3.5. Implicit Differentiation

If a function like $y = f(x)$, is called explicitly. Otherwise, we call it implicitly

Eg. $x^2 + y^2 = 9$

$\Rightarrow 2x + 2y \cdot y' = 0$

$\Rightarrow y' = -\frac{x}{y} = \pm \frac{x}{\sqrt{9 - x^2}}$

Eg. $y^5 + 3x^2y^2 + 5x^4 = 12$

$\Rightarrow 5y^4y' + 6xy^2 + 6x^2yy' + 20x^3 = 0$

$\Rightarrow y'(5y^4 + 6x^2y) = -20x^3 - 6xy^2$

$\Rightarrow y' = \frac{-20x^3 - 6xy^2}{5y^4 + 6x^2y}$

Eg. $\sin(x+y) = y^2\cos x$, find y'

$\Rightarrow [\cos(x+y)](1+y') = 2yy'\cos x - y^2\sin x$

$\Rightarrow y' = \frac{-y^2\sin x - \cos(x+y)}{\cos(x+y) - 2y\cos x}$

3.6. Derivatives of Log and Inverse Trigonometric Functions

1. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$

pf.: Let $y = \log_b x \Rightarrow b^y = x \Rightarrow b^y \cdot \ln b \cdot y' = 1 \Rightarrow y' = \frac{1}{x \cdot \ln b}$

2. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Eg. $\frac{d}{dx}\ln(\sin x)$

$= \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x)$

$= \frac{\cos x}{\sin x} = \cot x$

Eg. $f(x) = \log_{10}(2 + \sin x)$

$f'(x) = \frac{1}{(2+\sin x)\ln 10} \cdot \frac{d}{dx}(2+\sin x) = \frac{\cos x}{(2+\sin x)\cdot \ln 10}$

Eg. $f(x) = \ln|x|$

$\Rightarrow f(x) = \begin{cases} \ln x, & \text{if } x > 0 \longrightarrow f'(x) = \frac{1}{x} \\ \ln(-x), & \text{if } x < 0 \longrightarrow f'(x) = \frac{1}{x} \end{cases}$

Log Differentiation

Eg. $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$

$\ln y = \ln \frac{x^{3/4} \cdot \sqrt{x^2+1}}{(3x+2)^5} = \ln x^{3/4} + \ln\sqrt{x^2+1} - \ln(3x+2)^5$

$= \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$

$\Rightarrow \frac{y'}{y} = \left(\frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2+1} - 5 \cdot \frac{3}{3x+2}\right) \Rightarrow y' = \frac{x^{3/4} \cdot \sqrt{x^2+1}}{(3x+2)^5} \times \left[\frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)\right]$

Eg. $f(x)=x^{\sqrt{x}}$

$\ln f(x)=\sqrt{x}\cdot\ln x$

$\Rightarrow \frac{f'(x)}{f(x)}=\frac{1}{2}x^{-\frac{1}{2}}\cdot\ln x+\frac{\sqrt{x}}{x}$

$\Rightarrow f'(x)=x^{\sqrt{x}}\cdot\left(\frac{1}{2}x^{-\frac{1}{2}}\ln x+\frac{\sqrt{x}}{x}\right)$

∴ If we want to solve the problem of log differentiation, then

① Take $\ln$ on both sides

② Do the differentiation of implicit function.

Inverse trigonometric function

Let $y=\sin^{-1}x$. We take sin of both sides, then $\sin y=x$

$\Rightarrow y'\cos y=1 \Rightarrow y'=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}$

$\begin{cases} y=\sin^{-1}x & \Rightarrow \text{inverse function} \\ y=(\sin x)^{-1}=\frac{1}{\sin x} & \Rightarrow \text{倒數關係} \end{cases}$

Let $y=\tan^{-1}x$

$\Rightarrow \tan y=x$

$\Rightarrow (\sec^2 y)\cdot y'=1$

$\Rightarrow y'=\frac{1}{\sec^2 y}=\frac{1}{1+x^2}$

Eg. $y=\frac{1}{\sin^{-1}x}=\frac{1}{\arcsin x}=(\sin^{-1}x)^{-1}$

$y'=-(\sin^{-1}x)^{-2}\left(\frac{1}{\sqrt{1-x^2}}\right)$

Eg. $f(x)=x\cdot\arctan\sqrt{x}$

$f'(x)=\arctan\sqrt{x}+\frac{x}{1+(\sqrt{x})^2}\times\frac{1}{2\sqrt{x}}$

$=\tan^{-1}\sqrt{x}+\frac{\sqrt{x}}{2(1+x)}$

3.8. Exponential Growth and decay

$\boxed{\frac{dy}{dt}=k\cdot y(t)} \Rightarrow y=Ce^{kt}$ $\begin{cases} k>0, \text{ exponential growth} \\ k<0, \text{ exponential decay} \end{cases}$

Eg. 1950年 ⇒ 2560 million 人

1960年 ⇒ 3040 million 人

Find 1993年 and 2025年的 population

Sol.: Let $P(t)$ be the population of the world t years after 1950年.

$P(0)=Ce^{k\cdot 0}=C=2560$

$P(10)=Ce^{k\cdot 10}=3040$

$\Rightarrow \ln Ce^{k\cdot 10}=\ln 2560e^{10k}=\ln 3040 \Rightarrow k=\frac{1}{10}\times\frac{\ln 3040}{\ln 2560}$

$\therefore P(t)=2560e^{\frac{t}{10}\times\frac{\ln 3040}{\ln 2560}}$

Eg. Let $m(t)$ be the mass of radium-226.

$m(1590)=\frac{1}{2}m(0)$. $m(t)=Ce^{kt}$. $m(0)=C=100$

(A) Therefore, $m(t)=100e^{kt}$

(B) $100e^{1590k}=\frac{1}{2}\cdot 100=50$. $\ln e^{1590k}=\ln\frac{1}{2} \Rightarrow k=\frac{\ln\frac{1}{2}}{1590}$

(C) $30=100e^{\frac{-\ln 2}{1590}t}$, 求 t

$\ln\frac{3}{10}=-\frac{\ln 2}{1590}t \Rightarrow t=1590\times\ln\frac{10}{3}\times\frac{1}{\ln 2}$

3.9. Related Rates.

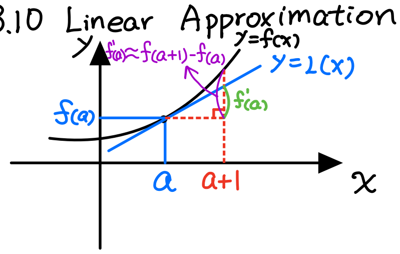
Eg. $V=\frac{4}{3}\pi r^3$

$$\frac{dV}{dt}=4\pi r^2\cdot\frac{dr}{dt}$$

If we put $\frac{dV}{dt}=100$ and $r=25$

$\Rightarrow 100=4\pi\cdot 25^2\cdot\frac{dr}{dt} \Rightarrow \frac{dr}{dt}=\frac{1}{25}$

3.10 Linear Approximation



Slope-point form (点斜式)

$L(x)-f(a)=f'(a)\cdot(x-a)$

$\boxed{y-f(a)=f'(a)\cdot(x-a)}$ Linearization of $f(x)$ at point $x=a$

$L(x)=f(a)+f'(a)\cdot(x-a)$

$y=f(a)+f'(a)\cdot(x-a)$

$\therefore$ We call $f(x)\approx f(a)+f'(a)\cdot(x-a)$ is the linear approximation

tangent line approximation

Eg. $x^3+25x^2-19x+200$

$f(103)-f(102)\approx f'(102)=3(102)^2+50\cdot(102)-19$

Eg. find an approximation of $\sqrt{3.98}$ and $\sqrt{4.05}$

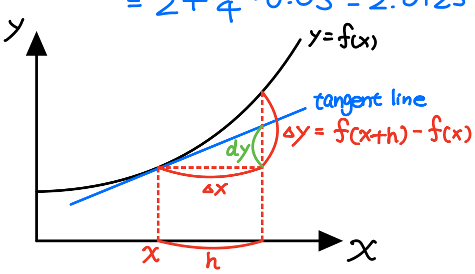
Let $f(x)=\sqrt{x}$ at $x=4$

$\sqrt{3.98}\approx\sqrt{4}+\frac{1}{2}(4)^{-\frac{1}{2}}\cdot(3.98-4)$

$=2+\frac{1}{4}\cdot(-0.02)=2-0.005=1.995$

$\sqrt{4.05}\approx\sqrt{4}+\frac{1}{2}(4)^{-\frac{1}{2}}\cdot(4.05-4)$

$=2+\frac{1}{4}\cdot 0.05=2.0125$



$\Delta y=f(x+h)-f(x)$

在極短範圍$(h>0)$內, $dy\approx\Delta y$, $dy=f'(x)\cdot dx$

Eg. $f(x)=x^3+x^2-2x+1$. Find Δy and dy when (a) $x=2$ to $x=2.05$

(a) $\Delta y=f(2.05)-f(2)=0.717625$

$dy=f'(2)\cdot(2.05-2)=14\cdot 0.05=0.7$