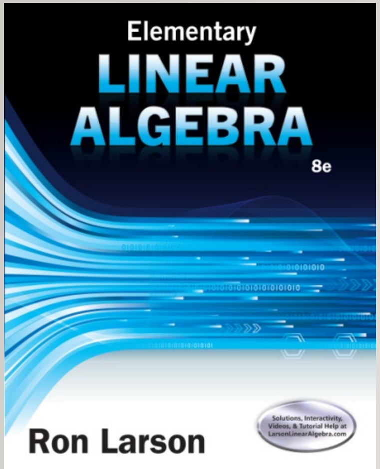


CHAPTER 4
VECTOR SPACES

4.1 Vectors in R^n

- An ordered n-tuple:

  a sequence of n real number $(x_1, x_2, \cdots, x_n)$

- n-space: R^n

  the set of all ordered n-tuple

- Ex:

$n = 1 \qquad R^1 = 1\text{-space}$

$\qquad\qquad = \text{set of all real number}$

$n = 2 \qquad R^2 = 2\text{-space}$

$\qquad\qquad = \text{set of all ordered pair of real numbers } (x_1, x_2)$

$n = 3 \qquad R^3 = 3\text{-space}$

$\qquad\qquad = \text{set of all ordered triple of real numbers } (x_1, x_2, x_3)$

$n = 4 \qquad R^4 = 4\text{-space}$

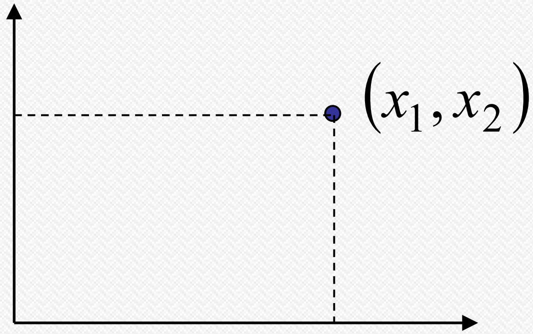
$\qquad\qquad = \text{set of all ordered quadruple of real numbers } (x_1, x_2, x_3, x_4)$

- Notes:

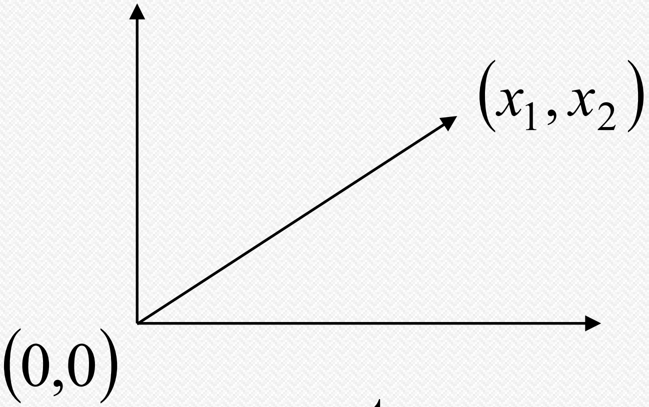
(1) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as <u>a point</u> in R^n with the x_i's as its coordinates.

(2) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as <u>a vector</u> $x = (x_1, x_2, \cdots, x_n)$ in R^n with the x_i's as its components.

- Ex:



a point



a vector

$\mathbf{u} = (u_1, u_2, \cdots, u_n)$, $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ (two vectors in R^n)

- Equal:

  $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1,\ u_2 = v_2, \cdots, u_n = v_n$

- Vector addition (the sum of **u** and **v**):

  $$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$$

- Scalar multiplication (the scalar multiple of **u** by c):

  $$c\mathbf{u} = (cu_1, cu_2, \cdots, cu_n)$$

- Notes:

  The sum of two vectors and the scalar multiple of a vector in R^n are called the standard operations in R^n.

- Negative:

$$-\mathbf{u} = (-u_1, -u_2, -u_3, \ldots, -u_n)$$

- Difference:

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3, \ldots, u_n - v_n)$$

- Zero vector:

$$\mathbf{0} = (0, 0, \ldots, 0)$$

- Notes:

(1) The zero vector $\mathbf{0}$ in R^n is called the **additive identity** in R^n.

(2) The vector $-\mathbf{v}$ is called the **additive inverse** of $\mathbf{v}$.

- Thm 4.2: (Properties of vector addition and scalar multiplication)

Let **u**, **v**, and **w** be vectors in R^n , and let c and d be scalars.

(1) $\mathbf{u}+\mathbf{v}$ is a vector in R^n

(2) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(3) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$

(4) $\mathbf{u}+\mathbf{0} = \mathbf{u}$

(5) $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$

(6) $c\mathbf{u}$ is a vector in R^n

(7) $\mathrm{c}(\mathbf{u}+\mathbf{v}) = \mathrm{c}\mathbf{u}+\mathrm{c}\mathbf{v}$

(8) $(c+d)\mathbf{u} = \mathrm{c}\mathbf{u}+\mathrm{d}\mathbf{u}$

(9) $c(\mathrm{d}\mathbf{u}) = (\mathrm{cd})\mathbf{u}$

(10) $1(\mathbf{u}) = \mathbf{u}$

- Ex 5: (Vector operations in R^4)

Let $\mathbf{u}$=(2, – 1, 5, 0), $\mathbf{v}$=(4, 3, 1, – 1), and $\mathbf{w}$=(– 6, 2, 0, 3) be vectors in R^4. Solve $\mathbf{x}$ for x in each of the following.

(a) $\mathbf{x} = 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w})$

(b) $3(\mathbf{x}+\mathbf{w}) = 2\mathbf{u} - \mathbf{v}+\mathbf{x}$

Sol: (a)
$$\begin{aligned}\mathbf{x} &= 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w}) \\ &= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\ &= (4, -2, 10, 0) - (4, 3, 1, -1) - (-18, 6, 0, 9) \\ &= (4-4+18, -2-3-6, 10-1-0, 0+1-9) \\ &= (18, -11, 9, -8).\end{aligned}$$

$$
\begin{aligned}
\text{(b)}\quad 3(\mathbf{x}+\mathbf{w}) &= 2\mathbf{u}-\mathbf{v}+\mathbf{x} \\
3\mathbf{x}+3\mathbf{w} &= 2\mathbf{u}-\mathbf{v}+\mathbf{x} \\
3\mathbf{x}-\mathbf{x} &= 2\mathbf{u}-\mathbf{v}-3\mathbf{w} \\
2\mathbf{x} &= 2\mathbf{u}-\mathbf{v}-3\mathbf{w} \\
\mathbf{x} &= \mathbf{u}-\tfrac{1}{2}\mathbf{v}-\tfrac{3}{2}\mathbf{w} \\
&= (2,1,5,0)+\left(-2,\tfrac{-3}{2},\tfrac{-1}{2},\tfrac{1}{2}\right)+\left(9,-3,0,\tfrac{-9}{2}\right) \\
&= \left(9,\tfrac{-11}{2},\tfrac{9}{2},-4\right)
\end{aligned}
$$

- Thm 4.3: (Properties of additive identity and additive inverse)

  Let $\mathbf{v}$ be a vector in R^n and c be a scalar. Then the following is true.

(1) The additive identity is unique. That is, if $\mathbf{u}+\mathbf{v}=\mathbf{v}$, then $\mathbf{u} = \mathbf{0}$

(2) The additive inverse of $\mathbf{v}$ is unique. That is, if $\mathbf{v}+\mathbf{u}=\mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$

(3) $0\mathbf{v}=\mathbf{0}$

(4) $c\mathbf{0}=\mathbf{0}$

(5) If $c\mathbf{v}=\mathbf{0}$, then $c=0$ or $\mathbf{v}=\mathbf{0}$

(6) $-(-\mathbf{v}) = \mathbf{v}$

- Linear combination:

  The vector **x** is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$,

  if it can be expressed in the form

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \qquad c_1, c_2, \cdots, c_n : \text{scalar}$$

- Ex 6:

  Given **x** = (– 1, – 2, – 2), **u** = (0,1,4), **v** = (– 1,1,2), and **w** = (3,1,2) in R^3, find a, b, and c such that **x** = a**u**+b**v**+c**w**.

Sol:

$$\begin{array}{rcrcrcr} & & -b & + & 3c & = & -1 \\ a & + & b & + & c & = & -2 \\ 4a & + & 2b & + & 2c & = & -2 \end{array}$$

$$\Rightarrow a = 1,\ b = -2,\ c = -1$$

Thus $\mathbf{x} = \mathbf{u} - 2\mathbf{v} - \mathbf{w}$

- Notes:

A vector $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ in R^n can be viewed as:

a $1\times n$ row matrix (row vector): $\mathbf{u}=[u_1,\ u_2,\cdots,u_n]$

or

a $n\times 1$ column matrix (column vector): $\mathbf{u}=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

(The matrix operations of addition and scalar multiplication give the same results as the corresponding vector operations)

Vector addition

$$\mathbf{u}+\mathbf{v}=(u_1, u_2, \cdots, u_n)+(v_1, v_2, \cdots, v_n)$$
$$=(u_1+v_1, u_2+v_2, \cdots, u_n+v_n)$$

$$\mathbf{u}+\mathbf{v}=[u_1, u_2, \cdots, u_n]+[v_1, v_2, \cdots, v_n]$$
$$=[u_1+v_1, u_2+v_2, \cdots, u_n+v_n]$$

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}+\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}=\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}$$

Scalar multiplication

$$c\mathbf{u}=c(u_1, u_2, \cdots, u_n)$$
$$=(cu_1, cu_2, \cdots, cu_n)$$

$$c\mathbf{u}=c[u_1, u_2, \cdots, u_n]$$
$$=[cu_1, cu_2, \cdots, cu_n]$$

$$c\mathbf{u}=c\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}=\begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$$

Key Learning in Section 4.1

- Represent a vector as a directed line segment.
- Perform basic vector operations in R^2 and represent them graphically.
- Perform basic vector operations in R^n.
- Write a vector as a linear combination of other vectors.

Keywords in Section 4.1

- ordered *n*-tuple：有序的*n*項
- *n*-space：*n*維空間
- equal：相等
- vector addition：向量加法
- scalar multiplication：純量乘法
- negative：負向量
- difference：向量差
- zero vector：零向量
- additive identity：加法單位元素
- additive inverse：加法反元素

4.2 Vector Spaces

- Vector spaces:

  Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every **u**, **v**, and **w** in V and every scalar (real number) c and d, then V is called a **vector space**.

Addition:

(1) $\mathbf{u} + \mathbf{v}$ is in V

(2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(3) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

(4) V has a zero vector $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u} + \mathbf{0} = \mathbf{u}$

(5) For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

Scalar multiplication:

(6) $c\mathbf{u}$ is in V.

(7) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(8) $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(10) $1(\mathbf{u}) = \mathbf{u}$

- Notes:

(1) A vector space consists of four entities:

a set of vectors, a set of scalars, and two operations

V : nonempty set

c : scalar

$+(\mathbf{u},\mathbf{v})=\mathbf{u}+\mathbf{v}$: vector addition

$\bullet(c,\mathbf{u})=c\mathbf{u}$: scalar multiplication

$(V, \quad +, \quad \bullet)$ is called a vector space

(2) $V=\{\mathbf{0}\}$: zero vector space

- Examples of vector spaces:

(1) n-tuple space: R^n

$$(u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$ vector addition

$$k(u_1,u_2,\cdots,u_n)=(ku_1,ku_2,\cdots,ku_n)$$ scalar multiplication

(2) Matrix space: $V = M_{m\times n}$ (the set of all $m\times n$ matrices with real values)

Ex: ：$(m=n=2)$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}+\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}=\begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$ vector addition

$$k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}=\begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$ scalar multiplication

(3) *n*-th degree polynomial space: $V = P_n(x)$

(the set of all real polynomials of degree *n* or less)

$$p(x)+q(x)=(a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$$

$$kp(x)=ka_0+ka_1x+\cdots+ka_nx^n$$

(4) Function space: $V = c(-\infty,\infty)$ （the set of all real-valued continuous functions defined on the entire real line.)

$$(f+g)(x)=f(x)+g(x)$$

$$(kf)(x)=kf(x)$$

- Notes: To show that a set is not a vector space, you need only find one axiom that is not satisfied.

- Ex 6: The set of all integers. VS or not?

Pf:

- Ex 7: The set of all second-degree polynomials. VS or not?

Pf:

- Ex 8:

  $V=R^2$=the set of all ordered pairs of real numbers

  vector addition: $(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$

  scalar multiplication: $c(u_1, u_2) = (cu_1, 0)$

  IS V a VS or not.

  Sol:

  $\because 1(1,1) = (1,0) \neq (1,1)$

  $\therefore$ the set (together with the two given operations) is not a vector space

Key Learning in Section 4.2

- Define a vector space and recognize some important vector spaces.
- Show that a given set is not a vector space.

Keywords in Section 4.2:

- vector space：向量空間
- *n*-space：*n*維空間
- matrix space：矩陣空間
- polynomial space：多項式空間
- function space：函數空間

4.3 Subspaces of Vector Spaces

- Subspace:

$(V,+,\bullet)$: a vector space

$$\left.\begin{array}{l} W \neq \phi \\ W \subseteq V \end{array}\right\}$$: a nonempty subset

$(W,+,\bullet)$: a vector space (under the operations of addition and scalar multiplication defined in V)

$\Rightarrow$ W is a subspace of V

- Trivial subspace:

Every vector space V has at least two subspaces.

(1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V.

(2) V is a subspace of V.

- Thm 4.5: (Test for a subspace)

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold.

(1) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W.

(2) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

- V has a zero vector $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u} + \mathbf{0} = \mathbf{u}$?
- For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$?

- Ex 2: (A subspace of $M_{2\times2}$)

Let W be the set of all 2×2 symmetric matrices. Determine whether W is a subspace of vector space $M_{2\times2}$ or not, with the standard operations of matrix addition and scalar multiplication.

Sol:

$$\because W \subseteq M_{2\times2} \qquad M_{2\times2}: \text{vector sapces}$$

$$\text{Let } A_1, A_2 \in W \quad (A_1^T = A_1, A_2^T = A_2)$$

$$A_1 \in W, A_2 \in W \Rightarrow (A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2 \qquad (A_1 + A_2 \in W)$$

$$k \in R, A \in W \Rightarrow (kA)^T = kA^T = kA \qquad (kA \in W)$$

$$\therefore W \text{ is a subspace of } M_{2\times2}$$

- Ex 3: (The set of singular matrices is not a subspace of $M_{2\times 2}$)

Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2\times 2}$ with the standard operations.

Sol:

$$A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\in W,\ B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\in W$$

$$\therefore A+B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\notin W$$

$$\therefore W_2 \text{ is not a subspace of } M_{2\times 2}$$

- Ex 4: (The set of first-quadrant vectors is not a subspace of R^2)

Show that $W = \{(x_1, x_2) : x_1 \geq 0 \text{ and } x_2 \geq 0\}$, with the standard operations, is not a subspace of R^2.

Sol:

Let $\mathbf{u} = (1, 1) \in W$

$\because (-1)\mathbf{u} = (-1)(1, 1) = (-1, -1) \notin W$

(not closed under scalar multiplication)

$\therefore W$ is not a subspace of R^2

- Ex 6: (Determining subspaces of R^2)

Which of the following two subsets is a subspace of R^2?

(a) The set of points on the line given by $x+2y=0$.

(b) The set of points on the line given by $x+2y=1$.

Sol:

(a) $W = \{(x, y) \mid x + 2y = 0\} = \{(-2t, t) \mid t \in R\}$

Let $v_1 = (-2t_1, t_1) \in W \qquad v_2 = (-2t_2, t_2) \in W$

$\because v_1 + v_2 = (-2(t_1 + t_2), t_1 + t_2) \in W$ (closed under addition)

$kv_1 = (-2(kt_1), kt_1) \in W$ (closed under scalar multiplication)

$\therefore W$ is a subspace of R^2

(b) $W = \{(x, y) \mid x + 2y = 1\}$ (Note: the zero vector is not on the line)

Let $v = (1,0) \in W$

$\because (-1)v = (-1,0) \notin W$

$\therefore W$ is not a subspace of R^2

- Ex 8: (Determining subspaces of R^3)

Which of the following subsets is a subspace of R^3?

(a) $W = \{(x_1, x_2, 1) \mid x_1, x_2 \in R\}$

(b) $W = \{(x_1, x_1 + x_3, x_3) \mid x_1, x_3 \in R\}$

Sol:

(a) Let $\mathbf{v} = (0,0,1) \in W$

$$\Rightarrow (-1)\mathbf{v} = (0,0,-1) \notin W$$

$\therefore W$ is not a subspace of R^3

(b) Let $\mathbf{v} = (\mathrm{v}_1, \mathrm{v}_1 + \mathrm{v}_3, \mathrm{v}_3) \in W$, $\mathbf{u} = (\mathrm{u}_1, \mathrm{u}_1 + \mathrm{u}_3, \mathrm{u}_3) \in W$

$$\because \mathbf{v} + \mathbf{u} = \left(\mathrm{v}_1 + \mathrm{u}_1, \left(\mathrm{v}_1 + \mathrm{u}_1\right) + \left(\mathrm{v}_3 + \mathrm{u}_3\right), \mathrm{v}_3 + \mathrm{u}_3\right) \in \mathrm{W}$$

$$k\mathbf{v} = \left(k\mathrm{v}_1, \left(k\mathrm{v}_1\right) + \left(k\mathrm{v}_3\right), k\mathrm{v}_3\right) \in \mathrm{W}$$

$\therefore W$ is a subspace of R^3

- Thm 4.6: (The intersection of two subspaces is a subspace)

If V and W are both subspaces of a vector space U, then the intersection of V and W (denoted by $V \cap U$) is also a subspace of U.

Key Learning in Section 4.3

- Determine whether a subset W of a vector space V is a subspace of V.
- Determine subspaces of R^n.

Keywords in Section 4.3:

- subspace：子空間
- trivial subspace：顯然子空間

4.4 Spanning Sets and Linear Independence

- Linear combination:

A vector $\mathbf{v}$ in a vector space V is called a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k \qquad c_1, c_2, \cdots, c_k : \text{scalars}$$

- Ex 2-3: (Finding a linear combination)

$$\mathbf{v}_1 = (1,2,3) \quad \mathbf{v}_2 = (0,1,2) \quad \mathbf{v}_3 = (-1,0,1)$$

Prove (a) $\mathbf{w} = (1,1,1)$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

(b) $\mathbf{w} = (1,-2,2)$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Sol:

(a) $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

$$\begin{aligned}(1,1,1) &= c_1(1,2,3) + c_2(0,1,2) + c_3(-1,0,1) \\ &= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3)\end{aligned}$$

$$\Rightarrow \begin{aligned} c_1 \qquad\quad - c_3 &= 1 \\ 2c_1 + \quad c_2 \qquad\quad &= 1 \\ 3c_1 + 2c_2 + c_3 &= 1 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{Guass}-\text{Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = 1+t\,,\ c_2 = -1-2t\,,\ c_3 = t$$

(this system has infinitely many solutions)

$$\overset{t=1}{\Rightarrow} \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$$

(b)

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{Guass}-\text{Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

$\Rightarrow$ this system has no solution $(\because 0 \neq 7)$

$$\Rightarrow \mathbf{w} \neq c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

- the span of a set: span (S)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then **the span of S** is the set of all linear combinations of the vectors in S,

$$\operatorname{span}(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \mid \forall c_i \in R\}$$

(the set of all linear combinations of vectors in S)

- a spanning set of a vector space:

If every vector in a given vector space can be written as a linear combination of vectors in a given set S, then S is called **a spanning set** of the vector space.

$$\operatorname{span}(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \mid \forall c_i \in R\} = V$$

- Notes:

$$\text{span}\,(S) = V$$

$\Rightarrow$ S spans (generates) V

V is spanned (generated) by S

S is a spanning set of V

- Ex 5: (A spanning set for R^3)

Show that the set $S = \{(1,2,3),(0,1,2),(-2,0,1)\}$ spans R^3

Sol:

We must determine whether an arbitrary vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.

$$\mathbf{u} \in R^3 \Rightarrow \mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\begin{aligned}
\Rightarrow \quad c_1 \qquad\quad - 2c_3 &= u_1 \\
2c_1 + c_2 \qquad\quad &= u_2 \\
3c_1 + 2c_2 + c_3 &= u_3
\end{aligned}$$

The problem thus reduces to determining whether this system is consistent for all values of u_1, u_2, and u_3.

$$\because |A| = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \neq 0$$

$\Rightarrow A\mathbf{x} = \mathbf{b}$ has exactly one solution for every u.

$$\Rightarrow span(S) = R^3$$

- Thm 4.7: (Span(S) is a subspace of V)

If $S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then

(a) span (S) is a subspace of V.

(b) span (S) is the smallest subspace of V that contains S.

(Every other subspace of V that contains S must contain span (S).)

- Linear Independent (L.I.) and Linear Dependent (L.D.):

$S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_k\}$: a set of vectors in a vector space V

$c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_k\mathbf{v}_k=\mathbf{0}$

(1) If the equation has only the trivial solution ($c_1=c_2=\cdots=c_k=0$) then S is called linearly independent.

(2) If the equation has a nontrivial solution (i.e., not all zeros), then S is called linearly dependent.

- Ex 8: (Testing for linearly independent)

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{\underset{\mathbf{v}_1}{(1, 2, 3)}, \underset{\mathbf{v}_2}{(0, 1, 2)}, \underset{\mathbf{v}_3}{(-2, 0, 1)}\}$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \Rightarrow \begin{aligned} c_1 \qquad\qquad - 2c_3 &= 0 \\ 2c_1 + c_2 + \qquad\quad &= 0 \\ 3c_1 + 2c_2 + c_3 &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{Gauss - Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = c_2 = c_3 = 0 \quad \text{(only the trivial solution)}$$

$$\Rightarrow S \text{ is linearly independent}$$

- Ex 9: (Testing for linearly independent)

Determine whether the following set of vectors in P_2 is L.I. or L.D.

$$S = \{\underset{\mathbf{v}_1}{1+x-2x^2}\ ,\ \underset{\mathbf{v}_2}{2+5x-x^2}\ ,\ \underset{\mathbf{v}_3}{x+x^2}\}$$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3 = \mathbf{0}$$

i.e. $c_1(1+x-2x^2) + c_2(2+5x-x^2) + c_3(x+x^2) = 0+0x+0x^2$

$$\Rightarrow \begin{aligned} c_1+2c_2 \qquad &= 0 \\ c_1+5c_2+c_3 &= 0 \\ -2c_1-c_2+c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{G.J.}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ This system has infinitely many solutions.
(i.e., This system has nontrivial solutions.)

$\Rightarrow$ S is linearly dependent. (Ex: $c_1=2$, $c_2=-1$, $c_3=3$)

- Ex 10: (Testing for linearly independent)

Determine whether the following set of vectors in 2×2 matrix space is L.I. or L.D.

$$S = \left\{ \underset{\mathbf{v}_1}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}, \underset{\mathbf{v}_2}{\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}}, \underset{\mathbf{v}_3}{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}} \right\}$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$$c_1\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2c_1+3c_2+\ c_3 &= 0 \\ c_1 \qquad\qquad &= 0 \\ 2c_2+2c_3 &= 0 \\ c_1+c_2 \qquad &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{Gauss - Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ $c_1 = c_2 = c_3 = 0$ (This system has only the trivial solution.)

$\Rightarrow$ S is linearly independent.

- Thm 4.8: (A property of linearly dependent sets)

  A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\boldsymbol{v}_j$ in S can be written as a linear combination of the other vectors in S.

Pf:

$(\Rightarrow)$ $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$

$\because S$ is linearly dependent

$\Rightarrow c_i \neq 0$ for some i

$$\Rightarrow \mathbf{v}_i = \frac{c_1}{c_\mathrm{i}}\mathbf{v}_1 + \cdots + \frac{c_{i-1}}{c_i}\mathbf{v}_{i-1} + \frac{c_{i+1}}{c_i}\mathbf{v}_{i+1} + \cdots + \frac{c_k}{c_i}\mathbf{v}_k$$

($\Leftarrow$)

Let $\mathbf{v}_i = d_1\mathbf{v}_1+\ldots+d_{i\text{-}1}\mathbf{v}_{i\text{-}1}+d_{i+1}\mathbf{v}_{i+1}+\ldots+d_k\mathbf{v}_k$

$\Rightarrow d_1\mathbf{v}_1+\ldots+d_{i\text{-}1}\mathbf{v}_{i\text{-}1}-\mathbf{v}_i+d_{i+1}\mathbf{v}_{i+1}+\ldots+d_k\mathbf{v}_k = \mathbf{0}$

$\Rightarrow c_1=d_1, \ldots, c_{i\text{-}1}=d_{i\text{-}1}, c_i=-1, c_{i+1}=d_{i+1}, \ldots, c_k=d_k$ (nontrivial solution)

$\Rightarrow S$ is linearly dependent

Key Learning in Section 4.4

- Write a linear combination of a set of vectors in a vector space V.
- Determine whether a set S of vectors in a vector space V is a spanning set of V.
- Determine whether a set of vectors in a vector space V is linearly independent.

Keywords in Section 4.4:

- linear combination：線性組合
- spanning set：生成集合
- trivial solution：顯然解
- linear independent：線性獨立
- linear dependent：線性相依