CHAPTER 11

Binomial Option Pricing: II

Chapter 10 introduced binomial option pricing, focusing on how the model can be used to compute European and American option prices for a variety of underlying assets. In this chapter we continue the discussion of binomial pricing, delving more deeply into the economics of the model and its underlying assumptions.

First, the binomial model can value options that may be early-exercised. We will examine early exercise in more detail, and see that the option pricing calculation reflects the economic determinants of early exercise discussed in Chapter 9.

Second, the binomial option pricing formula can be interpreted as the expected option payoff one period hence, discounted at the risk-free rate. In Chapter 10 we referred to this calculation as *risk-neutral pricing*. This calculation appears to be inconsistent with standard discounted cash flow valuation, in which expected cash flows are discounted at a risk-adjusted rate, not the risk-free rate. We show that, in fact, the binomial pricing formula (and, hence, risk-neutral valuation) is consistent with option valuation using standard discounted cash flow techniques.

Third, we modeled the stock price by using volatility (σ) to determine the magnitude of the up and down stock price movements. In this chapter we explain this calculation in more detail. What is the economic meaning of this assumption? In constructing the binomial tree, why is volatility multiplied by the square root of time ($\sigma\sqrt{h}$)? How should we estimate volatility?

Finally, we saw how to price options on stock indices where the dividend is continuous. In this chapter we adapt the binomial model to price options on stocks that pay discrete dividends.

11.1 Understanding Early Exercise

In deciding whether to early-exercise an option, the option holder compares the value of exercising immediately with the value of continuing to hold the option, and exercises if immediate exercise is more valuable. This is the comparison we performed at each binomial node when we valued American options in Chapter 10.

We obtain an economic perspective on the early-exercise decision by considering the costs and benefits of early exercise. As discussed in Section 9.3, there are three economic considerations governing the decision to exercise early. By exercising, the option holder

- Receives the stock and therefore receives future dividends,
- Pays the strike price prior to expiration (this has an interest cost), and
- Loses the insurance implicit in the call. By holding the call instead of exercising, the option holder is protected against the possibility that the stock price will be less than the strike price at expiration. Once the option is exercised, this protection no longer exists.

Consider an example where a call option has a strike price of \$100, the interest rate is 5%, and the stock pays continuous dividends of 5%. If the stock price is \$200, the net effect of dividends and interest encourages early exercise. Annual dividends are approximately 5% of \$200, or $0.05 \times \$200 = \$10$. The annual interest saved by deferring exercise is approximately $0.05 \times \$100 = \$5$. Thus, for a stock price of \$200 (indeed, for any stock price above \$100) dividends lost by not exercising exceed interest saved by deferring exercise.

The only reason in this case not to exercise early is the implicit insurance the option owner loses by exercising. This implicit insurance arises from the fact that the option holder could exercise and then the stock price could fall below the strike price of \$100. Leaving the option unexercised protects against this scenario. The early-exercise calculation for a call therefore implicitly weighs dividends, which encourage early exercise, against interest and insurance, which discourage early exercise.

If volatility is zero, then the value of insurance is zero, and it is simple to find the optimal exercise policy as long as r and δ are constant. It is optimal to defer exercise as long as interest savings on the strike exceed dividends lost, or

$$rK > \delta S$$

It is optimal to exercise when this is not true, or

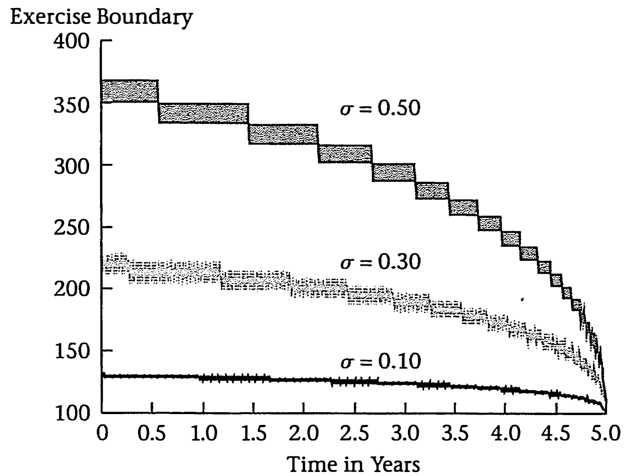
$$S > \frac{rK}{\delta}$$

In the special case when $r = \delta$ and $\sigma = 0$, any in-the-money option should be exercised immediately. If $\delta = 0.5r$, then we exercise when the stock price is twice the exercise price.

The decision to exercise is more complicated when volatility is positive. In this case the implicit insurance has value that varies with time to expiration. Figure 11.1 displays the price above which early exercise is optimal for a 5-year option with $K = \$100$, $r = 5\%$, and $\delta = 5\%$, for three different volatilities, computed using 500 binomial steps. Recall from Chapter 9 that if it is optimal to exercise a call at a given stock price, then it is optimal to exercise at all higher stock prices. Figure 11.1 thus shows the *lowest* stock price at which exercise is optimal. The oscillation in this lowest price, which is evident in the figure, is due to the up and down binomial movements that approximate the behavior of the stock; with an infinite number of binomial steps the early-exercise schedule would be smooth and continuously decreasing. Comparing the three lines, we observe a significant volatility effect. A 5-year option with a volatility of 50% should only be exercised if the stock price exceeds about \$360. If volatility is 10%, the boundary

FIGURE 11.1

Early-exercise boundaries for volatilities of 10%, 30%, and 50% for a 5-year American call option. In all cases $K = \$100$, $r = 5\%$, and $\delta = 5\%$.

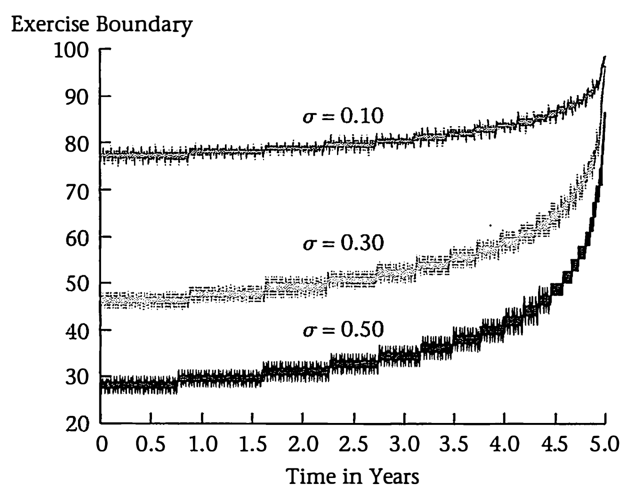


drops to \$130. This volatility effect stems from the fact that the insurance value lost by early-exercising is greater when volatility is greater.

Figure 11.2 performs the same experiment for put options with the same inputs. The picture is similar, as is the logic: The advantage of early exercise is receiving the strike price sooner rather than later. The disadvantages are the dividends lost by giving up the stock, and the loss of insurance against the stock price exceeding the strike price.

FIGURE 11.2

Early-exercise boundaries for volatilities of 10%, 30%, and 50% for a 5-year American put option. In all cases $K = \$100$, $r = 5\%$, and $\delta = 5\%$.



Figures 11.1 and 11.2 also show that, other things equal, early-exercise criteria become less stringent closer to expiration. This occurs because the value of insurance diminishes as the options approach expiration.

While these pictures are constructed for the special case where $\delta = r$, the overall conclusion holds generally.

11.2 UNDERSTANDING RISK-NEUTRAL PRICING

In Chapter 10, we saw that the binomial option pricing formula can be written

$$C = e^{-rh}[p^* C_u + (1 - p^*) C_d] \tag{11.1}$$

where

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} \tag{11.2}$$

We labeled p^* the *risk-neutral probability* that the stock will go up. Equation (11.1) has the appearance of a discounted expected value, where the expected value calculation uses p^* and discounting is done at the risk-free rate.

The idea that an option price is the result of a present value calculation is reassuring, but at the same time equation (11.1) is puzzling. A standard discounted cash flow calculation would require computing an expected value using the true probability that the stock price would go up. Discounting would then be done using the expected return on an asset of equivalent risk, not the risk-free rate. Moreover, what is p^*? Is it really a probability?

We will begin our exploration of risk-neutral pricing by interpreting p^*, showing that it is not the true probability that the stock goes up, but rather the probability that gives the stock an expected rate of return equal to the risk-free rate. We will then show that it *is* possible to compute an option price using standard discounted cash flow calculations using the true probability that the stock goes up, but that doing so is cumbersome.

The Risk-Neutral Probability

It is common in finance to emphasize that investors are risk averse. To see what risk aversion means, suppose you are offered either (a) \$1000, or (b) \$2000 with probability 0.5, and \$0 with probability 0.5. A **risk-averse** investor prefers (a), since alternative (b) is risky and has the same expected value as (a). This kind of investor will require a premium to bear risk when expected values are equal.

A **risk-neutral** investor is indifferent between a sure thing and a risky bet with an expected payoff equal to the value of the sure thing. A risk-neutral investor, for example, will be equally happy with alternative (a) or (b).

Before proceeding, we need to emphasize that *at no point are we assuming that investors are risk-neutral. Now and throughout the book, the pricing calculations are consistent with investors being risk-averse.*

Having said this, let's consider what an imaginary world populated by risk-neutral investors would be like. In such a world, investors care only about expected returns and not about riskiness. Assets would have no risk premium since investors would be willing to hold assets with an expected return equal to the risk-free rate.

In this hypothetical risk-neutral world, we can solve for the probability of the stock going up, p^*, such that the stock is expected to earn the risk-free rate. In the binomial model we assume that the stock can go up to uS or down to dS. If the stock is to earn the risk-free return on average, then the probability that the stock will go up, p^*, must satisfy

$$p^* u S e^{\delta h} + (1 - p^*) d S e^{\delta h} = e^{rh} S$$

Solving for p^* gives

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

This is exactly the definition of p^* in equation (11.2). This is why we refer to p^* as the *risk-neutral probability that the stock price will go up.* It is the probability that the stock price would increase in a risk-neutral world.

Not only would the risk-neutral probability, equation (11.2), be used in a risk-neutral world, but also all discounting would take place at the risk-free rate. Thus, the option pricing formula, equation (11.1), can be said to price options *as if* investors are risk-neutral. At the risk of being repetitious, we are not assuming that investors are actually risk-neutral, and we are not assuming that risky assets are actually expected to earn the risk-free rate of return. Rather, *risk-neutral pricing is an* interpretation *of the formulas above.* Those formulas in turn arise from finding the cost of the portfolio that replicates the option payoff.

Interestingly, this interpretation of the option-pricing procedure has great practical importance; risk-neutral pricing can sometimes be used where other pricing methods are too difficult. We will see in Chapter 19 that risk-neutral pricing is the basis for Monte Carlo valuation, in which asset prices are simulated under the assumption that assets earn the risk-free rate, and these simulated prices are used to value the option.

Pricing an Option Using Real Probabilities

We are left with the question: Is option pricing consistent with standard discounted cash flow calculations? The answer is yes. We can use the true distribution for the future stock price in computing the expected payoff to the option. This expected payoff can then be discounted with a rate based on the stock's required return.

Discounted cash flow is not used in practice to price options because there is no reason to do so: It is necessary to compute the option price in order to compute the correct discount rate. However, we present two examples of valuing an option using real probabilities to see the difficulty in using real probabilities, and also to understand how to determine the risk of an option.

Suppose that the continuously compounded expected return on the stock is α and that the stock does not pay dividends. Then if p is the true probability of the stock going up, p must be consistent with u, d, and α:

$$puS + (1 - p)dS = e^{\alpha h} S \tag{11.3}$$

Solving for p gives us

$$p = \frac{e^{\alpha h} - d}{u - d} \tag{11.4}$$

For probabilities to be between 0 and 1, we must have $u > e^{\alpha h} > d$. Using p, the actual expected payoff to the option one period hence is

$$pC_u + (1 - p)C_d = \frac{e^{\alpha h} - d}{u - d}C_u + \frac{u - e^{\alpha h}}{u - d}C_d \tag{11.5}$$

Now we face the problem with using real as opposed to risk-neutral probabilities: At what rate do we discount this expected payoff? It is not correct to discount the option at the expected return on the stock, α, because the option is equivalent to a leveraged investment in the stock and, hence, is riskier than the stock.

Denote the appropriate per-period discount rate for the option as γ. To compute γ, we can use the fact that the required return on any portfolio is the weighted average of the returns on the assets in the portfolio.[1] In Chapter 10, we saw that an option is equivalent to holding a portfolio consisting of Δ shares of stock and B bonds. The expected return on this portfolio is

$$e^{\gamma h} = \frac{S\Delta}{S\Delta + B}e^{\alpha h} + \frac{B}{S\Delta + B}e^{rh} \tag{11.6}$$

We can now compute the option price as the expected option payoff, equation (11.5), discounted at the appropriate discount rate, given by equation (11.6). This gives

$$e^{-\gamma h}\left[\frac{e^{\alpha h} - d}{u - d}C_u + \frac{u - e^{\alpha h}}{u - d}C_d\right] \tag{11.7}$$

It turns out that *this gives us the same option price as performing the risk-neutral calculation.* Appendix 11.A demonstrates algebraically that equation (11.7) is equivalent to the risk-neutral calculation, equation (11.1).

The calculations leading to equation (11.7) started with the assumption that the expected return on the stock is α. We then derived a consistent probability, p, and discount rate for the option, γ. You may be wondering if it matters whether we have the "correct" value of α to start with. The answer is that it does not matter: *Any* consistent pair of α and γ will give the same option price. Risk-neutral pricing is valuable because setting $\alpha = r$ results in the *simplest* pricing procedure.

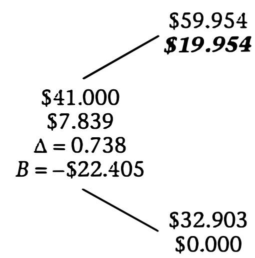
A one-period example To see how to value an option using true probabilities, we will compute two examples. First, consider the one-period binomial example in Figure 11.3. Suppose that the continuously compounded expected return on XYZ is $\alpha = 15\%$. Then the true probability of the stock going up, from equation (11.4), is

$$p = \frac{e^{0.15} - 0.8025}{1.4623 - 0.8025} = 0.5446$$

[1] See, for example, Brealey and Myers (2003, ch. 9).

FIGURE 11.3

Binomial tree for pricing a European call option; assumes $S = \$41.00$, $K = \$40.00$, $\sigma = 0.30$, $r = 0.08$, $T = 1.00$ years, $\delta = 0.00$, and $h = 1.000$. This is the same as Figure 10.3.



The expected payoff to the option in one period, from equation (11.5) is

$$0.5446 \times \$19.954 + (1 - 0.5446) \times \$0 = \$10.867$$

The replicating portfolio, Δ and B, does not depend on p or α. In this example, $\Delta = 0.738$ and $B = -\$22.405$. The discount rate, γ, from equation (11.6) is given by

$$\begin{aligned} e^{\gamma h} &= \frac{0.738 \times \$41}{0.738 \times \$41 - \$22.405} e^{0.15} + \frac{-\$22.405}{0.738 \times \$41 - \$22.405} e^{0.08} \\ &= 1.386 \end{aligned}$$

Thus, $\gamma = \ln(1.386) = 32.64\%$. The option price is then given by equation (11.7):

$$e^{-0.3264} \times \$10.867 = \$7.839$$

This is exactly the price we obtained before.

Notice that in order to compute the discount rate, we first had to compute Δ and B. But once we have computed Δ and B, we can simply compute the option price as $\Delta S + B$. There is no need for further computations. It can be helpful to know the actual expected return on an option, but for valuation it is pointless.

A multi-period example To demonstrate that this method of valuation works over multiple periods, Figure 11.4 presents the same binomial tree as Figure 10.5, with the addition that the true discount rate for the option, γ, is reported at each node. Given the 15% continuously compounded discount rate, the true probability of an up move in Figure 11.4 is

$$\frac{e^{0.15 \times 1/3} - 0.8637}{1.2212 - 0.8637} = 0.5247$$

To compute the price at the node where the stock price is \$61.149, we discount the expected option price the next period at 26.9%. This gives

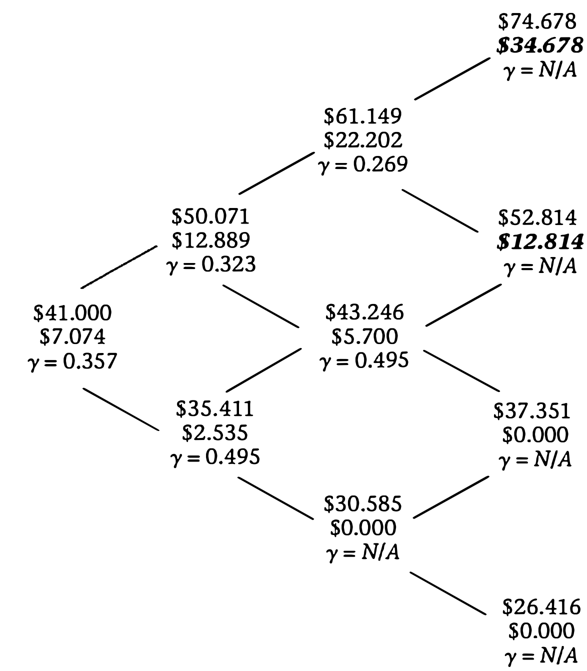
$$e^{-0.269 \times 1/3} \left[0.5247 \times \$34.678 + (1 - 0.5247) \times \$12.814\right] = \$22.202$$

When the stock price is \$43.246, the discount rate is 49.5%, and the option price is

$$e^{-0.495 \times 1/3} \left[0.5247 \times \$12.814 + (1 - 0.5247) \times \$0\right] = \$5.700$$

FIGURE 11.4

Binomial tree for pricing an American call option; assumes $S = \$41.00$, $K = \$40.00$, $\sigma = 0.30$, $r = 0.08$, $T = 1.00$ years, $\delta = 0.00$, and $h = 0.333$. The continuously compounded true expected return on the stock, α, is 15%. At each node the stock price, option price, and continuously compounded true discount rate for the option, γ, are given.



These are both the same option prices as in Figure 10.5, where we used risk-neutral pricing.

We continue by working back through the tree. To compute the price at the node where the stock price is \$50.071, we discount the expected option price the next period at 32.3%. Thus,

$$e^{-0.323\times 1/3}\left[0.5247 \times \$22.202 + (1 - 0.5247) \times \$5.700\right] = \$12.889$$

Again, this is the same price at this node as in Figure 10.5.

The actual discount rate for the option changes as we move down the tree at a point in time and also over time. The required return on the option is less when the stock price is \$61.149 (26.9%) than when it is \$43.246 (49.5%). The discount rate increases as the stock price decreases because the option is equivalent to a leveraged position in the stock, and the degree of leverage increases as the option moves out of the money.

These examples illustrate that it is possible to obtain option prices using standard discounted-cash-flow techniques. Generally, however, there is no reason to do so. Moreover, the fact that risk-neutral pricing works means that it is not necessary to estimate α, the expected return on the stock, when pricing an option. Since expected returns are hard to estimate precisely, this makes option pricing a great deal easier.

Appendix 11.B goes into more detail about risk-neutral pricing.

11.3 THE BINOMIAL TREE AND LOGNORMALITY

The usefulness of the binomial pricing model hinges on the binomial tree providing a reasonable representation of the stock price distribution. In this section we discuss the motivation for and plausibility of the binomial tree. We will define a lognormal distribution and see that the binomial tree approximates this distribution.

The Random Walk Model

It is often said that stock prices follow a random walk. In this section we will explain what a random walk is. In the next section we will apply the random walk model to stock prices.

To understand a random walk, imagine that we flip a coin repeatedly. Let the random variable Y denote the outcome of the flip. If the coin lands displaying a head, $Y = 1$. If the coin lands displaying a tail, $Y = -1$. If the probability of a head is 50%, we say the coin is fair. After n flips, with the ith flip denoted Y_i, the cumulative total, Z_n, is

$$Z_n = \sum_{i=1}^{n} Y_i \tag{11.8}$$

It turns out that the more times we flip, on average, the farther we will move from where we start. We can understand intuitively why with more flips the average distance from the starting point increases. Think about the first flip and imagine you get a head. You move to $+1$, and as far as the remaining flips are concerned, *this is your new starting point*. After the second flip, you will either be at 0 or $+2$. If you are at zero, it is as if you started over; however if you are at $+2$ you are starting at $+2$. Continuing in this way your average distance from the starting point increases with the number of flips.[2]

Another way to represent the process followed by Z_n is in terms of the *change* in Z_n:

$$Z_n - Z_{n-1} = Y_n$$

We can rewrite this more explicitly as

$$\text{Heads:} \quad Z_n - Z_{n-1} = +1 \tag{11.9}$$

$$\text{Tails:} \quad Z_n - Z_{n-1} = -1 \tag{11.10}$$

[2]After n flips, the average squared distance from the starting point will be n. Conditional on the first flip being a head, your average squared distance is $0.5 \times 0 + 0.5 \times 2^2 = 2$. If your first flip had been a tail, your average squared distance after two moves would also be 2. Thus, the unconditional average squared distance is 2 after 2 flips. If D_n^2 represents your squared distance from the starting point, then

$$D_n^2 = 0.5 \times (D_{n-1} + 1)^2 + 0.5 \times (D_{n-1} - 1)^2 = D_{n-1}^2 + 1$$

Since $D_0^2 = 0$, this implies that $D_n^2 = n$. This idea that with a random walk you drift increasingly farther from the starting point is an important concept later in the book.

With heads, the *change* in Z is 1, and with tails, the change in Z is -1. This random walk is illustrated in Figure 11.5.

The idea that asset prices should follow a random walk was articulated in Samuelson (1965). In efficient markets, an asset price should reflect all available information. By definition, new information is a surprise. In response to new information the price is equally likely to move up or down, as with the coin flip. The price after a period of time is the initial price plus the cumulative up and down movements due to informational surprises.

Modeling Stock Prices as a Random Walk

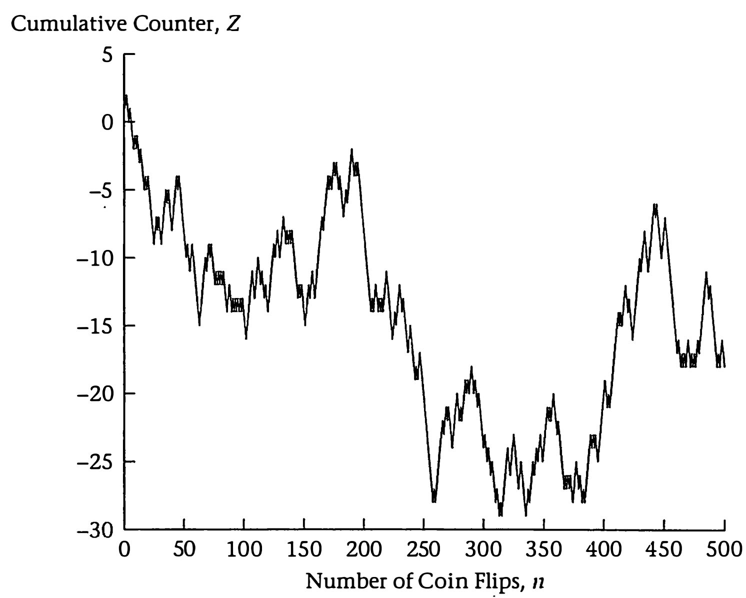
The idea that stock prices move up or down randomly makes sense; however, the description of a random walk in the previous section is not a satisfactory description of stock price movements. Suppose we take the random walk model in Figure 11.5 literally. Assume the beginning stock price is \$100, and the stock price will move up or down \$1 each time we flip the coin. There are at least three problems with this model:

1. If by chance we get enough cumulative down movements, the stock price will become negative. Because stockholders have limited liability (they can walk away from a bankrupt firm), a stock price will never be negative.

2. The magnitude of the move (\$1) should depend upon how quickly the coin flips occur and the level of the stock price. If we flip coins once a second, \$1 moves

FIGURE 11.5

Illustration of a random walk, where the counter, Z, increases by one when a fair coin flip comes up heads, and decreases by one with tails.



are excessive; in real life, a \$100 stock will not typically have 60 \$1 up or down movements in 1 minute. Also, if a \$1 move is appropriate for a \$100 stock, it likely isn't appropriate for a \$5 stock.

3. The stock on average should have a positive return. The random walk model taken literally does not permit this.

It turns out that the binomial model is a variant of the random walk model that solves all of these problems at once. The binomial model assumes that *continuously compounded returns are a random walk.* Thus, before proceeding, we first review some properties of continuously compounded returns.

Continuously Compounded Returns

Here is a summary of the important properties of continuously compounded returns. (See also Appendix B at the end of this book.)

The logarithmic function computes returns from prices Let S_t and S_{t+h} be stock prices at times t and $t+h$. The continuously compounded return between t and $t+h$, $r_{t,t+h}$ is then

$$r_{t,t+h} = \ln(S_{t+h}/S_t) \tag{11.11}$$

The exponential function computes prices from returns If we know the continuously compounded return, we can obtain S_{t+h} by exponentiating both sides of equation (11.11). This gives

$$S_{t+h} = S_t e^{r_{t,t+h}} \tag{11.12}$$

Continuously compounded returns are additive Suppose we have continuously compounded returns over a number of periods—for example, $r_{t,t+h}$, $r_{t+h,t+2h}$, etc. The continuously compounded return over a long period is the *sum* of continuously compounded returns over the shorter periods, i.e.,

$$r_{t,t+nh} = \sum_{i=1}^{n} r_{t+(i-1)h,t+ih} \tag{11.13}$$

Continuously compounded returns can be less than −100% A continuously compounded return that is a large negative number still gives a positive stock price. The reason is that e^r is positive for any r. Thus, if the log of the stock price follows a random walk, the stock price cannot become negative.

Here are some examples illustrating these statements.

Example 11.1 Suppose the stock price on four consecutive days is \$100, \$103, \$97, and \$98. The daily continuously compounded returns are

$$\ln(103/100) = 0.02956; \quad \ln(97/103) = -0.06002; \quad \ln(98/97) = 0.01026$$

The continuously compounded return from day 1 to day 4 is $\ln(98/100) = -0.0202$. This is also the sum of the daily continuously compounded returns:

$$r_{1,2} + r_{2,3} + r_{3,4} = 0.02956 + (-0.06002) + 0.01026 = -0.0202$$

Example 11.2 Suppose that the stock price today is \$100 and that 1 year from today it is \$10. The percentage return is $(10 - 100)/100 = -0.9 = -90\%$. However, the continuously compounded return is $\ln(10/100) = -2.30$, a continuously compounded return of -230%.

Example 11.3 Suppose that the stock price today is \$100 and that over 1 year the continuously compounded return is -500%. Using equation (11.12), the end-of-year price will be small but positive: $S_1 = 100e^{-5.00} = \$0.6738$. The percentage return is $0.6738/100 - 1 = -99.326\%$.

The Standard Deviation of Returns

Suppose the continuously compounded return over month i is $r_{\text{monthly},i}$. From equation (11.13), we can sum continuously compounded returns. Thus, the annual return is

$$r_{\text{annual}} = \sum_{i=1}^{12} r_{\text{monthly},i}$$

The variance of the annual return is therefore

$$\text{Var}(r_{\text{annual}}) = \text{Var}\left(\sum_{i=1}^{12} r_{\text{monthly},i}\right) \tag{11.14}$$

Now suppose that returns are uncorrelated over time; that is, the realization of the return in one period does not affect the expected returns in subsequent periods. With this assumption, the variance of a sum is the sum of the variances. Also suppose that each month has the same variance of returns. If we let σ^2 denote the annual variance, then from equation (11.14) we have

$$\sigma^2 = 12 \times \sigma^2_{\text{monthly}}$$

Taking the square root of both sides and rearranging, we can express the monthly standard deviation in terms of the annual standard deviation:

$$\sigma_{\text{monthly}} = \frac{\sigma}{\sqrt{12}}$$

If we split the year into n periods of length h (so that $h = 1/n$), the standard deviation over the period of length h, σ_h, is

$$\boxed{\sigma_h = \sigma\sqrt{h}} \tag{11.15}$$

The standard deviation therefore scales with the square root of time. This is why $\sigma\sqrt{h}$ appears in the binomial pricing model.

The Binomial Model

We are now in a position to better understand the binomial model, which is

$$S_{t+h} = S_t e^{(r-\delta)h \pm \sigma\sqrt{h}}$$

Taking logs, we obtain

$$\ln(S_{t+h}/S_t) = (r - \delta)h \pm \sigma\sqrt{h} \qquad (11.16)$$

Since $\ln(S_{t+h}/S_t)$ is the continuously compounded return from t to $t + h$, $r_{t,t+h}$, the binomial model is simply a particular way to model the continuously compounded return. That return has two parts, one of which is certain $[(r - \delta)h]$, and the other of which is uncertain and generates the up and down stock price moves ($\pm\sigma\sqrt{h}$).

Let's see how equation (11.16) solves the three problems in the random walk discussed earlier:

1. The stock price cannot become negative. Even if we move down the binomial tree many times in a row, the resulting large, negative, continuously compounded return will give us a positive price.
2. As stock price moves occur more frequently, h gets smaller, therefore up and down moves get smaller. By construction, annual volatility is the same no matter how many binomial periods there are. Since returns follow a random walk, the percentage price change is the same whether the stock price is \$100 or \$5.
3. There is a $(r - \delta)h$ term, and we can choose the probability of an up move, so we can guarantee that the expected change in the stock price is positive.

To illustrate that the binomial tree can be thought of as a random walk, Figure 11.6 illustrates the stock price that results when the continuously compounded return follows a random walk. The figure is one particular path through a 500-step binomial tree, with the particular path generated by the same sequence of coin flips as in Figure 11.5.

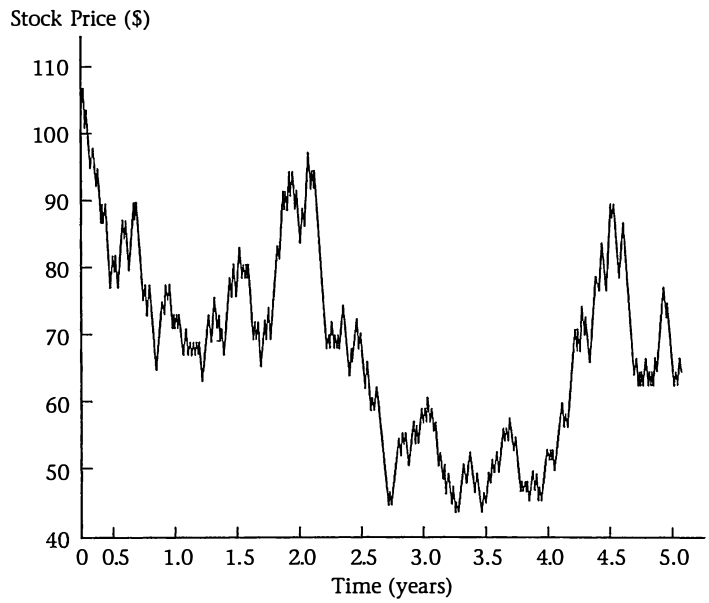
Lognormality and the Binomial Model

The binomial tree approximates a lognormal distribution, which is commonly used to model stock prices.

First, what is the lognormal distribution? The lognormal distribution is the probability distribution that arises from the assumption that *continuously compounded returns on the stock are normally distributed*. When we traverse the binomial tree, we are implicitly adding up binomial random return components of $(r - \delta)h \pm \sigma\sqrt{h}$. In the limit (as $n \to \infty$ or, the same thing, $h \to 0$), the sum of binomial random variables is normally distributed. Thus, continuously compounded returns in a binomial tree are

FIGURE 11.6

Illustration of a particular path through a 500-step binomial tree, where the up and down moves are the same as in Figure 11.5. Assumes $S_0 = \$100$, $r = 6\%$, $\sigma = 30\%$, $T = 5$ years, and $h = 0.01$.

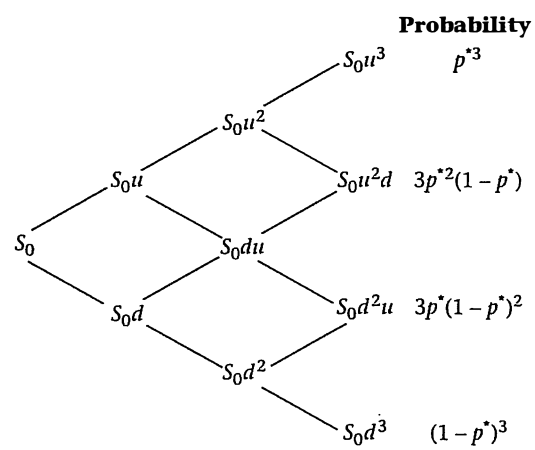


(approximately) normally distributed, which means that the stock is lognormally distributed. We defer a more complete discussion of this to Chapters 18 and 20, but we can see with an example how it works.

The binomial model implicitly assigns probabilities to the various nodes. Figure 11.7 depicts the construction of a tree for three binomial periods, along with the risk-

FIGURE 11.7

Construction of a binomial tree depicting stock price paths, along with risk-neutral probabilities of reaching the various terminal prices.



neutral probability of reaching each final period node. There is only one path—sequence of up and down moves—reaching the top or bottom node (*uuu* or *ddd*), but there are three paths reaching each intermediate node. For example, the first node below the top (S_0u^2d) can be reached by the sequences *uud*, *udu*, or *duu*. Thus, there are more paths that reach the intermediate nodes than the extreme nodes.

We can take the probabilities and outcomes from the binomial tree and plot them against a lognormal distribution with the same parameters. Figure 11.8 compares a three-period binomial approximation with a lognormal distribution assuming that the initial stock price is \$100, volatility is 30%, the expected return on the stock is 10%, and the time horizon is 1 year. Because we need different scales for the discrete and continuous distributions, lognormal probabilities are graphed on the left vertical axis and binomial probabilities on the right vertical axis.

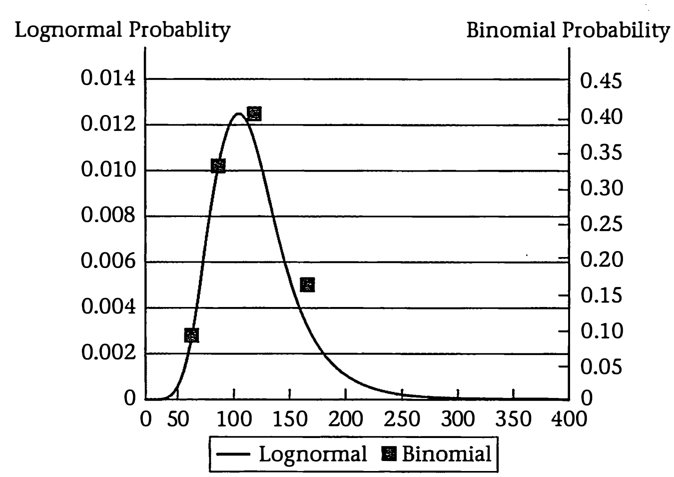
Suppose that a binomial tree has n periods and the risk-neutral probability of an up move is p^*. To reach the top node, we must go up n times in a row, which occurs with a probability of $(p^*)^n$. The price at the top node is Su^n. There is only one path through the tree by which we can reach the top node. To reach the first node below the top node, we must go up $n-1$ times and down once, for a probability of $(p^*)^{n-1} \times (1-p^*)$. The price at that node is $Su^{n-1}d$. Since the single down move can occur in any of the n periods, there are n ways this can happen. The probability of reaching the ith node below the top is $(p^*)^{n-i} \times (1-p^*)^i$. The price at this node is $Su^{n-i}d^i$. The number of ways to reach this node is

$$\text{Number of ways to reach } i\text{th node} = \frac{n!}{(n-i)!\,i!} = \binom{n}{i}$$

where $n! = n \times (n-1) \times \cdots \times 1$.[3]

FIGURE 11.8

Comparison of lognormal distribution with three-period binomial approximation.



[3]The expression $\binom{n}{i}$ can be computed in Excel using the combinatorial function, *Combin*(n, i).

We can construct the implied probability distribution in the binomial tree by plotting the stock price at each final period node, $Su^{n-i}d^i$, against the probability of reaching that node. The probability of reaching any given node is the probability of one path reaching that node times the number of paths reaching that node:

$$\text{Probability of reaching } i^{\text{th}} \text{ node} = p^{*n-i}(1-p^*)^i \frac{n!}{(n-i)!\, i!} \tag{11.17}$$

Figure 11.9 compares the probability distribution for a 25-period binomial tree with the corresponding lognormal distribution. The two distributions appear close; as a practical matter, a 25-period approximation works fairly well for an option expiring in a few months.

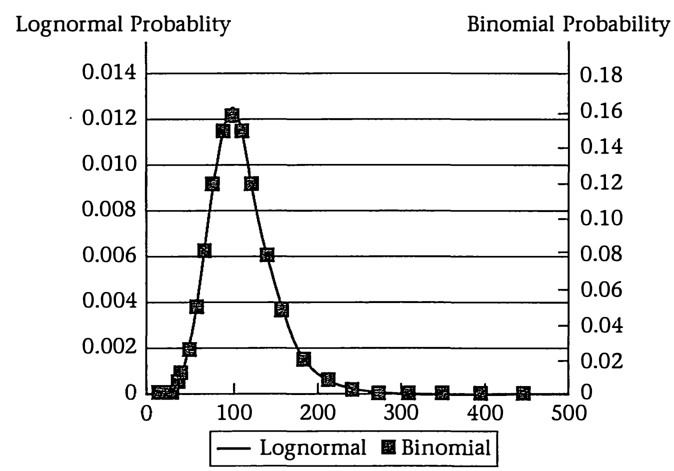
Figures 11.8 and 11.9 show you what the lognormal distribution for the stock price looks like. The stock price is positive, and the distribution is skewed to the right; that is, there is a chance that extremely high stock prices will occur.

Alternative Binomial Trees

There are other ways besides equation (11.16) to construct a binomial tree that approximates a lognormal distribution. An acceptable tree must match the standard deviation of the continuously compounded return on the asset and must generate an appropriate distribution as the length of the binomial period, h, goes to 0. Different methods of constructing the binomial tree will result in different u and d stock movements. No matter how we construct the tree, however, we use equation (10.5) to determine the risk-neutral probability and equation (10.6) to determine the option value.

FIGURE 11.9

Comparison of lognormal distribution with 25-period binomial approximation.



The Cox-Ross-Rubinstein binomial tree The best-known way to construct a binomial tree is that in Cox et al. (1979), in which the tree is constructed as

$$
\begin{aligned}
u &= e^{\sigma\sqrt{h}} \\
d &= e^{-\sigma\sqrt{h}}
\end{aligned}
\tag{11.18}
$$

The Cox-Ross-Rubinstein approach is often used in practice. A problem with this approach, however, is that if h is large or σ is small, it is possible that $e^{rh} > e^{\sigma\sqrt{h}}$, in which case the binomial tree violates the restriction in equation (10.4). In real applications h would be small, so this problem does not occur. In any event, the tree based on the forward price never violates equation (10.4).

The lognormal tree Another alternative is to construct the tree using

$$
\begin{aligned}
u &= e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}} \\
d &= e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}}
\end{aligned}
\tag{11.19}
$$

This procedure for generating a tree was proposed by Jarrow and Rudd (1983) and is sometimes called the Jarrow-Rudd binomial model. It has a very natural motivation that you will understand after we discuss lognormality in Chapter 18. You will find in computing equation (10.5) that the risk-neutral probability of an up-move is generally close to 0.5.

All three methods of constructing a binomial tree yield different option prices for finite n, but approach the same price as $n \to \infty$. Also, while the different binomial trees all have different up and down movements, all have the same ratio of u to d:

$$
\frac{u}{d} = e^{2\sigma\sqrt{h}} \qquad \text{or} \qquad \ln(u/d) = 2\sigma\sqrt{h}
$$

This is the sense in which, however the tree is constructed, the proportional distance between u and d measures volatility.

Is the Binomial Model Realistic?

Any option pricing model relies on an assumption about the behavior of stock prices. As we have seen in this section, the binomial model is a form of the random walk model, adapted to modeling stock prices. The lognormal random walk model in this section assumes, among other things, that volatility is constant, that "large" stock price movements do not occur, and that returns are independent over time. All of these assumptions appear to be violated in the data.

We will discuss the behavior of volatility in Chapters 18 and 23. However, there is evidence that volatility changes over time (see Bollerslev et al., 1994). It also appears that on occasion stocks move by a large amount. The binomial model has the property that stock price movements become smaller as the period length, h, becomes smaller. Occasional large price movements—"jumps"—are therefore a feature of the data inconsistent with the binomial model. We will also discuss such moves in Chapters 19 and 21. Finally, there is some evidence that stock returns are correlated across time, with positive correlations at the short to medium term and negative correlation at long horizons (see Campbell et al., 1997, ch. 2).

The random walk model is a useful starting point for thinking about stock price behavior, and it is widely used because of its elegant simplicity. However, it is not sacrosanct.

11.4 ESTIMATING VOLATILITY

In practice we need to figure out what parameters to use in the binomial model. The most important decision is the value we assign to σ, which we cannot observe directly. One possibility is to measure σ by computing the standard deviation of continuously compounded historical returns. Volatility computed from historical stock returns is **historical volatility.**

Table 11.1 lists 13 weeks of Wednesday closing prices for the S&P 500 composite index and for IBM, along with the standard deviation of the continuously compounded returns, computed using the *StDev* function in Excel.[4]

Over the 13-week period in the table, the weekly standard deviation was 0.0309 and 0.0365 for the S&P 500 index and IBM, respectively. These are weekly standard deviations since they are computed from weekly returns; they therefore measure the variability in weekly returns. We obtain annualized standard deviations by multiplying the weekly standard deviations by $\sqrt{52}$, giving annual standard deviations of 22.32% for the S&P 500 index and 26.32% for IBM.

We can now use these annualized standard deviations to construct binomial trees with the binomial period, h, set to whatever is appropriate. Don't be misled by the fact that the standard deviations were estimated with weekly data. Once we annualize the estimated standard deviations by multiplying by $\sqrt{52}$, we can then multiply again by $\sqrt{h}$ to adapt the annual standard deviation to any size binomial step.

The procedure outlined above is a reasonable way to estimate volatility when continuously compounded returns are independent and identically distributed, as in the logarithmic random walk model in Section 11.3. However, if returns are not independent—as with some commodities, for example—volatility estimation becomes more complicated. If a high price of oil today leads to decreased demand and increased supply, we would expect prices in the future to come down. In this case, the volatility over T years will be less than $\sigma\sqrt{T}$, reflecting the tendency of prices to revert from extreme values. Extra care is required with volatility if the random walk model is not a plausible economic model of the asset's price behavior.

[4]We use weekly rather than daily data because computing daily statistics is complicated by weekends and holidays. In theory the standard deviation over the 3 days from Friday to Monday should be greater than over the 1 day from Monday to Tuesday. Using weekly data avoids this kind of complication. Further, using Wednesdays avoids most holidays.

TABLE 11.1 Weekly prices and continuously compounded returns for the S&P 500 index and IBM, from 3/5/03 to 5/28/03.

Date	S&P 500		IBM	
	Price	$\ln(S_t/S_{t-1})$	Price	$\ln(S_t/S_{t-1})$
03/05/03	829.85	—	77.73	—
03/12/03	804.19	−0.0314	75.18	−0.0334
03/19/03	874.02	0.0833	82.00	0.0868
03/26/03	869.95	−0.0047	81.55	−0.0055
04/02/03	880.90	0.0125	81.46	−0.0011
04/09/03	865.99	−0.0171	78.71	−0.0343
04/16/03	879.91	0.0159	82.88	0.0516
04/23/03	919.02	0.0435	85.75	0.0340
04/30/03	916.92	−0.0023	84.90	−0.0100
05/07/03	929.62	0.0138	86.68	0.0207
05/14/03	939.28	0.0103	88.70	0.0230
05/21/03	923.42	−0.0170	86.18	−0.0288
05/28/03	953.22	0.0318	87.57	0.0160
Std. deviation		0.0309	—	0.0365
Std. deviation $\times\sqrt{52}$		0.2232	—	0.2632

11.5 STOCKS PAYING DISCRETE DIVIDENDS

Although it may be reasonable to assume that a stock index pays dividends continuously, individual stocks pay dividends in discrete lumps, quarterly or annually. In addition, over short horizons it is frequently possible to predict the amount of the dividend. How should we price an option when the stock will pay a known dollar dividend during the life of the option? The procedure we have already developed for creating a binomial tree can accommodate this case. However, we will also discuss a preferable alternative due to Schroder (1988).

Modeling Discrete Dividends

When no dividend will be paid between time t and $t + h$, we create the binomial tree as in Chapter 10. Suppose that a dividend will be paid between times t and $t + h$ and that its future value at time $t + h$ is D. The time t forward price for delivery at $t + h$ is then

$$F_{t,t+h} = S_t e^{rh} - D$$

Since the stock price at time $t+h$ will be ex-dividend, we create the up and down moves based on the ex-dividend stock price:

$$\begin{aligned} S_t^u &= \left(S_t e^{rh} - D\right) e^{\sigma\sqrt{h}} \\ S_t^d &= \left(S_t e^{rh} - D\right) e^{-\sigma\sqrt{h}} \end{aligned} \tag{11.20}$$

How does option replication work when a dividend is imminent? When a dividend is paid, we have to account for the fact that the stock earns the dividend. Thus, we have

$$\begin{aligned} \left(S_t^u + D\right) \Delta + e^{rh} B &= C_u \\ \left(S_t^d + D\right) \Delta + e^{rh} B &= C_d \end{aligned}$$

The solution is

$$\Delta = \frac{C_u - C_d}{S_t^u - S_t^d}$$

$$B = e^{-rh}\left[\frac{S_t^u C_d - S_t^d C_u}{S_t^u - S_t^d}\right] - \Delta D e^{-rh}$$

Because the dividend is known, we decrease the bond position by the present value of the certain dividend. (When the dividend is proportional to the stock price, as with a stock index, we reduce the stock position, equation (10.1).) The expression for the option price is given by equation (10.17).

Problems with the Discrete Dividend Tree

The practical problem with this procedure is that the tree does not completely recombine after a discrete dividend. In all previous cases we have examined, we reached the same price after a given number of up and down movements, regardless of the order of the movements.

Figure 11.10, in which a dividend with a period-2 value of \$5 is paid between periods 1 and 2, demonstrates that with a discrete dividend, the order of up and down movements affects the price. In the third binomial period there are six rather than four possible stock prices.

To see how the tree is constructed, period-1 prices are

$$\$41e^{0.08\times1/3+0.3\times\sqrt{1/3}} = \$50.071$$

$$\$41e^{0.08\times1/3-0.3\times\sqrt{1/3}} = \$35.411$$

The period-2 prices from the \$50.071 node are

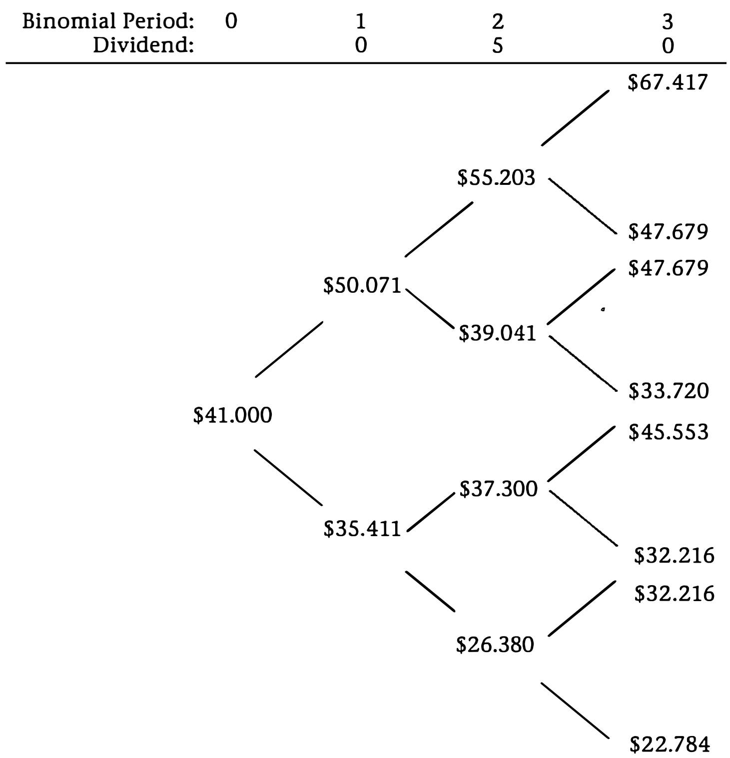
$$(\$50.071e^{0.08\times1/3} - 5) \times e^{0.3\times\sqrt{1/3}} = \$55.203$$

$$(\$50.071e^{0.08\times1/3} - 5) \times e^{-0.3\times\sqrt{1/3}} = \$39.041$$

Repeating this procedure for the node $S = \$35.411$ gives prices of \$37.300 and \$26.380. You can see that there are now four prices instead of three after two binomial steps: The

FIGURE 11.10

The tree depicts a discrete $5 dividend paid between the first and second binomial periods. There are eight discrete terminal nodes (six of them distinct) rather than four. Assumes $S = \$41$, $\sigma = 0.3$, $r = 0.08$, $t = 1$ year, and $h = 0.333$.



ud and *du* nodes do not recombine. There are six distinct prices in the final period as each set of ex-dividend prices generates a distinct tree (three prices arise from the top two prices in period 2 and three prices arise from the bottom two prices in period 2). Each discrete dividend causes the tree to bifurcate.

There is also a conceptual problem with equation (11.20). Since the amount of the dividend is fixed, the stock price could in principle become negative if there have been large downward moves in the stock prior to the dividend.

This example demonstrates that handling fixed dividends requires care. We now turn to a method that is computationally easier than constructing a tree using equation (11.20) and that will not generate negative stock prices.

A Binomial Tree Using the Prepaid Forward

Schroder (1988) presents an elegant method of constructing a tree for a dividend-paying stock that solves both problems encountered with the method in Figure 11.10. The key insight for this method is that if we know for certain that a stock will pay a fixed dividend, then we can view the stock price as being the sum of two components: the dividend, which is like a zero-coupon bond with zero volatility, and the present value of the ex-dividend value of the stock—in other words, the prepaid forward price. Since the dividend is known, all volatility is attributed to the prepaid forward component of the stock price.

Suppose we know that a stock will pay a dividend D at time $T_D < T$, where T is the expiration date of the option. Then we base stock price movements on the prepaid forward price, $F_{t,T}^P = S_t - De^{-r(T_D-t)}$. The one-period forward price for the prepaid forward is $F_{t,t+h} = F_{t,T}^P e^{rh}$. As before, this gives us up and down movements of

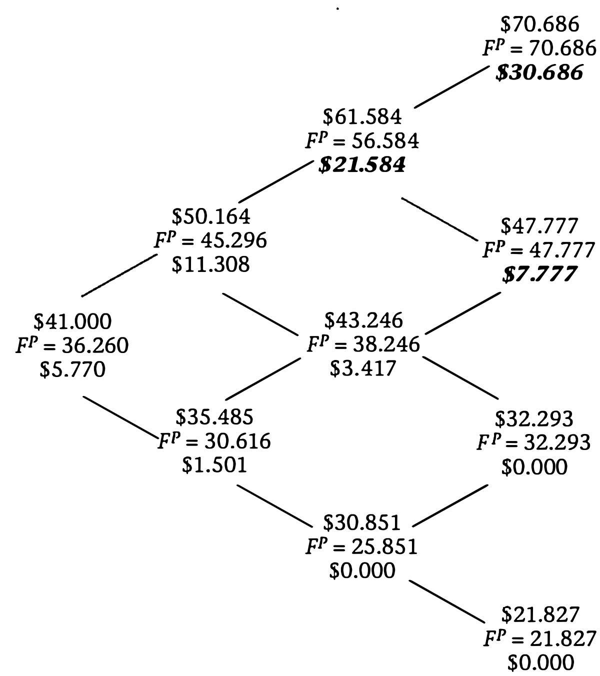
$$u = e^{rh+\sigma\sqrt{h}} \qquad d = e^{rh-\sigma\sqrt{h}}$$

However, the actual stock price at each node is given by $S_t = F_{t,T}^P + De^{-r(T_D-t)}$.

Figure 11.11 shows the construction of the binomial tree for this case. Both the observed stock price and the stock price less the present value of dividends (the prepaid forward price) are included in the figure. Note that the volatility is 0.3392 rather than 0.3 as in Figure 10.5. The reason for this difference is that the random walk is assumed to apply to the prepaid forward price. If the actual stock price is observed to have a volatility of 30%, then the prepaid forward price, which is less than the stock price, must

FIGURE 11.11

Binomial tree for pricing an American call option on a stock paying a discrete dividend of \$5 in 8 months; assumes $S = \$41.00$, $K = \$40.00$, $\sigma = 0.3392$, $r = 0.08$, $T = 1.00$ years, $\delta = 0.00$, and $h = 0.333$. At each node the stock price, prepaid forward price, and option price are given. Option prices in ***bold italic*** signify that exercise is optimal at that node.



have a greater volatility. We use the approximate correction

$$\begin{aligned}\sigma_F &= \sigma_S \times \frac{S}{F^P} \\ &= 0.3 \times \frac{\$41}{\$36.26} = 0.3392\end{aligned}$$

You may be wondering exactly how the dividend affects Figure 11.11. Note first that $u = 1.2492$. Look at the node where the stock price is \$61.584. This is a *cum-dividend* price, just before the dividend is paid. The nodes in the last period are constructed based on the *ex-dividend* price, for example

$$(\$61.584 - \$5) \times 1.2492 = \$70.686$$

As a final point, we obtain risk-neutral probabilities for the tree in the same way as in the absence of dividends. It is important to realize that we construct the binomial tree for the *prepaid forward*, which pays no dividends. Thus, the risk-neutral probability of an up move in the prepaid forward price is given by equation (10.5), just as in the case of a nondividend paying stock.

CHAPTER SUMMARY

Both call and put options may be rationally exercised prior to expiration. The early-exercise decision weighs three considerations: dividends on the underlying asset, interest on the strike price, and the insurance value of keeping the option alive. Calls will be early-exercised in order to capture dividends on the underlying stock; interest and insurance weigh against early exercise. Puts will be early-exercised in order to capture interest on the strike price; dividends and insurance weigh against early exercise. For both calls and puts, the early-exercise criterion becomes less stringent as the option has less time to maturity.

Risk-neutral option valuation is consistent with valuation using more traditional discounted cash flow methods. With risk-neutral pricing it is not necessary to estimate the expected return on the stock in order to price an option. With traditional discounted cash flow methods, the correct discount rate for the option varies along the binomial tree; thus, valuation is considerably more complicated than with risk-neutral pricing.

The binomial model, which approximates the lognormal distribution, is a random walk model adapted to modeling stock prices. The model assumes that the continuously compounded return on the stock follows a random walk. The volatility needed for the binomial model can be estimated by computing the standard deviation of continuously compounded returns and annualizing the result.

The binomial model can be adapted to price options on a stock that pays discrete dividends. Discrete dividends can lead to a nonrecombining binomial tree. If we assume that the prepaid forward price follows a binomial process instead of the stock price, the tree becomes recombining.

FURTHER READING

The binomial model can be used to derive the Black-Scholes model, which we discuss in Chapter 12. The practical importance of risk-neutral pricing will become evident in Chapter 19, when we see that Monte Carlo valuation hinges upon risk-neutral pricing. In that chapter we will also reexamine Figure 11.4 and show how the option price may be computed as an expected value using only stock prices in the final period.

The issue of how the stock price is distributed will also arise frequently in later chapters. Chapter 18 discusses lognormality in more detail and presents evidence that stock prices are not exactly lognormally distributed. Chapter 20 will examine in more detail the question of how the stock price moves, in particular what happens when h gets very small in the binomial model.

We will return to the determinants of early exercise in Chapter 17, when we discuss real options.

The literature on risk-neutral pricing is fairly technical. Cox and Ross (1976) was the first paper to use risk-neutral pricing and Harrison and Kreps (1979) studied the economic underpinnings. Two good treatments of this topic are Huang and Litzenberger (1988, ch. 8)—their treatment inspired Appendix 11.B—and Baxter and Rennie (1996).

Campbell et al. (1997) and Cochrane (2001) summarize evidence on the distribution of stock prices. The original Samuelson work on asset prices following a random walk (Samuelson, 1965) remains a classic, modern empirical evidence notwithstanding.

Broadie and Detemple (1996) discuss the computation of American option prices, and also discuss alternative binomial approaches and their relative numerical efficiency.

PROBLEMS

Many (but not all) of these questions can be answered with the help of the *BinomCall* and *BinomPut* functions available on the spreadsheets accompanying this book.

11.1. Consider a one-period binomial model with $h = 1$, where $S = \$100$, $r = 0$, $\sigma = 30\%$, and $\delta = 0.08$. Compute American call option prices for $K = \$70$, \$80, \$90, and \$100.

- **a.** At which strike(s) does early exercise occur?
- **b.** Use put-call parity to explain why early exercise does not occur at the higher strikes.
- **c.** Use put-call parity to explain why early exercise is sure to occur for all lower strikes than that in your answer to (a).

11.2. Repeat Problem 11.1, only assume that $r = 0.08$. What is the greatest strike price at which early exercise will occur? What condition related to put-call parity is satisfied at this strike price?

11.3. Repeat Problem 11.1, only assume that $r = 0.08$ and $\delta = 0$. Will early exercise ever occur? Why?

11.4. Consider a one-period binomial model with $h = 1$, where $S = \$100$, $r = 0.08$, $\sigma = 30\%$, and $\delta = 0$. Compute American put option prices for $K = \$100$, \$110, \$120, and \$130.

a. At which strike(s) does early exercise occur?

b. Use put-call parity to explain why early exercise does not occur at the other strikes.

c. Use put-call parity to explain why early exercise is sure to occur for all strikes greater than that in your answer to (a).

11.5. Repeat Problem 11.4, only set $\delta = 0.08$. What is the lowest strike price at which early exercise will occur? What condition related to put-call parity is satisfied at this strike price?

11.6. Repeat Problem 11.4, only set $r = 0$ and $\delta = 0.08$. What is the lowest strike price (if there is one) at which early exercise will occur? If early exercise never occurs, explain why not.

For the following problems, note that the *BinomCall* and *BinomPut* functions are array functions that return the option delta (Δ) as well as the price. If you know Δ, you can compute B as $C - S\Delta$.

11.7. Let $S = \$100$, $K = \$100$, $\sigma = 30\%$, $r = 0.08$, $t = 1$, and $\delta = 0$. Let $n = 10$. Suppose the stock has an expected return of 15%.

a. What is the expected return on a European call option? A European put option?

b. What happens to the expected return if you increase the volatility to 50%?

11.8. Let $S = \$100$, $\sigma = 30\%$, $r = 0.08$, $t = 1$, and $\delta = 0$. Suppose the true expected return on the stock is 15%. Set $n = 10$. Compute European call prices, Δ, and B for strikes of \$70, \$80, \$90, \$100, \$110, \$120, and \$130. For each strike, compute the expected return on the option. What effect does the strike have on the option's expected return?

11.9. Repeat the previous problem, except that for each strike price, compute the expected return on the option for times to expiration of 3 months, 6 months, 1 year, and 2 years. What effect does time to maturity have on the option's expected return?

11.10. Let $S = \$100$, $\sigma = 30\%$, $r = 0.08$, $t = 1$, and $\delta = 0$. Suppose the true expected return on the stock is 15%. Set $n = 10$. Compute European put prices, Δ, and B for strikes of \$70, \$80, \$90, \$100, \$110, \$120, and \$130. For each strike, compute the expected return on the option. What effect does the strike have on the option's expected return?

11.11. Repeat the previous problem, except that for each strike price, compute the expected return on the option for times to expiration of 3 months, 6 months, 1 year, and 2 years. What effect does time to maturity have on the option's expected return?

11.12. Let $S = \$100$, $\sigma = 0.30$, $r = 0.08$, $t = 1$, and $\delta = 0$. Using equation (11.17) to compute the probability of reaching a terminal node and $Su^i d^{n-i}$ to compute the price at that node, plot the risk-neutral distribution of year-1 stock prices as in Figures 11.8 and 11.9 for $n = 3$ and $n = 10$.

11.13. Repeat the previous problem for $n = 50$. What is the risk-neutral probability that $S_1 < \$80$? $S_1 > \$120$?

11.14. We saw in Section 10.1 that the undiscounted risk-neutral expected stock price equals the forward price. We will verify this using the binomial tree in Figure 11.4.

- **a.** Using $S = \$100$, $r = 0.08$, and $\delta = 0$, what are the 4-month, 8-month, and 1-year forward prices?
- **b.** Verify your answers in (a) by computing the risk-neutral expected stock price in the first, second, and third binomial period. Use equation (11.17) to determine the probability of reaching each node.

11.15. Compute the 1-year forward price using the 50-step binomial tree in Problem 11.13.

11.16. Suppose $S = \$100$, $K = \$95$, $r = 8\%$ (continuously compounded), $t = 1$, $\sigma = 30\%$, and $\delta = 5\%$. Explicitly construct an 8-period binomial tree using the Cox-Ross-Rubinstein expressions for u and d:

$$u = e^{\sigma\sqrt{h}} \qquad d = e^{-\sigma\sqrt{h}}$$

Compute the prices of European and American calls and puts.

11.17. Suppose $S = \$100$, $K = \$95$, $r = 8\%$ (continuously compounded), $t = 1$, $\sigma = 30\%$, and $\delta = 5\%$. Explicitly construct an 8-period binomial tree using the lognormal expressions for u and d:

$$u = e^{(r-\delta-.5\sigma^2)h+\sigma\sqrt{h}} \qquad d = e^{(r-\delta-.5\sigma^2)h-\sigma\sqrt{h}}$$

Compute the prices of European and American calls and puts.

11.18. Obtain at least 5 years' worth of daily or weekly stock price data for a stock of your choice.

- **a.** Compute annual volatility using all the data.
- **b.** Compute annual volatility for each calendar year in your data. How does volatility vary over time?
- **c.** Compute annual volatility for the first and second half of each year in your data. How much variation is there in your estimate?

11.19. Obtain at least 5 years of daily data for at least three stocks and, if you can, one currency. Estimate annual volatility for each year for each asset in your data.

What do you observe about the pattern of historical volatility over time? Does historical volatility move in tandem for different assets?

11.20. Suppose that $S = \$50$, $K = \$45$, $\sigma = 0.30$, $r = 0.08$, and $t = 1$. The stock will pay a \$4 dividend in exactly 3 months. Compute the price of European and American call options using a four-step binomial tree.

APPENDIX 11.A: PRICING OPTIONS WITH TRUE PROBABILITIES

In this appendix we demonstrate algebraically that computing the option price in a consistent way using α as the expected return on the stock gives the correct option price. Using the definition of γ, equation (11.6), we can rewrite equation (11.7) as

$$(\Delta S + B)\left(\frac{1}{e^{\alpha h}\Delta S + e^{rh}B}\left[\frac{e^{rh}-d}{u-d}C_u + \frac{u-e^{rh}}{u-d}C_d + \frac{e^{\alpha h}-e^{rh}}{u-d}(C_u - C_d)\right]\right)$$

Since $\Delta S + B$ is the call price, we need only show that the expression in large parentheses is equal to one. From the definitions of Δ and B we have

$$\frac{e^{rh}-d}{u-d}C_u + \frac{u-e^{rh}}{u-d}C_d = e^{rh}(\Delta S + B)$$

We can rewrite (11.4) as

$$(\Delta S + B)\left(\frac{1}{e^{\alpha h}\Delta S + e^{rh}B}\left[e^{rh}(\Delta S + B) + (e^{\alpha h} - e^{rh})\Delta S\right]\right) = \Delta S + B$$

This follows since the expression in large parentheses equals one.

APPENDIX 11.B: WHY DOES RISK-NEUTRAL PRICING WORK?

There is a large and highly technical literature on risk-neutral pricing. The underlying economic idea is fairly easy to understand, however.

Utility-Based Valuation

The starting point is that the well-being of investors is not measured in dollars, but in *utility*. Utility is a measure of satisfaction. Economists say that investors exhibit *declining marginal utility*: Starting from a given level of wealth, the utility gained from adding \$1 to wealth is less than the utility lost from taking \$1 away from wealth. Thus, we expect that more dollars will make an investor happier, but that if we keep adding dollars, each additional dollar will make the investor less happy than the previous dollars.

Declining marginal utility implies that investors are risk-averse, which means that an investor will prefer a safer investment to a riskier investment that has the same expected return. Since losses are more costly than gains are beneficial, a risk-averse investor will avoid a fair bet, which by definition has equal expected gains and losses.[5]

To illustrate risk-neutral pricing, we imagine a world where there are two assets, a risky stock and a risk-free bond. Investors are risk-averse. Suppose the economy in one period will be in one of two states, a high state and a low state. How do we value assets in such a world? We need to know three things:

1. What utility value, expressed in terms of dollars today, does an investor attach to the marginal dollar received in each state in the future? Denote the values of \$1 received in the high and low states as U_H and U_L, respectively.[6] Because the investor is risk-averse, \$1 received in the high state is worth less than \$1 received in the low state, hence, $U_H < U_L$.
2. How many dollars will an asset pay in each state? Denote the payoffs to the risky stock in each state C_H and C_L.
3. What is the probability of each state occurring? Denote the probability of the high state as p.

We begin by defining a state price as the price of a security that pays \$1 only when a particular state occurs. Let Q_H be the price of a security that pays \$1 when the high state occurs, and Q_L the price of a security paying \$1 when the low state occurs.[7] Since U_H and U_L are the value today of \$1 in each state, the price we would pay is just the value times the probability that state is reached:

$$\begin{aligned} Q_H &= p \times U_H \\ Q_L &= (1-p) \times U_L \end{aligned} \tag{11.21}$$

Since there are only two possible states, we can value any future cash flow using these state prices.

The price of the risky stock, S_0 is

$$\text{Price of stock} = Q_H \times C_H + Q_L \times C_L \tag{11.22}$$

[5]This is an example of *Jensen's Inequality* (see Appendix C at the end of this book). A risk-averse investor has a concave utility function, which implies that

$$E[U(x)] < U[E(x)]$$

The expected utility associated with a gamble, $E[U(x)]$, is less than the utility from receiving the expected value of the gamble for sure, $U[E(x)]$.

[6]Technically U_H and U_L are ratios of marginal utilities, discounted by the rate of time preference. However, you can think of them as simply converting future dollars in a particular state into dollars today.

[7]These are often called "Arrow-Debreu" securities, named after Nobel-prize–winning economists Kenneth Arrow and Gerard Debreu.

Since the risk-free bond pays \$1 in each state, we have

$$\text{Price of bond} = Q_H \times 1 + Q_L \times 1 \tag{11.23}$$

We can calculate rates of return by dividing expected cash flows by the price. Thus, the risk-free rate is

$$\begin{aligned} 1 + r &= \frac{1}{\text{Price of bond}} \\ &= \frac{1}{Q_H + Q_L} \end{aligned} \tag{11.24}$$

The expected return on the stock is

$$\begin{aligned} 1 + \alpha &= \frac{p \times C_H + (1 - p)C_L}{\text{Price of stock}} \\ &= \frac{p \times C_H + (1 - p)C_L}{Q_H \times C_H + Q_L \times C_L} \end{aligned} \tag{11.25}$$

Standard Discounted Cash Flow

The standard discounted cash flow calculation entails computing the security price by discounting the expected cash flow at the expected rate of return. In the case of the stock, this gives us

$$\frac{p \times C_H + (1 - p)C_L}{1 + \alpha} = \text{Price of stock}$$

This is simply a rewriting of equation (11.25); hence, it is obviously correct. Similarly, the bond price is

$$\frac{1}{1 + r} = \text{Price of bond}$$

Risk-Neutral Pricing

The point of risk-neutral pricing is to sidestep the utility calculations above. We are looking for probabilities such that when we use those probabilities to compute expected cash flows *without* explicit utility adjustments, and discount that expectation at the risk-free rate, then we will get the correct answer.

The trick is the following: Instead of utility-weighting the cash flows and computing expectations, *we utility-weight the probabilities*, creating new "risk-neutral" probabilities. Now we will see how to perform risk-neutral pricing in this context. Use the state prices in equation (11.21) to define the risk-neutral probability of the high state, p^*, as

$$p^* = \frac{p \times U_H}{p \times U_H + (1 - p) \times U_L} = \frac{Q_H}{Q_H + Q_L}$$

Now we compute the stock price by using the risk-neutral probabilities to compute expected cash flow, and then discounting at the risk-free rate. We have

$$\frac{p^*C_H + (1-p^*)C_L}{1+r} = \frac{\frac{Q_H}{Q_H+Q_L}C_H + \frac{Q_L}{Q_H+Q_L}C_L}{1+r}$$
$$= \frac{Q_HC_H + Q_LC_L}{(Q_H+Q_L)(1+r)}$$
$$= Q_HC_H + Q_LC_L$$

which is the price of stock, from equation (11.22). This shows that we can construct risk-neutral probabilities and use them to price risky assets.

Example

Table 11.2 contains assumptions for a numerical example.

State prices Using equation 11.21, the state prices are $Q_H = 0.52 \times \$0.87 = \$0.4524$, and $Q_L = 0.48 \times \$0.98 = \$0.4704$.

Valuing the risk-free bond The risk-free bond pays \$1 in each state. Thus, using equation (11.23) the risk-free bond price, B_0, is

$$B_0 = Q_H + Q_L = \$0.4524 + \$0.4704 = \$0.9228 \qquad (11.26)$$

The risk-free rate is

$$r = \frac{1}{0.9228} - 1 = 8.366\%$$

Valuing the risky stock using real probabilities Using equation (11.22) the price of the stock is

$$S_0 = 0.4524 \times \$180 + 0.4704 \times \$30 = \$95.544 \qquad (11.27)$$

TABLE 11.2 Probabilities, utility weights, and equity cash flows in high and low states of the economy.

	High State	Low State
Cash flow to risk-free bond	$C_H = \$1$	$C_L = \$1$
Cash flow to stock	$C_H = \$180$	$C_L = \$30$
Probability	$p = 0.52$	$p = 0.48$
Value of \$1	$U_H = \$0.87$	$U_L = \$0.98$

The expected cash flow on the stock in one period is

$$E(S_1) = 0.52 \times \$180 + 0.48 \times \$30 = \$108$$

The expected return on the stock is therefore

$$\alpha = \frac{\$108}{\$95.544} - 1 = 13.037\%$$

By definition, if we discount $E(S_1)$ at the rate 13.037%, we will get the price \$95.544.

Risk-neutral valuation of the stock The risk-neutral probability is

$$\begin{aligned} p^* &= \frac{\$0.4524}{\$0.4524 + \$0.4704} \\ &= 49.025\% \end{aligned}$$

Now we can value the stock using p^* instead of the true probabilities, and discount at the risk-free rate:

$$\begin{aligned} S_0 &= \frac{0.49025 \times \$180 + (1 - 0.49025) \times \$30}{1.08366} \\ &= \$95.544 \end{aligned}$$

We can also verify that a call option on the stock can be valued using risk-neutral pricing. Suppose the call has a strike of \$130. Then the value computed using true probabilities and utility weights is

$$\begin{aligned} C &= 0.52 \times 0.87 \times \max(0, \$180 - \$130) + 0.48 \times 0.98 \times \max(0, \$30 - \$130) \\ &= \$22.62 \end{aligned}$$

Using risk-neutral pricing, we obtain

$$\begin{aligned} C &= \frac{[0.49025 \times \max(0, \$180 - \$130) + (1 - 0.49025) \times \max(0, \$30 - \$130)]}{1.08366} \\ &= \$22.62 \end{aligned}$$

Why Risk-Neutral Pricing Works

Risk-neutral pricing works in the above example because the same utility weights and probabilities are used to value both the stock and risk-free bond. As long as this is true, risk-neutral pricing formulas can be obtained simply by rewriting the more complicated valuation formulas that take account of utility.

A basic result from portfolio theory states that as long as investors are optimally choosing their portfolios, they will use the same utility weights for an additional dollar of investment in all assets. Thus, in an economy with well-functioning capital markets, risk-neutral pricing is possible for derivatives on traded assets.

When would risk-neutral pricing not work? Suppose you have an asset you cannot trade or hedge, or you have a nontradable asset with cash flows that cannot be replicated by the cash flows of traded assets. If you cannot trade or offset the risk of the asset, then there is no guarantee that the marginal utility you use to value payoffs from this asset in a given state will be the same as for other assets. In other words, U_H and U_L will differ across assets. If the same U_H and U_L are not used to value the stock and bond, the calculations in this appendix fail. Valuing the nontradable stream of cash flows then requires computing the utility value of the payoffs. The point of risk-neutral pricing is to avoid having to do this.