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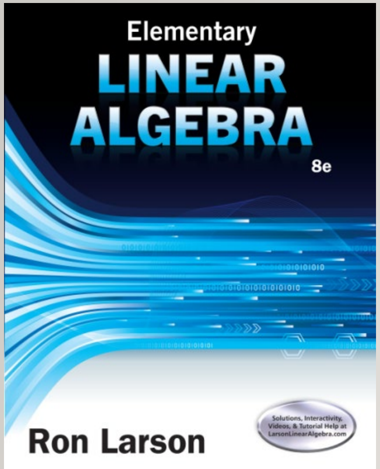


CHAPTER 5
INNER PRODUCT SPACES

5.1 Length and Dot Product in R^n
5.2 Inner Product Spaces
5.3 Orthonormal Bases: Gram-Schmidt Process
5.4 Mathematical Models and Least Square Analysis
5.5 Applications of Inner Product Space

5.1 Length and Dot Product in R^n

- Length:

The length of a vector $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ in R^n is given by

$$\|\mathbf{v}\| = \sqrt{v_1^{\ 2} + v_2^{\ 2} + \cdots + v_n^{\ 2}}$$

- Notes: The length of a vector is also called its **norm**.

- Notes: Properties of length

$$(1) \quad \|\mathbf{v}\| \geq 0$$

$(2) \quad \|\mathbf{v}\| = 1 \Rightarrow \mathbf{v}$ is called a **unit vector**.

$$(3) \quad \|\mathbf{v}\| = 0 \ \text{ iff } \ \mathbf{v} = 0$$

$$(4) \quad \|c\mathbf{v}\| = |c|\|\mathbf{v}\|$$

- Ex 1:

(a) In R^5, the length of $\mathbf{v}=(0,-2,1,4,-2)$ is given by

$$\|\mathbf{v}\|=\sqrt{0^2+(-2)^2+1^2+4^2+(-2)^2}=\sqrt{25}=5$$

(b) In R^3 the length of $\mathbf{v}=(\frac{2}{\sqrt{17}},\frac{-2}{\sqrt{17}},\frac{3}{\sqrt{17}})$ is given by

$$\|\mathbf{v}\|=\sqrt{\left(\frac{2}{\sqrt{17}}\right)^2+\left(\frac{-2}{\sqrt{17}}\right)^2+\left(\frac{3}{\sqrt{17}}\right)^2}=\sqrt{\frac{17}{17}}=1$$

($\mathbf{v}$ is a unit vector)

- A standard unit vector in R^n:

$$\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} = \{(1,0,\cdots,0),(0,1,\cdots,0),(0,0,\cdots,1)\}$$

- Ex:

the standard unit vector in R^2: $\{i, j\} = \{(1,0),(0,1)\}$

the standard unit vector in R^3: $\{i, j, k\} = \{(1,0,0),(0,1,0),(0,0,1)\}$

- Notes: (Two nonzero vectors are parallel)

$$\mathbf{u} = c\mathbf{v}$$

(1) $c > 0 \Rightarrow$ **u** and **v** have the same direction

(2) $c < 0 \Rightarrow$ **u** and **v** have the opposite direction

- Thm 5.1: (Length of a scalar multiple)

Let $\mathbf{v}$ be a vector in R^n and c be a scalar. Then

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$$

Pf:

$$\mathbf{v} = (v_1\,, v_2\,, \cdots, v_n)$$

$$\Rightarrow c\mathbf{v} = (cv_1\,, cv_2\,, \cdots, cv_n)$$

$$\|c\mathbf{v}\| = \|\,(cv_1\,, cv_2\,, \cdots, cv_n)\,\|$$

$$= \sqrt{(cv_1)^2 + (cv_2)^2 + \cdots + (cv_n)^2}$$

$$= \sqrt{c^2(v_1{}^2 + v_2{}^2 + \cdots + v_n{}^2)}$$

$$= |c|\sqrt{v_1{}^2 + v_2{}^2 + \cdots + v_n{}^2}$$

$$= |c|\,\|\mathbf{v}\|$$

- Thm 5.2: (Unit vector in the direction of **v**)

  If **v** is a nonzero vector in R^n, then the vector $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$

  has length 1 and has the same direction as **v**. This vector **u** is called the **unit vector in the direction of v**.

Pf:

$$\mathbf{v} \text{ is nonzero } \Rightarrow \|\mathbf{v}\| \neq 0 \Rightarrow \frac{1}{\|\mathbf{v}\|} > 0$$

$$\Rightarrow \mathbf{u} = \frac{\mathbf{1}}{\|\mathbf{v}\|}\mathbf{v} \qquad (\mathbf{u} \text{ has the same direction as } \mathbf{v})$$

$$\|\mathbf{u}\| = \left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1 \qquad (\text{u has length 1 })$$

- Notes:

  (1) The vector $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is called the unit vector in the direction of **v**.

  (2) The process of finding the unit vector in the direction of **v** is called **normalizing** the vector **v.**

- Ex 2: (Finding a unit vector)

  Find the unit vector in the direction of $\mathbf{v}=(3\,,-1\,,2)$,

  and verify that this vector has length 1.

Sol:

$$\mathbf{v}=(3\,,-1\,,2) \quad \Rightarrow \|\mathbf{v}\|=\sqrt{3^2+(-1)^2+2^2}=\sqrt{14}$$

$$\Rightarrow \frac{\mathbf{v}}{\|\mathbf{v}\|}=\frac{(3\,,-1\,,2)}{\sqrt{3^2+(-1)^2+2^2}}=\frac{1}{\sqrt{14}}(3\,,-1\,,2)=\left(\frac{3}{\sqrt{14}}\,,\frac{-1}{\sqrt{14}}\,,\frac{2}{\sqrt{14}}\right)$$

$$\because \quad \sqrt{\left(\frac{3}{\sqrt{14}}\right)^2+\left(\frac{-1}{\sqrt{14}}\right)^2+\left(\frac{2}{\sqrt{14}}\right)^2}=\sqrt{\frac{14}{14}}=1$$

$$\therefore \quad \frac{\mathbf{v}}{\|\mathbf{v}\|} \quad \text{is a unit vector.}$$

- Distance between two vectors:

The **distance** between two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

- Notes: (Properties of distance)

(1) $d(\mathbf{u}, \mathbf{v}) \geq 0$

(2) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

(3) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

- Ex 3: (Finding the distance between two vectors)

The distance between **u** = (0, 2, 2) and **v** = (2, 0, 1) is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(0-2, 2-0, 2-1)\|$$
$$= \sqrt{(-2)^2 + 2^2 + 1^2} = 3$$

- Dot product in R^n:

  The **dot product** of $\mathbf{u}=(u_1\ ,u_2\ ,\cdots,u_n)$ and $\mathbf{v}=(v_1\ ,v_2\ ,\cdots,v_n)$ is the scalar quantity

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+\cdots+u_nv_n$$

- Ex 4: (Finding the dot product of two vectors)

  The dot product of **u**=(1, 2, 0, -3) and **v**=(3, -2, 4, 2) is

$$\mathbf{u}\cdot\mathbf{v}=(1)(3)+(2)(-2)+(0)(4)+(-3)(2)=-7$$

- Thm 5.3: (Properties of the dot product)

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n and c is a scalar, then the following properties are true.

(1) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(2) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(3) $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$

(4) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

(5) $\mathbf{v} \cdot \mathbf{v} \geq 0$, and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\boldsymbol{v} = 0$

- Euclidean *n*-space:

  R^n was defined to be the *set* of all order n-tuples of real numbers. When R^n is combined with the standard operations of vector addition, scalar multiplication, vector length, and the dot product, the resulting vector space is called **Euclidean *n*-space**.

- Ex 5: (Finding dot products)

$$\mathbf{u} = (2\,,-2)\,,\ \mathbf{v} = (5\,,8),\ \mathbf{w} = (-4\,,3)$$

(a) $\mathbf{u}\cdot\mathbf{v}$ (b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ (c) $\mathbf{u}\cdot(2\mathbf{v})$ (d) $\|\mathbf{w}\|^2$ (e) $\mathbf{u}\cdot(\mathbf{v}-2\mathbf{w})$

Sol:

(a) $\mathbf{u}\cdot\mathbf{v} = (2)(5) + (-2)(8) = -6$

(b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{w} = -6\mathbf{w} = -6(-4\,,3) = (24\,,-18)$

(c) $\mathbf{u}\cdot(2\mathbf{v}) = 2(\mathbf{u}\cdot\mathbf{v}) = 2(-6) = -12$

(d) $\|\mathbf{w}\|^2 = \mathbf{w}\cdot\mathbf{w} = (-4)(-4) + (3)(3) = 25$

(e) $\mathbf{v} - 2\mathbf{w} = (5-(-8)\,,8-6) = (13\,,2)$

$\mathbf{u}\cdot(\mathbf{v}-2\mathbf{w}) = (2)(13) + (-2)(2) = 26 - 4 = 22$

- Ex 6: (Using the properties of the dot product)

Given $\mathbf{u}\cdot\mathbf{u}=39 \quad \mathbf{u}\cdot\mathbf{v}=-3 \qquad \mathbf{v}\cdot\mathbf{v}=79$

Find $(\mathbf{u}+2\mathbf{v})\cdot(3\mathbf{u}+\mathbf{v})$

Sol:

$$\begin{aligned}
(\mathbf{u}+2\mathbf{v})\cdot(3\mathbf{u}+\mathbf{v}) &= \mathbf{u}\cdot(3\mathbf{u}+\mathbf{v})+2\mathbf{v}\cdot(3\mathbf{u}+\mathbf{v}) \\
&= \mathbf{u}\cdot(3\mathbf{u})+\mathbf{u}\cdot\mathbf{v}+(2\mathbf{v})\cdot(3\mathbf{u})+(2\mathbf{v})\cdot\mathbf{v} \\
&= 3(\mathbf{u}\cdot\mathbf{u})+\mathbf{u}\cdot\mathbf{v}+6(\mathbf{v}\cdot\mathbf{u})+2(\mathbf{v}\cdot\mathbf{v}) \\
&= 3(\mathbf{u}\cdot\mathbf{u})+7(\mathbf{u}\cdot\mathbf{v})+2(\mathbf{v}\cdot\mathbf{v}) \\
&= 3(39)+7(-3)+2(79)=254
\end{aligned}$$

- Thm 5.4: (The Cauchy - Schwarz inequality)

  If **u** and **v** are vectors in R^n, then

  $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$ ($|\mathbf{u}\cdot\mathbf{v}|$ denotes the absolute value of $\mathbf{u}\cdot\mathbf{v}$)

- Ex 7: (An example of the Cauchy - Schwarz inequality)

  Verify the Cauchy - Schwarz inequality for **u**=(1, -1, 3) and **v**=(2, 0, -1)

Sol: $\mathbf{u}\cdot\mathbf{v} = -1, \quad \mathbf{u}\cdot\mathbf{u} = 11, \quad \mathbf{v}\cdot\mathbf{v} = 5$

$$\Rightarrow |\mathbf{u}\cdot\mathbf{v}| = |-1| = 1$$

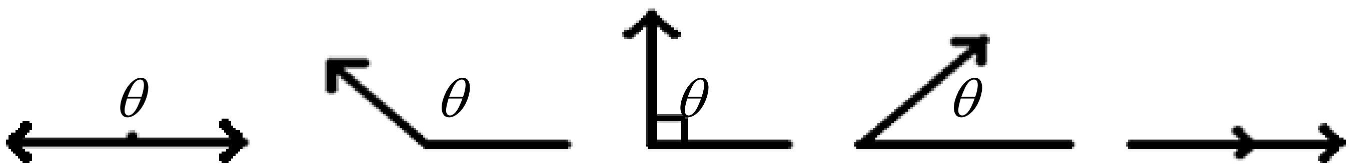
$$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}}\cdot\sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{11}\cdot\sqrt{5} = \sqrt{55}$$

$$\therefore |\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$$

- The angle between two vectors in R^n:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|},\ 0 \le \theta \le \pi$$

Opposite direction	$\mathbf{u}\cdot\mathbf{v} < \mathbf{0}$	$\mathbf{u}\cdot\mathbf{v} = \mathbf{0}$	$\mathbf{u}\cdot\mathbf{v} > \mathbf{0}$	Same direction



$\theta = \pi$	$\frac{\pi}{2} < \theta < \pi$	$\theta = \frac{\pi}{2}$	$0 < \theta < \frac{\pi}{2}$	$\theta = 0$
$\cos = -1$	$\cos < 0$	$\cos = 0$	$\cos > 0$	$\cos = 1$

- Note:

  The angle between the zero vector and another vector is not defined.

- Ex 8: (Finding the angle between two vectors)

$$\mathbf{u} = (-4\,,0\,,2\,,-2) \qquad \mathbf{v} = (2\,,0\,,-1\,,1)$$

Sol:

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{(-4)^2 + 0^2 + 2^2 + (-2)^2} = \sqrt{24}$$

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{2^2 + (0)^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$\mathbf{u}\cdot\mathbf{v} = (-4)(2) + (0)(0) + (2)(-1) + (-2)(1) = -12$$

$$\Rightarrow \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-12}{\sqrt{24}\sqrt{6}} = -\frac{12}{\sqrt{144}} = -1$$

$$\Rightarrow \theta = \pi \qquad \therefore\ \mathbf{u} \text{ and } \mathbf{v} \text{ have opposite directions.} \quad (\mathbf{u} = -2\mathbf{v})$$

- Orthogonal vectors:

  Two vectors **u** and **v** in R^n are orthogonal if

  $$\mathbf{u} \cdot \mathbf{v} = 0$$

- Note:

  The vector **0** is said to be orthogonal to every vector.

- Ex 10: (Finding orthogonal vectors)

Determine all vectors in R^n that are orthogonal to **u**=(4, 2).

Sol:

$$\mathbf{u} = (4\,,2) \quad \text{Let} \quad \mathbf{v} = (v_1\,,v_2)$$

$$\begin{aligned}\Rightarrow \quad \mathbf{u}\cdot\mathbf{v} &= (4\,,2)\cdot(v_1\,,v_2)\\ &= 4v_1 + 2v_2\\ &= 0\end{aligned}$$

$$\begin{bmatrix}4 & 2 & 0\end{bmatrix} \rightarrow \begin{bmatrix}1 & \frac{1}{2} & 0\end{bmatrix}$$

$$\Rightarrow \quad v_1 = \frac{-t}{2} \quad , \quad v_2 = t$$

$$\therefore \quad \mathbf{v} = \left(\frac{-t}{2}, t\right), \qquad t \in R$$

- Thm 5.5: (The triangle inequality)

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$

Pf:

$$
\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\
&= \mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) + \mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v} \\
&= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 \quad \le \|\mathbf{u}\|^2 + 2\,|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2 \\
&\le \|\mathbf{u}\|^2 + 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2
\end{aligned}
$$

$$\therefore \|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$

- Note:

  Equality occurs in the triangle inequality if and only if the vectors $\mathbf{u}$ and $\mathbf{v}$ have the same direction.

- Thm 5.6: (The Pythagorean theorem)

  If **u** and **v** are vectors in R^n, then **u** and **v** are orthogonal if and only if

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

- Dot product and matrix multiplication:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

(A vector $\mathbf{u} = (u_1\ , u_2\ , \cdots , u_n)$ in R^n is represented as an $n \times 1$ column matrix)

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = [u_1 \quad u_2 \quad \cdots \quad u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = [u_1 v_1 + u_2 v_2 + \cdots + u_n v_n]$$

Key Learning in Section 5.1

- Find the length of a vector and find a unit vector.
- Find the distance between two vectors.
- Find a dot product and the angle between two vectors, determine orthogonality, and verify the Cauchy-Schwarz Inequality, the triangle inequality, and the Pythagorean Theorem.
- Use a matrix product to represent a dot product.

Keywords in Section 5.1

- length: 長度
- norm: 範數
- unit vector: 單位向量
- standard unit vector : 標準單位向量
- normalizing: 單範化
- distance: 距離
- dot product: 點積
- Euclidean *n*-space: 歐基里德*n*維空間
- Cauchy-Schwarz inequality: 科西-舒瓦茲不等式
- angle: 夾角
- triangle inequality: 三角不等式
- Pythagorean theorem: 畢氏定理

5.2 Inner Product Spaces

- Inner product:

Let **u**, **v**, and **w** be vectors in a vector space V, and let c be any scalar. An inner product on V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors **u** and **v** and satisfies the following axioms.

(1) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

(2) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

(3) $c \langle \mathbf{u}, \mathbf{v} \rangle = \langle c\mathbf{u}, \mathbf{v} \rangle$

(4) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$

- Note:

$$\mathbf{u} \cdot \mathbf{v} = \text{dot product (Euclidean inner product for } R^n\text{)}$$

$$< \mathbf{u}\ ,\ \mathbf{v} >= \text{general inner product for vector space } V$$

- Note:

A vector space V with an inner product is called an inner product space.

Vector space: $(V,\ +,\ \bullet)$

Inner product space: $(V,\ +,\ \bullet,\ <,>)$

- Ex 1: (The Euclidean inner product for R^n)

Show that the dot product in R^n satisfies the four axioms of an inner product.

Sol:

$$\mathbf{u}=(u_1\ ,u_2\ ,\cdots,u_n)\quad ,\quad \mathbf{v}=(v_1\ ,v_2\ ,\cdots,v_n)$$

$$\langle\, \mathbf{u}\ ,\ \mathbf{v}\,\rangle=\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+\cdots+u_nv_n$$

By Theorem 5.3, this dot product satisfies the required four axioms. Thus it is an inner product on R^n.

Thm 5.3: (Properties of the dot product)

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n and c is a scalar, then the following properties are true.

(1) $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$

(2) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$

(3) $c(\mathbf{u}\cdot\mathbf{v})=(c\mathbf{u})\cdot\mathbf{v}=\mathbf{u}\cdot(c\mathbf{v})$

(4) $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2$

(5) $\mathbf{v}\cdot\mathbf{v}\geq 0$, and $\mathbf{v}\cdot\mathbf{v}=0$ if and only if $\boldsymbol{v}=0$

- Ex 2: (A different inner product for R^n)

Show that the function defines an inner product on R^2, where $\mathbf{u}=(u_1, u_2)$ and $\mathbf{v}=(v_1, v_2)$.

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 2u_2 v_2$$

Sol:

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 2u_2 v_2 = v_1 u_1 + 2v_2 u_2 = \langle \mathbf{v}, \mathbf{u} \rangle$

(b) $\mathbf{w}=(w_1, w_2)$

$$\begin{aligned} \Rightarrow \langle \mathbf{u}, \mathbf{v}+\mathbf{w} \rangle &= u_1(v_1+w_1)+2u_2(v_2+w_2) \\ &= u_1v_1+u_1w_1+2u_2v_2+2u_2w_2 \\ &= (u_1v_1+2u_2v_2)+(u_1w_1+2u_2w_2) \\ &= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle \end{aligned}$$

$$(c) \quad c\langle \mathbf{u}, \mathbf{v}\rangle = c(u_1v_1 + 2u_2v_2) = (cu_1)v_1 + 2(cu_2)v_2 = \langle c\mathbf{u}, \mathbf{v}\rangle$$

$$(d) \quad \langle \mathbf{v}, \mathbf{v}\rangle = v_1^{\ 2} + 2v_2^{\ 2} \geq 0$$

$$\langle \mathbf{v}, \mathbf{v}\rangle = 0 \Rightarrow v_1^{\ 2} + 2v_2^{\ 2} = 0 \qquad \Rightarrow \quad v_1 = v_2 = 0 \quad (\mathbf{v} = 0)$$

- Note: (An inner product on R^n)

$$\langle \mathbf{u}, \mathbf{v}\rangle = c_1u_1v_1 + c_2u_2v_2 + \cdots + c_nu_nv_n\ , \qquad c_i > 0$$

- Ex 3: (A function that is not an inner product)

Show that the following function is not an inner product on R^3.

$$\langle \mathbf{u} \cdot \mathbf{v} \rangle = u_1 v_1 - 2u_2 v_2 + u_3 v_3$$

Sol:

Let $\mathbf{v} = (1, 2, 1)$

Then $< \mathbf{v}, \mathbf{v} >= (1)(1) - 2(2)(2) + (1)(1) = -6 < 0$

Axiom 4 is not satisfied.

Thus this function is not an inner product on R^3.

- Thm 5.7: (Properties of inner products)

  Let **u, v**, and **w** be vectors in an inner product space V, and let c be any real number.

  (1) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$

  (2) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

  (3) $\langle \mathbf{u}, c\mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$

- Norm (length) of **u**:

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

- Note:

$$\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$$

- Distance between **u** and **v**:

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

- Angle between two nonzero vectors **u** and **v**:

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|}, \quad 0 \le \theta \le \pi$$

- Orthogonal: $(\mathbf{u} \perp \mathbf{v})$

u and **v** are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

- Notes:

(1) If $\|\mathbf{v}\| = 1$, then v is called a unit vector.

(2) $\begin{array}{l} \|\mathbf{v}\| \neq 1 \\ \mathbf{v} \neq 0 \end{array} \xrightarrow{\text{Normalizing}} \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ (the unit vector in the direction of **v**)

not a unit vector

- Ex 6: (Finding inner product)

$$\langle p\,,q\rangle = a_0b_0 + a_1b_1 + \cdots + a_nb_n \quad \text{is an inner product}$$

Let $p(x) = 1 - 2x^2,\ q(x) = 4 - 2x + x^2$ be polynomials in $P_2(x)$

$(a)\ \langle p\,,q\rangle = ? \qquad (b)\ \ \|q\| = ? \qquad (c)\ \ d(p\,,q) = ?$

Sol:

(a) $\langle p\,,q\rangle = (1)(4) + (0)(-2) + (-2)(1) = 2$

(b) $\|q\| = \sqrt{\langle q\,,q\rangle} = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{21}$

(c) $\because\ p - q = -3 + 2x - 3x^2$

$$\therefore d(p\,,q) = \|p - q\| = \sqrt{\langle p-q, p-q\rangle}$$

$$= \sqrt{(-3)^2 + 2^2 + (-3)^2} = \sqrt{22}$$

- Properties of norm:

(1) $\|\mathbf{u}\| \geq 0$

(2) $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$

(3) $\|c\mathbf{u}\| = |c|\,\|\mathbf{u}\|$

- Properties of distance:

(1) $d(\mathbf{u}, \mathbf{v}) \geq 0$

(2) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

(3) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

- Thm 5.8：

Let **u** and **v** be vectors in an inner product space *V*.

(1) Cauchy-Schwarz inequality:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{u}\| \, \|\mathbf{v}\| \qquad \text{Theorem 5.4}$$

(2) Triangle inequality:

$$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\| \qquad \text{Theorem 5.5}$$

(3) Pythagorean theorem :

u and **v** are orthogonal if and only if

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \qquad \text{Theorem 5.6}$$

- Orthogonal projections in inner product spaces:

  Let **u** and **v** be two vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Then the **orthogonal projection of u onto v** is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}$$

- Note:

  If **v** is a init vector, then $\langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{v}\|^2 = 1.$

  The formula for the orthogonal projection of **u** onto **v** takes the following simpler form.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{v}$$

- Orthogonal projections in inner product spaces:

  Let **u** and **v** be two vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Then the **orthogonal projection of u onto v** is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}$$

- proof:

- Ex 10: (Finding an orthogonal projection in R^3)

Use the Euclidean inner product in R^3 to find the orthogonal projection of u=(6, 2, 4) onto v=(1, 2, 0).

Sol:

$$\because \langle \mathbf{u}\,,\,\mathbf{v}\rangle = (6)(1)+(2)(2)+(4)(0) = 10$$

$$\langle \mathbf{v}\,,\,\mathbf{v}\rangle = 1^2 + 2^2 + 0^2 = 5$$

$$\therefore \operatorname{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \tfrac{10}{5}(1\,,2\,,0) = (2\,,4\,,0)$$

- Note:

$\boldsymbol{u} - \operatorname{proj}_v \boldsymbol{u} = (6, 2, 4) - (2, 4, 0) = (4, -2, 4)$ is orthogonal to $v = (1,2,0)$.

- Thm 5.9: (Orthogonal projection and distance)

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Then

$$d(\mathbf{u}\,,\, \text{proj}_{\mathbf{v}}\mathbf{u}) < d(\mathbf{u}\,,\, c\mathbf{v})\,, \qquad c \neq \frac{\langle \mathbf{u}\,,\, \mathbf{v}\rangle}{\langle \mathbf{v}\,,\, \mathbf{v}\rangle}$$

Key Learning in Section 5.2

- Determine whether a function defines an inner product, and find the inner product of two vectors in R^n, $M_{m,n}$, P_n and C[a, b].
- Find an orthogonal projection of a vector onto another vector in an inner product space.

Keywords in Section 5.2

- inner product: 內積
- inner product space: 內積空間
- norm: 範數
- distance: 距離
- angle: 夾角
- orthogonal: 正交
- unit vector: 單位向量
- normalizing: 單範化
- Cauchy – Schwarz inequality: 科西 - 舒瓦茲不等式
- triangle inequality: 三角不等式
- Pythagorean theorem: 畢氏定理
- orthogonal projection: 正交投影

5.3 Orthonormal Bases: Gram-Schmidt Process

- Orthogonal:

  A set S of vectors in an inner product space V is called an **orthogonal set** if every pair of vectors in the set is orthogonal.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subseteq V$$

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \qquad i \neq j$$

- Orthonormal:

  An orthogonal set in which each vector is a unit vector is called **orthonormal**.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subseteq V$$

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- Note:

  If S is a basis, then it is called an orthogonal basis or an orthonormal basis.

- Ex 1: (A nonstandard orthonormal basis for R^3)

Show that the following set is an orthonormal basis.

$$S = \left\{ \overset{\mathbf{v}_1}{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}, \quad \overset{\mathbf{v}_2}{\left(-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right)}, \quad \overset{\mathbf{v}_3}{\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)} \right\}$$

Sol:

Show that the three vectors are mutually orthogonal.

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = -\tfrac{1}{6} + \tfrac{1}{6} + 0 = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0$$

$$\mathbf{v}_2 \cdot \mathbf{v}_3 = -\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0$$

Show that each vector is of length 1.

$$\|\mathbf{v}_1\| = \sqrt{\mathbf{v}_1 \cdot \mathbf{v}_1} = \sqrt{\tfrac{1}{2} + \tfrac{1}{2} + 0} = 1$$

$$\|\mathbf{v}_2\| = \sqrt{\mathbf{v}_2 \cdot \mathbf{v}_2} = \sqrt{\tfrac{2}{36} + \tfrac{2}{36} + \tfrac{8}{9}} = 1$$

$$\|\mathbf{v}_3\| = \sqrt{\mathbf{v}_3 \cdot \mathbf{v}_3} = \sqrt{\tfrac{4}{9} + \tfrac{4}{9} + \tfrac{1}{9}} = 1$$

Thus S is an orthonormal set.

- Ex 2: (An orthonormal basis for $P_3(x)$)

In $P_3(x)$, with the inner product

$$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$$

The standard basis $B = \{1, x, x^2\}$ is orthonormal.

Sol:

$$\mathbf{v}_1 = 1 + 0x + 0x^2, \quad \mathbf{v}_2 = 0 + x + 0x^2, \quad \mathbf{v}_3 = 0 + 0x + x^2,$$

Then

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = (1)(0) + (0)(1) + (0)(0) = 0,$$
$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = (1)(0) + (0)(0) + (0)(1) = 0,$$
$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = (0)(0) + (1)(0) + (0)(1) = 0$$

$$\|\mathbf{v}_1\| = \sqrt{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} = \sqrt{(1)(1) + (0)(0) + (0)(0)} = 1,$$

$$\|\mathbf{v}_2\| = \sqrt{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} = \sqrt{(0)(0) + (1)(1) + (0)(0)} = 1,$$

$$\|\mathbf{v}_3\| = \sqrt{\langle \mathbf{v}_3, \mathbf{v}_3 \rangle} = \sqrt{(0)(0) + (0)(0) + (1)(1)} = 1$$

- Thm 5.10: (Orthogonal sets are linearly independent)

  If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ is an orthogonal set of *nonzero* vectors in an inner product space *V*, then *S* is linearly independent.

Pf:

S is an orthogonal set of nonzero vectors

i.e. $\langle\mathbf{v}_i,\mathbf{v}_j\rangle=0 \quad i\neq j$ and $\langle\mathbf{v}_i,\mathbf{v}_i\rangle>0$

Let $c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_n\mathbf{v}_n=0$

$$\Rightarrow \quad \langle c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_n\mathbf{v}_n,\mathbf{v}_i\rangle=\langle 0,\mathbf{v}_i\rangle=0 \qquad \forall i$$

$$\Rightarrow \quad c_1\langle\mathbf{v}_1,\mathbf{v}_i\rangle+c_2\langle\mathbf{v}_2,\mathbf{v}_i\rangle+\cdots+c_i\langle\mathbf{v}_i,\mathbf{v}_i\rangle+\cdots+c_n\langle\mathbf{v}_n,\mathbf{v}_i\rangle$$
$$=c_i\langle\mathbf{v}_i,\mathbf{v}_i\rangle$$

$\because\langle\mathbf{v}_i,\mathbf{v}_i\rangle\neq 0 \;\Rightarrow c_i=0 \quad \forall i \qquad \therefore S$ is linearly independent.

- Corollary to Thm 5.10:

  If V is an inner product space of dimension n, then any orthogonal set of n nonzero vectors is a basis for V.

- Ex 4: (Using orthogonality to test for a basis)

Show that the following set is a basis for R^4.

$$S=\{\underset{\mathbf{v}_1}{(2,3,2,-2)},\underset{\mathbf{v}_2}{(1,0,0,1)},\underset{\mathbf{v}_3}{(-1,0,2,1)},\underset{\mathbf{v}_4}{(-1,2,-1,1)}\}$$

Sol:

$\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4$: nonzero vectors

$$\mathbf{v}_1\cdot\mathbf{v}_2=2+0+0-2=0 \qquad \mathbf{v}_2\cdot\mathbf{v}_3=-1+0+0+1=0$$

$$\mathbf{v}_1\cdot\mathbf{v}_3=-2+0+4-2=0 \qquad \mathbf{v}_2\cdot\mathbf{v}_4=-1+0+0+1=0$$

$$\mathbf{v}_1\cdot\mathbf{v}_4=-2+6-2-2=0 \qquad \mathbf{v}_3\cdot\mathbf{v}_4=1+0-2+1=0$$

$\Rightarrow S$ is orthogonal.

$\Rightarrow S$ is a basis for R^4 (by Corollary to Theorem 5.10)

- Thm 5.11: (Coordinates relative to an orthonormal basis)

  If $B=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, then the coordinate representation of a vector $\mathbf{w}$ with respect to B is

$$\mathbf{w}=\langle \mathbf{w}, \mathbf{v}_1\rangle \mathbf{v}_1+\langle \mathbf{w}, \mathbf{v}_2\rangle \mathbf{v}_2+\cdots+\langle \mathbf{w}, \mathbf{v}_n\rangle \mathbf{v}_n$$

Pf:

$B=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for V

$\mathbf{w}\in V$

$\mathbf{w}=k_1\mathbf{v}_1+k_2\mathbf{v}_2+\cdots+k_n\mathbf{v}_n$ (unique representation)

$\because B=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is orthonormal

$$\Rightarrow \langle \mathbf{v}_i, \mathbf{v}_j\rangle=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$$

$$\langle \mathbf{w}, \mathbf{v}_i \rangle = \langle (k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n), \mathbf{v}_i \rangle$$

$$= k_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + \cdots + k_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle + \cdots + k_n\langle \mathbf{v}_n, \mathbf{v}_i \rangle$$

$$= k_i \qquad \forall i$$

$$\Rightarrow \mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{w}, \mathbf{v}_n \rangle \mathbf{v}_n$$

- Note:

If $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthonormal basis for V and $\mathbf{w} \in V$,

Then the corresponding coordinate matrix of $\mathbf{w}$ relative to B is

$$[\mathbf{w}]_B = \begin{bmatrix} \langle \mathbf{w}, \mathbf{v}_1 \rangle \\ \langle \mathbf{w}, \mathbf{v}_2 \rangle \\ \vdots \\ \langle \mathbf{w}, \mathbf{v}_n \rangle \end{bmatrix}$$

- Ex 5: (Representing vectors relative to an orthonormal basis)

  Find the coordinates of $\mathbf{w} = (5, -5, 2)$ relative to the following orthonormal basis for R^3.

$$B = \{(\tfrac{3}{5}, \tfrac{4}{5}, 0), (-\tfrac{4}{5}, \tfrac{3}{5}, 0), (0, 0, 1)\}$$

Sol:

$$\langle \mathbf{w}, \mathbf{v}_1 \rangle = \mathbf{w} \cdot \mathbf{v}_1 = (5, -5, 2) \cdot (\tfrac{3}{5}, \tfrac{4}{5}, 0) = -1$$

$$\langle \mathbf{w}, \mathbf{v}_2 \rangle = \mathbf{w} \cdot \mathbf{v}_2 = (5, -5, 2) \cdot (-\tfrac{4}{5}, \tfrac{3}{5}, 0) = -7$$

$$\langle \mathbf{w}, \mathbf{v}_3 \rangle = \mathbf{w} \cdot \mathbf{v}_3 = (5, -5, 2) \cdot (0, 0, 1) = 2$$

$$\Rightarrow [\mathbf{w}]_B = \begin{bmatrix} -1 \\ -7 \\ 2 \end{bmatrix}$$

- Gram-Schmidt orthonormalization process:

$B=\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is a basis for an inner product space V

Let $\mathbf{v}_1=\mathbf{u}_1$

$\mathbf{w}_1 = span(\{\mathbf{v}_1\})$

$$\mathbf{v}_2=\mathbf{u}_2-\text{proj}_{\mathbf{W}_1}\mathbf{u}_2=\mathbf{u}_2-\frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1$$

$\mathbf{w}_2 = span(\{\mathbf{v}_1, \mathbf{v}_2\})$

$$\mathbf{v}_3=\mathbf{u}_3-\text{proj}_{\mathbf{W}_2}\mathbf{u}_3=\mathbf{u}_3-\frac{\langle \mathbf{u}_3, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1-\frac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\mathbf{v}_2$$

$$\vdots$$

$$\mathbf{v}_n=\mathbf{u}_n-\text{proj}_{\mathbf{W}_{n-1}}\mathbf{u}_n=\mathbf{u}_n-\sum_{i=1}^{n-1}\frac{\langle \mathbf{v}_n, \mathbf{v}_i\rangle}{\langle \mathbf{v}_i, \mathbf{v}_i\rangle}\mathbf{v}_i$$

$\Rightarrow B'=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthogonal basis.

$\Rightarrow B''=\{\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \cdots, \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}\}$ is an orthonormal basis.

- Ex 7: (Applying the Gram-Schmidt orthonormalization process)

Apply the Gram-Schmidt process to the following basis.

$$B = \{\underset{\mathbf{u}_1}{(1, 1, 0)}, \quad \underset{\mathbf{u}_2}{(1, 2, 0)}, \quad \underset{\mathbf{u}_3}{(0, 1, 2)}\}$$

Sol: $\mathbf{v}_1 = \mathbf{u}_1 = (1, 1, 0)$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (1, 2, 0) - \frac{3}{2}(1, 1, 0) = (-\frac{1}{2}, \frac{1}{2}, 0)$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

$$= (0, 1, 2) - \frac{1}{2}(1, 1, 0) - \frac{1/2}{1/2}(-\frac{1}{2}, \frac{1}{2}, 0) = (0, 0, 2)$$

Orthogonal basis

$$\Rightarrow B' = \{\mathbf{v}_1\ , \mathbf{v}_2\ , \mathbf{v}_3\} = \{(1, 1, 0), (\frac{-1}{2}, \frac{1}{2}, 0), (0, 0, 2)\}$$

Orthonormal basis

$$\Rightarrow B'' = \{\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|}\} = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (0, 0, 1)\}$$

- Ex 10: (Alternative form of Gram-Schmidt orthonormalization process)

Find an orthonormal basis for the solution space of the homogeneous system of linear equations.

$$\begin{aligned} x_1 + x_2 \qquad\qquad + 7x_4 &= 0 \\ 2x_1 + x_2 + 2x_3 + 6x_4 &= 0 \end{aligned}$$

Sol:

$$\begin{bmatrix} 1 & 1 & 0 & 7 & 0 \\ 2 & 1 & 2 & 6 & 0 \end{bmatrix} \xrightarrow{G.J.E} \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -2 & 8 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s+t \\ 2s-8t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -8 \\ 0 \\ 1 \end{bmatrix}$$

Thus one basis for the solution space is

$$B=\{\mathbf{v}_1\ ,\ \mathbf{v}_2\}=\{(-2\ ,\ 2\ ,\ 1\ ,\ 0)\ ,\ (1\ ,\ -8\ ,\ 0\ ,\ 1)\}$$

$$\mathbf{v}_1=\mathbf{u}_1=(-2,2,1,0)$$

$$\mathbf{v}_2=\mathbf{u}_2-\frac{\langle\mathbf{u}_2,\mathbf{v}_1\rangle}{\langle\mathbf{v}_1,\mathbf{v}_1\rangle}\mathbf{v}_1=(1,-8,0,1)-\frac{-18}{9}(-2,2,1,0)$$

$$=(-3,-4,2,1)$$

$$\Rightarrow B'=\{(-2,2,1,0)(-3,-4,2,1)\}$$ (orthogonal basis)

$$\Rightarrow B''=\left\{\left(\frac{-2}{3},\frac{2}{3},\frac{1}{3},0\right),\left(\frac{-3}{\sqrt{30}},\frac{-4}{\sqrt{30}},\frac{2}{\sqrt{30}},\frac{1}{\sqrt{30}}\right)\right\}$$

(orthonormal basis)

Key Learning in Section 5.3

- Show that a set of vectors is orthogonal and forms an orthonormal basis, and represent a vector relative to an orthonormal basis.
- Apply the Gram-Schmidt orthonormalization process.

Keywords in Section 5.3

- orthogonal set: 正交集合
- orthonormal set: 單範正交集合
- orthogonal basis: 正交基底
- orthonormal basis: 單範正交基底
- linear independent: 線性獨立
- Gram-Schmidt Process: Gram-Schmidt過程

- Orthogonal subspaces:

  The subspaces W_1 and W_2 of an inner product space V are orthogonal if $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = 0$ for all $\boldsymbol{v}_1$ in W_1 and all $\boldsymbol{v}_2$ in W_2.

- Ex 2: (Orthogonal subspaces)

  The subspaces

$$W_1 = \text{span}(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}) \text{ and } W_2 = \text{span}(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix})$$

  are orthogonal because $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = 0$ for any vector in W_1 and any vector in W_2 is zero.

- Orthogonal complement of W:

Let W be a subspace of an inner product space V.

(a) A vector $\mathbf{u}$ in V is said to **orthogonal to W**,

if $\mathbf{u}$ is orthogonal to every vector in W.

(b) The set of all vectors in V that are orthogonal to every

vector in W is called the **orthogonal complement of W**.

$$W^{\perp} = \{\mathbf{v} \in V \mid \langle \mathbf{v}\,,\mathbf{w} \rangle = 0\,, \forall\, \mathbf{w} \in W\}$$

$W^{\perp}$ (read " W perp")

- Notes:

(1) $(\{0\})^{\perp} = V$ (2) $V^{\perp} = \{0\}$

- Notes:

W is a subspace of V

(1) $W^{\perp}$ is a subspace of V

(2) $W \cap W^{\perp} = \{\mathbf{0}\}$

(3) $(W^{\perp})^{\perp} = W$

- Ex:

If $V = R^2$, $W = x-\text{axis}$

Then (1) $W^{\perp} = y\text{-axis}$ is a subspace of R^2

(2) $W \cap W^{\perp} = \{(0,0)\}$

(3) $(W^{\perp})^{\perp} = W$

- Direct sum:

  Let W_1 and W_2 be two subspaces of R^n. If each vector $\mathbf{x} \in R^n$ can be uniquely written as a sum of a vector $\mathbf{w}_1$ from W_1 and a vector $\mathbf{w}_2$ from W_2, $\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2$, then R^n is the direct sum of W_1 and W_2, and you can write .

  $$R^n = W_1 \oplus W_2$$

- Thm 5.13: (Properties of orthogonal subspaces)

  Let W be a subspace of R^n. Then the following properties are true.

  (1) $\dim(W) + \dim(W^{\perp}) = n$

  (2) $R^n = W \oplus W^{\perp}$

  (3) $(W^{\perp})^{\perp} = W$

- Thm 5.14: (Projection onto a subspace)

  If $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_t\}$ is an orthonormal basis for the subspace S of V, and $\mathbf{v} \in V$, then

$$\text{proj}_W \mathbf{v} = \langle \mathbf{v}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{v}, \mathbf{u}_2 \rangle \mathbf{u}_2 + \cdots + \langle \mathbf{v}, \mathbf{u}_t \rangle \mathbf{u}_t$$

Pf:

$\because \text{proj}_W \boldsymbol{v} \in \text{W}$ and $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_t\}$ is an orthonormal basis for W

$$\Rightarrow \text{proj}_W \boldsymbol{v} = \langle \text{proj}_W \boldsymbol{v}, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 + \cdots + \langle \text{proj}_W \boldsymbol{v}, \boldsymbol{u}_t \rangle \boldsymbol{u}_t$$

$$= \langle \boldsymbol{v} - \text{proj}_{W^\perp} \boldsymbol{v}, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 + \cdots + \langle \boldsymbol{v} - \text{proj}_{W^\perp} \boldsymbol{v}, \boldsymbol{u}_t \rangle \boldsymbol{u}_t$$

$$(\because \text{proj}_W \boldsymbol{v} = \boldsymbol{v} - \text{proj}_{W^\perp} \boldsymbol{v})$$

$$= \langle \boldsymbol{v}, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 + \cdots + \langle \boldsymbol{v}, \boldsymbol{u}_t \rangle \boldsymbol{u}_t \quad (\because \langle \text{proj}_{W^\perp} \boldsymbol{v}, \boldsymbol{u}_i \rangle = 0, \forall\, i)$$

- Ex 5: (Projection onto a subspace)

$$\mathbf{w}_1 = (0, 3, 1), \mathbf{w}_2 = (2, 0, 0), \mathbf{v} = (1, 1, 3)$$

Find the projection of the vector **v** onto the subspace W.

Sol:

$$W = span(\{\mathbf{w}_1, \mathbf{w}_2\})$$

$\{\mathbf{w}_1, \mathbf{w}_2\}$: an orthogonal basis for W

$$\Rightarrow \{\mathbf{u}_1, \mathbf{u}_2\} = \left\{\frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}, \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}\right\} = \left\{(0, \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}), (1,0,0)\right\} :$$

an orthonormal basis for W

$$\text{proj}_W \boldsymbol{v} = \langle \boldsymbol{v}, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 + \langle \boldsymbol{v}, \boldsymbol{u}_2 \rangle \boldsymbol{u}_2$$

$$= \frac{6}{\sqrt{10}}(0, \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}) + (1,0,0) = (1, \frac{9}{5}, \frac{3}{5})$$

- Find by the other method:

$$A=[\mathbf{w}_1,\mathbf{w}_2],\quad \mathbf{b}=\mathbf{v}$$

$$Ax=\mathbf{b}$$

$$\Rightarrow x=(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\mathbf{b}$$

$$\Rightarrow \operatorname{proj}_{cs(A)}\mathbf{b}=Ax=A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\mathbf{b}$$

- Thm 5.15: (Orthogonal projection and distance)

Let W be a subspace of an inner product space V, and $\mathbf{v} \in V$

Then for all $\mathbf{w} \in W, \ \mathbf{w} \neq \text{proj}_W \mathbf{v}$

$$\| \mathbf{v} - \text{proj}_W \mathbf{v} \| < \| \mathbf{v} - \mathbf{w} \|$$

or $$\| \boldsymbol{v} - \text{proj}_W \boldsymbol{v} \| = \min_{\boldsymbol{w} \in \text{W}} \| \boldsymbol{v} - \boldsymbol{w} \|$$

($\text{proj}_W \mathbf{v}$ is the best approximation to $\mathbf{v}$ from W)

Pf:

$$\mathbf{v}-\mathbf{w}=(\mathbf{v}-\operatorname{proj}_W\mathbf{v})+(\operatorname{proj}_W\mathbf{v}-\mathbf{w})$$

$$(\mathbf{v}-\operatorname{proj}_W\mathbf{v})\perp(\operatorname{proj}_W\mathbf{v}-\mathbf{w})$$

By the Pythagorean theorem

$$\Rightarrow \|\mathbf{v}-\mathbf{w}\|^2=\|\mathbf{v}-\operatorname{proj}_W\mathbf{v}\|^2+\|\operatorname{proj}_W\mathbf{v}-\mathbf{w}\|^2$$

$$\mathbf{w}\neq\operatorname{proj}_W\mathbf{v}\Rightarrow\left\|\operatorname{proj}_W\mathbf{v}-\mathbf{w}\right\|>0$$

$$\Rightarrow \|\mathbf{v}-\mathbf{w}\|^2>\|\mathbf{v}-\operatorname{proj}_W\mathbf{v}\|^2$$

$$\Rightarrow \|\mathbf{v}-\operatorname{proj}_W\mathbf{v}\|<\|\mathbf{v}-\mathbf{w}\|$$

- Notes:

  (1) Among all the scalar multiples of a vector $\mathbf{u}$, the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$ is the one that is closest to $\mathbf{v}$. (p.250 Thm 5.9)

  (2) Among all the vectors in the subspace W, the vector $\text{proj}_W \mathbf{v}$ is the closest vector to $\mathbf{v}$.

- Thm 5.16: (Fundamental subspaces of a matrix)

If A is an $m \times n$ matrix, then

(1) $(CS(A))^{\perp} = NS(A^{\mathrm{T}})$

$(NS(A^{\mathrm{T}}))^{\perp} = CS(A)$

(2) $(CS(A^{\mathrm{T}}))^{\perp} = NS(A)$

$(NS(A))^{\perp} = CS(A^{\mathrm{T}})$

(3) $CS(A) \oplus NS(A^{T}) = R^{m}$ $\quad CS(A) \oplus (NS(A))^{\perp} = R^{m}$

(4) $CS(A^{T}) \oplus NS(A) = R^{n}$ $\quad CS(A^{T}) \oplus (CS(A^{\mathrm{T}}))^{\perp} = R^{n}$

- Ex 6: (Fundamental subspaces)

Find the four fundamental subspaces of the matrix.

$$A=\begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \qquad \text{(reduced row-echelon form)}$$

Sol:

$$CS(A)=\text{span}(\{(1,0,0,0)\ \ (0,1,0,0)\})\ \text{ is a subspace of } R^4$$

$$CS(A^{\mathrm{T}})=RS(A)=\text{span}(\{(1,2,0)\ \ (0,0,1)\})\ \text{ is a subspace of } R^3$$

$$NS(A)=\text{span}(\{(-2,1,0)\})\ \text{ is a subspace of } R^3$$

$$A^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim R = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} & & s & t \end{array} \end{array}$$

$NS(A^{\mathrm{T}}) = \operatorname{span}(\{(0,0,1,0) \quad (0,0,0,1)\})$ is a subspace of R^4

- Check:

$$(CS(A))^{\perp} = NS(A^{\mathrm{T}})$$

$$(CS(A^{\mathrm{T}}))^{\perp} = NS(A)$$

$$CS(A) \oplus NS(A^T) = R^4$$

$$CS(A^T) \oplus NS(A) = R^3$$

- Ex 3 & Ex 4:

$$W=\text{span}(\{\mathbf{w}_1,\mathbf{w}_2\})$$

Let W is a subspace of R^4 and $\mathbf{w}_1=(1,2,1,0),\ \ \mathbf{w}_2=(0,0,0,1)$.

(a) Find a basis for W

(b) Find a basis for the orthogonal complement of W.

Sol:

$$A=\begin{array}{c}\begin{bmatrix}1&0\\2&0\\1&0\\0&1\end{bmatrix}\\ \mathbf{w}_1\ \ \mathbf{w}_2\end{array}\sim R=\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}\quad\text{(reduced row-echelon form)}$$

$(a)\quad W = CS(A)$

$\Rightarrow \quad \{(1,2,1,0), (0,0,0,1)\}$ is a basis for W

$(b)\quad W^{\perp} = (CS(A))^{\perp} = NS(A^{\mathrm{T}})$

$$\because A^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s-t \\ s \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow \{(-2,1,0,0)\ \ (-1,0,1,0)\}$ is a basis for $W^{\perp}$

- Notes:

  (1) $\dim(W) + \dim(W^{\perp}) = \dim(R^4)$

  (2) $W \oplus W^{\perp} = R^4$

- Least squares problem:

$$\underset{m\times n}{A}\;\underset{n\times 1}{\mathbf{x}} = \underset{m\times 1}{\mathbf{b}} \qquad \text{(A system of linear equations)}$$

(1) When the system is consistent, we can use the Gaussian elimination with back-substitution to solve for $\mathbf{x}$

(2) When the system is inconsistent, how to find the "best possible" solution of the system. That is, the value of $\mathbf{x}$ for which the difference between $A\mathbf{x}$ and $\mathbf{b}$ is small.

- Least squares solution:

Given a system $A\mathbf{x} = \mathbf{b}$ of m linear equations in n unknowns, the least squares problem is to find a vector $\mathbf{x}$ in R^n that minimizes $\|A\mathbf{x} - \mathbf{b}\|$ with respect to the Euclidean inner product on R^n. Such a vector is called a least squares solution of $A\mathbf{x} = \mathbf{b}$.

- Notes:

The least square problem is to find a vector $\hat{\boldsymbol{x}}$ in R^n such that $A\hat{\boldsymbol{x}} = \operatorname{proj}_{CS(A)}\boldsymbol{b}$ in the column space of A (i.e., $A\hat{\boldsymbol{x}} \in CS(A)$) is as close as possible to $\boldsymbol{b}$. That is,

$$\|\boldsymbol{b} - \operatorname{proj}_{CS(A)}\boldsymbol{b}\| = \|\boldsymbol{b} - A\hat{\boldsymbol{x}}\| = \min_{\boldsymbol{x} \in R^n}\|\boldsymbol{b} - A\boldsymbol{x}\|$$

$$A \in M_{m\times n}$$

$$\boldsymbol{x} \in R^n$$

$$A\boldsymbol{x} \in CS(A) \quad (CS(A) \text{ is a subspace of } R^m)$$

$$\because \mathrm{b} \notin CS(A) \quad (A\boldsymbol{x} = \mathrm{b} \text{ is an inconsistent system})$$

Let $A\hat{\boldsymbol{x}} = proj_{CS(A)}\boldsymbol{b}$

$$\Rightarrow (\boldsymbol{b} - A\hat{\boldsymbol{x}}) \perp CS(A)$$

$$\Rightarrow \boldsymbol{b} - A\hat{\boldsymbol{x}} \in (CS(A))^{\perp} = NS(A^{\mathrm{T}})$$

$$\Rightarrow A^{\mathrm{T}}(\boldsymbol{b} - A\hat{\boldsymbol{x}}) = 0$$

i.e. $A^{\mathrm{T}}A\hat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ (the normal system associated with $A\mathbf{x} = \mathbf{b}$)

- Note: ($A\mathbf{x} = \mathbf{b}$ is an inconsistent system)

  The problem of finding the least squares solution of $A\mathbf{x} = \mathbf{b}$

  is equal to he problem of finding an exact solution of the

  associated normal system $A^{\mathrm{T}} A\hat{\mathbf{x}} = A^{\mathrm{T}}\mathbf{b}$.

- Ex 7: (Solving the normal equations)

  Find the least squares solution of the following system

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

(this system is inconsistent)

and find the orthogonal projection of **b** on the column space of A.

Sol:

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

the associated normal system

$$A^T A \hat{\boldsymbol{x}} = A^T \boldsymbol{b}$$

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

the least squares solution of $A\mathbf{x} = \mathbf{b}$

$$\hat{\boldsymbol{x}} = \begin{bmatrix} -\frac{5}{3} \\ \frac{3}{2} \end{bmatrix}$$

the orthogonal projection of $\mathbf{b}$ on the column space of A

$$\text{proj}_{CS(A)}\boldsymbol{b} = A\hat{\boldsymbol{x}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{-5}{3} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{-1}{6} \\ \frac{8}{6} \\ \frac{17}{6} \end{bmatrix}$$

Key Learning in Section 5.4

- Define the least squares problem.
- Find the orthogonal complement of a subspace and the projection of a vector onto a subspace.
- Find the four fundamental subspaces of a matrix.
- Solve a least squares problem.
- Use least squares for mathematical modeling.

Keywords in Section 5.4

- orthogonal to W: 正交於W
- orthogonal complement: 正交補集
- direct sum: 直和
- projection onto a subspace: 在子空間的投影
- fundamental subspaces: 基本子空間
- least squares problem: 最小平方問題
- normal equations: 一般方程式

5.5 Applications of Inner Product Spaces

- The cross product of two vectors in R^3

A vector product that yields a vector in R^3 is orthogonal to two vectors. This vector product is called the cross product, and it is most conveniently defined and calculated with vectors written in standard unit vector form

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \qquad \mathbf{i} = (1,0,0), \mathbf{j} = (0,1,0), \mathbf{k} = (0,0,1)$$

- Cross product of two vectors in R^3:

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in R^3.
The cross product of **u** and ***v*** is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{array}{l} \\ \longleftarrow \text{Components of } \mathbf{u} \\ \longleftarrow \text{Components of } \mathbf{v} \end{array}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$= (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}$$

- Notes:

(1) The cross product is defined only for vectors in R^3.

(2) The cross product of two vectors in R^3 is orthogonal to two vectors.

(3) The cross product of two vectors in R^n, n ≠ 3 is not defined here.

- Ex 1: (Finding the Cross Product of Two Vectors)

$$\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \qquad \mathbf{v} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

Sol:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}\mathbf{k} = 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{k} = -3\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$$

$$\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 2 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -2 \\ 3 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

- Thm 5.17: (Algebraic Properties of the Cross Product)

If **u**, **v**, and **w** are vectors in R^3 and c is a scalar, then the following properties are true.

$$1.\ \mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$$

$$2.\ \mathbf{u}\times(\mathbf{v}+\mathbf{w}) = (\mathbf{u}\times\mathbf{v})+(\mathbf{u}\times\mathbf{w})$$

$$3.\ c(\mathbf{u}\times\mathbf{v}) = c\mathbf{u}\times\mathbf{v} = \mathbf{u}\times c\mathbf{v}$$

$$4.\ \mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \mathbf{0}$$

$$5.\ \mathbf{u}\times\mathbf{u} = \mathbf{0}$$

$$6.\ \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$$

Pf:

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \qquad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

$$\mathbf{v}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = (v_2u_3 - v_3u_2)\mathbf{i} - (v_1u_3 - v_3u_1)\mathbf{j} + (v_1u_2 - v_2u_1)\mathbf{k}$$

$$= -(u_2v_3 - u_3v_2)\mathbf{i} + (u_1v_3 - u_3v_1)\mathbf{j} - (u_1v_2 - u_2v_1)\mathbf{k}$$

$$= -(\mathbf{v}\times\mathbf{u})$$

- Note:

  The vectors $\mathbf{u}\times\mathbf{v}$ and $\mathbf{v}\times\mathbf{u}$ have equal lengths but opposite directions.

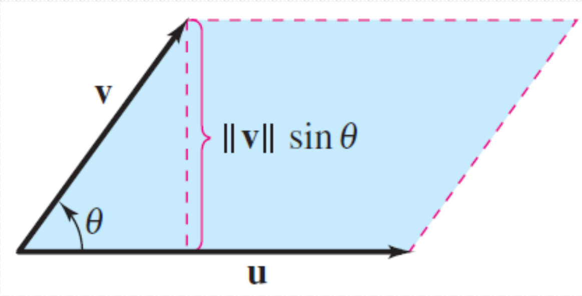
- Thm 5.18: (Geometric Properties of the Cross Product)

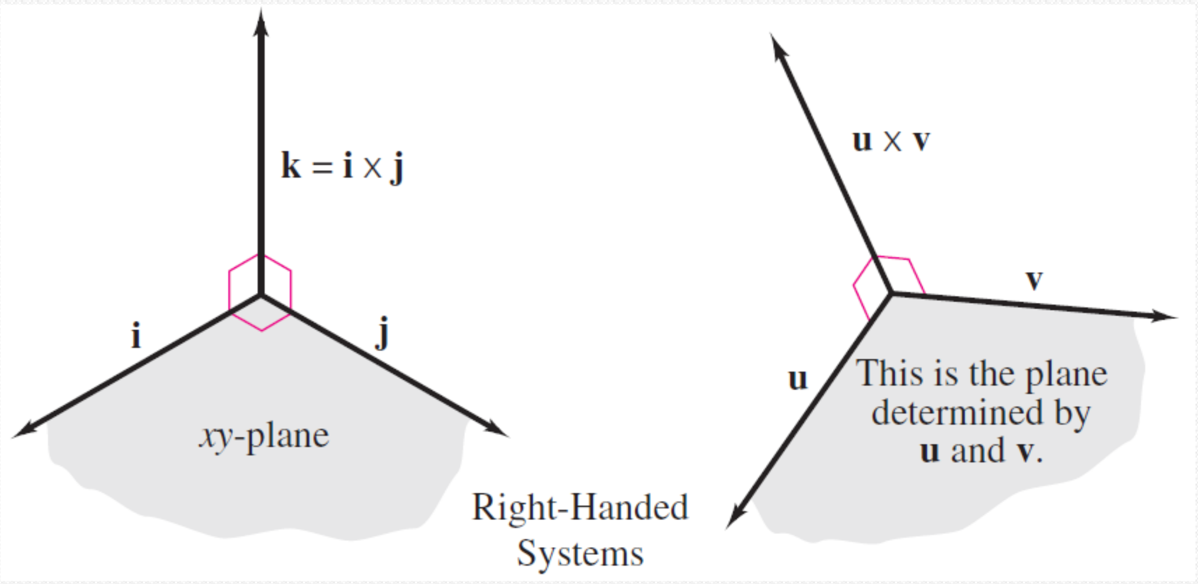
If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^3, then the following properties are true.

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
2. The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is given by $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta$.
3. $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
4. The parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides has an area of $\|\mathbf{u} \times \mathbf{v}\|$.

Pf:

$$\text{Area} = \underbrace{\|\mathbf{u}\|}_{\text{Base}}\,\underbrace{\|\mathbf{v}\|\sin\theta}_{\text{Height}} = \|\mathbf{u}\times\mathbf{v}\|$$





- Notes:

  (1) The three vectors **u**, **v**, and $\mathbf{u}\times\mathbf{v}$ form a right-handed system.

  (2) The three vectors **u**, **v**, and $\mathbf{v}\times\mathbf{u}$ form a left-handed system.

Ex 2: (Finding a Vector Orthogonal to Two Given Vectors)

$$\mathbf{u} = \mathbf{i} - 4\mathbf{j} + \mathbf{k} \qquad \mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$$

Sol:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 1 \\ 2 & 3 & 0 \end{vmatrix} = -3\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}$$

length

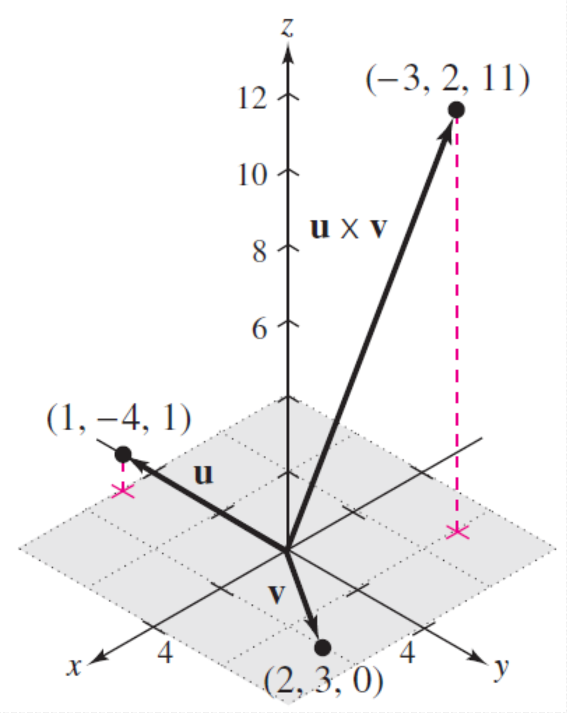
$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + 11^2} = \sqrt{134}$$

unit vector

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{-3}{\sqrt{134}}\mathbf{i} + \frac{2}{\sqrt{134}}\mathbf{j} + \frac{11}{\sqrt{134}}\mathbf{k}$$

$$\left(\frac{-3}{\sqrt{134}}, \frac{2}{\sqrt{134}}, \frac{11}{\sqrt{134}}\right) \cdot (1, -4, 1) = 0$$

$$\left(\frac{-3}{\sqrt{134}}, \frac{2}{\sqrt{134}}, \frac{11}{\sqrt{134}}\right) \cdot (2, 3, 0) = 0$$



Ex 3: (Finding the Area of a Parallelogram)

$$\mathbf{u} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k} \qquad \mathbf{v} = -2\mathbf{j} + 6\mathbf{k}$$

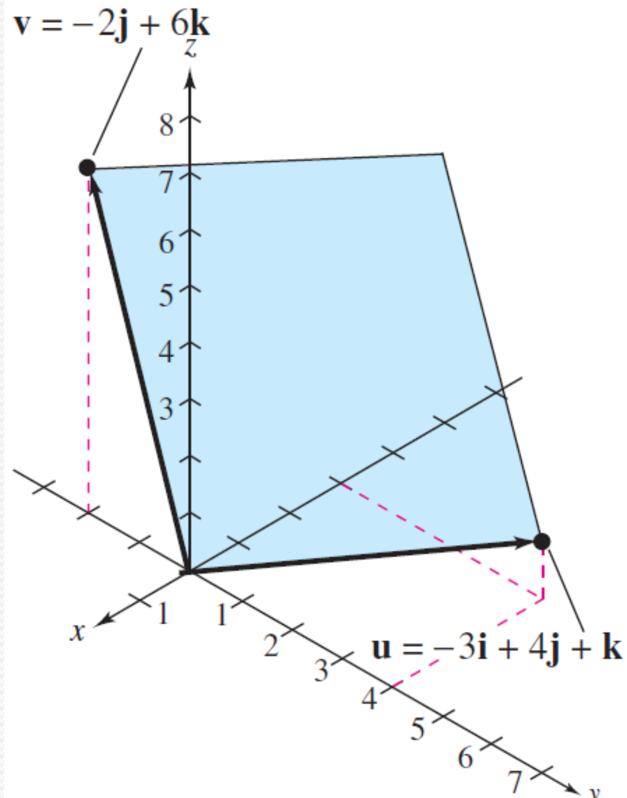
Sol:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix} = 26\mathbf{i} + 18\mathbf{j} + 6\mathbf{k}$$

area

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{26^2 + 18^2 + 6^2}$$

$$= \sqrt{1036} \approx 32.19$$



The area of the parallelogram is $\|\mathbf{u} \times \mathbf{v}\| = \sqrt{1036}$.

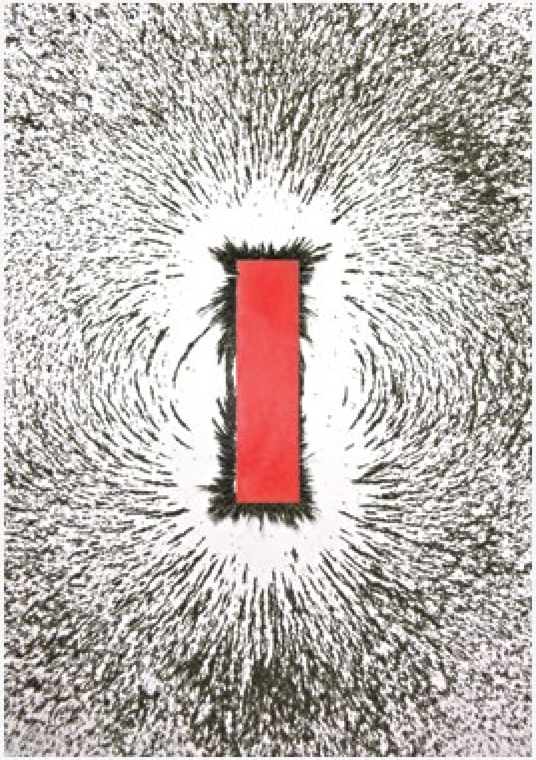
Key Learning in Section 5.5

- Find the cross product of two vectors in R^3.
- Find the linear or quadratic least squares approximation of a function.
- Find the nth-order Fourier approximation of a function.

Keywords in Section 5.5

- cross product: 外積
- parallelogram: 平行四邊形

- Electric/Magnetic Flux



Electrical engineers can use the dot product to calculate electric or magnetic *flux*, which is a measure of the strength of the electric or magnetic field penetrating a surface. Consider an arbitrarily shaped surface with an element of area dA, normal (perpendicular) vector $d\mathbf{A}$, electric field vector $\mathbf{E}$ and magnetic field vector $\mathbf{B}$. The electric flux Φ_e can be found using the surface integral $\Phi_e = \int \mathbf{E}\cdot d\mathbf{A}$ and the magnetic flux can be found using the surface integral $\Phi_e = \int \mathbf{B}\cdot d\mathbf{A}$. It is interesting to note that for a given closed surface that surrounds an electrical charge, the net electric flux is proportional to the charge, but the net magnetic flux is zero. This is because electric fields initiate at positive charges and terminate at negative charges, but magnetic fields form closed loops, so they do not initiate or terminate at any point. This means that the magnetic field entering a closed surface must equal the magnetic field leaving the closed surface.

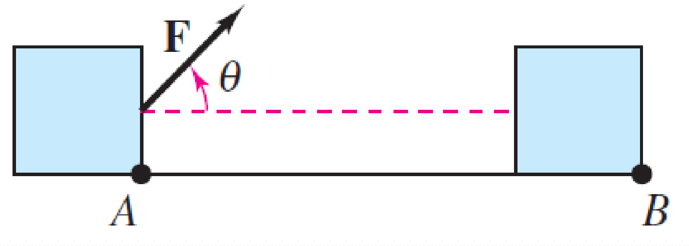
- Work

The concept of work is important to scientists and engineers for determining the energy needed to perform various jobs. If a constant force **F** acts at an angle θ with the line of motion of an object to move the object from point A to point B (see figure below), then the work done by the force is given by

$$W = (\cos\theta)\|\mathbf{F}\|\|\overrightarrow{AB}\|$$

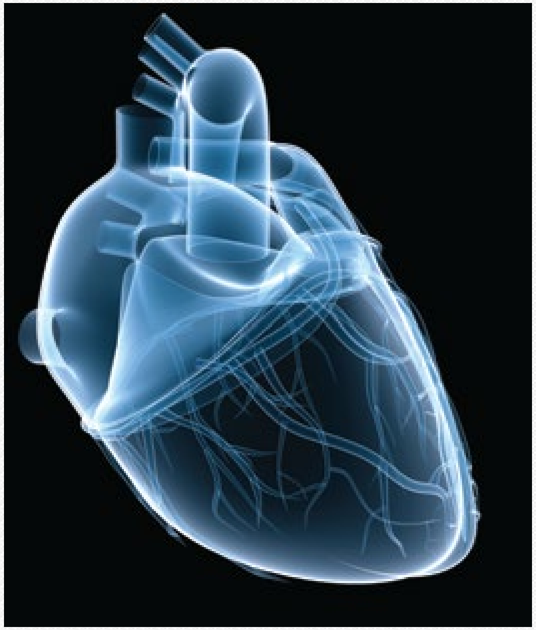
$$= \mathbf{F} \cdot \overrightarrow{AB}$$

where $\overrightarrow{AB}$ represents the directed line segment from A to B. The quantity $(\cos\theta)\|\mathbf{F}\|$ is the length of the orthogonal projection of **F** onto $\overrightarrow{AB}$ Orthogonal projections are discussed on the next page.



5.3 Linear Algebra Applied

- Heart Rhythm Analysis



Time-frequency analysis of irregular physiological signals, such as beat-to-beat cardiac rhythm variations (also known as heart rate variability or HRV), can be difficult. This is because the structure of a signal can include multiple periodic, nonperiodic, and pseudo-periodic components. Researchers have proposed and validated a simplified HRV analysis method called orthonormal-basis partitioning and time-frequency representation (OPTR). This method can detect both abrupt and slow changes in the HRV signal's structure, divide a nonstationary HRV signal into segments that are "less nonstationary," and determine patterns in the HRV. The researchers found that although it had poor time resolution with signals that changed gradually, the OPTR method accurately represented multicomponent and abrupt changes in both real-life and simulated HRV signals.

- Revenues

The least squares problem has a wide variety of real-life applications. To illustrate, in Examples 9 and 10 and Exercises 39, 40, and 41, are all least squares analysis problems, and they involve such diverse subject matter as world population, astronomy, master's degrees awarded, company revenues, and galloping speeds of animals. In each of these applications, you will be given a set of data and you are asked to come up with mathematical model(s) for the data. For example, in Exercise 40, you are given the annual revenues from 2008 through 2013 for General Dynamics Corporation. You are asked to find the least squares regression quadratic and cubic polynomials for the data, to predict the revenue for the year 2018, and to decide which of the models appears to be more accurate for predicting future revenues.

- Torque

In physics, the cross product can be used to measure torque—the moment **M** of a force **F** about a point A as shown in the figure below. When the point of application of the force is B, the moment of **F** about A is given by

$$\mathbf{M} = \overrightarrow{AB} \times \mathbf{F}$$

where $\overrightarrow{AB}$ represents the vector whose initial point is A and whose terminal point is B. The magnitude of the moment **M** measures the tendency of $\overrightarrow{AB}$ to rotate counterclockwise about an axis directed along the vector **M.**