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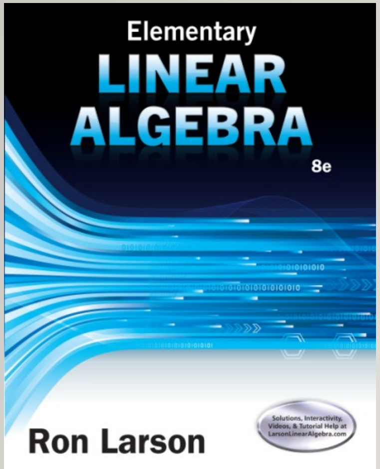

CHAPTER 7 EIGENVALUES AND EIGENVECTORS

7.1 Eigenvalues and Eigenvectors

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7.3 Symmetric Matrices and Orthogonal Diagonalization

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7.1 Eigenvalues and Eigenvectors

- Eigenvalue problem:

  If A is an $n\times n$ matrix, do there exist nonzero vectors $\boldsymbol{x}$ in R^n such that $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$?

- Eigenvalue and eigenvector:

  A : an $n\times n$ matrix

  λ : a scalar

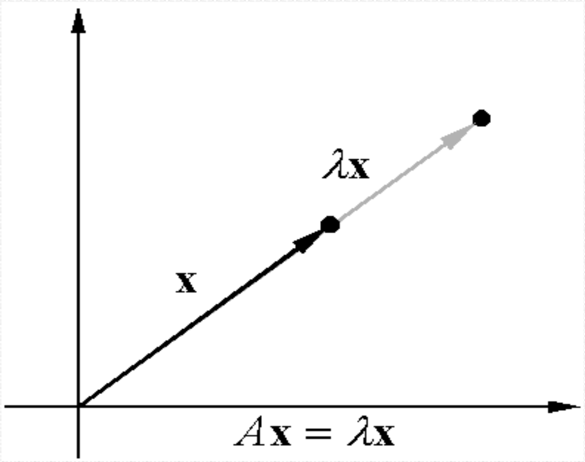
  $\boldsymbol{x}$: a nonzero vector in R^n

Eigenvalue

$$A\mathbf{x} = \lambda\mathbf{x}$$

Eigenvector

- Geometrical Interpretation



- Ex 1: (Verifying eigenvalues and eigenvectors)

$$A=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix} \quad x_1=\begin{bmatrix}1\\ 0\end{bmatrix} \quad x_2=\begin{bmatrix}0\\ 1\end{bmatrix}$$

$$Ax_1=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}2\\ 0\end{bmatrix}=\overset{\substack{\text{Eigenvalue}\\ \downarrow}}{2}\underset{\substack{\uparrow\\ \text{Eigenvector}}}{\begin{bmatrix}1\\ 0\end{bmatrix}}=2x_1$$

$$Ax_2=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ -1\end{bmatrix}=\overset{\substack{\text{Eigenvalue}\\ \downarrow}}{-1}\underset{\substack{\uparrow\\ \text{Eigenvector}}}{\begin{bmatrix}0\\ 1\end{bmatrix}}=(-1)x_2$$

- Thm 7.1: (The eigenspace of A corresponding to λ)

If A is an $n\times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ together with the zero vector is a subspace of R^n. This subspace is called **the eigenspace of λ** .

Pf:

x_1 and x_2 are eigenvectors corresponding to λ

$$(i.e.\ \ Ax_1 = \lambda x_1,\ \ Ax_2 = \lambda x_2)$$

$$(1)\ A(x_1 + x_2) = Ax_1 + Ax_2 = \lambda x_1 + \lambda x_2 = \lambda(x_1 + x_2)$$

$(i.e.\ \ x_1 + x_2$ is an eigenvector corresponding to $\lambda)$

$$(2)\ A(cx_1) = c(Ax_1) = c(\lambda x_1) = \lambda(cx_1)$$

$(i.e.\ \ cx_1$ is an eigenvector corresponding to $\lambda)$

- Ex 3: (An example of eigenspaces in the plane)

Find the eigenvalues and corresponding eigenspaces of

$$A=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$$

Sol:

If $\quad \mathrm{v}=(x, y)$

$$A\mathrm{v}=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-x\\ y\end{bmatrix}$$

For a vector on the x-axis

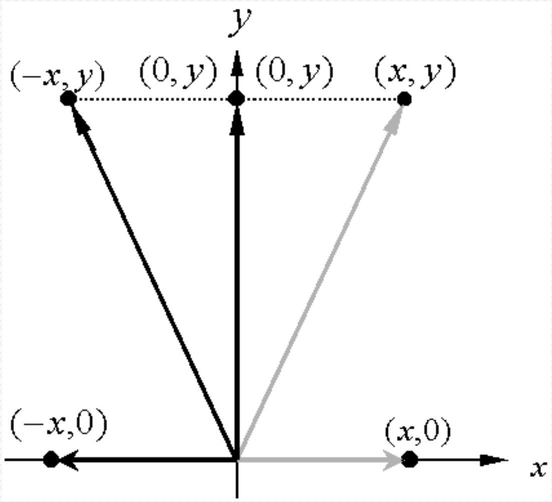
Eigenvalue $\lambda_1=-1$

$$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ 0\end{bmatrix}=\begin{bmatrix}-x\\ 0\end{bmatrix}=-1\begin{bmatrix}x\\ 0\end{bmatrix}$$

For a vector on the y-axis

Eigenvalue $\lambda_2 = 1$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1 \begin{bmatrix} 0 \\ y \end{bmatrix}$$



Geometrically, multiplying a vector (x, y) in R^2 by the matrix A corresponds to a reflection in the y-axis.

The eigenspace corresponding to $\lambda_1 = -1$ is the x-axis.

The eigenspace corresponding to $\lambda_2 = 1$ is the y-axis.

- Thm 7.2: (Finding eigenvalues and eigenvectors of a matrix $A \in M_{n\times n}$)

  Let A be an $n\times n$ matrix.

  (1) The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda \mathrm{I} - A)x = 0$.

  (2) An eigenvalue of A is a scalar λ such that $\det(\lambda \mathrm{I} - A) = 0$.

- Note:

  $Ax = \lambda x \;\Rightarrow\; (\lambda \mathrm{I} - A)x = 0$ (homogeneous system)

  If $(\lambda \mathrm{I} - A)x = 0$ has nonzero solutions iff $\det(\lambda \mathrm{I} - A) = 0$.

- Characteristic polynomial of $A \in M_{n\times n}$:

$$\det(\lambda \mathrm{I} - A) = \left|(\lambda \mathrm{I} - A)\right| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

- Characteristic equation of A:

$$\det(\lambda \mathrm{I} - A) = 0$$

- Ex 4: (Finding eigenvalues and eigenvectors)

$$A=\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Sol: Characteristic equation:

$$\det(\lambda \mathrm{I}-A)=\begin{vmatrix} \lambda-2 & 12 \\ -1 & \lambda+5 \end{vmatrix}$$

$$=\lambda^2+3\lambda+2=(\lambda+1)(\lambda+2)=0$$

$$\Rightarrow \lambda=-1,-2$$

$$\text{Eigenvalues}: \ \lambda_1=-1, \lambda_2=-2$$

$$(1)\lambda_1 = -1 \Rightarrow (\lambda_1 \mathrm{I} - A)x = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix},\ t \neq 0$$

$$(2)\lambda_2 = -2 \Rightarrow (\lambda_2 \mathrm{I} - A)x = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix},\ t \neq 0$$

$$\text{Check}: \ Ax = \lambda_i x$$

- Ex 5: (Finding eigenvalues and eigenvectors)

  Find the eigenvalues and corresponding eigenvectors for the matrix A. What is the dimension of the eigenspace of each eigenvalue?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I} - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

Eigenvalue: $\lambda = 2$

The eigenspace of A corresponding to $\lambda = 2$:

$$(\lambda \mathrm{I} - A)x = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

$$\left\{ s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| s, t \in R \right\} : \text{the eigenspace of A corresponding to } \lambda = 2$$

Thus, the dimension of its eigenspace is 2.

- Notes:

(1) If an eigenvalue λ_1 occurs as a multiple root (*k times*) for the characteristic polynominal, then λ_1 has multiplicity k.

(2) The multiplicity of an eigenvalue is greater than or equal to the dimension of its eigenspace.

- Ex 6：Find the eigenvalues of the matrix A and find a basis for each of the corresponding eigenspaces.

$$A=\begin{bmatrix}1&0&0&0\\0&1&5&-10\\1&0&2&0\\1&0&0&3\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I}-A|=\begin{vmatrix}\lambda-1&0&0&0\\0&\lambda-1&-5&10\\-1&0&\lambda-2&0\\-1&0&0&\lambda-3\end{vmatrix}$$

$$=(\lambda-1)^2(\lambda-2)(\lambda-3)=0$$

$$\text{Eigenvalues}: \lambda_1=1,\ \lambda_2=2,\ \lambda_3=3$$

$$(1)\lambda_1 = 1$$

$$\Rightarrow (\lambda_1 \mathrm{I} - A)x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 1$

$$(2)\lambda_2 = 2$$

$$\Rightarrow (\lambda_2 \mathrm{I} - A)x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \ t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 2$

$$(3)\lambda_3 = 3$$

$$\Rightarrow (\lambda_3 \mathrm{I} - A)x = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -5t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \ t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 3$

- Thm 7.3: (Eigenvalues of triangular matrices)

  If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

- Ex 7: (Finding eigenvalues for diagonal and triangular matrices)

$$(a) A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix} \quad (b) A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Sol:

$$(a) \ |\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3)$$

$$\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -3$$

$$(b) \ \lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 0, \lambda_4 = -4, \lambda_5 = 3$$

- Eigenvalues and eigenvectors of linear transformations:

  A number λ is called an eigenvalue of a linear transformation $T : V \rightarrow V$ if there is a nonzero vector $\mathbf{x}$ such that
  $T(\mathbf{x}) = \lambda\mathbf{x}$.
  The vector $\mathbf{x}$ is called an eigenvector of T corresponding to λ, and the set of all eigenvectors of λ (with the zero vector) is called the eigenspace of λ.

- Ex 8: (Finding eigenvalues and eigenspaces)

Find the eigenvalues and corresponding eigenspaces

$$A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}.$$

Sol:

$$\left|\lambda I-A\right|=\begin{vmatrix}\lambda-1 & -3 & 0\\ -3 & \lambda-1 & 0\\ 0 & 0 & \lambda+2\end{vmatrix}=(\lambda+2)^2(\lambda-4)$$

eigenvalues : $\lambda_1=4,\ \lambda_2=-2$

The eigenspaces for these two eigenvalues are as follows.

$B_1=\{(1,1,0)\}$ Basis for $\lambda_1=4$

$B_2=\{(1,-1,0),(0,0,1)\}$ Basis for $\lambda_2=-2$

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

- Notes:

(1) Let $T: R^3 \to R^3$ be the linear transformation whose standard matrix is A in Ex. 8, and let B' be the basis of R^3 made up of three linear independent eigenvectors found in Ex. 8. Then A', the matrix of T relative to the basis B', is diagonal.

$$B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$$

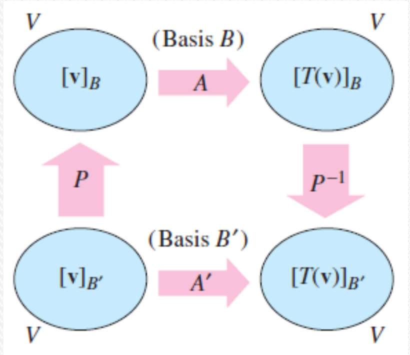
Eigenvectors of A

$$A' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Eigenvalues of A

(2) The main diagonal entries of the matrix A' are the eigenvalues of A.

7.2 Diagonalization



$$\boxed{A' = P^{-1}AP}$$

- Diagonalization problem:

  For a square matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?

- Diagonalizable matrix:

  A square matrix A is called **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is **a diagonal matrix**.

  (P diagonalizes A)

- Notes:

  (1) If there exists an invertible matrix P such that $B = P^{-1}AP$, then two square matrices A and B are called **similar**.

  (2) The eigenvalue problem is related closely to the diagonalization problem.

- Thm 7.4: (Similar matrices have the same eigenvalues)

If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.

Pf:

$$A \text{ and } B \text{ are similar} \Rightarrow B = P^{-1}AP$$

$$\begin{aligned}
\left|\lambda \mathrm{I} - B\right| &= \left|\lambda \mathrm{I} - P^{-1}AP\right| = \left|P^{-1}\lambda \mathrm{I}P - P^{-1}AP\right| = \left|P^{-1}(\lambda \mathrm{I} - A)P\right| \\
&= \left|P^{-1}\right|\left|\lambda \mathrm{I} - A\right|\left|P\right| = \left|P^{-1}\right|\left|P\right|\left|\lambda \mathrm{I} - A\right| = \left|P^{-1}P\right|\left|\lambda \mathrm{I} - A\right| \\
&= \left|\lambda \mathrm{I} - A\right|
\end{aligned}$$

A and B have the same characteristic polynomial.

Thus A and B have the same eigenvalues.

- Ex 1: (A diagonalizable matrix)

$$A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I}-A|=\begin{vmatrix}\lambda-1 & -3 & 0\\ -3 & \lambda-1 & 0\\ 0 & 0 & \lambda+2\end{vmatrix}=(\lambda-4)(\lambda+2)^2=0$$

$$\text{Eigenvalues}: \ \lambda_1=4, \lambda_2=-2, \lambda_3=-2$$

$$(1)\lambda=4 \Rightarrow \text{Eigenvector}: p_1=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$$(2)\lambda = -2 \Rightarrow \text{Eigenvector}: p_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Notes:

$$(1)\ P = [p_2 \quad p_1 \quad p_3] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$(2)\ P = [p_2 \quad p_3 \quad p_1] = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

- Thm 7.5: (Condition for diagonalization)

An $n\times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Pf:

$(\Rightarrow) A$ is diagonalizable

there exists an invertible P s.t. $D = P^{-1}AP$ is diagonal

Let $P = [\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n]$ and $D = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$

$$PD = [\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= [\lambda_1\mathbf{p}_1 \mid \lambda_2\mathbf{p}_2 \mid \cdots \mid \lambda_n\mathbf{p}_n]$$

$$AP = A[\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n] = [A\mathbf{p}_1 \mid A\mathbf{p}_2 \mid \cdots \mid A\mathbf{p}_n]$$

$\because AP = PD$

$\therefore A\mathbf{p}_i = \lambda_i \mathbf{p}_i,\ i = 1, 2, \ldots, n$

($i.e.$ the column vector $\mathbf{p}_i$ of P are eigenvectors of A)

$\because P$ is invertible $\Rightarrow \mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$ are linearly independent.

$\therefore$ A has n linearly independent eigenvectors.

$(\Leftarrow)$ A has n linearly independent eigenvectors $p_1, p_2, \cdots p_n$

with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$

i.e. $A\mathbf{p}_i = \lambda_i \mathbf{p}_i,\ i = 1, 2, \ldots, n$

Let $P = [\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n]$

$$
\begin{aligned}
AP &= A[\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n] \\
&= [A\mathbf{p}_1 \mid A\mathbf{p}_2 \mid \cdots \mid A\mathbf{p}_n] \\
&= [\lambda_1\mathbf{p}_1 \mid \lambda_2\mathbf{p}_2 \mid \cdots \mid \lambda_n\mathbf{p}_n] \\
&= [\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD
\end{aligned}
$$

$\because \mathbf{p}_1, \mathbf{p}_1, \cdots, \mathbf{p}_n$ are linearly independent $\Rightarrow P$ is invertible

$\therefore P^{-1}AP = D$

$\Rightarrow$ A is diagonalizable

Note: If n linearly independent vectors do not exist, then an $n \times n$ matrix A is not diagonalizable.

- Ex 4: (A matrix that is not diagonalizable)

Show that the following matrix is not diagonalizable.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I} - A| = \begin{vmatrix} \lambda - 1 & -2 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0$$

$$\text{Eigenvalue}: \ \lambda_1 = 1$$

$$\lambda \mathrm{I} - A = I - A = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{Eigenvector}: \ p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

A does not have two (n=2) linearly independent eigenvectors, so A is not diagonalizable.

- Steps for diagonalizing an $n\times n$ square matrix:

Step 1: Find n linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$ for A with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$

Step 2: Let $P = [\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n]$

Step 3:

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \text{ where } A\mathbf{p}_i = \lambda_i \mathbf{p}_i,\ i = 1, 2, \ldots, n$$

Note:

The order of the eigenvalues used to form P will determine the order in which the eigenvalues appear on the main diagonal of D.

- Ex 5: (Diagonalizing a matrix)

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

Sol: Characteristic equation:

$$|\lambda \mathrm{I} - A| = \begin{vmatrix} \lambda - 1 & 1 & 1 \\ -1 & \lambda - 3 & -1 \\ 3 & -1 & \lambda + 1 \end{vmatrix} = (\lambda - 2)(\lambda + 2)(\lambda - 3) = 0$$

$$\text{Eigenvalues}: \ \lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 3$$

$$\lambda_1 = 2$$

$$\Rightarrow \lambda_1 \mathrm{I} - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{Eigenvector}: \ p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -2$$

$$\Rightarrow \lambda_2 \mathrm{I} - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ -\frac{1}{4}t \\ t \end{bmatrix} = \frac{1}{4}t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \Rightarrow \text{Eigenvector}: \ p_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$\lambda_3 = 3$

$$\Rightarrow \lambda_3 \mathrm{I} - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \text{Eigenvector}: \; p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Let } P = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

$$\Rightarrow \quad P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- Notes: k is a positive integer

$$(1)\ D=\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow \ D^k=\begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

$$(2)\ D=P^{-1}AP$$

$$\begin{aligned} \Rightarrow\ D^k &= (P^{-1}AP)^k \\ &= (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP) \\ &= P^{-1}A(PP^{-1})A(PP^{-1})\cdots(PP^{-1})AP \\ &= P^{-1}AA\cdots AP \\ &= P^{-1}A^kP \end{aligned}$$

$$\therefore A^k = PD^kP^{-1}$$

- Thm 7.6: (Sufficient conditions for diagonalization)

  If an $n \times n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

- Proof

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n distinct eigenvalues of A with corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. To begin, assume the set of eigenvectors is linearly dependent. Moreover, consider the eigenvectors to be ordered so that the first m eigenvectors are linearly independent, but the first $m + 1$ are linearly dependent, where $m < n$. Then $\mathbf{x}_{m+1}$ can be written as a linear combination of the first m eigenvectors:

$$\mathbf{x}_{m+1} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m \qquad \text{Equation 1}$$

where the c_i's are not all zero. Multiplication of both sides of Equation 1 by A yields

$$A\mathbf{x}_{m+1} = Ac_1\mathbf{x}_1 + Ac_2\mathbf{x}_2 + \cdots + Ac_m\mathbf{x}_m.$$

- Proof $\mathbf{x}_{m+1} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdot \cdot \cdot + c_m\mathbf{x}_m$ Equation 1

$$A\mathbf{x}_{m+1} = Ac_1\mathbf{x}_1 + Ac_2\mathbf{x}_2 + \cdot \cdot \cdot + Ac_m\mathbf{x}_m.$$

Now $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$, $i = 1, 2, \ldots, m + 1$, so you have

$$\lambda_{m+1}\mathbf{x}_{m+1} = c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdot \cdot \cdot + c_m\lambda_m\mathbf{x}_m.$$ Equation 2

Multiplication of Equation 1 by λ_{m+1} yields

$$\lambda_{m+1}\mathbf{x}_{m+1} = c_1\lambda_{m+1}\mathbf{x}_1 + c_2\lambda_{m+1}\mathbf{x}_2 + \cdot \cdot \cdot + c_m\lambda_{m+1}\mathbf{x}_m.$$ Equation 3

$$c_1(\lambda_{m+1} - \lambda_1)\mathbf{x}_1 + c_2(\lambda_{m+1} - \lambda_2)\mathbf{x}_2 + \cdot \cdot \cdot + c_m(\lambda_{m+1} - \lambda_m)\mathbf{x}_m = \mathbf{0}$$ Eq3-Eq2

$$c_1(\lambda_{m+1} - \lambda_1) = c_2(\lambda_{m+1} - \lambda_2) = \cdot \cdot \cdot = c_m(\lambda_{m+1} - \lambda_m) = 0.$$

All the eigenvalues are distinct, so it follows that $c_i = 0$, $i = 1, 2, \ldots, m$.

- Proof

All the eigenvalues are distinct, so it follows that $c_i = 0$, $i = 1, 2, \ldots, m$. But this result contradicts our assumption that $\mathbf{x}_{m+1}$ can be written as a linear combination of the first m eigenvectors. So, the set of eigenvectors is linearly independent, and from Theorem 7.5, you can conclude that A is diagonalizable. ∎

- Ex 7: (Determining whether a matrix is diagonalizable)

$$A=\begin{bmatrix}1 & -2 & 1\\ 0 & 0 & 1\\ 0 & 0 & -3\end{bmatrix}$$

Sol: Because A is a triangular matrix,

its eigenvalues are the main diagonal entries.

$$\lambda_1=1, \lambda_2=0, \lambda_3=-3$$

These three values are distinct, so A is diagonalizable. (Thm.7.6)

- Ex 8: (Finding a diagonalizing matrix for a linear transformation)

  Let $T: R^3 \to R^3$ be the linear transformation given by

  $$T(x_1, x_2, x_3) = (x_1 - x_2 - x_3,\ x_1 + 3x_2 + x_3,\ -3x_1 + x_2 - x_3)$$

  Find a basis B for R^3 such that the matrix for T relative to B is diagonal.

Sol: The standard matrix for T is given by

$$A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

From Ex. 5, there are three distinct eigenvalues

$$\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 3$$

so A is diagonalizable. (Thm. 7.6)

Thus, the three linearly independent eigenvectors found in Ex. 5

$$p_1 = (-1, 0, 1), p_2 = (1, -1, 4), p_3 = (-1, 1, 1)$$

can be used to form the basis B. That is

$$B = \{p_1, p_2, p_3\} = \{(-1, 0, 1), (1, -1, 4), (-1, 1, 1)\}$$

The matrix for T relative to this basis is

$$\begin{aligned} D &= \left[[T(p_1)]_B \ [T(p_2)]_B \ [T(p_3)]_B\right] \\ &= \left[[Ap_1]_B \ [Ap_2]_B \ [Ap_3]_B\right] \\ &= \left[[\lambda_1 p_1]_B \ [\lambda_2 p_2]_B \ [\lambda_3 p_3]_B\right] \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \end{aligned}$$

7.3 Symmetric Matrices and Orthogonal Diagonalization

- Symmetric matrix:

A square matrix A is **symmetric** if it is equal to its transpose:

$$A = A^T$$

- Ex 1: (Symmetric matrices and nonsymetric matrices)

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix}$$ (symmetric)

$$B = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$$ (symmetric)

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$ (nonsymmetric)

- Thm 7.7: (Eigenvalues of symmetric matrices)

If A is an $n \times n$ symmetric matrix, then the following properties are true.

(1) A is diagonalizable.

(2) All eigenvalues of A are real.

(3) If λ is an eigenvalue of A with multiplicity k, then λ has k linearly independent eigenvectors. That is, the eigenspace of λ has dimension k.

- Ex 2:

Prove that a symmetric matrix is diagonalizable.

$$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

Pf: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix} = \lambda^2 - (a+b)\lambda + ab - c^2 = 0$$

As a quadratic in λ, this polynomial has a discriminant of

$$\begin{aligned}(a+b)^2 - 4(ab - c^2) &= a^2 + 2ab + b^2 - 4ab + 4c^2 \\ &= a^2 - 2ab + b^2 + 4c^2 \\ &= \underline{(a-b)^2 + 4c^2} \ \geq 0\end{aligned}$$

$$(1)\ \ (a-b)^2+4c^2=0$$

$$\Rightarrow\ a=b,\ c=0$$

$$A=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$ is a matrix of diagonal.

$$(2)\ \ (a-b)^2+4c^2>0$$

The characteristic polynomial of A has two distinct real roots, which implies that A has two distinct real eigenvalues. Thus, A is diagonalizable.

- Orthogonal matrix:

A square matrix P is called **orthogonal** if it is invertible and

$$P^{-1} = P^{T}$$

- Ex 4: (Orthogonal matrices)

(a) $P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is orthogonal because $P^{-1} = P^{T} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(b) $P = \begin{bmatrix} \frac{3}{5} & 0 & \frac{-4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$ is orthogonal because $P^{-1} = P^{T} = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \\ \frac{-4}{5} & 0 & \frac{3}{5} \end{bmatrix}$.

- Thm 7.8: (Properties of orthogonal matrices)

An $n \times n$ matrix P is orthogonal if and only if its column vectors form an orthogonal set.

- Ex 5: (An orthogonal matrix)

$$P=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

Sol: If P is a orthogonal matrix, then $P^{-1}=P^T \Rightarrow PP^T=\mathrm{I}$

$$PP^T=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{-2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{-4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\mathrm{I}$$

$$\text{Let } \mathbf{p}_1=\begin{bmatrix} \frac{1}{3} \\ \frac{-2}{\sqrt{5}} \\ \frac{-2}{3\sqrt{5}} \end{bmatrix}, \mathbf{p}_2=\begin{bmatrix} \frac{2}{3} \\ \frac{1}{\sqrt{5}} \\ \frac{-4}{3\sqrt{5}} \end{bmatrix}, \mathbf{p}_3=\begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$$

produces

$$\mathbf{p}_1 \cdot \mathbf{p}_2 = \mathbf{p}_1 \cdot \mathbf{p}_3 = \mathbf{p}_2 \cdot \mathbf{p}_3 = 0$$

$$\|\mathbf{p}_1\| = \|\mathbf{p}_2\| = \|\mathbf{p}_3\| = 1$$

$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is an orthonormal set.

- Thm 7.9: (Properties of symmetric matrices)

Let A be an $n\times n$ symmetric matrix. If λ_1 and λ_2 are distinct eigenvalues of A, then their corresponding eigenvectors x_1 and x_2 are orthogonal.

- Ex 6: (Eigenvectors of a symmetric matrix)

$$\text{Show that any two eigenvectors of } A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

corresponding to distinct eigenvalues are orthogonal.

- Sol: Characteristic function

$$|\lambda \mathrm{I}-A|=\begin{vmatrix}\lambda-3 & -1\\ -1 & \lambda-3\end{vmatrix}=\lambda^2-6\lambda+8=(\lambda-2)(\lambda-4)=0$$

$$\Rightarrow \text{Eigenvalues}:\ \lambda_1=2,\lambda_2=4$$

$$(1)\ \lambda_1=2\Rightarrow \lambda_1\mathrm{I}-A=\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}\sim\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\quad\Rightarrow \mathrm{x}_1=s\begin{bmatrix}-1\\ 1\end{bmatrix},\ s\neq 0$$

$$(2)\ \lambda_2=4\Rightarrow \lambda_2\mathrm{I}-A=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\sim\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}\quad\Rightarrow \mathrm{x}_2=t\begin{bmatrix}1\\ 1\end{bmatrix},\ t\neq 0$$

$$\mathrm{x}_1\cdot\mathrm{x}_2=\begin{bmatrix}-s\\ s\end{bmatrix}\cdot\begin{bmatrix}t\\ t\end{bmatrix}=st-st=0\ \Rightarrow\ \mathrm{x}_1 \text{ and } \mathrm{x}_2 \text{ are orthogonal.}$$

- Thm 7.10: (Fundamental theorem of symmetric matrices)

  Let A be an $n \times n$ matrix. Then A is orthogonally diagonalizable and has real eigenvalue if and only if A is symmetric.

- Orthogonal diagonalization of a symmetric matrix:

  Let A be an $n \times n$ symmetric matrix.

  (1) Find all eigenvalues of A and determine the multiplicity of each.

  (2) For each eigenvalue of multiplicity 1, choose a unit eigenvector.

  (3) For each eigenvalue of multiplicity $k \geq 2$, find a set of k linearly independent eigenvectors. If this set is not orthonormal, apply Gram-Schmidt orthonormalization process.

  (4) The composite of steps 2 and 3 produces <u>an orthonormal set of n eigenvectors</u>. Use these eigenvectors to form the columns of P. The matrix $P^{-1}AP = P^{T}AP = D$ will be diagonal.

- Ex 7: (Determining whether a matrix is orthogonally diagonalizable)

	Symmetric matrix	Orthogonally diagonalizable
$A_1=\begin{bmatrix}1&1&1\\1&0&1\\1&1&1\end{bmatrix}$	○	○
$A_2=\begin{bmatrix}5&2&1\\2&1&8\\-1&8&0\end{bmatrix}$	×	×
$A_3=\begin{bmatrix}3&2&0\\2&0&1\end{bmatrix}$	×	×
$A_4=\begin{bmatrix}0&0\\0&-2\end{bmatrix}$	○	○

- Ex 9: (Orthogonal diagonalization)

Find an orthogonal matrix P that diagonalizes A.

$$A=\begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$$

Sol:

(1) $\left|\lambda I - A\right| = (\lambda-3)^2(\lambda+6)=0$

$\lambda_1 = -6, \lambda_2 = 3$ (has a multiplicity of 2)

(2) $\lambda_1 = -6, \quad v_1 = (1,-2,2) \Rightarrow u_1 = \dfrac{v_1}{\left\|v_1\right\|} = (\tfrac{1}{3}, \tfrac{-2}{3}, \tfrac{2}{3})$

(3) $\lambda_2 = 3, \quad v_2 = (2,1,0), \; v_3 = (-2,0,1)$

Linear Independent

Gram-Schmidt Process:

$$w_2 = v_2 = (2, 1, 0), \quad w_3 = v_3 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 = (\tfrac{-2}{5}, \tfrac{4}{5}, 1)$$

$$u_2 = \frac{w_2}{\|w_2\|} = (\tfrac{2}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}, 0), \quad u_3 = \frac{w_3}{\|w_3\|} = (\tfrac{-2}{3\sqrt{5}}, \tfrac{4}{3\sqrt{5}}, \tfrac{5}{3\sqrt{5}})$$

$$(4)\ P = [p_1\ p_2\ p_3] = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} \\ \frac{-2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

$$\Rightarrow \quad P^{-1}AP = P^{T}AP = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Key Learning in Section 7.3

- Recognize, and apply properties of, symmetric matrices.
- Recognize, and apply properties of, orthogonal matrices.
- Find an orthogonal matrix P that orthogonally diagonalizes a symmetric matrix A.

Keywords in Section 7.3

- symmetric matrix: 對稱矩陣
- orthogonal matrix: 正交矩陣
- orthonormal set: 單範正交集
- orthogonal diagonalization: 正交對角化