic risk-free rate	Commodity lease rate
Coupon bond	Domestic risk-free rate	Yield on bond

CHAPTER SUMMARY

In order to price options, we must make an assumption about the probability distribution of the underlying asset. The binomial distribution provides a particularly simple stock price distribution: At any point in time, the stock price can go from S up to uS or down to dS, where the movement factors u and d are given by equation (10.10).

Given binomial stock price movements, the option can be replicated by holding Δ shares of stock and B bonds. The option price is the cost of this replicating portfolio, $\Delta S + B$. For a call option, $\Delta > 0$ and $B < 0$, so the option is replicated by borrowing to buy shares. For a put, $\Delta < 0$ and $B > 0$. If the option price does not equal this theoretical price, arbitrage is possible. The replicating portfolio is dynamic, changing as the stock price moves up or down. Thus it is unlike the replicating portfolio for a forward contract, which is fixed.

The binomial option pricing formula has an interpretation as a discounted expected value, with the risk-neutral probability (equation 10.5) used to compute the expected payoff to the option and the risk-free rate used to discount the expected payoff. This is known as risk-neutral pricing.

The binomial model can be used to price American and European calls and puts on a variety of underlying assets, including stocks, indexes, futures, currencies, commodities, and bonds.

FURTHER READING

This chapter has focused on the *mechanics* of binomial option pricing. Some of the underlying concepts will be discussed in more detail in Chapter 11. There we will have more to say about risk-neutral pricing, the link between the binomial tree and the assumed stock price distribution, how to estimate volatility, and how to price options when the stock pays a discrete dividend.

The binomial model provides a foundation for much of what we will do in later chapters. We will see in Chapter 12, for example, that the binomial option pricing formula gives results equivalent to the Black-Scholes formula when h becomes small. Consequently, if you thoroughly understand binomial pricing, you also understand the Black-Scholes formula. In Chapter 22, we will see how to generalize binomial trees to handle two sources of uncertainty.

In addition to the original papers by Cox et al. (1979) and Rendleman and Bartter (1979), Cox and Rubinstein (1985) provides an excellent exposition of the binomial model.

PROBLEMS

In these problems, n refers to the number of binomial periods. Assume all rates are continuously compounded unless the problem explicitly states otherwise.

10.1. Let $S = \$100$, $K = \$105$, $r = 8\%$, $T = 0.5$, and $\delta = 0$. Let $u = 1.3$, $d = 0.8$, and $n = 1$.

a. What are the premium, Δ, and B for a European call?

b. What are the premium, Δ, and B for a European put?

10.2. Let $S = \$100$, $K = \$95$, $r = 8\%$, $T = 0.5$, and $\delta = 0$. Let $u = 1.3$, $d = 0.8$, and $n = 1$.

a. Verify that the price of a European call is \$16.196.

b. Suppose you observe a call price of \$17. What is the arbitrage?

c. Suppose you observe a call price of \$15.50. What is the arbitrage?

10.3. Let $S = \$100$, $K = \$95$, $r = 8\%$, $T = 0.5$, and $\delta = 0$. Let $u = 1.3$, $d = 0.8$, and $n = 1$.

a. Verify that the price of a European put is \$7.471.

b. Suppose you observe a put price of \$8. What is the arbitrage?

c. Suppose you observe a put price of \$6. What is the arbitrage?

10.4. Let $S = \$100$, $K = \$95$, $\sigma = 30\%$, $r = 8\%$, $T = 1$, and $\delta = 0$. Let $u = 1.3$, $d = 0.8$, and $n = 2$. Construct the binomial tree for a call option. At each node provide the premium, Δ, and B.

10.5. Repeat the option price calculation in the previous question for stock prices of \$80, \$90, \$110, \$120, and \$130, keeping everything else fixed. What happens to the initial option Δ as the stock price increases?

10.6. Let $S = \$100$, $K = \$95$, $\sigma = 30\%$, $r = 8\%$, $T = 1$, and $\delta = 0$. Let $u = 1.3$, $d = 0.8$, and $n = 2$. Construct the binomial tree for a European put option. At each node provide the premium, Δ, and B.

10.7. Repeat the option price calculation in the previous question for stock prices of \$80, \$90, \$110, \$120, and \$130, keeping everything else fixed. What happens to the inital put Δ as the stock price increases?

10.8. Let $S = \$100$, $K = \$95$, $\sigma = 30\%$, $r = 8\%$, $T = 1$, and $\delta = 0$. Let $u = 1.3$, $d = 0.8$, and $n = 2$. Construct the binomial tree for an American put option. At each node provide the premium, Δ, and B.

10.9. Suppose $S_0 = \$100$, $K = \$50$, $r = 7.696\%$ (continuously compounded), $\delta = 0$, and $T = 1$.

a. Suppose that for $h = 1$, we have $u = 1.2$ and $d = 1.05$. What is the binomial option price for a call option that lives one period? Is there any problem with having $d > 1$?

b. Suppose now that $u = 1.4$ and $d = 0.6$. Before computing the option price, what is your guess about how it will change from your previous answer? Does it change? How do you account for the result? Interpret your answer using put-call parity.

c. Now let $u = 1.4$ and $d = 0.4$. How do you think the call option price will change from (a)? How does it change? How do you account for this? Use put-call parity to explain your answer.

10.10. Let $S = \$100$, $K = \$95$, $r = 8\%$ (continuously compounded), $\sigma = 30\%$, $\delta = 0$, $T = 1$ year, and $n = 3$.

a. Verify that the binomial option price for an American call option is \$18.283. Verify that there is never early exercise; hence, a European call would have the same price.

b. Show that the binomial option price for a European put option is \$5.979. Verify that put-call parity is satisfied.

c. Verify that the price of an American put is \$6.678.

10.11. Repeat the previous problem assuming that the stock pays a continuous dividend of 8% per year (continuously compounded). Calculate the prices of the American and European puts and calls. Which options are early-exercised?

10.12. Let $S = \$40$, $K = \$40$, $r = 8\%$ (continuously compounded), $\sigma = 30\%$, $\delta = 0$, $T = 0.5$ year, and $n = 2$.

a. Construct the binomial tree for the stock. What are u and d?

b. Show that the call price is \$4.110.

c. Compute the prices of American and European puts.

10.13. Use the same data as in the previous problem, only suppose that the call price is \$5 instead of \$4.110.

a. At time 0, assume you write the option and form the replicating portfolio to offset the written option. What is the replicating portfolio and what are the net cash flows from selling the overpriced call and buying the synthetic equivalent?

b. What are the cash flows in the next binomial period (3 months later) if the call at that time is fairly priced and you liquidate the position? What would you do if the option continues to be overpriced the next period?

c. What would you do if the option is underpriced the next period?

10.14. Suppose that the exchange rate is \$0.92/€. Let $r_\$ = 4\%$, and $r_€ = 3\%$, $u = 1.2$, $d = 0.9$, $T = 0.75$, $n = 3$, and $K = \$0.85$.

a. What is the price of a 9-month European call?

b. What is the price of a 9-month American call?

10.15. Use the same inputs as in the previous problem, except that $K = \$1.00$.

a. What is the price of a 9-month European put?

b. What is the price of a 9-month American put?

10.16. Suppose that the exchange rate is 1 dollar for 120 yen. The dollar interest rate is 5% (continuously compounded) and the yen rate is 1% (continuously compounded). Consider an at-the-money American dollar call that is yen-denominated (i.e., the call permits you to buy 1 dollar for 120 yen). The option has 1 year to expiration and the exchange rate volatility is 10%. Let $n = 3$.

a. What is the price of a European call? An American call?

b. What is the price of a European put? An American put?

c. How do you account for the pattern of early exercise across the two options?

10.17. An option has a gold futures contract as the underlying asset. The current 1-year gold futures price is \$300/oz., the strike price is \$290, the risk-free rate is 6%, volatility is 10%, and time to expiration is 1 year. Suppose $n = 1$. What is the price of a call option on gold? What is the replicating portfolio for the call option? Evaluate the statement: "Replicating a call option always entails borrowing to buy the underlying asset."

10.18. Suppose the S&P 500 futures price is 1000, $\sigma = 30\%$, $r = 5\%$, $\delta = 5\%$, $T = 1$, and $n = 3$.

a. What are the prices of European calls and puts for $K = \$1000$? Why do you find the prices to be equal?

b. What are the prices of American calls and puts for $K = \$1000$?

c. What are the time-0 replicating portfolios for the European call and put?

10.19. For a stock index, $S = \$100$, $\sigma = 30\%$, $r = 5\%$, $\delta = 3\%$, and $T = 3$. Let $n = 3$.

a. What is the price of a European call option with a strike of \$95?

b. What is the price of a European put option with a strike of \$95?

c. Now let $S = \$95$, $K = \$100$, $\sigma = 30\%$, $r = 3\%$, and $\delta = 5\%$. (You have exchanged values for the stock price and strike price and for the interest rate and dividend yield.) Value both options again. What do you notice?

10.20. Repeat the previous problem calculating prices for American options instead of European. What happens?

10.21. Suppose that $u < e^{(r-\delta)h}$. Show that there is an arbitrage opportunity. Now suppose that $d > e^{(r-\delta)h}$. Show again that there is an arbitrage opportunity.

APPENDIX 10.A: TAXES AND OPTION PRICES

It is possible to solve for a binomial price when there are taxes. Suppose that each form of income is taxed at a different rate: interest at the rate τ_i, capital gains on a stock at the rate τ_g, capital gains on options at the rate τ_O, and dividends at the rate τ_d. We assume that taxes on all forms of income are paid on an accrual basis, and that there is no limit on the ability to deduct losses or to offset losses on one form of income against gains on another form of income.

We then choose Δ_t and B_t by requiring that the *after-tax* return on the stock/bond portfolio equal the *after-tax* return on the option in both the up and down states. Thus we require that

$$\begin{aligned}\left[S_{t+h} - \tau_g(S_{t+h} - S_t) + \delta S_t(1 - \tau_d)\right]\Delta_t + [1 + r_h(1 - \tau)]\,B_t \\ = \phi_{t+h}(S_{t+h}) - \tau_O\left[\phi_{t+h}(S_{t+h}) - \phi_t(S_t)\right]\end{aligned} \tag{10.18}$$

The solutions for Δ and B are then

$$\Delta = \frac{1-\tau_O}{1-\tau_g}\frac{\phi_1(S_1^+) - \phi_1(S_1^-)}{S_1^+ - S_1^-}$$

$$B = \frac{1}{1 + r_h\dfrac{1-\tau_i}{1-\tau_O}}\left(\frac{u\phi_1(S_1^-) - d\phi_1(S_1^+)}{u-d} - \frac{\Delta}{1-\tau_O}S_0\left[\frac{\tau_g - \tau_O}{1-\tau_g} + \delta(1-\tau_d)\right]\right)$$

This gives an option price of

$$\phi_t = \frac{1}{1 + r_h\dfrac{1-\tau_i}{1-\tau_O}}\left[p^*\phi_{t+h}(S_{t+h}^+) + (1-p^*)\phi_{t+h}(S_{t+h}^-)\right] \tag{10.19}$$

where

$$p^* = \frac{1 + r_h\dfrac{1-\tau_i}{1-\tau_g} - \delta\dfrac{1-\tau_d}{1-\tau_O} - d}{u-d} \tag{10.20}$$

In practice, dealers are marked-to-market for tax purposes and face the same tax rate on all forms of income. In this case taxes drop out of all the option-pricing expressions. When dealers are the effective price-setters in a market, taxes should not affect prices.