3 ☐ DIFFERENTIATION RULES

3.1 Derivatives of Polynomials and Exponential Functions

1. (a) e is the number such that $\lim\limits_{h\to 0} \dfrac{e^h - 1}{h} = 1$.

(b)

x	$\dfrac{2.7^x - 1}{x}$
-0.001	0.9928
-0.0001	0.9932
0.001	0.9937
0.0001	0.9933

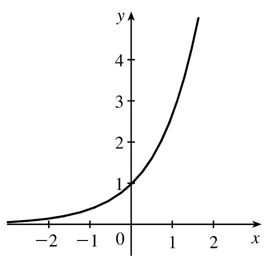
x	$\dfrac{2.8^x - 1}{x}$
-0.001	1.0291
-0.0001	1.0296
0.001	1.0301
0.0001	1.0297

From the tables (to two decimal places),

$\lim\limits_{h\to 0} \dfrac{2.7^h - 1}{h} = 0.99$ and $\lim\limits_{h\to 0} \dfrac{2.8^h - 1}{h} = 1.03$.

Since $0.99 < 1 < 1.03$, $2.7 < e < 2.8$.

2. (a)



The function value at $x = 0$ is 1 and the slope at $x = 0$ is 1.

(b) $f(x) = e^x$ is an exponential function and $g(x) = x^e$ is a power function. $\dfrac{d}{dx}(e^x) = e^x$ and $\dfrac{d}{dx}(x^e) = ex^{e-1}$.

(c) $f(x) = e^x$ grows more rapidly than $g(x) = x^e$ when x is large.

3. $g(x) = 4x + 7 \quad\Rightarrow\quad g'(x) = 4(1) + 0 = 4$

4. $g(t) = 5t + 4t^2 \quad\Rightarrow\quad g'(t) = 5(1) + 4(2t^{2-1}) = 5(1) + 4(2t) = 5 + 8t$

5. $f(x) = x^{75} - x + 3 \quad\Rightarrow\quad f'(x) = 75x^{75-1} - 1(1) + 0 = 75x^{74} - 1$

6. $g(x) = \frac{7}{4}x^2 - 3x + 12 \quad\Rightarrow\quad g'(x) = \frac{7}{4}(2x^{2-1}) - 3(1) + 0 = \frac{7}{4}(2x) - 3 = \frac{7}{2}x - 3$

7. $f(t) = -2e^t \quad\Rightarrow\quad f'(t) = -2(e^t) = -2e^t$

8. $F(t) = t^3 + e^3 \quad\Rightarrow\quad F'(t) = 3t^2 + 0 = 3t^2$ [Note that e^3 is constant, so its derivative is zero.]

9. $W(v) = 1.8v^{-3} \quad\Rightarrow\quad W'(v) = 1.8(-3v^{-3-1}) = 1.8(-3v^{-4}) = -5.4v^{-4}$

10. $r(z) = z^{-5} - z^{1/2} \quad\Rightarrow\quad r'(z) = -5z^{-6} - \frac{1}{2}z^{-1/2}$

11. $f(x) = x^{3/2} + x^{-3} \quad\Rightarrow\quad f'(x) = \frac{3}{2}x^{1/2} + (-3x^{-4}) = \frac{3}{2}x^{1/2} - 3x^{-4}$

12. $V(t) = t^{-3/5} + t^4 \quad\Rightarrow\quad V'(t) = -\frac{3}{5}t^{-8/5} + 4t^3$

13. $s(t) = \dfrac{1}{t} + \dfrac{1}{t^2} = t^{-1} + t^{-2} \quad\Rightarrow\quad s'(t) = -t^{-2} + \left(-2t^{-3}\right) = -t^{-2} - 2t^{-3} = -\dfrac{1}{t^2} - \dfrac{2}{t^3}$

14. $r(t) = \dfrac{a}{t^2} + \dfrac{b}{t^4} = at^{-2} + bt^{-4} \quad \Rightarrow \quad r'(t) = a(-2t^{-3}) + b(-4t^{-5}) = -2at^{-3} - 4bt^{-5} = -\dfrac{2a}{t^3} - \dfrac{4b}{t^5}$

15. $y = 2x + \sqrt{x} = 2x + x^{1/2} \quad \Rightarrow \quad y' = 2(1) + \frac{1}{2}x^{-1/2} = 2 + \frac{1}{2}x^{-1/2}$ or $2 + \dfrac{1}{2\sqrt{x}}$

16. $h(w) = \sqrt{2}\,w - \sqrt{2} \quad \Rightarrow \quad h'(w) = \sqrt{2}\,(1) - 0 = \sqrt{2}$

17. $g(x) = \dfrac{1}{\sqrt{x}} + \sqrt[4]{x} = x^{-1/2} + x^{1/4} \quad \Rightarrow \quad g'(x) = -\frac{1}{2}x^{-3/2} + \frac{1}{4}x^{-3/4}$ or $-\dfrac{1}{2x\sqrt{x}} + \dfrac{1}{4\sqrt[4]{x^3}}$

18. $W(t) = \sqrt{t} - 2e^t = t^{1/2} - 2e^t \quad \Rightarrow \quad W'(t) = \frac{1}{2}t^{-1/2} - 2(e^t) = \frac{1}{2}t^{-1/2} - 2e^t$ or $\dfrac{1}{2\sqrt{t}} - 2e^t$

19. $f(x) = x^3(x+3) = x^4 + 3x^3 \quad \Rightarrow \quad f'(x) = 4x^3 + 3(3x^2) = 4x^3 + 9x^2$

20. $F(t) = (2t-3)^2 = 4t^2 - 12t + 9 \quad \Rightarrow \quad F'(t) = 4(2t) - 12(1) + 0 = 8t - 12$

21. $y = 3e^x + \dfrac{4}{\sqrt[3]{x}} = 3e^x + 4x^{-1/3} \quad \Rightarrow \quad y' = 3(e^x) + 4(-\frac{1}{3})x^{-4/3} = 3e^x - \frac{4}{3}x^{-4/3}$ or $3e^x - \dfrac{4}{3x\sqrt[3]{x}}$

22. $S(R) = 4\pi R^2 \quad \Rightarrow \quad S'(R) = 4\pi(2R) = 8\pi R$

23. $f(x) = \dfrac{3x^2 + x^3}{x} = \dfrac{3x^2}{x} + \dfrac{x^3}{x} = 3x + x^2 \quad \Rightarrow \quad f'(x) = 3(1) + 2x = 3 + 2x$

24. $y = \dfrac{\sqrt{x} + x}{x^2} = \dfrac{\sqrt{x}}{x^2} + \dfrac{x}{x^2} = x^{1/2-2} + x^{1-2} = x^{-3/2} + x^{-1} \quad \Rightarrow \quad y' = -\frac{3}{2}x^{-5/2} + (-1x^{-2}) = -\frac{3}{2}x^{-5/2} - x^{-2}$

25. $G(r) = \dfrac{3r^{3/2} + r^{5/2}}{r} = \dfrac{3r^{3/2}}{r} + \dfrac{r^{5/2}}{r} = 3r^{3/2-2/2} + r^{5/2-2/2} = 3r^{1/2} + r^{3/2} \quad \Rightarrow$

$G'(r) = 3\left(\frac{1}{2}r^{-1/2}\right) + \frac{3}{2}r^{1/2} = \frac{3}{2}r^{-1/2} + \frac{3}{2}r^{1/2}$ or $\dfrac{3}{2\sqrt{r}} + \dfrac{3}{2}\sqrt{r}$

26. $G(t) = \sqrt{5t} + \dfrac{\sqrt{7}}{t} = \sqrt{5}\,t^{1/2} + \sqrt{7}\,t^{-1} \quad \Rightarrow \quad G'(t) = \sqrt{5}\left(\frac{1}{2}t^{-1/2}\right) + \sqrt{7}\left(-1t^{-2}\right) = \dfrac{\sqrt{5}}{2\sqrt{t}} - \dfrac{\sqrt{7}}{t^2}$

27. $j(x) = x^{2.4} + e^{2.4} \quad \Rightarrow \quad j'(x) = 2.4x^{1.4} + 0 = 2.4x^{1.4}$

28. $k(r) = e^r + r^e \quad \Rightarrow \quad k'(r) = e^r + er^{e-1}$

29. $F(z) = \dfrac{A + Bz + Cz^2}{z^2} = \dfrac{A}{z^2} + \dfrac{Bz}{z^2} + \dfrac{Cz^2}{z^2} = Az^{-2} + Bz^{-1} + C \quad \Rightarrow$

$F'(z) = A(-2z^{-3}) + B(-1z^{-2}) + 0 = -2Az^{-3} - Bz^{-2} = -\dfrac{2A}{z^3} - \dfrac{B}{z^2}$ or $-\dfrac{2A + Bz}{z^3}$

30. $G(q) = \left(1 + q^{-1}\right)^2 = 1 + 2q^{-1} + q^{-2} \quad \Rightarrow \quad G'(q) = 0 + 2(-1q^{-2}) + (-2q^{-3}) = -2q^{-2} - 2q^{-3}$

31. $D(t) = \dfrac{1 + 16t^2}{(4t)^3} = \dfrac{1 + 16t^2}{64t^3} = \dfrac{1}{64}t^{-3} + \dfrac{1}{4}t^{-1} \quad \Rightarrow$

$D'(t) = \frac{1}{64}(-3t^{-4}) + \frac{1}{4}(-1t^{-2}) = -\frac{3}{64}t^{-4} - \frac{1}{4}t^{-2}$ or $-\dfrac{3}{64t^4} - \dfrac{1}{4t^2}$ or $-\dfrac{3 + 16t^2}{64t^4}$

32. $f(v) = \dfrac{\sqrt[3]{v} - 2ve^v}{v} = \dfrac{\sqrt[3]{v}}{v} - \dfrac{2ve^v}{v} = v^{-2/3} - 2e^v \quad\Rightarrow\quad f'(v) = -\frac{2}{3}v^{-5/3} - 2e^v$

33. $P(w) = \dfrac{2w^2 - w + 4}{\sqrt{w}} = \dfrac{2w^2}{\sqrt{w}} - \dfrac{w}{\sqrt{w}} + \dfrac{4}{\sqrt{w}} = 2w^{2-1/2} - w^{1-1/2} + 4w^{-1/2} = 2w^{3/2} - w^{1/2} + 4w^{-1/2} \quad\Rightarrow$

$P'(w) = 2\left(\frac{3}{2}w^{1/2}\right) - \frac{1}{2}w^{-1/2} + 4\left(-\frac{1}{2}w^{-3/2}\right) = 3w^{1/2} - \frac{1}{2}w^{-1/2} - 2w^{-3/2}$ or $3\sqrt{w} - \dfrac{1}{2\sqrt{w}} - \dfrac{2}{w\sqrt{w}}$

34. $y = e^{x+1} + 1 = e^x e^1 + 1 = e \cdot e^x + 1 \quad\Rightarrow\quad y' = e \cdot e^x = e^{x+1}$

35. $y = tx^2 + t^3x$.

To find dy/dx, we treat t as a constant and x as a variable to get $dy/dx = t(2x) + t^3(1) = 2tx + t^3$.

To find dy/dt, we treat x as a constant and t as a variable to get $dy/dt = (1)x^2 + (3t^2)x = x^2 + 3t^2x$.

36. $y = \dfrac{t}{x^2} + \dfrac{x}{t} = tx^{-2} + xt^{-1}$.

To find dy/dx, we treat t as a constant and x as a variable to get $dy/dx = t(-2x^{-3}) + (1)t^{-1} = -2tx^{-3} + t^{-1}$ or $-\dfrac{2t}{x^3} + \dfrac{1}{t}$.

To find dy/dt, we treat x as a constant and t as a variable to get $dy/dt = (1)x^{-2} + x(-1t^{-2}) = x^{-2} - xt^{-2}$ or $\dfrac{1}{x^2} - \dfrac{x}{t^2}$.

37. $y = 2x^3 - x^2 + 2 \quad\Rightarrow\quad y' = 6x^2 - 2x$. At $(1, 3)$, $y' = 6(1)^2 - 2(1) = 4$ and an equation of the tangent line is $y - 3 = 4(x - 1)$ or $y = 4x - 1$.

38. $y = 2e^x + x \quad\Rightarrow\quad y' = 2e^x + 1$. At $(0, 2)$, $y' = 2e^0 + 1 = 3$ and an equation of the tangent line is $y - 2 = 3(x - 0)$ or $y = 3x + 2$.

39. $y = x + \dfrac{2}{x} = x + 2x^{-1} \quad\Rightarrow\quad y' = 1 - 2x^{-2}$. At $(2, 3)$, $y' = 1 - 2(2)^{-2} = \frac{1}{2}$ and an equation of the tangent line is $y - 3 = \frac{1}{2}(x - 2)$ or $y = \frac{1}{2}x + 2$.

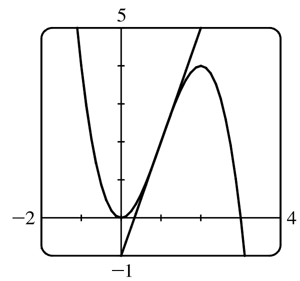
40. $y = \sqrt[4]{x} - x = x^{1/4} - x \quad\Rightarrow\quad y' = \frac{1}{4}x^{-3/4} - 1 = \dfrac{1}{4\sqrt[4]{x^3}} - 1$. At $(1, 0)$, $y' = \frac{1}{4} - 1 = -\frac{3}{4}$ and an equation of the tangent line is $y - 0 = -\frac{3}{4}(x - 1)$ or $y = -\frac{3}{4}x + \frac{3}{4}$.

41. $y = x^4 + 2e^x \quad\Rightarrow\quad y' = 4x^3 + 2e^x$. At $(0, 2)$, $y' = 2$ and an equation of the tangent line is $y - 2 = 2(x - 0)$ or $y = 2x + 2$. The slope of the normal line is $-\frac{1}{2}$ (the negative reciprocal of 2) and an equation of the normal line is $y - 2 = -\frac{1}{2}(x - 0)$ or $y = -\frac{1}{2}x + 2$.

42. $y = x^{3/2} \quad\Rightarrow\quad y' = \frac{3}{2}x^{1/2}$. At $(1, 1)$, $y' = \frac{3}{2}$ and an equation of the tangent line is $y - 1 = \frac{3}{2}(x - 1)$ or $y = \frac{3}{2}x - \frac{1}{2}$. The slope of the normal line is $-\frac{2}{3}$ $\left(\text{the negative reciprocal of } \frac{3}{2}\right)$ and an equation of the normal line is $y - 1 = -\frac{2}{3}(x - 1)$ or $y = -\frac{2}{3}x + \frac{5}{3}$.

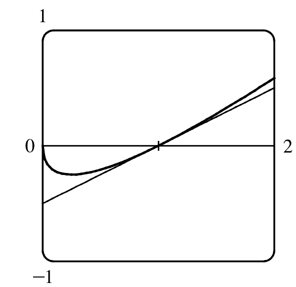
43. $y = 3x^2 - x^3 \quad\Rightarrow\quad y' = 6x - 3x^2$.

At $(1, 2)$, $y' = 6 - 3 = 3$, so an equation of the tangent line is $y - 2 = 3(x - 1)$ or $y = 3x - 1$.



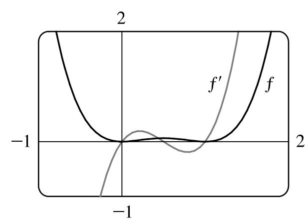
44. $y = x - \sqrt{x} \quad\Rightarrow\quad y' = 1 - \frac{1}{2}x^{-1/2} = 1 - \dfrac{1}{2\sqrt{x}}$.

At $(1, 0)$, $y' = \frac{1}{2}$, so an equation of the tangent line is $y - 0 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x - \frac{1}{2}$.



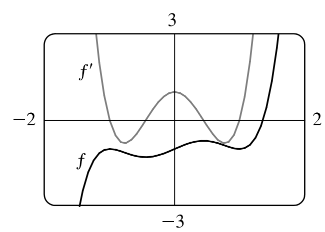
45. $f(x) = x^4 - 2x^3 + x^2 \quad\Rightarrow\quad f'(x) = 4x^3 - 6x^2 + 2x$

Note that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

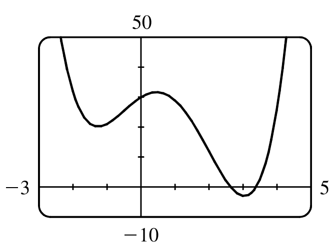


46. $f(x) = x^5 - 2x^3 + x - 1 \quad\Rightarrow\quad f'(x) = 5x^4 - 6x^2 + 1$

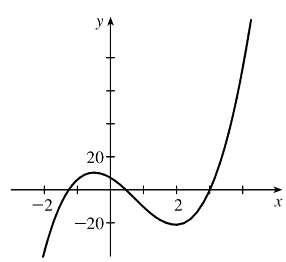
Note that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.



47. (a)

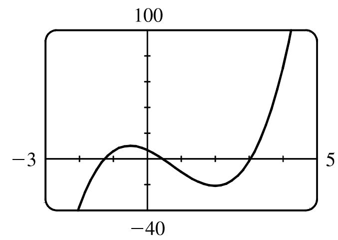


(b) From the graph in part (a), it appears that f' is zero at $x_1 \approx -1.25$, $x_2 \approx 0.5$, and $x_3 \approx 3$. The slopes are negative (so f' is negative) on $(-\infty, x_1)$ and (x_2, x_3). The slopes are positive (so f' is positive) on (x_1, x_2) and (x_3, ∞).

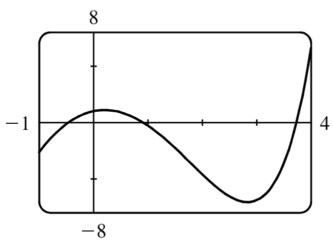


(c) $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30 \quad \Rightarrow$

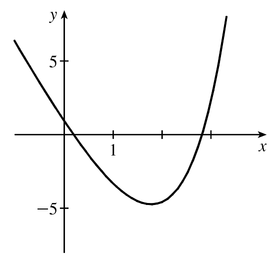
$f'(x) = 4x^3 - 9x^2 - 12x + 7$



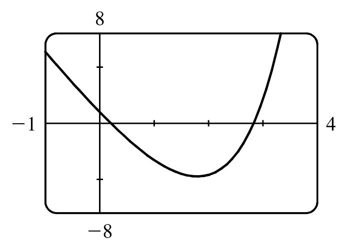
48. (a)



(b) From the graph in part (a), it appears that f' is zero at $x_1 \approx 0.2$ and $x_2 \approx 2.8$. The slopes are positive (so f' is positive) on $(-\infty, x_1)$ and (x_2, ∞). The slopes are negative (so f' is negative) on (x_1, x_2).



(c) $g(x) = e^x - 3x^2 \quad \Rightarrow \quad g'(x) = e^x - 6x$

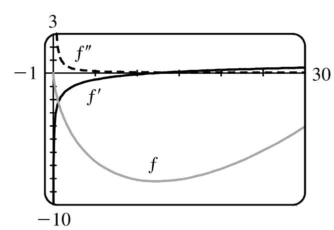


49. $f(x) = 0.001x^5 - 0.02x^3 \quad \Rightarrow \quad f'(x) = 0.005x^4 - 0.06x^2 \quad \Rightarrow \quad f''(x) = 0.02x^3 - 0.12x$

50. $G(r) = \sqrt{r} + \sqrt[3]{r} \quad \Rightarrow \quad G'(r) = \frac{1}{2}r^{-1/2} + \frac{1}{3}r^{-2/3} \quad \Rightarrow \quad G''(r) = -\frac{1}{4}r^{-3/2} - \frac{2}{9}r^{-5/3}$

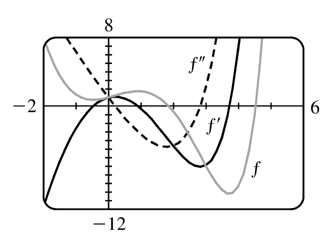
51. $f(x) = 2x - 5x^{3/4} \quad \Rightarrow \quad f'(x) = 2 - \frac{15}{4}x^{-1/4} \quad \Rightarrow \quad f''(x) = \frac{15}{16}x^{-5/4}$

Note that f' is negative when f is decreasing and positive when f is increasing. f'' is always positive since f' is always increasing.



52. $f(x) = e^x - x^3 \quad \Rightarrow \quad f'(x) = e^x - 3x^2 \quad \Rightarrow \quad f''(x) = e^x - 6x$

Note that $f'(x) = 0$ when f has a horizontal tangent and that $f''(x) = 0$ when f' has a horizontal tangent.



53. (a) $s = t^3 - 3t \quad \Rightarrow \quad v(t) = s'(t) = 3t^2 - 3 \quad \Rightarrow \quad a(t) = v'(t) = 6t$

(b) $a(2) = 6(2) = 12 \text{ m/s}^2$

(c) $v(t) = 3t^2 - 3 = 0$ when $t^2 = 1$, that is, $t = 1$ $[t \geq 0]$ and $a(1) = 6 \text{ m/s}^2$.

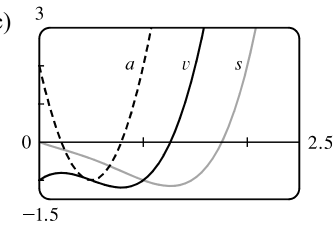
54. (a) $s = t^4 - 2t^3 + t^2 - t \quad \Rightarrow$

$v(t) = s'(t) = 4t^3 - 6t^2 + 2t - 1 \quad \Rightarrow$

$a(t) = v'(t) = 12t^2 - 12t + 2$

(b) $a(1) = 12(1)^2 - 12(1) + 2 = 2\,\text{m/s}^2$

(c)



55. $L = 0.0155A^3 - 0.372A^2 + 3.95A + 1.21 \quad \Rightarrow \quad \dfrac{dL}{dA} = 0.0465A^2 - 0.744A + 3.95$, so

$\left.\dfrac{dL}{dA}\right|_{A=12} = 0.117(12)^2 - 1.89(12) + 10.03 = 4.198$. The derivative is the instantaneous rate of change of the length (in centimeters) of an Alaskan rockfish with respect to its age when its age is 12 years. Its units are centimeters/year.

56. $S(A) = 0.882A^{0.842} \quad \Rightarrow \quad S'(A) = 0.882(0.842A^{-0.158}) = 0.742644A^{-0.158}$, so

$S'(100) = 0.742644(100)^{-0.158} \approx 0.36$. The derivative is the instantaneous rate of change of the number of tree species with respect to area. Its units are number of species per square meter.

57. (a) $P = \dfrac{k}{V}$ and $P = 50$ when $V = 0.106$, so $k = PV = 50(0.106) = 5.3$. Thus, $P = \dfrac{5.3}{V}$ and $V = \dfrac{5.3}{P}$.

(b) $V = 5.3P^{-1} \quad \Rightarrow \quad \dfrac{dV}{dP} = 5.3(-1P^{-2}) = -\dfrac{5.3}{P^2}$. When $P = 50$, $\dfrac{dV}{dP} = -\dfrac{5.3}{50^2} = -0.00212$. The derivative is the instantaneous rate of change of the volume with respect to the pressure at 25°C. Its units are m^3/kPa.

58. (a) $L = aP^2 + bP + c$, where $a \approx -0.00920$, $b \approx 4.5519$, and $c \approx -434.58914$.

(b) $\frac{dL}{dP} = 2aP + b$. When $P = 200$, $\frac{dL}{dP} \approx 0.88$, and when $P = 300$, $\frac{dL}{dP} \approx -0.96$. The derivative is the instantaneous rate of change of tire life with respect to pressure. Its units are (thousands of kilometers)/kPa. When $\frac{dL}{dP}$ is positive, tire life is increasing, and when $\frac{dL}{dP} < 0$, tire life is decreasing.

59. $y = x^3 + 3x^2 - 9x + 10 \quad \Rightarrow \quad y' = 3x^2 + 3(2x) - 9(1) + 0 = 3x^2 + 6x - 9$. Horizontal tangents occur where $y' = 0$. Thus, $3x^2 + 6x - 9 = 0 \quad \Rightarrow \quad 3(x^2 + 2x - 3) = 0 \quad \Rightarrow \quad 3(x+3)(x-1) = 0 \quad \Rightarrow \quad x = -3$ or $x = 1$. The corresponding points are $(-3, 37)$ and $(1, 5)$.

60. $f(x) = e^x - 2x \quad \Rightarrow \quad f'(x) = e^x - 2$. $f'(x) = 0 \quad \Rightarrow \quad e^x = 2 \quad \Rightarrow \quad x = \ln 2$, so f has a horizontal tangent when $x = \ln 2$.

61. $y = 2e^x + 3x + 5x^3 \quad \Rightarrow \quad y' = 2e^x + 3 + 15x^2$. Since $2e^x > 0$ and $15x^2 \geq 0$, we must have $y' > 0 + 3 + 0 = 3$, so no tangent line can have slope 2.

62. $y = x^4 + 1 \quad \Rightarrow \quad y' = 4x^3$. The slope of the line $32x - y = 15$ (or $y = 32x - 15$) is 32, so the slope of any line parallel to it is also 32. Thus, $y' = 32 \quad \Leftrightarrow \quad 4x^3 = 32 \quad \Leftrightarrow \quad x^3 = 8 \quad \Leftrightarrow \quad x = 2$, which is the x-coordinate of the point on the curve at which the slope is 32. The y-coordinate is $2^4 + 1 = 17$, so an equation of the tangent line is $y - 17 = 32(x - 2)$ or $y = 32x - 47$.

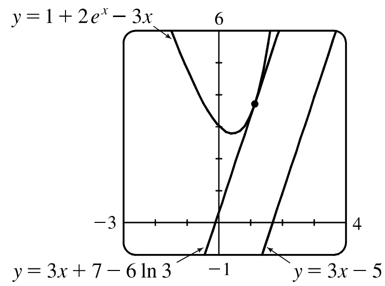
63. The slope of the line $3x - y = 15$ (or $y = 3x - 15$) is 3, so the slope of both tangent lines to the curve is 3. $y = x^3 - 3x^2 + 3x - 3 \quad\Rightarrow\quad y' = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2$. Thus, $3(x-1)^2 = 3 \quad\Rightarrow$ $(x-1)^2 = 1 \quad\Rightarrow\quad x - 1 = \pm 1 \quad\Rightarrow\quad x = 0$ or 2, which are the x-coordinates at which the tangent lines have slope 3. The points on the curve are $(0, -3)$ and $(2, -1)$, so the tangent line equations are $y - (-3) = 3(x - 0)$ or $y = 3x - 3$ and $y - (-1) = 3(x - 2)$ or $y = 3x - 7$.

64. The slope of $y = 1 + 2e^x - 3x$ is given by $m = y' = 2e^x - 3$.

The slope of $3x - y = 5 \quad\Leftrightarrow\quad y = 3x - 5$ is 3.

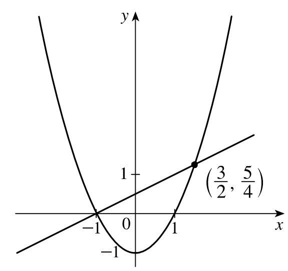
$m = 3 \quad\Rightarrow\quad 2e^x - 3 = 3 \quad\Rightarrow\quad e^x = 3 \quad\Rightarrow\quad x = \ln 3$.

This occurs at the point $(\ln 3, 7 - 3\ln 3) \approx (1.1, 3.7)$.

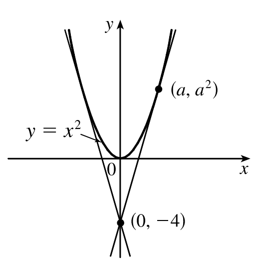


65. The slope of $y = \sqrt{x}$ is given by $y = \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$. The slope of $2x + y = 1$ (or $y = -2x + 1$) is -2, so the desired normal line must have slope -2, and hence, the tangent line to the curve must have slope $\frac{1}{2}$. This occurs if $\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2} \quad\Rightarrow$ $\sqrt{x} = 1 \quad\Rightarrow\quad x = 1$. When $x = 1$, $y = \sqrt{1} = 1$, and an equation of the normal line is $y - 1 = -2(x - 1)$ or $y = -2x + 3$.

66. $y = f(x) = x^2 - 1 \quad\Rightarrow\quad f'(x) = 2x$. So $f'(-1) = -2$, and the slope of the normal line is $\frac{1}{2}$. The equation of the normal line at $(-1, 0)$ is $y - 0 = \frac{1}{2}[x - (-1)]$ or $y = \frac{1}{2}x + \frac{1}{2}$. Substituting this into the equation of the parabola, we obtain $\frac{1}{2}x + \frac{1}{2} = x^2 - 1 \quad\Leftrightarrow\quad x + 1 = 2x^2 - 2 \quad\Leftrightarrow$ $2x^2 - x - 3 = 0 \quad\Leftrightarrow\quad (2x - 3)(x + 1) = 0 \quad\Leftrightarrow\quad x = \frac{3}{2}$ or -1. Substituting $\frac{3}{2}$ into the equation of the normal line gives us $y = \frac{5}{4}$. Thus, the second point of intersection is $\left(\frac{3}{2}, \frac{5}{4}\right)$, as shown in the sketch.



67.



Let (a, a^2) be a point on the parabola at which the tangent line passes through the point $(0, -4)$. The tangent line has slope $2a$ and equation $y - (-4) = 2a(x - 0) \quad\Leftrightarrow\quad y = 2ax - 4$. Since (a, a^2) also lies on the line, $a^2 = 2a(a) - 4$, or $a^2 = 4$. So $a = \pm 2$ and the points are $(2, 4)$ and $(-2, 4)$.

68. (a) If $y = x^2 + x$, then $y' = 2x + 1$. If the point at which a tangent meets the parabola is $(a, a^2 + a)$, then the slope of the tangent is $2a + 1$. But since it passes through $(2, -3)$, the slope must also be $\dfrac{\Delta y}{\Delta x} = \dfrac{a^2 + a + 3}{a - 2}$.

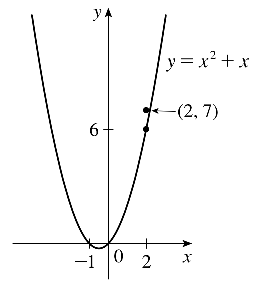
Therefore, $2a + 1 = \dfrac{a^2 + a + 3}{a - 2}$. Solving this equation for a we get $a^2 + a + 3 = 2a^2 - 3a - 2 \quad\Leftrightarrow$

$a^2 - 4a - 5 = (a-5)(a+1) = 0 \quad \Leftrightarrow \quad a = 5$ or -1. If $a = -1$, the point is $(-1, 0)$ and the slope is -1, so the equation is $y - 0 = (-1)(x+1)$ or $y = -x - 1$. If $a = 5$, the point is $(5, 30)$ and the slope is 11, so the equation is $y - 30 = 11(x - 5)$ or $y = 11x - 25$.

(b) As in part (a), but using the point $(2, 7)$, we get the equation

$$2a + 1 = \frac{a^2 + a - 7}{a - 2} \quad \Rightarrow \quad 2a^2 - 3a - 2 = a^2 + a - 7 \quad \Leftrightarrow \quad a^2 - 4a + 5 = 0.$$

The last equation has no real solution (discriminant $= -16 < 0$), so there is no line through the point $(2, 7)$ that is tangent to the parabola. The diagram shows that the point $(2, 7)$ is "inside" the parabola, but tangent lines to the parabola do not pass through points inside the parabola.



69. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = \lim\limits_{h \to 0} \dfrac{x - (x+h)}{hx(x+h)} = \lim\limits_{h \to 0} \dfrac{-h}{hx(x+h)} = \lim\limits_{h \to 0} \dfrac{-1}{x(x+h)} = -\dfrac{1}{x^2}$

70. (a) $f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1} \quad \Rightarrow \quad f''(x) = n(n-1)x^{n-2} \quad \Rightarrow \quad \cdots \quad \Rightarrow$

$f^{(n)}(x) = n(n-1)(n-2)\cdots 2 \cdot 1x^{n-n} = n!$

(b) $f(x) = x^{-1} \quad \Rightarrow \quad f'(x) = (-1)x^{-2} \quad \Rightarrow \quad f''(x) = (-1)(-2)x^{-3} \quad \Rightarrow \quad \cdots \quad \Rightarrow$

$f^{(n)}(x) = (-1)(-2)(-3)\cdots(-n)x^{-(n+1)} = (-1)^n n! x^{-(n+1)}$ or $\dfrac{(-1)^n\, n!}{x^{n+1}}$

71. Let $P(x) = ax^2 + bx + c$. Then $P'(x) = 2ax + b$ and $P''(x) = 2a$. $P''(2) = 2 \quad \Rightarrow \quad 2a = 2 \quad \Rightarrow \quad a = 1$.

$P'(2) = 3 \quad \Rightarrow \quad 2(1)(2) + b = 3 \quad \Rightarrow \quad 4 + b = 3 \quad \Rightarrow \quad b = -1$.

$P(2) = 5 \quad \Rightarrow \quad 1(2)^2 + (-1)(2) + c = 5 \quad \Rightarrow \quad 2 + c = 5 \quad \Rightarrow \quad c = 3$. So $P(x) = x^2 - x + 3$.

72. $y = Ax^2 + Bx + C \quad \Rightarrow \quad y' = 2Ax + B \quad \Rightarrow \quad y'' = 2A$. We substitute these expressions into the equation $y'' + y' - 2y = x^2$ to get

$$\begin{aligned} (2A) + (2Ax + B) - 2(Ax^2 + Bx + C) &= x^2 \\ 2A + 2Ax + B - 2Ax^2 - 2Bx - 2C &= x^2 \\ (-2A)x^2 + (2A - 2B)x + (2A + B - 2C) &= (1)x^2 + (0)x + (0) \end{aligned}$$

The coefficients of x^2 on each side must be equal, so $-2A = 1 \quad \Rightarrow \quad A = -\frac{1}{2}$. Similarly, $2A - 2B = 0 \quad \Rightarrow$ $A = B = -\frac{1}{2}$ and $2A + B - 2C = 0 \quad \Rightarrow \quad -1 - \frac{1}{2} - 2C = 0 \quad \Rightarrow \quad C = -\frac{3}{4}$.

73. $y = f(x) = ax^3 + bx^2 + cx + d \quad \Rightarrow \quad f'(x) = 3ax^2 + 2bx + c$. The point $(-2, 6)$ is on f, so $f(-2) = 6 \quad \Rightarrow$ $-8a + 4b - 2c + d = 6$ **(1)**. The point $(2, 0)$ is on f, so $f(2) = 0 \quad \Rightarrow \quad 8a + 4b + 2c + d = 0$ **(2)**. Since there are horizontal tangents at $(-2, 6)$ and $(2, 0)$, $f'(\pm 2) = 0$. $f'(-2) = 0 \quad \Rightarrow \quad 12a - 4b + c = 0$ **(3)** and $f'(2) = 0 \quad \Rightarrow$ $12a + 4b + c = 0$ **(4)**. Subtracting equation **(3)** from **(4)** gives $8b = 0 \quad \Rightarrow \quad b = 0$. Adding **(1)** and **(2)** gives $8b + 2d = 6$, so $d = 3$ since $b = 0$. From **(3)** we have $c = -12a$, so **(2)** becomes $8a + 4(0) + 2(-12a) + 3 = 0 \quad \Rightarrow \quad 3 = 16a \quad \Rightarrow$ $a = \frac{3}{16}$. Now $c = -12a = -12\left(\frac{3}{16}\right) = -\frac{9}{4}$ and the desired cubic function is $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$.

74. $y = ax^2 + bx + c \quad \Rightarrow \quad y'(x) = 2ax + b$. The parabola has slope 4 at $x = 1$ and slope -8 at $x = -1$, so $y'(1) = 4 \quad \Rightarrow$ $2a + b = 4$ **(1)** and $y'(-1) = -8 \quad \Rightarrow \quad -2a + b = -8$ **(2)**. Adding **(1)** and **(2)** gives us $2b = -4 \quad \Leftrightarrow \quad b = -2$. From **(1)**, $2a - 2 = 4 \quad \Leftrightarrow \quad a = 3$. Thus, the equation of the parabola is $y = 3x^2 - 2x + c$. Since it passes through the point $(2, 15)$, we have $15 = 3(2)^2 - 2(2) + c \quad \Rightarrow \quad c = 7$, so the equation is $y = 3x^2 - 2x + 7$.

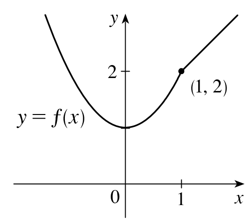
75. $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \geq 1 \end{cases}$

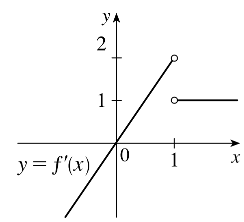
Calculate the left- and right-hand derivatives as defined in Exercise 2.8.64:

$$f'_-(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{[(1+h)^2 + 1] - (1+1)}{h} = \lim_{h \to 0^-} \frac{h^2 + 2h}{h} = \lim_{h \to 0^-} (h + 2) = 2 \text{ and}$$

$$f'_+(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{[(1+h) + 1] - (1+1)}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1.$$

Since the left and right limits are different, $\displaystyle\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$ does not exist, that is, $f'(1)$ does not exist. Therefore, f is not differentiable at 1.





76. $g(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ 2x - x^2 & \text{if } 0 < x < 2 \\ 2 - x & \text{if } x \geq 2 \end{cases}$

Investigate the left- and right-hand derivatives at $x = 0$ and $x = 2$:

$$g'_-(0) = \lim_{h \to 0^-} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^-} \frac{2h - 2(0)}{h} = 2 \text{ and}$$

$$g'_+(0) = \lim_{h \to 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^+} \frac{(2h - h^2) - 2(0)}{h} = \lim_{h \to 0^+} (2 - h) = 2, \text{ so } g \text{ is differentiable at } x = 0.$$

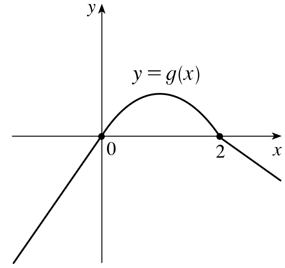
$$g'_-(2) = \lim_{h \to 0^-} \frac{g(2+h) - g(2)}{h} = \lim_{h \to 0^-} \frac{2(2+h) - (2+h)^2 - (2-2)}{h} = \lim_{h \to 0^-} \frac{-2h - h^2}{h} = \lim_{h \to 0^-} (-2 - h) = -2$$

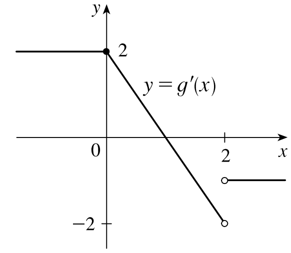
and

$$g'_+(2) = \lim_{h \to 0^+} \frac{g(2+h) - g(2)}{h} = \lim_{h \to 0^+} \frac{[2 - (2+h)] - (2-2)}{h} = \lim_{h \to 0^+} \frac{-h}{h} = \lim_{h \to 0^+} (-1) = -1,$$

so g is not differentiable at $x = 2$. Thus, a formula for g' is

$$g'(x) = \begin{cases} 2 & \text{if } x \leq 0 \\ 2 - 2x & \text{if } 0 < x < 2 \\ -1 & \text{if } x > 2 \end{cases}$$





77. (a) Note that $x^2 - 9 < 0$ for $x^2 < 9 \quad\Leftrightarrow\quad |x| < 3 \quad\Leftrightarrow\quad -3 < x < 3$. So

$$f(x) = \begin{cases} x^2 - 9 & \text{if } x \le -3 \\ -x^2 + 9 & \text{if } -3 < x < 3 \\ x^2 - 9 & \text{if } x \ge 3 \end{cases} \quad\Rightarrow\quad f'(x) = \begin{cases} 2x & \text{if } x < -3 \\ -2x & \text{if } -3 < x < 3 \\ 2x & \text{if } x > 3 \end{cases} = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

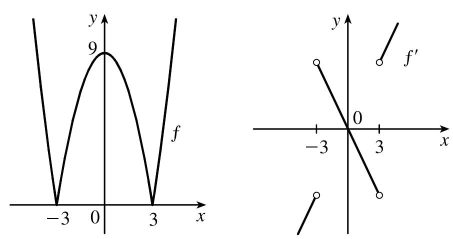
To show that $f'(3)$ does not exist we investigate $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ by computing the left- and right-hand derivatives defined in Exercise 2.8.64.

$$f'_-(3) = \lim_{h \to 0^-} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0^-} \frac{[-(3+h)^2 + 9] - 0}{h} = \lim_{h \to 0^-} (-6 - h) = -6 \quad \text{and}$$

$$f'_+(3) = \lim_{h \to 0^+} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0^+} \frac{[(3+h)^2 - 9] - 0}{h} = \lim_{h \to 0^+} \frac{6h + h^2}{h} = \lim_{h \to 0^+} (6 + h) = 6.$$

Since the left and right limits are different, $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ does not exist, that is, $f'(3)$ does not exist. Similarly, $f'(-3)$ does not exist. Therefore, f is not differentiable at 3 or at -3.

(b)



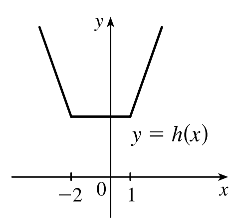
78. If $x \ge 1$, then $h(x) = |x - 1| + |x + 2| = x - 1 + x + 2 = 2x + 1$.

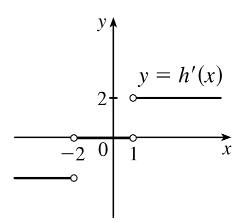
If $-2 < x < 1$, then $h(x) = -(x - 1) + x + 2 = 3$.

If $x \le -2$, then $h(x) = -(x - 1) - (x + 2) = -2x - 1$. Therefore,

$$h(x) = \begin{cases} -2x - 1 & \text{if } x \le -2 \\ 3 & \text{if } -2 < x < 1 \\ 2x + 1 & \text{if } x \ge 1 \end{cases} \quad\Rightarrow\quad h'(x) = \begin{cases} -2 & \text{if } x < -2 \\ 0 & \text{if } -2 < x < 1 \\ 2 & \text{if } x > 1 \end{cases}$$

To see that $h'(1) = \lim\limits_{x \to 1} \dfrac{h(x) - h(1)}{x - 1}$ does not exist, observe that $\lim\limits_{x \to 1^-} \dfrac{h(x) - h(1)}{x - 1} = \lim\limits_{x \to 1^-} \dfrac{3 - 3}{3 - 1} = 0$ but $\lim\limits_{x \to 1^+} \dfrac{h(x) - h(1)}{x - 1} = \lim\limits_{x \to 1^+} \dfrac{2x - 2}{x - 1} = 2$. Similarly, $h'(-2)$ does not exist.





79. Substituting $x = 1$ and $y = 1$ into $y = ax^2 + bx$ gives us $a + b = 1$ **(1)**. The slope of the tangent line $y = 3x - 2$ is 3 and the slope of the tangent to the parabola at (x, y) is $y' = 2ax + b$. At $x = 1$, $y' = 3 \quad\Rightarrow\quad 3 = 2a + b$ **(2)**. Subtracting **(1)** from **(2)** gives us $2 = a$ and it follows that $b = -1$. The parabola has equation $y = 2x^2 - x$.

80. $y = x^4 + ax^3 + bx^2 + cx + d \quad\Rightarrow\quad y(0) = d$. Since the tangent line $y = 2x + 1$ is equal to 1 at $x = 0$, we must have $d = 1$. $y' = 4x^3 + 3ax^2 + 2bx + c \quad\Rightarrow\quad y'(0) = c$. Since the slope of the tangent line $y = 2x + 1$ at $x = 0$ is 2, we must have $c = 2$. Now $y(1) = 1 + a + b + c + d = a + b + 4$ and the tangent line $y = 2 - 3x$ at $x = 1$ has y-coordinate -1, so $a + b + 4 = -1$ or $a + b = -5$ **(1)**. Also, $y'(1) = 4 + 3a + 2b + c = 3a + 2b + 6$ and the slope of the tangent line $y = 2 - 3x$ at $x = 1$ is -3, so $3a + 2b + 6 = -3$ or $3a + 2b = -9$ **(2)**. Adding -2 times **(1)** to **(2)** gives us $a = 1$ and hence, $b = -6$. The curve has equation $y = x^4 + x^3 - 6x^2 + 2x + 1$.

81. $y = f(x) = ax^2 \quad\Rightarrow\quad f'(x) = 2ax$. So the slope of the tangent to the parabola at $x = 2$ is $m = 2a(2) = 4a$. The slope of the given line, $2x + y = b \quad\Leftrightarrow\quad y = -2x + b$, is seen to be -2, so we must have $4a = -2 \quad\Leftrightarrow\quad a = -\frac{1}{2}$. So when $x = 2$, the point in question has y-coordinate $-\frac{1}{2} \cdot 2^2 = -2$. Now we simply require that the given line, whose equation is $2x + y = b$, pass through the point $(2, -2)$: $2(2) + (-2) = b \quad\Leftrightarrow\quad b = 2$. So we must have $a = -\frac{1}{2}$ and $b = 2$.

82. The slope of the curve $y = c\sqrt{x}$ is $y' = \dfrac{c}{2\sqrt{x}}$ and the slope of the tangent line $y = \frac{3}{2}x + 6$ is $\frac{3}{2}$. These must be equal at the point of tangency $\left(a, c\sqrt{a}\right)$, so $\dfrac{c}{2\sqrt{a}} = \dfrac{3}{2} \quad\Rightarrow\quad c = 3\sqrt{a}$. The y-coordinates must be equal at $x = a$, so $c\sqrt{a} = \frac{3}{2}a + 6 \quad\Rightarrow\quad \left(3\sqrt{a}\right)\sqrt{a} = \frac{3}{2}a + 6 \quad\Rightarrow\quad 3a = \frac{3}{2}a + 6 \quad\Rightarrow\quad \frac{3}{2}a = 6 \quad\Rightarrow\quad a = 4$. Since $c = 3\sqrt{a}$, we have $c = 3\sqrt{4} = 6$.

83. The line $y = 2x + 3$ has slope 2. The parabola $y = cx^2 \quad\Rightarrow\quad y' = 2cx$ has slope $2ca$ at $x = a$. Equating slopes gives us $2ca = 2$, or $ca = 1$. Equating y-coordinates at $x = a$ gives us $ca^2 = 2a + 3 \quad\Leftrightarrow\quad (ca)a = 2a + 3 \quad\Leftrightarrow\quad 1a = 2a + 3 \quad\Leftrightarrow\quad a = -3$. Thus, $c = \dfrac{1}{a} = -\dfrac{1}{3}$.

84. $f(x) = ax^2 + bx + c \quad\Rightarrow\quad f'(x) = 2ax + b$. The slope of the tangent line at $x = p$ is $2ap + b$, the slope of the tangent line at $x = q$ is $2aq + b$, and the average of those slopes is $\dfrac{(2ap + b) + (2aq + b)}{2} = ap + aq + b$. The midpoint of the interval $[p, q]$ is $\dfrac{p + q}{2}$ and the slope of the tangent line at the midpoint is $2a\left(\dfrac{p + q}{2}\right) + b = a(p + q) + b$. This is equal to $ap + aq + b$, as required.

85. f is clearly differentiable for $x < 2$ and for $x > 2$. For $x < 2$, $f'(x) = 2x$, so $f'_-(2) = 4$. For $x > 2$, $f'(x) = m$, so $f'_+(2) = m$. For f to be differentiable at $x = 2$, we need $4 = f'_-(2) = f'_+(2) = m$. So $f(x) = 4x + b$. We must also have continuity at $x = 2$, so $4 = f(2) = \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (4x + b) = 8 + b$. Hence, $b = -4$.

86. We have $g(x) = \begin{cases} ax^3 - 3x & \text{if } x \leq 1 \\ bx^2 + 2 & \text{if } x > 1 \end{cases}$

For $x < 1$, $g'(x) = a(3x^2) - 3(1) = 3ax^2 - 3$, so $g'_-(1) = 3a(1)^2 - 3 = 3a - 3$. For $x > 1$, $g'(x) = b(2x) + 0 = 2bx$, so $g'_+(1) = 2b(1) = 2b$. For g to be differentiable at $x = 1$, we need $g'_-(1) = g'_+(1)$, so $3a - 3 = 2b$, or $b = \dfrac{3a - 3}{2}$. For g to be continuous at $x = 1$, we need $g_-(1) = a - 3$ equal to $g_+(1) = b + 2$. So we have the system of two equations: $a - 3 = b + 2$, $b = \dfrac{3a - 3}{2}$. Substituting the second equation into the first equation we have $a - 3 = \dfrac{3a - 3}{2} + 2 \quad\Rightarrow\quad 2a - 6 = 3a - 3 + 4 \quad\Rightarrow\quad a = -7$ and $b = \dfrac{3(-7) - 3}{2} = -12$.

87. *Solution 1:* Let $f(x) = x^{1000}$. Then, by the definition of a derivative, $f'(1) = \lim\limits_{x \to 1} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1} \dfrac{x^{1000} - 1}{x - 1}$.

But this is just the limit we want to find, and we know (from the Power Rule) that $f'(x) = 1000x^{999}$, so

$f'(1) = 1000(1)^{999} = 1000$. So $\lim\limits_{x \to 1} \dfrac{x^{1000} - 1}{x - 1} = 1000$.

Solution 2: Note that $(x^{1000} - 1) = (x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)$. So

$$\lim_{x \to 1} \frac{x^{1000} - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)}{x - 1} = \lim_{x \to 1} (x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)$$
$$= \underbrace{1 + 1 + 1 + \cdots + 1 + 1 + 1}_{1000 \text{ ones}} = 1000, \text{ as above.}$$

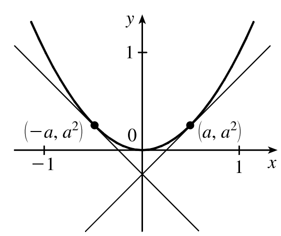
88. (a) $xy = c \quad \Rightarrow \quad y = \dfrac{c}{x}$. Let $P = \left(a, \dfrac{c}{a}\right)$. The slope of the tangent line at $x = a$ is $y'(a) = -\dfrac{c}{a^2}$. Its equation is

$y - \dfrac{c}{a} = -\dfrac{c}{a^2}(x - a)$ or $y = -\dfrac{c}{a^2}x + \dfrac{2c}{a}$, so its y-intercept is $\dfrac{2c}{a}$. Setting $y = 0$ gives $x = 2a$, so the x-intercept is $2a$.

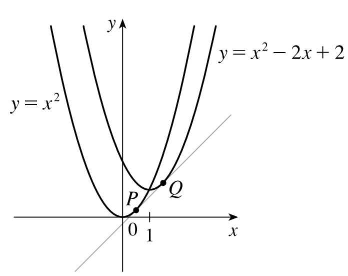
The midpoint of the line segment joining $\left(0, \dfrac{2c}{a}\right)$ and $(2a, 0)$ is $\left(a, \dfrac{c}{a}\right) = P$.

(b) We know the x- and y-intercepts of the tangent line from part (a), so the area of the triangle bounded by the axes and the tangent is $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}xy = \frac{1}{2}(2a)(2c/a) = 2c$, a constant.

89. In order for the two tangents to intersect on the y-axis, the points of tangency must be at equal distances from the y-axis, since the parabola $y = x^2$ is symmetric about the y-axis. Say the points of tangency are (a, a^2) and $(-a, a^2)$, for some $a > 0$. Then since the derivative of $y = x^2$ is $dy/dx = 2x$, the left-hand tangent has slope $-2a$ and equation $y - a^2 = -2a(x + a)$, or $y = -2ax - a^2$, and similarly the right-hand tangent line has equation $y - a^2 = 2a(x - a)$, or $y = 2ax - a^2$. So the two lines intersect at $(0, -a^2)$. Now if the lines are perpendicular, then the product of their slopes is -1, so $(-2a)(2a) = -1 \quad \Leftrightarrow \quad a^2 = \frac{1}{4} \quad \Leftrightarrow \quad a = \frac{1}{2}$. So the lines intersect at $\left(0, -\frac{1}{4}\right)$.



90.



From the sketch, it appears that there may be a line that is tangent to both curves. The slope of the line through the points $P(a, a^2)$ and $Q(b, b^2 - 2b + 2)$ is $\dfrac{b^2 - 2b + 2 - a^2}{b - a}$. The slope of the tangent line at P is $2a$ $\quad [y' = 2x]$ $\quad$ and at Q is $2b - 2$ $\quad [y' = 2x - 2]$. All three slopes are equal, so $2a = 2b - 2 \quad \Leftrightarrow \quad a = b - 1$.

Also, $2b - 2 = \dfrac{b^2 - 2b + 2 - a^2}{b - a} \quad \Rightarrow \quad 2b - 2 = \dfrac{b^2 - 2b + 2 - (b - 1)^2}{b - (b - 1)} \quad \Rightarrow \quad 2b - 2 = b^2 - 2b + 2 - b^2 + 2b - 1 \quad \Rightarrow$

$2b = 3 \quad \Rightarrow \quad b = \frac{3}{2}$ and $a = \frac{3}{2} - 1 = \frac{1}{2}$. Thus, an equation of the tangent line at P is $y - \left(\frac{1}{2}\right)^2 = 2\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)$ or

$y = x - \frac{1}{4}$.

91. $y = x^2 \quad \Rightarrow \quad y' = 2x$, so the slope of a tangent line at the point (a, a^2) is $y' = 2a$ and the slope of a normal line is $-1/(2a)$, for $a \neq 0$. The slope of the normal line through the points (a, a^2) and $(0, c)$ is $\dfrac{a^2 - c}{a - 0}$, so $\dfrac{a^2 - c}{a} = -\dfrac{1}{2a} \quad \Rightarrow$ $a^2 - c = -\frac{1}{2} \quad \Rightarrow \quad a^2 = c - \frac{1}{2}$. The last equation has two solutions if $c > \frac{1}{2}$, one solution if $c = \frac{1}{2}$, and no solution if $c < \frac{1}{2}$. Since the y-axis is normal to $y = x^2$ regardless of the value of c (this is the case for $a = 0$), we have *three* normal lines if $c > \frac{1}{2}$ and *one* normal line if $c \leq \frac{1}{2}$.

APPLIED PROJECT Building a Better Roller Coaster

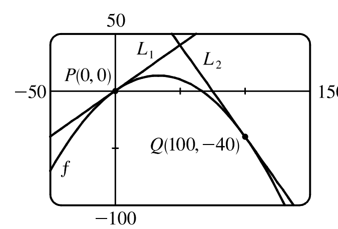
1. (a) $f(x) = ax^2 + bx + c \quad \Rightarrow \quad f'(x) = 2ax + b$.

The origin is at P:	$f(0) = 0$	$\Rightarrow$	$c = 0$
The slope of the ascent is 0.8:	$f'(0) = 0.8$	$\Rightarrow$	$b = 0.8$
The slope of the drop is -1.6:	$f'(100) = -1.6$	$\Rightarrow$	$200a + b = -1.6$

(b) $b = 0.8$, so $200a + b = -1.6 \quad \Rightarrow \quad 200a + 0.8 = -1.6 \quad \Rightarrow \quad 200a = -2.4 \quad \Rightarrow \quad a = -\dfrac{2.4}{200} = -0.012$.

Thus, $f(x) = -0.012x^2 + 0.8x$.

(c) Since L_1 passes through the origin with slope 0.8, it has equation $y = 0.8x$. The horizontal distance between P and Q is 100, so the y-coordinate at Q is $f(100) = -0.012(100)^2 + 0.8(100) = -40$. Since L_2 passes through the point $(100, -40)$ and has slope -1.6, it has equation $y + 40 = -1.6(x - 100)$ or $y = -1.6x + 120$.



(d) The difference in elevation between $P(0, 0)$ and $Q(100, -40)$ is $0 - (-40) = 40$ feet.

2. (a)

Interval	Function	First Derivative	Second Derivative
$(-\infty, 0)$	$L_1(x) = 0.8x$	$L_1'(x) = 0.8$	$L_1''(x) = 0$
$[0, 10)$	$g(x) = kx^3 + lx^2 + mx + n$	$g'(x) = 3kx^2 + 2lx + m$	$g''(x) = 6kx + 2l$
$[10, 90]$	$q(x) = ax^2 + bx + c$	$q'(x) = 2ax + b$	$q''(x) = 2a$
$(90, 100]$	$h(x) = px^3 + qx^2 + rx + s$	$h'(x) = 3px^2 + 2qx + r$	$h''(x) = 6px + 2q$
$(100, \infty)$	$L_2(x) = -1.6x + 120$	$L_2'(x) = -1.6$	$L_2''(x) = 0$

There are 4 values of x (0, 10, 90, and 100) for which we must make sure the function values are equal, the first derivative values are equal, and the second derivative values are equal. The third column in the following table contains the value of each side of the condition — these are found after solving the system in part (b).

[continued]

At $x =$	Condition	Value	Resulting Equation
0	$g(0) = L_1(0)$	0	$n = 0$
	$g'(0) = L_1'(0)$	$\frac{4}{5}$	$m = 0.8$
	$g''(0) = L_1''(0)$	0	$2l = 0$
10	$g(10) = q(10)$	$\frac{68}{9}$	$1000k + 100l + 10m + n = 100a + 10b + c$
	$g'(10) = q'(10)$	$\frac{2}{3}$	$300k + 20l + m = 20a + b$
	$g''(10) = q''(10)$	$-\frac{2}{75}$	$60k + 2l = 2a$
90	$h(90) = q(90)$	$-\frac{220}{9}$	$729{,}000p + 8100q + 90r + s = 8100a + 90b + c$
	$h'(90) = q'(90)$	$-\frac{22}{15}$	$24{,}300p + 180q + r = 180a + b$
	$h''(90) = q''(90)$	$-\frac{2}{75}$	$540p + 2q = 2a$
100	$h(100) = L_2(100)$	-40	$1{,}000{,}000p + 10{,}000q + 100r + s = -40$
	$h'(100) = L_2'(100)$	$-\frac{8}{5}$	$30{,}000p + 200q + r = -1.6$
	$h''(100) = L_2''(100)$	0	$600p + 2q = 0$

(b) We can arrange our work in a 12×12 matrix as follows.

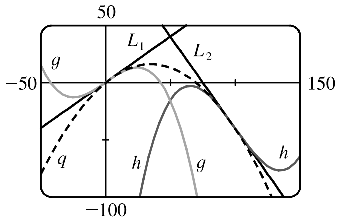
a	b	c	k	l	m	n	p	q	r	s	constant
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0.8
0	0	0	0	2	0	0	0	0	0	0	0
-100	-10	-1	1000	100	10	1	0	0	0	0	0
-20	-1	0	300	20	1	0	0	0	0	0	0
-2	0	0	60	2	0	0	0	0	0	0	0
-8100	-90	-1	0	0	0	0	729,000	8100	90	1	0
-180	-1	0	0	0	0	0	24,300	180	1	0	0
-2	0	0	0	0	0	0	540	2	0	0	0
0	0	0	0	0	0	0	1,000,000	10,000	100	1	-40
0	0	0	0	0	0	0	30,000	200	1	0	-1.6
0	0	0	0	0	0	0	600	2	0	0	0

Solving the system gives us the formulas for q, g, and h.

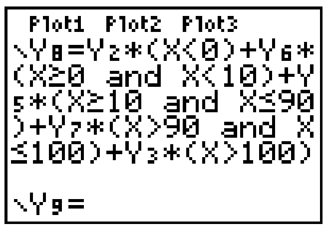
$$\left.\begin{aligned} a &= -0.01\overline{3} = -\tfrac{1}{75} \\ b &= 0.9\overline{3} = \tfrac{14}{15} \\ c &= -0.\overline{4} = -\tfrac{4}{9} \end{aligned}\right\} q(x) = -\tfrac{1}{75}x^2 + \tfrac{14}{15}x - \tfrac{4}{9} \qquad \left.\begin{aligned} k &= -0.000\overline{4} = -\tfrac{1}{2250} \\ l &= 0 \\ m &= 0.8 = \tfrac{4}{5} \\ n &= 0 \end{aligned}\right\} g(x) = -\tfrac{1}{2250}x^3 + \tfrac{4}{5}x$$

$$\left.\begin{aligned} p &= 0.000\overline{4} = \tfrac{1}{2250} \\ q &= -0.1\overline{3} = -\tfrac{2}{15} \\ r &= 11.7\overline{3} = \tfrac{176}{15} \\ s &= -324.\overline{4} = -\tfrac{2920}{9} \end{aligned}\right\} h(x) = \tfrac{1}{2250}x^3 - \tfrac{2}{15}x^2 + \tfrac{176}{15}x - \tfrac{2920}{9}$$

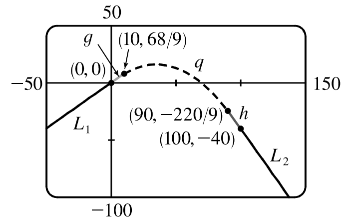
(c) Graph of L_1, q, g, h, and L_2:



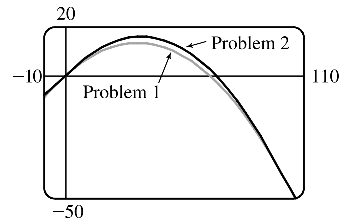
This is the piecewise-defined function assignment on a TI-83/4 Plus calculator, where $\mathrm{Y}_2 = L_1$, $\mathrm{Y}_6 = g$, $\mathrm{Y}_5 = q$, $\mathrm{Y}_7 = h$, and $\mathrm{Y}_3 = L_2$.



The graph of the five functions as a piecewise-defined function:



A comparison of the graphs in part 1(c) and part 2(c):



3.2 The Product and Quotient Rules

1. Product Rule: $f(x) = (1 + 2x^2)(x - x^2) \quad \Rightarrow$

$$f'(x) = (1 + 2x^2)(1 - 2x) + (x - x^2)(4x) = 1 - 2x + 2x^2 - 4x^3 + 4x^2 - 4x^3 = 1 - 2x + 6x^2 - 8x^3.$$

Multiplying first: $f(x) = (1 + 2x^2)(x - x^2) = x - x^2 + 2x^3 - 2x^4 \quad \Rightarrow \quad f'(x) = 1 - 2x + 6x^2 - 8x^3$ (equivalent).

2. Quotient Rule: $F(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2} = \dfrac{x^4 - 5x^3 + x^{1/2}}{x^2} \quad \Rightarrow$

$$F'(x) = \frac{x^2(4x^3 - 15x^2 + \frac{1}{2}x^{-1/2}) - (x^4 - 5x^3 + x^{1/2})(2x)}{(x^2)^2} = \frac{4x^5 - 15x^4 + \frac{1}{2}x^{3/2} - 2x^5 + 10x^4 - 2x^{3/2}}{x^4}$$

$$= \frac{2x^5 - 5x^4 - \frac{3}{2}x^{3/2}}{x^4} = 2x - 5 - \tfrac{3}{2}x^{-5/2}$$

Simplifying first: $F(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2} = x^2 - 5x + x^{-3/2} \quad \Rightarrow \quad F'(x) = 2x - 5 - \frac{3}{2}x^{-5/2}$ (equivalent).

For this problem, simplifying first seems to be the better method.

3. By the Product Rule, $y = (4x^2 + 3)(2x + 5) \quad \Rightarrow$

$$y' = (4x^2 + 3)(2x + 5)' + (2x + 5)(4x^2 + 3)' = (4x^2 + 3)(2) + (2x + 5)(8x)$$
$$= 8x^2 + 6 + 16x^2 + 40x = 24x^2 + 40x + 6$$

4. By the Product Rule, $y = (10x^2 + 7x - 2)(2 - x^2) \quad \Rightarrow$

$$y' = (10x^2 + 7x - 2)(2 - x^2)' + (2 - x^2)(10x^2 + 7x - 2)' = (10x^2 + 7x - 2)(-2x) + (2 - x^2)(20x + 7)$$
$$= -20x^3 - 14x^2 + 4x + 40x + 14 - 20x^3 - 7x^2 = -40x^3 - 21x^2 + 44x + 14$$

The notations $\overset{\text{PR}}{\Rightarrow}$ and $\overset{\text{QR}}{\Rightarrow}$ indicate the use of the Product and Quotient Rules, respectively.

5. $y = x^3 e^x \overset{\text{PR}}{\Rightarrow} y' = x^3(e^x)' + e^x(x^3)' = x^3 e^x + e^x \cdot 3x^2 = e^x(x^3 + 3x^2)$

6. $y = (e^x + 2)(2e^x - 1) \overset{\text{PR}}{\Rightarrow}$

$$y' = (e^x + 2)(2e^x - 1)' + (2e^x - 1)(e^x + 2)' = (e^x + 2)(2e^x) + (2e^x - 1)(e^x)$$
$$= 2e^{2x} + 4e^x + 2e^{2x} - e^x = 4e^{2x} + 3e^x \text{ or } e^x(4e^x + 3)$$

7. $f(x) = (3x^2 - 5x)e^x \overset{\text{PR}}{\Rightarrow}$

$$f'(x) = (3x^2 - 5x)(e^x)' + e^x(3x^2 - 5x)' = (3x^2 - 5x)e^x + e^x(6x - 5)$$
$$= e^x[(3x^2 - 5x) + (6x - 5)] = e^x(3x^2 + x - 5)$$

8. $g(x) = (x + 2\sqrt{x}\,)\, e^x \overset{\text{PR}}{\Rightarrow}$

$$g'(x) = (x + 2\sqrt{x}\,)(e^x)' + e^x(x + 2\sqrt{x}\,)' = (x + 2\sqrt{x}\,)e^x + e^x\left(1 + 2 \cdot \tfrac{1}{2}x^{-1/2}\right)$$
$$= e^x\left[(x + 2\sqrt{x}\,) + \left(1 + 1/\sqrt{x}\right)\right] = e^x\left(x + 2\sqrt{x} + 1 + 1/\sqrt{x}\right)$$

9. By the Quotient Rule, $y = \dfrac{x}{e^x} \Rightarrow y' = \dfrac{e^x(1) - x(e^x)}{(e^x)^2} = \dfrac{e^x(1 - x)}{(e^x)^2} = \dfrac{1 - x}{e^x}$.

10. By the Quotient Rule, $y = \dfrac{e^x}{1 - e^x} \Rightarrow y' = \dfrac{(1 - e^x)e^x - e^x(-e^x)}{(1 - e^x)^2} = \dfrac{e^x - e^{2x} + e^{2x}}{(1 - e^x)^2} = \dfrac{e^x}{(1 - e^x)^2}$.

11. $g(t) = \dfrac{3 - 2t}{5t + 1} \overset{\text{QR}}{\Rightarrow} g'(t) = \dfrac{(5t + 1)(-2) - (3 - 2t)(5)}{(5t + 1)^2} = \dfrac{-10t - 2 - 15 + 10t}{(5t + 1)^2} = -\dfrac{17}{(5t + 1)^2}$

12. $G(u) = \dfrac{6u^4 - 5u}{u + 1} \overset{\text{QR}}{\Rightarrow}$

$$G'(u) = \frac{(u + 1)(24u^3 - 5) - (6u^4 - 5u)(1)}{(u + 1)^2} = \frac{24u^4 - 5u + 24u^3 - 5 - 6u^4 + 5u}{(u + 1)^2} = \frac{18u^4 + 24u^3 - 5}{(u + 1)^2}$$

13. $f(t) = \dfrac{5t}{t^3 - t - 1} \overset{\text{QR}}{\Rightarrow} f'(t) = \dfrac{(t^3 - t - 1)(5) - (5t)(3t^2 - 1)}{(t^3 - t - 1)^2} = \dfrac{5t^3 - 5t - 5 - 15t^3 + 5t}{(t^3 - t - 1)^2} = -\dfrac{10t^3 + 5}{(t^3 - t - 1)^2}$

14. $F(x) = \dfrac{1}{2x^3 - 6x^2 + 5} \overset{\text{QR}}{\Rightarrow} F'(x) = \dfrac{(2x^3 - 6x^2 + 5)(0) - 1(6x^2 - 12x)}{(2x^3 - 6x^2 + 5)^2} = -\dfrac{6x^2 - 12x}{(2x^3 - 6x^2 + 5)^2}$

15. $y = \dfrac{s - \sqrt{s}}{s^2} = \dfrac{s}{s^2} - \dfrac{\sqrt{s}}{s^2} = s^{-1} - s^{-3/2} \Rightarrow y' = -s^{-2} + \tfrac{3}{2}s^{-5/2} = \dfrac{-1}{s^2} + \dfrac{3}{2s^{5/2}} = \dfrac{3 - 2\sqrt{s}}{2s^{5/2}}$

16. $y = \dfrac{\sqrt{x}}{\sqrt{x} + 1} \overset{\text{QR}}{\Rightarrow} y' = \dfrac{\left(\sqrt{x} + 1\right)\left(\dfrac{1}{2\sqrt{x}}\right) - \sqrt{x}\left(\dfrac{1}{2\sqrt{x}}\right)}{\left(\sqrt{x} + 1\right)^2} = \dfrac{\dfrac{1}{2} + \dfrac{1}{2\sqrt{x}} - \dfrac{1}{2}}{\left(\sqrt{x} + 1\right)^2} = \dfrac{1}{2\sqrt{x}\left(\sqrt{x} + 1\right)^2}$

17. $J(u) = \left(\frac{1}{u} + \frac{1}{u^2}\right)\left(u + \frac{1}{u}\right) = (u^{-1} + u^{-2})(u + u^{-1}) \overset{\text{PR}}{\Rightarrow}$

$$J'(u) = (u^{-1} + u^{-2})(u + u^{-1})' + (u + u^{-1})(u^{-1} + u^{-2})' = (u^{-1} + u^{-2})(1 - u^{-2}) + (u + u^{-1})(-u^{-2} - 2u^{-3})$$
$$= u^{-1} - u^{-3} + u^{-2} - u^{-4} - u^{-1} - 2u^{-2} - u^{-3} - 2u^{-4} = -u^{-2} - 2u^{-3} - 3u^{-4} = -\left(\frac{1}{u^2} + \frac{2}{u^3} + \frac{3}{u^4}\right)$$

18. $h(w) = (w^2 + 3w)(w^{-1} - w^{-4}) \overset{\text{PR}}{\Rightarrow}$

$$h'(w) = (w^2 + 3w)(-w^{-2} + 4w^{-5}) + (w^{-1} - w^{-4})(2w + 3)$$
$$= -1 + 4w^{-3} - 3w^{-1} + 12w^{-4} + 2 + 3w^{-1} - 2w^{-3} - 3w^{-4} = 1 + 2w^{-3} + 9w^{-4}$$

Alternate solution: An easier method is to simplify first and then differentiate as follows:

$$h(w) = (w^2 + 3w)(w^{-1} - w^{-4}) = w - w^{-2} + 3 - 3w^{-3} \quad \Rightarrow \quad h'(w) = 1 + 2w^{-3} + 9w^{-4}$$

19. $H(u) = (u - \sqrt{u}\,)(u + \sqrt{u}\,) \overset{\text{PR}}{\Rightarrow}$

$$H'(u) = (u - \sqrt{u}\,)\left(1 + \frac{1}{2\sqrt{u}}\right) + (u + \sqrt{u}\,)\left(1 - \frac{1}{2\sqrt{u}}\right) = u + \tfrac{1}{2}\sqrt{u} - \sqrt{u} - \tfrac{1}{2} + u - \tfrac{1}{2}\sqrt{u} + \sqrt{u} - \tfrac{1}{2} = 2u - 1.$$

Alternate solution: An easier method is to simplify first and then differentiate as follows:

$$H(u) = (u - \sqrt{u}\,)(u + \sqrt{u}\,) = u^2 - (\sqrt{u}\,)^2 = u^2 - u \quad \Rightarrow \quad H'(u) = 2u - 1$$

20. $f(z) = (1 - e^z)(z + e^z) \overset{\text{PR}}{\Rightarrow}$

$$f'(z) = (1 - e^z)(1 + e^z) + (z + e^z)(-e^z) = 1^2 - (e^z)^2 - ze^z - (e^z)^2 = 1 - ze^z - 2e^{2z}$$

21. $V(t) = (t + 2e^t)\sqrt{t} \overset{\text{PR}}{\Rightarrow}$

$$V'(t) = (t + 2e^t)\frac{1}{2\sqrt{t}} + \sqrt{t}\,(1 + 2e^t) = \frac{t}{2\sqrt{t}} + \frac{2e^t}{2\sqrt{t}} + \sqrt{t} + 2\sqrt{t}\,e^t = \frac{t + 2e^t + 2t + 4te^t}{2\sqrt{t}} = \frac{3t + 2e^t + 4te^t}{2\sqrt{t}}$$

22. $W(t) = e^t(1 + te^t) \overset{\text{PR}}{\Rightarrow}$

$$W'(t) = e^t[0 + (te^t + e^t(1))] + (1 + te^t)(e^t) \qquad \text{[factor out } e^t\text{]}$$
$$= e^t(te^t + e^t + 1 + te^t) = e^t(1 + e^t + 2te^t)$$

23. $y = e^p(p + p\sqrt{p}\,) = e^p(p + p^{3/2}) \overset{\text{PR}}{\Rightarrow} \quad y' = e^p\left(1 + \frac{3}{2}p^{1/2}\right) + (p + p^{3/2})e^p = e^p\left(1 + \frac{3}{2}\sqrt{p} + p + p\sqrt{p}\,\right)$

24. $h(r) = \dfrac{ae^r}{b + e^r} \overset{\text{QR}}{\Rightarrow} \quad h'(r) = \dfrac{(b + e^r)(ae^r) - (ae^r)(e^r)}{(b + e^r)^2} = \dfrac{abe^r + ae^{2r} - ae^{2r}}{(b + e^r)^2} = \dfrac{abe^r}{(b + e^r)^2}$

25. $f(t) = \dfrac{\sqrt[3]{t}}{t - 3} \overset{\text{QR}}{\Rightarrow}$

$$f'(t) = \frac{(t - 3)\left(\frac{1}{3}t^{-2/3}\right) - t^{1/3}(1)}{(t - 3)^2} = \frac{\frac{1}{3}t^{1/3} - t^{-2/3} - t^{1/3}}{(t - 3)^2} = \frac{-\frac{2}{3}t^{1/3} - t^{-2/3}}{(t - 3)^2}$$
$$= \frac{\dfrac{-2t}{3t^{2/3}} - \dfrac{3}{3t^{2/3}}}{(t - 3)^2} = \frac{-2t - 3}{3t^{2/3}(t - 3)^2}$$

26. $y = (z^2 + e^z)\sqrt{z} \overset{\text{PR}}{\Rightarrow}$

$$y' = (z^2 + e^z)\left(\frac{1}{2\sqrt{z}}\right) + \sqrt{z}\,(2z + e^z) = \frac{z^2}{2\sqrt{z}} + \frac{e^z}{2\sqrt{z}} + 2z\sqrt{z} + \sqrt{z}\,e^z$$

$$= \frac{z^2 + e^z + 4z^2 + 2ze^z}{2\sqrt{z}} = \frac{5z^2 + e^z + 2ze^z}{2\sqrt{z}}$$

27. $f(x) = \dfrac{x^2 e^x}{x^2 + e^x} \overset{\text{QR}}{\Rightarrow}$

$$f'(x) = \frac{(x^2 + e^x)\left[x^2 e^x + e^x(2x)\right] - x^2 e^x(2x + e^x)}{(x^2 + e^x)^2} = \frac{x^4 e^x + 2x^3 e^x + x^2 e^{2x} + 2xe^{2x} - 2x^3 e^x - x^2 e^{2x}}{(x^2 + e^x)^2}$$

$$= \frac{x^4 e^x + 2xe^{2x}}{(x^2 + e^x)^2} = \frac{xe^x(x^3 + 2e^x)}{(x^2 + e^x)^2}$$

28. $F(t) = \dfrac{At}{Bt^2 + Ct^3} = \dfrac{A}{Bt + Ct^2} \overset{\text{QR}}{\Rightarrow}$

$$F'(t) = \frac{(Bt + Ct^2)(0) - A(B + 2Ct)}{(Bt + Ct^2)^2} = \frac{-A(B + 2Ct)}{(t)^2(B + Ct)^2} = -\frac{A(B + 2Ct)}{t^2(B + Ct)^2}$$

29. $f(x) = \dfrac{x}{x + \dfrac{c}{x}} \overset{\text{QR}}{\Rightarrow} f'(x) = \dfrac{(x + c/x)(1) - x(1 - c/x^2)}{\left(x + \dfrac{c}{x}\right)^2} = \dfrac{x + c/x - x + c/x}{\left(\dfrac{x^2 + c}{x}\right)^2} = \dfrac{2c/x}{\dfrac{(x^2 + c)^2}{x^2}} \cdot \dfrac{x^2}{x^2} = \dfrac{2cx}{(x^2 + c)^2}$

30. $f(x) = \dfrac{ax + b}{cx + d} \overset{\text{QR}}{\Rightarrow} f'(x) = \dfrac{(cx + d)(a) - (ax + b)(c)}{(cx + d)^2} = \dfrac{acx + ad - acx - bc}{(cx + d)^2} = \dfrac{ad - bc}{(cx + d)^2}$

31. $f(x) = x^2 e^x \overset{\text{PR}}{\Rightarrow} f'(x) = x^2 e^x + e^x(2x) = e^x(x^2 + 2x)$

Using the Product Rule and $f'(x) = e^x(x^2 + 2x)$, we get

$$f''(x) = e^x(2x + 2) + (x^2 + 2x)e^x = e^x(2x + 2 + x^2 + 2x) = e^x(x^2 + 4x + 2)$$

32. $f(x) = \sqrt{x}\,e^x \overset{\text{PR}}{\Rightarrow} f'(x) = \sqrt{x}\,e^x + e^x\left(\dfrac{1}{2\sqrt{x}}\right) = \left(\sqrt{x} + \dfrac{1}{2\sqrt{x}}\right)e^x = \dfrac{2x + 1}{2\sqrt{x}}\,e^x.$

Using the Product Rule and $f'(x) = \left(x^{1/2} + \frac{1}{2}x^{-1/2}\right)e^x$, we get

$$f''(x) = \left(x^{1/2} + \tfrac{1}{2}x^{-1/2}\right)e^x + e^x\left(\tfrac{1}{2}x^{-1/2} - \tfrac{1}{4}x^{-3/2}\right) = \left(x^{1/2} + x^{-1/2} - \tfrac{1}{4}x^{-3/2}\right)e^x = \frac{4x^2 + 4x - 1}{4x^{3/2}}\,e^x$$

33. $f(x) = \dfrac{x}{x^2 - 1} \overset{\text{QR}}{\Rightarrow} f'(x) = \dfrac{(x^2 - 1)(1) - x(2x)}{(x^2 - 1)^2} = \dfrac{x^2 - 1 - 2x^2}{(x^2 - 1)^2} = \dfrac{-x^2 - 1}{(x^2 - 1)^2} \Rightarrow$

$$f''(x) = \frac{(x^2 - 1)^2(-2x) - (-x^2 - 1)(x^4 - 2x^2 + 1)'}{[(x^2 - 1)^2]^2} = \frac{(x^2 - 1)^2(-2x) + (x^2 + 1)(4x^3 - 4x)}{(x^2 - 1)^4}$$

$$= \frac{(x^2 - 1)^2(-2x) + (x^2 + 1)(4x)(x^2 - 1)}{(x^2 - 1)^4} = \frac{(x^2 - 1)[(x^2 - 1)(-2x) + (x^2 + 1)(4x)]}{(x^2 - 1)^4}$$

$$= \frac{-2x^3 + 2x + 4x^3 + 4x}{(x^2 - 1)^3} = \frac{2x^3 + 6x}{(x^2 - 1)^3}$$

34. $f(x) = \dfrac{x}{1+\sqrt{x}} \overset{\text{QR}}{\Rightarrow}$

$$f'(x) = \frac{\left(1+\sqrt{x}\right)(1) - x\left(\dfrac{1}{2\sqrt{x}}\right)}{\left(1+\sqrt{x}\right)^2} = \frac{1+\sqrt{x}-\frac{1}{2}\sqrt{x}}{1+2\sqrt{x}+x} = \frac{1+\frac{1}{2}\sqrt{x}}{1+2\sqrt{x}+x} = \frac{\dfrac{2+\sqrt{x}}{2}}{1+2\sqrt{x}+x}$$

$$= \frac{2+\sqrt{x}}{2\left(1+2\sqrt{x}+x\right)} = \frac{2+\sqrt{x}}{2+4\sqrt{x}+2x}$$

Using the Quotient Rule and $f'(x) = \dfrac{2+\sqrt{x}}{2+4\sqrt{x}+2x}$, we get

$$f''(x) = \frac{\left(2+4\sqrt{x}+2x\right)\left(\dfrac{1}{2\sqrt{x}}\right) - \left(2+\sqrt{x}\right)\left(\dfrac{4}{2\sqrt{x}}+2\right)}{\left(2+4\sqrt{x}+2x\right)^2}$$

$$= \frac{\dfrac{1}{\sqrt{x}} + 2 + \sqrt{x} - \dfrac{4}{\sqrt{x}} - 4 - 2 - 2\sqrt{x}}{\left(2+4\sqrt{x}+2x\right)^2}$$

$$= \frac{\dfrac{-3-4\sqrt{x}-x}{\sqrt{x}}}{\left(2+4\sqrt{x}+2x\right)^2} = -\frac{3+4\sqrt{x}+x}{\sqrt{x}\left(2+4\sqrt{x}2x\right)^2}$$

35. $y = \dfrac{x^2}{1+x} \quad\Rightarrow\quad y' = \dfrac{(1+x)(2x)-x^2(1)}{(1+x)^2} = \dfrac{2x+2x^2-x^2}{(1+x)^2} = \dfrac{x^2+2x}{(1+x)^2}$

At $\left(1,\frac{1}{2}\right)$, $y' = \dfrac{1^2+2\,(1)}{(1+1)^2} = \dfrac{3}{4}$, and an equation of the tangent line is $y - \frac{1}{2} = \frac{3}{4}(x-1)$, or $y = \frac{3}{4}x - \frac{1}{4}$.

36. $y = \dfrac{1+x}{1+e^x} \quad\Rightarrow\quad y' = \dfrac{(1+e^x)(1)-(1+x)e^x}{(1+e^x)^2} = \dfrac{1+e^x-e^x-xe^x}{(1+e^x)^2} = \dfrac{1-xe^x}{(1+e^x)^2}$

At $\left(0,\frac{1}{2}\right)$, $y' = \dfrac{1}{(1+1)^2} = \dfrac{1}{4}$, and an equation of the tangent line is $y - \frac{1}{2} = \frac{1}{4}(x-0)$ or $y = \frac{1}{4}x + \frac{1}{2}$.

37. $y = \dfrac{3x}{1+5x^2} \quad\Rightarrow\quad y' = \dfrac{(1+5x^2)(3)-3x(10x)}{(1+5x^2)^2} = \dfrac{3+15x^2-30x^2}{(1+5x^2)^2} = \dfrac{3-15x^2}{(1+5x^2)^2}$

At $\left(1,\frac{1}{2}\right)$, $y' = \dfrac{3-15(1^2)}{(1+5\cdot 1^2)^2} = \dfrac{-12}{6^2} = -\dfrac{1}{3}$, and an equation of the tangent line is $y - \frac{1}{2} = -\frac{1}{3}(x-1)$, or $y = -\frac{1}{3}x + \frac{5}{6}$.

The slope of the normal line is 3, so an equation of the normal line is $y - \frac{1}{2} = 3(x-1)$, or $y = 3x - \frac{5}{2}$.

38. $y = x + xe^x \quad\Rightarrow\quad y' = 1 + (xe^x + e^x\cdot 1) = 1 + e^x(x+1)$

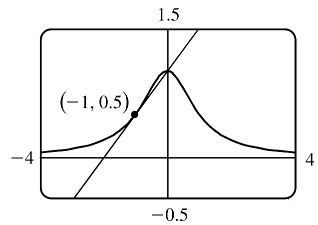
At $(0,0)$, $y' = 1 + e^0(0+1) = 1 + 1\cdot 1 = 2$, and an equation of the tangent line is $y - 0 = 2(x-0)$, or $y = 2x$.

The slope of the normal line is $-\frac{1}{2}$, so an equation of the normal line is $y - 0 = -\frac{1}{2}(x-0)$, or $y = -\frac{1}{2}x$.

39. (a) $y = f(x) = \dfrac{1}{1+x^2} \quad \Rightarrow$

$f'(x) = \dfrac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2} = \dfrac{-2x}{(1+x^2)^2}$. So the slope of the tangent line at the point $\left(-1, \frac{1}{2}\right)$ is $f'(-1) = \dfrac{2}{2^2} = \frac{1}{2}$ and its equation is $y - \frac{1}{2} = \frac{1}{2}(x+1)$ or $y = \frac{1}{2}x + 1$.

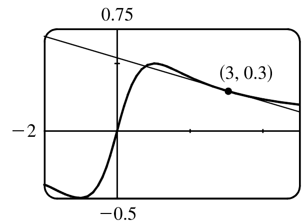
(b)



40. (a) $y = f(x) = \dfrac{x}{1+x^2} \quad \Rightarrow$

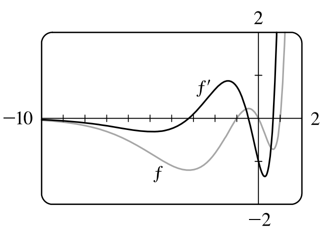
$f'(x) = \dfrac{(1+x^2)1 - x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$. So the slope of the tangent line at the point $(3, 0.3)$ is $f'(3) = \frac{-8}{100}$ and its equation is $y - 0.3 = -0.08(x-3)$ or $y = -0.08x + 0.54$.

(b)



41. (a) $f(x) = (x^3 - x)e^x \quad \Rightarrow \quad f'(x) = (x^3 - x)e^x + e^x(3x^2 - 1) = e^x(x^3 + 3x^2 - x - 1)$

(b)



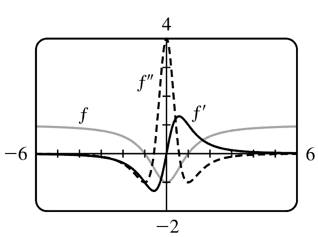
$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

42. (a) $f(x) = \dfrac{x^2 - 1}{x^2 + 1} \quad \Rightarrow$

$$f'(x) = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} = \frac{(2x)[(x^2+1) - (x^2-1)]}{(x^2+1)^2} = \frac{(2x)(2)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2} \quad \Rightarrow$$

$$\begin{aligned} f''(x) &= \frac{(x^2+1)^2(4) - 4x(x^4 + 2x^2 + 1)'}{[(x^2+1)^2]^2} \\ &= \frac{4(x^2+1)^2 - 4x(4x^3 + 4x)}{(x^2+1)^4} = \frac{4(x^2+1)^2 - 16x^2(x^2+1)}{(x^2+1)^4} \\ &= \frac{4(x^2+1)[(x^2+1) - 4x^2]}{(x^2+1)^4} = \frac{4(1-3x^2)}{(x^2+1)^3} \end{aligned}$$

(b)



$f' = 0$ when f has a horizontal tangent and $f'' = 0$ when f' has a horizontal tangent. f' is negative when f is decreasing and positive when f is increasing. f'' is negative when f' is decreasing and positive when f' is increasing. f'' is negative when f is concave down and positive when f is concave up.

43. $f(x) = \dfrac{x^2}{1+x} \quad \Rightarrow \quad f'(x) = \dfrac{(1+x)(2x) - x^2(1)}{(1+x)^2} = \dfrac{2x + 2x^2 - x^2}{(1+x)^2} = \dfrac{x^2 + 2x}{x^2 + 2x + 1} \quad \Rightarrow$

$$f''(x) = \frac{(x^2+2x+1)(2x+2) - (x^2+2x)(2x+2)}{(x^2+2x+1)^2} = \frac{(2x+2)(x^2+2x+1-x^2-2x)}{[(x+1)^2]^2}$$

$$= \frac{2(x+1)(1)}{(x+1)^4} = \frac{2}{(x+1)^3},$$

so $f''(1) = \dfrac{2}{(1+1)^3} = \dfrac{2}{8} = \dfrac{1}{4}$.

44. $g(x) = \dfrac{x}{e^x} \quad \Rightarrow \quad g'(x) = \dfrac{e^x \cdot 1 - x \cdot e^x}{(e^x)^2} = \dfrac{e^x(1-x)}{(e^x)^2} = \dfrac{1-x}{e^x} \quad \Rightarrow$

$$g''(x) = \frac{e^x \cdot (-1) - (1-x)e^x}{(e^x)^2} = \frac{e^x[-1-(1-x)]}{(e^x)^2} = \frac{x-2}{e^x} \quad \Rightarrow$$

$$g'''(x) = \frac{e^x \cdot 1 - (x-2)e^x}{(e^x)^2} = \frac{e^x[1-(x-2)]}{(e^x)^2} = \frac{3-x}{e^x} \quad \Rightarrow$$

$$g^{(4)}(x) = \frac{e^x \cdot (-1) - (3-x)e^x}{(e^x)^2} = \frac{e^x[-1-(3-x)]}{(e^x)^2} = \frac{x-4}{e^x}.$$

The pattern suggests that $g^{(n)}(x) = \dfrac{(x-n)(-1)^n}{e^x}$. (We could use mathematical induction to prove this formula.)

45. We are given that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$.

(a) $(fg)'(5) = f(5)g'(5) + g(5)f'(5) = (1)(2) + (-3)(6) = 2 - 18 = -16$

(b) $\left(\dfrac{f}{g}\right)'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{(-3)(6) - (1)(2)}{(-3)^2} = -\dfrac{20}{9}$

(c) $\left(\dfrac{g}{f}\right)'(5) = \dfrac{f(5)g'(5) - g(5)f'(5)}{[f(5)]^2} = \dfrac{(1)(2) - (-3)(6)}{(1)^2} = 20$

46. We are given that $f(4) = 2$, $g(4) = 5$, $f'(4) = 6$, and $g'(4) = -3$.

(a) $h(x) = 3f(x) + 8g(x) \quad \Rightarrow \quad h'(x) = 3f'(x) + 8g'(x)$, so

$h'(4) = 3f'(4) + 8g'(4) = 3(6) + 8(-3) = 18 - 24 = -6$.

(b) $h(x) = f(x)\,g(x) \quad \Rightarrow \quad h'(x) = f(x)\,g'(x) + g(x)\,f'(x)$, so

$h'(4) = f(4)\,g'(4) + g(4)\,f'(4) = 2(-3) + 5(6) = -6 + 30 = 24$.

(c) $h(x) = \dfrac{f(x)}{g(x)} \quad \Rightarrow \quad h'(x) = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2}$, so

$h'(4) = \dfrac{g(4)\,f'(4) - f(4)\,g'(4)}{[g(4)]^2} = \dfrac{5(6) - 2(-3)}{5^2} = \dfrac{30+6}{25} = \dfrac{36}{25}$.

(d) $h(x) = \dfrac{g(x)}{f(x) + g(x)} \quad \Rightarrow$

$$h'(4) = \frac{[f(4)+g(4)]\,g'(4) - g(4)\,[f'(4)+g'(4)]}{[f(4)+g(4)]^2} = \frac{(2+5)\,(-3) - 5\,[6+(-3)]}{(2+5)^2} = \frac{-21-15}{7^2} = -\frac{36}{49}$$

47. $f(x) = e^x g(x) \quad \Rightarrow \quad f'(x) = e^x g'(x) + g(x)e^x = e^x[g'(x) + g(x)]$. $f'(0) = e^0[g'(0) + g(0)] = 1(5 + 2) = 7$

48. $\dfrac{d}{dx}\left[\dfrac{h(x)}{x}\right] = \dfrac{xh'(x) - h(x)\cdot 1}{x^2} \quad \Rightarrow \quad \dfrac{d}{dx}\left[\dfrac{h(x)}{x}\right]_{x=2} = \dfrac{2h'(2) - h(2)}{2^2} = \dfrac{2(-3) - (4)}{4} = \dfrac{-10}{4} = -2.5$

49. $g(x) = xf(x) \quad \Rightarrow \quad g'(x) = xf'(x) + f(x)\cdot 1$. Now $g(3) = 3f(3) = 3\cdot 4 = 12$ and $g'(3) = 3f'(3) + f(3) = 3(-2) + 4 = -2$. Thus, an equation of the tangent line to the graph of g at the point where $x = 3$ is $y - 12 = -2(x - 3)$, or $y = -2x + 18$.

50. $f'(x) = x^2 f(x) \quad \Rightarrow \quad f''(x) = x^2 f'(x) + f(x)\cdot 2x$. Now $f'(2) = 2^2 f(2) = 4(10) = 40$, so $f''(2) = 2^2(40) + 10(4) = 200$.

51. (a) From the graphs of f and g, we obtain the following values: $f(1) = 2$ since the point $(1, 2)$ is on the graph of f; $g(1) = 3$ since the point $(1, 3)$ is on the graph of g; $f'(1) = \frac{1}{3}$ since the slope of the line segment between $(1, 2)$ and $(4, 3)$ is $\dfrac{3-2}{4-1} = \dfrac{1}{3}$; $g'(1) = 1$ since the slope of the line segment between $(-2, 0)$ and $(2, 4)$ is $\dfrac{4-0}{2-(-2)} = \dfrac{4}{4} = 1$.

Now $u(x) = f(x)\,g(x)$, so $u'(1) = f(1)\,g'(1) + g(1)\,f'(1) = 2\cdot 1 + 3\cdot \frac{1}{3} = 3$.

(b) From the graphs of f and g, we obtain the following values: $f(4) = 3$ since the point $(4, 3)$ is on the graph of f; $g(4) = 2$ since the point $(4, 2)$ is on the graph of g; $f'(4) = f'(1) = \frac{1}{3}$ from the part (a); $g'(4) = 1$ since the slope of the line segment between $(3, 1)$ and $(5, 3)$ is $\dfrac{3-1}{5-3} = \dfrac{2}{2} = 1$.

$v(x) = \dfrac{f(x)}{g(x)}$, so $v'(4) = \dfrac{g(4)f'(4) - f(4)g'(4)}{[g(4)]^2} = \dfrac{2\cdot\frac{1}{3} - 3\cdot 1}{2^2} = \dfrac{-\frac{7}{3}}{4} = -\dfrac{7}{12}$

52. (a) $P(x) = F(x)\,G(x)$, so $P'(2) = F(2)\,G'(2) + G(2)\,F'(2) = 3\cdot\frac{2}{4} + 2\cdot 0 = \frac{3}{2}$.

(b) $Q(x) = \dfrac{F(x)}{G(x)}$, so $Q'(7) = \dfrac{G(7)\,F'(7) - F(7)\,G'(7)}{[G(7)]^2} = \dfrac{1\cdot\frac{1}{4} - 5\cdot\left(-\frac{2}{3}\right)}{1^2} = \dfrac{1}{4} + \dfrac{10}{3} = \dfrac{43}{12}$

53. (a) $y = xg(x) \quad \Rightarrow \quad y' = xg'(x) + g(x)\cdot 1 = xg'(x) + g(x)$

(b) $y = \dfrac{x}{g(x)} \quad \Rightarrow \quad y' = \dfrac{g(x)\cdot 1 - xg'(x)}{[g(x)]^2} = \dfrac{g(x) - xg'(x)}{[g(x)]^2}$

(c) $y = \dfrac{g(x)}{x} \quad \Rightarrow \quad y' = \dfrac{xg'(x) - g(x)\cdot 1}{(x)^2} = \dfrac{xg'(x) - g(x)}{x^2}$

54. (a) $y = x^2 f(x) \quad \Rightarrow \quad y' = x^2 f'(x) + f(x)(2x)$

(b) $y = \dfrac{f(x)}{x^2} \quad \Rightarrow \quad y' = \dfrac{x^2 f'(x) - f(x)(2x)}{(x^2)^2} = \dfrac{xf'(x) - 2f(x)}{x^3}$

(c) $y = \dfrac{x^2}{f(x)} \quad \Rightarrow \quad y' = \dfrac{f(x)(2x) - x^2 f'(x)}{[f(x)]^2}$

(d) $y = \dfrac{1 + xf(x)}{\sqrt{x}} \quad \Rightarrow$

$$y' = \frac{\sqrt{x}\,[xf'(x) + f(x)] - [1 + xf(x)]\,\dfrac{1}{2\sqrt{x}}}{\left(\sqrt{x}\,\right)^2}$$

$$= \frac{x^{3/2}f'(x) + x^{1/2}f(x) - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}f(x)}{x} \cdot \frac{2x^{1/2}}{2x^{1/2}} = \frac{xf(x) + 2x^2f'(x) - 1}{2x^{3/2}}$$

55. If $y = f(x) = \dfrac{x}{x+1}$, then $f'(x) = \dfrac{(x+1)(1) - x(1)}{(x+1)^2} = \dfrac{1}{(x+1)^2}$. When $x = a$, the equation of the tangent line is

$y - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(x-a)$. This line passes through $(1, 2)$ when $2 - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(1-a) \quad \Leftrightarrow$

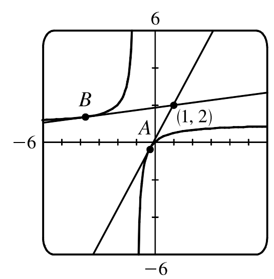
$2(a+1)^2 - a(a+1) = 1 - a \quad \Leftrightarrow \quad 2a^2 + 4a + 2 - a^2 - a - 1 + a = 0 \quad \Leftrightarrow \quad a^2 + 4a + 1 = 0.$

The quadratic formula gives the solutions of this equation as $a = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} = \dfrac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$,

so there are two such tangent lines. Since

$$f\left(-2 \pm \sqrt{3}\,\right) = \frac{-2 \pm \sqrt{3}}{-2 \pm \sqrt{3} + 1} = \frac{-2 \pm \sqrt{3}}{-1 \pm \sqrt{3}} \cdot \frac{-1 \mp \sqrt{3}}{-1 \mp \sqrt{3}}$$

$$= \frac{2 \pm 2\sqrt{3} \mp \sqrt{3} - 3}{1 - 3} = \frac{-1 \pm \sqrt{3}}{-2} = \frac{1 \mp \sqrt{3}}{2},$$



the lines touch the curve at $A\left(-2 + \sqrt{3}, \frac{1 - \sqrt{3}}{2}\right) \approx (-0.27, -0.37)$

and $B\left(-2 - \sqrt{3}, \frac{1 + \sqrt{3}}{2}\right) \approx (-3.73, 1.37)$.

56. $y = \dfrac{x-1}{x+1} \quad \Rightarrow \quad y' = \dfrac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$. If the tangent intersects

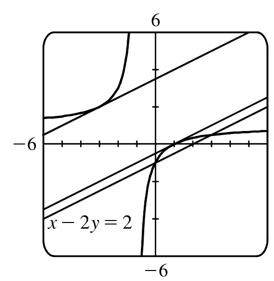
the curve when $x = a$, then its slope is $2/(a+1)^2$. But if the tangent is parallel to

$x - 2y = 2$, that is, $y = \frac{1}{2}x - 1$, then its slope is $\frac{1}{2}$. Thus, $\dfrac{2}{(a+1)^2} = \dfrac{1}{2} \quad \Rightarrow$

$(a+1)^2 = 4 \quad \Rightarrow \quad a + 1 = \pm 2 \quad \Rightarrow \quad a = 1$ or -3. When $a = 1$, $y = 0$ and the

equation of the tangent is $y - 0 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x - \frac{1}{2}$. When $a = -3$, $y = 2$ and

the equation of the tangent is $y - 2 = \frac{1}{2}(x + 3)$ or $y = \frac{1}{2}x + \frac{7}{2}$.



57. $R = \dfrac{f}{g} \quad \Rightarrow \quad R' = \dfrac{gf' - fg'}{g^2}$. For $f(x) = x - 3x^3 + 5x^5$, $f'(x) = 1 - 9x^2 + 25x^4$,

and for $g(x) = 1 + 3x^3 + 6x^6 + 9x^9$, $g'(x) = 9x^2 + 36x^5 + 81x^8$.

Thus, $R'(0) = \dfrac{g(0)f'(0) - f(0)g'(0)}{[g(0)]^2} = \dfrac{1 \cdot 1 - 0 \cdot 0}{1^2} = \dfrac{1}{1} = 1$.

58. $Q = \dfrac{f}{g} \quad \Rightarrow \quad Q' = \dfrac{gf' - fg'}{g^2}$. For $f(x) = 1 + x + x^2 + xe^x$, $f'(x) = 1 + 2x + xe^x + e^x$,

and for $g(x) = 1 - x + x^2 - xe^x$, $g'(x) = -1 + 2x - xe^x - e^x$.

Thus, $Q'(0) = \dfrac{g(0)f'(0) - f(0)g'(0)}{[g(0)]^2} = \dfrac{1 \cdot 2 - 1 \cdot (-2)}{1^2} = \dfrac{4}{1} = 4$.

59. If $P(t)$ denotes the population at time t and $A(t)$ denotes the average annual income, then $T(t) = P(t)\,A(t)$ is the total personal income. The rate at which $T(t)$ is rising is given by $T'(t) = P(t)\,A'(t) + A(t)\,P'(t) \quad \Rightarrow$

$$\begin{aligned} T'(2015) &= P(2015)\,A'(2015) + A(2015)\,P'(2015) = (107{,}350)\,(\$2250/\text{year}) + (\$60{,}220)\,(1960/\text{year}) \\ &= \$241{,}537{,}500/\text{year} + \$118{,}031{,}200/\text{year} = \$359{,}568{,}700/\text{year} \end{aligned}$$

So the total personal income in Boulder was rising by about \$360 million per year in 2015.

The term $P(t)\,A'(t) \approx \$242$ million represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)\,P'(t) \approx \$118$ million represents the portion of the rate of change of total income due to increasing population.

60. (a) $f(20) = 10{,}000$ means that when the price of the fabric is \$20/meter, 10,000 meters will be sold.

$f'(20) = -350$ means that as the price of the fabric increases past \$20/meter, the amount of fabric which will be sold is decreasing at a rate of 350 meters per (dollar per meter).

(b) $R(p) = pf(p) \quad \Rightarrow \quad R'(p) = pf'(p) + f(p) \cdot 1 \quad \Rightarrow \quad R'(20) = 20f'(20) + f(20) \cdot 1 = 20(-350) + 10{,}000 = 3000$.

This means that as the price of the fabric increases past \$20/meter, the total revenue is increasing at \$3000/(\$/meter). Note that the Product Rule indicates that we will lose \$7000/(\$/meter) due to selling less fabric, but this loss is more than made up for by the additional revenue due to the increase in price.

61. $v = \dfrac{0.14[\text{S}]}{0.015 + [\text{S}]} \quad \Rightarrow \quad \dfrac{dv}{d[\text{S}]} = \dfrac{(0.015 + [\text{S}])(0.14) - (0.14[\text{S}])(1)}{(0.015 + [\text{S}])^2} = \dfrac{0.0021}{(0.015 + [\text{S}])^2}$.

$dv/d[\text{S}]$ represents the rate of change of the rate of an enzymatic reaction with respect to the concentration of a substrate S.

62. $B(t) = N(t)\,M(t) \quad \Rightarrow \quad B'(t) = N(t)\,M'(t) + M(t)\,N'(t)$, so

$B'(4) = N(4)\,M'(4) + M(4)\,N'(4) = 820(0.14) + 1.2(50) = 174.8$ g/week.

63. (a) $(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = (f'g + fg')h + (fg)h' = f'gh + fg'h + fgh'$

(b) Putting $f = g = h$ in part (a), we have $\dfrac{d}{dx}[f(x)]^3 = (fff)' = f'ff + ff'f + fff' = 3fff' = 3[f(x)]^2 f'(x)$.

(c) $\dfrac{d}{dx}\left(e^{3x}\right) = \dfrac{d}{dx}\left(e^x\right)^3 = 3(e^x)^2 e^x = 3e^{2x}e^x = 3e^{3x}$

64. (a) We use the Product Rule repeatedly: $F = fg \quad \Rightarrow \quad F' = f'g + fg' \quad \Rightarrow$

$F'' = (f''g + f'g') + (f'g' + fg'') = f''g + 2f'g' + fg''$.

(b) $F''' = f'''g + f''g' + 2\,(f''g' + f'g'') + f'g'' + fg''' = f'''g + 3f''g' + 3f'g'' + fg''' \quad \Rightarrow$

$$\begin{aligned} F^{(4)} &= f^{(4)}g + f'''g' + 3\,(f'''g' + f''g'') + 3\,(f''g'' + f'g''') + f'g''' + fg^{(4)} \\ &= f^{(4)}g + 4f'''g' + 6f''g'' + 4f'g''' + fg^{(4)} \end{aligned}$$

(c) By analogy with the Binomial Theorem, we make the guess:

$$F^{(n)} = f^{(n)}g + nf^{(n-1)}g' + \binom{n}{2}f^{(n-2)}g'' + \cdots + \binom{n}{k}f^{(n-k)}g^{(k)} + \cdots + nf'g^{(n-1)} + fg^{(n)},$$

where $\displaystyle\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$.

65. For $f(x) = x^2e^x$, $f'(x) = x^2e^x + e^x(2x) = e^x(x^2 + 2x)$. Similarly, we have

$$\begin{aligned} f''(x) &= e^x(x^2 + 4x + 2) \\ f'''(x) &= e^x(x^2 + 6x + 6) \\ f^{(4)}(x) &= e^x(x^2 + 8x + 12) \\ f^{(5)}(x) &= e^x(x^2 + 10x + 20) \end{aligned}$$

It appears that the coefficient of x in the quadratic term increases by 2 with each differentiation. The pattern for the constant terms seems to be $0 = 1 \cdot 0$, $2 = 2 \cdot 1$, $6 = 3 \cdot 2$, $12 = 4 \cdot 3$, $20 = 5 \cdot 4$. So a reasonable guess is that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

Proof: Let S_n be the statement that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

1. S_1 is true because $f'(x) = e^x(x^2 + 2x)$.
2. Assume that S_k is true; that is, $f^{(k)}(x) = e^x[x^2 + 2kx + k(k-1)]$. Then

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}\left[f^{(k)}(x)\right] = e^x(2x + 2k) + [x^2 + 2kx + k(k-1)]e^x \\ &= e^x[x^2 + (2k+2)x + (k^2 + k)] = e^x[x^2 + 2(k+1)x + (k+1)k] \end{aligned}$$

   This shows that S_{k+1} is true.
3. Therefore, by mathematical induction, S_n is true for all n; that is, $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$ for every positive integer n.

66. (a) $\displaystyle\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{g(x)\cdot\frac{d}{dx}(1) - 1\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}$ [Quotient Rule] $\displaystyle = \frac{g(x)\cdot 0 - 1\cdot g'(x)}{[g(x)]^2} = \frac{0 - g'(x)}{[g(x)]^2} = -\frac{g'(x)}{[g(x)]^2}$

(b) $\displaystyle\frac{d}{dx}\left(\frac{1}{2x^3 - 6x^2 + 5}\right) = -\frac{(2x^3 - 6x^2 + 5)'}{(2x^3 - 6x^2 + 5)^2} = -\frac{6x^2 - 12x}{(2x^3 - 6x^2 + 5)^2}$

(c) $\displaystyle\frac{d}{dx}\left(x^{-n}\right) = \frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{(x^n)'}{(x^n)^2}$ [by the Reciprocal Rule] $\displaystyle = -\frac{nx^{n-1}}{x^{2n}} = -nx^{n-1-2n} = -nx^{-n-1}$

3.3 Derivatives of Trigonometric Functions

1. $f(x) = 3\sin x - 2\cos x \quad \Rightarrow \quad f'(x) = 3(\cos x) - 2(-\sin x) = 3\cos x + 2\sin x$

2. $f(x) = \tan x - 4\sin x \quad \Rightarrow \quad f'(x) = \sec^2 x - 4(\cos x) = \sec^2 x - 4\cos x$

3. $y = x^2 + \cot x \quad \Rightarrow \quad y' = 2x + (-\csc^2 x) = 2x - \csc^2 x$

4. $y = 2\sec x - \csc x \quad \Rightarrow \quad y' = 2(\sec x\ \tan x) - (-\csc x\ \cot x) = 2\sec x\ \tan x + \csc x\ \cot x$

5. $h(\theta) = \theta^2 \sin\theta \quad \overset{\text{PR}}{\Rightarrow} \quad h'(\theta) = \theta^2(\cos\theta) + (\sin\theta)(2\theta) = \theta(\theta\cos\theta + 2\sin\theta)$

6. $g(x) = 3x + x^2\cos x \quad \Rightarrow \quad g'(x) = 3 + x^2(-\sin x) + (\cos x)(2x) = 3 - x^2\sin x + 2x\cos x$

7. $y = \sec\theta\ \tan\theta \quad \Rightarrow \quad y' = \sec\theta\,(\sec^2\theta) + \tan\theta\,(\sec\theta\tan\theta) = \sec\theta\,(\sec^2\theta + \tan^2\theta)$. Using the identity $1 + \tan^2\theta = \sec^2\theta$, we can write alternative forms of the answer as $\sec\theta\,(1 + 2\tan^2\theta)$ or $\sec\theta\,(2\sec^2\theta - 1)$.

8. $y = \sin\theta\ \cos\theta \quad \Rightarrow \quad y' = \sin\theta(-\sin\theta) + \cos\theta(\cos\theta) = \cos^2\theta - \sin^2\theta \quad$ [or $\cos 2\theta$]

9. $f(\theta) = (\theta - \cos\theta)\ \sin\theta \quad \overset{\text{PR}}{\Rightarrow} \quad f'(\theta) = (\theta - \cos\theta)(\cos\theta) + (\sin\theta)(1 + \sin\theta) = \theta\cos\theta - \cos^2\theta + \sin\theta + \sin^2\theta$

10. $g(\theta) = e^\theta(\tan\theta - \theta) \quad \Rightarrow \quad g'(\theta) = e^\theta(\sec^2\theta - 1) + (\tan\theta - \theta)e^\theta = e^\theta(\sec^2\theta - 1 + \tan\theta - \theta)$

11. $H(t) = \cos^2 t = \cos t \cdot \cos t \quad \overset{\text{PR}}{\Rightarrow} \quad H'(t) = \cos t\,(-\sin t) + \cos t\,(-\sin t) = -2\sin t\ \cos t$. Using the identity $\sin 2t = 2\sin t\ \cos t$, we can write an alternative form of the answer as $-\sin 2t$.

12. $f(x) = e^x \sin x + \cos x \Rightarrow f'(x) = e^x(\cos x) + \sin x \cdot e^x + (-\sin x) = e^x(\cos x + \sin x) - \sin x$

13. $f(\theta) = \dfrac{\sin\theta}{1 + \cos\theta} \quad \Rightarrow$

$$f'(\theta) = \frac{(1 + \cos\theta)\cos\theta - (\sin\theta)(-\sin\theta)}{(1 + \cos\theta)^2} = \frac{\cos\theta + \cos^2\theta + \sin^2\theta}{(1 + \cos\theta)^2} = \frac{\cos\theta + 1}{(1 + \cos\theta)^2} = \frac{1}{1 + \cos\theta}$$

14. $y = \dfrac{\cos x}{1 - \sin x} \quad \Rightarrow$

$$y' = \frac{(1 - \sin x)(-\sin x) - \cos x(-\cos x)}{(1 - \sin x)^2} = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2} = \frac{-\sin x + 1}{(1 - \sin x)^2} = \frac{1}{1 - \sin x}$$

15. $y = \dfrac{x}{2 - \tan x} \quad \Rightarrow \quad y' = \dfrac{(2 - \tan x)(1) - x(-\sec^2 x)}{(2 - \tan x)^2} = \dfrac{2 - \tan x + x\sec^2 x}{(2 - \tan x)^2}$

16. $f(t) = \dfrac{\cot t}{e^t} \quad \Rightarrow \quad f'(t) = \dfrac{e^t(-\csc^2 t) - (\cot t)e^t}{(e^t)^2} = \dfrac{e^t(-\csc^2 t - \cot t)}{(e^t)^2} = -\dfrac{\csc^2 t + \cot t}{e^t}$

17. $f(w) = \dfrac{1 + \sec w}{1 - \sec w} \quad \overset{\text{QR}}{\Rightarrow}$

$$f'(w) = \frac{(1 - \sec w)(\sec w\ \tan w) - (1 + \sec w)(-\sec w\ \tan w)}{(1 - \sec w)^2}$$

$$= \frac{\sec w\ \tan w - \sec^2 w\ \tan w + \sec w\ \tan w + \sec^2 w\ \tan w}{(1 - \sec w)^2} = \frac{2\sec w\ \tan w}{(1 - \sec w)^2}$$

18. $y = \dfrac{\sin t}{1 + \tan t} \quad \Rightarrow$

$$y' = \frac{(1 + \tan t)\cos t - (\sin t)\sec^2 t}{(1 + \tan t)^2} = \frac{\cos t + \sin t - \dfrac{\sin t}{\cos^2 t}}{(1 + \tan t)^2} = \frac{\cos t + \sin t - \tan t \sec t}{(1 + \tan t)^2}$$

19. $y = \dfrac{t \sin t}{1 + t} \quad \Rightarrow$

$$y' = \frac{(1 + t)(t \cos t + \sin t) - t \sin t(1)}{(1 + t)^2} = \frac{t \cos t + \sin t + t^2 \cos t + t \sin t - t \sin t}{(1 + t)^2} = \frac{(t^2 + t)\cos t + \sin t}{(1 + t)^2}$$

20. $g(z) = \dfrac{z}{\sec z + \tan z} \quad \overset{\text{QR}}{\Rightarrow}$

$$g'(z) = \frac{(\sec z + \tan z)(1) - z(\sec z \tan z + \sec^2 z)}{(\sec z + \tan z)^2} = \frac{(\sec z + \tan z)(1) - z \sec z(\tan z + \sec z)}{(\sec z + \tan z)^2}$$

$$= \frac{(1 - z \sec z)(\sec z + \tan z)}{(\sec z + \tan z)^2} = \frac{1 - z \sec z}{\sec z + \tan z}$$

21. Using Exercise 3.2.63(a), $f(\theta) = \theta \cos\theta \, \sin\theta \quad \Rightarrow$

$$f'(\theta) = 1 \cos\theta \, \sin\theta + \theta(-\sin\theta)\sin\theta + \theta\cos\theta(\cos\theta) = \cos\theta \, \sin\theta - \theta \sin^2\theta + \theta\cos^2\theta$$

$$= \sin\theta \, \cos\theta + \theta(\cos^2\theta - \sin^2\theta) = \tfrac{1}{2}\sin 2\theta + \theta \cos 2\theta \quad \text{[using double-angle formulas]}$$

22. Using Exercise 3.2.63(a), $f(t) = te^t \cot t \quad \Rightarrow$

$$f'(t) = 1e^t \cot t + te^t \cot t + te^t(-\csc^2 t) = e^t(\cot t + t \cot t - t \csc^2 t)$$

23. $$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \, \cot x$$

24. $$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \, \tan x$$

25. $$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

26. $f(x) = \cos x \quad \Rightarrow$

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h} = \lim_{h \to 0} \frac{\cos x \, \cosh - \sin x \, \sin h - \cos x}{h}$$

$$= \lim_{h \to 0}\left(\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h}\right) = \cos x \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \to 0} \frac{\sin h}{h}$$

$$= (\cos x)(0) - (\sin x)(1) = -\sin x$$

27. $y = \sin x + \cos x \quad \Rightarrow \quad y' = \cos x - \sin x$, so $y'(0) = \cos 0 - \sin 0 = 1 - 0 = 1$. An equation of the tangent line to the curve $y = \sin x + \cos x$ at the point $(0, 1)$ is $y - 1 = 1(x - 0)$ or $y = x + 1$.

28. $y = x + \sin x \quad \Rightarrow \quad y' = 1 + \cos x$, so $y'(\pi) = 1 + \cos\pi = 1 + (-1) = 0$. An equation of the tangent line to the curve $y = x + \sin x$ at the point (π, π) is $y - \pi = 0(x - \pi)$ or $y = \pi$.

29. $y = e^x \cos x + \sin x \quad \Rightarrow \quad y' = e^x(-\sin x) + (\cos x)(e^x) + \cos x = e^x(\cos x - \sin x) + \cos x$, so

$y'(0) = e^0(\cos 0 - \sin 0) + \cos 0 = 1(1 - 0) + 1 = 2$. An equation of the tangent line to the curve $y = e^x \cos x + \sin x$ at the point $(0, 1)$ is $y - 1 = 2(x - 0)$ or $y = 2x + 1$.

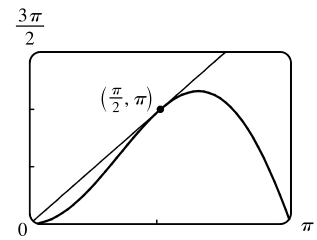
30. $y = \dfrac{1 + \sin x}{\cos x} \quad \overset{\text{QR}}{\Rightarrow} \quad y' = \dfrac{\cos x(\cos x) - (1 + \sin x)(-\sin x)}{\cos^2 x} = \dfrac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x}$

Using the identity $\cos^2 x + \sin^2 x = 1$, we can write $y' = \dfrac{1 + \sin x}{\cos^2 x}$.

$y'(\pi) = \dfrac{1 + \sin \pi}{\cos^2 \pi} = \dfrac{1 + 0}{(-1)^2} = 1$. An equation of the tangent line to the curve $y = \dfrac{1 + \sin x}{\cos x}$ at the point $(\pi, -1)$ is $y - (-1) = 1(x - \pi)$ or $y = x - \pi - 1$.

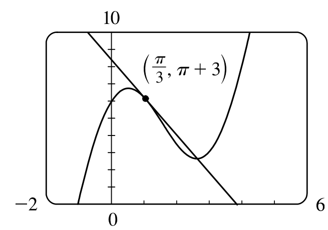
31. (a) $y = 2x \sin x \quad \Rightarrow \quad y' = 2(x \cos x + \sin x \cdot 1)$. At $\left(\frac{\pi}{2}, \pi\right)$, $y' = 2\left(\frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2}\right) = 2(0 + 1) = 2$, and an equation of the tangent line is $y - \pi = 2\left(x - \frac{\pi}{2}\right)$, or $y = 2x$.

(b)



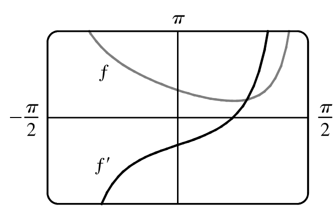
32. (a) $y = 3x + 6 \cos x \quad \Rightarrow \quad y' = 3 - 6 \sin x$. At $\left(\frac{\pi}{3}, \pi + 3\right)$, $y' = 3 - 6 \sin \frac{\pi}{3} = 3 - 6 \frac{\sqrt{3}}{2} = 3 - 3\sqrt{3}$, and an equation of the tangent line is $y - (\pi + 3) = \left(3 - 3\sqrt{3}\right)\left(x - \frac{\pi}{3}\right)$, or $y = \left(3 - 3\sqrt{3}\right)x + 3 + \pi\sqrt{3}$.

(b)



33. (a) $f(x) = \sec x - x \quad \Rightarrow \quad f'(x) = \sec x \tan x - 1$

(b)

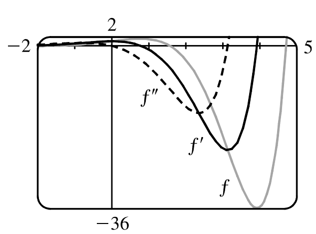


Note that $f' = 0$ where f has a minimum. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

34. (a) $f(x) = e^x \cos x \quad \Rightarrow \quad f'(x) = e^x(-\sin x) + (\cos x)\, e^x = e^x(\cos x - \sin x) \quad \Rightarrow$

$f''(x) = e^x(-\sin x - \cos x) + (\cos x - \sin x)\, e^x = e^x(-\sin x - \cos x + \cos x - \sin x) = -2e^x \sin x$

(b)



Note that $f' = 0$ where f has a minimum and $f'' = 0$ where f' has a minimum. Also note that f' is negative when f is decreasing and f'' is negative when f' is decreasing.

35. $g(\theta) = \dfrac{\sin\theta}{\theta} \overset{\text{QR}}{\Rightarrow} g'(\theta) = \dfrac{\theta(\cos\theta) - (\sin\theta)(1)}{\theta^2} = \dfrac{\theta\cos\theta - \sin\theta}{\theta^2}$

Using the Quotient Rule and $g'(\theta) = \dfrac{\theta\cos\theta - \sin\theta}{\theta^2}$, we get

$$\begin{aligned} g''(\theta) &= \frac{\theta^2\{[\theta(-\sin\theta) + (\cos\theta)(1)] - \cos\theta\} - (\theta\cos\theta - \sin\theta)(2\theta)}{(\theta^2)^2} \\ &= \frac{-\theta^3\sin\theta + \theta^2\cos\theta - \theta^2\cos\theta - 2\theta^2\cos\theta + 2\theta\sin\theta}{\theta^4} = \frac{\theta(-\theta^2\sin\theta - 2\theta\cos\theta + 2\sin\theta)}{\theta\cdot\theta^3} \\ &= \frac{-\theta^2\sin\theta - 2\theta\cos\theta + 2\sin\theta}{\theta^3} \end{aligned}$$

36. $f(t) = \sec t \quad\Rightarrow\quad f'(t) = \sec t\,\tan t \quad\Rightarrow\quad f''(t) = (\sec t)\sec^2 t + (\tan t)\sec t\,\tan t = \sec^3 t + \sec t\,\tan^2 t$, so $f''\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^3 + \sqrt{2}(1)^2 = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$.

37. (a) $f(x) = \dfrac{\tan x - 1}{\sec x} \quad\Rightarrow$

$$f'(x) = \frac{\sec x(\sec^2 x) - (\tan x - 1)(\sec x\tan x)}{(\sec x)^2} = \frac{\sec x(\sec^2 x - \tan^2 x + \tan x)}{\sec^2 x} = \frac{1 + \tan x}{\sec x}$$

(b) $f(x) = \dfrac{\tan x - 1}{\sec x} = \dfrac{\dfrac{\sin x}{\cos x} - 1}{\dfrac{1}{\cos x}} = \dfrac{\dfrac{\sin x - \cos x}{\cos x}}{\dfrac{1}{\cos x}} = \sin x - \cos x \quad\Rightarrow\quad f'(x) = \cos x - (-\sin x) = \cos x + \sin x$

(c) From part (a), $f'(x) = \dfrac{1 + \tan x}{\sec x} = \dfrac{1}{\sec x} + \dfrac{\tan x}{\sec x} = \cos x + \sin x$, which is the expression for $f'(x)$ in part (b).

38. (a) $g(x) = f(x)\sin x \quad\Rightarrow\quad g'(x) = f(x)\cos x + \sin x\cdot f'(x)$, so

$g'\left(\frac{\pi}{3}\right) = f\left(\frac{\pi}{3}\right)\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cdot f'\left(\frac{\pi}{3}\right) = 4\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot(-2) = 2 - \sqrt{3}$

(b) $h(x) = \dfrac{\cos x}{f(x)} \quad\Rightarrow\quad h'(x) = \dfrac{f(x)\cdot(-\sin x) - \cos x\cdot f'(x)}{[f(x)]^2}$, so

$$h'\left(\tfrac{\pi}{3}\right) = \frac{f\left(\frac{\pi}{3}\right)\cdot\left(-\sin\frac{\pi}{3}\right) - \cos\frac{\pi}{3}\cdot f'\left(\frac{\pi}{3}\right)}{\left[f\left(\frac{\pi}{3}\right)\right]^2} = \frac{4\left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)(-2)}{4^2} = \frac{-2\sqrt{3} + 1}{16} = \frac{1 - 2\sqrt{3}}{16}$$

39. $f(x) = x + 2\sin x$ has a horizontal tangent when $f'(x) = 0 \quad\Leftrightarrow\quad 1 + 2\cos x = 0 \quad\Leftrightarrow\quad \cos x = -\frac{1}{2} \quad\Leftrightarrow$ $x = \frac{2\pi}{3} + 2\pi n$ or $\frac{4\pi}{3} + 2\pi n$, where n is an integer. Note that $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$ are $\pm\frac{\pi}{3}$ units from π. This allows us to write the solutions in the more compact equivalent form $(2n + 1)\pi \pm \frac{\pi}{3}$, n an integer.

40. $f(x) = e^x\cos x$ has a horizontal tangent when $f'(x) = 0$. $f'(x) = e^x(-\sin x) + (\cos x)e^x = e^x(\cos x - \sin x)$. $f'(x) = 0 \quad\Leftrightarrow\quad \cos x - \sin x = 0 \quad\Leftrightarrow\quad \cos x = \sin x \quad\Leftrightarrow\quad \tan x = 1 \quad\Leftrightarrow\quad x = \frac{\pi}{4} + n\pi$, n an integer.

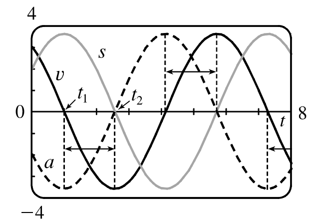
41. (a) $x(t) = 8\sin t \quad\Rightarrow\quad v(t) = x'(t) = 8\cos t \quad\Rightarrow\quad a(t) = x''(t) = -8\sin t$

(b) The mass at time $t = \frac{2\pi}{3}$ has position $x\left(\frac{2\pi}{3}\right) = 8\sin\frac{2\pi}{3} = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$, velocity $v\left(\frac{2\pi}{3}\right) = 8\cos\frac{2\pi}{3} = 8\left(-\frac{1}{2}\right) = -4$, and acceleration $a\left(\frac{2\pi}{3}\right) = -8\sin\frac{2\pi}{3} = -8\left(\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}$. Since $v\left(\frac{2\pi}{3}\right) < 0$, the particle is moving to the left.

42. (a) $s(t) = 2\cos t + 3\sin t \quad \Rightarrow \quad v(t) = -2\sin t + 3\cos t \quad \Rightarrow$

$a(t) = -2\cos t - 3\sin t$

(b)

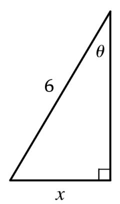


(c) $s = 0 \quad \Rightarrow \quad t_2 \approx 2.55$. So the mass passes through the equilibrium position for the first time when $t \approx 2.55$ s.

(d) $v = 0 \quad \Rightarrow \quad t_1 \approx 0.98$, $s(t_1) \approx 3.61$ cm. So the mass travels a maximum of about 3.6 cm (upward and downward) from its equilibrium position.

(e) The speed $|v|$ is greatest when $s = 0$, that is, when $t = t_2 + n\pi$, n a positive integer.

43.

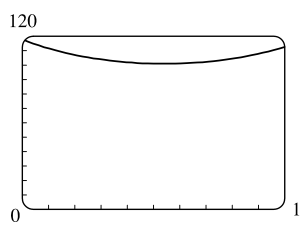


From the diagram we can see that $\sin\theta = x/6 \quad \Leftrightarrow \quad x = 6\sin\theta$. We want to find the rate of change of x with respect to θ, that is, $dx/d\theta$. Taking the derivative of $x = 6\sin\theta$, we get $dx/d\theta = 6(\cos\theta)$. So when $\theta = \frac{\pi}{3}$, $\frac{dx}{d\theta} = 6\cos\frac{\pi}{3} = 6\left(\frac{1}{2}\right) = 3\ \text{m/rad}$.

44. (a) $F = \dfrac{\mu mg}{\mu\sin\theta + \cos\theta} \quad \Rightarrow \quad \dfrac{dF}{d\theta} = \dfrac{(\mu\sin\theta + \cos\theta)(0) - \mu mg(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = \dfrac{\mu mg(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)^2}$

(b) $\dfrac{dF}{d\theta} = 0 \quad \Leftrightarrow \quad \mu mg(\sin\theta - \mu\cos\theta) = 0 \quad \Leftrightarrow \quad \sin\theta = \mu\cos\theta \quad \Leftrightarrow \quad \tan\theta = \mu \quad \Leftrightarrow \quad \theta = \tan^{-1}\mu$

(c)



From the graph of $F = \dfrac{0.6(20)(9.8)}{0.6\sin\theta + \cos\theta}$ for $0 \le \theta \le 1$, we see that $\dfrac{dF}{d\theta} = 0 \quad \Rightarrow \quad \theta \approx 0.54$. Checking this with part (b) and $\mu = 0.6$, we calculate $\theta = \tan^{-1} 0.6 \approx 0.54$. So the value from the graph is consistent with the value in part (b).

45. $\displaystyle \lim_{x\to 0}\frac{\sin 5x}{3x} = \lim_{x\to 0}\frac{5}{3}\left(\frac{\sin 5x}{5x}\right) = \frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x} = \frac{5}{3}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} \quad \left[\begin{array}{l}\text{where } \theta = 5x,\\ \text{using Equation 5}\end{array}\right] \quad = \frac{5}{3}\cdot 1 = \frac{5}{3}$

46. $\displaystyle \lim_{x\to 0}\frac{\sin x}{\sin \pi x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\pi x}{\sin \pi x}\cdot\frac{1}{\pi} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin\theta}\cdot\frac{1}{\pi} \quad \left[\begin{array}{l}\text{where } \theta = \pi x,\\ \text{using Equation 5}\end{array}\right]$

$\displaystyle = 1\cdot\lim_{\theta\to 0}\frac{1}{\dfrac{\sin\theta}{\theta}}\cdot\frac{1}{\pi} = 1\cdot 1\cdot\frac{1}{\pi} = \frac{1}{\pi}$

47. $\displaystyle \lim_{t\to 0}\frac{\sin 3t}{\sin t} = \lim_{t\to 0}\frac{\sin 3t}{3t}\cdot\frac{t}{\sin t}\cdot 3 = \lim_{t\to 0}\frac{\sin 3t}{3t}\cdot\lim_{t\to 0}\frac{t}{\sin t}\cdot\lim_{t\to 0}3 = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{t\to 0}\frac{1}{\dfrac{\sin t}{t}}\cdot\lim_{t\to 0}3 \qquad [\theta = 3t]$

$= 1\cdot 1\cdot 3 = 3$

48. $\displaystyle \lim_{x\to 0}\frac{\sin^2 3x}{x} = \lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{\sin 3x}{3x}\cdot 3\cdot 3\cdot x = \lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\lim_{x\to 0}3\cdot 3\cdot x$

$\displaystyle = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{x\to 0}9x \qquad [\theta = 3x]$

$= 1\cdot 1\cdot 0 = 0$

49. $\displaystyle \lim_{x\to 0}\frac{\sin x-\sin x\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x\,(1-\cos x)}{x^2}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1-\cos x}{x}$

$=1\cdot 0$ [by Equations 5 and 6] $=0$

50. $\displaystyle \lim_{x\to 0}\frac{1-\sec x}{2x}=\lim_{x\to 0}\frac{1-(1/\cos x)}{2x}\cdot\frac{\cos x}{\cos x}=\lim_{x\to 0}\frac{\cos x-1}{2x\cos x}=\lim_{x\to 0}\frac{\cos x-1}{x}\cdot\lim_{x\to 0}\frac{1}{2\cos x}$

$=0\cdot\frac{1}{2}$ [using Equation 6] $=0$

51. $\displaystyle \lim_{x\to 0}\frac{\tan 2x}{x}=\lim_{x\to 0}\frac{\dfrac{\sin 2x}{\cos 2x}}{x}=\lim_{x\to 0}\frac{\sin 2x}{x\cos 2x}=\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{2}{\cos 2x}$

$\displaystyle =\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\lim_{x\to 0}\frac{2}{\cos 2x}=\lim_{x\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{x\to 0}\frac{2}{\cos 2x}\qquad[\theta=2x]$

$\displaystyle =1\cdot\frac{2}{1}=2$

52. $\displaystyle \lim_{\theta\to 0}\frac{\sin\theta}{\tan 7\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{\dfrac{\sin 7\theta}{\cos 7\theta}}=\lim_{\theta\to 0}\sin\theta\cdot\frac{\cos 7\theta}{\sin 7\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\cos 7\theta\cdot\frac{7\theta}{\sin 7\theta}\cdot\frac{1}{7}$

$\displaystyle =\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\cos 7\theta\cdot\lim_{\theta\to 0}\frac{1}{\dfrac{\sin 7\theta}{7\theta}}\cdot\frac{1}{7}=1\cdot 1\cdot\lim_{x\to 0}\frac{1}{\dfrac{\sin x}{x}}\cdot\frac{1}{7}\qquad[x=7\theta]$

$\displaystyle =1\cdot 1\cdot 1\cdot\frac{1}{7}=\frac{1}{7}$

53. $\displaystyle \lim_{x\to 0}\frac{\sin 3x}{5x^3-4x}=\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\cdot\frac{3}{5x^2-4}\right)=\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\lim_{x\to 0}\frac{3}{5x^2-4}=1\cdot\left(\frac{3}{-4}\right)=-\frac{3}{4}$

54. $\displaystyle \lim_{x\to 0}\frac{\sin 3x\sin 5x}{x^2}=\lim_{x\to 0}\left(\frac{3\sin 3x}{3x}\cdot\frac{5\sin 5x}{5x}\right)=\lim_{x\to 0}\frac{3\sin 3x}{3x}\cdot\lim_{x\to 0}\frac{5\sin 5x}{5x}$

$\displaystyle =3\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot 5\lim_{x\to 0}\frac{\sin 5x}{5x}=3(1)\cdot 5(1)=15$

55. Divide numerator and denominator by θ. [$\sin\theta$ also works.]

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta}=\lim_{\theta\to 0}\frac{\dfrac{\sin\theta}{\theta}}{1+\dfrac{\sin\theta}{\theta}\cdot\dfrac{1}{\cos\theta}}=\frac{\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}{1+\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\lim_{\theta\to 0}\frac{1}{\cos\theta}}=\frac{1}{1+1\cdot 1}=\frac{1}{2}$$

56. $\displaystyle \lim_{x\to 0}\csc x\,\sin(\sin x)=\lim_{x\to 0}\frac{\sin(\sin x)}{\sin x}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$ [As $x\to 0$, $\theta=\sin x\to 0$.] $=1$

57. $\displaystyle \lim_{\theta\to 0}\frac{\cos\theta-1}{2\theta^2}=\lim_{\theta\to 0}\frac{\cos\theta-1}{2\theta^2}\cdot\frac{\cos\theta+1}{\cos\theta+1}=\lim_{\theta\to 0}\frac{\cos^2\theta-1}{2\theta^2(\cos\theta+1)}=\lim_{\theta\to 0}\frac{-\sin^2\theta}{2\theta^2(\cos\theta+1)}$

$\displaystyle =-\frac{1}{2}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta+1}=-\frac{1}{2}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{1}{\cos\theta+1}$

$\displaystyle =-\frac{1}{2}\cdot 1\cdot 1\cdot\frac{1}{1+1}=-\frac{1}{4}$

58. $\displaystyle \lim_{x\to 0}\frac{\sin(x^2)}{x}=\lim_{x\to 0}\left[x\cdot\frac{\sin(x^2)}{x\cdot x}\right]=\lim_{x\to 0}x\cdot\lim_{x\to 0}\frac{\sin(x^2)}{x^2}=0\cdot\lim_{y\to 0^+}\frac{\sin y}{y}$ [where $y=x^2$] $=0\cdot 1=0$

59. $\displaystyle \lim_{x\to\pi/4} \frac{1-\tan x}{\sin x - \cos x} = \lim_{x\to\pi/4} \frac{\left(1 - \dfrac{\sin x}{\cos x}\right)\cdot \cos x}{(\sin x - \cos x)\cdot \cos x} = \lim_{x\to\pi/4} \frac{\cos x - \sin x}{(\sin x - \cos x)\cos x} = \lim_{x\to\pi/4} \frac{-1}{\cos x} = \frac{-1}{1/\sqrt{2}} = -\sqrt{2}$

60. $\displaystyle \lim_{x\to 1} \frac{\sin(x-1)}{x^2+x-2} = \lim_{x\to 1} \frac{\sin(x-1)}{(x+2)(x-1)} = \lim_{x\to 1} \frac{1}{x+2} \lim_{x\to 1} \frac{\sin(x-1)}{x-1} = \frac{1}{3}\cdot 1 = \frac{1}{3}$

61. $\displaystyle \frac{d}{dx}(\sin x) = \cos x \;\Rightarrow\; \frac{d^2}{dx^2}(\sin x) = -\sin x \;\Rightarrow\; \frac{d^3}{dx^3}(\sin x) = -\cos x \;\Rightarrow\; \frac{d^4}{dx^4}(\sin x) = \sin x.$

The derivatives of $\sin x$ occur in a cycle of four. Since $99 = 4(24)+3$, we have $\displaystyle \frac{d^{99}}{dx^{99}}(\sin x) = \frac{d^3}{dx^3}(\sin x) = -\cos x.$

62. Let $f(x) = x\sin x$ and $h(x) = \sin x$, so $f(x) = xh(x)$. Then $f'(x) = h(x) + xh'(x)$,

$f''(x) = h'(x) + h'(x) + xh''(x) = 2h'(x) + xh''(x)$,

$f'''(x) = 2h''(x) + h''(x) + xh'''(x) = 3h''(x) + xh'''(x), \cdots, f^{(n)}(x) = nh^{(n-1)}(x) + xh^{(n)}(x).$

Since $34 = 4(8)+2$, we have $h^{(34)}(x) = h^{(2)}(x) = \dfrac{d^2}{dx^2}(\sin x) = -\sin x$ and $h^{(35)}(x) = -\cos x$.

Thus, $\dfrac{d^{35}}{dx^{35}}(x\sin x) = 35h^{(34)}(x) + xh^{(35)}(x) = -35\sin x - x\cos x.$

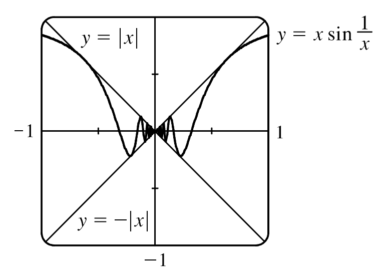
63. $y = A\sin x + B\cos x \;\Rightarrow\; y' = A\cos x - B\sin x \;\Rightarrow\; y'' = -A\sin x - B\cos x$. Substituting these expressions for y, y', and y'' into the given differential equation $y'' + y' - 2y = \sin x$ gives us

$(-A\sin x - B\cos x) + (A\cos x - B\sin x) - 2(A\sin x + B\cos x) = \sin x \;\Leftrightarrow$

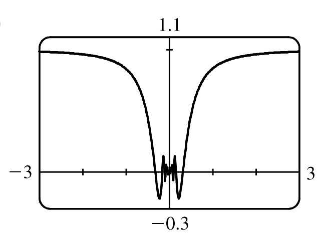
$-3A\sin x - B\sin x + A\cos x - 3B\cos x = \sin x \;\Leftrightarrow\; (-3A - B)\sin x + (A - 3B)\cos x = 1\sin x$, so we must have $-3A - B = 1$ and $A - 3B = 0$ (since 0 is the coefficient of $\cos x$ on the right side). Solving for A and B, we add the first equation to three times the second to get $B = -\frac{1}{10}$ and $A = -\frac{3}{10}$.

64. (a) Let $\theta = \dfrac{1}{x}$. Then as $x \to \infty$, $\theta \to 0^+$, and $\displaystyle \lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{\theta\to 0^+} \frac{1}{\theta}\sin\theta = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$

(b) Since $-1 \le \sin(1/x) \le 1$, we have (as illustrated in the figure) $-|x| \le x\sin(1/x) \le |x|$. We know that $\lim_{x\to 0}(|x|) = 0$ and $\lim_{x\to 0}(-|x|) = 0$; so by the Squeeze Theorem, $\lim_{x\to 0} x\sin(1/x) = 0.$



(c)



65. (a) $\displaystyle \frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} \;\Rightarrow\; \sec^2 x = \frac{\cos x\,\cos x - \sin x\,(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$ So $\displaystyle \sec^2 x = \frac{1}{\cos^2 x}.$

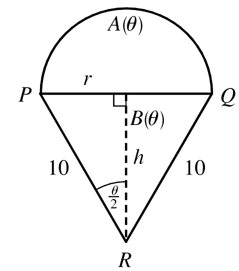
(b) $\dfrac{d}{dx}\sec x = \dfrac{d}{dx}\dfrac{1}{\cos x} \quad\Rightarrow\quad \sec x\,\tan x = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x}$. So $\sec x\,\tan x = \dfrac{\sin x}{\cos^2 x}$.

(c) $\dfrac{d}{dx}(\sin x + \cos x) = \dfrac{d}{dx}\dfrac{1+\cot x}{\csc x} \quad\Rightarrow$

$$\cos x - \sin x = \frac{\csc x\,(-\csc^2 x) - (1+\cot x)(-\csc x\,\cot x)}{\csc^2 x} = \frac{\csc x\,[-\csc^2 x + (1+\cot x)\,\cot x]}{\csc^2 x}$$
$$= \frac{-\csc^2 x + \cot^2 x + \cot x}{\csc x} = \frac{-1+\cot x}{\csc x}$$

So $\cos x - \sin x = \dfrac{\cot x - 1}{\csc x}$.

66. We get the following formulas for r and h in terms of θ:



$$\sin\frac{\theta}{2} = \frac{r}{10} \quad\Rightarrow\quad r = 10\sin\frac{\theta}{2} \quad\text{and}\quad \cos\frac{\theta}{2} = \frac{h}{10} \quad\Rightarrow\quad h = 10\cos\frac{\theta}{2}$$

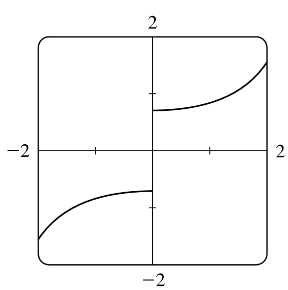
Now $A(\theta) = \frac{1}{2}\pi r^2$ and $B(\theta) = \frac{1}{2}(2r)h = rh$. So

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0^+}\frac{\frac{1}{2}\pi r^2}{rh} = \tfrac{1}{2}\pi\lim_{\theta\to 0^+}\frac{r}{h} = \tfrac{1}{2}\pi\lim_{\theta\to 0^+}\frac{10\sin(\theta/2)}{10\cos(\theta/2)}$$
$$= \tfrac{1}{2}\pi\lim_{\theta\to 0^+}\tan(\theta/2) = 0$$

67. By the definition of radian measure, $s = r\theta$, where r is the radius of the circle. By drawing the bisector of the angle θ, we can see that $\sin\dfrac{\theta}{2} = \dfrac{d/2}{r} \quad\Rightarrow\quad d = 2r\sin\dfrac{\theta}{2}$. So $\displaystyle\lim_{\theta\to 0^+}\frac{s}{d} = \lim_{\theta\to 0^+}\frac{r\theta}{2r\sin(\theta/2)} = \lim_{\theta\to 0^+}\frac{2\cdot(\theta/2)}{2\sin(\theta/2)} = \lim_{\theta\to 0}\frac{\theta/2}{\sin(\theta/2)} = 1$.

[This is just the reciprocal of the limit $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ combined with the fact that as $\theta\to 0$, $\frac{\theta}{2}\to 0$ also.]

68. (a)



It appears that $f(x) = \dfrac{x}{\sqrt{1-\cos 2x}}$ has a jump discontinuity at $x = 0$.

(b) Using the identity $\cos 2x = 1 - \sin^2 x$, we have $\dfrac{x}{\sqrt{1-\cos 2x}} = \dfrac{x}{\sqrt{1-(1-2\sin^2 x)}} = \dfrac{x}{\sqrt{2\sin^2 x}} = \dfrac{x}{\sqrt{2}\,|\sin x|}$.

Thus,
$$\lim_{x\to 0^-}\frac{x}{\sqrt{1-\cos 2x}} = \lim_{x\to 0^-}\frac{x}{\sqrt{2}\,|\sin x|} = \frac{1}{\sqrt{2}}\lim_{x\to 0^-}\frac{x}{-(\sin x)}$$
$$= -\frac{1}{\sqrt{2}}\lim_{x\to 0^-}\frac{1}{\sin x/x} = -\frac{1}{\sqrt{2}}\cdot\frac{1}{1} = -\frac{\sqrt{2}}{2}$$

Evaluating $\lim\limits_{x\to 0^+} f(x)$ is similar, but $|\sin x| = +\sin x$, so we get $\frac{1}{2}\sqrt{2}$. These values appear to be reasonable values for the graph, so they confirm our answer to part (a).

Another method: Multiply numerator and denominator by $\sqrt{1+\cos 2x}$.

3.4 The Chain Rule

1. Let $u = g(x) = 5 - x^4$ and $y = f(u) = u^3$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (3u^2)(-4x^3) = 3(5-x^4)^2(-4x^3) = -12x^3(5-x^4)^2$.

2. Let $u = g(x) = x^3 + 2$ and $y = f(u) = \sqrt{u}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{1}{2\sqrt{u}} \cdot 3x^2 = \dfrac{1}{2\sqrt{x^3+2}} \cdot 3x^2 = \dfrac{3x^2}{2\sqrt{x^3+2}}$.

3. Let $u = g(x) = \cos x$ and $y = f(u) = \sin u$. Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\cos u)(-\sin x) = (\cos(\cos x))(-\sin x) = -\sin x\,\cos(\cos x).$$

4. Let $u = g(x) = x^2$ and $y = f(u) = \tan u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\sec^2 u)(2x) = (\sec^2(x^2))(2x) = 2x\sec^2(x^2)$.

5. Let $u = g(x) = \sqrt{x}$ and $y = f(u) = e^u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (e^u)\left(\tfrac{1}{2}x^{-1/2}\right) = e^{\sqrt{x}} \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$.

6. Let $u = g(x) = e^x + 1$ and $y = f(u) = \sqrt[3]{u} = u^{1/3}$. Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{1}{3}u^{-2/3}\right)(e^x) = \left(\frac{1}{3\sqrt[3]{(e^x+1)^2}}\right)(e^x) = \frac{e^x}{3\sqrt[3]{(e^x+1)^2}}.$$

Note: The notation $\overset{\text{CR}}{\Rightarrow}$ indicates the use of the Chain Rule.

7. $f(x) = (2x^3 - 5x^2 + 4)^5 \quad \overset{\text{CR}}{\Rightarrow}$

$$\begin{aligned} f'(x) &= 5(2x^3 - 5x^2 + 4)^4 \cdot \frac{d}{dx}(2x^3 - 5x^2 + 4) = 5(2x^3 - 5x^2 + 4)^4(6x^2 - 10x) \\ &= 5(2x^3 - 5x^2 + 4)^4 \cdot 2x(3x - 5) = 10x(2x^3 - 5x^2 + 4)^4(3x - 5) \end{aligned}$$

8. $f(x) = (x^5 + 3x^2 - x)^{50} \quad \overset{\text{CR}}{\Rightarrow}$

$$f'(x) = 50(x^5 + 3x^2 - x)^{49} \cdot \frac{d}{dx}(x^5 + 3x^2 - x) = 50(x^5 + 3x^2 - x)^{49}(5x^4 + 6x - 1)$$

9. $f(x) = \sqrt{5x+1} = (5x+1)^{1/2} \quad \overset{\text{CR}}{\Rightarrow} \quad f'(x) = \tfrac{1}{2}(5x+1)^{-1/2} \cdot \dfrac{d}{dx}(5x+1) = \tfrac{1}{2}(5x+1)^{-1/2}(5) = \dfrac{5}{2\sqrt{5x+1}}$

10. $f(x) = \dfrac{1}{\sqrt[3]{x^2-1}} = (x^2-1)^{-1/3} \quad \overset{\text{CR}}{\Rightarrow} \quad f'(x) = -\dfrac{1}{3}(x^2-1)^{-4/3}(2x) = \dfrac{-2x}{3(x^2-1)^{4/3}}$

11. $g(t) = \dfrac{1}{(2t+1)^2} = (2t+1)^{-2} \quad \overset{\text{CR}}{\Rightarrow} \quad g'(t) = -2(2t+1)^{-3} \cdot \dfrac{d}{dt}(2t+1) = -2(2t+1)^{-3}(2) = -\dfrac{4}{(2t+1)^3}$

12. $F(t) = \left(\dfrac{1}{2t+1}\right)^4 = [(2t+1)^{-1}]^4 = (2t+1)^{-4} \quad \overset{\text{CR}}{\Rightarrow}$

$$F'(t) = -4(2t+1)^{-5} \cdot \frac{d}{dt}(2t+1) = -4(2t+1)^{-5}(2) = -\frac{8}{(2t+1)^5}$$

13. $f(\theta) = \cos(\theta^2) \quad \overset{\text{CR}}{\Rightarrow} \quad f'(\theta) = -\sin(\theta^2) \cdot \dfrac{d}{d\theta}(\theta^2) = -\sin(\theta^2) \cdot (2\theta) = -2\theta\sin(\theta^2)$

14. $g(\theta) = \cos^2\theta = (\cos\theta)^2 \overset{\text{CR}}{\Rightarrow} g'(\theta) = 2\,(\cos\theta)^1\,(-\sin\theta) = -2\sin\theta\cos\theta = -\sin 2\theta$

15. $g(x) = e^{x^2-x} \overset{\text{CR}}{\Rightarrow} g'(x) = e^{x^2-x}\cdot\dfrac{d}{dx}(x^2-x) = e^{x^2-x}(2x-1)$

16. Using Formula 5 and the Chain Rule, $y = 5^{\sqrt{x}} \Rightarrow y' = 5^{\sqrt{x}}\ln 5\cdot\dfrac{d}{dx}\left(\sqrt{x}\right) = 5^{\sqrt{x}}\ln 5\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{5^{\sqrt{x}}\ln 5}{2\sqrt{x}}$.

17. Using the Product Rule and the Chain Rule, $y = x^2e^{-3x} \Rightarrow$

$y' = x^2e^{-3x}(-3) + e^{-3x}(2x) = e^{-3x}(-3x^2+2x) = xe^{-3x}(2-3x)$.

18. Using the Product Rule and the Chain Rule, $f(t) = t\sin\pi t \Rightarrow f'(t) = t(\cos\pi t)\cdot\pi + (\sin\pi t)\cdot 1 = \pi t\cos\pi t + \sin\pi t$.

19. $f(t) = e^{at}\sin bt \Rightarrow f'(t) = e^{at}(\cos bt)\cdot b + (\sin bt)e^{at}\cdot a = e^{at}(b\cos bt + a\sin bt)$

20. $A(r) = \sqrt{r}\cdot e^{r^2+1} \Rightarrow$

$$A'(r) = \sqrt{r}\cdot e^{r^2+1}\cdot\frac{d}{dr}\left(r^2+1\right) + e^{r^2+1}\cdot\frac{d}{dr}\left(\sqrt{r}\right) = \sqrt{r}\cdot e^{r^2+1}\cdot 2r + e^{r^2+1}\cdot\frac{1}{2\sqrt{r}}$$
$$= e^{r^2+1}\left(2r\sqrt{r} + \frac{1}{2\sqrt{r}}\right) \text{ or } e^{r^2+1}\left(\frac{4r^2+1}{2\sqrt{r}}\right)$$

21. $F(x) = (4x+5)^3(x^2-2x+5)^4 \Rightarrow$

$$\begin{aligned} F'(x) &= (4x+5)^3\cdot 4(x^2-2x+5)^3(2x-2) + (x^2-2x+5)^4\cdot 3(4x+5)^2\cdot 4 \\ &= 4(4x+5)^2(x^2-2x+5)^3\left[(4x+5)\,(2x-2) + (x^2-2x+5)\cdot 3\right] \\ &= 4(4x+5)^2(x^2-2x+5)^3(8x^2+2x-10+3x^2-6x+15) \\ &= 4(4x+5)^2(x^2-2x+5)^3(11x^2-4x+5) \end{aligned}$$

22. $G(z) = (1-4z)^2\sqrt{z^2+1} \Rightarrow$

$$\begin{aligned} G'(z) &= (1-4z)^2\cdot\frac{1}{2\sqrt{z^2+1}}\cdot 2z + \sqrt{z^2+1}\cdot 2(1-4z)^1(-4) = 2(1-4z)\left[\frac{(1-4z)z}{2\sqrt{z^2+1}} - 4\sqrt{z^2+1}\right] \\ &= 2(1-4z)\left[\frac{(1-4z)z}{2\sqrt{z^2+1}} - \frac{8(z^2+1)}{2\sqrt{z^2+1}}\right] = 2(1-4z)\left(\frac{z-4z^2-8z^2-8}{2\sqrt{z^2+1}}\right) \\ &= (1-4z)\left(\frac{-12z^2+z-8}{\sqrt{z^2+1}}\right) \text{ or } (4z-1)\left(\frac{12z^2-z+8}{\sqrt{z^2+1}}\right) \end{aligned}$$

23. $y = \sqrt{\dfrac{x}{x+1}} = \left(\dfrac{x}{x+1}\right)^{1/2} \Rightarrow$

$$\begin{aligned} y' &= \frac{1}{2}\left(\frac{x}{x+1}\right)^{-1/2}\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{1}{2}\,\frac{x^{-1/2}}{(x+1)^{-1/2}}\,\frac{(x+1)(1)-x(1)}{(x+1)^2} \\ &= \frac{1}{2}\,\frac{(x+1)^{1/2}}{x^{1/2}}\,\frac{1}{(x+1)^2} = \frac{1}{2\sqrt{x}(x+1)^{3/2}} \end{aligned}$$

24. $y = \left(x+\dfrac{1}{x}\right)^5 \Rightarrow y' = 5\left(x+\dfrac{1}{x}\right)^4\dfrac{d}{dx}\left(x+\dfrac{1}{x}\right) = 5\left(x+\dfrac{1}{x}\right)^4\left(1-\dfrac{1}{x^2}\right)$.

Another form of the answer is $\dfrac{5(x^2+1)^4(x^2-1)}{x^6}$.

25. $y = e^{\tan\theta} \quad\Rightarrow\quad y' = e^{\tan\theta}\dfrac{d}{d\theta}(\tan\theta) = (\sec^2\theta)e^{\tan\theta}$

26. Using Formula 5 and the Chain Rule, $f(t) = 2^{t^3} \quad\Rightarrow\quad f'(t) = 2^{t^3}\ln 2\,\dfrac{d}{dt}\,(t^3) = 3(\ln 2)t^2 2^{t^3}$.

27. $g(u) = \left(\dfrac{u^3-1}{u^3+1}\right)^8 \quad\Rightarrow$

$$g'(u) = 8\left(\frac{u^3-1}{u^3+1}\right)^7 \frac{d}{du}\frac{u^3-1}{u^3+1} = 8\,\frac{(u^3-1)^7}{(u^3+1)^7}\,\frac{(u^3+1)(3u^2)-(u^3-1)(3u^2)}{(u^3+1)^2}$$

$$= 8\,\frac{(u^3-1)^7}{(u^3+1)^7}\,\frac{3u^2[(u^3+1)-(u^3-1)]}{(u^3+1)^2} = 8\frac{(u^3-1)^7}{(u^3+1)^7}\,\frac{3u^2(2)}{(u^3+1)^2} = \frac{48u^2(u^3-1)^7}{(u^3+1)^9}$$

28. $s(t) = \sqrt{\dfrac{1+\sin t}{1+\cos t}} = \left(\dfrac{1+\sin t}{1+\cos t}\right)^{1/2} \quad\Rightarrow$

$$s'(t) = \frac{1}{2}\left(\frac{1+\sin t}{1+\cos t}\right)^{-1/2} \frac{(1+\cos t)\cos t - (1+\sin t)(-\sin t)}{(1+\cos t)^2}$$

$$= \frac{1}{2}\,\frac{(1+\sin t)^{-1/2}}{(1+\cos t)^{-1/2}}\,\frac{\cos t+\cos^2 t+\sin t+\sin^2 t}{(1+\cos t)^2} = \frac{\cos t+\sin t+1}{2\sqrt{1+\sin t}\,(1+\cos t)^{3/2}}$$

29. Using Formula 5 and the Chain Rule, $r(t) = 10^{2\sqrt{t}} \quad\Rightarrow$

$$r'(t) = 10^{2\sqrt{t}}\ln 10\,\frac{d}{dt}\left(2\sqrt{t}\right) = 10^{2\sqrt{t}}\ln 10\left(2\cdot\tfrac{1}{2}t^{-1/2}\right) = \frac{(\ln 10)\,10^{2\sqrt{t}}}{\sqrt{t}}.$$

30. $f(z) = e^{z/(z-1)} \quad\Rightarrow\quad f'(z) = e^{z/(z-1)}\dfrac{d}{dz}\dfrac{z}{z-1} = e^{z/(z-1)}\dfrac{(z-1)(1)-z(1)}{(z-1)^2} = -\dfrac{e^{z/(z-1)}}{(z-1)^2}$

31. $H(r) = \dfrac{(r^2-1)^3}{(2r+1)^5} \quad\Rightarrow$

$$H'(r) = \frac{(2r+1)^5\cdot 3(r^2-1)^2(2r) - (r^2-1)^3\cdot 5(2r+1)^4(2)}{[(2r+1)^5]^2} = \frac{2(2r+1)^4(r^2-1)^2[3r(2r+1)-5(r^2-1)]}{(2r+1)^{10}}$$

$$= \frac{2(r^2-1)^2(6r^2+3r-5r^2+5)}{(2r+1)^6} = \frac{2(r^2-1)^2(r^2+3r+5)}{(2r+1)^6}$$

32. $J(\theta) = \tan^2(n\theta) = [\tan(n\theta)]^2 \quad\Rightarrow$

$$J'(\theta) = 2\,[\tan(n\theta)]^1\,\frac{d}{d\theta}\tan(n\theta) = 2\tan(n\theta)\,\sec^2(n\theta)\cdot n = 2n\tan(n\theta)\,\sec^2(n\theta)$$

33. $F(t) = e^{t\sin 2t} \quad\Rightarrow$

$$F'(t) = e^{t\sin 2t}(t\sin 2t)' = e^{t\sin 2t}(t\cdot 2\cos 2t + \sin 2t\cdot 1) = e^{t\sin 2t}(2t\cos 2t+\sin 2t)$$

34. $F(t) = \dfrac{t^2}{\sqrt{t^3+1}} \quad\Rightarrow$

$$F'(t) = \frac{(t^3+1)^{1/2}(2t) - t^2\cdot\frac{1}{2}(t^3+1)^{-1/2}(3t^2)}{\left(\sqrt{t^3+1}\right)^2} = \frac{t(t^3+1)^{-1/2}\left[2(t^3+1)-\frac{3}{2}t^3\right]}{(t^3+1)^1} = \frac{t\left(\frac{1}{2}t^3+2\right)}{(t^3+1)^{3/2}} = \frac{t(t^3+4)}{2(t^3+1)^{3/2}}$$

35. Using Formula 5 and the Chain Rule, $G(x) = 4^{C/x} \quad \Rightarrow$

$$G'(x) = 4^{C/x}(\ln 4)\,\frac{d}{dx}\frac{C}{x} \quad \left[\frac{C}{x} = Cx^{-1}\right] \quad = 4^{C/x}(\ln 4)\left(-Cx^{-2}\right) = -C(\ln 4)\,\frac{4^{C/x}}{x^2}.$$

36. $U(y) = \left(\dfrac{y^4+1}{y^2+1}\right)^5 \quad \Rightarrow$

$$U'(y) = 5\left(\frac{y^4+1}{y^2+1}\right)^4 \frac{(y^2+1)(4y^3)-(y^4+1)(2y)}{(y^2+1)^2} = \frac{5(y^4+1)^4 2y[2y^2(y^2+1)-(y^4+1)]}{(y^2+1)^4(y^2+1)^2}$$

$$= \frac{10y(y^4+1)^4(y^4+2y^2-1)}{(y^2+1)^6}$$

37. $f(x) = \sin x\, \cos(1-x^2) \quad \Rightarrow$

$$f'(x) = \sin x\left[-\sin(1-x^2)(-2x)\right] + \cos(1-x^2)\cdot\cos x = 2x\sin x\,\sin(1-x^2) + \cos x\,\cos(1-x^2)$$

38. $g(x) = e^{-x}\cos(x^2) \quad \Rightarrow \quad g'(x) = e^{-x}[-\sin(x^2)]\cdot 2x + \cos(x^2)\cdot e^{-x}(-1) = -e^{-x}[2x\sin(x^2)+\cos(x^2)]$

39. $F(t) = \tan\sqrt{1+t^2} \Rightarrow F'(t) = \sec^2\sqrt{1+t^2}\cdot\dfrac{1}{2\sqrt{1+t^2}}\cdot 2t = \dfrac{t\sec^2\sqrt{1+t^2}}{\sqrt{1+t^2}}$

40. $G(z) = (1+\cos^2 z)^3 \Rightarrow G'(z) = 3(1+\cos^2 z)^2\,[2(\cos z)(-\sin z)] = -6\cos z\,\sin z(1+\cos^2 z)^2$

41. $y = \sin^2(x^2+1) \quad \Rightarrow \quad y' = 2\sin(x^2+1)\cdot\cos(x^2+1)\cdot 2x = 4x\sin(x^2+1)\,\cos(x^2+1)$

42. $y = e^{\sin 2x} + \sin(e^{2x}) \quad \Rightarrow$

$$y' = e^{\sin 2x}\frac{d}{dx}\sin 2x + \cos(e^{2x})\frac{d}{dx}e^{2x} = e^{\sin 2x}(\cos 2x)\cdot 2 + \cos(e^{2x})\,e^{2x}\cdot 2 = 2\cos 2x\,e^{\sin 2x} + 2e^{2x}\cos(e^{2x})$$

43. $g(x) = \sin\left(\dfrac{e^x}{1+e^x}\right) \quad \Rightarrow$

$$g'(x) = \cos\left(\frac{e^x}{1+e^x}\right)\cdot\frac{(1+e^x)e^x - e^x(e^x)}{(1+e^x)^2} = \cos\left(\frac{e^x}{1+e^x}\right)\cdot\frac{e^x(1+e^x-e^x)}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}\cos\left(\frac{e^x}{1+e^x}\right)$$

44. $f(t) = e^{1/t}\sqrt{t^2-1} \quad \Rightarrow$

$$f'(t) = e^{1/t}\cdot\frac{1}{2\sqrt{t^2-1}}\cdot 2t + \sqrt{t^2-1}\cdot e^{1/t}\cdot\left(-\frac{1}{t^2}\right) \quad \left[\frac{1}{t} = t^{-1};\ \frac{d}{dt}\left(t^{-1}\right) = -t^{-2} = -\frac{1}{t^2}\right]$$

$$= e^{1/t}\left(\frac{t}{\sqrt{t^2-1}} - \frac{\sqrt{t^2-1}}{t^2}\right) \text{ or } e^{1/t}\left(\frac{t^3-t^2+1}{t^2\sqrt{t^2-1}}\right)$$

45. $f(t) = \tan(\sec(\cos t)) \quad \Rightarrow$

$$f'(t) = \sec^2(\sec(\cos t))\,\frac{d}{dt}\,\sec(\cos t) = \sec^2(\sec(\cos t))[\sec(\cos t)\,\tan(\cos t)]\,\frac{d}{dt}\,\cos t$$

$$= -\sec^2(\sec(\cos t))\,\sec(\cos t)\,\tan(\cos t)\,\sin t$$

46. $y = \sqrt{x+\sqrt{x+\sqrt{x}}} \quad \Rightarrow \quad y' = \frac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-1/2}\left[1+\frac{1}{2}\left(x+\sqrt{x}\right)^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right)\right]$

47. $f(x) = e^{\sin^2(x^2)} \Rightarrow f'(x) = e^{\sin^2(x^2)}\cdot 2\sin(x^2)\cdot\cos(x^2)\cdot 2x = 4x\sin(x^2)\,\cos(x^2)e^{\sin^2(x^2)}$

48. $y = 2^{3^{4^x}} \quad \Rightarrow$

$y' = 2^{3^{4^x}}(\ln 2)\dfrac{d}{dx}\,3^{4^x} = 2^{3^{4^x}}(\ln 2)3^{4^x}(\ln 3)\,\dfrac{d}{dx}\,4^x = 2^{3^{4^x}}(\ln 2)3^{4^x}(\ln 3)4^x(\ln 4) = (\ln 2)(\ln 3)(\ln 4)4^x 3^{4^x} 2^{3^{4^x}}$

49. $y = \left(3^{\cos(x^2)} - 1\right)^4 \quad \Rightarrow$

$y' = 4\left(3^{\cos(x^2)} - 1\right)^3 \cdot 3^{\cos(x^2)} \ln 3 \cdot (-\sin(x^2)) \cdot 2x = -8x(\ln 3)\sin(x^2)\,3^{\cos(x^2)}\left(3^{\cos(x^2)} - 1\right)^3$

50. $y = \sin(\theta + \tan(\theta + \cos\theta)) \quad \Rightarrow \quad y' = \cos(\theta + \tan(\theta + \cos\theta)) \cdot [1 + \sec^2(\theta + \cos\theta) \cdot (1 - \sin\theta)]$

51. $y = \cos\sqrt{\sin(\tan \pi x)} = \cos(\sin(\tan \pi x))^{1/2} \quad \Rightarrow$

$y' = -\sin(\sin(\tan \pi x))^{1/2} \cdot \dfrac{d}{dx}\,(\sin(\tan \pi x))^{1/2} = -\sin(\sin(\tan \pi x))^{1/2} \cdot \tfrac{1}{2}(\sin(\tan \pi x))^{-1/2} \cdot \dfrac{d}{dx}\,(\sin(\tan \pi x))$

$= \dfrac{-\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}} \cdot \cos(\tan \pi x) \cdot \dfrac{d}{dx}\tan \pi x = \dfrac{-\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}} \cdot \cos(\tan \pi x) \cdot \sec^2(\pi x) \cdot \pi$

$= \dfrac{-\pi\cos(\tan \pi x)\sec^2(\pi x)\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}}$

52. $y = \sin^3(\cos(x^2)) \quad \Rightarrow$

$y' = 3\sin^2(\cos(x^2)) \cdot \cos(\cos(x^2)) \cdot [-\sin(x^2) \cdot 2x] = -6x\sin(x^2)\,\sin^2(\cos(x^2))\,\cos(\cos(x^2))$

53. $y = \cos(\sin 3\theta) \quad \Rightarrow \quad y' = -\sin(\sin 3\theta) \cdot (\cos 3\theta) \cdot 3 = -3\cos 3\theta\,\sin(\sin 3\theta) \quad \Rightarrow$

$y'' = -3\left[(\cos 3\theta)\cos(\sin 3\theta)(\cos 3\theta) \cdot 3 + \sin(\sin 3\theta)(-\sin 3\theta) \cdot 3\right] = -9\cos^2(3\theta)\cos(\sin 3\theta) + 9(\sin 3\theta)\sin(\sin 3\theta)$

54. $y = \left(1 + \sqrt{x}\right)^3 \quad \Rightarrow \quad y' = 3\left(1 + \sqrt{x}\right)^2\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{3\left(1 + \sqrt{x}\right)^2}{2\sqrt{x}} \quad \Rightarrow$

$$y'' = \frac{2\sqrt{x} \cdot 3 \cdot 2\left(1 + \sqrt{x}\right)^1 \cdot \dfrac{1}{2\sqrt{x}} - 3\left(1 + \sqrt{x}\right)^2 \cdot 2 \cdot \dfrac{1}{2\sqrt{x}}}{\left(2\sqrt{x}\right)^2}$$

$$= \frac{6\sqrt{x}\left(1 + \sqrt{x}\right) \cdot \dfrac{1}{\sqrt{x}} - 3\left(1 + 2\sqrt{x} + x\right) \cdot \dfrac{1}{\sqrt{x}}}{4x} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{6\sqrt{x} + 6x - 3 - 6\sqrt{x} - 3x}{4x\sqrt{x}}$$

$$= \frac{3x - 3}{4x\sqrt{x}} \text{ or } \frac{3(x-1)}{4x^{3/2}}$$

55. $y = \sqrt{\cos x} \quad \Rightarrow \quad y' = \dfrac{1}{2\sqrt{\cos x}}(-\sin x) = -\dfrac{\sin x}{2\sqrt{\cos x}}$. With $y' = \dfrac{-\sin x}{2\sqrt{\cos x}}$, we get

$$y'' = \frac{2\sqrt{\cos x} \cdot (-\cos x) - (-\sin x)\left(2 \cdot \dfrac{1}{2\sqrt{\cos x}}\,(-\sin x)\right)}{\left(2\sqrt{\cos x}\,\right)^2} = \frac{-2\cos x\sqrt{\cos x} - \dfrac{\sin^2 x}{\sqrt{\cos x}}}{4\cos x} \cdot \frac{\sqrt{\cos x}}{\sqrt{\cos x}}$$

$$= \frac{-2\cos x \cdot \cos x - \sin^2 x}{4\cos x\sqrt{\cos x}} = -\frac{2\cos^2 x + \sin^2 x}{4(\cos x)^{3/2}}$$

Using the identity $\sin^2 x + \cos^2 x = 1$, the answer may be written as $-\dfrac{1 + \cos^2 x}{4(\cos x)^{3/2}}$.

56. $y = e^{e^x} \;\Rightarrow\; y' = e^{e^x} \cdot (e^x)' = e^{e^x} \cdot e^x \;\Rightarrow$

$y'' = e^{e^x} \cdot (e^x)' + e^x \cdot \left(e^{e^x}\right)' = e^{e^x} \cdot e^x + e^x \cdot e^{e^x} \cdot e^x = e^{e^x} \cdot e^x(1+e^x)$ or $e^{e^x+x}(1+e^x)$

57. $y = 2^x \;\Rightarrow\; y' = 2^x \ln 2$. At $(0,1)$, $y' = 2^0 \ln 2 = \ln 2$, and an equation of the tangent line is $y - 1 = (\ln 2)(x - 0)$ or $y = (\ln 2)x + 1$.

58. $y = \sqrt{1+x^3} = (1+x^3)^{1/2} \;\Rightarrow\; y' = \frac{1}{2}(1+x^3)^{-1/2} \cdot 3x^2 = \dfrac{3x^2}{2\sqrt{1+x^3}}$. At $(2,3)$, $y' = \dfrac{3 \cdot 4}{2\sqrt{9}} = 2$, and an equation of the tangent line is $y - 3 = 2(x-2)$, or $y = 2x - 1$.

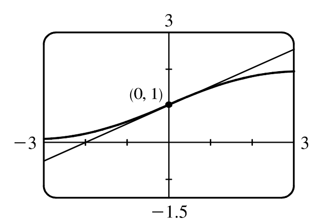
59. $y = \sin(\sin x) \;\Rightarrow\; y' = \cos(\sin x) \cdot \cos x$. At $(\pi, 0)$, $y' = \cos(\sin \pi) \cdot \cos \pi = \cos(0) \cdot (-1) = 1(-1) = -1$, and an equation of the tangent line is $y - 0 = -1(x - \pi)$, or $y = -x + \pi$.

60. $y = xe^{-x^2} \;\Rightarrow\; y' = xe^{-x^2}(-2x) + e^{-x^2}(1) = e^{-x^2}(-2x^2+1)$. At $(0,0)$, $y' = e^0(1) = 1$, and an equation of the tangent line is $y - 0 = 1(x-0)$ or $y = x$.

61. (a) $y = \dfrac{2}{1+e^{-x}} \;\Rightarrow\; y' = \dfrac{(1+e^{-x})(0) - 2(-e^{-x})}{(1+e^{-x})^2} = \dfrac{2e^{-x}}{(1+e^{-x})^2}$.

At $(0,1)$, $y' = \dfrac{2e^0}{(1+e^0)^2} = \dfrac{2(1)}{(1+1)^2} = \dfrac{2}{2^2} = \dfrac{1}{2}$. So an equation of the tangent line is $y - 1 = \frac{1}{2}(x-0)$ or $y = \frac{1}{2}x + 1$.

(b)



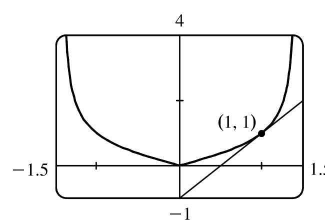
62. (a) For $x > 0$, $|x| = x$, and $y = f(x) = \dfrac{x}{\sqrt{2-x^2}} \;\Rightarrow$

$$f'(x) = \frac{\sqrt{2-x^2}\,(1) - x\left(\frac{1}{2}\right)(2-x^2)^{-1/2}(-2x)}{\left(\sqrt{2-x^2}\right)^2} \cdot \frac{(2-x^2)^{1/2}}{(2-x^2)^{1/2}}$$

$$= \frac{(2-x^2)+x^2}{(2-x^2)^{3/2}} = \frac{2}{(2-x^2)^{3/2}}$$

So at $(1,1)$, the slope of the tangent line is $f'(1) = 2$ and its equation is $y - 1 = 2(x-1)$ or $y = 2x - 1$.

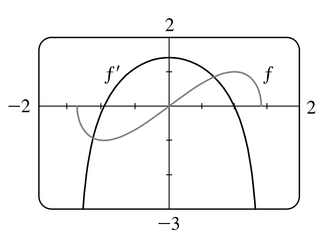
(b)



63. (a) $f(x) = x\sqrt{2-x^2} = x(2-x^2)^{1/2} \;\Rightarrow$

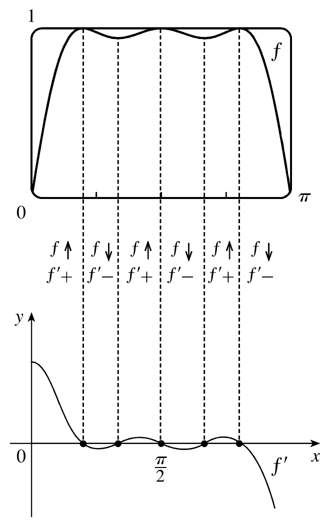
$$f'(x) = x \cdot \tfrac{1}{2}(2-x^2)^{-1/2}(-2x) + (2-x^2)^{1/2} \cdot 1 = (2-x^2)^{-1/2}\left[-x^2 + (2-x^2)\right] = \frac{2-2x^2}{\sqrt{2-x^2}}$$

(b)



$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

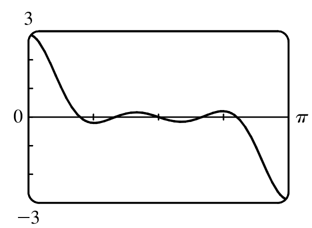
64. (a)



From the graph of f, we see that there are 5 horizontal tangents, so there must be 5 zeros on the graph of f'. From the symmetry of the graph of f, we must have the graph of f' as high at $x = 0$ as it is low at $x = \pi$. The intervals of increase and decrease as well as the signs of f' are indicated in the figure.

(b) $f(x) = \sin(x + \sin 2x) \quad \Rightarrow$

$$f'(x) = \cos(x + \sin 2x) \cdot \frac{d}{dx}\,(x + \sin 2x) = \cos(x + \sin 2x)(1 + 2\cos 2x)$$



65. For the tangent line to be horizontal, $f'(x) = 0$. $f(x) = 2\sin x + \sin^2 x \quad \Rightarrow \quad f'(x) = 2\cos x + 2\sin x\,\cos x = 0 \quad \Leftrightarrow$ $2\cos x(1 + \sin x) = 0 \quad \Leftrightarrow \quad \cos x = 0$ or $\sin x = -1$, so $x = \frac{\pi}{2} + 2n\pi$ or $\frac{3\pi}{2} + 2n\pi$, where n is any integer. Now $f\left(\frac{\pi}{2}\right) = 3$ and $f\left(\frac{3\pi}{2}\right) = -1$, so the points on the curve with a horizontal tangent are $\left(\frac{\pi}{2} + 2n\pi, 3\right)$ and $\left(\frac{3\pi}{2} + 2n\pi, -1\right)$, where n is any integer.

66. $y = \sqrt{1 + 2x} \quad \Rightarrow \quad y' = \frac{1}{2}(1 + 2x)^{-1/2} \cdot 2 = \dfrac{1}{\sqrt{1 + 2x}}$. The line $6x + 2y = 1$ $\left(\text{or } y = -3x + \frac{1}{2}\right)$ has slope -3, so the tangent line perpendicular to it must have slope $\frac{1}{3}$. Thus, $\dfrac{1}{3} = \dfrac{1}{\sqrt{1 + 2x}} \quad \Leftrightarrow \quad \sqrt{1 + 2x} = 3 \quad \Rightarrow \quad 1 + 2x = 9 \quad \Leftrightarrow$ $2x = 8 \quad \Leftrightarrow \quad x = 4$. When $x = 4$, $y = \sqrt{1 + 2(4)} = 3$, so the point is $(4, 3)$.

67. $F(x) = f(g(x)) \quad \Rightarrow \quad F'(x) = f'(g(x)) \cdot g'(x)$, so $F'(5) = f'(g(5)) \cdot g'(5) = f'(-2) \cdot 6 = 4 \cdot 6 = 24$.

68. $h(x) = \sqrt{4 + 3f(x)} \quad \Rightarrow \quad h'(x) = \frac{1}{2}(4 + 3f(x))^{-1/2} \cdot 3f'(x)$, so

$h'(1) = \frac{1}{2}(4 + 3f(1))^{-1/2} \cdot 3f'(1) = \frac{1}{2}(4 + 3 \cdot 7)^{-1/2} \cdot 3 \cdot 4 = \frac{6}{\sqrt{25}} = \frac{6}{5}$.

69. (a) $h(x) = f(g(x)) \quad \Rightarrow \quad h'(x) = f'(g(x)) \cdot g'(x)$, so $h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30$.

(b) $H(x) = g(f(x)) \quad \Rightarrow \quad H'(x) = g'(f(x)) \cdot f'(x)$, so $H'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36$.

70. (a) $F(x) = f(f(x)) \quad \Rightarrow \quad F'(x) = f'(f(x)) \cdot f'(x)$, so $F'(2) = f'(f(2)) \cdot f'(2) = f'(1) \cdot 5 = 4 \cdot 5 = 20$.

(b) $G(x) = g(g(x)) \quad \Rightarrow \quad G'(x) = g'(g(x)) \cdot g'(x)$, so $G'(3) = g'(g(3)) \cdot g'(3) = g'(2) \cdot 9 = 7 \cdot 9 = 63$.

71. (a) From the graphs of f and g, we obtain the following values: $g(1) = 4$ since the point $(1, 4)$ is on the graph of g; $f'(4) = -\frac{1}{4}$ since the slope of the line segment between $(2, 4)$ and $(6, 3)$ is $\dfrac{3-4}{6-2} = -\dfrac{1}{4}$; and $g'(1) = -1$ since the slope of the line segment between $(0, 5)$ and $(3, 2)$ is $\dfrac{2-5}{3-0} = -1$. Now $u(x) = f(g(x))$, so

$u'(1) = f'(g(1))\,g'(1) = f'(4)\,g'(1) = -\frac{1}{4}(-1) = \frac{1}{4}$.

(b) From the graphs of f and g, we obtain the following values: $f(1) = 2$ since the point $(1, 2)$ is on the graph of f; $g'(2) = g'(1) = -1$ [see part (a)]; and $f'(1) = 2$ since the slope of the line segment between $(0, 0)$ and $(2, 4)$ is $\dfrac{4-0}{2-0} = 2$. Now $v(x) = g(f(x))$, so $v'(1) = g'(f(1))f'(1) = g'(2)f'(1) = -1(2) = -2$.

(c) From part (a), we have $g(1) = 4$ and $g'(1) = -1$. From the graph of g we obtain $g'(4) = \frac{1}{2}$ since the slope of the line segment between $(3, 2)$ and $(7, 4)$ is $\dfrac{4-2}{7-3} = \dfrac{1}{2}$. Now $w(x) = g(g(x))$, so

$w'(1) = g'(g(1))\,g'(1) = g'(4)\,g'(1) = \frac{1}{2}(-1) = -\frac{1}{2}$.

72. (a) $h(x) = f(f(x)) \quad\Rightarrow\quad h'(x) = f'(f(x))f'(x)$. So $h'(2) = f'(f(2))f'(2) = f'(1)f'(2) \approx (-1)(-1) = 1$.

(b) $g(x) = f(x^2) \quad\Rightarrow\quad g'(x) = f'(x^2) \cdot \dfrac{d}{dx}\,(x^2) = f'(x^2)(2x)$. So $g'(2) = f'(2^2)(2 \cdot 2) = 4f'(4) \approx 4(2) = 8$.

73. The point $(3, 2)$ is on the graph of f, so $f(3) = 2$. The tangent line at $(3, 2)$ has slope $\dfrac{\Delta y}{\Delta x} = \dfrac{-4}{6} = -\dfrac{2}{3}$.

$g(x) = \sqrt{f(x)} \quad\Rightarrow\quad g'(x) = \frac{1}{2}[f(x)]^{-1/2} \cdot f'(x) \quad\Rightarrow$

$g'(3) = \frac{1}{2}[f(3)]^{-1/2} \cdot f'(3) = \frac{1}{2}(2)^{-1/2}(-\frac{2}{3}) = -\dfrac{1}{3\sqrt{2}}$ or $-\dfrac{1}{6}\sqrt{2}$.

74. (a) $F(x) = f(x^\alpha) \quad\Rightarrow\quad F'(x) = f'(x^\alpha)\dfrac{d}{dx}\,(x^\alpha) = f'(x^\alpha)\alpha x^{\alpha-1}$

(b) $G(x) = [f(x)]^\alpha \quad\Rightarrow\quad G'(x) = \alpha\,[f(x)]^{\alpha-1}\,f'(x)$

75. (a) $F(x) = f(e^x) \quad\Rightarrow\quad F'(x) = f'(e^x)\dfrac{d}{dx}\,(e^x) = f'(e^x)e^x$

(b) $G(x) = e^{f(x)} \quad\Rightarrow\quad G'(x) = e^{f(x)}\dfrac{d}{dx}\,f(x) = e^{f(x)}f'(x)$

76. (a) $g(x) = e^{cx} + f(x) \quad\Rightarrow\quad g'(x) = e^{cx} \cdot c + f'(x) \quad\Rightarrow\quad g'(0) = e^0 \cdot c + f'(0) = c + 5$.

$g'(x) = ce^{cx} + f'(x) \quad\Rightarrow\quad g''(x) = ce^{cx} \cdot c + f''(x) \quad\Rightarrow\quad g''(0) = c^2e^0 + f''(0) = c^2 - 2$.

(b) $h(x) = e^{kx}f(x) \quad\Rightarrow\quad h'(x) = e^{kx}f'(x) + f(x) \cdot ke^{kx} \quad\Rightarrow\quad h'(0) = e^0f'(0) + f(0) \cdot ke^0 = 5 + 3k$.

An equation of the tangent line to the graph of h at the point $(0, h(0)) = (0, f(0)) = (0, 3)$ is

$y - 3 = (5 + 3k)(x - 0)$ or $y = (5 + 3k)x + 3$.

77. $r(x) = f(g(h(x))) \quad\Rightarrow\quad r'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$, so

$r'(1) = f'(g(h(1))) \cdot g'(h(1)) \cdot h'(1) = f'(g(2)) \cdot g'(2) \cdot 4 = f'(3) \cdot 5 \cdot 4 = 6 \cdot 5 \cdot 4 = 120$

78. $f(x) = xg(x^2) \quad \Rightarrow \quad f'(x) = xg'(x^2)\,2x + g(x^2)\cdot 1 = 2x^2 g'(x^2) + g(x^2) \quad \Rightarrow$

$f''(x) = 2x^2 g''(x^2)\,2x + g'(x^2)\,4x + g'(x^2)\,2x = 4x^3 g''(x^2) + 4xg'(x^2) + 2xg'(x^2) = 6xg'(x^2) + 4x^3 g''(x^2)$

79. $F(x) = f(3f(4f(x))) \quad \Rightarrow$

$$\begin{aligned} F'(x) &= f'(3f(4f(x)))\cdot \frac{d}{dx}(3f(4f(x))) = f'(3f(4f(x)))\cdot 3f'(4f(x))\cdot \frac{d}{dx}(4f(x)) \\ &= f'(3f(4f(x)))\cdot 3f'(4f(x))\cdot 4f'(x), \quad \text{so} \end{aligned}$$

$F'(0) = f'(3f(4f(0)))\cdot 3f'(4f(0))\cdot 4f'(0) = f'(3f(4\cdot 0))\cdot 3f'(4\cdot 0)\cdot 4\cdot 2 = f'(3\cdot 0)\cdot 3\cdot 2\cdot 4\cdot 2 = 2\cdot 3\cdot 2\cdot 4\cdot 2 = 96.$

80. $F(x) = f(xf(xf(x))) \quad \Rightarrow$

$$\begin{aligned} F'(x) &= f'(xf(xf(x)))\cdot \frac{d}{dx}\,(xf(xf(x))) = f'(xf(xf(x)))\cdot \left[x\cdot f'(xf(x))\cdot \frac{d}{dx}(xf(x)) + f(xf(x))\cdot 1\right] \\ &= f'(xf(xf(x)))\cdot [xf'(xf(x))\cdot (xf'(x) + f(x)\cdot 1) + f(xf(x))], \quad \text{so} \end{aligned}$$

$$\begin{aligned} F'(1) &= f'(f(f(1)))\cdot [f'(f(1))\cdot (f'(1) + f(1)) + f(f(1))] = f'(f(2))\cdot [f'(2)\cdot (4+2) + f(2)] \\ &= f'(3)\cdot [5\cdot 6 + 3] = 6\cdot 33 = 198. \end{aligned}$$

81. $y = e^{2x}(A\cos 3x + B\sin 3x) \quad \Rightarrow$

$$\begin{aligned} y' &= e^{2x}(-3A\sin 3x + 3B\cos 3x) + (A\cos 3x + B\sin 3x)\cdot 2e^{2x} \\ &= e^{2x}(-3A\sin 3x + 3B\cos 3x + 2A\cos 3x + 2B\sin 3x) \\ &= e^{2x}[(2A+3B)\cos 3x + (2B-3A)\sin 3x] \quad \Rightarrow \end{aligned}$$

$$\begin{aligned} y'' &= e^{2x}[-3(2A+3B)\sin 3x + 3(2B-3A)\cos 3x] + [(2A+3B)\cos 3x + (2B-3A)\sin 3x]\cdot 2e^{2x} \\ &= e^{2x}\{[-3(2A+3B) + 2(2B-3A)]\sin 3x + [3(2B-3A) + 2(2A+3B)]\cos 3x\} \\ &= e^{2x}[(-12A-5B)\sin 3x + (-5A+12B)\cos 3x] \end{aligned}$$

Substitute the expressions for y, y', and y'' in $y'' - 4y' + 13y$ to get

$$\begin{aligned} y'' - 4y' + 13y &= e^{2x}[(-12A-5B)\sin 3x + (-5A+12B)\cos 3x] \\ &\qquad - 4e^{2x}[(2A+3B)\cos 3x + (2B-3A)\sin 3x] + 13e^{2x}(A\cos 3x + B\sin 3x) \\ &= e^{2x}[(-12A - 5B - 8B + 12A + 13B)\sin 3x + (-5A + 12B - 8A - 12B + 13A)\cos 3x] \\ &= e^{2x}[(0)\sin 3x + (0)\cos 3x] = 0 \end{aligned}$$

Thus, the function y satisfies the differential equation $y'' - 4y' + 13y = 0$.

82. $y = e^{rx} \quad \Rightarrow \quad y' = re^{rx} \quad \Rightarrow \quad y'' = r^2 e^{rx}$. Substituting y, y', and y'' into $y'' - 4y' + y = 0$ gives us $r^2 e^{rx} - 4re^{rx} + e^{rx} = 0 \quad \Rightarrow \quad e^{rx}(r^2 - 4r + 1) = 0$. Since $e^{rx} \neq 0$, we must have $r^2 - 4r + 1 = 0 \quad \Rightarrow \quad r = \dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$.

83. The use of D, D^2, ..., D^n is just a derivative notation (see text page 159). In general, $Df(2x) = 2f'(2x)$, $D^2 f(2x) = 4f''(2x)$, ..., $D^n f(2x) = 2^n f^{(n)}(2x)$. Since $f(x) = \cos x$ and $50 = 4(12) + 2$, we have $f^{(50)}(x) = f^{(2)}(x) = -\cos x$, so $D^{50} \cos 2x = -2^{50} \cos 2x$.

84. $f(x) = xe^{-x}$, $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$, $f''(x) = -e^{-x} + (1-x)(-e^{-x}) = (x-2)e^{-x}$. Similarly, $f'''(x) = (3-x)e^{-x}$, $f^{(4)}(x) = (x-4)e^{-x}$, ..., $f^{(1000)}(x) = (x - 1000)e^{-x}$.

85. $s(t) = 10 + \frac{1}{4}\sin(10\pi t) \quad \Rightarrow \quad$ the velocity after t seconds is $v(t) = s'(t) = \frac{1}{4}\cos(10\pi t)(10\pi) = \frac{5\pi}{2}\cos(10\pi t)$ cm/s.

86. (a) $s = A\cos(\omega t + \delta) \quad \Rightarrow \quad$ velocity $= s' = -\omega A \sin(\omega t + \delta)$.

(b) If $A \neq 0$ and $\omega \neq 0$, then $s' = 0 \quad \Leftrightarrow \quad \sin(\omega t + \delta) = 0 \quad \Leftrightarrow \quad \omega t + \delta = n\pi \quad \Leftrightarrow \quad t = \dfrac{n\pi - \delta}{\omega}$, n an integer.

87. (a) $B(t) = 4.0 + 0.35 \sin \dfrac{2\pi t}{5.4} \quad \Rightarrow \quad \dfrac{dB}{dt} = \left(0.35 \cos \dfrac{2\pi t}{5.4}\right)\left(\dfrac{2\pi}{5.4}\right) = \dfrac{0.7\pi}{5.4} \cos \dfrac{2\pi t}{5.4} = \dfrac{7\pi}{54} \cos \dfrac{2\pi t}{5.4}$

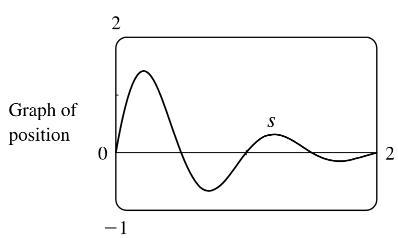
(b) At $t = 1$, $\dfrac{dB}{dt} = \dfrac{7\pi}{54} \cos \dfrac{2\pi}{5.4} \approx 0.16$.

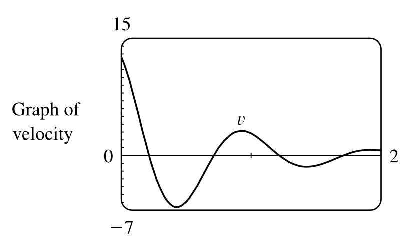
88. $L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right] \quad \Rightarrow \quad L'(t) = 2.8 \cos\left[\frac{2\pi}{365}(t-80)\right]\left(\frac{2\pi}{365}\right)$.

On March 21, $t = 80$, and $L'(80) \approx 0.0482$ hours per day. On May 21, $t = 141$, and $L'(141) \approx 0.02398$, which is approximately one-half of $L'(80)$.

89. $s(t) = 2e^{-1.5t} \sin 2\pi t \quad \Rightarrow$

$v(t) = s'(t) = 2[e^{-1.5t}(\cos 2\pi t)(2\pi) + (\sin 2\pi t)e^{-1.5t}(-1.5)] = 2e^{-1.5t}(2\pi \cos 2\pi t - 1.5 \sin 2\pi t)$

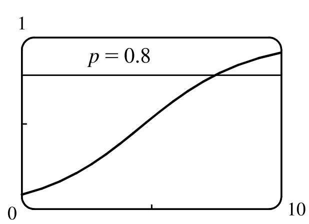




90. (a) $\displaystyle\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{1}{1 + ae^{-kt}} = \frac{1}{1 + a \cdot 0} = 1$, since $k > 0 \quad \Rightarrow \quad -kt \to -\infty \quad \Rightarrow \quad e^{-kt} \to 0$. As time increases, the proportion of the population that has heard the rumor approaches 1; that is, everyone in the population has heard the rumor.

(b) $p(t) = (1 + ae^{-kt})^{-1} \quad \Rightarrow \quad \dfrac{dp}{dt} = -(1 + ae^{-kt})^{-2}(-kae^{-kt}) = \dfrac{kae^{-kt}}{(1 + ae^{-kt})^2}$

(c)



From the graph of $p(t) = (1 + 10e^{-0.5t})^{-1}$, it seems that $p(t) = 0.8$ (indicating that 80% of the population has heard the rumor) when $t \approx 7.4$ hours.

91. (a) Use $C(t) = ate^{bt}$ with $a = 0.00225$ and $b = -0.0467$ to get $C'(t) = a(t \cdot e^{bt} \cdot b + e^{bt} \cdot 1) = a(bt + 1)e^{bt}$. $C'(10) = 0.00225(0.533)e^{-0.467} \approx 0.00075$, so the BAC was increasing at approximately $0.00075\ (\text{g/dL})/\text{min}$ after 10 minutes.

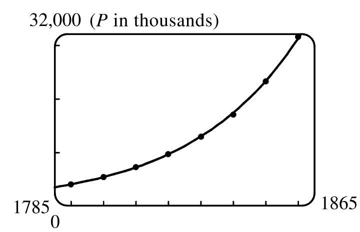
(b) A half an hour later gives us $t = 10 + 30 = 40$. $C'(40) = 0.00225(-0.868)e^{-1.868} \approx -0.00030$, so the BAC was decreasing at approximately $0.00030\ (\text{g/dL})/\text{min}$ after 40 minutes.

92. (a) The derivative dV/dr represents the rate of change of the volume with respect to the radius and the derivative dV/dt represents the rate of change of the volume with respect to time.

(b) Since $V = \frac{4}{3}\pi r^3$, $\dfrac{dV}{dt} = \dfrac{dV}{dr}\dfrac{dr}{dt} = 4\pi r^2 \dfrac{dr}{dt}$.

93. By the Chain Rule, $a(t) = \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = \dfrac{dv}{ds}v(t) = v(t)\dfrac{dv}{ds}$. The derivative dv/dt is the rate of change of the velocity with respect to time (in other words, the acceleration) whereas the derivative dv/ds is the rate of change of the velocity with respect to the displacement.

94. (a) $P = ab^t$ with $a = 4.502714 \times 10^{-20}$ and $b = 1.029953851$, where P is measured in thousands of people. The fit appears to be very good.



(b) **For 1800:** $m_1 = \dfrac{5308 - 3929}{1800 - 1790} = 137.9$, $m_2 = \dfrac{7240 - 5308}{1810 - 1800} = 193.2$.

So $P'(1800) \approx (m_1 + m_2)/2 = 165.55$ thousand people/year.

For 1850: $m_1 = \dfrac{23{,}192 - 17{,}063}{1850 - 1840} = 612.9$, $m_2 = \dfrac{31{,}443 - 23{,}192}{1860 - 1850} = 825.1$.

So $P'(1850) \approx (m_1 + m_2)/2 = 719$ thousand people/year.

(c) Using $P'(t) = ab^t \ln b$ (from Formula 5) with the values of a and b from part (a), we get $P'(1800) \approx 156.85$ and $P'(1850) \approx 686.07$. These estimates are somewhat less than the ones in part (b).

(d) $P(1870) \approx 41{,}946.56$. The difference of 3.4 million people is most likely due to the Civil War (1861–1865).

95. (a) If f is even, then $f(x) = f(-x)$. Using the Chain Rule to differentiate this equation, we get $f'(x) = f'(-x)\dfrac{d}{dx}(-x) = -f'(-x)$. Thus, $f'(-x) = -f'(x)$, so f' is odd.

(b) If f is odd, then $f(x) = -f(-x)$. Differentiating this equation, we get $f'(x) = -f'(-x)(-1) = f'(-x)$, so f' is even.

96.
$$\left[\frac{f(x)}{g(x)}\right]' = \left\{f(x)\,[g(x)]^{-1}\right\}' = f'(x)\,[g(x)]^{-1} + (-1)\,[g(x)]^{-2}\,g'(x)f(x)$$
$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

This is an alternative derivation of the *formula* in the Quotient Rule. But part of the purpose of the Quotient Rule is to show that if f and g are differentiable, so is f/g. The proof in Section 3.2 does that; this one doesn't.

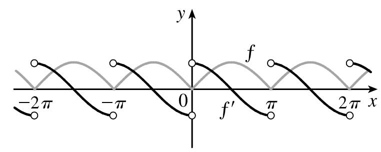
97. Since $\theta° = \left(\frac{\pi}{180}\right)\theta$ rad, we have $\dfrac{d}{d\theta}(\sin\theta°) = \dfrac{d}{d\theta}\left(\sin\frac{\pi}{180}\theta\right) = \frac{\pi}{180}\cos\frac{\pi}{180}\theta = \frac{\pi}{180}\cos\theta°$.

98. (a) $f(x) = |x| = \sqrt{x^2} = (x^2)^{1/2} \quad\Rightarrow\quad f'(x) = \frac{1}{2}(x^2)^{-1/2}(2x) = x/\sqrt{x^2} = x/|x|$ for $x \neq 0$.

f is not differentiable at $x = 0$.

(b) $f(x) = |\sin x| = \sqrt{\sin^2 x} \quad\Rightarrow$

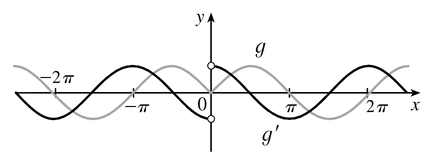
$$f'(x) = \tfrac{1}{2}(\sin^2 x)^{-1/2}\, 2\sin x\,\cos x = \frac{\sin x}{|\sin x|}\cos x = \begin{cases} \cos x & \text{if } \sin x > 0 \\ -\cos x & \text{if } \sin x < 0 \end{cases}$$



f is not differentiable when $x = n\pi$, n an integer.

(c) $g(x) = \sin|x| = \sin\sqrt{x^2} \quad\Rightarrow$

$$g'(x) = \cos|x|\cdot\frac{x}{|x|} = \frac{x}{|x|}\cos x = \begin{cases} \cos x & \text{if } x > 0 \\ -\cos x & \text{if } x < 0 \end{cases}$$



g is not differentiable at 0.

99. $y = b^x \quad\Rightarrow\quad y' = b^x \ln b$, so the slope of the tangent line to the curve $y = b^x$ at the point (a, b^a) is $b^a \ln b$. An equation of this tangent line is then $y - b^a = b^a \ln b\,(x - a)$. If c is the x-intercept of this tangent line, then $0 - b^a = b^a \ln b\,(c - a) \quad\Rightarrow$ $-1 = \ln b\,(c - a) \quad\Rightarrow\quad \dfrac{-1}{\ln b} = c - a \quad\Rightarrow\quad |c - a| = \left|\dfrac{-1}{\ln b}\right| = \dfrac{1}{|\ln b|}$. The distance between $(a, 0)$ and $(c, 0)$ is $|c - a|$, and this distance is the constant $\dfrac{1}{|\ln b|}$ for any a. [*Note*: The absolute value is needed for the case $0 < b < 1$ because $\ln b$ is negative there. If $b > 1$, we can write $a - c = 1/(\ln b)$ as the constant distance between $(a, 0)$ and $(c, 0)$.]

100. $y = b^x \quad\Rightarrow\quad y' = b^x \ln b$, so the slope of the tangent line to the curve $y = b^x$ at the point (x_0, y_0) is $b^{x_0} \ln b$. An equation of this tangent line is then $y - y_0 = b^{x_0} \ln b\,(x - x_0)$. Since this tangent line must pass through $(0, 0)$, we have $0 - y_0 = b^{x_0} \ln b\,(0 - x_0)$, or $y_0 = b^{x_0}(\ln b)\,x_0$. Since (x_0, y_0) is a point on the exponential curve $y = b^x$, we also have $y_0 = b^{x_0}$. Equating the expressions for y_0 gives $b^{x_0} = b^{x_0}(\ln b)\,x_0 \quad\Rightarrow\quad 1 = (\ln b)\,x_0 \quad\Rightarrow\quad x_0 = 1/(\ln b)$.

So $y_0 = b^{x_0} = e^{x_0 \ln b}$ [by Formula 1.5.10] $= e^{(1/(\ln b))\ln b} = e^1 = e$.

101. Let $j(x) = g(h(x))$ so that $F(x) = f(g(h(x))) = f(j(x))$. By the Chain Rule, we have $j'(x) = g'(h(x))\cdot h'(x)$ and, by the Chain Rule and substitution, we have $F'(x) = f'(j(x))\cdot j'(x) = f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)$.

102. $F(x) = f(g(x)) \quad\Rightarrow\quad F'(x) = f'(g(x))\cdot g'(x)$ by the Chain Rule. By the Product Rule and Chain Rule we have

$$\begin{aligned} F''(x) &= f'(g(x))\cdot g''(x) + g'(x)\cdot\frac{d}{dx}f'(g(x)) = f'(g(x))\cdot g''(x) + g'(x)\cdot f''(g(x))\,g'(x) \\ &= f''(g(x))[g'(x)]^2 + f'(g(x))\cdot g''(x) \end{aligned}$$

APPLIED PROJECT Where Should a Pilot Start Descent?

1. Condition (i) will hold if and only if all of the following four conditions hold:

(α) $P(0) = 0$

(β) $P'(0) = 0$ (for a smooth landing)

(γ) $P'(\ell) = 0$ (since the plane is cruising horizontally when it begins its descent)

(δ) $P(\ell) = h$.

First of all, condition α implies that $P(0) = d = 0$, so $P(x) = ax^3 + bx^2 + cx \quad \Rightarrow \quad P'(x) = 3ax^2 + 2bx + c$. But $P'(0) = c = 0$ by condition β. So $P'(\ell) = 3a\ell^2 + 2b\ell = \ell\,(3a\ell + 2b)$. Now by condition γ, $3a\ell + 2b = 0 \quad \Rightarrow \quad a = -\dfrac{2b}{3\ell}$.

Therefore, $P(x) = -\dfrac{2b}{3\ell}x^3 + bx^2$. Setting $P(\ell) = h$ for condition δ, we get $P(\ell) = -\dfrac{2b}{3\ell}\ell^3 + b\ell^2 = h \quad \Rightarrow$

$-\dfrac{2}{3}b\ell^2 + b\ell^2 = h \quad \Rightarrow \quad \dfrac{1}{3}b\ell^2 = h \quad \Rightarrow \quad b = \dfrac{3h}{\ell^2} \quad \Rightarrow \quad a = -\dfrac{2h}{\ell^3}$. So $y = P(x) = -\dfrac{2h}{\ell^3}x^3 + \dfrac{3h}{\ell^2}x^2$.

2. By condition (ii), $\dfrac{dx}{dt} = -v$ for all t, so $x(t) = \ell - vt$. Condition (iii) states that $\left|\dfrac{d^2y}{dt^2}\right| \le k$. By the Chain Rule,

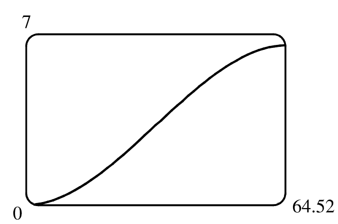
we have $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = -\dfrac{2h}{\ell^3}\left(3x^2\right)\dfrac{dx}{dt} + \dfrac{3h}{\ell^2}(2x)\dfrac{dx}{dt} = \dfrac{6hx^2v}{\ell^3} - \dfrac{6hxv}{\ell^2}$ (for $x \le \ell$) $\Rightarrow$

$\dfrac{d^2y}{dt^2} = \dfrac{6hv}{\ell^3}(2x)\dfrac{dx}{dt} - \dfrac{6hv}{\ell^2}\dfrac{dx}{dt} = -\dfrac{12hv^2}{\ell^3}x + \dfrac{6hv^2}{\ell^2}$. In particular, when $t = 0$, $x = \ell$ and so

$\left.\dfrac{d^2y}{dt^2}\right|_{t=0} = -\dfrac{12hv^2}{\ell^3}\ell + \dfrac{6hv^2}{\ell^2} = -\dfrac{6hv^2}{\ell^2}$. Thus, $\left|\dfrac{d^2y}{dt^2}\right|_{t=0} = \dfrac{6hv^2}{\ell^2} \le k$. (This condition also follows from taking $x = 0$.)

3. We substitute $k = 1385$ km/h^2, h $= 11,000$ m $= 11$ km, and $v = 480$ km/h into the result of part (b): $\frac{6(11)(480)^2}{l^2} \le$ $1385 \implies l \ge 480\sqrt{\frac{66}{1385}} \approx 104.8$ kilometers.

4. Substituting the values of h and ℓ in Problem 3 into $P(x) = -\dfrac{2h}{\ell^3}x^3 + \dfrac{3h}{\ell^2}x^2$ gives us $P(x) = ax^3 + bx^2$, where $a \approx -4.937 \times 10^{-5}$ and $b \approx 4.78 \times 10^{-3}$.



3.5 Implicitly Defined Functions

1. (a) $\dfrac{d}{dx}\left(5x^2 - y^3\right) = \dfrac{d}{dx}(7) \quad \Rightarrow \quad 10x - 3y^2\,y' = 0 \quad \Rightarrow \quad 3y^2\,y' = 10x \quad \Rightarrow \quad y' = \dfrac{10x}{3y^2}$

(b) $5x^2 - y^3 = 7 \quad \Rightarrow \quad y^3 = 5x^2 - 7 \quad \Rightarrow \quad y = \sqrt[3]{5x^2 - 7}$, so $y' = \frac{1}{3}(5x^2 - 7)^{-2/3}(10x) = \dfrac{10x}{3(5x^2 - 7)^{2/3}}$

(c) From part (a), $y' = \dfrac{10x}{3y^2} = \dfrac{10x}{3(y^3)^{2/3}} = \dfrac{10x}{3(5x^2 - 7)^{2/3}}$, which agrees with part (b).

2. (a) $\frac{d}{dx}(6x^4 + y^5) = \frac{d}{dx}(2x) \Rightarrow 24x^3 + 5y^4 y' = 2 \Rightarrow 5y^4\, y' = 2 - 24x^3 \Rightarrow y' = \frac{2 - 24x^3}{5y^4}$

(b) $6x^4 + y^5 = 2x \Rightarrow y^5 = 2x - 6x^4 \Rightarrow y = \sqrt[5]{2x - 6x^4}$, so $y' = \frac{1}{5}(2x - 6x^4)^{-4/5}(2 - 24x^3) = \frac{2 - 24x^3}{5(2x - 6x^4)^{4/5}}$

(c) From part (a), $y' = \frac{2 - 24x^3}{5y^4} = \frac{2 - 24x^2}{5(y^5)^{4/5}} = \frac{2 - 24x^3}{5(2x - 6x^4)^{4/5}}$, which agrees with part (b).

3. (a) $\frac{d}{dx}\left(\sqrt{x} + \sqrt{y}\right) = \frac{d}{dx}(1) \Rightarrow \frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0 \Rightarrow \frac{1}{2\sqrt{y}}\, y' = -\frac{1}{2\sqrt{x}} \Rightarrow y' = -\frac{\sqrt{y}}{\sqrt{x}}$

(b) $\sqrt{x} + \sqrt{y} = 1 \Rightarrow \sqrt{y} = 1 - \sqrt{x} \Rightarrow y = \left(1 - \sqrt{x}\,\right)^2 \Rightarrow y = 1 - 2\sqrt{x} + x$, so

$y' = -2 \cdot \frac{1}{2}x^{-1/2} + 1 = 1 - \frac{1}{\sqrt{x}}$.

(c) From part (a), $y' = -\frac{\sqrt{y}}{\sqrt{x}} = -\frac{1 - \sqrt{x}}{\sqrt{x}}$ [from part (b)] $= -\frac{1}{\sqrt{x}} + 1$, which agrees with part (b).

4. (a) $\frac{d}{dx}\left(\frac{2}{x} - \frac{1}{y}\right) = \frac{d}{dx}(4) \Rightarrow -2x^{-2} + y^{-2}y' = 0 \Rightarrow \frac{1}{y^2}\, y' = \frac{2}{x^2} \Rightarrow y' = \frac{2y^2}{x^2}$

(b) $\frac{2}{x} - \frac{1}{y} = 4 \Rightarrow \frac{1}{y} = \frac{2}{x} - 4 \Rightarrow \frac{1}{y} = \frac{2 - 4x}{x} \Rightarrow y = \frac{x}{2 - 4x}$, so

$y' = \frac{(2 - 4x)(1) - x(-4)}{(2 - 4x)^2} = \frac{2}{(2 - 4x)^2}$ $\left[\text{or } \frac{1}{2(1 - 2x)^2}\right]$.

(c) From part (a), $y' = \frac{2y^2}{x^2} = \frac{2\left(\frac{x}{2 - 4x}\right)^2}{x^2}$ [from part (b)] $= \frac{2x^2}{x^2(2 - 4x)^2} = \frac{2}{(2 - 4x)^2}$, which agrees with part (b).

5. $\frac{d}{dx}(x^2 - 4xy + y^2) = \frac{d}{dx}(4) \Rightarrow 2x - 4[xy' + y(1)] + 2y\, y' = 0 \Rightarrow 2y\, y' - 4xy' = 4y - 2x \Rightarrow$

$y'(y - 2x) = 2y - x \Rightarrow y' = \frac{2y - x}{y - 2x}$

6. $\frac{d}{dx}(2x^2 + xy - y^2) = \frac{d}{dx}(2) \Rightarrow 4x + xy' + y(1) - 2y\, y' = 0 \Rightarrow xy' - 2y\, y' = -4x - y \Rightarrow$

$(x - 2y)y' = -4x - y \Rightarrow y' = \frac{-4x - y}{x - 2y}$

7. $\frac{d}{dx}(x^4 + x^2y^2 + y^3) = \frac{d}{dx}(5) \Rightarrow 4x^3 + x^2 \cdot 2y\, y' + y^2 \cdot 2x + 3y^2y' = 0 \Rightarrow 2x^2y\, y' + 3y^2y' = -4x^3 - 2xy^2 \Rightarrow$

$(2x^2y + 3y^2)y' = -4x^3 - 2xy^2 \Rightarrow y' = \frac{-4x^3 - 2xy^2}{2x^2y + 3y^2} = -\frac{2x(2x^2 + y^2)}{y(2x^2 + 3y)}$

8. $\frac{d}{dx}(x^3 - xy^2 + y^3) = \frac{d}{dx}(1) \Rightarrow 3x^2 - x \cdot 2y\, y' - y^2 \cdot 1 + 3y^2y' = 0 \Rightarrow 3y^2y' - 2x\, y\, y' = y^2 - 3x^2 \Rightarrow$

$(3y^2 - 2xy)\, y' = y^2 - 3x^2 \Rightarrow y' = \frac{y^2 - 3x^2}{3y^2 - 2xy} = \frac{y^2 - 3x^2}{y(3y - 2x)}$

9. $\dfrac{d}{dx}\left(\dfrac{x^2}{x+y}\right) = \dfrac{d}{dx}(y^2+1) \quad\Rightarrow\quad \dfrac{(x+y)(2x) - x^2(1+y')}{(x+y)^2} = 2y\,y' \quad\Rightarrow$

$2x^2 + 2xy - x^2 - x^2\,y' = 2y(x+y)^2\,y' \quad\Rightarrow\quad x^2 + 2xy = 2y(x+y)^2\,y' + x^2\,y' \quad\Rightarrow$

$x(x+2y) = [2y(x^2+2xy+y^2) + x^2]\,y' \quad\Rightarrow\quad y' = \dfrac{x(x+2y)}{2x^2y + 4xy^2 + 2y^3 + x^2}$

Or: Start by clearing fractions and then differentiate implicitly.

10. $\dfrac{d}{dx}(xe^y) = \dfrac{d}{dx}(x-y) \quad\Rightarrow\quad xe^y\,y' + e^y\cdot 1 = 1 - y' \quad\Rightarrow\quad xe^y\,y' + y' = 1 - e^y \quad\Rightarrow\quad y'(xe^y+1) = 1 - e^y \quad\Rightarrow$

$y' = \dfrac{1-e^y}{xe^y+1}$

11. $\dfrac{d}{dx}(\sin x + \cos y) = \dfrac{d}{dx}(2x - 3y) \quad\Rightarrow\quad \cos x - \sin y\cdot y' = 2 - 3y' \quad\Rightarrow\quad 3y' - \sin y\cdot y' = 2 - \cos x \quad\Rightarrow$

$y'(3 - \sin y) = 2 - \cos x \quad\Rightarrow\quad y' = \dfrac{2-\cos x}{3-\sin y}$

12. $\dfrac{d}{dx}(e^x \sin y) = \dfrac{d}{dx}(x+y) \quad\Rightarrow\quad e^x\cos y\cdot y' + \sin y\cdot e^x = 1 + y' \quad\Rightarrow e^x\cos y\cdot y' - y' = 1 - e^x\sin y \quad\Rightarrow$

$y'(e^x\cos y - 1) = 1 - e^x\sin y \quad\Rightarrow\quad y' = \dfrac{1-e^x\sin y}{e^x\cos y - 1}$

13. $\dfrac{d}{dx}\sin(x+y) = \dfrac{d}{dx}(\cos x + \cos y) \quad\Rightarrow\quad \cos(x+y)\cdot(1+y') = -\sin x - \sin y\cdot y' \quad\Rightarrow$

$\cos(x+y) + y'\cos(x+y) = -\sin x - \sin y\cdot y' \quad\Rightarrow\quad y'\cos(x+y) + \sin y\cdot y' = -\sin x - \cos(x+y) \quad\Rightarrow$

$y'[\cos(x+y) + \sin y] = -[\sin x + \cos(x+y)] \quad\Rightarrow\quad y' = -\dfrac{\cos(x+y)+\sin x}{\cos(x+y)+\sin y}$

14. $\dfrac{d}{dx}\tan(x-y) = \dfrac{d}{dx}(2xy^3+1) \quad\Rightarrow\quad \sec^2(x-y)\cdot(1-y') = 2x(3y^2\,y') + y^3\cdot 2 \quad\Rightarrow$

$\sec^2(x-y) - y'\sec^2(x-y) = 6xy^2y' + 2y^3 \quad\Rightarrow\quad 6xy^2y' + y'\sec^2(x-y) = \sec^2(x-y) - 2y^3 \quad\Rightarrow$

$y'[6xy^2 + \sec^2(x-y)] = \sec^2(x-y) - 2y^3 \quad\Rightarrow\quad y' = \dfrac{\sec^2(x-y) - 2y^3}{\sec^2(x-y) + 6xy^2}$

15. $\dfrac{d}{dx}(y\cos x) = \dfrac{d}{dx}(x^2+y^2) \quad\Rightarrow\quad y(-\sin x) + \cos x\cdot y' = 2x + 2y\,y' \quad\Rightarrow\quad \cos x\cdot y' - 2y\,y' = 2x + y\sin x \quad\Rightarrow$

$y'(\cos x - 2y) = 2x + y\sin x \quad\Rightarrow\quad y' = \dfrac{2x + y\sin x}{\cos x - 2y}$

16. $\dfrac{d}{dx}\sin(xy) = \dfrac{d}{dx}\cos(x+y) \quad\Rightarrow\quad \cos(xy)\cdot(xy' + y\cdot 1) = -\sin(x+y)\cdot(1+y') \quad\Rightarrow$

$x\cos(xy)\,y' + y\cos(xy) = -\sin(x+y) - y'\sin(x+y) \quad\Rightarrow$

$x\cos(xy)\,y' + y'\sin(x+y) = -y\cos(xy) - \sin(x+y) \quad\Rightarrow$

$[x\cos(xy) + \sin(x+y)]\,y' = -1\,[y\cos(xy) + \sin(x+y)] \quad\Rightarrow\quad y' = -\dfrac{y\cos(xy)+\sin(x+y)}{x\cos(xy)+\sin(x+y)}$

17. $\dfrac{d}{dx}\left(2xe^y + ye^x\right) = \dfrac{d}{dx}(3) \;\Rightarrow\; (2x \cdot e^y y' + e^y \cdot 2) + (ye^x + e^x\, y') = 0 \;\Rightarrow\; 2xe^y\, y' + e^x\, y' = -2e^y - ye^x \;\Rightarrow$

$y'\left(2xe^y + e^x\right) = -\left(2e^y + ye^x\right) \;\Rightarrow\; y' = -\dfrac{2e^y + ye^x}{2xe^y + e^x}$

18. $\dfrac{d}{dx}(\sin x\, \cos y) = \dfrac{d}{dx}\left(x^2 - 5y\right) \;\Rightarrow\; \sin x\,(-\sin y) \cdot y' + \cos y\,(\cos x) = 2x - 5y' \;\Rightarrow$

$5y' - \sin x\, \sin y \cdot y' = 2x - \cos x\, \cos y \;\Rightarrow\; y'\,(5 - \sin x\, \sin y) = 2x - \cos x\, \cos y \;\Rightarrow\; y' = \dfrac{2x - \cos x\, \cos y}{5 - \sin x\, \sin y}$

19. $\dfrac{d}{dx}\sqrt{x+y} = \dfrac{d}{dx}\left(x^4 + y^4\right) \;\Rightarrow\; \frac{1}{2}(x+y)^{-1/2}(1 + y') = 4x^3 + 4y^3 y' \;\Rightarrow$

$\dfrac{1}{2\sqrt{x+y}} + \dfrac{1}{2\sqrt{x+y}}\, y' = 4x^3 + 4y^3 y' \;\Rightarrow\; \dfrac{1}{2\sqrt{x+y}} - 4x^3 = 4y^3 y' - \dfrac{1}{2\sqrt{x+y}} y' \;\Rightarrow$

$\dfrac{1 - 8x^3\sqrt{x+y}}{2\sqrt{x+y}} = \dfrac{8y^3\sqrt{x+y} - 1}{2\sqrt{x+y}}\, y' \;\Rightarrow\; y' = \dfrac{1 - 8x^3\sqrt{x+y}}{8y^3\sqrt{x+y} - 1}$

20. $\dfrac{d}{dx}(xy) = \dfrac{d}{dx}\sqrt{x^2+y^2} \;\Rightarrow\; xy' + y(1) = \frac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x + 2y\,y') \;\Rightarrow$

$xy' + y = \dfrac{x}{\sqrt{x^2+y^2}} + \dfrac{y}{\sqrt{x^2+y^2}}\, y' \;\Rightarrow\; xy' - \dfrac{y}{\sqrt{x^2+y^2}}\, y' = \dfrac{x}{\sqrt{x^2+y^2}} - y \;\Rightarrow$

$\dfrac{x\sqrt{x^2+y^2} - y}{\sqrt{x^2+y^2}}\, y' = \dfrac{x - y\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} \;\Rightarrow\; y' = \dfrac{x - y\sqrt{x^2+y^2}}{x\sqrt{x^2+y^2} - y}$

21. $\dfrac{d}{dx}(e^{x/y}) = \dfrac{d}{dx}(x - y) \;\Rightarrow\; e^{x/y} \cdot \dfrac{d}{dx}\left(\dfrac{x}{y}\right) = 1 - y' \;\Rightarrow\; e^{x/y} \cdot \dfrac{y \cdot 1 - x \cdot y'}{y^2} = 1 - y' \;\Rightarrow$

$e^{x/y} \cdot \dfrac{1}{y} - \dfrac{xe^{x/y}}{y^2} \cdot y' = 1 - y' \;\Rightarrow\; y' - \dfrac{xe^{x/y}}{y^2} \cdot y' = 1 - \dfrac{e^{x/y}}{y} \;\Rightarrow\; y'\!\left(1 - \dfrac{xe^{x/y}}{y^2}\right) = \dfrac{y - e^{x/y}}{y} \;\Rightarrow$

$$y' = \dfrac{\dfrac{y - e^{x/y}}{y}}{\dfrac{y^2 - xe^{x/y}}{y^2}} = \dfrac{y(y - e^{x/y})}{y^2 - xe^{x/y}}$$

22. $\dfrac{d}{dx}\cos(x^2 + y^2) = \dfrac{d}{dx}(xe^y) \;\Rightarrow\; -\sin(x^2+y^2) \cdot (2x + 2y\,y') = xe^y\, y' + e^y \cdot 1 \;\Rightarrow$

$-2x\sin(x^2+y^2) - 2y\,y'\sin(x^2+y^2) = xe^y\,y' + e^y \;\Rightarrow\; -e^y - 2x\sin(x^2+y^2) = xe^y\,y' + 2y\,y'\sin(x^2+y^2) \;\Rightarrow$

$-[e^y + 2x\sin(x^2+y^2)] = y'[xe^y + 2y\sin(x^2+y^2)] \;\Rightarrow\; y' = -\dfrac{e^y + 2x\sin(x^2+y^2)}{xe^y + 2y\sin(x^2+y^2)}$

23. $\dfrac{d}{dx}\left\{f(x) + x^2[f(x)]^3\right\} = \dfrac{d}{dx}(10) \;\Rightarrow\; f'(x) + x^2 \cdot 3[f(x)]^2 \cdot f'(x) + [f(x)]^3 \cdot 2x = 0.$ If $x = 1$, we have

$f'(1) + 1^2 \cdot 3[f(1)]^2 \cdot f'(1) + [f(1)]^3 \cdot 2(1) = 0 \;\Rightarrow\; f'(1) + 1 \cdot 3 \cdot 2^2 \cdot f'(1) + 2^3 \cdot 2 = 0 \;\Rightarrow$

$f'(1) + 12f'(1) = -16 \;\Rightarrow\; 13f'(1) = -16 \;\Rightarrow\; f'(1) = -\frac{16}{13}.$

24. $\dfrac{d}{dx}\left[g(x)+x\sin g(x)\right]=\dfrac{d}{dx}\left(x^2\right) \;\Rightarrow\; g'(x)+x\cos g(x)\cdot g'(x)+\sin g(x)\cdot 1=2x.$ If $x=0$, we have

$g'(0)+0+\sin g(0)=2(0) \;\Rightarrow\; g'(0)+\sin 0=0 \;\Rightarrow\; g'(0)+0=0 \;\Rightarrow\; g'(0)=0.$

25. $\dfrac{d}{dy}(x^4y^2-x^3y+2xy^3)=\dfrac{d}{dy}(0) \;\Rightarrow\; x^4\cdot 2y+y^2\cdot 4x^3\,x'-(x^3\cdot 1+y\cdot 3x^2\,x')+2(x\cdot 3y^2+y^3\cdot x')=0 \;\Rightarrow$

$4x^3y^2\,x'-3x^2y\,x'+2y^3\,x'=-2x^4y+x^3-6xy^2 \;\Rightarrow\; (4x^3y^2-3x^2y+2y^3)\,x'=-2x^4y+x^3-6xy^2 \;\Rightarrow$

$$x'=\frac{dx}{dy}=\frac{-2x^4y+x^3-6xy^2}{4x^3y^2-3x^2y+2y^3}$$

26. $\dfrac{d}{dy}(y\sec x)=\dfrac{d}{dy}(x\tan y) \;\Rightarrow\; y\cdot\sec x\,\tan x\cdot x'+\sec x\cdot 1=x\cdot\sec^2 y+\tan y\cdot x' \;\Rightarrow$

$y\sec x\,\tan x\cdot x'-\tan y\cdot x'=x\sec^2 y-\sec x \;\Rightarrow\; (y\sec x\,\tan x-\tan y)\,x'=x\sec^2 y-\sec x \;\Rightarrow$

$$x'=\frac{dx}{dy}=\frac{x\sec^2 y-\sec x}{y\sec x\,\tan x-\tan y}$$

27. $ye^{\sin x}=x\cos y \;\Rightarrow\; ye^{\sin x}\cdot\cos x+e^{\sin x}\cdot y'=x(-\sin y\cdot y')+\cos y\cdot 1 \;\Rightarrow$

$y'\,e^{\sin x}+y'\,x\sin y=\cos y-y\cos x\,e^{\sin x} \;\Rightarrow\; y'\,(e^{\sin x}+x\sin y)=\cos y-y\cos x\,e^{\sin x} \;\Rightarrow$

$$y'=\frac{\cos y-y\cos x\,e^{\sin x}}{x\sin y+e^{\sin x}}.$$

When $x=0$ and $y=0$, we have $y'=\dfrac{\cos 0-0\cdot\cos 0\cdot e^{\sin 0}}{0\cdot\sin 0+e^{\sin 0}}=\dfrac{1-0}{0+1}=1$, so an equation of the tangent line is

$y-0=1\,(x-0)$, or $y=x$.

28. $\tan(x+y)+\sec(x-y)=2 \;\Rightarrow\; \sec^2(x+y)\cdot(1+y')+\sec(x-y)\tan(x-y)\cdot(1-y')=0 \;\Rightarrow$

$\sec^2(x+y)+y'\cdot\sec^2(x+y)+\sec(x-y)\tan(x-y)-y'\cdot\sec(x-y)\tan(x-y)=0 \;\Rightarrow$

$\sec^2(x+y)+\sec(x-y)\tan(x-y)=y'\cdot\sec(x-y)\tan(x-y)-y'\cdot\sec^2(x+y) \;\Rightarrow$

$\sec^2(x+y)+\sec(x-y)\tan(x-y)=y'\left[\sec(x-y)\tan(x-y)-\sec^2(x+y)\right] \;\Rightarrow$

$$y'=\frac{\sec(x-y)\,\tan(x-y)+\sec^2(x+y)}{\sec(x-y)\,\tan(x-y)-\sec^2(x+y)}.$$

When $x=\frac{\pi}{8}$ and $y=\frac{\pi}{8}$, we have

$y'=\dfrac{\sec\left(\frac{\pi}{8}-\frac{\pi}{8}\right)\tan\left(\frac{\pi}{8}-\frac{\pi}{8}\right)+\sec^2\left(\frac{\pi}{8}+\frac{\pi}{8}\right)}{\sec\left(\frac{\pi}{8}-\frac{\pi}{8}\right)\tan\left(\frac{\pi}{8}-\frac{\pi}{8}\right)-\sec^2\left(\frac{\pi}{8}+\frac{\pi}{8}\right)}=\dfrac{\sec 0\,\tan 0+\sec^2\left(\frac{\pi}{4}\right)}{\sec 0\,\tan 0-\sec^2\left(\frac{\pi}{4}\right)}=\dfrac{0+\left(\sqrt{2}\right)^2}{0-\left(\sqrt{2}\right)^2}=-1$, so an equation of the

tangent line is $y-\frac{\pi}{8}=-1\left(x-\frac{\pi}{8}\right)$, or $y=-x+\frac{\pi}{4}$.

29. $x^{2/3} + y^{2/3} = 4 \quad \Rightarrow \quad \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0 \quad \Rightarrow \quad \dfrac{1}{\sqrt[3]{x}} + \dfrac{y'}{\sqrt[3]{y}} = 0 \quad \Rightarrow \quad y' = -\dfrac{\sqrt[3]{y}}{\sqrt[3]{x}}.$

When $x = -3\sqrt{3}$ and $y = 1$, we have $y' = -\dfrac{1}{\left(-3\sqrt{3}\right)^{1/3}} = -\dfrac{1}{(-3^{3/2})^{1/3}} = \dfrac{1}{3^{1/2}} = \dfrac{1}{\sqrt{3}}$, so an equation of the tangent line is $y - 1 = \frac{1}{\sqrt{3}}\left(x + 3\sqrt{3}\right)$ or $y = \frac{1}{\sqrt{3}}x + 4$.

30. $y^2(6-x) = x^3 \quad \Rightarrow \quad y^2(-1) + (6-x)\cdot 2yy' = 3x^2 \quad \Rightarrow \quad y' \cdot 2y(6-x) = 3x^2 + y^2 \quad \Rightarrow \quad y' = \dfrac{3x^2+y^2}{2y(6-x)}.$

When $x = 2$ and $y = \sqrt{2}$, we have $y' = \dfrac{3(2)^2 + \left(\sqrt{2}\right)^2}{2\sqrt{2}(6-2)} = \dfrac{12+2}{2\sqrt{2}\cdot 4} = \dfrac{14}{8\sqrt{2}} = \dfrac{7}{4\sqrt{2}}$, so an equation of the tangent line is

$y - \sqrt{2} = \dfrac{7}{4\sqrt{2}}(x-2)$, or $y = \dfrac{7}{4\sqrt{2}}x - \dfrac{7}{2\sqrt{2}} + \dfrac{4}{2\sqrt{2}} = \dfrac{7}{4\sqrt{2}}x - \dfrac{3}{2\sqrt{2}}.$

31. $x^2 - xy - y^2 = 1 \quad \Rightarrow \quad 2x - (xy' + y\cdot 1) - 2y\,y' = 0 \quad \Rightarrow \quad 2x - xy' - y - 2y\,y' = 0 \quad \Rightarrow \quad 2x - y = xy' + 2y\,y' \quad \Rightarrow$

$2x - y = (x+2y)\,y' \quad \Rightarrow \quad y' = \dfrac{2x-y}{x+2y}.$

When $x = 2$ and $y = 1$, we have $y' = \dfrac{4-1}{2+2} = \dfrac{3}{4}$, so an equation of the tangent line is $y - 1 = \frac{3}{4}(x-2)$, or $y = \frac{3}{4}x - \frac{1}{2}$.

32. $x^2 + 2xy + 4y^2 = 12 \quad \Rightarrow \quad 2x + 2x\,y' + 2y + 8y\,y' = 0 \quad \Rightarrow \quad 2x\,y' + 8y\,y' = -2x - 2y \quad \Rightarrow$

$(x+4y)\,y' = -x - y \quad \Rightarrow \quad y' = -\dfrac{x+y}{x+4y}.$

When $x = 2$ and $y = 1$, we have $y' = -\dfrac{2+1}{2+4} = -\dfrac{1}{2}$, so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x-2)$ or $y = -\frac{1}{2}x + 2$.

33. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \quad \Rightarrow \quad 2x + 2y\,y' = 2(2x^2 + 2y^2 - x)(4x + 4y\,y' - 1).$

When $x = 0$ and $y = \frac{1}{2}$, we have $0 + y' = 2(\frac{1}{2})(2y' - 1) \quad \Rightarrow \quad y' = 2y' - 1 \quad \Rightarrow \quad y' = 1$, so an equation of the tangent line is $y - \frac{1}{2} = 1(x - 0)$ or $y = x + \frac{1}{2}$.

34. $x^2y^2 = (y+1)^2(4-y^2) \quad \Rightarrow \quad x^2 \cdot 2y\,y' + y^2 \cdot 2x = (y+1)^2(-2y\,y') + (4-y^2)\cdot 2(y+1)\,y' \quad \Rightarrow$

$2x^2y\,y' + 2xy^2 = -2y\,y'(y+1)^2 + 2y'(4-y^2)(y+1) \quad \Rightarrow$

$2xy^2 = 2y'(4-y^2)(y+1) - 2y\,y'\,(y+1)^2 - 2x^2y\,y' \quad \Rightarrow \quad 2xy^2 = y'[2(4-y^2)(y+1) - 2y(y+1)^2 - 2x^2y] \quad \Rightarrow$

$$y' = \frac{2xy^2}{2(4-y^2)(y+1) - 2y(y+1)^2 - 2x^2y} = \frac{xy^2}{(4-y^2)(y+1) - y(y+1)^2 - x^2y}$$

When $x = 2\sqrt{3}$ and $y = 1$, we have $y' = \dfrac{2\sqrt{3}\cdot 1^2}{(4-1^2)(1+1) - 1(1+1)^2 - \left(2\sqrt{3}\right)^2\cdot 1} = \dfrac{2\sqrt{3}}{6-4-12} = -\dfrac{2\sqrt{3}}{10} = -\dfrac{\sqrt{3}}{5}$,

so an equation of the tangent line is $y - 1 = -\dfrac{\sqrt{3}}{5}\left(x - 2\sqrt{3}\right)$, or $y = -\dfrac{\sqrt{3}}{5}x + \dfrac{11}{5}$.

35. $2(x^2+y^2)^2 = 25(x^2-y^2) \Rightarrow 4(x^2+y^2)(2x+2y\,y') = 25(2x-2y\,y') \Rightarrow$

$4(x+y\,y')(x^2+y^2) = 25(x-y\,y') \Rightarrow 4y\,y'(x^2+y^2)+25y\,y' = 25x-4x(x^2+y^2) \Rightarrow$

$y' = \dfrac{25x-4x(x^2+y^2)}{25y+4y(x^2+y^2)}$.

When $x=3$ and $y=1$, we have $y' = \frac{75-120}{25+40} = -\frac{45}{65} = -\frac{9}{13}$, so an equation of the tangent line is $y-1 = -\frac{9}{13}(x-3)$ or $y = -\frac{9}{13}x+\frac{40}{13}$.

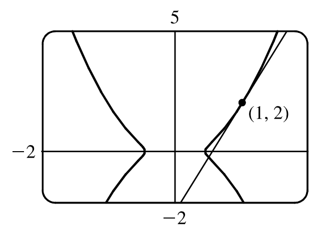
36. $y^2(y^2-4) = x^2(x^2-5) \Rightarrow y^4-4y^2 = x^4-5x^2 \Rightarrow 4y^3\,y'-8y\,y' = 4x^3-10x$.

When $x=0$ and $y=-2$, we have $-32y'+16y'=0 \Rightarrow -16y'=0 \Rightarrow y'=0$, so an equation of the tangent line is $y+2=0(x-0)$ or $y=-2$.

37. (a) $y^2 = 5x^4-x^2 \Rightarrow 2y\,y' = 5(4x^3)-2x \Rightarrow y' = \dfrac{10x^3-x}{y}$.

So at the point $(1,2)$ we have $y' = \dfrac{10(1)^3-1}{2} = \dfrac{9}{2}$, and an equation of the tangent line is $y-2=\frac{9}{2}(x-1)$ or $y=\frac{9}{2}x-\frac{5}{2}$.

(b)



38. (a) $y^2 = x^3+3x^2 \Rightarrow 2y\,y' = 3x^2+3(2x) \Rightarrow y' = \dfrac{3x^2+6x}{2y}$. So at the point $(1,-2)$ we have

$y' = \dfrac{3(1)^2+6(1)}{2(-2)} = -\dfrac{9}{4}$, and an equation of the tangent line is $y+2=-\frac{9}{4}(x-1)$ or $y=-\frac{9}{4}x+\frac{1}{4}$.

(b) The curve has a horizontal tangent where $y'=0 \Leftrightarrow$

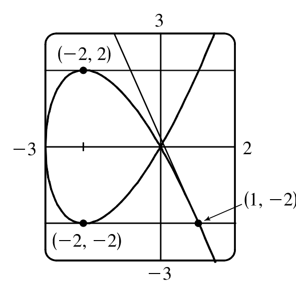
$3x^2+6x=0 \Leftrightarrow 3x(x+2)=0 \Leftrightarrow x=0$ or $x=-2$.

But note that at $x=0$, $y=0$ also, so the derivative does not exist.

At $x=-2$, $y^2=(-2)^3+3(-2)^2=-8+12=4$, so $y=\pm 2$.

So the two points at which the curve has a horizontal tangent are $(-2,-2)$ and $(-2,2)$.

(c)



39. $x^2+4y^2=4 \Rightarrow 2x+8y\,y'=0 \Rightarrow y'=-x/(4y) \Rightarrow$

$y'' = -\dfrac{1}{4}\dfrac{y\cdot 1-x\cdot y'}{y^2} = -\dfrac{1}{4}\dfrac{y-x[-x/(4y)]}{y^2} = -\dfrac{1}{4}\dfrac{4y^2+x^2}{4y^3} = -\dfrac{1}{4}\dfrac{4}{4y^3}$ $\left[\begin{array}{c}\text{since } x \text{ and } y \text{ must satisfy the}\\ \text{original equation } x^2+4y^2=4\end{array}\right]$

Thus, $y'' = -\dfrac{1}{4y^3}$.

40. $x^2+xy+y^2=3 \Rightarrow 2x+xy'+y+2y\,y'=0 \Rightarrow (x+2y)y'=-2x-y \Rightarrow y'=\dfrac{-2x-y}{x+2y}$.

Differentiating $2x+xy'+y+2y\,y'=0$ to find y'' gives $2+xy''+y'+y'+2y\,y''+2y'y'=0 \Rightarrow$

$$(x+2y)\,y'' = -2 - 2y' - 2(y')^2 = -2\left[1 - \frac{2x+y}{x+2y} + \left(\frac{2x+y}{x+2y}\right)^2\right] \quad \Rightarrow$$

$$y'' = -\frac{2}{x+2y}\left[\frac{(x+2y)^2 - (2x+y)(x+2y) + (2x+y)^2}{(x+2y)^2}\right]$$

$$= -\frac{2}{(x+2y)^3}\,(x^2 + 4xy + 4y^2 - 2x^2 - 4xy - xy - 2y^2 + 4x^2 + 4xy + y^2)$$

$$= -\frac{2}{(x+2y)^3}\,(3x^2 + 3xy + 3y^2) = -\frac{2}{(x+2y)^3}\,(9) \qquad \left[\begin{array}{c}\text{since } x \text{ and } y \text{ must satisfy the}\\ \text{original equation } x^2 + xy + y^2 = 3\end{array}\right]$$

Thus, $y'' = -\dfrac{18}{(x+2y)^3}$.

41. $\sin y + \cos x = 1 \quad \Rightarrow \quad \cos y \cdot y' - \sin x = 0 \quad \Rightarrow \quad y' = \dfrac{\sin x}{\cos y} \quad \Rightarrow$

$$y'' = \frac{\cos y\,\cos x - \sin x(-\sin y)\,y'}{(\cos y)^2} = \frac{\cos y\,\cos x + \sin x\,\sin y(\sin x/\cos y)}{\cos^2 y}$$

$$= \frac{\cos^2 y\,\cos x + \sin^2 x\,\sin y}{\cos^2 y\,\cos y} = \frac{\cos^2 y\,\cos x + \sin^2 x\,\sin y}{\cos^3 y}$$

42. $x^3 - y^3 = 7 \quad \Rightarrow \quad 3x^2 - 3y^2 y' = 0 \quad \Rightarrow \quad y' = \dfrac{x^2}{y^2} \quad \Rightarrow$

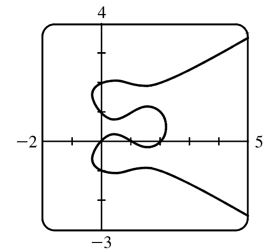
$$y'' = \frac{y^2(2x) - x^2(2y\,y')}{(y^2)^2} = \frac{2xy[y - x(x^2/y^2)]}{y^4} = \frac{2x(y - x^3/y^2)}{y^3} = \frac{2x(y^3 - x^3)}{y^3 y^2} = \frac{2x(-7)}{y^5} = \frac{-14x}{y^5}$$

43. If $x = 0$ in $xy + e^y = e$, then we get $0 + e^y = e$, so $y = 1$ and the point where $x = 0$ is $(0, 1)$. Differentiating implicitly with respect to x gives us $x\,y' + y \cdot 1 + e^y\,y' = 0$. Substituting 0 for x and 1 for y gives us $0 + 1 + e\,y' = 0 \quad \Rightarrow$ $e\,y' = -1 \quad \Rightarrow \quad y' = -1/e$. Differentiating $x\,y' + y + e^y\,y' = 0$ implicitly with respect to x gives us $x\,y'' + y' \cdot 1 + y' + e^y\,y'' + y' \cdot e^y\,y' = 0$. Now substitute 0 for x, 1 for y, and $-1/e$ for y'.

$$0 + \left(-\frac{1}{e}\right) + \left(-\frac{1}{e}\right) + e\,y'' + \left(-\frac{1}{e}\right)(e)\left(-\frac{1}{e}\right) = 0 \quad \Rightarrow \quad -\frac{2}{e} + e\,y'' + \frac{1}{e} = 0 \quad \Rightarrow \quad e\,y'' = \frac{1}{e} \quad \Rightarrow \quad y'' = \frac{1}{e^2}.$$

44. If $x = 1$ in $x^2 + xy + y^3 = 1$, then we get $1 + y + y^3 = 1 \quad \Rightarrow \quad y^3 + y = 0 \quad \Rightarrow \quad y(y^2 + 1) \quad \Rightarrow \quad y = 0$, so the point where $x = 1$ is $(1, 0)$. Differentiating implicitly with respect to x gives us $2x + x\,y' + y \cdot 1 + 3y^2 \cdot y' = 0$. Substituting 1 for x and 0 for y gives us $2 + y' + 0 + 0 = 0 \quad \Rightarrow \quad y' = -2$. Differentiating $2x + x\,y' + y + 3y^2\,y' = 0$ implicitly with respect to x gives us $2 + x\,y'' + y' \cdot 1 + y' + 3(y^2\,y'' + y' \cdot 2yy') = 0$. Now substitute 1 for x, 0 for y, and -2 for y'.
$2 + y'' + (-2) + (-2) + 3(0 + 0) = 0 \quad \Rightarrow \quad y'' = 2$. Differentiating $2 + x\,y'' + 2\,y' + 3y^2\,y'' + 6y(y')^2 = 0$ implicitly with respect to x gives us $x\,y''' + y'' \cdot 1 + 2\,y'' + 3(y^2\,y''' + y'' \cdot 2y\,y') + 6[y \cdot 2y'\,y'' + (y')^2\,y'] = 0$. Now substitute 1 for x, 0 for y, -2 for y', and 2 for y''. $y''' + 2 + 4 + 3(0 + 0) + 6[0 + (-8)] = 0 \quad \Rightarrow \quad y''' = -2 - 4 + 48 = 42$.

45. (a) There are eight points with horizontal tangents: four at $x \approx 1.57735$ and four at $x \approx 0.42265$.

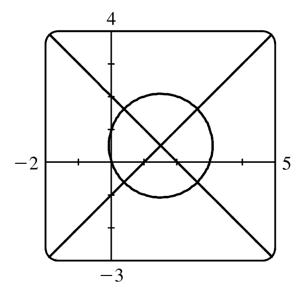


(b) $y' = \dfrac{3x^2 - 6x + 2}{2(2y^3 - 3y^2 - y + 1)} \quad \Rightarrow \quad y' = -1$ at $(0, 1)$ and $y' = \frac{1}{3}$ at $(0, 2)$.

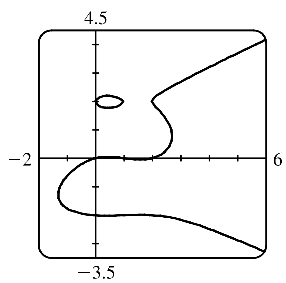
Equations of the tangent lines are $y = -x + 1$ and $y = \frac{1}{3}x + 2$.

(c) $y' = 0 \quad \Rightarrow \quad 3x^2 - 6x + 2 = 0 \quad \Rightarrow \quad x = 1 \pm \frac{1}{3}\sqrt{3}$

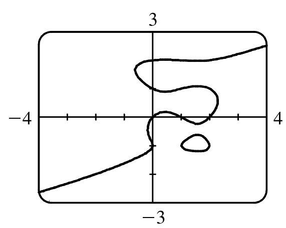
(d) By multiplying the right side of the equation by $x - 3$, we obtain the first graph. By modifying the equation in other ways, we can generate the other graphs.



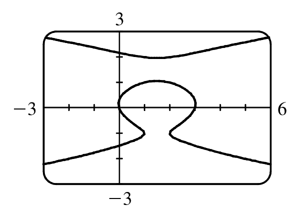
$y(y^2 - 1)(y - 2)$
$= x(x - 1)(x - 2)(x - 3)$



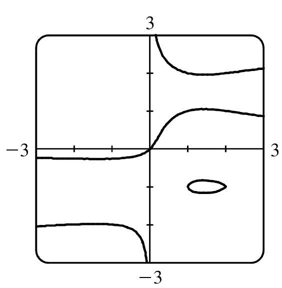
$y(y^2 - 4)(y - 2)$
$= x(x - 1)(x - 2)$



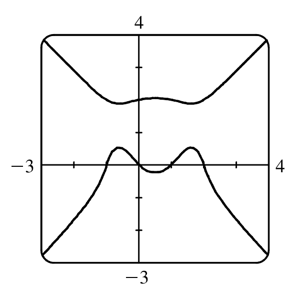
$y(y + 1)(y^2 - 1)(y - 2)$
$= x(x - 1)(x - 2)$



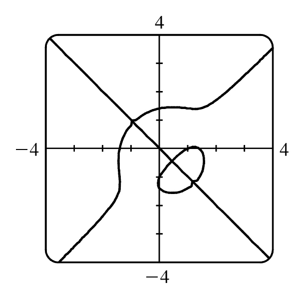
$(y + 1)(y^2 - 1)(y - 2)$
$= (x - 1)(x - 2)$



$x(y + 1)(y^2 - 1)(y - 2)$
$= y(x - 1)(x - 2)$

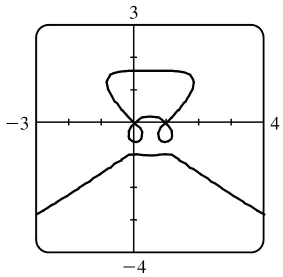


$y(y^2 + 1)(y - 2)$
$= x(x^2 - 1)(x - 2)$



$y(y + 1)(y^2 - 2)$
$= x(x - 1)(x^2 - 2)$

46. (a)



(b) $\frac{d}{dx}(2y^3 + y^2 - y^5) = \frac{d}{dx}(x^4 - 2x^3 + x^2) \quad \Rightarrow$

$6y^2 y' + 2y\, y' - 5y^4 y' = 4x^3 - 6x^2 + 2x \quad \Rightarrow$

$y' = \dfrac{2x(2x^2 - 3x + 1)}{6y^2 + 2y - 5y^4} = \dfrac{2x(2x-1)(x-1)}{y(6y + 2 - 5y^3)}$. From the graph and the values for which $y' = 0$, we speculate that there are 9 points with horizontal tangents: 3 at $x = 0$, 3 at $x = \frac{1}{2}$, and 3 at $x = 1$. The three horizontal tangents along the top of the wagon are hard to find, but by limiting the y-range of the graph (to $[1.6, 1.7]$, for example) they are distinguishable.

47. From Exercise 35, a tangent to the lemniscate will be horizontal if $y' = 0 \quad \Rightarrow \quad 25x - 4x(x^2 + y^2) = 0 \quad \Rightarrow$ $x[25 - 4(x^2 + y^2)] = 0 \quad \Rightarrow \quad x^2 + y^2 = \frac{25}{4}$ **(1)**. (Note that when x is 0, y is also 0, and there is no horizontal tangent at the origin.) Substituting $\frac{25}{4}$ for $x^2 + y^2$ in the equation of the lemniscate, $2(x^2 + y^2)^2 = 25(x^2 - y^2)$, we get $x^2 - y^2 = \frac{25}{8}$ **(2)**. Solving **(1)** and **(2)**, we have $x^2 = \frac{75}{16}$ and $y^2 = \frac{25}{16}$, so the four points are $\left(\pm\frac{5\sqrt{3}}{4}, \pm\frac{5}{4}\right)$.

48. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \quad \Rightarrow \quad y' = -\dfrac{b^2 x}{a^2 y} \quad \Rightarrow$ an equation of the tangent line at (x_0, y_0) is $y - y_0 = \dfrac{-b^2 x_0}{a^2 y_0}(x - x_0)$. Multiplying both sides by $\dfrac{y_0}{b^2}$ gives $\dfrac{y_0 y}{b^2} - \dfrac{y_0^2}{b^2} = -\dfrac{x_0 x}{a^2} + \dfrac{x_0^2}{a^2}$. Since (x_0, y_0) lies on the ellipse, we have $\dfrac{x_0 x}{a^2} + \dfrac{y_0 y}{b^2} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} = 1$.

49. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad \dfrac{2x}{a^2} - \dfrac{2yy'}{b^2} = 0 \quad \Rightarrow \quad y' = \dfrac{b^2 x}{a^2 y} \quad \Rightarrow$ an equation of the tangent line at (x_0, y_0) is $y - y_0 = \dfrac{b^2 x_0}{a^2 y_0}(x - x_0)$. Multiplying both sides by $\dfrac{y_0}{b^2}$ gives $\dfrac{y_0 y}{b^2} - \dfrac{y_0^2}{b^2} = \dfrac{x_0 x}{a^2} - \dfrac{x_0^2}{a^2}$. Since (x_0, y_0) lies on the hyperbola, we have $\dfrac{x_0 x}{a^2} - \dfrac{y_0 y}{b^2} = \dfrac{x_0^2}{a^2} - \dfrac{y_0^2}{b^2} = 1$.

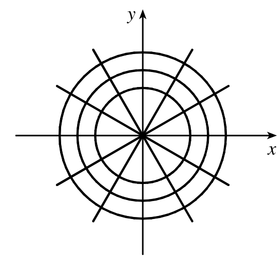
50. $\sqrt{x} + \sqrt{y} = \sqrt{c} \quad \Rightarrow \quad \dfrac{1}{2\sqrt{x}} + \dfrac{y'}{2\sqrt{y}} = 0 \quad \Rightarrow \quad y' = -\dfrac{\sqrt{y}}{\sqrt{x}} \quad \Rightarrow$ an equation of the tangent line at (x_0, y_0) is $y - y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$. Now $x = 0 \quad \Rightarrow \quad y = y_0 - \dfrac{\sqrt{y_0}}{\sqrt{x_0}}(-x_0) = y_0 + \sqrt{x_0}\sqrt{y_0}$, so the y-intercept is $y_0 + \sqrt{x_0}\sqrt{y_0}$. And $y = 0 \quad \Rightarrow \quad -y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0) \quad \Rightarrow \quad x - x_0 = \dfrac{y_0\sqrt{x_0}}{\sqrt{y_0}} \quad \Rightarrow \quad x = x_0 + \sqrt{x_0}\sqrt{y_0}$, so the x-intercept is $x_0 + \sqrt{x_0}\sqrt{y_0}$. The sum of the intercepts is

$$\left(y_0 + \sqrt{x_0}\sqrt{y_0}\right) + \left(x_0 + \sqrt{x_0}\sqrt{y_0}\right) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 = \left(\sqrt{c}\right)^2 = c.$$

51. If the circle has radius r, its equation is $x^2 + y^2 = r^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -\dfrac{x}{y}$, so the slope of the tangent line at $P(x_0, y_0)$ is $-\dfrac{x_0}{y_0}$. The negative reciprocal of that slope is $\dfrac{-1}{-x_0/y_0} = \dfrac{y_0}{x_0}$, which is the slope of OP, so the tangent line at P is perpendicular to the radius OP.

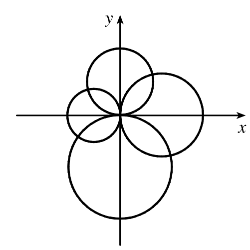
52. $y^q = x^p \Rightarrow qy^{q-1}y' = px^{p-1} \Rightarrow y' = \dfrac{px^{p-1}}{qy^{q-1}} = \dfrac{px^{p-1}y}{qy^q} = \dfrac{px^{p-1}x^{p/q}}{qx^p} = \dfrac{p}{q}x^{(p/q)-1}$

53. $x^2 + y^2 = r^2$ is a circle with center O and $ax + by = 0$ is a line through O [assume a and b are not both zero]. $x^2 + y^2 = r^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -x/y$, so the slope of the tangent line at $P_0\,(x_0, y_0)$ is $-x_0/y_0$. The slope of the line OP_0 is y_0/x_0, which is the negative reciprocal of $-x_0/y_0$. Hence, the curves are orthogonal, and the families of curves are orthogonal trajectories of each other.

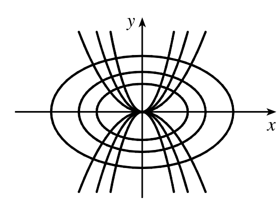


54. The circles $x^2 + y^2 = ax$ and $x^2 + y^2 = by$ intersect at the origin where the tangents are vertical and horizontal [assume a and b are both nonzero]. If (x_0, y_0) is the other point of intersection, then $x_0^2 + y_0^2 = ax_0$ (**1**) and $x_0^2 + y_0^2 = by_0$ (**2**).

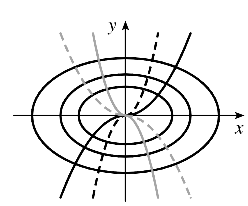
Now $x^2 + y^2 = ax \Rightarrow 2x + 2yy' = a \Rightarrow y' = \dfrac{a - 2x}{2y}$ and $x^2 + y^2 = by \Rightarrow$ $2x + 2yy' = by' \Rightarrow y' = \dfrac{2x}{b - 2y}$. Thus, the curves are orthogonal at $(x_0, y_0) \Leftrightarrow$ $\dfrac{a - 2x_0}{2y_0} = -\dfrac{b - 2y_0}{2x_0} \Leftrightarrow 2ax_0 - 4x_0^2 = 4y_0^2 - 2by_0 \Leftrightarrow ax_0 + by_0 = 2(x_0^2 + y_0^2)$, which is true by (**1**) and (**2**).



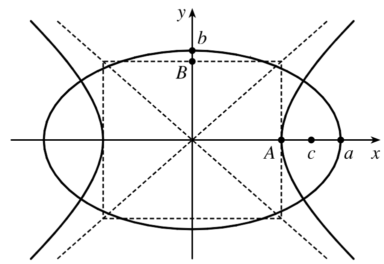
55. $y = cx^2 \Rightarrow y' = 2cx$ and $x^2 + 2y^2 = k$ [assume $k > 0$] $\Rightarrow 2x + 4yy' = 0 \Rightarrow$ $2yy' = -x \Rightarrow y' = -\dfrac{x}{2(y)} = -\dfrac{x}{2(cx^2)} = -\dfrac{1}{2cx}$, so the curves are orthogonal if $c \neq 0$. If $c = 0$, then the horizontal line $y = cx^2 = 0$ intersects $x^2 + 2y^2 = k$ orthogonally at $\left(\pm\sqrt{k}, 0\right)$, since the ellipse $x^2 + 2y^2 = k$ has vertical tangents at those two points.



56. $y = ax^3 \Rightarrow y' = 3ax^2$ and $x^2 + 3y^2 = b$ [assume $b > 0$] $\Rightarrow 2x + 6yy' = 0 \Rightarrow$ $3yy' = -x \Rightarrow y' = -\dfrac{x}{3(y)} = -\dfrac{x}{3(ax^3)} = -\dfrac{1}{3ax^2}$, so the curves are orthogonal if $a \neq 0$. If $a = 0$, then the horizontal line $y = ax^3 = 0$ intesects $x^2 + 3y^2 = b$ orthogonally at $\left(\pm\sqrt{b}, 0\right)$, since the ellipse $x^2 + 3y^2 = b$ has vertical tangents at those two points.



57. Since $A^2 < a^2$, we are assured that there are four points of intersection.



(1) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \;\Rightarrow\; \dfrac{yy'}{b^2} = -\dfrac{x}{a^2} \;\Rightarrow$

$y' = m_1 = -\dfrac{xb^2}{ya^2}$.

(2) $\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1 \;\Rightarrow\; \dfrac{2x}{A^2} - \dfrac{2yy'}{B^2} = 0 \;\Rightarrow\; \dfrac{yy'}{B^2} = \dfrac{x}{A^2} \;\Rightarrow$

$y' = m_2 = \dfrac{xB^2}{yA^2}$.

Now $m_1 m_2 = -\dfrac{xb^2}{ya^2} \cdot \dfrac{xB^2}{yA^2} = -\dfrac{b^2B^2}{a^2A^2} \cdot \dfrac{x^2}{y^2}$ **(3)**. Subtracting equations, **(1)** − **(2)**, gives us $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} = 0 \;\Rightarrow$ $\dfrac{y^2}{b^2} + \dfrac{y^2}{B^2} = \dfrac{x^2}{A^2} - \dfrac{x^2}{a^2} \;\Rightarrow\; \dfrac{y^2B^2 + y^2b^2}{b^2B^2} = \dfrac{x^2a^2 - x^2A^2}{A^2a^2} \;\Rightarrow\; \dfrac{y^2(b^2 + B^2)}{b^2B^2} = \dfrac{x^2(a^2 - A^2)}{a^2A^2}$ **(4)**. Since $a^2 - b^2 = A^2 + B^2$, we have $a^2 - A^2 = b^2 + B^2$. Thus, equation **(4)** becomes $\dfrac{y^2}{b^2B^2} = \dfrac{x^2}{A^2a^2} \;\Rightarrow\; \dfrac{x^2}{y^2} = \dfrac{A^2a^2}{b^2B^2}$, and substituting for $\dfrac{x^2}{y^2}$ in equation **(3)** gives us $m_1m_2 = -\dfrac{b^2B^2}{a^2A^2} \cdot \dfrac{a^2A^2}{b^2B^2} = -1$. Hence, the ellipse and hyperbola are orthogonal trajectories.

58. $y = (x+c)^{-1} \;\Rightarrow\; y' = -(x+c)^{-2}$ and $y = a(x+k)^{1/3} \;\Rightarrow\; y' = \frac{1}{3}a(x+k)^{-2/3}$, so the curves are othogonal if the product of the slopes is -1, that is, $\dfrac{-1}{(x+c)^2} \cdot \dfrac{a}{3(x+k)^{2/3}} = -1 \;\Rightarrow\; a = 3(x+c)^2(x+k)^{2/3} \;\Rightarrow$ $a = 3\left(\dfrac{1}{y}\right)^2 \left(\dfrac{y}{a}\right)^2$ [since $y^2 = (x+c)^{-2}$ and $y^2 = a^2(x+k)^{2/3}$] $\Rightarrow\; a = 3\left(\dfrac{1}{a^2}\right) \;\Rightarrow\; a^3 = 3 \;\Rightarrow\; a = \sqrt[3]{3}$.

59. (a) $\left(P + \dfrac{n^2a}{V^2}\right)(V - nb) = nRT \;\Rightarrow\; PV - Pnb + \dfrac{n^2a}{V} - \dfrac{n^3ab}{V^2} = nRT \;\Rightarrow$

$$\frac{d}{dP}(PV - Pnb + n^2aV^{-1} - n^3abV^{-2}) = \frac{d}{dP}(nRT) \;\Rightarrow$$

$$PV' + V \cdot 1 - nb - n^2aV^{-2} \cdot V' + 2n^3abV^{-3} \cdot V' = 0 \;\Rightarrow\; V'(P - n^2aV^{-2} + 2n^3abV^{-3}) = nb - V \;\Rightarrow$$

$$V' = \frac{nb - V}{P - n^2aV^{-2} + 2n^3abV^{-3}} \text{ or } \frac{dV}{dP} = \frac{V^3(nb - V)}{PV^3 - n^2aV + 2n^3ab}$$

(b) Using the last expression for dV/dP from part (a), we get

$$\frac{dV}{dP} = \frac{(10\text{ L})^3[(1\text{ mole})(0.04267\text{ L/mole}) - 10\text{ L}]}{\left[\begin{array}{l}(2.5\text{ atm})(10\text{ L})^3 - (1\text{ mole})^2(3.592\text{ L}^2\text{-atm/mole}^2)(10\text{ L}) \\ \qquad + 2(1\text{ mole})^3(3.592\text{ L}^2\text{-atm/mole}^2)(0.04267\text{ L/mole})\end{array}\right]}$$

$$= \frac{-9957.33\text{ L}^4}{2464.386541\text{ L}^3\text{-atm}} \approx -4.04\text{ L/atm}.$$

60. (a) $x^2 + xy + y^2 + 1 = 0 \quad \Rightarrow \quad 2x + xy' + y \cdot 1 + 2yy' + 0 = 0 \quad \Rightarrow \quad y'(x + 2y) = -2x - y \quad \Rightarrow \quad y' = \dfrac{-2x - y}{x + 2y}$

(b) Plotting the curve in part (a) gives us an empty graph, that is, there are no points that satisfy the equation. If there were any points that satisfied the equation, then x and y would have opposite signs; otherwise, all the terms are positive and their sum can not equal 0. $x^2 + xy + y^2 + 1 = 0 \quad \Rightarrow \quad x^2 + 2xy + y^2 - xy + 1 = 0 \quad \Rightarrow \quad (x + y)^2 = xy - 1$. The left side of the last equation is nonnegative, but the right side is at most -1, so that proves there are no points that satisfy the equation.

Another solution: $x^2 + xy + y^2 + 1 = \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + 1 = \frac{1}{2}(x^2 + 2xy + y^2) + \frac{1}{2}(x^2 + y^2) + 1$

$$= \tfrac{1}{2}(x + y)^2 + \tfrac{1}{2}(x^2 + y^2) + 1 \geq 1$$

Another solution: Regarding $x^2 + xy + y^2 + 1 = 0$ as a quadratic in x, the discriminant is $y^2 - 4(y^2 + 1) = -3y^2 - 4$. This is negative, so there are no real solutions.

(c) The expression for y' in part (a) is meaningless; that is, since the equation in part (a) has no solution, it does not implicitly define a function y of x, and therefore it is meaningless to consider y'.

61. To find the points at which the ellipse $x^2 - xy + y^2 = 3$ crosses the x-axis, let $y = 0$ and solve for x.

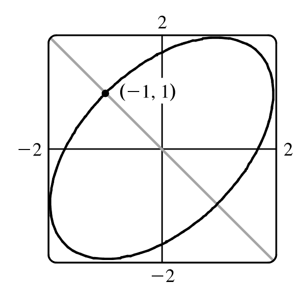
$y = 0 \quad \Rightarrow \quad x^2 - x(0) + 0^2 = 3 \quad \Leftrightarrow \quad x = \pm\sqrt{3}$. So the graph of the ellipse crosses the x-axis at the points $\left(\pm\sqrt{3}, 0\right)$.

Using implicit differentiation to find y', we get $2x - xy' - y + 2yy' = 0 \quad \Rightarrow \quad y'(2y - x) = y - 2x \quad \Leftrightarrow \quad y' = \dfrac{y - 2x}{2y - x}$.

So y' at $\left(\sqrt{3}, 0\right)$ is $\dfrac{0 - 2\sqrt{3}}{2(0) - \sqrt{3}} = 2$ and y' at $\left(-\sqrt{3}, 0\right)$ is $\dfrac{0 + 2\sqrt{3}}{2(0) + \sqrt{3}} = 2$. Thus, the tangent lines at these points are parallel.

62. (a) We use implicit differentiation to find $y' = \dfrac{y - 2x}{2y - x}$ as in Exercise 61. The slope of the tangent line at $(-1, 1)$ is $m = \dfrac{1 - 2(-1)}{2(1) - (-1)} = \dfrac{3}{3} = 1$, so the slope of the normal line is $-\dfrac{1}{m} = -1$, and its equation is $y - 1 = -1(x + 1) \quad \Leftrightarrow \quad y = -x$. Substituting $-x$ for y in the equation of the ellipse, we get $x^2 - x(-x) + (-x)^2 = 3 \quad \Rightarrow \quad 3x^2 = 3 \quad \Leftrightarrow \quad x = \pm 1$. So the normal line must intersect the ellipse again at $x = 1$, and since the equation of the line is $y = -x$, the other point of intersection must be $(1, -1)$.

(b)



63. $x^2y^2 + xy = 2 \quad \Rightarrow \quad x^2 \cdot 2yy' + y^2 \cdot 2x + x \cdot y' + y \cdot 1 = 0 \quad \Leftrightarrow \quad y'(2x^2y + x) = -2xy^2 - y \quad \Leftrightarrow$

$y' = -\dfrac{2xy^2 + y}{2x^2y + x}$. So $-\dfrac{2xy^2 + y}{2x^2y + x} = -1 \quad \Leftrightarrow \quad 2xy^2 + y = 2x^2y + x \quad \Leftrightarrow \quad y(2xy + 1) = x(2xy + 1) \quad \Leftrightarrow$

$y(2xy + 1) - x(2xy + 1) = 0 \quad \Leftrightarrow \quad (2xy + 1)(y - x) = 0 \quad \Leftrightarrow \quad xy = -\frac{1}{2}$ or $y = x$. But $xy = -\frac{1}{2} \quad \Rightarrow$

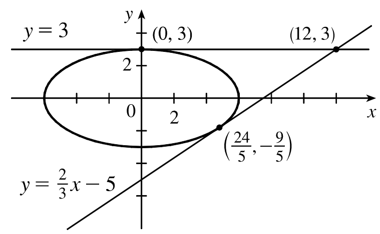
$x^2y^2 + xy = \frac{1}{4} - \frac{1}{2} \neq 2$, so we must have $x = y$. Then $x^2y^2 + xy = 2 \quad \Rightarrow \quad x^4 + x^2 = 2 \quad \Leftrightarrow \quad x^4 + x^2 - 2 = 0 \quad \Leftrightarrow$

$(x^2 + 2)(x^2 - 1) = 0$. So $x^2 = -2$, which is impossible, or $x^2 = 1 \quad \Leftrightarrow \quad x = \pm 1$. Since $x = y$, the points on the curve where the tangent line has a slope of -1 are $(-1, -1)$ and $(1, 1)$.

64. $x^2 + 4y^2 = 36 \quad\Rightarrow\quad 2x + 8yy' = 0 \quad\Rightarrow\quad y' = -\dfrac{x}{4y}$. Let (a, b) be a point on $x^2 + 4y^2 = 36$ whose tangent line passes through $(12, 3)$. The tangent line is then $y - 3 = -\dfrac{a}{4b}(x - 12)$, so $b - 3 = -\dfrac{a}{4b}(a - 12)$. Multiplying both sides by $4b$ gives $4b^2 - 12b = -a^2 + 12a$, so $4b^2 + a^2 = 12(a + b)$. But $4b^2 + a^2 = 36$, so $36 = 12(a + b) \quad\Rightarrow\quad a + b = 3 \quad\Rightarrow\quad b = 3 - a$. Substituting $3 - a$ for b into $a^2 + 4b^2 = 36$ gives $a^2 + 4(3 - a)^2 = 36 \quad\Leftrightarrow\quad a^2 + 36 - 24a + 4a^2 = 36 \quad\Leftrightarrow\quad 5a^2 - 24a = 0 \quad\Leftrightarrow\quad a(5a - 24) = 0$, so $a = 0$ or $a = \frac{24}{5}$. If $a = 0$, $b = 3 - 0 = 3$, and if $a = \frac{24}{5}$, $b = 3 - \frac{24}{5} = -\frac{9}{5}$.

So the two points on the ellipse are $(0, 3)$ and $\left(\frac{24}{5}, -\frac{9}{5}\right)$. Using $y - 3 = -\dfrac{a}{4b}(x - 12)$ with $(a, b) = (0, 3)$ gives us the tangent line $y - 3 = 0$ or $y = 3$. With $(a, b) = \left(\frac{24}{5}, -\frac{9}{5}\right)$, we have $y - 3 = -\frac{24/5}{4(-9/5)}(x - 12) \quad\Leftrightarrow\quad y - 3 = \frac{2}{3}(x - 12) \quad\Leftrightarrow\quad y = \frac{2}{3}x - 5$.

A graph of the ellipse and the tangent lines confirms our results.



65. For $\dfrac{x}{y} = y^2 + 1$, $y \neq 0$, we have $\dfrac{d}{dx}\left(\dfrac{x}{y}\right) = \dfrac{d}{dx}(y^2 + 1) \quad\Rightarrow\quad \dfrac{y \cdot 1 - x \cdot y'}{y^2} = 2y\,y' \quad\Rightarrow\quad y - x\,y' = 2y^3\,y' \quad\Rightarrow$

$2y^3\,y' + x\,y' = y \quad\Rightarrow\quad y'\,(2y^3 + x) = y \quad\Rightarrow\quad y' = \dfrac{y}{2y^3 + x}$.

For $x = y^3 + y$, $y \neq 0$, we have $\dfrac{d}{dx}(x) = \dfrac{d}{dx}(y^3 + y) \quad\Rightarrow\quad 1 = 3y^2y' + y' \quad\Rightarrow\quad 1 = y'\,(3y^2 + 1) \quad\Rightarrow$

$y' = \dfrac{1}{3y^2 + 1}$.

From part (a), $y' = \dfrac{y}{2y^3 + x}$. Since $y \neq 0$, we substitute $y^3 + y$ for x to get

$\dfrac{y}{2y^3 + x} = \dfrac{y}{2y^3 + (y^3 + y)} = \dfrac{y}{3y^3 + y} = \dfrac{y}{y(3y^2 + 1)} = \dfrac{1}{3y^2 + 1}$, which agrees with part (b).

66. (a) $y = J(x)$ and $xy'' + y' + xy = 0 \quad\Rightarrow\quad xJ''(x) + J'(x) + xJ(x) = 0$. If $x = 0$, we have $0 + J'(0) + 0 = 0$, so $J'(0) = 0$.

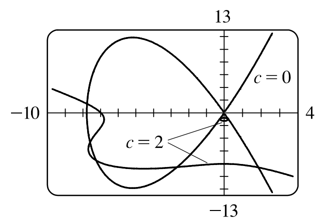
(b) Differentiating $xy'' + y' + xy = 0$ implicitly, we get $xy''' + y'' \cdot 1 + y'' + xy' + y \cdot 1 = 0 \quad\Rightarrow$ $xy''' + 2y'' + xy' + y = 0$, so $xJ'''(x) + 2J''(x) + xJ'(x) + J(x) = 0$. If $x = 0$, we have $0 + 2J''(0) + 0 + 1 \quad [J(0) = 1 \text{ is given}] \quad = 0 \quad\Rightarrow\quad 2J''(0) = -1 \quad\Rightarrow\quad J''(0) = -\frac{1}{2}$.

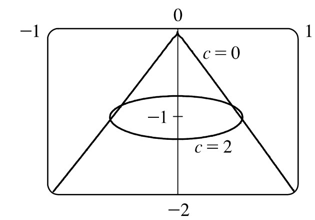
67. $x^2 + 4y^2 = 5 \quad\Rightarrow\quad 2x + 4(2yy') = 0 \quad\Rightarrow\quad y' = -\dfrac{x}{4y}$. Now let h be the height of the lamp, and let (a, b) be the point of tangency of the line passing through the points $(3, h)$ and $(-5, 0)$. This line has slope $(h - 0)/[3 - (-5)] = \frac{1}{8}h$. But the slope of the tangent line through the point (a, b) can be expressed as $y' = -\dfrac{a}{4b}$, or as $\dfrac{b - 0}{a - (-5)} = \dfrac{b}{a + 5}$ [since the line passes through $(-5, 0)$ and (a, b)], so $-\dfrac{a}{4b} = \dfrac{b}{a + 5} \quad\Leftrightarrow\quad 4b^2 = -a^2 - 5a \quad\Leftrightarrow\quad a^2 + 4b^2 = -5a$. But $a^2 + 4b^2 = 5$

[since (a, b) is on the ellipse], so $5 = -5a \quad \Leftrightarrow \quad a = -1$. Then $4b^2 = -a^2 - 5a = -1 - 5(-1) = 4 \quad \Rightarrow \quad b = 1$, since the point is on the top half of the ellipse. So $\dfrac{h}{8} = \dfrac{b}{a+5} = \dfrac{1}{-1+5} = \dfrac{1}{4} \quad \Rightarrow \quad h = 2$. So the lamp is located 2 units above the x-axis.

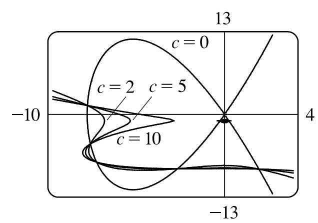
DISCOVERY PROJECT Families of Implicit Curves

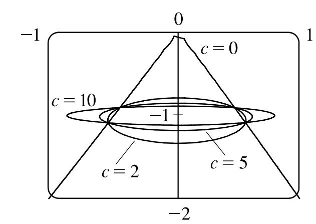
1. (a) There appear to be nine points of intersection. The "inner four" near the origin are about $(\pm 0.2, -0.9)$ and $(\pm 0.3, -1.1)$. The "outer five" are about $(2.0, -8.9)$, $(-2.8, -8.8)$, $(-7.5, -7.7)$, $(-7.8, -4.7)$, and $(-8.0, 1.5)$.





(b) We see from the graphs with $c = 5$ and $c = 10$, and for other values of c, that the curves change shape but the nine points of intersection are the same.



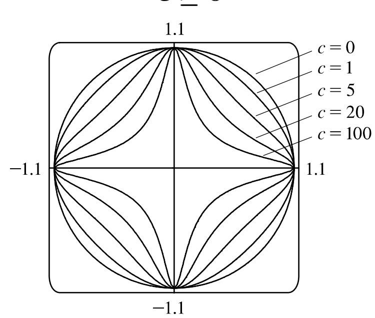


2. (a) If $c = 0$, the graph is the unit circle. As c increases, the graph looks more diamondlike and then more crosslike (see the graph for $c \geq 0$).

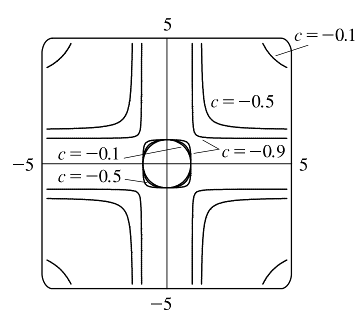
For $-1 < c < 0$ (see the graph), there are four hyperboliclike branches as well as an ellipticlike curve bounded by $|x| \leq 1$ and $|y| \leq 1$ for values of c close to 0. As c gets closer to -1, the branches and the curve become more rectangular, approaching the lines $|x| = 1$ and $|y| = 1$.

For $c = -1$, we get the lines $x = \pm 1$ and $y = \pm 1$. As c decreases, we get four test-tubelike curves (see the graph) that are bounded by $|x| = 1$ and $|y| = 1$, and get thinner as $|c|$ gets larger.

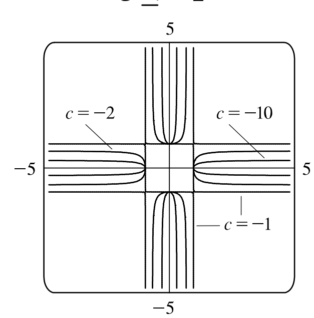
$\boldsymbol{c \geq 0}$



$\boldsymbol{-1 < c < 0}$



$\boldsymbol{c \leq -1}$



(b) The curve for $c = -1$ is described in part (a). When $c = -1$, we get

$x^2 + y^2 - x^2y^2 = 1 \quad\Leftrightarrow\quad 0 = x^2y^2 - x^2 - y^2 + 1 \quad\Leftrightarrow\quad 0 = (x^2-1)(y^2-1) \quad\Leftrightarrow\quad x = \pm 1$ or $y = \pm 1$, which algebraically proves that the graph consists of the stated lines.

(c) $\dfrac{d}{dx}(x^2 + y^2 + cx^2y^2) = \dfrac{d}{dx}(1) \quad\Rightarrow\quad 2x + 2y\,y' + c(x^2 \cdot 2y\,y' + y^2 \cdot 2x) = 0 \quad\Rightarrow$

$2y\,y' + 2cx^2y\,y' = -2x - 2cxy^2 \quad\Rightarrow\quad 2y(1 + cx^2)y' = -2x(1 + cy^2) \quad\Rightarrow\quad y' = -\dfrac{x(1+cy^2)}{y(1+cx^2)}.$

For $c = -1$, $y' = -\dfrac{x(1-y^2)}{y(1-x^2)} = -\dfrac{x(1+y)(1-y)}{y(1+x)(1-x)}$, so $y' = 0$ when $y = \pm 1$ or $x = 0$ (which leads to $y = \pm 1$) and y' is undefined when $x = \pm 1$ or $y = 0$ (which leads to $x = \pm 1$). Since the graph consists of the lines $x = \pm 1$ and $y = \pm 1$, the slope at any point on the graph is undefined or 0, which is consistent with the expression found for y'.

3.6 Derivatives of Logarithmic and Inverse Trigonometric Functions

1. The differentiation formula for logarithmic functions, $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x\ln a}$, is simplest when $a = e$ because $\ln e = 1$.

2. $g(t) = \ln(3 + t^2) \quad\Rightarrow\quad g'(t) = \dfrac{1}{3+t^2} \cdot \dfrac{d}{dt}(3+t^2) = \dfrac{1}{3+t^2} \cdot 2t = \dfrac{2t}{3+t^2}$

3. $f(x) = \ln(x^2 + 3x + 5) \quad\Rightarrow\quad f'(x) = \dfrac{1}{x^2+3x+5} \cdot \dfrac{d}{dx}(x^2+3x+5) = \dfrac{1}{x^2+3x+5} \cdot (2x+3) = \dfrac{2x+3}{x^2+3x+5}$

4. $f(x) = x\ln x - x \quad\Rightarrow\quad f'(x) = x \cdot \dfrac{1}{x} + (\ln x) \cdot 1 - 1 = 1 + \ln x - 1 = \ln x$

5. $f(x) = \sin(\ln x) \quad\Rightarrow\quad f'(x) = \cos(\ln x) \cdot \dfrac{d}{dx}\ln x = \cos(\ln x) \cdot \dfrac{1}{x} = \dfrac{\cos(\ln x)}{x}$

6. $f(x) = \ln(\sin^2 x) = \ln(\sin x)^2 = 2\ln|\sin x| \quad\Rightarrow\quad f'(x) = 2 \cdot \dfrac{1}{\sin x} \cdot \cos x = 2\cot x$

7. $f(x) = \ln\dfrac{1}{x} \quad\Rightarrow\quad f'(x) = \dfrac{1}{1/x}\,\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = x\left(-\dfrac{1}{x^2}\right) = -\dfrac{1}{x}.$

Another solution: $f(x) = \ln\dfrac{1}{x} = \ln 1 - \ln x = -\ln x \quad\Rightarrow\quad f'(x) = -\dfrac{1}{x}.$

8. $y = \dfrac{1}{\ln x} = (\ln x)^{-1} \quad\Rightarrow\quad y' = -1(\ln x)^{-2} \cdot \dfrac{1}{x} = \dfrac{-1}{x(\ln x)^2}$

9. $g(x) = \ln(xe^{-2x}) = \ln x + \ln e^{-2x} = \ln x - 2x \quad\Rightarrow\quad g'(x) = \dfrac{1}{x} - 2$

10. $g(t) = \sqrt{1 + \ln t} \quad\Rightarrow\quad g'(t) = \frac{1}{2}(1 + \ln t)^{-1/2}\,\dfrac{d}{dt}(1 + \ln t) = \dfrac{1}{2\sqrt{1+\ln t}} \cdot \dfrac{1}{t} = \dfrac{1}{2t\sqrt{1+\ln t}}$

11. $F(t) = (\ln t)^2 \sin t \quad\Rightarrow\quad F'(t) = (\ln t)^2 \cos t + \sin t \cdot 2\ln t \cdot \dfrac{1}{t} = \ln t\left(\ln t\,\cos t + \dfrac{2\sin t}{t}\right)$

12. $p(t) = \ln\sqrt{t^2+1} \quad \Rightarrow \quad p'(t) = \dfrac{1}{\sqrt{t^2+1}} \cdot \dfrac{d}{dt}\left(\sqrt{t^2+1}\right) = \dfrac{1}{\sqrt{t^2+1}} \cdot \dfrac{2t}{2\sqrt{t^2+1}} = \dfrac{t}{t^2+1}$

Or: $p(t) = \ln\sqrt{t^2+1} = \ln(t^2+1)^{1/2} = \frac{1}{2}\ln(t^2+1) \quad \Rightarrow \quad p'(t) = \dfrac{1}{2} \cdot \dfrac{1}{t^2+1} \cdot 2t = \dfrac{t}{t^2+1}$

13. $y = \log_8(x^2+3x) \quad \Rightarrow \quad y' = \dfrac{1}{(x^2+3x)\ln 8} \cdot \dfrac{d}{dx}(x^2+3x) = \dfrac{1}{(x^2+3x)\ln 8} \cdot (2x+3) = \dfrac{2x+3}{(x^2+3x)\ln 8}$

14. $y = \log_{10}\sec x \quad \Rightarrow \quad y' = \dfrac{1}{\sec x\,(\ln 10)} \cdot \dfrac{d}{dx}(\sec x) = \dfrac{1}{\sec x\,(\ln 10)} \cdot \sec x\, \tan x = \dfrac{\tan x}{\ln 10}$

15. $F(s) = \ln\ln s \quad \Rightarrow \quad F'(s) = \dfrac{1}{\ln s}\dfrac{d}{ds}\ln s = \dfrac{1}{\ln s} \cdot \dfrac{1}{s} = \dfrac{1}{s\ln s}$

16. $P(v) = \dfrac{\ln v}{1-v} \quad \Rightarrow \quad P'(v) = \dfrac{(1-v)(1/v) - (\ln v)(-1)}{(1-v)^2} \cdot \dfrac{v}{v} = \dfrac{1 - v + v\ln v}{v(1-v)^2}$

17. $T(z) = 2^z \log_2 z \quad \Rightarrow \quad T'(z) = 2^z \dfrac{1}{z\ln 2} + \log_2 z \cdot 2^z \ln 2 = 2^z\left(\dfrac{1}{z\ln 2} + \log_2 z\,(\ln 2)\right).$

Note that $\log_2 z\,(\ln 2) = \dfrac{\ln z}{\ln 2}(\ln 2) = \ln z$ by the change of base formula. Thus, $T'(z) = 2^z\left(\dfrac{1}{z\ln 2} + \ln z\right).$

18. $g(t) = \ln\dfrac{t(t^2+1)^4}{\sqrt[3]{2t-1}} = \ln t + \ln(t^2+1)^4 - \ln\sqrt[3]{2t-1} = \ln t + 4\ln(t^2+1) - \frac{1}{3}\ln(2t-1) \quad \Rightarrow$

$g'(t) = \dfrac{1}{t} + 4 \cdot \dfrac{1}{t^2+1} \cdot 2t - \dfrac{1}{3} \cdot \dfrac{1}{2t-1} \cdot 2 = \dfrac{1}{t} + \dfrac{8t}{t^2+1} - \dfrac{2}{3(2t-1)}$

19. $y = \ln\left|3 - 2x^5\right| \quad \Rightarrow \quad y' = \dfrac{1}{3-2x^5} \cdot (-10x^4) = \dfrac{-10x^4}{3-2x^5}$

20. $y = \ln(\csc x - \cot x) \quad \Rightarrow$

$y' = \dfrac{1}{\csc x - \cot x}\dfrac{d}{dx}(\csc x - \cot x) = \dfrac{1}{\csc x - \cot x}(-\csc x\, \cot x + \csc^2 x) = \dfrac{\csc x(\csc x - \cot x)}{\csc x - \cot x} = \csc x$

21. $y = \ln(e^{-x} + xe^{-x}) = \ln(e^{-x}(1+x)) = \ln(e^{-x}) + \ln(1+x) = -x + \ln(1+x) \quad \Rightarrow$

$y' = -1 + \dfrac{1}{1+x} = \dfrac{-1-x+1}{1+x} = -\dfrac{x}{1+x}$

22. $g(x) = e^{x^2\ln x} \quad \Rightarrow$

$g'(x) = e^{x^2\ln x} \cdot \dfrac{d}{dx}(x^2\ln x) = e^{x^2\ln x}\left[x^2 \cdot \dfrac{1}{x} + (\ln x) \cdot 2x\right] = e^{x^2\ln x}(x + 2x\ln x) = x\,e^{x^2\ln x}(1 + 2\ln x)$

23. $h(x) = e^{x^2+\ln x} = e^{x^2} \cdot e^{\ln x} = e^{x^2} \cdot x = xe^{x^2} \Rightarrow$

$$\begin{aligned} h'(x) &= x \cdot \frac{d}{dx}\left(e^{x^2}\right) + e^{x^2} \cdot \frac{d}{dx}(x) = x \cdot e^{x^2} \cdot \frac{d}{dx}(x^2) + e^{x^2} \cdot 1 = x \cdot e^{x^2} \cdot 2x + e^{x^2} \\ &= 2x^2e^{x^2} + e^{x^2} = e^{x^2}(2x^2+1) \end{aligned}$$

24. $y = \ln\sqrt{\dfrac{1+2x}{1-2x}} = \ln\sqrt{1+2x} - \ln\sqrt{1-2x} = \frac{1}{2}\ln(1+2x) - \frac{1}{2}\ln(1-2x) \quad\Rightarrow$

$$y' = \frac{1}{2}\cdot\frac{1}{1+2x}\cdot 2 - \frac{1}{2}\cdot\frac{1}{1-2x}\cdot(-2) = \frac{1}{1+2x} + \frac{1}{1-2x}$$

25. $y = \ln\dfrac{x^a}{b^x} = \ln x^a - \ln b^x = a\ln x - x\ln b \quad\Rightarrow\quad y' = a\cdot\dfrac{1}{x} - \ln b = \dfrac{a}{x} - \ln b$

26. $y = \log_2(x\log_5 x) \quad\Rightarrow$

$$y' = \frac{1}{(x\log_5 x)(\ln 2)}\,\frac{d}{dx}\,(x\log_5 x) = \frac{1}{(x\log_5 x)(\ln 2)}\left(x\cdot\frac{1}{x\ln 5} + \log_5 x\right) = \frac{1}{(x\log_5 x)(\ln 5)(\ln 2)} + \frac{1}{x(\ln 2)}.$$

Note that $\log_5 x(\ln 5) = \dfrac{\ln x}{\ln 5}(\ln 5) = \ln x$ by the change of base formula. Thus, $y' = \dfrac{1}{x\ln x\ln 2} + \dfrac{1}{x\ln 2} = \dfrac{1+\ln x}{x\ln x\ln 2}$.

27.
$$\frac{d}{dx}\ln\!\left(x+\sqrt{x^2+1}\right) = \frac{1}{x+\sqrt{x^2+1}}\cdot\frac{d}{dx}\left(x+\sqrt{x^2+1}\right) = \frac{1}{x+\sqrt{x^2+1}}\cdot\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)$$
$$= \frac{1}{x+\sqrt{x^2+1}}\cdot\left(\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} + \frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{x+\sqrt{x^2+1}}\cdot\left(\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}$$

28.
$$\frac{d}{dx}\ln\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{d}{dx}\left(\ln\sqrt{1-\cos x} - \ln\sqrt{1+\cos x}\right) = \frac{d}{dx}\left[\frac{1}{2}\ln(1-\cos x) - \frac{1}{2}\ln(1+\cos x)\right]$$
$$= \frac{1}{2}\cdot\frac{1}{1-\cos x}\cdot\sin x - \frac{1}{2}\cdot\frac{1}{1+\cos x}\cdot(-\sin x)$$
$$= \frac{1}{2}\left(\frac{\sin x}{1-\cos x} + \frac{\sin x}{1+\cos x}\right) = \frac{1}{2}\left[\frac{\sin x\,(1+\cos x) + \sin x\,(1-\cos x)}{(1-\cos x)(1+\cos x)}\right]$$
$$= \frac{1}{2}\left(\frac{\sin x + \sin x\,\cos x + \sin x - \sin x\,\cos x}{1-\cos^2 x}\right) = \frac{1}{2}\left(\frac{2\sin x}{\sin^2 x}\right) = \frac{1}{\sin x} = \csc x$$

29. $y = \sqrt{x}\,\ln x \quad\Rightarrow\quad y' = \sqrt{x}\cdot\dfrac{1}{x} + (\ln x)\,\dfrac{1}{2\sqrt{x}} = \dfrac{2+\ln x}{2\sqrt{x}} \quad\Rightarrow$

$$y'' = \frac{2\sqrt{x}\,(1/x) - (2+\ln x)(1/\sqrt{x}\,)}{(2\sqrt{x}\,)^2} = \frac{2/\sqrt{x} - (2+\ln x)(1/\sqrt{x}\,)}{4x} = \frac{2-(2+\ln x)}{\sqrt{x}(4x)} = -\frac{\ln x}{4x\sqrt{x}}$$

30. $y = \dfrac{\ln x}{1+\ln x} \quad\Rightarrow\quad y' = \dfrac{(1+\ln x)(1/x) - (\ln x)(1/x)}{(1+\ln x)^2} = \dfrac{1}{x(1+\ln x)^2} \quad\Rightarrow$

$$y'' = -\frac{\dfrac{d}{dx}[x(1+\ln x)^2]}{[x(1+\ln x)^2]^2}\quad\text{[Reciprocal Rule]}\quad = -\frac{x\cdot 2(1+\ln x)\cdot(1/x) + (1+\ln x)^2}{x^2(1+\ln x)^4}$$
$$= -\frac{(1+\ln x)[2+(1+\ln x)]}{x^2(1+\ln x)^4} = -\frac{3+\ln x}{x^2(1+\ln x)^3}$$

31. $y = \ln|\sec x| \quad\Rightarrow\quad y' = \dfrac{1}{\sec x}\,\dfrac{d}{dx}\sec x = \dfrac{1}{\sec x}\sec x\,\tan x = \tan x \quad\Rightarrow\quad y'' = \sec^2 x$

32. $y = \ln(1+\ln x) \quad\Rightarrow\quad y' = \dfrac{1}{1+\ln x}\cdot\dfrac{1}{x} = \dfrac{1}{x(1+\ln x)} \quad\Rightarrow$

$$y'' = -\frac{\dfrac{d}{dx}[x(1+\ln x)]}{[x(1+\ln x)]^2}\quad\text{[Reciprocal Rule]}\quad = -\frac{x(1/x) + (1+\ln x)(1)}{x^2(1+\ln x)^2} = -\frac{1+1+\ln x}{x^2(1+\ln x)^2} = -\frac{2+\ln x}{x^2(1+\ln x)^2}$$

33. $f(x) = \dfrac{x}{1-\ln(x-1)} \quad \Rightarrow$

$$f'(x) = \frac{[1-\ln(x-1)]\cdot 1 - x\cdot \dfrac{-1}{x-1}}{[1-\ln(x-1)]^2} = \frac{\dfrac{(x-1)[1-\ln(x-1)]+x}{x-1}}{[1-\ln(x-1)]^2} = \frac{x-1-(x-1)\ln(x-1)+x}{(x-1)[1-\ln(x-1)]^2}$$
$$= \frac{2x-1-(x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}$$

$\text{Dom}(f) = \{x \mid x-1>0 \quad \text{and} \quad 1-\ln(x-1)\neq 0\} = \{x \mid x>1 \quad \text{and} \quad \ln(x-1)\neq 1\}$
$= \{x \mid x>1 \quad \text{and} \quad x-1\neq e^1\} = \{x \mid x>1 \quad \text{and} \quad x\neq 1+e\} = (1,1+e)\cup(1+e,\infty)$

34. $f(x) = \sqrt{2+\ln x} = (2+\ln x)^{1/2} \quad \Rightarrow \quad f'(x) = \frac{1}{2}(2+\ln x)^{-1/2}\cdot\frac{1}{x} = \dfrac{1}{2x\sqrt{2+\ln x}}$

$\text{Dom}(f) = \{x \mid 2+\ln x \geq 0\} = \{x \mid \ln x \geq -2\} = \{x \mid x\geq e^{-2}\} = [e^{-2},\infty)$.

35. $f(x) = \ln(x^2-2x) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x^2-2x}(2x-2) = \dfrac{2(x-1)}{x(x-2)}$.

$\text{Dom}(f) = \{x \mid x(x-2)>0\} = (-\infty,0)\cup(2,\infty)$.

36. $f(x) = \ln\ln\ln x \quad \Rightarrow \quad f'(x) = \dfrac{1}{\ln\ln x}\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x}$.

$\text{Dom}(f) = \{x \mid \ln\ln x>0\} = \{x \mid \ln x>1\} = \{x \mid x>e\} = (e,\infty)$.

37. $f(x) = \ln(x+\ln x) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x+\ln x}\dfrac{d}{dx}(x+\ln x) = \dfrac{1}{x+\ln x}\left(1+\dfrac{1}{x}\right)$.

Substitute 1 for x to get $f'(1) = \dfrac{1}{1+\ln 1}\left(1+\dfrac{1}{1}\right) = \dfrac{1}{1+0}(1+1) = 1\cdot 2 = 2$.

38. $f(x) = \cos(\ln x^2) \quad \Rightarrow \quad f'(x) = -\sin(\ln x^2)\dfrac{d}{dx}\ln x^2 = -\sin(\ln x^2)\dfrac{1}{x^2}(2x) = -\dfrac{2\sin(\ln x^2)}{x}$.

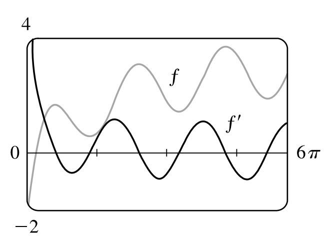
Substitute 1 for x to get $f'(1) = -\dfrac{2\sin(\ln 1^2)}{1} = -2\sin 0 = 0$.

39. $y = \ln(x^2-3x+1) \quad \Rightarrow \quad y' = \dfrac{1}{x^2-3x+1}\cdot(2x-3) \quad \Rightarrow \quad y'(3) = \frac{1}{1}\cdot 3 = 3$, so an equation of a tangent line at $(3,0)$ is $y-0 = 3(x-3)$, or $y = 3x-9$.

40. $y = x^2\ln x \quad \Rightarrow \quad y' = x^2\cdot\dfrac{1}{x}+(\ln x)(2x) \quad \Rightarrow \quad y'(1) = 1+0 = 1$, so an equation of a tangent line at $(1,0)$ is $y-0 = 1(x-1)$, or $y = x-1$.

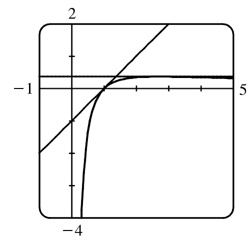
41. $f(x) = \sin x + \ln x \quad \Rightarrow \quad f'(x) = \cos x + 1/x$.

This is reasonable, because the graph shows that f increases when f' is positive, and $f'(x) = 0$ when f has a horizontal tangent.



42. $y = \dfrac{\ln x}{x} \quad \Rightarrow \quad y' = \dfrac{x(1/x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$.

$y'(1) = \dfrac{1-0}{1^2} = 1$ and $y'(e) = \dfrac{1-1}{e^2} = 0 \quad \Rightarrow \quad$ equations of tangent lines are $y - 0 = 1(x-1)$ or $y = x - 1$ and $y - 1/e = 0(x - e)$ or $y = 1/e$.



43. $f(x) = cx + \ln(\cos x) \quad \Rightarrow \quad f'(x) = c + \dfrac{1}{\cos x} \cdot (-\sin x) = c - \tan x$.

$f'(\frac{\pi}{4}) = 6 \quad \Rightarrow \quad c - \tan\frac{\pi}{4} = 6 \quad \Rightarrow \quad c - 1 = 6 \quad \Rightarrow \quad c = 7$.

44. $f(x) = \log_b(3x^2 - 2) \quad \Rightarrow \quad f'(x) = \dfrac{1}{(3x^2-2)\ln b} \cdot 6x$. $f'(1) = 3 \quad \Rightarrow \quad \dfrac{1}{\ln b} \cdot 6 = 3 \quad \Rightarrow \quad 2 = \ln b \quad \Rightarrow \quad b = e^2$.

45. $y = (x^2+2)^2(x^4+4)^4 \quad \Rightarrow \quad \ln y = \ln[(x^2+2)^2(x^4+4)^4] \quad \Rightarrow \quad \ln y = 2\ln(x^2+2) + 4\ln(x^4+4) \quad \Rightarrow$

$\dfrac{1}{y}\,y' = 2 \cdot \dfrac{1}{x^2+2} \cdot 2x + 4 \cdot \dfrac{1}{x^4+4} \cdot 4x^3 \quad \Rightarrow \quad y' = y\left(\dfrac{4x}{x^2+2} + \dfrac{16x^3}{x^4+4}\right) \quad \Rightarrow$

$y' = (x^2+2)^2(x^4+4)^4\left(\dfrac{4x}{x^2+2} + \dfrac{16x^3}{x^4+4}\right)$

46. $y = \dfrac{e^{-x}\cos^2 x}{x^2+x+1} \quad \Rightarrow \quad \ln y = \ln \dfrac{e^{-x}\cos^2 x}{x^2+x+1} \quad \Rightarrow$

$\ln y = \ln e^{-x} + \ln|\cos x|^2 - \ln(x^2+x+1) = -x + 2\ln|\cos x| - \ln(x^2+x+1) \quad \Rightarrow$

$\dfrac{1}{y}\,y' = -1 + 2 \cdot \dfrac{1}{\cos x}(-\sin x) - \dfrac{1}{x^2+x+1}(2x+1) \quad \Rightarrow \quad y' = y\left(-1 - 2\tan x - \dfrac{2x+1}{x^2+x+1}\right) \quad \Rightarrow$

$y' = -\dfrac{e^{-x}\cos^2 x}{x^2+x+1}\left(1 + 2\tan x + \dfrac{2x+1}{x^2+x+1}\right)$

47. $y = \sqrt{\dfrac{x-1}{x^4+1}} \quad \Rightarrow \quad \ln y = \ln\left(\dfrac{x-1}{x^4+1}\right)^{1/2} \quad \Rightarrow \quad \ln y = \dfrac{1}{2}\ln(x-1) - \dfrac{1}{2}\ln(x^4+1) \quad \Rightarrow$

$\dfrac{1}{y}\,y' = \dfrac{1}{2}\dfrac{1}{x-1} - \dfrac{1}{2}\dfrac{1}{x^4+1} \cdot 4x^3 \quad \Rightarrow \quad y' = y\left(\dfrac{1}{2(x-1)} - \dfrac{2x^3}{x^4+1}\right) \quad \Rightarrow \quad y' = \sqrt{\dfrac{x-1}{x^4+1}}\left(\dfrac{1}{2x-2} - \dfrac{2x^3}{x^4+1}\right)$

48. $y = \sqrt{x}\,e^{x^2-x}(x+1)^{2/3} \quad \Rightarrow \quad \ln y = \ln\left[x^{1/2}e^{x^2-x}(x+1)^{2/3}\right] \quad \Rightarrow$

$\ln y = \frac{1}{2}\ln x + (x^2 - x) + \frac{2}{3}\ln(x+1) \quad \Rightarrow \quad \dfrac{1}{y}\,y' = \dfrac{1}{2} \cdot \dfrac{1}{x} + 2x - 1 + \dfrac{2}{3} \cdot \dfrac{1}{x+1} \quad \Rightarrow$

$y' = y\left(\dfrac{1}{2x} + 2x - 1 + \dfrac{2}{3x+3}\right) \quad \Rightarrow \quad y' = \sqrt{x}\,e^{x^2-x}(x+1)^{2/3}\left(\dfrac{1}{2x} + 2x - 1 + \dfrac{2}{3x+3}\right)$

49. $y = x^x \quad \Rightarrow \quad \ln y = \ln x^x \quad \Rightarrow \quad \ln y = x\ln x \quad \Rightarrow \quad y'/y = x(1/x) + (\ln x) \cdot 1 \quad \Rightarrow \quad y' = y(1 + \ln x) \quad \Rightarrow$

$y' = x^x(1 + \ln x)$

50. $y = x^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x}\ln x \quad \Rightarrow \quad \dfrac{y'}{y} = -\dfrac{1}{x^2}\ln x + \dfrac{1}{x}\left(\dfrac{1}{x}\right) \quad \Rightarrow \quad y' = x^{1/x}\dfrac{1 - \ln x}{x^2}$

51. $y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} \Rightarrow \ln y = \sin x \ln x \Rightarrow \dfrac{y'}{y} = (\sin x) \cdot \dfrac{1}{x} + (\ln x)(\cos x) \Rightarrow$

$y' = y\left(\dfrac{\sin x}{x} + \ln x \cos x\right) \Rightarrow y' = x^{\sin x}\left(\dfrac{\sin x}{x} + \ln x \cos x\right)$

52. $y = \left(\sqrt{x}\right)^x \Rightarrow \ln y = \ln\left(\sqrt{x}\right)^x \Rightarrow \ln y = x \ln x^{1/2} \Rightarrow \ln y = \frac{1}{2} x \ln x \Rightarrow \dfrac{1}{y} y' = \dfrac{1}{2} x \cdot \dfrac{1}{x} + \ln x \cdot \dfrac{1}{2} \Rightarrow$

$y' = y\left(\frac{1}{2} + \frac{1}{2} \ln x\right) \Rightarrow y' = \frac{1}{2}\left(\sqrt{x}\right)^x (1 + \ln x)$

53. $y = (\cos x)^x \Rightarrow \ln y = \ln(\cos x)^x \Rightarrow \ln y = x \ln \cos x \Rightarrow \dfrac{1}{y} y' = x \cdot \dfrac{1}{\cos x} \cdot (-\sin x) + \ln \cos x \cdot 1 \Rightarrow$

$y' = y\left(\ln \cos x - \dfrac{x \sin x}{\cos x}\right) \Rightarrow y' = (\cos x)^x (\ln \cos x - x \tan x)$

54. $y = (\sin x)^{\ln x} \Rightarrow \ln y = \ln(\sin x)^{\ln x} \Rightarrow \ln y = \ln x \cdot \ln \sin x \Rightarrow \dfrac{1}{y} y' = \ln x \cdot \dfrac{1}{\sin x} \cdot \cos x + \ln \sin x \cdot \dfrac{1}{x} \Rightarrow$

$y' = y\left(\ln x \cdot \dfrac{\cos x}{\sin x} + \dfrac{\ln \sin x}{x}\right) \Rightarrow y' = (\sin x)^{\ln x}\left(\ln x \cot x + \dfrac{\ln \sin x}{x}\right)$

55. $y = x^{\ln x} \Rightarrow \ln y = \ln x \ln x = (\ln x)^2 \Rightarrow \dfrac{y'}{y} = 2 \ln x \left(\dfrac{1}{x}\right) \Rightarrow y' = x^{\ln x}\left(\dfrac{2 \ln x}{x}\right)$

56. $y = (\ln x)^{\cos x} \Rightarrow \ln y = \cos x \ln(\ln x) \Rightarrow \dfrac{y'}{y} = \cos x \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x} + (\ln \ln x)(-\sin x) \Rightarrow$

$y' = (\ln x)^{\cos x}\left(\dfrac{\cos x}{x \ln x} - \sin x \ln \ln x\right)$

57. $y = \ln(x^2 + y^2) \Rightarrow y' = \dfrac{1}{x^2 + y^2} \dfrac{d}{dx}(x^2 + y^2) \Rightarrow y' = \dfrac{2x + 2yy'}{x^2 + y^2} \Rightarrow x^2 y' + y^2 y' = 2x + 2yy' \Rightarrow$

$x^2 y' + y^2 y' - 2yy' = 2x \Rightarrow (x^2 + y^2 - 2y)y' = 2x \Rightarrow y' = \dfrac{2x}{x^2 + y^2 - 2y}$

58. $x^y = y^x \Rightarrow y \ln x = x \ln y \Rightarrow y \cdot \dfrac{1}{x} + (\ln x) \cdot y' = x \cdot \dfrac{1}{y} \cdot y' + \ln y \Rightarrow y' \ln x - \dfrac{x}{y} y' = \ln y - \dfrac{y}{x} \Rightarrow$

$y' = \dfrac{\ln y - y/x}{\ln x - x/y}$

59. $f(x) = \ln(x - 1) \Rightarrow f'(x) = \dfrac{1}{(x - 1)} = (x - 1)^{-1} \Rightarrow f''(x) = -(x - 1)^{-2} \Rightarrow f'''(x) = 2(x - 1)^{-3} \Rightarrow$

$f^{(4)}(x) = -2 \cdot 3(x - 1)^{-4} \Rightarrow \cdots \Rightarrow f^{(n)}(x) = (-1)^{n-1} \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n - 1)(x - 1)^{-n} = (-1)^{n-1} \dfrac{(n - 1)!}{(x - 1)^n}$

60. $y = x^8 \ln x$, so $D^9 y = D^8 y' = D^8(8x^7 \ln x + x^7)$. But the eighth derivative of x^7 is 0, so we now have

$D^8(8x^7 \ln x) = D^7(8 \cdot 7x^6 \ln x + 8x^6) = D^7(8 \cdot 7x^6 \ln x) = D^6(8 \cdot 7 \cdot 6x^5 \ln x) = \cdots = D(8!\, x^0 \ln x) = 8!/x.$

61. If $f(x) = \ln(1+x)$, then $f'(x) = \dfrac{1}{1+x}$, so $f'(0) = 1$.

Thus, $\displaystyle\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = f'(0) = 1$.

62. Let $m = n/x$. Then $n = xm$, and as $n \to \infty$, $m \to \infty$.

Therefore, $\displaystyle\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx} = \left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right]^x = e^x$ by Equation 6.

63. $f(x) = \sin^{-1}(5x) \Rightarrow f'(x) = \dfrac{1}{\sqrt{1-(5x)^2}} \cdot \dfrac{d}{dx}(5x) = \dfrac{5}{\sqrt{1-25x^2}}$

64. $g(x) = \sec^{-1}(e^x) \Rightarrow g'(x) = \dfrac{1}{e^x\sqrt{(e^x)^2-1}} \cdot \dfrac{d}{dx}(e^x) = \dfrac{1}{e^x\sqrt{e^{2x}-1}} \cdot e^x = \dfrac{1}{\sqrt{e^{2x}-1}}$

65. $y = \tan^{-1}\sqrt{x-1} \quad \Rightarrow$

$$y' = \frac{1}{1+\left(\sqrt{x-1}\right)^2} \cdot \frac{d}{dt}\left(\sqrt{x-1}\right) = \frac{1}{1+(x-1)} \cdot \frac{1}{2\sqrt{x-1}} = \frac{1}{x} \cdot \frac{1}{2\sqrt{x-1}} = \frac{1}{2x\sqrt{x-1}}$$

66. $y = \tan^{-1}(x^2) \quad \Rightarrow \quad y' = \dfrac{1}{1+(x^2)^2} \cdot \dfrac{d}{dx}(x^2) = \dfrac{1}{1+x^4} \cdot 2x = \dfrac{2x}{1+x^4}$

67. $y = (\tan^{-1} x)^2 \quad \Rightarrow \quad y' = 2(\tan^{-1} x)^1 \cdot \dfrac{d}{dx}(\tan^{-1} x) = 2\tan^{-1} x \cdot \dfrac{1}{1+x^2} = \dfrac{2\tan^{-1} x}{1+x^2}$

68. $g(x) = \arccos\sqrt{x} \quad \Rightarrow \quad g'(x) = -\dfrac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\,\dfrac{d}{dx}\sqrt{x} = -\dfrac{1}{\sqrt{1-x}}\left(\dfrac{1}{2}x^{-1/2}\right) = -\dfrac{1}{2\sqrt{x}\,\sqrt{1-x}}$

69. $h(x) = (\arcsin x)\ln x \quad \Rightarrow \quad h'(x) = (\arcsin x) \cdot \dfrac{1}{x} + (\ln x) \cdot \dfrac{1}{\sqrt{1-x^2}} = \dfrac{\arcsin x}{x} + \dfrac{\ln x}{\sqrt{1-x^2}}$

70. $g(t) = \ln(\arctan(t^4)) \quad \Rightarrow$

$$g'(t) = \frac{1}{\arctan(t^4)} \cdot \frac{d}{dt}(\arctan(t^4)) = \frac{1}{\arctan(t^4)} \cdot \frac{1}{1+(t^4)^2} \cdot \frac{d}{dt}(t^4)$$
$$= \frac{1}{\arctan(t^4)} \cdot \frac{1}{1+t^8} \cdot 4t^3 = \frac{4t^3}{(1+t^8)\arctan(t^4)}$$

71. $f(z) = e^{\arcsin(z^2)} \quad \Rightarrow \quad f'(z) = e^{\arcsin(z^2)} \cdot \dfrac{d}{dz}[\arcsin(z^2)] = e^{\arcsin(z^2)} \cdot \dfrac{1}{\sqrt{1-(z^2)^2}} \cdot 2z = \dfrac{2ze^{\arcsin(z^2)}}{\sqrt{1-z^4}}$

72. $y = \tan^{-1}\left(x - \sqrt{1+x^2}\right) \quad \Rightarrow$

$$y' = \frac{1}{1+\left(x-\sqrt{x^2+1}\right)^2}\left(1 - \frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{1+x^2-2x\sqrt{x^2+1}+x^2+1}\left(\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}}\right)$$
$$= \frac{\sqrt{x^2+1}-x}{2\left(1+x^2-x\sqrt{x^2+1}\right)\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}-x}{2\left[\sqrt{x^2+1}\,(1+x^2)-x(x^2+1)\right]} = \frac{\sqrt{x^2+1}-x}{2\left[(1+x^2)\left(\sqrt{x^2+1}-x\right)\right]}$$
$$= \frac{1}{2(1+x^2)}$$

73. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t) \quad \Rightarrow$

$$h'(t) = -\frac{1}{1+t^2} - \frac{1}{1+(1/t)^2} \cdot \frac{d}{dt}\frac{1}{t} = -\frac{1}{1+t^2} - \frac{t^2}{t^2+1} \cdot \left(-\frac{1}{t^2}\right) = -\frac{1}{1+t^2} + \frac{1}{t^2+1} = 0.$$

Note that this makes sense because $h(t) = \dfrac{\pi}{2}$ for $t > 0$ and $h(t) = \dfrac{3\pi}{2}$ for $t < 0$.

74. $R(t) = \arcsin(1/t) \quad \Rightarrow$

$$R'(t) = \frac{1}{\sqrt{1-(1/t)^2}}\,\frac{d}{dt}\,\frac{1}{t} = \frac{1}{\sqrt{1-1/t^2}}\left(-\frac{1}{t^2}\right) = -\frac{1}{\sqrt{1-1/t^2}}\,\frac{1}{\sqrt{t^4}} = -\frac{1}{\sqrt{t^4-t^2}}$$
$$= -\frac{1}{\sqrt{t^2(t^2-1)}} = -\frac{1}{|\,t\,|\sqrt{t^2-1}}$$

75. $y = x\sin^{-1}x + \sqrt{1-x^2} \quad \Rightarrow$

$$y' = x \cdot \frac{1}{\sqrt{1-x^2}} + (\sin^{-1}x)(1) + \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x - \frac{x}{\sqrt{1-x^2}} = \sin^{-1}x$$

76. $y = \cos^{-1}(\sin^{-1}t) \quad \Rightarrow \quad y' = -\dfrac{1}{\sqrt{1-(\sin^{-1}t)^2}} \cdot \dfrac{d}{dt}\sin^{-1}t = -\dfrac{1}{\sqrt{1-(\sin^{-1}t)^2}} \cdot \dfrac{1}{\sqrt{1-t^2}}$

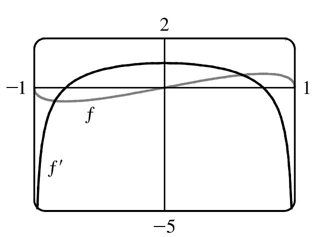
77. $y = \tan^{-1}\left(\dfrac{x}{a}\right) + \ln\sqrt{\dfrac{x-a}{x+a}} = \tan^{-1}\left(\dfrac{x}{a}\right) + \dfrac{1}{2}\ln\left(\dfrac{x-a}{x+a}\right) \quad \Rightarrow$

$$y' = \frac{1}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1}{a} + \frac{1}{2} \cdot \frac{1}{\frac{x-a}{x+a}} \cdot \frac{(x+a)\cdot 1 - (x-a)\cdot 1}{(x+a)^2} = \frac{1}{a+\frac{x^2}{a}} + \frac{1}{2} \cdot \frac{x+a}{x-a} \cdot \frac{2a}{(x+a)^2}$$
$$= \frac{1}{a+\frac{x^2}{a}} \cdot \frac{a}{a} + \frac{a}{(x-a)(x+a)} = \frac{a}{x^2+a^2} + \frac{a}{x^2-a^2}$$

78. $y = \arctan\sqrt{\dfrac{1-x}{1+x}} = \arctan\left(\dfrac{1-x}{1+x}\right)^{1/2} \quad \Rightarrow$

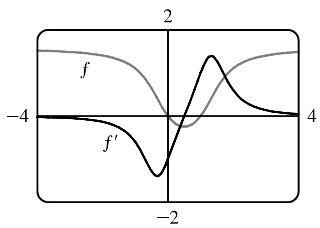
$$y' = \frac{1}{1+\left(\sqrt{\frac{1-x}{1+x}}\right)^2} \cdot \frac{d}{dx}\left(\frac{1-x}{1+x}\right)^{1/2} = \frac{1}{1+\frac{1-x}{1+x}} \cdot \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2} \cdot \frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}$$
$$= \frac{1}{\frac{1+x}{1+x} + \frac{1-x}{1+x}} \cdot \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{1/2} \cdot \frac{-2}{(1+x)^2} = \frac{1+x}{2} \cdot \frac{1}{2} \cdot \frac{(1+x)^{1/2}}{(1-x)^{1/2}} \cdot \frac{-2}{(1+x)^2}$$
$$= \frac{-1}{2(1-x)^{1/2}(1+x)^{1/2}} = \frac{-1}{2\sqrt{1-x^2}}$$

79. $f(x) = \sqrt{1-x^2}\arcsin x \quad \Rightarrow \quad f'(x) = \sqrt{1-x^2} \cdot \dfrac{1}{\sqrt{1-x^2}} + \arcsin x \cdot \dfrac{1}{2}\left(1-x^2\right)^{-1/2}(-2x) = 1 - \dfrac{x\arcsin x}{\sqrt{1-x^2}}$



Note that $f' = 0$ where the graph of f has a horizontal tangent. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

80. $f(x) = \arctan(x^2 - x) \quad \Rightarrow \quad f'(x) = \dfrac{1}{1 + (x^2 - x)^2} \cdot \dfrac{d}{dx}(x^2 - x) = \dfrac{2x - 1}{1 + (x^2 - x)^2}$



Note that $f' = 0$ where the graph of f has a horizontal tangent. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

81. Let $y = \cos^{-1} x$. Then $\cos y = x$ and $0 \le y \le \pi \quad \Rightarrow \quad -\sin y \, \dfrac{dy}{dx} = 1 \quad \Rightarrow$

$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - \cos^2 y}} = -\frac{1}{\sqrt{1 - x^2}}. \quad \text{[Note that } \sin y \ge 0 \text{ for } 0 \le y \le \pi.]$$

82. (a) Let $y = \sec^{-1} x$. Then $\sec y = x$ and $y \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$. Differentiate with respect to x: $\sec y \, \tan y \left(\dfrac{dy}{dx}\right) = 1 \quad \Rightarrow$

$\dfrac{dy}{dx} = \dfrac{1}{\sec y \, \tan y} = \dfrac{1}{\sec y \sqrt{\sec^2 y - 1}} = \dfrac{1}{x \sqrt{x^2 - 1}}$. Note that $\tan^2 y = \sec^2 y - 1 \quad \Rightarrow \quad \tan y = \sqrt{\sec^2 y - 1}$

since $\tan y > 0$ when $0 < y < \frac{\pi}{2}$ or $\pi < y < \frac{3\pi}{2}$.

(b) $y = \sec^{-1} x \quad \Rightarrow \quad \sec y = x \quad \Rightarrow \quad \sec y \, \tan y \, \dfrac{dy}{dx} = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{\sec y \, \tan y}$. Now $\tan^2 y = \sec^2 y - 1 = x^2 - 1$,

so $\tan y = \pm\sqrt{x^2 - 1}$. For $y \in \left[0, \frac{\pi}{2}\right)$, $x \ge 1$, so $\sec y = x = |x|$ and $\tan y \ge 0 \quad \Rightarrow$

$\dfrac{dy}{dx} = \dfrac{1}{x \sqrt{x^2 - 1}} = \dfrac{1}{|x| \sqrt{x^2 - 1}}$. For $y \in \left(\frac{\pi}{2}, \pi\right]$, $x \le -1$, so $|x| = -x$ and $\tan y = -\sqrt{x^2 - 1} \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{1}{\sec y \, \tan y} = \frac{1}{x\left(-\sqrt{x^2 - 1}\right)} = \frac{1}{(-x) \sqrt{x^2 - 1}} = \frac{1}{|x| \sqrt{x^2 - 1}}.$$

83. If $y = f^{-1}(x)$, then $f(y) = x$. Differentiating implicitly with respect to x and remembering that y is a function of x,

we get $f'(y) \dfrac{dy}{dx} = 1$, so $\dfrac{dy}{dx} = \dfrac{1}{f'(y)} \quad \Rightarrow \quad \left(f^{-1}\right)'(x) = \dfrac{1}{f'(f^{-1}(x))}$.

84. $f(4) = 5 \quad \Rightarrow \quad f^{-1}(5) = 4$. By Exercise 83, $\left(f^{-1}\right)'(5) = \dfrac{1}{f'(f^{-1}(5))} = \dfrac{1}{f'(4)} = \dfrac{1}{2/3} = \dfrac{3}{2}$.

85. $f(x) = x + e^x \quad \Rightarrow \quad f'(x) = 1 + e^x$. Observe that $f(0) = 1$, so that $f^{-1}(1) = 0$. By Exercise 83, we have

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{1 + e^0} = \frac{1}{1 + 1} = \frac{1}{2}.$$

86. $f(x) = x^3 + 3\sin x + 2\cos x \quad \Rightarrow \quad f'(x) = 3x^2 + 3\cos x - 2\sin x$. Observe that $f(0) = 2$, so that $f^{-1}(2) = 0$.

By Exercise 83, we have $(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(0)} = \dfrac{1}{3(0)^2 + 3\cos 0 - 2\sin 0} = \dfrac{1}{3(1)} = \dfrac{1}{3}$.

87. $h = f^g \quad\Rightarrow\quad \ln h = \ln f^g \quad\Rightarrow\quad \ln h = g \ln f \quad\Rightarrow\quad \dfrac{1}{h}\,h' = g \cdot \dfrac{1}{f}\,f\,' + (\ln f)\,g' \quad\Rightarrow$

$$h' = h\left(g \cdot \frac{1}{f}\,f\,' + (\ln f)\,g'\right) = f^g\left(g \cdot \frac{1}{f}\,f\,' + (\ln f)\,g'\right) = g \cdot \frac{f^g}{f} \cdot f\,' + (\ln f) \cdot f^g \cdot g' \quad\Rightarrow$$

$$h' = g \cdot f^{g-1} \cdot f\,' + (\ln f) \cdot f^g \cdot g'$$

88. (a) With $h(x) = x^3$, we have $f(x) = x$ and $g(x) = 3$ in the formula in Exercise 87. That formula gives

$h'(x) = 3 \cdot x^{3-1} \cdot 1 + (\ln x) \cdot x^3 \cdot 0 = 3x^2$.

(b) With $h(x) = 3^x$, we have $f(x) = 3$ and $g(x) = x$ in the formula in Exercise 87. That formula gives

$h'(x) = x \cdot 3^{x-1} \cdot 0 + (\ln 3) \cdot 3^x \cdot 1 = 3^x(\ln 3)$.

(c) With $h(x) = (\sin x)^x$, we have $f(x) = \sin x$ and $g(x) = x$ in the formula in Exercise 87. That formula gives

$h'(x) = x \cdot (\sin x)^{x-1} \cdot \cos x + (\ln \sin x) \cdot (\sin x)^x \cdot 1 = x \cos x\,(\sin x)^{x-1} + (\sin x)^x \ln \sin x$. Further simplification

gives $x \cos x\,(\sin x)^{x-1} + (\sin x)^x \ln \sin x = \dfrac{x \cos x\,(\sin x)^x}{\sin x} + (\sin x)^x\,\ln \sin x = (\sin x)^x\,(x \cot x + \ln \sin x)$.

3.7 Rates of Change in the Natural and Social Sciences

1. (a) $s = f(t) = t^3 - 9t^2 + 24t$ (in m) $\quad\Rightarrow\quad v(t) = f'(t) = 3t^2 - 18t + 24$ (in m/s)

(b) $v(1) = 3(1)^2 - 18(1) + 24 = 9$ m/s

(c) The particle is at rest when $v(t) = 0$: $3t^2 - 18t + 24 = 0 \quad\Leftrightarrow\quad 3(t-2)(t-4) = 0 \quad\Leftrightarrow\quad t = 2$ s or $t = 4$ s.

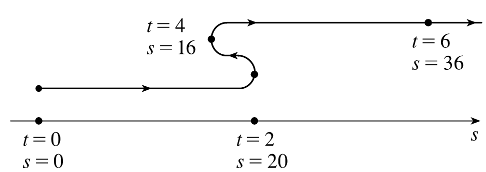
(d) The particle is moving in the positive direction when $v(t) > 0$. $3(t-2)(t-4) > 0 \quad\Rightarrow\quad 0 \le t < 2$ or $t > 4$.

(e) Because the particle changes direction when $t = 2$ and $t = 4$, we need to calculate the distances traveled in the intervals $[0, 2]$, $[2, 4]$, and $[4, 6]$ separately.

$$|f(2) - f(0)| = |20 - 0| = 20$$
$$|f(4) - f(2)| = |16 - 20| = 4$$
$$|f(6) - f(4)| = |36 - 16| = 20$$

The total distance is $20 + 4 + 20 = 44$ ft.

(f)

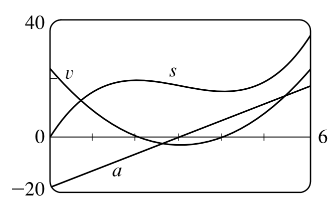


(g) $v(t) = 3t^2 - 18t + 24 \quad\Rightarrow$

$a(t) = v'(t) = 6t - 18$ (in (m/s)/s or m/s^2).

$a(1) = 6(1) - 18 = -12$ m/s^2

(h)



(i) The particle is speeding up when v and a have the same sign. This occurs when $2 < t < 3$ (v and a are both negative) and when $t > 4$ (v and a are both positive). It is slowing down when v and a have opposite signs; that is, when $0 \le t < 2$ and when $3 < t < 4$.

2. (a) $s = f(t) = \dfrac{9t}{t^2+9}$ (in m) $\Rightarrow$ $v(t) = f'(t) = \dfrac{(t^2+9)(9) - 9t(2t)}{(t^2+9)^2} = \dfrac{-9t^2+81}{(t^2+9)^2} = \dfrac{-9(t^2-9)}{(t^2+9)^2}$ (in m/s)

(b) $v(1) = \dfrac{-9(1-9)}{(1+9)^2} = \dfrac{72}{100} = 0.72$ m/s

(c) The particle is at rest when $v(t) = 0$. $\dfrac{-9(t^2-9)}{(t^2+9)^2} = 0 \quad \Leftrightarrow \quad t^2 - 9 = 0 \quad \Rightarrow \quad t = 3$ s [since $t \geq 0$].

(d) The particle is moving in the positive direction when $v(t) > 0$.

$\dfrac{-9(t^2-9)}{(t^2+9)^2} > 0 \quad \Rightarrow \quad -9(t^2-9) > 0 \quad \Rightarrow \quad t^2 - 9 < 0 \quad \Rightarrow \quad t^2 < 9 \quad \Rightarrow \quad 0 \leq t < 3.$

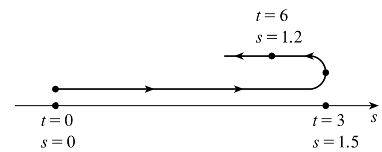
(e) Since the particle is moving in the positive direction and in the negative direction, we need to calculate the distance traveled in the intervals $[0, 3]$ and$[3, 6]$, respectively.

$|f(3) - f(0)| = \left|\frac{27}{18} - 0\right| = \frac{3}{2}$

$|f(6) - f(3)| = \left|\frac{54}{45} - \frac{27}{18}\right| = \frac{3}{10}$

The total distance is $\frac{3}{2} + \frac{3}{10} = \frac{9}{5}$ or 1.8 m.

(f)

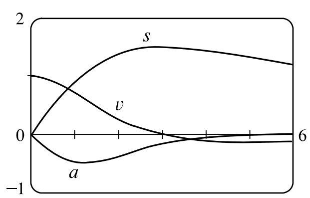


(g) $v(t) = -9\,\dfrac{t^2-9}{(t^2+9)^2} \quad \Rightarrow$

$$a(t) = v'(t) = -9\,\frac{(t^2+9)^2(2t) - (t^2-9)2(t^2+9)(2t)}{[(t^2+9)^2]^2} = -9\,\frac{2t(t^2+9)[(t^2+9) - 2(t^2-9)]}{(t^2+9)^4} = \frac{18t(t^2-27)}{(t^2+9)^3}.$$

$a(1) = \dfrac{18(-26)}{10^3} = -0.468$ m/s^2

(h)



(i) The particle is speeding up when v and a have the same sign. a is negative for $0 < t < \sqrt{27}$ [≈ 5.2], so from the figure in part (h), we see that v and a are both negative for $3 < t < 3\sqrt{3}$. The particle is slowing down when v and a have opposite signs. This occurs when $0 < t < 3$ and when $t > 3\sqrt{3}$.

3. (a) $s = f(t) = \sin(\pi t/2)$ (in m) $\Rightarrow$ $v(t) = f'(t) = \cos(\pi t/2) \cdot (\pi/2) = \frac{\pi}{2}\cos(\pi t/2)$ (in m/s)

(b) $v(1) = \frac{\pi}{2}\cos\frac{\pi}{2} = \frac{\pi}{2}(0) = 0$ m/s

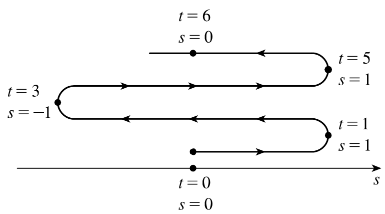
(c) The particle is at rest when $v(t) = 0$. $\frac{\pi}{2}\cos\frac{\pi}{2}t = 0 \quad \Leftrightarrow \quad \cos\frac{\pi}{2}t = 0 \quad \Leftrightarrow \quad \frac{\pi}{2}t = \frac{\pi}{2} + n\pi \quad \Leftrightarrow \quad t = 1 + 2n$, where n is a nonnegative integer since $t \geq 0$.

(d) The particle is moving in the positive direction when $v(t) > 0$. From part (c), we see that v changes sign at every positive odd integer. v is positive when $0 < t < 1$, $3 < t < 5$, $7 < t < 9$, and so on.

(e) v changes sign at $t = 1$, 3, and 5 in the interval $[0, 6]$. The total distance traveled during the first 6 seconds is

$$\begin{aligned} |f(1) - f(0)| + |f(3) - f(1)| + |f(5) - f(3)| + |f(6) - f(5)| &= |1 - 0| + |-1 - 1| + |1 - (-1)| + |0 - 1| \\ &= 1 + 2 + 2 + 1 = 6 \text{ m} \end{aligned}$$

(f)

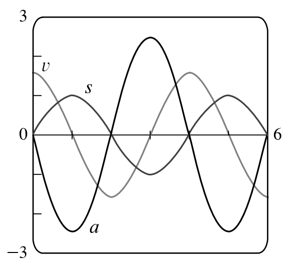


(g) $v(t) = \frac{\pi}{2}\cos(\pi t/2) \quad \Rightarrow$

$$\begin{aligned} a(t) = v'(t) &= \tfrac{\pi}{2}\left[-\sin(\pi t/2) \cdot (\pi/2)\right] \\ &= (-\pi^2/4)\sin(\pi t/2) \text{ m/s}^2 \end{aligned}$$

$$a(1) = (-\pi^2/4)\sin(\pi/2) = -\pi^2/4 \text{ m/s}^2$$

(h)



(i) The particle is speeding up when v and a have the same sign. From the figure in part (h), we see that v and a are both positive when $3 < t < 4$ and both negative when $1 < t < 2$ and $5 < t < 6$. Thus, the particle is speeding up when $1 < t < 2$, $3 < t < 4$, and $5 < t < 6$. The particle is slowing down when v and a have opposite signs; that is, when $0 < t < 1$, $2 < t < 3$, and $4 < t < 5$.

4. (a) $s = f(t) = t^2 e^{-t}$ (in m) $\quad \Rightarrow \quad v(t) = f'(t) = t^2(-e^{-t}) + e^{-t}(2t) = te^{-t}(-t + 2)$ (in m/s)

(b) $v(1) = (1)e^{-1}(-1 + 2) = 1/e$ m/s

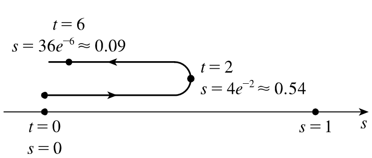
(c) The particle is at rest when $v(t) = 0$. $v(t) = 0 \quad \Leftrightarrow \quad t = 0$ or 2 s.

(d) The particle is moving in the positive direction when $v(t) > 0 \quad \Leftrightarrow \quad te^{-t}(-t + 2) > 0 \quad \Leftrightarrow \quad t(-t + 2) > 0 \quad \Leftrightarrow$ $0 < t < 2$.

(e) v changes sign at $t = 2$ in the interval $[0, 6]$. The total distance traveled during the first 6 seconds is

$$\begin{aligned} |f(2) - f(0)| + |f(6) - f(2)| &= \left|4e^{-2} - 0\right| + \left|36e^{-6} - 4e^{-2}\right| = 4e^{-2} + 4e^{-2} - 36e^{-6} \\ &= 8e^{-2} - 36e^{-6} \approx 0.99 \text{ m} \end{aligned}$$

(f)

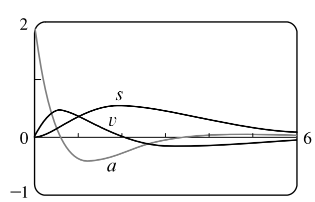


(g) $v(t) = (2t - t^2)e^{-t} \quad \Rightarrow$

$$\begin{aligned} a(t) = v'(t) &= (2t - t^2)(-e^{-t}) + e^{-t}(2 - 2t) \\ &= e^{-t}\left[-(2t - t^2) + (2 - 2t)\right] \\ &= e^{-t}(t^2 - 4t + 2) \text{ m/s}^2 \end{aligned}$$

$$a(1) = e^{-1}(1 - 4 + 2) = -1/e \text{ m/s}^2$$

(h)



(i) $a(t) = 0 \quad \Leftrightarrow \quad t^2 - 4t + 2 = 0 \quad [e^{-t} \neq 0] \quad \Leftrightarrow \quad t = \dfrac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$ [$\approx$ 0.6 and 3.4]. The particle is speeding up when v and a have the same sign. Using the previous information and the figure in part (h), we see that v and a are both positive when $0 < t < 2 - \sqrt{2}$ and both negative when $2 < t < 2 + \sqrt{2}$. The particle is slowing down when v and a have opposite signs. This occurs when $2 - \sqrt{2} < t < 2$ and $t > 2 + \sqrt{2}$.

5. (a) From the figure, the velocity v is positive on the interval $(0, 2)$ and negative on the interval $(2, 3)$. The acceleration a is positive (negative) when the slope of the tangent line is positive (negative), so the acceleration is positive on the interval $(0, 1)$, and negative on the interval $(1, 3)$. The particle is speeding up when v and a have the same sign, that is, on the interval $(0, 1)$ when $v > 0$ and $a > 0$, and on the interval $(2, 3)$ when $v < 0$ and $a < 0$. The particle is slowing down when v and a have opposite signs, that is, on the interval $(1, 2)$ when $v > 0$ and $a < 0$.

(b) $v > 0$ on $(0, 3)$ and $v < 0$ on $(3, 4)$. $a > 0$ on $(1, 2)$ and $a < 0$ on $(0, 1)$ and $(2, 4)$. The particle is speeding up on $(1, 2)$ $[v > 0, a > 0]$ and on $(3, 4)$ $[v < 0, a < 0]$. The particle is slowing down on $(0, 1)$ and $(2, 3)$ $[v > 0, a < 0]$.

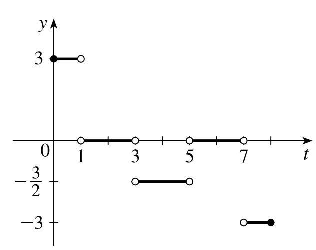
6. (a) The velocity v is positive when s is increasing, that is, on the intervals $(0, 1)$ and $(3, 4)$; and it is negative when s is decreasing, that is, on the interval $(1, 3)$. The acceleration a is positive when the graph of s is concave upward (v is increasing), that is, on the interval $(2, 4)$; and it is negative when the graph of s is concave downward (v is decreasing), that is, on the interval $(0, 2)$. The particle is speeding up on the interval $(1, 2)$ $[v < 0, a < 0]$ and on $(3, 4)$ $[v > 0, a > 0]$. The particle is slowing down on the interval $(0, 1)$ $[v > 0, a < 0]$ and on $(2, 3)$ $[v < 0, a > 0]$.

(b) The velocity v is positive on $(3, 4)$ and negative on $(0, 3)$. The acceleration a is positive on $(0, 1)$ and $(2, 4)$ and negative on $(1, 2)$. The particle is speeding up on the interval $(1, 2)$ $[v < 0, a < 0]$ and on $(3, 4)$ $[v > 0, a > 0]$. The particle is slowing down on the interval $(0, 1)$ $[v < 0, a > 0]$ and on $(2, 3)$ $[v < 0, a > 0]$.

7. The particle is traveling forward when its velocity is positive. From the graph, this occurs when $0 < t < 5$. The particle is traveling backward when its velocity is negative. From the graph, this occurs when $7 < t < 8$. When $5 < t < 7$, its velocity is zero and the particle is not moving.

8. The graph of the acceleration function is the graph of the derivative of the velocity function. Since the velocity function is piecewise linear, its derivative, where it exists, equals the slope of the corresponding piece of the velocity graph. Thus, we obtain the following table and graph of the acceleration function.

Interval	Acceleration (slope of velocity graph)
$0 < t < 1$	3
$1 < t < 3$	0
$3 < t < 5$	$-\frac{3}{2}$
$5 < t < 7$	0
$7 < t < 8$	-3



[continued]

The particle speeds up when its velocity and acceleration have the same sign. This occurs when $0 < t < 1$ ($v > 0$ and $a > 0$) and when $7 < t < 8$ ($v < 0$ and $a < 0$). The particle slows down when its velocity and acceleration have opposite signs. This occurs when $3 < t < 5$ ($v > 0$ and $a < 0$). The particle travels at a constant speed when its acceleration is zero. This occurs when $1 < t < 3$ and when $5 < t < 7$.

9. (a) $h(t) = 2 + 24.5t - 4.9t^2 \quad \Rightarrow \quad v(t) = h'(t) = 24.5 - 9.8t$. The velocity after 2 s is $v(2) = 24.5 - 9.8(2) = 4.9$ m/s and after 4 s is $v(4) = 24.5 - 9.8(4) = -14.7$ m/s.

(b) The projectile reaches its maximum height when the velocity is zero. $\quad v(t) = 0 \quad \Leftrightarrow \quad 24.5 - 9.8t = 0 \quad \Leftrightarrow$

$$t = \frac{24.5}{9.8} = 2.5 \text{ s.}$$

(c) The maximum height occurs when $t = 2.5$. $\quad h(2.5) = 2 + 24.5(2.5) - 4.9(2.5)^2 = 32.625 \text{ m} \ \left[\text{or } 32\tfrac{5}{8} \text{ m}\right]$.

(d) The projectile hits the ground when $h = 0 \quad \Leftrightarrow \quad 2 + 24.5t - 4.9t^2 = 0 \quad \Leftrightarrow$

$$t = \frac{-24.5 \pm \sqrt{24.5^2 - 4(-4.9)(2)}}{2(-4.9)} \quad \Rightarrow \quad t = t_f \approx 5.08 \text{ s} \ \ [\text{since } t \geq 0].$$

(e) The projectile hits the ground when $t = t_f$. Its velocity is $v(t_f) = 24.5 - 9.8t_f \approx -25.3$ m/s [downward].

10. (a) At maximum height, the velocity of the ball is 0 m/s. $v(t) = s'(t) = 24.5 - 9.8t \iff 9.8t = 24.5 \iff t = \frac{5}{2}$. So the maximum height is $s(\frac{5}{2}) = 24.5(\frac{5}{2}) - 4.9(\frac{5}{2})^2 = 30.625$ m.

(b) $s(t) = 24.5t - 4.9t^2 = 29.4 \iff 4.9t^2 - 24.5 + 29.4 = 0 \iff 4.9(t^2 - 5t + 6) = 0 \iff 4.9(t-2)(t-3) = 0$. So the ball has a height of 29.4 m on the way up when $t = 2$ and on the way down at $t = 3$. At these times the velocities are $v(2) = 24.5 - 9.8(2) = 4.9$ m/s and $v(3) = 24.5 - 9.8(3) = -4.9$ m/s, respectively.

11. (a) $h(t) = 15t - 1.86t^2 \quad \Rightarrow \quad v(t) = h'(t) = 15 - 3.72t$. The velocity after 2 s is $v(2) = 15 - 3.72(2) = 7.56$ m/s.

(b) $25 = h \quad \Leftrightarrow \quad 1.86t^2 - 15t + 25 = 0 \quad \Leftrightarrow \quad t = \dfrac{15 \pm \sqrt{15^2 - 4(1.86)(25)}}{2(1.86)} \quad \Leftrightarrow \quad t = t_1 \approx 2.35$ or $t = t_2 \approx 5.71$.

The velocities are $v(t_1) = 15 - 3.72t_1 \approx 6.24$ m/s [upward] and $v(t_2) = 15 - 3.72t_2 \approx -6.24$ m/s [downward].

12. (a) $s(t) = t^4 - 4t^3 - 20t^2 + 20t \quad \Rightarrow \quad v(t) = s'(t) = 4t^3 - 12t^2 - 40t + 20. \quad v = 20 \quad \Leftrightarrow$

$4t^3 - 12t^2 - 40t + 20 = 20 \quad \Leftrightarrow \quad 4t^3 - 12t^2 - 40t = 0 \quad \Leftrightarrow \quad 4t(t^2 - 3t - 10) = 0 \quad \Leftrightarrow$

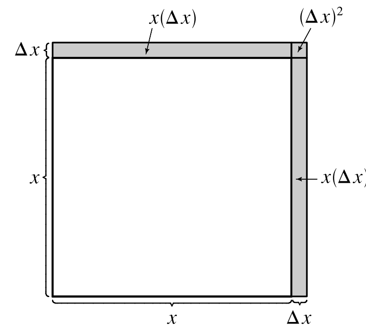
$4t(t-5)(t+2) = 0 \quad \Leftrightarrow \quad t = 0$ s or 5 s [for $t \geq 0$].

(b) $a(t) = v'(t) = 12t^2 - 24t - 40. \quad a = 0 \quad \Leftrightarrow \quad 12t^2 - 24t - 40 = 0 \quad \Leftrightarrow \quad 4(3t^2 - 6t - 10) = 0 \quad \Leftrightarrow$

$t = \dfrac{6 \pm \sqrt{6^2 - 4(3)(-10)}}{2(3)} = 1 \pm \dfrac{1}{3}\sqrt{39} \approx 3.08$ s [for $t \geq 0$]. At this time, the acceleration changes from negative to positive and the velocity attains its minimum value.

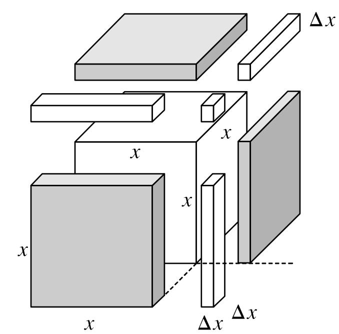
13. (a) $A(x) = x^2 \quad \Rightarrow \quad A'(x) = 2x$. $A'(15) = 30\text{ mm}^2/\text{mm}$ is the rate at which the area is increasing with respect to the side length as x reaches 15 mm.

(b) The perimeter is $P(x) = 4x$, so $A'(x) = 2x = \frac{1}{2}(4x) = \frac{1}{2}P(x)$. The figure suggests that if Δx is small, then the change in the area of the square is approximately half of its perimeter (2 of the 4 sides) times Δx. From the figure, $\Delta A = 2x\,(\Delta x) + (\Delta x)^2$. If Δx is small, then $\Delta A \approx 2x\,(\Delta x)$ and so $\Delta A/\Delta x \approx 2x$.



14. (a) $V(x) = x^3 \quad \Rightarrow \quad \dfrac{dV}{dx} = 3x^2$. $\left.\dfrac{dV}{dx}\right|_{x=3} = 3(3)^2 = 27\text{ mm}^3/\text{mm}$ is the rate at which the volume is increasing as x increases past 3 mm.

(b) The surface area is $S(x) = 6x^2$, so $V'(x) = 3x^2 = \frac{1}{2}(6x^2) = \frac{1}{2}S(x)$. The figure suggests that if Δx is small, then the change in the volume of the cube is approximately half of its surface area (the area of 3 of the 6 faces) times Δx. From the figure, $\Delta V = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3$. If Δx is small, then $\Delta V \approx 3x^2(\Delta x)$ and so $\Delta V/\Delta x \approx 3x^2$.



15. (a) Using $A(r) = \pi r^2$, we find that the average rate of change is:

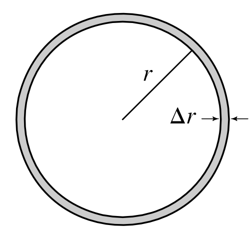
(i) $\dfrac{A(3) - A(2)}{3 - 2} = \dfrac{9\pi - 4\pi}{1} = 5\pi$

(ii) $\dfrac{A(2.5) - A(2)}{2.5 - 2} = \dfrac{6.25\pi - 4\pi}{0.5} = 4.5\pi$

(iii) $\dfrac{A(2.1) - A(2)}{2.1 - 2} = \dfrac{4.41\pi - 4\pi}{0.1} = 4.1\pi$

(b) $A(r) = \pi r^2 \quad \Rightarrow \quad A'(r) = 2\pi r$, so $A'(2) = 4\pi$.

(c) The circumference is $C(r) = 2\pi r = A'(r)$. The figure suggests that if Δr is small, then the change in the area of the circle (a ring around the outside) is approximately equal to its circumference times Δr. Straightening out this ring gives us a shape that is approximately rectangular with length $2\pi r$ and width Δr, so $\Delta A \approx 2\pi r(\Delta r)$. Algebraically, $\Delta A = A(r + \Delta r) - A(r) = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r(\Delta r) + \pi(\Delta r)^2$. So we see that if Δr is small, then $\Delta A \approx 2\pi r(\Delta r)$ and therefore, $\Delta A/\Delta r \approx 2\pi r$.



16. After t seconds the radius is $r = 60t$, so the area is $A(t) = \pi(60t)^2 = 3600\pi t^2 \quad \Rightarrow \quad A'(t) = 7200\pi t \quad \Rightarrow$

(a) $A'(1) = 7200\pi\text{ cm}^2/\text{s}$ (b) $A'(3) = 21{,}600\pi\text{ cm}^2/\text{s}$ (c) $A'(5) = 36{,}000\pi\text{ cm}^2/\text{s}$

As time goes by, the area grows at an increasing rate. In fact, the rate of change is linear with respect to time.

17. $S(r) = 4\pi r^2 \quad \Rightarrow \quad S'(r) = 8\pi r \quad \Rightarrow$

(a) $S'(20) = 160\pi\text{ m}^2/\text{m}$ (b) $S'(40) = 320\pi\text{ m}^2/\text{m}$ (c) $S'(60) = 480\pi\text{ m}^2/\text{m}$

As the radius increases, the surface area grows at an increasing rate. In fact, the rate of change is linear with respect to the radius.

18. (a) Using $V(r) = \frac{4}{3}\pi r^3$, we find that the average rate of change is:

(i) $\dfrac{V(8) - V(5)}{8 - 5} = \dfrac{\frac{4}{3}\pi(512) - \frac{4}{3}\pi(125)}{3} = 172\pi\ \mu\text{m}^3/\mu\text{m}$

(ii) $\dfrac{V(6) - V(5)}{6 - 5} = \dfrac{\frac{4}{3}\pi(216) - \frac{4}{3}\pi(125)}{1} = 121.\overline{3}\pi\ \mu\text{m}^3/\mu\text{m}$

(iii) $\dfrac{V(5.1) - V(5)}{5.1 - 5} = \dfrac{\frac{4}{3}\pi(5.1)^3 - \frac{4}{3}\pi(5)^3}{0.1} = 102.01\overline{3}\pi\ \mu\text{m}^3/\mu\text{m}$

(b) $V'(r) = 4\pi r^2$, so $V'(5) = 100\pi\ \mu\text{m}^3/\mu\text{m}$.

(c) $V(r) = \frac{4}{3}\pi r^3 \quad\Rightarrow\quad V'(r) = 4\pi r^2 = S(r)$. By analogy with Exercise 15(c), we can say that the change in the volume of the spherical shell, ΔV, is approximately equal to its thickness, Δr, times the surface area of the inner sphere. Thus, $\Delta V \approx 4\pi r^2(\Delta r)$ and so $\Delta V/\Delta r \approx 4\pi r^2$.

19. The mass is $f(x) = 3x^2$, so the linear density at x is $\rho(x) = f'(x) = 6x$.

(a) $\rho(1) = 6\ \text{kg/m}$ (b) $\rho(2) = 12\ \text{kg/m}$ (c) $\rho(3) = 18\ \text{kg/m}$

Since ρ is an increasing function, the density will be the highest at the right end of the rod and lowest at the left end.

20. $V(t) = 5000\left(1 - \frac{1}{40}t\right)^2 \quad\Rightarrow\quad V'(t) = 5000 \cdot 2\left(1 - \frac{1}{40}t\right)\left(-\frac{1}{40}\right) = -250\left(1 - \frac{1}{40}t\right)$

(a) $V'(5) = -250\left(1 - \frac{5}{40}\right) = -218.75\ \text{L/min}$ (b) $V'(10) = -250\left(1 - \frac{10}{40}\right) = -187.5\ \text{L/min}$

(c) $V'(20) = -250\left(1 - \frac{20}{40}\right) = -125\ \text{L/min}$ (d) $V'(40) = -250\left(1 - \frac{40}{40}\right) = 0\ \text{L/min}$

The water is flowing out the fastest at the beginning—when $t = 0$, $V'(t) = -250\ \text{L/min}$. The water is flowing out the slowest at the end—when $t = 40$, $V'(t) = 0$. As the tank empties, the water flows out more slowly.

21. The quantity of charge is $Q(t) = t^3 - 2t^2 + 6t + 2$, so the current is $Q'(t) = 3t^2 - 4t + 6$.

(a) $Q'(0.5) = 3(0.5)^2 - 4(0.5) + 6 = 4.75\ \text{A}$ (b) $Q'(1) = 3(1)^2 - 4(1) + 6 = 5\ \text{A}$

The current is lowest when Q' has a minimum. $Q''(t) = 6t - 4 < 0$ when $t < \frac{2}{3}$. So the current decreases when $t < \frac{2}{3}$ and increases when $t > \frac{2}{3}$. Thus, the current is lowest at $t = \frac{2}{3}$ s.

22. (a) $F = \dfrac{GmM}{r^2} = (GmM)r^{-2} \quad\Rightarrow\quad \dfrac{dF}{dr} = -2(GmM)r^{-3} = -\dfrac{2GmM}{r^3}$, which is the rate of change of the force with respect to the distance between the bodies. The minus sign indicates that as the distance r between the bodies increases, the magnitude of the force F exerted by the body of mass m on the body of mass M is decreasing.

(b) Given $F'(20{,}000) = -2$, find $F'(10{,}000)$. $-2 = -\dfrac{2GmM}{20{,}000^3} \quad\Rightarrow\quad GmM = 20{,}000^3$.

$$F'(10,000) = -\frac{2(20{,}000^3)}{10{,}000^3} = -2 \cdot 2^3 = -16\ \text{N/km}$$

23. With $m = m_0\left(1 - \dfrac{v^2}{c^2}\right)^{-1/2}$,

$$F = \frac{d}{dt}(mv) = m\,\frac{d}{dt}(v) + v\,\frac{d}{dt}(m) = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}\cdot a + v\cdot m_0\left[-\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\right]\left(-\frac{2v}{c^2}\right)\frac{d}{dt}(v)$$

$$= m_0\left(1-\frac{v^2}{c^2}\right)^{-3/2}\cdot a\left[\left(1-\frac{v^2}{c^2}\right)+\frac{v^2}{c^2}\right] = \frac{m_0 a}{(1-v^2/c^2)^{3/2}}$$

Note that we factored out $(1 - v^2/c^2)^{-3/2}$ since $-3/2$ was the lesser exponent. Also note that $\dfrac{d}{dt}(v) = a$.

24. (a) $D(t) = 7 + 5\cos[0.503(t - 6.75)] \quad\Rightarrow\quad D'(t) = -5\sin[0.503(t - 6.75)](0.503) = -2.515\sin[0.503(t - 6.75)]$.

At 3:00 AM, $t = 3$, and $D'(3) = -2.515\sin[0.503(-3.75)] \approx 2.39$ m/h (rising).

(b) At 6:00 AM, $t = 6$, and $D'(6) = -2.515\sin[0.503(-0.75)] \approx 0.93$ m/h (rising).

(c) At 9:00 AM, $t = 9$, and $D'(9) = -2.515\sin[0.503(2.25)] \approx -2.28$ m/h (falling).

(d) At noon, $t = 12$, and $D'(12) = -2.515\sin[0.503(5.25)] \approx -1.21$ m/h (falling).

25. (a) To find the rate of change of volume with respect to pressure, we first solve for V in terms of P.

$$PV = C \quad\Rightarrow\quad V = \frac{C}{P} \quad\Rightarrow\quad \frac{dV}{dP} = -\frac{C}{P^2}.$$

(b) From the formula for dV/dP in part (a), we see that as P increases, the absolute value of dV/dP decreases.

Thus, the volume is decreasing more rapidly at the beginning of the 10 minutes.

(c) $\beta = -\dfrac{1}{V}\dfrac{dV}{dP} = -\dfrac{1}{V}\left(-\dfrac{C}{P^2}\right)$ [from part (a)] $= \dfrac{C}{(PV)P} = \dfrac{C}{CP} = \dfrac{1}{P}$

26. (a) $[\mathrm{C}] = \dfrac{a^2kt}{akt+1} \quad\Rightarrow\quad$ rate of reaction $= \dfrac{d[\mathrm{C}]}{dt} = \dfrac{(akt+1)(a^2k) - (a^2kt)(ak)}{(akt+1)^2} = \dfrac{a^2k(akt+1-akt)}{(akt+1)^2} = \dfrac{a^2k}{(akt+1)^2}$

(b) If $x = [\mathrm{C}]$, then $a - x = a - \dfrac{a^2kt}{akt+1} = \dfrac{a^2kt + a - a^2kt}{akt+1} = \dfrac{a}{akt+1}$.

So $k(a-x)^2 = k\left(\dfrac{a}{akt+1}\right)^2 = \dfrac{a^2k}{(akt+1)^2} = \dfrac{d[\mathrm{C}]}{dt}$ [from part (a)] $= \dfrac{dx}{dt}$.

(c) As $t \to \infty$, $[\mathrm{C}] = \dfrac{a^2kt}{akt+1} = \dfrac{(a^2kt)/t}{(akt+1)/t} = \dfrac{a^2k}{ak + (1/t)} \to \dfrac{a^2k}{ak} = a$ moles/L.

(d) As $t \to \infty$, $\dfrac{d[\mathrm{C}]}{dt} = \dfrac{a^2k}{(akt+1)^2} \to 0$.

(e) As t increases, nearly all of the reactants A and B are converted into product C. In practical terms, the reaction virtually stops.

27. In Example 6, the population function was $n = 2^t n_0$. Since we are tripling instead of doubling and the initial population is 400, the population function is $n(t) = 400 \cdot 3^t$. The rate of growth is $n'(t) = 400 \cdot 3^t \cdot \ln 3$, so the rate of growth after 2.5 hours is $n'(2.5) = 400 \cdot 3^{2.5} \cdot \ln 3 \approx 6850$ bacteria/hour.

28. $n = f(t) = \dfrac{a}{1 + be^{-0.7t}} \quad \Rightarrow \quad n' = -\dfrac{a \cdot be^{-0.7t}(-0.7)}{(1 + be^{-0.7t})^2}$ [Reciprocal Rule]. When $t = 0$, $n = 20$ and $n' = 12$.

$f(0) = 20 \quad \Rightarrow \quad 20 = \dfrac{a}{1 + b} \quad \Rightarrow \quad a = 20(1 + b)$. $\quad f'(0) = 12 \quad \Rightarrow \quad 12 = \dfrac{0.7ab}{(1 + b)^2} \quad \Rightarrow \quad 12 = \dfrac{0.7(20)(1 + b)b}{(1 + b)^2} \quad \Rightarrow$

$\dfrac{12}{14} = \dfrac{b}{1 + b} \quad \Rightarrow \quad 6(1 + b) = 7b \quad \Rightarrow \quad 6 + 6b = 7b \quad \Rightarrow \quad b = 6$ and $a = 20(1 + 6) = 140$. For the long run, we let t increase without bound. $\displaystyle\lim_{t \to \infty} f(t) = \lim_{t \to \infty} \frac{140}{1 + 6e^{-0.7t}} = \frac{140}{1 + 6 \cdot 0} = 140$, indicating that in the long run, the yeast population stabilizes at 140 cells.

29. (a) **1920:** $m_1 = \dfrac{1860 - 1750}{1920 - 1910} = \dfrac{110}{10} = 11$, $m_2 = \dfrac{2070 - 1860}{1930 - 1920} = \dfrac{210}{10} = 21$,

$(m_1 + m_2)/2 = (11 + 21)/2 = 16$ million/year

1980: $m_1 = \dfrac{4450 - 3710}{1980 - 1970} = \dfrac{740}{10} = 74$, $m_2 = \dfrac{5280 - 4450}{1990 - 1980} = \dfrac{830}{10} = 83$,

$(m_1 + m_2)/2 = (74 + 83)/2 = 78.5$ million/year

(b) $P(t) = at^3 + bt^2 + ct + d$ (in millions of people), where $a \approx -0.000\,284\,900\,3$, $b \approx 0.522\,433\,122\,43$, $c \approx -6.395\,641\,396$, and $d \approx 1720.586\,081$.

(c) $P(t) = at^3 + bt^2 + ct + d \quad \Rightarrow \quad P'(t) = 3at^2 + 2bt + c$ (in millions of people per year)

(d) 1920 corresponds to $t = 20$ and $P'(20) \approx 14.16$ million/year. 1980 corresponds to $t = 80$ and $P'(80) \approx 71.72$ million/year. These estimates are smaller than the estimates in part (a).

(e) $f(t) = pq^t$ (where $p = 1.43653 \times 10^9$ and $q = 1.01395$) $\quad \Rightarrow \quad f'(t) = pq^t \ln q$ (in millions of people per year)

(f) $f'(20) \approx 26.25$ million/year [much larger than the estimates in part (a) and (d)].

$f'(80) \approx 60.28$ million/year [much smaller than the estimates in parts (a) and (d)].

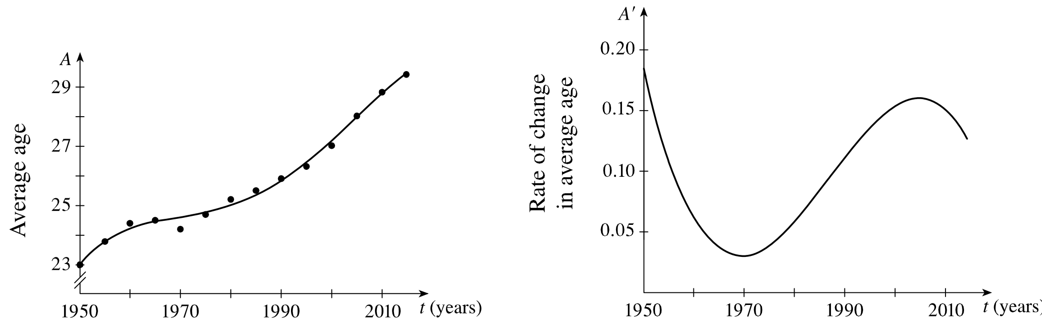
(g) $P'(85) \approx 76.24$ million/year and $f'(85) \approx 64.61$ million/year. The first estimate is probably more accurate.

30. (a) $A(t) = at^4 + bt^3 + ct^2 + dt + e$ years of age, where $a \approx -1.404\,771\,699 \times 10^{-6}$, $b \approx 0.011\,167\,331\,7$, $c \approx -33.288\,096\,21$, $d \approx 44{,}097.25101$, $e \approx -21{,}904{,}396.36$.

(b) $A(t) = at^4 + bt^3 + ct^2 + dt + e \quad \Rightarrow \quad A'(t) = 4at^3 + 3bt^2 + 2ct + d$ (in years of age per year).

(c) $A'(1990) \approx 0.11$, so the rate of change of first marriage age for Japanese women in 1990 was approximately 0.11 years of age per year.

(d) The model for A and the data points are shown on the left, and a graph for $A'(t)$ is shown on the right:



31. (a) Using $v = \dfrac{P}{4\eta l}(R^2 - r^2)$ with $R = 0.01$, $l = 3$, $P = 3000$, and $\eta = 0.027$, we have v as a function of r:

$v(r) = \dfrac{3000}{4(0.027)3}(0.01^2 - r^2)$. $v(0) = 0.\overline{925}$ cm/s, $v(0.005) = 0.69\overline{4}$ cm/s, $v(0.01) = 0$.

(b) $v(r) = \dfrac{P}{4\eta l}(R^2 - r^2) \;\;\Rightarrow\;\; v'(r) = \dfrac{P}{4\eta l}(-2r) = -\dfrac{Pr}{2\eta l}$. When $l = 3$, $P = 3000$, and $\eta = 0.027$, we have

$v'(r) = -\dfrac{3000r}{2(0.027)3}$. $v'(0) = 0$, $v'(0.005) = -92.\overline{592}$ (cm/s)/cm, and $v'(0.01) = -185.\overline{185}$ (cm/s)/cm.

(c) The velocity is greatest where $r = 0$ (at the center) and the velocity is changing most where $r = R = 0.01$ cm (at the edge).

32. (a) (i) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-1} \;\;\Rightarrow\;\; \dfrac{df}{dL} = -\left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-2} = -\dfrac{1}{2L^2}\sqrt{\dfrac{T}{\rho}}$

(ii) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{1/2} \;\;\Rightarrow\;\; \dfrac{df}{dT} = \dfrac{1}{2}\left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{-1/2} = \dfrac{1}{4L\sqrt{T\rho}}$

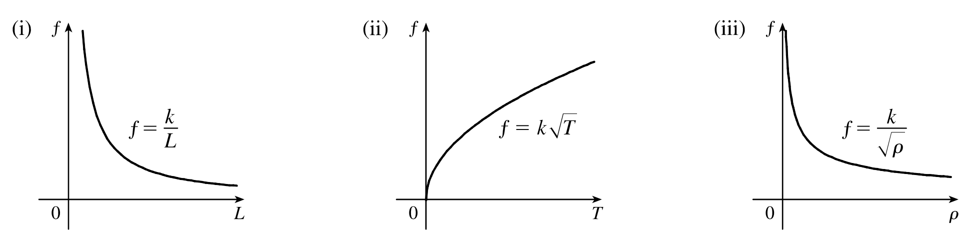
(iii) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-1/2} \;\;\Rightarrow\;\; \dfrac{df}{d\rho} = -\dfrac{1}{2}\left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-3/2} = -\dfrac{\sqrt{T}}{4L\rho^{3/2}}$

(b) *Note:* Illustrating tangent lines on the generic figures may help to explain the results.

(i) $\dfrac{df}{dL} < 0$ and L is decreasing $\Rightarrow$ f is increasing $\Rightarrow$ higher note

(ii) $\dfrac{df}{dT} > 0$ and T is increasing $\Rightarrow$ f is increasing $\Rightarrow$ higher note

(iii) $\dfrac{df}{d\rho} < 0$ and ρ is increasing $\Rightarrow$ f is decreasing $\Rightarrow$ lower note



33. (a) $C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3 \quad \Rightarrow \quad C'(x) = 0 + 3(1) + 0.01(2x) + 0.0002(3x^2) = 3 + 0.02x + 0.0006x^2$

(b) $C'(100) = 3 + 0.02(100) + 0.0006(100)^2 = 3 + 2 + 6 = \$11/\text{pair}$. $C'(100)$ is the rate at which the cost is increasing as the 100th pair of jeans is produced. It predicts the (approximate) cost of the 101st pair.

(c) The cost of manufacturing the 101st pair of jeans is

$C(101) - C(100) = 2611.0702 - 2600 = 11.0702 \approx \$11.07$. This is close to the marginal cost from part (b).

34. (a) $C(q) = 84 + 0.16q - 0.0006q^2 + 0.000003q^3 \quad \Rightarrow \quad C'(q) = 0.16 - 0.0012q + 0.000009q^2$, and

$C'(100) = 0.16 - 0.0012(100) + 0.000009(100)^2 = 0.13$. This is the rate at which the cost is increasing as the 100th item is produced.

(b) The actual cost of producing the 101st item is $C(101) - C(100) = 97.13030299 - 97 \approx \$0.13$

35. (a) $A(x) = \dfrac{p(x)}{x} \quad \Rightarrow \quad A'(x) = \dfrac{xp'(x) - p(x) \cdot 1}{x^2} = \dfrac{xp'(x) - p(x)}{x^2}$.

$A'(x) > 0 \quad \Rightarrow \quad A(x)$ is increasing; that is, the average productivity increases as the size of the workforce increases.

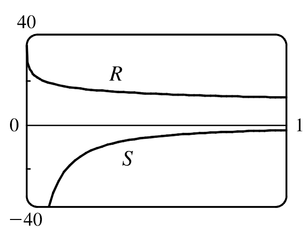
(b) $p'(x)$ is greater than the average productivity $\quad \Rightarrow \quad p'(x) > A(x) \quad \Rightarrow \quad p'(x) > \dfrac{p(x)}{x} \quad \Rightarrow \quad xp'(x) > p(x) \quad \Rightarrow$

$xp'(x) - p(x) > 0 \quad \Rightarrow \quad \dfrac{xp'(x) - p(x)}{x^2} > 0 \quad \Rightarrow \quad A'(x) > 0.$

36. (a) $R = \dfrac{40 + 24x^{0.4}}{1 + 4x^{0.4}} \quad \Rightarrow \quad S = \dfrac{dR}{dx} = \dfrac{(1 + 4x^{0.4})(9.6x^{-0.6}) - (40 + 24x^{0.4})(1.6x^{-0.6})}{(1 + 4x^{0.4})^2}$

$$= \frac{9.6x^{-0.6} + 38.4x^{-0.2} - 64x^{-0.6} - 38.4x^{-0.2}}{(1 + 4x^{0.4})^2} = -\frac{54.4x^{-0.6}}{(1 + 4x^{0.4})^2}$$

(b)



At low levels of brightness, R is quite large $[R(0) = 40]$ and is quickly decreasing, that is, S is negative with large absolute value. This is to be expected: at low levels of brightness, the eye is more sensitive to slight changes than it is at higher levels of brightness.

37. $t = \ln\left(\dfrac{3c + \sqrt{9c^2 - 8c}}{2}\right) = \ln\left(3c + \sqrt{9c^2 - 8c}\right) - \ln 2 \quad \Rightarrow$

$$\frac{dt}{dc} = \frac{1}{3c + \sqrt{9c^2 - 8c}} \frac{d}{dc}\left(3c + \sqrt{9c^2 - 8c}\right) - 0 = \frac{3 + \frac{1}{2}(9c^2 - 8c)^{-1/2}(18c - 8)}{3c + \sqrt{9c^2 - 8c}}$$

$$= \frac{3 + \dfrac{9c - 4}{\sqrt{9c^2 - 8c}}}{3c + \sqrt{9c^2 - 8c}} = \frac{3\sqrt{9c^2 - 8c} + 9c - 4}{\sqrt{9c^2 - 8c}\left(3c + \sqrt{9c^2 - 8c}\right)}.$$

This derivative represents the rate of change of duration of dialysis required with respect to the initial urea concentration.

38. $f(r) = 2\sqrt{Dr} \quad\Rightarrow\quad f'(r) = 2 \cdot \frac{1}{2}(Dr)^{-1/2} \cdot D = \dfrac{D}{\sqrt{Dr}} = \sqrt{\dfrac{D}{r}}$. $f'(r)$ is the rate of change of the wave speed with respect to the reproductive rate.

39. $PV = nRT \quad\Rightarrow\quad T = \dfrac{PV}{nR} = \dfrac{PV}{(10)(0.0821)} = \dfrac{1}{0.821}(PV)$. Using the Product Rule, we have

$$\frac{dT}{dt} = \frac{1}{0.821}\left[P(t)V'(t) + V(t)P'(t)\right] = \frac{1}{0.821}\left[(8)(-0.15) + (10)(0.10)\right] \approx -0.2436 \text{ K/min}.$$

40. (a) If $dP/dt = 0$, the population is stable (it is constant).

(b) $\dfrac{dP}{dt} = 0 \quad\Rightarrow\quad \beta P = r_0\left(1 - \dfrac{P}{P_c}\right)P \quad\Rightarrow\quad \dfrac{\beta}{r_0} = 1 - \dfrac{P}{P_c} \quad\Rightarrow\quad \dfrac{P}{P_c} = 1 - \dfrac{\beta}{r_0} \quad\Rightarrow\quad P = P_c\left(1 - \dfrac{\beta}{r_0}\right)$.

If $P_c = 10{,}000$, $r_0 = 5\% = 0.05$, and $\beta = 4\% = 0.04$, then $P = 10{,}000\left(1 - \frac{4}{5}\right) = 2000$.

(c) If $\beta = 0.05$, then $P = 10{,}000\left(1 - \frac{5}{5}\right) = 0$. There is no stable population.

41. (a) If the populations are stable, then the growth rates are neither positive nor negative; that is, $\dfrac{dC}{dt} = 0$ and $\dfrac{dW}{dt} = 0$.

(b) "The caribou go extinct" means that the population is zero, or mathematically, $C = 0$.

(c) We have the equations $\dfrac{dC}{dt} = aC - bCW$ and $\dfrac{dW}{dt} = -cW + dCW$. Let $dC/dt = dW/dt = 0$, $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$ to obtain $0.05C - 0.001CW = 0$ **(1)** and $-0.05W + 0.0001CW = 0$ **(2)**. Adding 10 times **(2)** to **(1)** eliminates the CW-terms and gives us $0.05C - 0.5W = 0 \quad\Rightarrow\quad C = 10W$. Substituting $C = 10W$ into **(1)** results in $0.05(10W) - 0.001(10W)W = 0 \quad\Leftrightarrow\quad 0.5W - 0.01W^2 = 0 \quad\Leftrightarrow\quad 50W - W^2 = 0 \quad\Leftrightarrow$ $W(50 - W) = 0 \quad\Leftrightarrow\quad W = 0$ or 50. Since $C = 10W$, $C = 0$ or 500. Thus, the population pairs (C, W) that lead to stable populations are $(0, 0)$ and $(500, 50)$. So it is possible for the two species to live in harmony.

42. (a) The rate of change of retention t days after a task is learned is given by $R'(t)$. $R(t) = a + b(1 + ct)^{-\beta} \quad\Rightarrow$ $R'(t) = b \cdot (-\beta)(1 + ct)^{-\beta-1} \cdot c = -\beta bc(1 + ct)^{-\beta-1}$ (expressed as a fraction of memory per day).

(b) We may write the rate of change as $R'(t) = -\dfrac{\beta bc}{(1 + ct)^{\beta+1}}$. The magnitude of this quantity decreases as t increases. Thus, you forget how to perform a task faster soon after learning it than a long time after you have learned it.

(c) $\displaystyle\lim_{t\to\infty} R(t) = \lim_{t\to\infty}\left[a + \frac{b}{(1 + ct)^{\beta}}\right] = a + 0 = a$, so the fraction of memory that is permanent is a.

43. (a) $I = \log_2\left(\dfrac{2D}{W}\right) \quad\Rightarrow\quad \dfrac{dI}{dD}\quad [W \text{ constant}] \quad = \dfrac{1}{\left(\dfrac{2D}{W}\right)\ln 2} \cdot \dfrac{2}{W} = \dfrac{1}{D\ln 2}$

As D increases, the rate of change of difficulty decreases.

(b) $I = \log_2\left(\dfrac{2D}{W}\right) \Rightarrow \dfrac{dI}{dW}$ [D constant] $= \dfrac{1}{\left(\dfrac{2D}{W}\right)\ln 2} \cdot \left(-2DW^{-2}\right) = \dfrac{W}{2D\ln 2} \cdot \dfrac{-2D}{W^2} = -\dfrac{1}{W\ln 2}$

The negative sign indicates that difficulty decreases with increasing width. While the magnitude of the rate of change decreases with increasing width $\left(\text{that is, } \left|-\dfrac{1}{W\ln 2}\right| = \dfrac{1}{W\ln 2} \text{ decreases as } W \text{ increases}\right)$, the rate of change itself increases (gets closer to zero from the negative side) with increasing values of W.

(c) The answers to (a) and (b) agree with intuition. For fixed width, the difficulty of acquiring a target increases, but less and less so, as the distance to the target increases. Similarly, for a fixed distance to a target, the difficulty of acquiring the target decreases, but less and less so, as the width of the target increases.

3.8 Exponential Growth and Decay

1. The relative growth rate is $\dfrac{1}{P}\dfrac{dP}{dt} = 0.4159$, so $\dfrac{dP}{dt} = 0.4159P$ and by Theorem 2, $P(t) = P(0)e^{0.4159t} = 3.8e^{0.4159t}$ million cells. Thus, $P(2) = 3.8e^{0.4159(2)} \approx 8.7$ million cells.

2. (a) By Theorem 2, $P(t) = P(0)e^{kt} = 50e^{kt}$. In 20 minutes $\left(\frac{1}{3}\text{ hour}\right)$, there are 100 cells, so $P\left(\frac{1}{3}\right) = 50e^{k/3} = 100 \Rightarrow$ $e^{k/3} = 2 \Rightarrow k/3 = \ln 2 \Rightarrow k = 3\ln 2 = \ln(2^3) = \ln 8$.

(b) $P(t) = 50e^{(\ln 8)t} = 50 \cdot 8^t$

(c) $P(6) = 50 \cdot 8^6 = 50 \cdot 2^{18} = 13{,}107{,}200$ cells

(d) $\dfrac{dP}{dt} = kP \Rightarrow P'(6) = kP(6) = (\ln 8)P(6) \approx 27{,}255{,}656$ cells/h

(e) $P(t) = 10^6 \Leftrightarrow 50 \cdot 8^t = 1{,}000{,}000 \Leftrightarrow 8^t = 20{,}000 \Leftrightarrow t\ln 8 = \ln 20{,}000 \Leftrightarrow t = \dfrac{\ln 20{,}000}{\ln 8} \approx 4.76$ h

3. (a) By Theorem 2, $P(t) = P(0)e^{kt} = 50e^{kt}$. Now $P(1.5) = 50e^{k(1.5)} = 975 \Rightarrow e^{1.5k} = \frac{975}{50} \Rightarrow$ $1.5k = \ln 19.5 \Rightarrow k = \frac{1}{1.5}\ln 19.5 \approx 1.9803$. So $P(t) \approx 50e^{1.9803t}$ cells.

(b) Using 1.9803 for k, we get $P(3) = 50e^{1.9803(3)} = 19{,}013.85 \approx 19{,}014$ cells.

(c) $\dfrac{dP}{dt} = kP \Rightarrow P'(3) = k \cdot P(3) = 1.9803 \cdot 19{,}014$ [from parts (a) and (b)] $= 37{,}653.4 \approx 37{,}653$ cells/h

(d) $P(t) = 50e^{1.9803t} = 250{,}000 \Rightarrow e^{1.9803t} = \dfrac{250{,}000}{50} \Rightarrow e^{1.9803t} = 5000 \Rightarrow 1.9803t = \ln 5000 \Rightarrow$ $t = \dfrac{\ln 5000}{1.9803} \approx 4.30$ h

4. (a) $y(t) = y(0)e^{kt} \Rightarrow y(2) = y(0)e^{2k} = 400$ and $y(6) = y(0)e^{6k} = 25{,}600$. Dividing these equations, we get $e^{6k}/e^{2k} = 25{,}600/400 \Rightarrow e^{4k} = 64 \Rightarrow 4k = \ln 2^6 = 6\ln 2 \Rightarrow k = \frac{3}{2}\ln 2 \approx 1.0397$, about 104% per hour.

(b) $400 = y(0)e^{2k} \Rightarrow y(0) = 400/e^{2k} \Rightarrow y(0) = 400/e^{3\ln 2} = 400/\left(e^{\ln 2}\right)^3 = 400/2^3 = 50$.

(c) $y(t) = y(0)e^{kt} = 50e^{(3/2)(\ln 2)t} = 50(e^{\ln 2})^{(3/2)t} \quad \Rightarrow \quad y(t) = 50(2)^{1.5t}$

(d) $y(4.5) = 50(2)^{1.5(4.5)} = 50(2)^{6.75} \approx 5382$ bacteria

(e) $\dfrac{dy}{dt} = ky = \left(\dfrac{3}{2}\ln 2\right)(50(2)^{6.75})$ [from parts (a) and (b)] $\approx 5596 \text{ bacteria/h}$

(f) $y(t) = 50{,}000 \quad \Rightarrow \quad 50{,}000 = 50(2)^{1.5t} \quad \Rightarrow \quad 1000 = (2)^{1.5t} \quad \Rightarrow \quad \ln 1000 = 1.5t \ln 2 \quad \Rightarrow$

$t = \dfrac{\ln 1000}{1.5 \ln 2} \approx 6.64$ h

5. (a) Let the population (in millions) in the year t be $P(t)$. Since the initial time is the year 1750, we substitute $t - 1750$ for t in Theorem 2, so the exponential model gives $P(t) = P(1750)e^{k(t-1750)}$. Then $P(1800) = 980 = 790e^{k(1800-1750)} \quad \Rightarrow$ $\frac{980}{790} = e^{k(50)} \quad \Rightarrow \quad \ln \frac{980}{790} = 50k \quad \Rightarrow \quad k = \frac{1}{50}\ln\frac{980}{790} \approx 0.0043104$. So with this model, we have $P(1900) = 790e^{k(1900-1750)} \approx 1508$ million, and $P(1950) = 790e^{k(1950-1750)} \approx 1871$ million. Both of these estimates are much too low.

(b) In this case, the exponential model gives $P(t) = P(1850)e^{k(t-1850)} \quad \Rightarrow \quad P(1900) = 1650 = 1260e^{k(1900-1850)} \quad \Rightarrow$ $\ln\frac{1650}{1260} = k(50) \quad \Rightarrow \quad k = \frac{1}{50}\ln\frac{1650}{1260} \approx 0.005393$. So with this model, we estimate $P(1950) = 1260e^{k(1950-1850)} \approx 2161$ million. This is still too low, but closer than the estimate of $P(1950)$ in part (a).

(c) The exponential model gives $P(t) = P(1900)e^{k(t-1900)} \quad \Rightarrow \quad P(1950) = 2560 = 1650e^{k(1950-1900)} \quad \Rightarrow$ $\ln\frac{2560}{1650} = k(50) \quad \Rightarrow \quad k = \frac{1}{50}\ln\frac{2560}{1650} \approx 0.008785$. With this model, we estimate $P(2000) = 1650e^{k(2000-1900)} \approx 3972$ million. This is much too low. The discrepancy is explained by the fact that the world birth rate (average yearly number of births per person) is about the same as always, whereas the mortality rate (especially the infant mortality rate) is much lower, owing mostly to advances in medical science and to the wars in the first part of the twentieth century. The exponential model assumes, among other things, that the birth and mortality rates will remain constant.

6. (a) Let $P(t)$ be the population (in millions) in the year t. Since the initial time is the year 1950, we substitute $t - 1950$ for t in Theorem 2, and find that the exponential model gives $P(t) = P(1950)e^{k(t-1950)} \quad \Rightarrow$ $P(1960) = 100 = 83e^{k(1960-1950)} \quad \Rightarrow \quad \frac{100}{83} = e^{10k} \quad \Rightarrow \quad k = \frac{1}{10}\ln\frac{100}{83} \approx 0.0186$. With this model, we estimate $P(1980) = 83e^{k(1980-1950)} = 83e^{30k} \approx 145$ million, which is an underestimate of the actual population of 150 million.

(b) As in part (a), $P(t) = P(1960)e^{k(t-1960)} \quad \Rightarrow \quad P(1980) = 150 = 100e^{20k} \quad \Rightarrow \quad 20k = \ln\frac{150}{100} \quad \Rightarrow$ $k = \frac{1}{20}\ln\frac{3}{2} \approx 0.0203$. Thus, $P(2000) = 100e^{40k} = 225$ million, which is an overestimate of the actual population of 214 million.

(c) As in part (a), $P(t) = P(1980)e^{k(t-1980)} \quad \Rightarrow \quad P(2000) = 214 = 150e^{20k} \quad \Rightarrow \quad 20k = \ln\frac{214}{150} \quad \Rightarrow$ $k = \frac{1}{20}\ln\frac{214}{150} \approx 0.0178$. Thus, $P(2010) = 150e^{30k} \approx 256$, which is an overestimate of the actual population of 243 million.

(d) Using the model in part (c), $P(2025) = 150e^{(2025-1980)k} = 150e^{45k} \approx 334$ million. This prediction is likely too high. The model gave an overestimate for 2010, and the amount of overestimation is likely to compound as time increases.

7. (a) If $y = [\mathrm{N_2O_5}]$ then by Theorem 2, $\dfrac{dy}{dt} = -0.0005y \;\Rightarrow\; y(t) = y(0)e^{-0.0005t} = Ce^{-0.0005t}$.

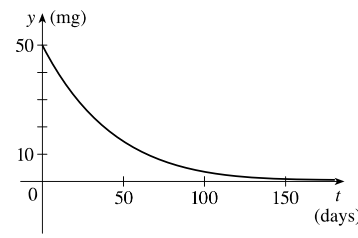
(b) $y(t) = Ce^{-0.0005t} = 0.9C \;\Rightarrow\; e^{-0.0005t} = 0.9 \;\Rightarrow\; -0.0005t = \ln 0.9 \;\Rightarrow\; t = -2000 \ln 0.9 \approx 211$ s

8. (a) The mass remaining after t days is $y(t) = y(0)\,e^{kt} = 50e^{kt}$. Since the half-life is 28 days, $y(28) = 50e^{28k} = 25 \;\Rightarrow$ $e^{28k} = \frac{1}{2} \;\Rightarrow\; 28k = \ln \frac{1}{2} \;\Rightarrow\; k = -(\ln 2)/28$, so $y(t) = 50e^{-(\ln 2)t/28} = 50 \cdot 2^{-t/28}$.

(b) $y(40) = 50 \cdot 2^{-40/28} \approx 18.6\,\mathrm{mg}$

(c) $y(t) = 2 \;\Rightarrow\; 2 = 50 \cdot 2^{-t/28} \;\Rightarrow\; \frac{2}{50} = 2^{-t/28} \;\Rightarrow$
$(-t/28)\ln 2 = \ln \frac{1}{25} \;\Rightarrow\; t = \left(-28 \ln \frac{1}{25}\right) / \ln 2 \approx 130$ days

(d)



9. (a) If $y(t)$ is the mass (in mg) remaining after t years, then $y(t) = y(0)e^{kt} = 100e^{kt}$.
$y(30) = 100e^{30k} = \frac{1}{2}(100) \;\Rightarrow\; e^{30k} = \frac{1}{2} \;\Rightarrow\; k = -(\ln 2)/30 \;\Rightarrow\; y(t) = 100e^{-(\ln 2)t/30} = 100 \cdot 2^{-t/30}$

(b) $y(100) = 100 \cdot 2^{-100/30} \approx 9.92$ mg

(c) $100e^{-(\ln 2)t/30} = 1 \;\Rightarrow\; -(\ln 2)t/30 = \ln \frac{1}{100} \;\Rightarrow\; t = -30\,\frac{\ln 0.01}{\ln 2} \approx 199.3$ years

10. (a) If $y(t)$ is the mass after t days and $y(0) = A$, then $y(t) = Ae^{kt}$.
$y(300) = Ae^{300k} = 0.643A \;\Rightarrow\; e^{300k} = 0.643 \;\Rightarrow\; k = \frac{1}{300}\ln 0.643$. To find the half-life, we set the mass after t days equal to one-half of the original mass. Hence, $Ae^{(1/300)(\ln 0.643)t} = \frac{1}{2}A \;\Leftrightarrow\; \frac{1}{300}(\ln 0.643)\,t = \ln \frac{1}{2} \;\Leftrightarrow$
$t = \dfrac{300 \ln \frac{1}{2}}{\ln 0.643} \approx 471$ days.

(b) $Ae^{(1/300)(\ln 0.643)t} = \frac{1}{3}A \;\Leftrightarrow\; \frac{1}{300}(\ln 0.643)\,t = \ln \frac{1}{3} \;\Leftrightarrow\; t = \dfrac{300 \ln \frac{1}{3}}{\ln 0.643} \approx 746$ days

11. Let $y(t)$ be the level of radioactivity. Thus, $y(t) = y(0)e^{-kt}$ and k is determined by using the half-life:

$$y(5730) = \tfrac{1}{2}y(0) \;\Rightarrow\; y(0)e^{-k(5730)} = \tfrac{1}{2}y(0) \;\Rightarrow\; e^{-5730k} = \tfrac{1}{2} \;\Rightarrow\; -5730k = \ln \tfrac{1}{2} \;\Rightarrow\; k = -\frac{\ln \frac{1}{2}}{5730} = \frac{\ln 2}{5730}.$$

If 74% of the $^{14}\mathrm{C}$ remains, then we know that $y(t) = 0.74y(0) \;\Rightarrow\; 0.74 = e^{-t(\ln 2)/5730} \;\Rightarrow\; \ln 0.74 = -\dfrac{t \ln 2}{5730} \;\Rightarrow$

$$t = -\frac{5730(\ln 0.74)}{\ln 2} \approx 2489 \approx 2500 \text{ years.}$$

12. From Exercise 11, we have the model $y(t) = y(0)e^{-kt}$ with $k = (\ln 2)/5730$. Thus, $y(68{,}000{,}000) = y(0)e^{-68{,}000{,}000k} \approx y(0) \cdot 0 = 0$. There would be an undetectable amount of $^{14}\mathrm{C}$ remaining for a 68-million-year-old dinosaur.

Now let $y(t) = 0.1\%\,y(0)$, so $0.001y(0) = y(0)e^{-kt} \;\Rightarrow\; 0.001 = e^{-kt} \;\Rightarrow\; \ln 0.001 = -kt \;\Rightarrow$

$$t = \frac{\ln 0.001}{-k} = \frac{\ln 0.001}{-(\ln 2)/5730} \approx 57{,}104,$$ which is the maximum age of a fossil that we could date using $^{14}\mathrm{C}$.

13. Let t measure time since a dinosaur died in millions of years, and let $y(t)$ be the amount of ^{40}K in the dinosaur's bones at time t. Then $y(t) = y(0)e^{-kt}$ and k is determined by the half-life: $y(1250) = \frac{1}{2}y(0) \quad \Rightarrow \quad y(0)e^{-k(1250)} = \frac{1}{2}y(0) \quad \Rightarrow$ $e^{-1250k} = \frac{1}{2} \quad \Rightarrow \quad -1250k = \ln\frac{1}{2} \quad \Rightarrow \quad k = -\dfrac{\ln\frac{1}{2}}{1250} = \dfrac{\ln 2}{1250}$. To determine if a dinosaur dating of 68 million years is possible, we find that $y(68) = y(0)e^{-k(68)} \approx 0.963y(0)$, indicating that about 96% of the ^{40}K is remaining, which is clearly detectable. To determine the maximum age of a fossil by using ^{40}K, we solve $y(t) = 0.1\%y(0)$ for t.

$y(0)e^{-kt} = 0.001y(0) \quad \Leftrightarrow \quad e^{-kt} = 0.001 \quad \Leftrightarrow \quad -kt = \ln 0.001 \quad \Leftrightarrow \quad t = \dfrac{\ln 0.001}{-(\ln 2)/1250} \approx$ 12,457 million, or 12.457 billion years.

14. From the information given, we know that $\dfrac{dy}{dx} = 2y \quad \Rightarrow \quad y = Ce^{2x}$ by Theorem 2. To calculate C we use the point $(0, 5)$: $5 = Ce^{2(0)} \quad \Rightarrow \quad C = 5$. Thus, the equation of the curve is $y = 5e^{2x}$.

15. (a) Using Newton's Law of Cooling, $\dfrac{dT}{dt} = k(T - T_s)$, we have $\dfrac{dT}{dt} = k(T - 22)$. Now let $y = T - 22$, so $y(0) = T(0) - 22 = 85 - 22 = 63$, so y is a solution of the initial-value problem $dy/dt = ky$ with $y(0) = 63$ and by Theorem 2 we have $y(t) = y(0)e^{kt} = 63\,e^{kt}$.

$y(30) = 63e^{30k} = 65 - 22 \implies e^{30k} = \frac{43}{63} \implies k = \frac{1}{30}\ln\frac{43}{63}$, so $y(t) = 63e^{\frac{1}{30}t\ln\frac{43}{63}}$ and $y(45) = 63e^{\frac{45}{30}\ln\frac{43}{63}} \approx$ 35.5°C. Thus, $T(45) \approx 35.5 + 22 = 57.5$°C.

(b) $T(t) = 40 \implies y(t) = 18$. $y(t) = 63e^{\frac{1}{30}t\ln\frac{43}{63}} = 18 \implies e^{\frac{1}{30}t\ln\frac{43}{63}} = \frac{18}{63} \implies \frac{1}{30}t\ln\frac{43}{63} = \ln\frac{2}{7} \implies t = \frac{30\ln\frac{2}{7}}{\ln\frac{43}{63}} \approx 98$ min.

16. Let $T(t)$ be the temperature of the body t hours after 1:30 PM. Then $T(0) = 32.5$ and $T(1) = 30.3$. Using Newton's Law of Cooling, $\dfrac{dT}{dt} = k(T - T_s)$, we have $\dfrac{dT}{dt} = k(T - 20)$. Now let $y = T - 20$, so $y(0) = T(0) - 20 = 32.5 - 20 = 12.5$, so y is a solution to the initial value problem $dy/dt = ky$ with $y(0) = 12.5$ and by Theorem 2, we have $y(t) = y(0)e^{kt} = 12.5e^{kt}$.

$y(1) = 30.3 - 20 \quad \Rightarrow \quad 10.3 = 12.5e^{k(1)} \quad \Rightarrow \quad e^k = \frac{10.3}{12.5} \quad \Rightarrow \quad k = \ln\frac{10.3}{12.5}$. The murder occurred when

$y(t) = 37 - 20 \quad \Rightarrow \quad 12.5e^{kt} = 17 \quad \Rightarrow \quad e^{kt} = \frac{17}{12.5} \quad \Rightarrow \quad kt = \ln\frac{17}{12.5} \quad \Rightarrow \quad t = \left(\ln\frac{17}{12.5}\right)/\ln\frac{10.3}{12.5} \approx -1.588$ h ≈ -95 minutes. Thus, the murder took place about 95 minutes before 1:30 PM, or 11:55 AM.

17. $\dfrac{dT}{dt} = k(T - 20)$. Letting $y = T - 20$, we get $\dfrac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$. $y(0) = T(0) - 20 = 5 - 20 = -15$, so $y(25) = y(0)e^{25k} = -15e^{25k}$, and $y(25) = T(25) - 20 = 10 - 20 = -10$, so $-15e^{25k} = -10 \quad \Rightarrow \quad e^{25k} = \frac{2}{3}$. Thus,

$25k = \ln\left(\frac{2}{3}\right)$ and $k = \frac{1}{25}\ln\left(\frac{2}{3}\right)$, so $y(t) = y(0)e^{kt} = -15e^{(1/25)\ln(2/3)t}$. More simply, $e^{25k} = \frac{2}{3} \Rightarrow e^k = \left(\frac{2}{3}\right)^{1/25} \Rightarrow$ $e^{kt} = \left(\frac{2}{3}\right)^{t/25} \Rightarrow y(t) = -15 \cdot \left(\frac{2}{3}\right)^{t/25}$.

(a) $T(50) = 20 + y(50) = 20 - 15 \cdot \left(\frac{2}{3}\right)^{50/25} = 20 - 15 \cdot \left(\frac{2}{3}\right)^2 = 20 - \frac{20}{3} = 13.\bar{3}\,^\circ\text{C}$

(b) $15 = T(t) = 20 + y(t) = 20 - 15 \cdot \left(\frac{2}{3}\right)^{t/25} \Rightarrow 15 \cdot \left(\frac{2}{3}\right)^{t/25} = 5 \Rightarrow \left(\frac{2}{3}\right)^{t/25} = \frac{1}{3} \Rightarrow$

$(t/25)\ln\left(\frac{2}{3}\right) = \ln\left(\frac{1}{3}\right) \Rightarrow t = 25\ln\left(\frac{1}{3}\right)/\ln\left(\frac{2}{3}\right) \approx 67.74$ min.

18. $\dfrac{dT}{dt} = k(T - 20)$. Let $y = T - 20$. Then $\dfrac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$. $y(0) = T(0) - 20 = 95 - 20 = 75$, so $y(t) = 75e^{kt}$. When $T(t) = 70$, $\dfrac{dT}{dt} = -1^\circ\text{C/min}$. Equivalently, $\dfrac{dy}{dt} = -1$ when $y(t) = 50$. Thus, $-1 = \dfrac{dy}{dt} = ky(t) = 50k$ and $50 = y(t) = 75e^{kt}$. The first relation implies $k = -1/50$, so the second relation says $50 = 75e^{-t/50}$. Thus, $e^{-t/50} = \frac{2}{3} \Rightarrow -t/50 = \ln\left(\frac{2}{3}\right) \Rightarrow t = -50\ln\left(\frac{2}{3}\right) \approx 20.27$ min.

19. (a) Let $P(h)$ be the pressure at altitude h. Then $dP/dh = kP \Rightarrow P(h) = P(0)e^{kh} = 101.3e^{kh}$.

$P(1000) = 101.3e^{1000k} = 87.14 \Rightarrow 1000k = \ln\left(\frac{87.14}{101.3}\right) \Rightarrow k = \frac{1}{1000}\ln\left(\frac{87.14}{101.3}\right) \Rightarrow$

$P(h) = 101.3\, e^{\frac{1}{1000}h\ln\left(\frac{87.14}{101.3}\right)}$, so $P(3000) = 101.3e^{3\ln\left(\frac{87.14}{101.3}\right)} \approx 64.5$ kPa.

(b) $P(6187) = 101.3\, e^{\frac{6187}{1000}\ln\left(\frac{87.14}{101.3}\right)} \approx 39.9$ kPa

20. (a) Using $A = A_0\left(1 + \dfrac{r}{n}\right)^{nt}$ with $A_0 = 2500$, $r = 0.045$, and $t = 3$, we have:

(i) Annually: $n = 1$ $\quad A = 2500\left(1 + \dfrac{0.045}{1}\right)^{1\cdot 3} = \$2852.92$

(ii) Quarterly: $n = 4$ $\quad A = 2500\left(1 + \dfrac{0.045}{4}\right)^{4\cdot 3} = \$2859.19$

(iii) Monthly: $n = 12$ $\quad A = 2500\left(1 + \dfrac{0.045}{12}\right)^{12\cdot 3} = \$2860.62$

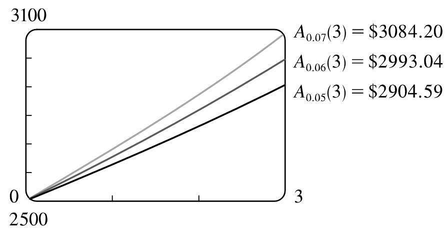
(iv) Weekly: $n = 52$ $\quad A = 2500\left(1 + \dfrac{0.045}{52}\right)^{52\cdot 3} = \$2861.17$

(v) Daily: $n = 365$ $\quad A = 2500\left(1 + \dfrac{0.045}{365}\right)^{365\cdot 3} = \$2861.32$

(vi) Hourly: $n = 365 \cdot 24$ $\quad A = 2500\left(1 + \dfrac{0.045}{365\cdot 24}\right)^{365\cdot 24\cdot 3} = \$2861.34$

(vii) Continuously: $\quad A = 2500e^{(0.045)3} = \$2861.34$

(b)



21. (a) Using $A = A_0\left(1 + \dfrac{r}{n}\right)^{nt}$ with $A_0 = 4000$, $r = 0.0175$, and $t = 5$, we have:

(i) Annually: $n = 1$ $\quad A = 4000\left(1 + \dfrac{0.0175}{1}\right)^{1 \cdot 5} = \$4362.47$

(ii) Semiannually: $n = 2$ $\quad A = 4000\left(1 + \dfrac{0.0175}{2}\right)^{2 \cdot 5} = \$4364.11$

(iii) Monthly: $n = 12$ $\quad A = 4000\left(1 + \dfrac{0.0175}{12}\right)^{12 \cdot 5} = \$4365.49$

(iv) Weekly: $n = 52$ $\quad A = 4000\left(1 + \dfrac{0.0175}{52}\right)^{52 \cdot 5} = \$4365.70$

(v) Daily: $n = 365$ $\quad A = 4000\left(1 + \dfrac{0.0175}{365}\right)^{365 \cdot 5} = \$4365.76$

(vi) Continuously: $\quad A = 4000e^{(0.0175)5} = \$4365.77$

(b) $dA/dt = 0.0175A$ and $A(0) = 4000$.

22. (a) $A_0e^{0.03t} = 2A_0 \quad \Leftrightarrow \quad e^{0.03t} = 2 \quad \Leftrightarrow \quad 0.03t = \ln 2 \quad \Leftrightarrow \quad t = \frac{100}{3}\ln 2 \approx 23.10$, so the investment will double in about 23.10 years.

(b) The annual interest rate in $A = A_0(1 + r)^t$ is r. From part (a), we have $A = A_0e^{0.03t}$. These amounts must be equal, so $(1 + r)^t = e^{0.03t} \quad \Rightarrow \quad 1 + r = e^{0.03} \quad \Rightarrow \quad r = e^{0.03} - 1 \approx 0.0305 = 3.05\%$, which is the equivalent annual interest rate.

APPLIED PROJECT Controlling Red Blood Cell Loss During Surgery

1. Let $R(t)$ be the volume of RBCs (in liters) at time t (in hours). Since the total volume of blood is 5 L, the concentration of RBCs is $R/5$. The patient bleeds 2 L of blood in 4 hours, so

$$\frac{dR}{dt} = -\frac{2L}{4h} \cdot \frac{R}{5} = -\frac{1}{10}R$$

From Section 3.8, we know that $dR/dt = kR$ has solution $R(t) = R(0)e^{kt}$. In this case, $R(0) = 45\%$ of $5 = \frac{9}{4}$ and $k = -\frac{1}{10}$, so $R(t) = \frac{9}{4}e^{-t/10}$. At the end of the operation, the volume of RBCs is $R(4) = \frac{9}{4}e^{-0.4} \approx 1.51$ L.

2. Let V be the volume of blood that is extracted and replaced with saline solution. Let $R_A(t)$ be the volume of RBCs with the ANH procedure. Then $R_A(0)$ is 45% of $(5 - V)$, or $\frac{9}{20}(5 - V)$, and hence $R_A(t) = \frac{9}{20}(5 - V)e^{-t/10}$. We want $R_A(4) \geq 25\%$ of 5 $\Leftrightarrow$ $\frac{9}{20}(5 - V)e^{-0.4} \geq \frac{5}{4}$ $\Leftrightarrow$ $5 - V \geq \frac{25}{9}e^{0.4}$ $\Leftrightarrow$ $V \leq 5 - \frac{25}{9}e^{0.4} \approx 0.86$ L. To maximize the effect of the ANH procedure, the surgeon should remove 0.86 L of blood and replace it with saline solution.

3. The RBC loss *without* the ANH procedure is $R(0) - R(4) = \frac{9}{4} - \frac{9}{4}e^{-0.4} \approx 0.74$ L. The RBC loss *with* the ANH procedure is $R_A(0) - R_A(4) = \frac{9}{20}(5 - V) - \frac{9}{20}(5 - V)e^{-0.4} = \frac{9}{20}(5 - V)(1 - e^{-0.4})$. Now let $V = 5 - \frac{25}{9}e^{0.4}$ [from Problem 2] to get $R_A(0) - R_A(4) = \frac{9}{20}\left[5 - \left(5 - \frac{25}{9}e^{0.4}\right)\right](1 - e^{0.4}) = \frac{9}{20} \cdot \frac{25}{9}e^{0.4}(1 - e^{0.4}) = \frac{5}{4}(e^{0.4} - 1) \approx 0.61$ L. Thus, the ANH procedure reduces the RBC loss by about $0.74 - 0.61 = 0.13$ L (about 4.4 fluid ounces).

3.9 Related Rates

1. (a) $V = x^3 \;\Rightarrow\; \dfrac{dV}{dt} = \dfrac{dV}{dx}\dfrac{dx}{dt} = 3x^2\,\dfrac{dx}{dt}$

(b) With $\dfrac{dx}{dt} = 4$ cm/s and $x = 15$ cm, we have $\dfrac{dV}{dt} = 3(15)^2 \cdot 4 = 2700\ \text{cm}^3/\text{s}$.

2. (a) $A = \pi r^2 \;\Rightarrow\; \dfrac{dA}{dt} = \dfrac{dA}{dr}\dfrac{dr}{dt} = 2\pi r\,\dfrac{dr}{dt}$

(b) With $\dfrac{dr}{dt} = 2$ m/s and $r = 30$ m, we have $\dfrac{dA}{dt} = 2\pi \cdot 30 \cdot 2 = 120\pi\ \text{m}^2/\text{s}$.

3. Let s denote the side of a square. The square's area A is given by $A = s^2$. Differentiating with respect to t gives us $\dfrac{dA}{dt} = 2s\,\dfrac{ds}{dt}$. When $A = 16$, $s = 4$. Substituting 4 for s and 6 for $\dfrac{ds}{dt}$ gives us $\dfrac{dA}{dt} = 2(4)(6) = 48\ \text{cm}^2/\text{s}$.

4. $V = \frac{4}{3}\pi r^3 \;\Rightarrow\; \dfrac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2\,\dfrac{dr}{dt} \;\Rightarrow\; \dfrac{dV}{dt} = 4\pi\left(\frac{1}{2} \cdot 80\right)^2(4) = 25{,}600\pi\ \text{mm}^3/\text{s}$.

5. $S = 4\pi r^2 \;\Rightarrow\; \dfrac{dS}{dt} = 4\pi \cdot 2r\,\dfrac{dr}{dt} \;\Rightarrow\; \dfrac{dS}{dt} = 4\pi \cdot 2 \cdot 8 \cdot 2 = 128\pi\ \text{cm}^2/\text{min}$.

6. $A = \ell w \;\Rightarrow\; \dfrac{dA}{dt} = \ell \cdot \dfrac{dw}{dt} + w \cdot \dfrac{d\ell}{dt} = 20(3) + 10(8) = 140\ \text{cm}^2/\text{s}$.

7. $V = \pi r^2 h = \pi(5)^2 h = 25\pi h \;\Rightarrow\; \dfrac{dV}{dt} = 25\pi\,\dfrac{dh}{dt} \;\Rightarrow\; 3 = 25\pi\,\dfrac{dh}{dt} \;\Rightarrow\; \dfrac{dh}{dt} = \dfrac{3}{25\pi}$ m/min.

8. (a) $A = \frac{1}{2}ab\sin\theta \;\Rightarrow\; \dfrac{dA}{dt} = \frac{1}{2}ab\cos\theta\,\dfrac{d\theta}{dt} = \frac{1}{2}(2)(3)\left(\cos\frac{\pi}{3}\right)(0.2) = 3\left(\frac{1}{2}\right)(0.2) = 0.3\ \text{cm}^2/\text{min}$.

(b) $A = \frac{1}{2}ab\sin\theta \;\Rightarrow$

$$\frac{dA}{dt} = \frac{1}{2}a\left(b\cos\theta\,\frac{d\theta}{dt} + \sin\theta\,\frac{db}{dt}\right) = \tfrac{1}{2}(2)\left[3\left(\cos\tfrac{\pi}{3}\right)(0.2) + \left(\sin\tfrac{\pi}{3}\right)(1.5)\right]$$
$$= 3\left(\tfrac{1}{2}\right)(0.2) + \tfrac{1}{2}\sqrt{3}\left(\tfrac{3}{2}\right) = 0.3 + \tfrac{3}{4}\sqrt{3}\ \text{cm}^2/\text{min}\quad [\approx 1.6]$$

(c) $A = \frac{1}{2}ab\sin\theta \quad \Rightarrow$

$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{da}{dt}\,b\sin\theta + a\,\frac{db}{dt}\sin\theta + ab\cos\theta\,\frac{d\theta}{dt}\right) \qquad \text{[by Exercise 3.2.63(a)]}$$

$$= \tfrac{1}{2}\left[(2.5)(3)\left(\tfrac{1}{2}\sqrt{3}\right) + (2)(1.5)\left(\tfrac{1}{2}\sqrt{3}\right) + (2)(3)\left(\tfrac{1}{2}\right)(0.2)\right]$$

$$= \left(\tfrac{15}{8}\sqrt{3} + \tfrac{3}{4}\sqrt{3} + 0.3\right) = \left(\tfrac{21}{8}\sqrt{3} + 0.3\right)\ \text{cm}^2/\text{min} \quad [\approx 4.85]$$

Note how this answer relates to the answer in part (a) [θ changing] and part (b) [b and θ changing].

9. (a) $\dfrac{d}{dt}\left(4x^2 + 9y^2\right) = \dfrac{d}{dt}(25) \quad \Rightarrow \quad 8x\,\dfrac{dx}{dt} + 18y\,\dfrac{dy}{dt} = 0 \quad \Rightarrow \quad 4x\,\dfrac{dx}{dt} + 9y\,\dfrac{dy}{dt} = 0 \quad \Rightarrow$

$4(2)\,\dfrac{dx}{dt} + 9(1)\cdot\frac{1}{3} = 0 \quad \Rightarrow \quad 8\,\dfrac{dx}{dt} + 3 = 0 \quad \Rightarrow \quad \dfrac{dx}{dt} = -\dfrac{3}{8}$

(b) $4x\,\dfrac{dx}{dt} + 9y\,\dfrac{dy}{dt} = 0 \quad \Rightarrow \quad 4(-2)(3) + 9(1)\cdot\dfrac{dy}{dt} = 0 \quad \Rightarrow \quad -24 + 9\,\dfrac{dy}{dt} = 0 \quad \Rightarrow \quad \dfrac{dy}{dt} = \dfrac{24}{9} = \dfrac{8}{3}$

10. $\dfrac{d}{dt}\left(x^2 + y^2 + z^2\right) = \dfrac{d}{dt}(9) \quad \Rightarrow \quad 2x\,\dfrac{dx}{dt} + 2y\,\dfrac{dy}{dt} + 2z\dfrac{dz}{dt} = 0 \quad \Rightarrow \quad x\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt} + z\,\dfrac{dz}{dt} = 0.$

If $\dfrac{dx}{dt} = 5$, $\dfrac{dy}{dt} = 4$ and $(x,y,z) = (2,2,1)$, then $2(5) + 2(4) + 1\,\dfrac{dz}{dt} = 0 \quad \Rightarrow \quad \dfrac{dz}{dt} = -18.$

11. $w = w_0\left(\frac{6370}{6370+h}\right)^2 = w_0\cdot 6370^2(6370+h)^{-2} \Rightarrow \frac{dw}{dt} = w_0\cdot 6370^2(-2)(6370+h)^{-3}\cdot\frac{dh}{dt}$. Then $w_0 = 580$ N, $h = 60$ km,

and $dh/dt = 19$ km/s $\Rightarrow \frac{dw}{dt} = 580\cdot 6370^2(-2)(6370+60)^{-3}(19) = -3.364012 \approx -3.364$ N/s.

12. $\dfrac{d}{dt}(xy) = \dfrac{d}{dt}(8) \quad \Rightarrow \quad x\,\dfrac{dy}{dt} + y\,\dfrac{dx}{dt} = 0.$ If $\dfrac{dy}{dt} = -3$ cm/s and $(x,y) = (4,2)$, then $4(-3) + 2\,\dfrac{dx}{dt} = 0 \quad \Rightarrow$

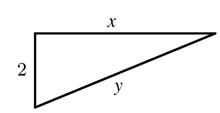
$\dfrac{dx}{dt} = 6$. Thus, the x-coordinate is increasing at a rate of 6 cm/s.

13. (a) Given: a plane flying horizontally at an altitude of 2 km and a speed of 800 km/h passes directly over a radar station.

If we let t be time (in hours) and x be the horizontal distance traveled by the plane (in km), then we are given that $dx/dt = 800$ km/h.

(b) Unknown: the rate at which the distance from the plane to the station is increasing when it is 3 km from the station. If we let y be the distance from the plane to the station, then we want to find dy/dt when $y = 3$ km.

(c)



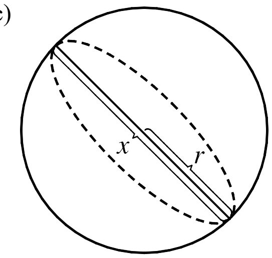
(d) By the Pythagorean Theorem, $y^2 = x^2 + 2^2 \quad \Rightarrow \quad 2y\,(dy/dt) = 2x\,(dx/dt)$.

(e) $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt} = \dfrac{x}{y}(800)$. Since $y^2 = x^2 + 4$, when $y = 3$, $x = \sqrt{5}$, so $\dfrac{dy}{dt} = \dfrac{\sqrt{5}}{3}(800) \approx 596$ km/h.

14. (a) Given: the rate of decrease of the surface area is 1 cm^2/min. If we let t be time (in minutes) and S be the surface area (in cm^2), then we are given that $dS/dt = -1\ \text{cm}^2/\text{s}$.

(b) Unknown: the rate of decrease of the diameter when the diameter is 10 cm. If we let x be the diameter, then we want to find dx/dt when $x = 10$ cm.

(c)



(d) If the radius is r and the diameter $x = 2r$, then $r = \frac{1}{2}x$ and

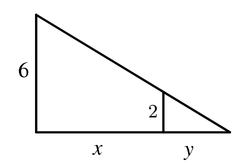
$$S = 4\pi r^2 = 4\pi\left(\tfrac{1}{2}x\right)^2 = \pi x^2 \quad\Rightarrow\quad \frac{dS}{dt} = \frac{dS}{dx}\frac{dx}{dt} = 2\pi x\,\frac{dx}{dt}.$$

(e) $-1 = \dfrac{dS}{dt} = 2\pi x\,\dfrac{dx}{dt} \quad\Rightarrow\quad \dfrac{dx}{dt} = -\dfrac{1}{2\pi x}$. When $x = 10$, $\dfrac{dx}{dt} = -\dfrac{1}{20\pi}$. So the rate of decrease is $\dfrac{1}{20\pi}$ cm/min.

15. (a) Given: a man 2 m tall walks away from a street light mounted on a 6 m tall pole at a rate of 1.5 ft/s. If we let t be time (in s) and x be the distance from the pole to the man (in m), then we are given that $dx/dt = 1.5$ m/s.

(b) Unknown: the rate at which the tip of his shadow is moving when he is 10 m from the pole. If we let y be the distance from the man to the tip of his shadow (in m), then we want to find $\dfrac{d}{dt}(x+y)$ when $x = 10$ m.

(c)



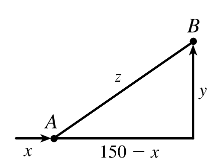
(d) By similar triangles, $\dfrac{6}{2} = \dfrac{x+y}{y} \quad\Rightarrow\quad 3y = x + y \quad\Rightarrow\quad 2y = x \quad\Rightarrow\quad y = \frac{x}{2}$.

(e) The tip of the shadow moves at a rate of $\frac{d}{dt}(x+y) = \frac{d}{dt}\left(x + \frac{x}{2}\right) = \frac{3}{2}\frac{dx}{dt} = \frac{3}{2}(1.5) = \frac{9}{4} = 2.25$ m/s

16. (a) Given: at noon, ship A is 150 km west of ship B; ship A is sailing east at 35 km/h, and ship B is sailing north at 25 km/h. If we let t be time (in hours), x be the distance traveled by ship A (in km), and y be the distance traveled by ship B (in km), then we are given that $dx/dt = 35$ km/h and $dy/dt = 25$ km/h.

(b) Unknown: the rate at which the distance between the ships is changing at 4:00 PM. If we let z be the distance between the ships, then we want to find dz/dt when $t = 4$ h.

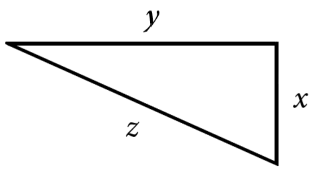
(c)



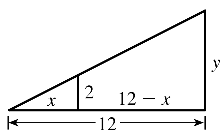
(d) $z^2 = (150 - x)^2 + y^2 \quad\Rightarrow\quad 2z\,\dfrac{dz}{dt} = 2(150 - x)\left(-\dfrac{dx}{dt}\right) + 2y\,\dfrac{dy}{dt}$

(e) At 4:00 PM, $x = 4(35) = 140$ and $y = 4(25) = 100 \quad\Rightarrow\quad z = \sqrt{(150 - 140)^2 + 100^2} = \sqrt{10{,}100}$.

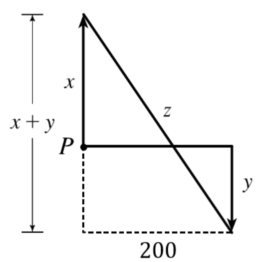
So $\dfrac{dz}{dt} = \dfrac{1}{z}\left[(x - 150)\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt}\right] = \dfrac{-10(35) + 100(25)}{\sqrt{10{,}100}} = \dfrac{215}{\sqrt{101}} \approx 21.4$ km/h.

17. 

We are given that $\frac{dx}{dt} = 30$ km/h and $\frac{dy}{dt} = 72$ km/h. $z^2 = x^2 + y^2 \Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} \Rightarrow \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$. After 2 hours, $x = 2(30) = 60$, and $y = 2(72) = 144 \Rightarrow z = \sqrt{60^2 + 144^2} = 156$, so $\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{60(30)+144(72)}{156} = 78$ km/h.

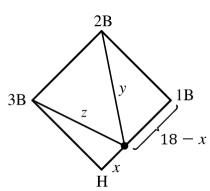
18. 

We are given that $\dfrac{dx}{dt} = 1.6$ m/s. By similar triangles, $\dfrac{y}{12} = \dfrac{2}{x} \Rightarrow y = \dfrac{24}{x} \Rightarrow \dfrac{dy}{dt} = -\dfrac{24}{x^2}\dfrac{dx}{dt} = -\dfrac{24}{x^2}(1.6)$. When $x = 8$, $\dfrac{dy}{dt} = -\dfrac{24(1.6)}{64} = -0.6$ m/s, so the shadow is decreasing at a rate of 0.6 m/s.

19. 

We are given that $\frac{dx}{dt} = 1.2$ m/s and $\frac{dy}{dt} = 1.6$ m/s. $z^2 = (x+y)^2 + 200^2 \Rightarrow 2z\frac{dz}{dt} = 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$. 15 minutes after the woman starts, we have $x = (1.2\text{ m/s})(20\text{ min})(60\text{ s/min}) = 1440$ m and $y = 1.6 \cdot 15 \cdot 60 = 1440 \Rightarrow z = \sqrt{(2 \cdot 1440)^2 + 200^2} = \sqrt{8,334,400}$, so $\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{1440+1440}{\sqrt{8,334,400}}(1.2 + 1.6) = \frac{8064}{10\sqrt{83,344}} \approx 2.79$ m/s.

20. We are given that $\frac{dx}{dt} = 7.5$ m/s.

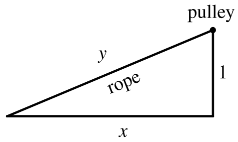
(a) 

$y^2 = (18-x)^2 + 18^2 \Rightarrow 2y\frac{dy}{dt} = 2(18-x)\left(-\frac{dx}{dt}\right)$. When $x = 9$, $y = \sqrt{9^2 + 18^2} = 9\sqrt{5}$, so $\frac{dy}{dt} = \frac{18-x}{y}\left(-\frac{dx}{dt}\right) = \frac{9}{9\sqrt{5}}(-7.5) = \frac{-7.5}{\sqrt{5}}$, so the distance from second base is decreasing at a rate of $\frac{7.5}{\sqrt{5}} \approx 3.35$ m/s.

(b) Due to the symmetric nature of the problem in part (a), we expect to get the same answer – and we do.

$z^2 = x^2 + 18 \Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt}$. When $x = 9$, $z = 9\sqrt{5}$, so $\frac{dz}{dt} = \frac{9}{9\sqrt{5}}(7.5) = \frac{7.5}{\sqrt{5}} \approx 3.35$ m/s.

21. $A = \frac{1}{2}bh$, where b is the base and h is the altitude. We are given that $\dfrac{dh}{dt} = 1$ cm/min and $\dfrac{dA}{dt} = 2$ cm^2/min. Using the Product Rule, we have $\dfrac{dA}{dt} = \dfrac{1}{2}\left(b\,\dfrac{dh}{dt} + h\,\dfrac{db}{dt}\right)$. When $h = 10$ and $A = 100$, we have $100 = \frac{1}{2}b(10) \Rightarrow \frac{1}{2}b = 10 \Rightarrow b = 20$, so $2 = \dfrac{1}{2}\left(20 \cdot 1 + 10\,\dfrac{db}{dt}\right) \Rightarrow 4 = 20 + 10\,\dfrac{db}{dt} \Rightarrow \dfrac{db}{dt} = \dfrac{4-20}{10} = -1.6$ cm/min.

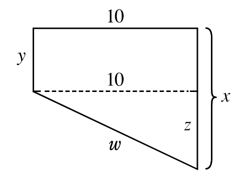
22. 

Given $\dfrac{dy}{dt} = -1$ m/s, find $\dfrac{dx}{dt}$ when $x = 8$ m. $y^2 = x^2 + 1 \quad \Rightarrow \quad 2y\,\dfrac{dy}{dt} = 2x\,\dfrac{dx}{dt} \quad \Rightarrow$ $\dfrac{dx}{dt} = \dfrac{y}{x}\dfrac{dy}{dt} = -\dfrac{y}{x}$. When $x = 8$, $y = \sqrt{65}$, so $\dfrac{dx}{dt} = -\dfrac{\sqrt{65}}{8}$. Thus, the boat approaches the dock at $\dfrac{\sqrt{65}}{8} \approx 1.01$ m/s.

23. Let x be the distance (in meters) the first dropped stone has traveled, and let y be the distance (in meters) the stone dropped one second later has traveled. Let t be the time (in seconds) since the woman drops the second stone. Using $d = 4.9t^2$, we have $x = 4.9(t+1)^2$ and $y = 4.9t^2$. Let z be the distance between the stones. Then $z = x - y$ and we have

$$\frac{dz}{dt} = \frac{dx}{dt} - \frac{dy}{dt} \quad \Rightarrow \quad \frac{dz}{dt} = 9.8(t+1) - 9.8t = 9.8 \text{ m/s}.$$

24. *Given:* Two men 10 m apart each drop a stone, the second one, one minute after the first. Let x be the distance (in meters) the first dropped stone has traveled, and let y be the distance (in meters) the second stone has traveled. Let t be the time (in seconds) since the man drops the second stone. Using $d = 4.9t^2$, we have $x = 4.9(t+1)^2$ and $y = 4.9t^2$. Let z be the vertical distance between the stones. Then $z = x - y \quad \Rightarrow$



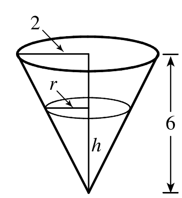
$$\frac{dz}{dt} = \frac{dx}{dt} - \frac{dy}{dt} \quad \Rightarrow \quad \frac{dz}{dt} = 9.8(t+1) - 9.8t = 9.8 \text{ m/s}.$$

By the Pythagorean Theorem, $w^2 = 10^2 + z^2$. Differentiating with respect to t, we obtain

$2w\,\dfrac{dw}{dt} = 2z\,\dfrac{dz}{dt} \quad \Rightarrow \quad \dfrac{dw}{dt} = \dfrac{z\,(dz/dt)}{w}$. One second after the second stone is dropped, $t = 1$, so

$z = x - y = 4.9(1+1)^2 - 4.9(1)^2 = 14.7$ m, and $w = \sqrt{10^2 + (14.7)^2} = \sqrt{316.09}$, so $\dfrac{dw}{dt} = \dfrac{14.7\,(9.8)}{\sqrt{316.09}} \approx 8.10$ m/s.

25. If $C =$ the rate at which water is pumped in, then $\dfrac{dV}{dt} = C - 10{,}000$, where $V = \frac{1}{3}\pi r^2 h$ is the volume at time t. By similar triangles, $\dfrac{r}{2} = \dfrac{h}{6} \quad \Rightarrow \quad r = \dfrac{1}{3}h \quad \Rightarrow$ $V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{\pi}{27}h^3 \quad \Rightarrow \quad \dfrac{dV}{dt} = \dfrac{\pi}{9}h^2\,\dfrac{dh}{dt}$. When $h = 200$ cm,



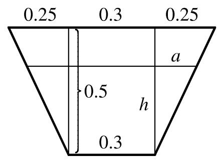
$\dfrac{dh}{dt} = 20$ cm/min, so $C - 10{,}000 = \dfrac{\pi}{9}(200)^2(20) \quad \Rightarrow \quad C = 10{,}000 + \dfrac{800{,}000}{9}\pi \approx 289{,}253$ cm^3/min.

26. The distance z of the particle to the origin is given by $z = \sqrt{x^2 + y^2}$, so $z^2 = x^2 + [2\sin(\pi x/2)]^2 \quad \Rightarrow$

$2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt} + 4 \cdot 2\sin\left(\dfrac{\pi}{2}x\right)\cos\left(\dfrac{\pi}{2}x\right) \cdot \dfrac{\pi}{2}\dfrac{dx}{dt} \quad \Rightarrow \quad z\,\dfrac{dz}{dt} = x\,\dfrac{dx}{dt} + 2\pi\sin\left(\dfrac{\pi}{2}x\right)\cos\left(\dfrac{\pi}{2}x\right)\dfrac{dx}{dt}$. When

$(x, y) = \left(\dfrac{1}{3}, 1\right)$, $z = \sqrt{\left(\dfrac{1}{3}\right)^2 + 1^2} = \sqrt{\dfrac{10}{9}} = \dfrac{1}{3}\sqrt{10}$, so $\dfrac{1}{3}\sqrt{10}\,\dfrac{dz}{dt} = \dfrac{1}{3}\sqrt{10} + 2\pi\,\sin\,\dfrac{\pi}{6}\,\cos\,\dfrac{\pi}{6} \cdot \sqrt{10} \quad \Rightarrow$

$$\frac{1}{3}\frac{dz}{dt} = \frac{1}{3} + 2\pi\left(\frac{1}{2}\right)\left(\frac{1}{2}\sqrt{3}\right) \quad \Rightarrow \quad \frac{dz}{dt} = 1 + \frac{3\sqrt{3}\,\pi}{2} \text{ cm/s}.$$

27. 

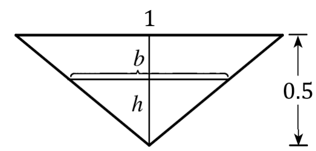
The figure is labeled in meters. The area A of a trapezoid is $\frac{1}{2}(\text{base}_1 + \text{base}_2)(\text{height})$, and the volume V of the 10-meter-long trough is $10A$. Thus, the volume of the trapezoid with height h is $V = (10)\frac{1}{2}[0.3 + (0.3 + 2a)]h$.

By similar triangles, $\dfrac{a}{h} = \dfrac{0.25}{0.5} = \dfrac{1}{2}$, so $2a = h \quad\Rightarrow\quad V = 5(0.6 + h)h = 3h + 5h^2$.

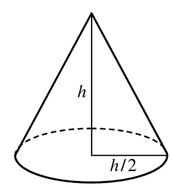
Now $\dfrac{dV}{dt} = \dfrac{dV}{dh}\dfrac{dh}{dt} \quad\Rightarrow\quad 0.2 = (3 + 10h)\dfrac{dh}{dt} \quad\Rightarrow\quad \dfrac{dh}{dt} = \dfrac{0.2}{3 + 10h}$. When $h = 0.3$,

$\dfrac{dh}{dt} = \dfrac{0.2}{3 + 10(0.3)} = \dfrac{0.2}{6}\text{ m/min} = \dfrac{1}{30}\text{ m/min or }\dfrac{10}{3}\text{ cm/min.}$

28. By similar triangles, $\frac{1}{0.5} = \frac{b}{h}$, so $b = 2h$. The trough has volume $V = \frac{1}{2}bh(6) = 3(2h)h = 6h^2 \Rightarrow 1.2 = \frac{dV}{dt} = 12h\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{10h}$. When $h = 0.3$, $\frac{dh}{dt} = \frac{1}{10 \cdot 0.3} = \frac{1}{3}$ m/min.



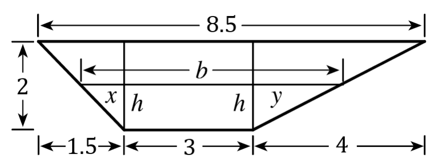
29. We are given that $\frac{dV}{dt} = 3\text{ m}^3/\text{min}$. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12} \Rightarrow \frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} \Rightarrow$ $3 = \frac{\pi h^2}{4}\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{12}{\pi h^2}$. When $h = 3$ m, $\frac{dh}{dt} = \frac{12}{3^2\pi} = \frac{4}{3\pi} \approx 0.4246$ m/min.



30. The figure is drawn without the top 1 meter.

$V = \frac{1}{2}(b+3)h(5) = \frac{5}{2}(b+3)h$ and, from similar triangles, $\frac{x}{h} = \frac{1.5}{2} = \frac{3}{4}$ and $\frac{y}{h} = \frac{4}{2} = 2$, so $b = x + 3 + y = \frac{3h}{4} + 3 + 2h = 3 + \frac{11h}{4}$. Thus, $V = \frac{5}{2}\left(6 + \frac{11h}{4}\right)h = 15h + \frac{55}{8}h^2$ and so $0.1 = \frac{dV}{dt} = \left(15 + \frac{55}{4}h\right)\frac{dh}{dt}$.

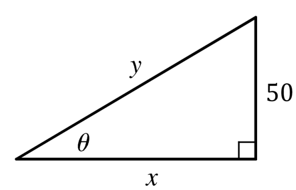
When $h = 1$, $\frac{dh}{dt} = \frac{0.1}{\left(15 + \frac{55}{4}\right)} = \frac{2}{575} \approx 0.00348$ m/min.

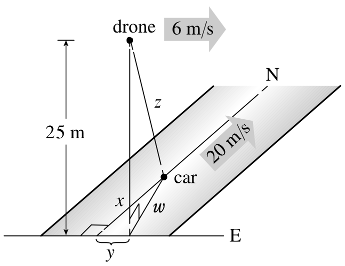


31. The area A of an equilateral triangle with side s is given by $A = \frac{1}{4}\sqrt{3}\,s^2$.

$\dfrac{dA}{dt} = \frac{1}{4}\sqrt{3} \cdot 2s\,\dfrac{ds}{dt} = \frac{1}{4}\sqrt{3} \cdot 2(30)(10) = 150\sqrt{3}\text{ cm}^2/\text{min.}$

32. We are given $dx/dt = 2$ m/s. $\cot\theta = \frac{x}{50} \Rightarrow x = 50\cot\theta \Rightarrow \frac{dx}{dt} = -50\csc^2\theta\,\frac{d\theta}{dt} \Rightarrow$ $\frac{d\theta}{dt} = -\frac{\sin^2\theta}{50} \cdot 2$. When $y = 100$, $\sin\theta = \frac{50}{100} = \frac{1}{2} \Rightarrow \frac{d\theta}{dt} = -\frac{(1/2)^2}{50} \cdot 2 = -\frac{1}{100}$ rad/s. The angle is decreasing at a rate of $\frac{1}{100}$ rad/s.



33. 

Let t be the time, in seconds, after the drone passes directly over the car. Given $\frac{dx}{dt} = 20$ m/s, $x = 20t$ m, $\frac{dy}{dt} = 6$ m/s, and $y = 6t$ m, find $\frac{dz}{dt}$ when $t = 5$.

By the Pythagorean Theorem, $w^2 = x^2 + y^2$ and $z^2 = 25^2 + w^2$. This gives $z^2 = 25^2 + x^2 + y^2 \quad \Rightarrow$

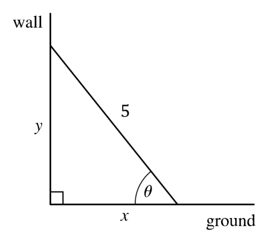
$$2z\,\frac{dz}{dt} = 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} \quad \Rightarrow \quad \frac{dz}{dt} = \frac{x\,(dx/dt) + y\,(dy/dt)}{z}$$

When $t = 5$, $x = 20(5) = 100$ and $y = 6(5) = 30$, so $z^2 = 25^2 + 100^2 + 30^2 \quad \Rightarrow \quad z = \sqrt{11{,}525}$ m.

$$\frac{dz}{dt} = \frac{100(20) + 30(6)}{\sqrt{11{,}525}} \approx 20.3 \text{ m/s}.$$

34. The area A of a sector of a circle with radius r and angle θ is given by $A = \frac{1}{2}r^2\theta$. Here r is constant and θ varies, so $\frac{dA}{dt} = \frac{1}{2}r^2\,\frac{d\theta}{dt}$. The minute hand rotates through $360^\circ = 2\pi$ radians each hour, so $\frac{d\theta}{dt} = 2\pi$ and $\frac{dA}{dt} = \frac{1}{2}r^2(2\pi) = \pi r^2$ cm^2/h. This answer makes sense because the minute hand sweeps through the full area of a circle, πr^2, each hour.

35. $\cos\theta = \frac{x}{5} \Rightarrow -\sin\theta\,\frac{d\theta}{dt} = \frac{1}{5}\,\frac{dx}{dt}$. From Example 2, $\frac{dx}{dt} = 1$ and when $x = 3$, $y = 4$, so $\sin\theta = \frac{4}{5}$. Thus, $-\frac{4}{5}\,\frac{d\theta}{dt} = \frac{1}{5}(1) \Rightarrow \frac{d\theta}{dt} = -\frac{1}{4}$ rad/s.



36. According to the model in Example 2, $\frac{dy}{dt} = -\frac{x}{y}\,\frac{dx}{dt} \to -\infty$ as $y \to 0$, which doesn't make physical sense. For example, the model predicts that for sufficiently small y, the tip of the ladder moves at a speed greater than the speed of light. Therefore the model is not appropriate for small values of y. What actually happens is that the tip of the ladder leaves the wall at some point in its descent. For a discussion of the true situation see the article "The Falling Ladder Paradox" by Paul Scholten and Andrew Simoson in *The College Mathematics Journal,* 27, (1), January 1996, pages 49–54. Also see "On Mathematical and Physical Ladders" by M. Freeman and P. Palffy-Muhoray in the *American Journal of Physics,* 53 (3), March 1985, pages 276–277.

37. Differentiating both sides of $PV = C$ with respect to t and using the Product Rule gives us $P\,\frac{dV}{dt} + V\,\frac{dP}{dt} = 0 \quad \Rightarrow$ $\frac{dV}{dt} = -\frac{V}{P}\,\frac{dP}{dt}$. When $V = 600$, $P = 150$ and $\frac{dP}{dt} = 20$, so we have $\frac{dV}{dt} = -\frac{600}{150}(20) = -80$. Thus, the volume is decreasing at a rate of 80 cm^3/min.

38. The volume of a hemisphere is $\frac{2}{3}\pi r^3$, so the volume of a hemispherical basin of radius 30 cm is $\frac{2}{3}\pi(30)^3 = 18{,}000\pi$ cm^3. If the basin is half full, then $V = \pi\left(rh^2 - \frac{1}{3}h^3\right) \quad \Rightarrow \quad 9000\pi = \pi\left(30h^2 - \frac{1}{3}h^3\right) \quad \Rightarrow \quad \frac{1}{3}h^3 - 30h^2 + 9000 = 0 \quad \Rightarrow$

$h = H \approx 19.58$ [from a graph or numerical rootfinder; the other two solutions are less than 0 and greater than 30].

$V = \pi\left(30h^2 - \frac{1}{3}h^3\right) \Rightarrow \dfrac{dV}{dt} = \pi\left(60h\,\dfrac{dh}{dt} - h^2\,\dfrac{dh}{dt}\right) \Rightarrow \left(2\,\dfrac{\text{L}}{\text{min}}\right)\left(1000\,\dfrac{\text{cm}^3}{\text{L}}\right) = \pi(60h - h^2)\,\dfrac{dh}{dt} \Rightarrow$

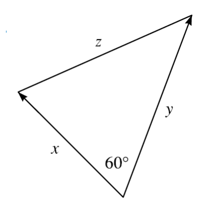
$\dfrac{dh}{dt} = \dfrac{2000}{\pi(60H - H^2)} \approx 0.804$ cm/min.

39. With $R_1 = 80$ and $R_2 = 100$, $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} = \dfrac{1}{80} + \dfrac{1}{100} = \dfrac{180}{8000} = \dfrac{9}{400}$, so $R = \dfrac{400}{9}$. Differentiating $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$ with respect to t, we have $-\dfrac{1}{R^2}\dfrac{dR}{dt} = -\dfrac{1}{R_1^2}\dfrac{dR_1}{dt} - \dfrac{1}{R_2^2}\dfrac{dR_2}{dt} \Rightarrow \dfrac{dR}{dt} = R^2\left(\dfrac{1}{R_1^2}\dfrac{dR_1}{dt} + \dfrac{1}{R_2^2}\dfrac{dR_2}{dt}\right)$. When $R_1 = 80$ and $R_2 = 100$, $\dfrac{dR}{dt} = \dfrac{400^2}{9^2}\left[\dfrac{1}{80^2}(0.3) + \dfrac{1}{100^2}(0.2)\right] = \dfrac{107}{810} \approx 0.132\ \Omega/\text{s}$.

40. $PV^{1.4} = C \Rightarrow P \cdot 1.4V^{0.4}\,\dfrac{dV}{dt} + V^{1.4}\,\dfrac{dP}{dt} = 0 \Rightarrow \dfrac{dV}{dt} = -\dfrac{V^{1.4}}{P \cdot 1.4V^{0.4}}\,\dfrac{dP}{dt} = -\dfrac{V}{1.4P}\,\dfrac{dP}{dt}$.

When $V = 400$, $P = 80$ and $\dfrac{dP}{dt} = -10$, so we have $\dfrac{dV}{dt} = -\dfrac{400}{1.4(80)}(-10) = \dfrac{250}{7}$. Thus, the volume is increasing at a rate of $\frac{250}{7} \approx 36\ \text{cm}^3/\text{min}$.

41.



We are given that $\frac{dx}{dt} = 60$ km/h and $\frac{dy}{dt} = 100$ km/h. By the Law of Cosines,

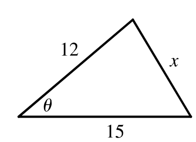
$z^2 = x^2 + y^2 - 2xy\cos 60^\circ = x^2 + y^2 - xy \Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - x\frac{dy}{dt} - y\frac{dx}{dt}$. At $t = \frac{1}{2}$ h, we have $x = 60(\frac{1}{2}) = 30$ and $y = 100(\frac{1}{2}) = 50 \Rightarrow z^2 = 30^2 + 50^2 - (30)(50) = 1900 \Rightarrow$

$z = \sqrt{1900}$ and $\frac{dz}{dt} = \frac{2(30)(60)+2(50)(100)-(30)(100)-(50)(60)}{2\sqrt{1900}} = \frac{7600}{2\sqrt{1900}} = 20\sqrt{19} \approx 87.2$ km/h.

42. We want to find $\dfrac{dB}{dt}$ when $L = 18$ using $B = 0.007W^{2/3}$ and $W = 0.12L^{2.53}$.

$$\frac{dB}{dt} = \frac{dB}{dW}\frac{dW}{dL}\frac{dL}{dt} = \left(0.007 \cdot \tfrac{2}{3}W^{-1/3}\right)\left(0.12 \cdot 2.53 \cdot L^{1.53}\right)\left(\frac{20-15}{10{,}000{,}000}\right)$$
$$= \left[0.007 \cdot \tfrac{2}{3}(0.12 \cdot 18^{2.53})^{-1/3}\right]\left(0.12 \cdot 2.53 \cdot 18^{1.53}\right)\left(\frac{5}{10^7}\right) \approx 1.045 \times 10^{-8}\ \text{g/yr}$$

43. We are given $d\theta/dt = 2^\circ/\text{min} = \frac{\pi}{90}$ rad/min. By the Law of Cosines,



$x^2 = 12^2 + 15^2 - 2(12)(15)\cos\theta = 369 - 360\cos\theta \Rightarrow$

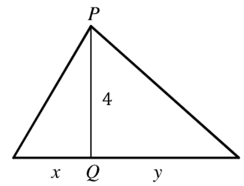
$2x\,\dfrac{dx}{dt} = 360\sin\theta\,\dfrac{d\theta}{dt} \Rightarrow \dfrac{dx}{dt} = \dfrac{180\sin\theta}{x}\,\dfrac{d\theta}{dt}$. When $\theta = 60^\circ$,

$x = \sqrt{369 - 360\cos 60^\circ} = \sqrt{189} = 3\sqrt{21}$, so $\dfrac{dx}{dt} = \dfrac{180\sin 60^\circ}{3\sqrt{21}}\,\dfrac{\pi}{90} = \dfrac{\pi\sqrt{3}}{3\sqrt{21}} = \dfrac{\sqrt{7}\,\pi}{21} \approx 0.396$ m/min.

44. Using Q for the origin, we are given $\frac{dx}{dt} = -0.5$ m/s and need to find $\frac{dy}{dt}$ when $x = -5$.

Using the Pythagorean Theorem twice, we have $\sqrt{x^2+4^2} + \sqrt{y^2+4^2} = 12$, the total length of the rope. Differentiating with respect to t, we get

$\frac{x}{\sqrt{x^2+4^2}}\frac{dx}{dt} + \frac{y}{\sqrt{y^2+4^2}}\frac{dy}{dt} = 0$, so $\frac{dy}{dt} = -\frac{x\sqrt{y^2+4^2}}{y\sqrt{x^2+4^2}}\frac{dx}{dt}$.



Now, when $x = -3$, $12 = \sqrt{(-3)^2+4^2} + \sqrt{y^2+4^2} = 5 + \sqrt{y^2+4^2} \Leftrightarrow \sqrt{y^2+4^2} = 7$, and $y = \sqrt{7^2-4^2} = \sqrt{33}$.

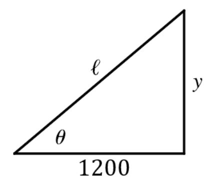
So when $x = -3$, $\frac{dy}{dt} = -\frac{(-3)(7)}{\sqrt{33}(5)}(-0.5) \approx -0.37$ m/s.

So cart B is moving towards Q at about 0.37 m/s.

45. (a) By the Pythagorean Theorem, $1200^2 + y^2 = l^2$. Differentiating with respect to t, we obtain $2y\frac{dy}{dt} = 2l\frac{dl}{dt}$. We know that $\frac{dy}{dt} = 200$ m/s, so when $y = 900$ m,

$l = \sqrt{1200^2+900^2} = \sqrt{2,250,000} = 1500$ m

and $\frac{dl}{dt} = \frac{y}{l}\frac{dy}{dt} = \frac{900}{1500}(200) = \frac{600}{5} = 120$ m/s.

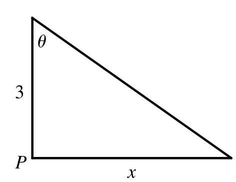


(b) Here $\tan\theta = \frac{y}{1200} \Rightarrow \frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{y}{1200}\right) \Rightarrow \sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{1200}\frac{dy}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{\cos^2\theta}{1200}\frac{dy}{dt}$. When $y = 900$ m, $\frac{dy}{dt} = 200$ m/s, $l = 1500$ and $\cos\theta = \frac{1200}{l} = \frac{1200}{1500} = \frac{4}{5}$, so $\frac{d\theta}{dt} = \frac{(4/5)^2}{1200}(200) = 0.10\dot{6}$ rad/s.

46. We are given that $\dfrac{d\theta}{dt} = 4(2\pi) = 8\pi$ rad/min. $x = 3\tan\theta \quad\Rightarrow$

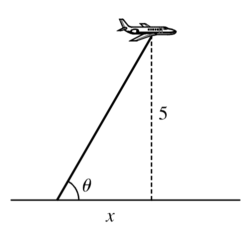
$\dfrac{dx}{dt} = 3\sec^2\theta\,\dfrac{d\theta}{dt}$. When $x = 1$, $\tan\theta = \frac{1}{3}$, so $\sec^2\theta = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9}$

and $\dfrac{dx}{dt} = 3\left(\frac{10}{9}\right)(8\pi) = \frac{80}{3}\pi \approx 83.8$ km/min.



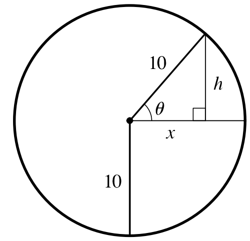
47. $\cot\theta = \dfrac{x}{5} \quad\Rightarrow\quad -\csc^2\theta\,\dfrac{d\theta}{dt} = \dfrac{1}{5}\dfrac{dx}{dt} \quad\Rightarrow\quad -\left(\csc\dfrac{\pi}{3}\right)^2\left(-\dfrac{\pi}{6}\right) = \dfrac{1}{5}\dfrac{dx}{dt} \quad\Rightarrow$

$\dfrac{dx}{dt} = \dfrac{5\pi}{6}\left(\dfrac{2}{\sqrt{3}}\right)^2 = \dfrac{10}{9}\pi$ km/min $[\approx 130$ mi/h$]$



48. We are given that $\dfrac{d\theta}{dt} = \dfrac{2\pi\text{ rad}}{2\text{ min}} = \pi$ rad/min. By the Pythagorean Theorem, when $h = 6$, $x = 8$, so $\sin\theta = \frac{6}{10}$ and $\cos\theta = \frac{8}{10}$. From the figure, $\sin\theta = \dfrac{h}{10} \quad\Rightarrow$

$h = 10\sin\theta$, so $\dfrac{dh}{dt} = 10\cos\theta\,\dfrac{d\theta}{dt} = 10\left(\dfrac{8}{10}\right)\pi = 8\pi$ m/min.

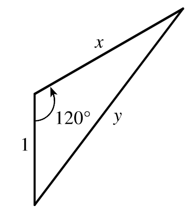


49. We are given that $\dfrac{dx}{dt} = 300$ km/h. By the Law of Cosines,

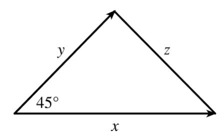
$y^2 = x^2 + 1^2 - 2(1)(x)\cos 120^\circ = x^2 + 1 - 2x\left(-\frac{1}{2}\right) = x^2 + x + 1$, so

$2y\,\dfrac{dy}{dt} = 2x\,\dfrac{dx}{dt} + \dfrac{dx}{dt} \quad\Rightarrow\quad \dfrac{dy}{dt} = \dfrac{2x+1}{2y}\,\dfrac{dx}{dt}$. After 1 minute, $x = \frac{300}{60} = 5$ km $\quad\Rightarrow$

$y = \sqrt{5^2+5+1} = \sqrt{31}$ km $\quad\Rightarrow\quad \dfrac{dy}{dt} = \dfrac{2(5)+1}{2\sqrt{31}}(300) = \dfrac{1650}{\sqrt{31}} \approx 296$ km/h.



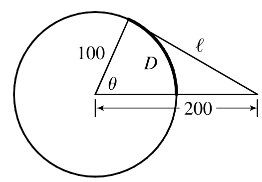
50. We are given that $\frac{dx}{dt} = 4$ km/h and $\frac{dy}{dt} = 2$ km/h. By the Law of Cosines, $z^2 =$ $x^2+y^2-2xy\cos 45^\circ = x^2+y^2-\sqrt{2}xy \Rightarrow 2z\,\frac{dz}{dt} = 2x\,\frac{dx}{dt} + 2\frac{dy}{dt} - \sqrt{2}x\,\frac{dy}{dt} - \sqrt{2}y\,\frac{dx}{dt}$.



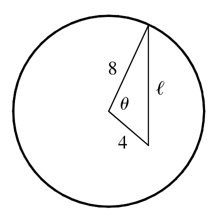
After 15 minutes $[= \frac{1}{4}$ h],

we have $x = \frac{4}{4} = 1$ and $y = \frac{2}{4} = \frac{1}{2} \Rightarrow z^2 = (1)^2 + (\frac{1}{2})^2 - \sqrt{2}(1)(\frac{1}{2}) \Rightarrow z = \frac{\sqrt{5-2\sqrt{2}}}{2}$ and $\frac{dz}{dt} = \frac{1}{\sqrt{5-2\sqrt{2}}}[2(1)4 + 2(\frac{1}{2})2 -$ $\sqrt{2}(1)2 - \sqrt{2}(\frac{1}{2})4] = \frac{1}{\sqrt{5-2\sqrt{2}}}(10 - 4\sqrt{2}) = 2\sqrt{5-2\sqrt{2}} \approx 2.947$ km/h.

51. Let the distance between the runner and the friend be ℓ. Then by the Law of Cosines, $\ell^2 = 200^2 + 100^2 - 2\cdot 200\cdot 100\cdot\cos\theta = 50{,}000 - 40{,}000\cos\theta$ $(\star)$. Differentiating implicitly with respect to t, we obtain $2\ell\,\dfrac{d\ell}{dt} = -40{,}000(-\sin\theta)\,\dfrac{d\theta}{dt}$. Now if D is the distance run when the angle is θ radians, then by the formula for the length of an arc on a circle, $s = r\theta$, we have $D = 100\theta$, so $\theta = \dfrac{1}{100}D \quad\Rightarrow\quad \dfrac{d\theta}{dt} = \dfrac{1}{100}\,\dfrac{dD}{dt} = \dfrac{7}{100}$. To substitute into the expression for $\dfrac{d\ell}{dt}$, we must know $\sin\theta$ at the time when $\ell = 200$, which we find from $(\star)$: $200^2 = 50{,}000 - 40{,}000\cos\theta \quad\Leftrightarrow$ $\cos\theta = \frac{1}{4} \quad\Rightarrow\quad \sin\theta = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$. Substituting, we get $2(200)\,\dfrac{d\ell}{dt} = 40{,}000\frac{\sqrt{15}}{4}\left(\frac{7}{100}\right) \quad\Rightarrow$ $d\ell/dt = \frac{7\sqrt{15}}{4} \approx 6.78$ m/s. Whether the distance between them is increasing or decreasing depends on the direction in which the runner is running.



52. The hour hand of a clock goes around once every 12 hours or, in radians per hour, $\frac{2\pi}{12} = \frac{\pi}{6}$ rad/h. The minute hand goes around once an hour, or at the rate of 2π rad/h. So the angle θ between them (measuring clockwise from the minute hand to the hour hand) is changing at the rate of $d\theta/dt = \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$ rad/h. Now, to relate θ to ℓ, we use the Law of Cosines: $\ell^2 = 4^2 + 8^2 - 2\cdot 4\cdot 8\cdot\cos\theta = 80 - 64\cos\theta$ $(\star)$.



Differentiating implicitly with respect to t, we get $2\ell\,\dfrac{d\ell}{dt} = -64(-\sin\theta)\,\dfrac{d\theta}{dt}$. At 1:00, the angle between the two hands is one-twelfth of the circle, that is, $\frac{2\pi}{12} = \frac{\pi}{6}$ radians. We use ($\star$) to find ℓ at 1:00: $\ell = \sqrt{80 - 64\cos\frac{\pi}{6}} = \sqrt{80 - 32\sqrt{3}}$.

Substituting, we get $2\ell\,\dfrac{d\ell}{dt} = 64\sin\dfrac{\pi}{6}\left(-\dfrac{11\pi}{6}\right) \quad\Rightarrow\quad \dfrac{d\ell}{dt} = \dfrac{64\left(\frac{1}{2}\right)\left(-\frac{11\pi}{6}\right)}{2\sqrt{80-32\sqrt{3}}} = -\dfrac{88\pi}{3\sqrt{80-32\sqrt{3}}} \approx -18.6.$

So at 1:00, the distance between the tips of the hands is decreasing at a rate of 18.6 mm/h ≈ 0.005 mm/s.

53. The volume of the snowball is given by $V = \frac{4}{3}\pi r^3$, so $\dfrac{dV}{dt} = \dfrac{4}{3}\pi \cdot 3r^2\,\dfrac{dr}{dt} = 4\pi r^2\,\dfrac{dr}{dt}$. Since the volume is proportional to the surface area S, with $S = 4\pi r^2$, we also have $\dfrac{dV}{dt} = k \cdot 4\pi r^2$ for some constant k. Equating the two expressions for $\dfrac{dV}{dt}$ gives $4\pi r^2\,\dfrac{dr}{dt} = k \cdot 4\pi r^2 \quad\Rightarrow\quad \dfrac{dr}{dt} = k$, that is, dr/dt is constant.

3.10 Linear Approximations and Differentials

1. $f(x) = x^3 - x^2 + 3 \quad\Rightarrow\quad f'(x) = 3x^2 - 2x$, so $f(-2) = -9$ and $f'(-2) = 16$. Thus,

$L(x) = f(-2) + f'(-2)(x-(-2)) = -9 + 16(x+2) = 16x + 23.$

2. $f(x) = e^{3x} \quad\Rightarrow\quad f'(x) = 3e^{3x}$, so $f(0) = 1$ and $f'(0) = 3$. Thus, $L(x) = f(0) + f'(0)(x-0) = 1 + 3(x-0) = 3x + 1$.

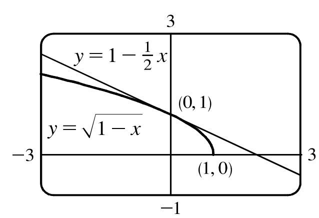
3. $f(x) = \sqrt[3]{x} \quad\Rightarrow\quad f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$, so $f(8) = 2$ and $f'(8) = \dfrac{1}{12}$. Thus,

$L(x) = f(8) + f'(8)(x-8) = 2 + \frac{1}{12}(x-8) = \frac{1}{12}x + \frac{4}{3}.$

4. $f(x) = \cos 2x \quad\Rightarrow\quad f'(x) = -2\sin 2x$, so $f\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $f'\left(\frac{\pi}{6}\right) = -\sqrt{3}$. Thus,

$L(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) = \frac{1}{2} - \sqrt{3}\left(x - \frac{\pi}{6}\right) = -\sqrt{3}\,x + (\sqrt{3}\,\pi)/6 + \frac{1}{2}.$

5. $f(x) = \sqrt{1-x} \quad\Rightarrow\quad f'(x) = \dfrac{-1}{2\sqrt{1-x}}$, so $f(0) = 1$ and $f'(0) = -\frac{1}{2}$.



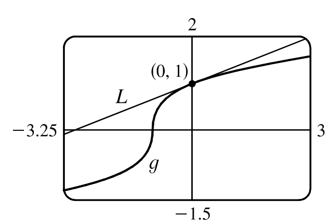
Therefore,

$\sqrt{1-x} = f(x) \approx f(0) + f'(0)(x-0) = 1 + \left(-\frac{1}{2}\right)(x-0) = 1 - \frac{1}{2}x.$

So $\sqrt{0.9} = \sqrt{1-0.1} \approx 1 - \frac{1}{2}(0.1) = 0.95$

and $\sqrt{0.99} = \sqrt{1-0.01} \approx 1 - \frac{1}{2}(0.01) = 0.995$.

6. $g(x) = \sqrt[3]{1+x} = (1+x)^{1/3} \quad\Rightarrow\quad g'(x) = \frac{1}{3}(1+x)^{-2/3}$, so $g(0) = 1$ and



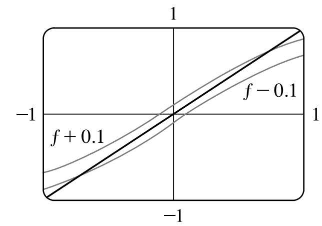
$g'(0) = \frac{1}{3}$. Therefore, $\sqrt[3]{1+x} = g(x) \approx g(0) + g'(0)(x-0) = 1 + \frac{1}{3}x$.

So $\sqrt[3]{0.95} = \sqrt[3]{1+(-0.05)} \approx 1 + \frac{1}{3}(-0.05) = 0.98\overline{3}$,

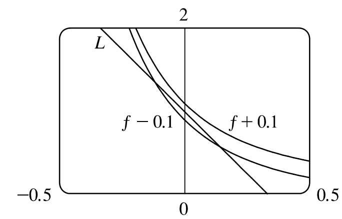
and $\sqrt[3]{1.1} = \sqrt[3]{1+0.1} \approx 1 + \frac{1}{3}(0.1) = 1.0\overline{3}$.

7. $f(x) = \tan^{-1} x \quad \Rightarrow \quad f'(x) = \dfrac{1}{1+x^2}$, so $f(0) = 0$ and $f'(0) = 1$.

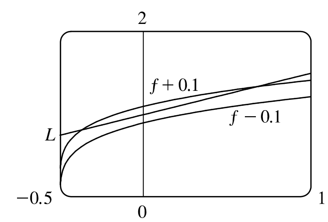
Thus, $f(x) \approx f(0) + f'(0)(x - 0) = x$. We need $\tan^{-1} x - 0.1 < x < \tan^{-1} x + 0.1$, which is true when $-0.732 < x < 0.732$. Note that to ensure the accuracy, we have rounded the smaller value up and the larger value down.



8. $f(x) = (1+x)^{-3} \quad \Rightarrow \quad f'(x) = -3(1+x)^{-4}$, so $f(0) = 1$ and $f'(0) = -3$. Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 - 3x$. We need $(1+x)^{-3} - 0.1 < 1 - 3x < (1+x)^{-3} + 0.1$, which is true when $-0.116 < x < 0.144$. Note that to ensure the accuracy, we have rounded the smaller value up and the larger value down.



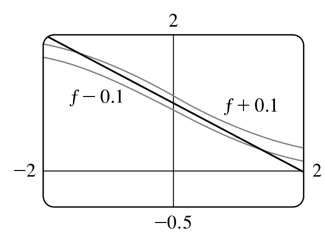
9. $f(x) = \sqrt[4]{1+2x} \quad \Rightarrow \quad f'(x) = \frac{1}{4}(1+2x)^{-3/4}(2) = \frac{1}{2}(1+2x)^{-3/4}$, so $f(0) = 1$ and $f'(0) = \frac{1}{2}$. Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 + \frac{1}{2}x$. We need $\sqrt[4]{1+2x} - 0.1 < 1 + \frac{1}{2}x < \sqrt[4]{1+2x} + 0.1$, which is true when $-0.368 < x < 0.677$. Note that to ensure the accuracy, we have rounded the smaller value up and the larger value down.



10. $f(x) = \dfrac{2}{1+e^x} \quad \Rightarrow \quad f'(x) = -\dfrac{2e^x}{(1+e^x)^2}$, so $f(0) = 1$ and $f'(0) = -\dfrac{1}{2}$.

Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 - \frac{1}{2}x$. We need

$\dfrac{2}{1+e^x} - 0.1 < 1 - \dfrac{1}{2}x < \dfrac{2}{1+e^x} + 0.1$, which is true when $-1.423 < x < 1.423$. Note that to ensure the accuracy, we have rounded the smaller value up and the larger value down.



11. The differential dy is defined in terms of dx by the equation $dy = f'(x)\,dx$. For $y = f(x) = e^{5x}$, $f'(x) = 5e^{5x}$, so $dy = 5e^{5x}\,dx$.

12. For $y = f(t) = \sqrt{1-t^4}$, $f'(t) = \frac{1}{2}(1-t^4)^{-1/2}(-4t^3) = -\dfrac{2t^3}{\sqrt{1-t^4}}$, so $dy = -\dfrac{2t^3}{\sqrt{1-t^4}}\,dt$.

13. For $y = f(u) = \dfrac{1+2u}{1+3u}$, $f'(u) = \dfrac{(1+3u)(2) - (1+2u)(3)}{(1+3u)^2} = \dfrac{-1}{(1+3u)^2}$, so $dy = \dfrac{-1}{(1+3u)^2}\,du$.

14. For $y = f(\theta) = \theta^2 \sin 2\theta$, $f'(\theta) = \theta^2(\cos 2\theta)(2) + (\sin 2\theta)(2\theta)$, so $dy = 2\theta(\theta\cos 2\theta + \sin 2\theta)\,d\theta$.

15. For $y = f(x) = \dfrac{1}{x^2 - 3x} = (x^2-3x)^{-1}$, $f'(x) = -(x^2-3x)^{-2} \cdot (2x-3) = -\dfrac{2x-3}{(x^2-3x)^2}$, so $dy = -\dfrac{2x-3}{(x^2-3x)^2}\,dx$.

16. For $y = f(\theta) = \sqrt{1+\cos\theta}$, $f'(\theta) = \frac{1}{2}(1+\cos\theta)^{-1/2}(-\sin\theta) = -\dfrac{\sin\theta}{2\sqrt{1+\cos\theta}}$, so $dy = -\dfrac{\sin\theta}{2\sqrt{1+\cos\theta}}\,d\theta$.

17. For $y = f(\theta) = \ln(\sin\theta)$, $f'(\theta) = \dfrac{1}{\sin\theta}\cos\theta = \cot\theta$, so $dy = \cot\theta\,d\theta$.

18. For $y = f(x) = \dfrac{e^x}{1-e^x}$, $f'(x) = \dfrac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2} = \dfrac{e^x[(1-e^x)-(-e^x)]}{(1-e^x)^2} = \dfrac{e^x}{(1-e^x)^2}$, so $dy = \dfrac{e^x}{(1-e^x)^2}\,dx$.

19. (a) $y = e^{x/10} \quad\Rightarrow\quad dy = e^{x/10}\cdot\frac{1}{10}\,dx = \frac{1}{10}e^{x/10}dx$

(b) $x = 0$ and $dx = 0.1 \quad\Rightarrow\quad dy = \frac{1}{10}e^{0/10}(0.1) = 0.01.$

20. (a) $y = \cos\pi x \quad\Rightarrow\quad dy = -\sin\pi x\cdot\pi\,dx = -\pi\,\sin\pi x\,dx$

(b) $x = \frac{1}{3}$ and $dx = -0.02 \quad\Rightarrow\quad dy = -\pi\,\sin\frac{\pi}{3}(-0.02) = \pi\left(\sqrt{3}/2\right)(0.02) = 0.01\pi\sqrt{3} \approx 0.054.$

21. (a) $y = \sqrt{3+x^2} \quad\Rightarrow\quad dy = \dfrac{1}{2}(3+x^2)^{-1/2}(2x)\,dx = \dfrac{x}{\sqrt{3+x^2}}\,dx$

(b) $x = 1$ and $dx = -0.1 \quad\Rightarrow\quad dy = \dfrac{1}{\sqrt{3+1^2}}(-0.1) = \dfrac{1}{2}(-0.1) = -0.05.$

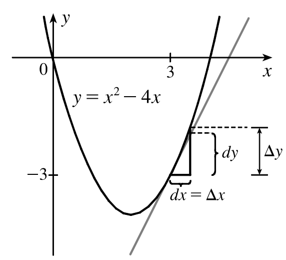
22. (a) $y = \dfrac{x+1}{x-1} \quad\Rightarrow\quad dy = \dfrac{(x-1)(1)-(x+1)(1)}{(x-1)^2}\,dx = \dfrac{-2}{(x-1)^2}\,dx$

(b) $x = 2$ and $dx = 0.05 \quad\Rightarrow\quad dy = \dfrac{-2}{(2-1)^2}(0.05) = -2(0.05) = -0.1.$

23. $y = f(x) = x^2 - 4x,\ x = 3,\ \Delta x = 0.5 \quad\Rightarrow$

$\Delta y = f(3.5) - f(3) = -1.75 - (-3) = 1.25$

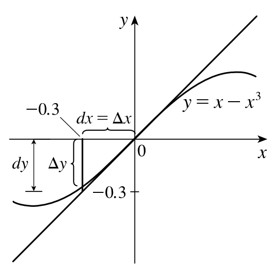
$dy = f'(x)\,dx = (2x-4)\,dx = (6-4)(0.5) = 1$



24. $y = f(x) = x - x^3,\ x = 0,\ \Delta x = -0.3 \quad\Rightarrow$

$\Delta y = f(-0.3) - f(0) = -0.273 - 0 = -0.273$

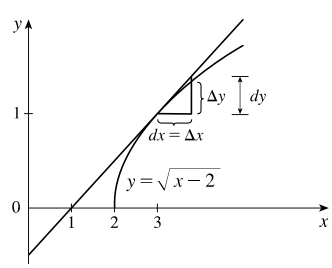
$dy = f'(x)\,dx = (1-3x^2)\,dx = (1-0)(-0.3) = -0.3$



25. $y = f(x) = \sqrt{x-2},\ x = 3,\ \Delta x = 0.8 \quad\Rightarrow$

$\Delta y = f(3.8) - f(3) = \sqrt{1.8} - 1 \approx 0.34$

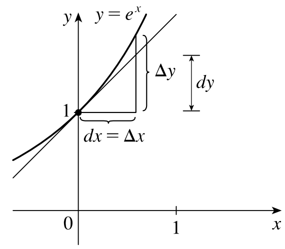
$dy = f'(x)\,dx = \dfrac{1}{2\sqrt{x-2}}\,dx = \dfrac{1}{2(1)}(0.8) = 0.4$



26. $y = f(x) = e^x,\ x = 0,\ \Delta x = 0.5 \quad \Rightarrow$

$\Delta y = f(0.5) - f(0) = \sqrt{e} - 1\ \ [\approx 0.65]$

$dy = e^x\,dx = e^0(0.5) = 0.5$



27. $y = f(x) = x^4 - x + 1$. If x changes from 1 to 1.05, $dx = \Delta x = 1.05 - 1 = 0.05 \quad \Rightarrow$

$\Delta y = f(1.05) - f(1) = 1.16550625 - 1 \approx 0.1655$ and $dy = (4x^3 - 1)\,dx = (4 \cdot 1^3 - 1)(0.05) = 3(0.05) = 0.15$.

If x changes from 1 to 1.01, $dx = \Delta x = 1.01 - 1 = 0.01 \quad \Rightarrow \quad \Delta y = f(1.01) - f(1) = 1.03060401 - 1 \approx 0.0306$

and $dy = (4x^3 - 1)\,dx = 3(0.01) = 0.03$.

With $\Delta x = 0.05$, $|dy - \Delta y| \approx |0.15 - 0.1655| = 0.0155$. With $\Delta x = 0.01$, $|dy - \Delta y| \approx |0.03 - 0.0306| = 0.0006$.

Since $|dy - \Delta y|$ is smaller for $\Delta x = 0.01$ than for $\Delta x = 0.05$, yes, the approximation $\Delta y \approx dy$ becomes better as Δx gets smaller.

28. $y = f(x) = e^{2x-2}$. If x changes from 1 to 1.05, $dx = \Delta x = 1.05 - 1 = 0.05 \quad \Rightarrow$

$\Delta y = f(1.05) - f(1) = e^{0.10} - 1 \approx 0.105$ and $dy = 2e^{2x-2}dx = 2e^{2(1)-2}(0.05) = 2(0.05) = 0.1$.

If x changes from 1 to 1.01, $dx = \Delta x = 1.01 - 1 = 0.01 \quad \Rightarrow \quad \Delta y = f(1.01) - f(1) = e^{0.02} - 1 \approx 0.0202$

and $dy = 2e^{2x-2}dx = 2(0.01) = 0.02$.

With $\Delta x = 0.05$, $|dy - \Delta y| \approx |0.1 - 0.105| = 0.005$. With $\Delta x = 0.01$, $|dy - \Delta y| \approx |0.02 - 0.0202| = 0.0002$.

Since $|dy - \Delta y|$ is smaller for $\Delta x = 0.01$ than for $\Delta x = 0.05$, yes, the approximation $\Delta y \approx dy$ becomes better as Δx gets smaller.

29. $y = f(x) = \sqrt{5 - x}$. If x changes from 1 to 1.05, $dx = \Delta x = 1.05 - 1 = 0.05 \quad \Rightarrow$

$\Delta y = f(1.05) - f(1) = \sqrt{3.95} - 2 \approx -0.012539$ and $dy = -\dfrac{1}{2\sqrt{5-x}}dx = -\dfrac{1}{2\sqrt{5-1}}dx = -\dfrac{1}{4}(0.05) = -0.0125$.

If x changes from 1 to 1.01, $dx = \Delta x = 1.01 - 1 = 0.01 \quad \Rightarrow \quad \Delta y = f(1.01) - f(1) = \sqrt{3.99} - 2 \approx -0.002502$

and $dy = -\frac{1}{4}(0.01) = -0.0025$.

With $\Delta x = 0.05$, $|dy - \Delta y| \approx |-0.0125 - (-0.012539)| = 0.000039$. With $\Delta x = 0.01$,

$|dy - \Delta y| \approx |-0.0025 - (-0.002502)| = 0.000002$. Since $|dy - \Delta y|$ is smaller for $\Delta x = 0.01$ than for $\Delta x = 0.05$, yes, the approximation $\Delta y \approx dy$ becomes better as Δx gets smaller.

30. $y = f(x) = \dfrac{1}{x^2 + 1}$. If x changes from 1 to 1.05, $dx = \Delta x = 1.05 - 1 = 0.05 \quad \Rightarrow$

$\Delta y = f(1.05) - f(1) = \dfrac{1}{(1.05)^2 - 1} - \dfrac{1}{2} \approx -0.02438$ and

$$dy = -\frac{2x}{(x^2+1)^2}dx = -\frac{2(1)}{(1^2+1)^2}\,(0.05) = -\frac{1}{2}\,(0.05) = -0.025.$$

If x changes from 1 to 1.01, $dx = \Delta x = 1.01 - 1 = 0.01 \quad \Rightarrow \quad \Delta y = f(1.01) - f(1) = \dfrac{1}{1.01^2 + 1} - \dfrac{1}{2} \approx -0.00498$

and $dy = -\frac{1}{2}(0.01) = -0.005$.

With $\Delta x = 0.05$, $|dy - \Delta y| \approx |-0.025 - (-0.02438)| = 0.00038$. With $\Delta x = 0.01$,

$|dy - \Delta y| \approx |-0.005 - (-0.00498)| = 0.00002$. Since $|dy - \Delta y|$ is smaller for $\Delta x = 0.01$ than for $\Delta x = 0.05$,

yes, the approximation $\Delta y \approx dy$ becomes better as Δx gets smaller.

31. To estimate $(1.999)^4$, we'll find the linearization of $f(x) = x^4$ at $a = 2$. Since $f'(x) = 4x^3$, $f(2) = 16$, and $f'(2) = 32$, we have $L(x) = 16 + 32(x - 2)$. Thus, $x^4 \approx 16 + 32(x - 2)$ when x is near 2, so $(1.999)^4 \approx 16 + 32(1.999 - 2) = 16 - 0.032 = 15.968$.

32. $y = f(x) = 1/x \quad \Rightarrow \quad dy = -1/x^2\,dx$. When $x = 4$ and $dx = 0.002$, $dy = -\frac{1}{16}(0.002) = -\frac{1}{8000}$, so $\frac{1}{4.002} \approx f(4) + dy = \frac{1}{4} - \frac{1}{8000} = \frac{1999}{8000} = 0.249875$.

33. $y = f(x) = \sqrt[3]{x} \quad \Rightarrow \quad dy = \frac{1}{3}x^{-2/3}\,dx$. When $x = 1000$ and $dx = 1$, $dy = \frac{1}{3}(1000)^{-2/3}(1) = \frac{1}{300}$, so $\sqrt[3]{1001} = f(1001) \approx f(1000) + dy = 10 + \frac{1}{300} = 10.00\overline{3} \approx 10.003$.

34. $y = f(x) = \sqrt{x} \quad \Rightarrow \quad dy = \frac{1}{2}x^{-1/2}\,dx$. When $x = 100$ and $dx = 0.5$, $dy = \frac{1}{2}(100)^{-1/2}\left(\frac{1}{2}\right) = \frac{1}{40}$, so $\sqrt{100.5} = f(100.5) \approx f(100) + dy = 10 + \frac{1}{40} = 10.025$.

35. $y = f(x) = e^x \quad \Rightarrow \quad dy = e^x\,dx$. When $x = 0$ and $dx = 0.1$, $dy = e^0(0.1) = 0.1$, so $e^{0.1} = f(0.1) \approx f(0) + dy = 1 + 0.1 = 1.1$.

36. $y = f(x) = \cos x \quad \Rightarrow \quad dy = -\sin x\,dx$. When $x = 30^\circ$ $[\pi/6]$ and $dx = -1^\circ$ $[-\pi/180]$, $dy = \left(-\sin\frac{\pi}{6}\right)\left(-\frac{\pi}{180}\right) = -\frac{1}{2}\left(-\frac{\pi}{180}\right) = \frac{\pi}{360}$, so $\cos 29^\circ = f(29^\circ) \approx f(30^\circ) + dy = \frac{1}{2}\sqrt{3} + \frac{\pi}{360} \approx 0.875$.

37. $y = f(x) = \ln x \quad \Rightarrow \quad f'(x) = 1/x$, so $f(1) = 0$ and $f'(1) = 1$. The linear approximation of f at 1 is $f(1) + f'(1)(x - 1) = x - 1$. Now $f(1.04) = \ln 1.04 \approx 1.04 - 1 = 0.04$, so the approximation is reasonable.

38. $y = f(x) = \sqrt{x} \quad \Rightarrow \quad f'(x) = 1/(2\sqrt{x})$, so $f(4) = 2$ and $f'(4) = \frac{1}{4}$. The linear approximation of f at 4 is $f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4)$. Now $f(4.02) = \sqrt{4.02} \approx 2 + \frac{1}{4}(0.02) = 2 + 0.005 = 2.005$, so the approximation is reasonable.

39. $y = f(x) = 1/x \quad \Rightarrow \quad f'(x) = -1/x^2$, so $f(10) = 0.1$ and $f'(10) = -0.01$. The linear approximation of f at 10 is $f(10) + f'(10)(x - 10) = 0.1 - 0.01(x - 10)$. Now $f(9.98) = 1/9.98 \approx 0.1 - 0.01(-0.02) = 0.1 + 0.0002 = 0.1002$, so the approximation is reasonable.

40. (a) $f(x) = (x - 1)^2 \quad \Rightarrow \quad f'(x) = 2(x - 1)$, so $f(0) = 1$ and $f'(0) = -2$.

Thus, $f(x) \approx L_f(x) = f(0) + f'(0)(x - 0) = 1 - 2x$.

$g(x) = e^{-2x} \quad \Rightarrow \quad g'(x) = -2e^{-2x}$, so $g(0) = 1$ and $g'(0) = -2$.

[continued]

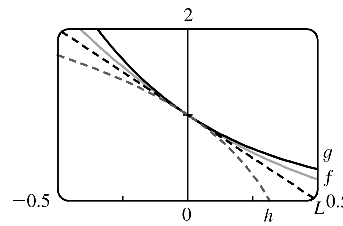
Thus, $g(x) \approx L_g(x) = g(0) + g'(0)(x - 0) = 1 - 2x$.

$h(x) = 1 + \ln(1 - 2x) \quad \Rightarrow \quad h'(x) = \dfrac{-2}{1 - 2x}$, so $h(0) = 1$ and $h'(0) = -2$.

Thus, $h(x) \approx L_h(x) = h(0) + h'(0)(x - 0) = 1 - 2x$.

Notice that $L_f = L_g = L_h$. This happens because f, g, and h have the same function values and the same derivative values at $a = 0$.

(b) The linear approximation appears to be the best for the function f since it is closer to f for a larger domain than it is to g and h. The approximation looks worst for h since h moves away from L faster than f and g do.



41. (a) If x is the edge length, then $V = x^3 \quad \Rightarrow \quad dV = 3x^2\,dx$. When $x = 30$ and $dx = 0.1$, $dV = 3(30)^2(0.1) = 270$, so the maximum possible error in computing the volume of the cube is about 270 cm^3. The relative error is calculated by dividing the change in V, ΔV, by V. We approximate ΔV with dV.

Relative error $= \dfrac{\Delta V}{V} \approx \dfrac{dV}{V} = \dfrac{3x^2\,dx}{x^3} = 3\,\dfrac{dx}{x} = 3\left(\dfrac{0.1}{30}\right) = 0.01$.

Percentage error = relative error $\times$ 100% = 0.01 $\times$ 100% = 1%.

(b) $S = 6x^2 \quad \Rightarrow \quad dS = 12x\,dx$. When $x = 30$ and $dx = 0.1$, $dS = 12(30)(0.1) = 36$, so the maximum possible error in computing the surface area of the cube is about 36 cm^2.

Relative error $= \dfrac{\Delta S}{S} \approx \dfrac{dS}{S} = \dfrac{12x\,dx}{6x^2} = 2\,\dfrac{dx}{x} = 2\left(\dfrac{0.1}{30}\right) = 0.00\overline{6}$.

Percentage error = relative error $\times$ 100% = $0.00\overline{6} \times 100\% = 0.\overline{6}\%$.

42. (a) $A = \pi r^2 \quad \Rightarrow \quad dA = 2\pi r\,dr$. When $r = 24$ and $dr = 0.2$, $dA = 2\pi(24)(0.2) = 9.6\pi$, so the maximum possible error in the calculated area of the disk is about $9.6\pi \approx 30$ cm^2.

(b) Relative error $= \dfrac{\Delta A}{A} \approx \dfrac{dA}{A} = \dfrac{2\pi r\,dr}{\pi r^2} = \dfrac{2\,dr}{r} = \dfrac{2(0.2)}{24} = \dfrac{0.2}{12} = \dfrac{1}{60} = 0.01\overline{6}$.

Percentage error = relative error $\times 100\% = 0.01\overline{6} \times 100\% = 1.\overline{6}\%$.

43. (a) For a sphere of radius r, the circumference is $C = 2\pi r$ and the surface area is $S = 4\pi r^2$, so

$r = \dfrac{C}{2\pi} \quad \Rightarrow \quad S = 4\pi\left(\dfrac{C}{2\pi}\right)^2 = \dfrac{C^2}{\pi} \quad \Rightarrow \quad dS = \dfrac{2}{\pi}C\,dC$. When $C = 84$ and $dC = 0.5$, $dS = \dfrac{2}{\pi}(84)(0.5) = \dfrac{84}{\pi}$,

so the maximum error is about $\dfrac{84}{\pi} \approx 27$ cm^2. Relative error $\approx \dfrac{dS}{S} = \dfrac{84/\pi}{84^2/\pi} = \dfrac{1}{84} \approx 0.012 = 1.2\%$

(b) $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{C}{2\pi}\right)^3 = \frac{C^3}{6\pi^2} \quad\Rightarrow\quad dV = \frac{1}{2\pi^2}C^2\,dC$. When $C = 84$ and $dC = 0.5$,

$dV = \frac{1}{2\pi^2}(84)^2(0.5) = \frac{1764}{\pi^2}$, so the maximum error is about $\frac{1764}{\pi^2} \approx 179\text{ cm}^3$.

The relative error is approximately $\frac{dV}{V} = \frac{1764/\pi^2}{(84)^3/(6\pi^2)} = \frac{1}{56} \approx 0.018 = 1.8\%$.

44. For a hemispherical dome, $V = \frac{2}{3}\pi r^3 \quad\Rightarrow\quad dV = 2\pi r^2\,dr$. When $r = \frac{1}{2}(50) = 25$ m and $dr = 0.05$ cm $= 0.0005$ m,

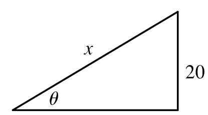
$dV = 2\pi(25)^2(0.0005) = \frac{5\pi}{8}$, so the amount of paint needed is about $\frac{5\pi}{8} \approx 2\text{ m}^3$.

45. (a) $V = \pi r^2 h \quad\Rightarrow\quad \Delta V \approx dV = 2\pi rh\,dr = 2\pi rh\,\Delta r$

(b) The error is

$$\Delta V - dV = [\pi(r+\Delta r)^2 h - \pi r^2 h] - 2\pi rh\,\Delta r = \pi r^2 h + 2\pi rh\,\Delta r + \pi(\Delta r)^2 h - \pi r^2 h - 2\pi rh\,\Delta r = \pi(\Delta r)^2 h.$$

46. (a) $\sin\theta = \frac{20}{x} \quad\Rightarrow\quad x = 20\csc\theta \quad\Rightarrow$



$dx = 20(-\csc\theta\,\cot\theta)\,d\theta = -20\csc 30^\circ\ \cot 30^\circ\ (\pm 1^\circ)$

$= -20(2)(\sqrt{3})\left(\pm\frac{\pi}{180}\right) = \pm\frac{2\sqrt{3}}{9}\pi$

So the maximum error is about $\pm\frac{2}{9}\sqrt{3}\,\pi \approx \pm 1.21$ cm.

(b) The relative error is $\frac{\Delta x}{x} \approx \frac{dx}{x} = \frac{\pm\frac{2}{9}\sqrt{3}\,\pi}{20(2)} = \pm\frac{\sqrt{3}}{180}\pi \approx \pm 0.03$, so the percentage error is approximately $\pm 3\%$.

47. $V = RI \quad\Rightarrow\quad I = \frac{V}{R} \quad\Rightarrow\quad dI = -\frac{V}{R^2}\,dR$. The relative error in calculating I is $\frac{\Delta I}{I} \approx \frac{dI}{I} = \frac{-(V/R^2)\,dR}{V/R} = -\frac{dR}{R}$.

Hence, the relative error in calculating I is approximately the same (in magnitude) as the relative error in R.

48. $F = kR^4 \quad\Rightarrow\quad dF = 4kR^3\,dR \quad\Rightarrow\quad \frac{dF}{F} = \frac{4kR^3\,dR}{kR^4} = 4\left(\frac{dR}{R}\right)$. Thus, the relative change in F is about 4 times the relative change in R. So a 5% increase in the radius corresponds to a 20% increase in blood flow.

49. (a) $dc = \frac{dc}{dx}\,dx = 0\,dx = 0$

(b) $d(cu) = \frac{d}{dx}(cu)\,dx = c\,\frac{du}{dx}\,dx = c\,du$

(c) $d(u+v) = \frac{d}{dx}(u+v)\,dx = \left(\frac{du}{dx} + \frac{dv}{dx}\right)dx = \frac{du}{dx}\,dx + \frac{dv}{dx}\,dx = du + dv$

(d) $d(uv) = \frac{d}{dx}(uv)\,dx = \left(u\,\frac{dv}{dx} + v\,\frac{du}{dx}\right)dx = u\,\frac{dv}{dx}\,dx + v\,\frac{du}{dx}\,dx = u\,dv + v\,du$

(e) $d\left(\frac{u}{v}\right) = \frac{d}{dx}\left(\frac{u}{v}\right)\,dx = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}\,dx = \frac{v\,\frac{du}{dx}\,dx - u\,\frac{dv}{dx}\,dx}{v^2} = \frac{v\,du - u\,dv}{v^2}$

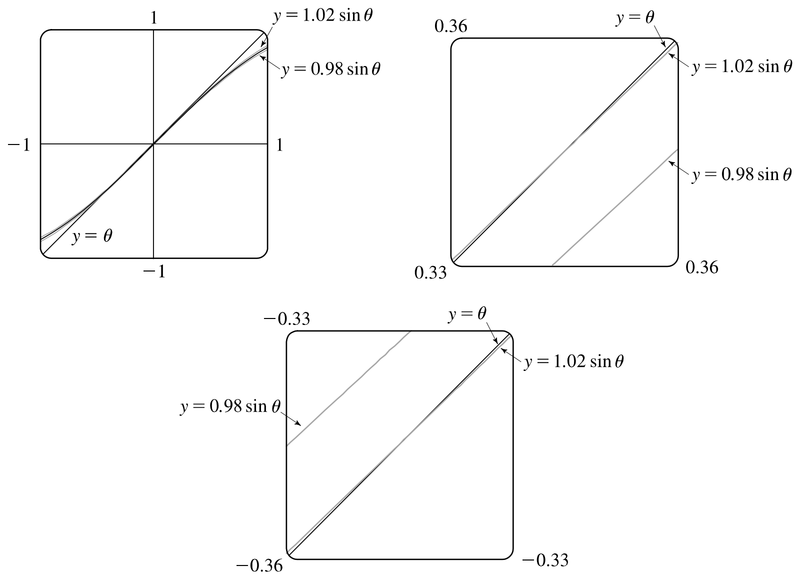
(f) $d\,(x^n) = \frac{d}{dx}\,(x^n)\,dx = nx^{n-1}\,dx$

50. (a) $f(\theta) = \sin\theta \quad\Rightarrow\quad f'(\theta) = \cos\theta$, so $f(0) = 0$ and $f'(0) = 1$.

Thus, $f(\theta) \approx f(0) + f'(0)(\theta - 0) = 0 + 1(\theta - 0) = \theta$.

(b) The relative error in approximating $\sin\theta$ by θ for $\theta = \pi/18$ is $\dfrac{\pi/18 - \sin(\pi/18)}{\sin(\pi/18)} \approx 0.0051$, so the percentage error is 0.51%.

(c)



We want to know the values of θ for which $y = \theta$ approximates $y = \sin\theta$ with less than a 2% difference; that is, the values of θ for which

$$\left|\frac{\theta - \sin\theta}{\sin\theta}\right| < 0.02 \quad\Leftrightarrow\quad -0.02 < \frac{\theta - \sin\theta}{\sin\theta} < 0.02 \quad\Leftrightarrow$$

$$\begin{cases} -0.02\sin\theta < \theta - \sin\theta < 0.02\sin\theta & \text{if } \sin\theta > 0 \\ -0.02\sin\theta > \theta - \sin\theta > 0.02\sin\theta & \text{if } \sin\theta < 0 \end{cases} \quad\Leftrightarrow\quad \begin{cases} 0.98\sin\theta < \theta < 1.02\sin\theta & \text{if } \sin\theta > 0 \\ 1.02\sin\theta < \theta < 0.98\sin\theta & \text{if } \sin\theta < 0 \end{cases}$$

In the first figure, we see that the graphs are very close to each other near $\theta = 0$. Changing the viewing rectangle and using an intersect feature (see the second figure) we find that $y = \theta$ intersects $y = 1.02\sin\theta$ at $\theta \approx 0.344$. By symmetry, they also intersect at $\theta \approx -0.344$ (see the third figure).

Thus, with θ measured in radians, $\sin\theta$ and θ differ by less than 2% for $-0.344 < \theta < 0.344$. Converting 0.344 radians to degrees, we get $0.344(180^\circ/\pi) \approx 19.7^\circ$, so the corresponding interval in degrees is approximately $-19.7^\circ < \theta < 19.7^\circ$.

51. (a) The graph shows that $f'(1) = 2$, so $L(x) = f(1) + f'(1)(x - 1) = 5 + 2(x - 1) = 2x + 3$.

$f(0.9) \approx L(0.9) = 4.8$ and $f(1.1) \approx L(1.1) = 5.2$.

(b) From the graph, we see that $f'(x)$ is positive and decreasing. This means that the slopes of the tangent lines are positive, but the tangents are becoming less steep. So the tangent lines lie *above* the curve. Thus, the estimates in part (a) are too large.

52. (a) $g'(x) = \sqrt{x^2+5} \;\Rightarrow\; g'(2) = \sqrt{9} = 3$. $g(1.95) \approx g(2) + g'(2)(1.95-2) = -4 + 3(-0.05) = -4.15$.

$g(2.05) \approx g(2) + g'(2)(2.05-2) = -4 + 3(0.05) = -3.85$.

(b) The formula $g'(x) = \sqrt{x^2+5}$ shows that $g'(x)$ is positive and increasing. This means that the slopes of the tangent lines are positive and the tangents are getting steeper. So the tangent lines lie *below* the graph of g. Hence, the estimates in part (a) are too small.

DISCOVERY PROJECT Polynomial Approximations

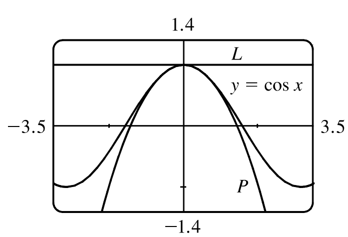
1. We first write the functions described in conditions (i), (ii), and (iii):

$$\begin{aligned} P(x) &= A + Bx + Cx^2 & \qquad f(x) &= \cos x \\ P'(x) &= B + 2Cx & f'(x) &= -\sin x \\ P''(x) &= 2C & f''(x) &= -\cos x \end{aligned}$$

So, taking $a = 0$, our three conditions become

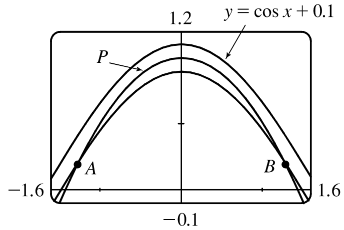
$$\begin{aligned} P(0) &= f(0): & A &= \cos 0 = 1 \\ P'(0) &= f'(0): & B &= -\sin 0 = 0 \\ P''(0) &= f''(0): & 2C &= -\cos 0 = -1 \;\Rightarrow\; C = -\tfrac{1}{2} \end{aligned}$$

The desired quadratic function is $P(x) = 1 - \frac{1}{2}x^2$, so the quadratic approximation is $\cos x \approx 1 - \frac{1}{2}x^2$.



The figure shows a graph of the cosine function together with its linear approximation $L(x) = 1$ and quadratic approximation $P(x) = 1 - \frac{1}{2}x^2$ near 0. You can see that the quadratic approximation is much better than the linear one.

2. Accuracy to within 0.1 means that $\left|\cos x - \left(1 - \frac{1}{2}x^2\right)\right| < 0.1 \;\Leftrightarrow\; -0.1 < \cos x - \left(1 - \frac{1}{2}x^2\right) < 0.1 \;\Leftrightarrow$ $0.1 > \left(1 - \frac{1}{2}x^2\right) - \cos x > -0.1 \;\Leftrightarrow\; \cos x + 0.1 > 1 - \frac{1}{2}x^2 > \cos x - 0.1 \;\Leftrightarrow\; \cos x - 0.1 < 1 - \frac{1}{2}x^2 < \cos x + 0.1$.



From the figure we see that this is true between A and B. Zooming in or using an intersect feature, we find that the x-coordinates of B and A are about ± 1.26. Thus, the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ is accurate to within 0.1 when $-1.26 < x < 1.26$.

3. If $P(x) = A + B(x-a) + C(x-a)^2$, then $P'(x) = B + 2C(x-a)$ and $P''(x) = 2C$. Applying the conditions (i), (ii), and (iii), we get

$$\begin{aligned} P(a) &= f(a): & A &= f(a) \\ P'(a) &= f'(a): & B &= f'(a) \\ P''(a) &= f''(a): & 2C &= f''(a) \quad \Rightarrow \quad C = \tfrac{1}{2}f''(a) \end{aligned}$$

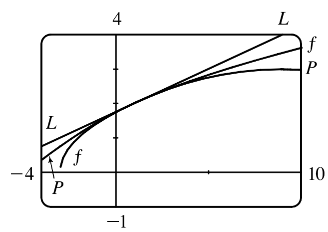
Thus, $P(x) = A + B(x-a) + C(x-a)^2$ can be written in the form $P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$.

4. From Example 3.10.1, we have $f(1) = 2$, $f'(1) = \frac{1}{4}$, and $f'(x) = \frac{1}{2}(x+3)^{-1/2}$.

So $f''(x) = -\frac{1}{4}(x+3)^{-3/2} \quad \Rightarrow \quad f''(1) = -\frac{1}{32}$.

From Problem 3, the quadratic approximation $P(x)$ is

$\sqrt{x+3} \approx f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 = 2 + \frac{1}{4}(x-1) - \frac{1}{64}(x-1)^2$.



The figure shows the function $f(x) = \sqrt{x+3}$ together with its linear approximation $L(x) = \frac{1}{4}x + \frac{7}{4}$ and its quadratic approximation $P(x)$. You can see that $P(x)$ is a better approximation than $L(x)$ and this is borne out by the numerical values in the following chart.

	from $L(x)$	actual value	from $P(x)$
$\sqrt{3.98}$	1.9950	1.99499373...	1.99499375
$\sqrt{4.05}$	2.0125	2.01246118...	2.01246094
$\sqrt{4.2}$	2.0500	2.04939015...	2.04937500

5. $T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$. If we put $x = a$ in this equation, then all terms after the first are 0 and we get $T_n(a) = c_0$. Now we differentiate $T_n(x)$ and obtain $T_n'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^{n-1}$. Substituting $x = a$ gives $T_n'(a) = c_1$. Differentiating again, we have $T_n''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a^2) + \cdots + (n-1)nc_n(x-a)^{n-2}$ and so $T_n''(a) = 2c_2$. Continuing in this manner, we get $T_n'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + \cdots + (n-2)(n-1)nc_n(x-a)^{n-3}$ and $T_n'''(a) = 2\cdot 3c_3$. By now we see the pattern. If we continue to differentiate and substitute $x = a$, we obtain $T_n^{(4)}(a) = 2\cdot 3\cdot 4c_4$ and in general, for any integer k between 1 and n, $T_n^{(k)}(a) = 2\cdot 3\cdot 4\cdot 5\cdot \cdots \cdot kc_k = k!\,c_k \quad \Rightarrow$

$c_k = \dfrac{T_n^{(k)}(a)}{k!}$. Because we want T_n and f to have the same derivatives at a, we require that $c_k = \dfrac{f^{(k)}(a)}{k!}$ for $k = 1, 2, \ldots, n$.

6. $T_n(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$. To compute the coefficients in this equation we need to calculate the derivatives of f at 0:

$$\begin{aligned} f(x) &= \cos x & f(0) &= \cos 0 = 1 \\ f'(x) &= -\sin x & f'(0) &= -\sin 0 = 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= \sin x & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1 \end{aligned}$$

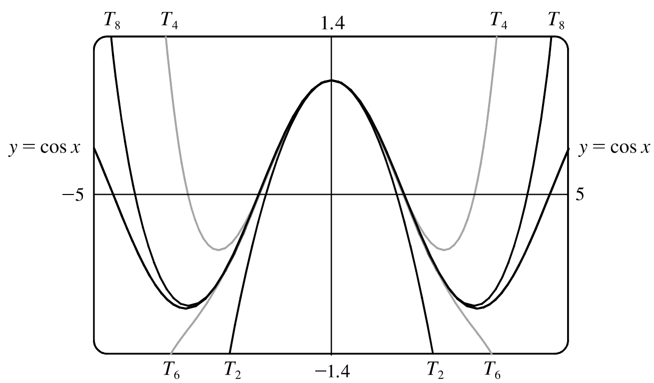
[continued]

We see that the derivatives repeat in a cycle of length 4, so $f^{(5)}(0) = 0$, $f^{(6)}(0) = -1$, $f^{(7)}(0) = 0$, and $f^{(8)}(0) = 1$.

From the original expression for $T_n(x)$, with $n = 8$ and $a = 0$, we have

$$\begin{aligned} T_8(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^{(8)}(0)}{8!}(x-0)^8 \\ &= 1 + 0\cdot x + \frac{-1}{2!}x^2 + 0\cdot x^3 + \frac{1}{4!}x^4 + 0\cdot x^5 + \frac{-1}{6!}x^6 + 0\cdot x^7 + \frac{1}{8!}x^8 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \end{aligned}$$

and the desired approximation is $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$. The Taylor polynomials T_2, T_4, and T_6 consist of the initial terms of T_8 up through degree 2, 4, and 6, respectively. Therefore, $T_2(x) = 1 - \frac{x^2}{2!}$, $T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, and $T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$. We graph T_2, T_4, T_6, T_8, and f:



Notice that $T_2(x)$ is a good approximation to $\cos x$ near 0, $T_4(x)$ is a good approximation on a larger interval, $T_6(x)$ is a better approximation, and $T_8(x)$ is better still. Each successive Taylor polynomial is a good approximation on a larger interval than the previous one.

3.11 Hyperbolic Functions

1. (a) $\sinh 0 = \frac{1}{2}(e^0 - e^{-0}) = 0$ (b) $\cosh 0 = \frac{1}{2}(e^0 + e^{-0}) = \frac{1}{2}(1+1) = 1$

2. (a) $\tanh 0 = \dfrac{(e^0 - e^{-0})/2}{(e^0 + e^{-0})/2} = 0$ (b) $\tanh 1 = \dfrac{e^1 - e^{-1}}{e^1 + e^{-1}} = \dfrac{e^2 - 1}{e^2 + 1} \approx 0.76159$

3. (a) $\cosh(\ln 5) = \frac{1}{2}(e^{\ln 5} + e^{-\ln 5}) = \frac{1}{2}\left(5 + (e^{\ln 5})^{-1}\right) = \frac{1}{2}(5 + 5^{-1}) = \frac{1}{2}\left(5 + \frac{1}{5}\right) = \frac{13}{5}$

(b) $\cosh 5 = \frac{1}{2}(e^5 + e^{-5}) \approx 74.20995$

4. (a) $\sinh 4 = \frac{1}{2}(e^4 - e^{-4}) \approx 27.28992$

(b) $\sinh(\ln 4) = \frac{1}{2}(e^{\ln 4} - e^{-\ln 4}) = \frac{1}{2}\left(4 - (e^{\ln 4})^{-1}\right) = \frac{1}{2}(4 - 4^{-1}) = \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}$

5. (a) $\operatorname{sech} 0 = \dfrac{1}{\cosh 0} = \dfrac{1}{1} = 1$ (b) $\cosh^{-1} 1 = 0$ because $\cosh 0 = 1$.

6. (a) $\sinh 1 = \frac{1}{2}(e^1 - e^{-1}) \approx 1.17520$

(b) Using Equation 3, we have $\sinh^{-1} 1 = \ln\left(1 + \sqrt{1^2 + 1}\right) = \ln\left(1 + \sqrt{2}\right) \approx 0.88137$.

7. $8 \sinh x + 5 \cosh x = 8\left(\dfrac{e^x - e^{-x}}{2}\right) + 5\left(\dfrac{e^x + e^{-x}}{2}\right) = \dfrac{8}{2}e^x - \dfrac{8}{2}e^{-x} + \dfrac{5}{2}e^x + \dfrac{5}{2}e^{-x} = \dfrac{13}{2}e^x - \dfrac{3}{2}e^{-x}$

8. $2e^{2x}+3e^{-2x}=a\sinh 2x+b\cosh 2x \quad\Rightarrow\quad 2e^{2x}+3e^{-2x}=a\left(\dfrac{e^{2x}-e^{-2x}}{2}\right)+b\left(\dfrac{e^{2x}+e^{-2x}}{2}\right) \quad\Rightarrow$

$2e^{2x}+3e^{-2x}=\dfrac{a}{2}e^{2x}-\dfrac{a}{2}e^{-2x}+\dfrac{b}{2}e^{2x}+\dfrac{b}{2}e^{-2x} \quad\Rightarrow\quad 2e^{2x}+3e^{-2x}=\dfrac{a+b}{2}e^{2x}+\dfrac{-a+b}{2}e^{-2x} \quad\Rightarrow$

$\dfrac{a+b}{2}=2$ and $\dfrac{-a+b}{2}=3 \quad\Rightarrow\quad a+b=4$ and $-a+b=6 \quad\Rightarrow\quad 2b=10 \quad\Rightarrow\quad b=5$ and $a=-1$.

Thus, $2e^{2x}+3e^{-2x}=-\sinh 2x+5\cosh 2x$.

9. $\sinh(\ln x)=\dfrac{1}{2}(e^{\ln x}-e^{-\ln x})=\dfrac{1}{2}\left(x-e^{\ln x^{-1}}\right)=\dfrac{1}{2}(x-x^{-1})=\dfrac{1}{2}\left(x-\dfrac{1}{x}\right)=\dfrac{1}{2}\left(\dfrac{x^2-1}{x}\right)=\dfrac{x^2-1}{2x}$

10. $\cosh(4\ln x)=\cosh(\ln x^4)=\dfrac{1}{2}\left(e^{\ln x^4}+e^{-\ln x^4}\right)=\dfrac{1}{2}\left(x^4+e^{\ln x^{-4}}\right)=\dfrac{1}{2}(x^4+x^{-4})$

$$=\frac{1}{2}\left(x^4+\frac{1}{x^4}\right)=\frac{1}{2}\left(\frac{x^8+1}{x^4}\right)=\frac{x^8+1}{2x^4}$$

11. $\sinh(-x)=\frac{1}{2}[e^{-x}-e^{-(-x)}]=\frac{1}{2}(e^{-x}-e^{x})=-\frac{1}{2}(e^{-x}-e^{x})=-\sinh x$

12. $\cosh(-x)=\frac{1}{2}[e^{-x}+e^{-(-x)}]=\frac{1}{2}(e^{-x}+e^{x})=\frac{1}{2}(e^{x}+e^{-x})=\cosh x$

13. $\cosh x+\sinh x=\frac{1}{2}(e^x+e^{-x})+\frac{1}{2}(e^x-e^{-x})=\frac{1}{2}(2e^x)=e^x$

14. $\cosh x-\sinh x=\frac{1}{2}(e^x+e^{-x})-\frac{1}{2}(e^x-e^{-x})=\frac{1}{2}(2e^{-x})=e^{-x}$

15. $\sinh x\,\cosh y+\cosh x\,\sinh y=\left[\frac{1}{2}(e^x-e^{-x})\right]\left[\frac{1}{2}(e^y+e^{-y})\right]+\left[\frac{1}{2}(e^x+e^{-x})\right]\left[\frac{1}{2}(e^y-e^{-y})\right]$

$$=\tfrac{1}{4}[(e^{x+y}+e^{x-y}-e^{-x+y}-e^{-x-y})+(e^{x+y}-e^{x-y}+e^{-x+y}-e^{-x-y})]$$

$$=\tfrac{1}{4}(2e^{x+y}-2e^{-x-y})=\tfrac{1}{2}[e^{x+y}-e^{-(x+y)}]=\sinh(x+y)$$

16. $\cosh x\,\cosh y+\sinh x\,\sinh y=\left[\frac{1}{2}(e^x+e^{-x})\right]\left[\frac{1}{2}(e^y+e^{-y})\right]+\left[\frac{1}{2}(e^x-e^{-x})\right]\left[\frac{1}{2}(e^y-e^{-y})\right]$

$$=\tfrac{1}{4}\left[(e^{x+y}+e^{x-y}+e^{-x+y}+e^{-x-y})+(e^{x+y}-e^{x-y}-e^{-x+y}+e^{-x-y})\right]$$

$$=\tfrac{1}{4}(2e^{x+y}+2e^{-x-y})=\tfrac{1}{2}\left[e^{x+y}+e^{-(x+y)}\right]=\cosh(x+y)$$

17. Divide both sides of the identity $\cosh^2 x-\sinh^2 x=1$ by $\sinh^2 x$:

$\dfrac{\cosh^2 x}{\sinh^2 x}-\dfrac{\sinh^2 x}{\sinh^2 x}=\dfrac{1}{\sinh^2 x} \quad\Leftrightarrow\quad \coth^2 x-1=\operatorname{csch}^2 x$.

18. $\tanh(x+y)=\dfrac{\sinh(x+y)}{\cosh(x+y)}=\dfrac{\sinh x\,\cosh y+\cosh x\,\sinh y}{\cosh x\,\cosh y+\sinh x\,\sinh y}=\dfrac{\dfrac{\sinh x\,\cosh y}{\cosh x\,\cosh y}+\dfrac{\cosh x\,\sinh y}{\cosh x\,\cosh y}}{\dfrac{\cosh x\,\cosh y}{\cosh x\,\cosh y}+\dfrac{\sinh x\,\sinh y}{\cosh x\,\cosh y}}$

$$=\frac{\tanh x+\tanh y}{1+\tanh x\,\tanh y}$$

19. Putting $y=x$ in the result from Exercise 15, we have

$\sinh 2x=\sinh(x+x)=\sinh x\,\cosh x+\cosh x\,\sinh x=2\sinh x\,\cosh x$.

20. Putting $y = x$ in the result from Exercise 16, we have

$$\cosh 2x = \cosh(x + x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x.$$

21. $\tanh(\ln x) = \dfrac{\sinh(\ln x)}{\cosh(\ln x)} = \dfrac{(e^{\ln x} - e^{-\ln x})/2}{(e^{\ln x} + e^{-\ln x})/2} = \dfrac{x - (e^{\ln x})^{-1}}{x + (e^{\ln x})^{-1}} = \dfrac{x - x^{-1}}{x + x^{-1}} = \dfrac{x - 1/x}{x + 1/x} = \dfrac{(x^2 - 1)/x}{(x^2 + 1)/x} = \dfrac{x^2 - 1}{x^2 + 1}$

22. $\dfrac{1 + \tanh x}{1 - \tanh x} = \dfrac{1 + (\sinh x)/\cosh x}{1 - (\sinh x)/\cosh x} = \dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{\frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x})} = \dfrac{e^x}{e^{-x}} = e^{2x}$

Or: Using the results of Exercises 13 and 14, $\dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{e^x}{e^{-x}} = e^{2x}$

23. By Exercise 13, $(\cosh x + \sinh x)^n = (e^x)^n = e^{nx} = \cosh nx + \sinh nx$.

24. $\coth x = \dfrac{1}{\tanh x} \;\Rightarrow\; \coth x = \dfrac{1}{\tanh x} = \dfrac{1}{12/13} = \dfrac{13}{12}$.

$\operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - \left(\frac{12}{13}\right)^2 = \frac{25}{169} \;\Rightarrow\; \operatorname{sech} x = \frac{5}{13}$ [sech, like cosh, is positive].

$\cosh x = \dfrac{1}{\operatorname{sech} x} \;\Rightarrow\; \cosh x = \dfrac{1}{5/13} = \dfrac{13}{5}$.

$\tanh x = \dfrac{\sinh x}{\cosh x} \;\Rightarrow\; \sinh x = \tanh x \cosh x \;\Rightarrow\; \sinh x = \dfrac{12}{13} \cdot \dfrac{13}{5} = \dfrac{12}{5}$.

$\operatorname{csch} x = \dfrac{1}{\sinh x} \;\Rightarrow\; \operatorname{csch} x = \dfrac{1}{12/5} = \dfrac{5}{12}$.

25. $\operatorname{sech} x = \dfrac{1}{\cosh x} \;\Rightarrow\; \operatorname{sech} x = \dfrac{1}{5/3} = \dfrac{3}{5}$.

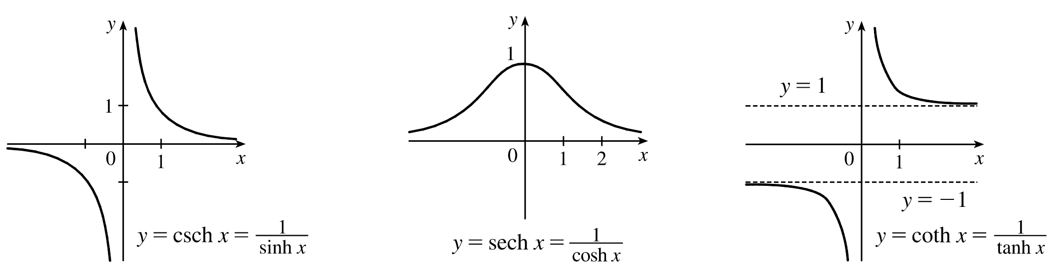
$\cosh^2 x - \sinh^2 x = 1 \;\Rightarrow\; \sinh^2 x = \cosh^2 x - 1 = \left(\frac{5}{3}\right)^2 - 1 = \frac{16}{9} \;\Rightarrow\; \sinh x = \frac{4}{3}$ [because $x > 0$].

$\operatorname{csch} x = \dfrac{1}{\sinh x} \;\Rightarrow\; \operatorname{csch} x = \dfrac{1}{4/3} = \dfrac{3}{4}$.

$\tanh x = \dfrac{\sinh x}{\cosh x} \;\Rightarrow\; \tanh x = \dfrac{4/3}{5/3} = \dfrac{4}{5}$.

$\coth x = \dfrac{1}{\tanh x} \;\Rightarrow\; \coth x = \dfrac{1}{4/5} = \dfrac{5}{4}$.

26. (a)



27. (a) $\displaystyle\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1 - 0}{1 + 0} = 1$

(b) $\displaystyle\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} = \lim_{x\to-\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{0 - 1}{0 + 1} = -1$

(c) $\lim\limits_{x\to\infty} \sinh x = \lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{2} = \infty$

(d) $\lim\limits_{x\to-\infty} \sinh x = \lim\limits_{x\to-\infty} \dfrac{e^x - e^{-x}}{2} = -\infty$

(e) $\lim\limits_{x\to\infty} \operatorname{sech} x = \lim\limits_{x\to\infty} \dfrac{2}{e^x + e^{-x}} = 0$

(f) $\lim\limits_{x\to\infty} \coth x = \lim\limits_{x\to\infty} \dfrac{e^x + e^{-x}}{e^x - e^{-x}} \cdot \dfrac{e^{-x}}{e^{-x}} = \lim\limits_{x\to\infty} \dfrac{1 + e^{-2x}}{1 - e^{-2x}} = \dfrac{1+0}{1-0} = 1$ [*Or:* Use part (a).]

(g) $\lim\limits_{x\to0^+} \coth x = \lim\limits_{x\to0^+} \dfrac{\cosh x}{\sinh x} = \infty$, since $\sinh x \to 0$ through positive values and $\cosh x \to 1$.

(h) $\lim\limits_{x\to0^-} \coth x = \lim\limits_{x\to0^-} \dfrac{\cosh x}{\sinh x} = -\infty$, since $\sinh x \to 0$ through negative values and $\cosh x \to 1$.

(i) $\lim\limits_{x\to-\infty} \operatorname{csch} x = \lim\limits_{x\to-\infty} \dfrac{2}{e^x - e^{-x}} = 0$

(j) $\lim\limits_{x\to\infty} \dfrac{\sinh x}{e^x} = \lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{2e^x} = \lim\limits_{x\to\infty} \dfrac{1 - e^{-2x}}{2} = \dfrac{1-0}{2} = \dfrac{1}{2}$

28. (a) $\dfrac{d}{dx}(\cosh x) = \dfrac{d}{dx}\left[\tfrac{1}{2}(e^x + e^{-x})\right] = \tfrac{1}{2}(e^x - e^{-x}) = \sinh x$

(b) $\dfrac{d}{dx}(\tanh x) = \dfrac{d}{dx}\left(\dfrac{\sinh x}{\cosh x}\right) = \dfrac{\cosh x\, \cosh x - \sinh x\, \sinh x}{\cosh^2 x} = \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

(c) $\dfrac{d}{dx}(\operatorname{csch} x) = \dfrac{d}{dx}\left(\dfrac{1}{\sinh x}\right) = -\dfrac{\cosh x}{\sinh^2 x} = -\dfrac{1}{\sinh x} \cdot \dfrac{\cosh x}{\sinh x} = -\operatorname{csch} x\, \coth x$

(d) $\dfrac{d}{dx}(\operatorname{sech} x) = \dfrac{d}{dx}\left(\dfrac{1}{\cosh x}\right) = -\dfrac{\sinh x}{\cosh^2 x} = -\dfrac{1}{\cosh x} \cdot \dfrac{\sinh x}{\cosh x} = -\operatorname{sech} x\, \tanh x$

(e) $\dfrac{d}{dx}(\coth x) = \dfrac{d}{dx}\left(\dfrac{\cosh x}{\sinh x}\right) = \dfrac{\sinh x\, \sinh x - \cosh x\, \cosh x}{\sinh^2 x} = \dfrac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\dfrac{1}{\sinh^2 x} = -\operatorname{csch}^2 x$

29. Let $y = \sinh^{-1} x$. Then $\sinh y = x$ and, by Example 1(a), $\cosh^2 y - \sinh^2 y = 1 \;\Rightarrow\;$ [with $\cosh y > 0$] $\cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}$. So by Exercise 13, $e^y = \sinh y + \cosh y = x + \sqrt{1 + x^2} \;\Rightarrow\; y = \ln\!\left(x + \sqrt{1 + x^2}\right)$.

30. Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \ge 0$, so $\sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}$. So, by Exercise 13, $e^y = \cosh y + \sinh y = x + \sqrt{x^2 - 1} \;\Rightarrow\; y = \ln\!\left(x + \sqrt{x^2 - 1}\right)$.

Another method: Write $x = \cosh y = \frac{1}{2}\left(e^y + e^{-y}\right)$ and solve a quadratic, as in Example 3.

31. (a) Let $y = \tanh^{-1} x$. Then $x = \tanh y = \dfrac{\sinh y}{\cosh y} = \dfrac{(e^y - e^{-y})/2}{(e^y + e^{-y})/2} \cdot \dfrac{e^y}{e^y} = \dfrac{e^{2y} - 1}{e^{2y} + 1} \;\Rightarrow\; xe^{2y} + x = e^{2y} - 1 \;\Rightarrow$

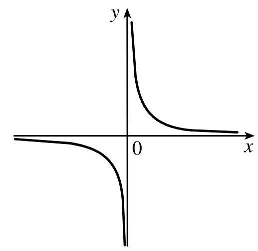
$1 + x = e^{2y} - xe^{2y} \;\Rightarrow\; 1 + x = e^{2y}(1 - x) \;\Rightarrow\; e^{2y} = \dfrac{1+x}{1-x} \;\Rightarrow\; 2y = \ln\!\left(\dfrac{1+x}{1-x}\right) \;\Rightarrow$

$y = \dfrac{1}{2}\ln\!\left(\dfrac{1+x}{1-x}\right)$.

(b) Let $y = \tanh^{-1} x$. Then $x = \tanh y$, so from Exercise 22 we have

$$e^{2y} = \frac{1 + \tanh y}{1 - \tanh y} = \frac{1 + x}{1 - x} \quad \Rightarrow \quad 2y = \ln\left(\frac{1 + x}{1 - x}\right) \quad \Rightarrow \quad y = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right).$$

32. (a) (i) $y = \operatorname{csch}^{-1} x \quad \Leftrightarrow \quad \operatorname{csch} y = x \quad (x \neq 0)$



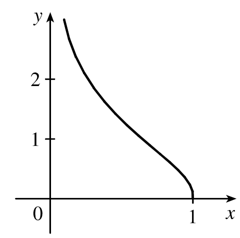
(ii) We sketch the graph of csch^{-1} by reflecting the graph of csch (see Exercise 26) about the line $y = x$.

(iii) Let $y = \operatorname{csch}^{-1} x$. Then $x = \operatorname{csch} y = \dfrac{2}{e^y - e^{-y}} \quad \Rightarrow \quad xe^y - xe^{-y} = 2 \quad \Rightarrow$

$x(e^y)^2 - 2e^y - x = 0 \quad \Rightarrow \quad e^y = \dfrac{1 \pm \sqrt{x^2 + 1}}{x}$. But $e^y > 0$, so for $x > 0$,

$e^y = \dfrac{1 + \sqrt{x^2 + 1}}{x}$ and for $x < 0$, $e^y = \dfrac{1 - \sqrt{x^2 + 1}}{x}$. Thus, $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{x^2 + 1}}{|x|}\right)$.

(b) (i) $y = \operatorname{sech}^{-1} x \quad \Leftrightarrow \quad \operatorname{sech} y = x$ and $y > 0$.



(ii) We sketch the graph of sech^{-1} by reflecting the graph of sech (see Exercise 26) about the line $y = x$.

(iii) Let $y = \operatorname{sech}^{-1} x$, so $x = \operatorname{sech} y = \dfrac{2}{e^y + e^{-y}} \quad \Rightarrow \quad xe^y + xe^{-y} = 2 \quad \Rightarrow$

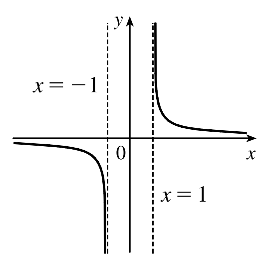
$x\,(e^y)^2 - 2e^y + x = 0 \quad \Leftrightarrow \quad e^y = \dfrac{1 \pm \sqrt{1 - x^2}}{x}$. But $y > 0 \quad \Rightarrow \quad e^y > 1$.

This rules out the minus sign because $\dfrac{1 - \sqrt{1 - x^2}}{x} > 1 \quad \Leftrightarrow \quad 1 - \sqrt{1 - x^2} > x \quad \Leftrightarrow \quad 1 - x > \sqrt{1 - x^2} \quad \Leftrightarrow$

$1 - 2x + x^2 > 1 - x^2 \quad \Leftrightarrow \quad x^2 > x \quad \Leftrightarrow \quad x > 1$, but $x = \operatorname{sech} y \leq 1$.

Thus, $e^y = \dfrac{1 + \sqrt{1 - x^2}}{x} \quad \Rightarrow \quad \operatorname{sech}^{-1} x = \ln\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right)$.

(c) (i) $y = \coth^{-1} x \quad \Leftrightarrow \quad \coth y = x$



(ii) We sketch the graph of $\coth^{-1}$ by reflecting the graph of coth (see Exercise 26) about the line $y = x$.

(iii) Let $y = \coth^{-1} x$. Then $x = \coth y = \dfrac{e^y + e^{-y}}{e^y - e^{-y}} \quad \Rightarrow$

$xe^y - xe^{-y} = e^y + e^{-y} \quad \Rightarrow \quad (x - 1)e^y = (x + 1)e^{-y} \quad \Rightarrow \quad e^{2y} = \dfrac{x + 1}{x - 1} \quad \Rightarrow$

$2y = \ln\dfrac{x + 1}{x - 1} \quad \Rightarrow \quad \coth^{-1} x = \dfrac{1}{2}\ln\dfrac{x + 1}{x - 1}$

33. (a) Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0 \quad \Rightarrow \quad \sinh y\,\dfrac{dy}{dx} = 1 \quad \Rightarrow$

$\dfrac{dy}{dx} = \dfrac{1}{\sinh y} = \dfrac{1}{\sqrt{\cosh^2 y - 1}} = \dfrac{1}{\sqrt{x^2 - 1}}$ [since $\sinh y \geq 0$ for $y \geq 0$]. *Or:* Use Formula 4.

(b) Let $y = \tanh^{-1} x$. Then $\tanh y = x \quad \Rightarrow \quad \operatorname{sech}^2 y\,\dfrac{dy}{dx} = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{\operatorname{sech}^2 y} = \dfrac{1}{1 - \tanh^2 y} = \dfrac{1}{1 - x^2}$.

Or: Use Formula 5.

(c) Let $y = \coth^{-1} x$. Then $\coth y = x \quad \Rightarrow \quad -\operatorname{csch}^2 y \,\dfrac{dy}{dx} = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch}^2 y} = \dfrac{1}{1 - \coth^2 y} = \dfrac{1}{1 - x^2}$ by Exercise 17.

34. (a) Let $y = \operatorname{sech}^{-1} x$. Then $\operatorname{sech} y = x \quad \Rightarrow \quad -\operatorname{sech} y \,\tanh y \,\dfrac{dy}{dx} = 1 \quad \Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{sech} y \,\tanh y} = -\dfrac{1}{\operatorname{sech} y \sqrt{1 - \operatorname{sech}^2 y}} = -\dfrac{1}{x\sqrt{1 - x^2}}$. [Note that $y > 0$ and so $\tanh y > 0$.]

(b) Let $y = \operatorname{csch}^{-1} x$. Then $\operatorname{csch} y = x \quad \Rightarrow \quad -\operatorname{csch} y \,\coth y \,\dfrac{dy}{dx} = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \,\coth y}$. By Exercise 17, $\coth y = \pm\sqrt{\operatorname{csch}^2 y + 1} = \pm\sqrt{x^2 + 1}$. If $x > 0$, then $\coth y > 0$, so $\coth y = \sqrt{x^2 + 1}$. If $x < 0$, then $\coth y < 0$, so $\coth y = -\sqrt{x^2 + 1}$. In either case we have $\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \,\coth y} = -\dfrac{1}{|x|\sqrt{x^2 + 1}}$.

35. $f(x) = \cosh 3x \quad \Rightarrow \quad f'(x) = \sinh(3x) \cdot \dfrac{d}{dx}(3x) = \sinh(3x) \cdot 3 = 3\sinh 3x$

36. $f(x) = e^x \cosh x \quad \overset{\text{PR}}{\Rightarrow} \quad f'(x) = e^x \sinh x + (\cosh x)e^x = e^x(\sinh x + \cosh x)$, or, using Exercise 13, $e^x(e^x) = e^{2x}$.

37. $h(x) = \sinh(x^2) \quad \Rightarrow \quad h'(x) = \cosh(x^2)\,\dfrac{d}{dx}(x^2) = 2x\cosh(x^2)$

38. $g(x) = \sinh^2 x = (\sinh x)^2 \quad \Rightarrow \quad g'(x) = 2(\sinh x)^1 \dfrac{d}{dx}(\sinh x) = 2\sinh x \,\cosh x$, or, using Exercise 19, $\sinh 2x$.

39. $G(t) = \sinh(\ln t) \quad \Rightarrow \quad G'(t) = \cosh(\ln t)\,\dfrac{d}{dt}\ln t = \dfrac{1}{2}\left(e^{\ln t} + e^{-\ln t}\right)\left(\dfrac{1}{t}\right) = \dfrac{1}{2t}\left(t + \dfrac{1}{t}\right) = \dfrac{1}{2t}\left(\dfrac{t^2 + 1}{t}\right) = \dfrac{t^2 + 1}{2t^2}$

Or: $G(t) = \sinh(\ln t) = \dfrac{1}{2}(e^{\ln t} - e^{-\ln t}) = \dfrac{1}{2}\left(t - \dfrac{1}{t}\right) \quad \Rightarrow \quad G'(t) = \dfrac{1}{2}\left(1 + \dfrac{1}{t^2}\right) = \dfrac{t^2 + 1}{2t^2}$

40. $F(t) = \ln(\sinh t) \quad \Rightarrow \quad F'(t) = \dfrac{1}{\sinh t}\,\dfrac{d}{dt}\sinh t = \dfrac{1}{\sinh t}\cosh t = \coth t$

41. $f(x) = \tanh\sqrt{x} \quad \Rightarrow \quad f'(x) = \operatorname{sech}^2\sqrt{x}\,\dfrac{d}{dx}\sqrt{x} = \operatorname{sech}^2\sqrt{x}\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{\operatorname{sech}^2\sqrt{x}}{2\sqrt{x}}$

42. $H(v) = e^{\tanh 2v} \quad \Rightarrow \quad H'(v) = e^{\tanh 2v} \cdot \dfrac{d}{dv}\tanh 2v = e^{\tanh 2v} \cdot \operatorname{sech}^2(2v) \cdot 2 = 2e^{\tanh 2v}\operatorname{sech}^2(2v)$

43. $y = \operatorname{sech} x \,\tanh x \quad \overset{\text{PR}}{\Rightarrow} \quad y' = \operatorname{sech} x \cdot \operatorname{sech}^2 x + \tanh x \cdot (-\operatorname{sech} x \,\tanh x) = \operatorname{sech}^3 x - \operatorname{sech} x \,\tanh^2 x$

44. $y = \operatorname{sech}(\tanh x) \quad \Rightarrow \quad y' = -\operatorname{sech}(\tanh x)\,\tanh(\tanh x) \cdot \dfrac{d}{dx}(\tanh x) = -\operatorname{sech}(\tanh x)\,\tanh(\tanh x) \cdot \operatorname{sech}^2 x$

45. $g(t) = t\coth\sqrt{t^2 + 1} \quad \overset{\text{PR}}{\Rightarrow}$

$$g'(t) = t\left[-\operatorname{csch}^2\sqrt{t^2 + 1}\left(\tfrac{1}{2}(t^2 + 1)^{-1/2} \cdot 2t\right)\right] + \left(\coth\sqrt{t^2 + 1}\right)(1) = \coth\sqrt{t^2 + 1} - \frac{t^2}{\sqrt{t^2 + 1}}\operatorname{csch}^2\sqrt{t^2 + 1}$$

46. $f(t) = \dfrac{1+\sinh t}{1-\sinh t} \overset{\text{QR}}{\Rightarrow}$

$$f'(t) = \frac{(1-\sinh t)\cosh t - (1+\sinh t)(-\cosh t)}{(1-\sinh t)^2} = \frac{\cosh t - \sinh t\cosh t + \cosh t + \sinh t\cosh t}{(1-\sinh t)^2}$$
$$= \frac{2\cosh t}{(1-\sinh t)^2}$$

47. $f(x) = \sinh^{-1}(-2x) \quad\Rightarrow\quad f'(x) = \dfrac{1}{\sqrt{1+(-2x)^2}} \cdot \dfrac{d}{dx}(-2x) = -\dfrac{2}{\sqrt{1+4x^2}}$

48. $g(x) = \tanh^{-1}(x^3) \quad\Rightarrow\quad g'(x) = \dfrac{1}{1-(x^3)^2} \cdot \dfrac{d}{dx}(x^3) = \dfrac{3x^2}{1-x^6}$

49. $y = \cosh^{-1}(\sec\theta) \quad\Rightarrow$

$$y' = \frac{1}{\sqrt{\sec^2\theta - 1}} \cdot \frac{d}{d\theta}(\sec\theta) = \frac{1}{\sqrt{\tan^2\theta}} \cdot \sec\theta\tan\theta = \frac{1}{\tan\theta}\cdot\sec\theta\tan\theta \quad [\text{since } 0 \le \theta < \pi/2] \quad = \sec\theta$$

50. $y = \operatorname{sech}^{-1}(\sin\theta) \quad\Rightarrow$

$$y' = -\frac{1}{\sin\theta\sqrt{1-\sin^2\theta}} \cdot \frac{d}{d\theta}(\sin\theta) = -\frac{1}{\sin\theta\sqrt{\cos^2\theta}} \cdot \cos\theta$$
$$= -\frac{1}{\sin\theta\cdot\cos\theta}\cdot\cos\theta \quad [\text{since } 0 < \theta < \pi/2] \quad = -\frac{1}{\sin\theta} = -\csc\theta$$

51. $G(u) = \cosh^{-1}\sqrt{1+u^2} \quad\Rightarrow$

$$G'(u) = \frac{1}{\sqrt{\left(\sqrt{1+u^2}\right)^2 - 1}} \cdot \frac{d}{du}\left(\sqrt{1+u^2}\right) = \frac{1}{\sqrt{(1+u^2)-1}} \cdot \frac{2u}{2\sqrt{1+u^2}} = \frac{u}{\sqrt{u^2}\cdot\sqrt{1+u^2}}$$
$$= \frac{u}{u\sqrt{1+u^2}} \quad [\text{since } u > 0] \quad = \frac{1}{\sqrt{1+u^2}}$$

52. $y = x\tanh^{-1}x + \ln\sqrt{1-x^2} = x\tanh^{-1}x + \frac{1}{2}\ln(1-x^2) \quad\Rightarrow$

$$y' = \tanh^{-1}x + \frac{x}{1-x^2} + \frac{1}{2}\left(\frac{1}{1-x^2}\right)(-2x) = \tanh^{-1}x$$

53. $y = x\sinh^{-1}(x/3) - \sqrt{9+x^2} \quad\Rightarrow$

$$y' = \sinh^{-1}\left(\frac{x}{3}\right) + x\frac{1/3}{\sqrt{1+(x/3)^2}} - \frac{2x}{2\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right) + \frac{x}{\sqrt{9+x^2}} - \frac{x}{\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right)$$

54. $\dfrac{1+\tanh x}{1-\tanh x} = \dfrac{1+(\sinh x)/\cosh x}{1-(\sinh x)/\cosh x} = \dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{e^x}{e^{-x}}$ [by Exercises 13 and 14] $= e^{2x}$, so

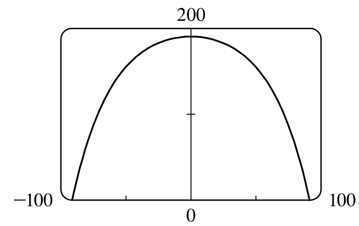
$\sqrt[4]{\dfrac{1+\tanh x}{1-\tanh x}} = \sqrt[4]{e^{2x}} = e^{x/2}$. Thus, $\dfrac{d}{dx}\sqrt[4]{\dfrac{1+\tanh x}{1-\tanh x}} = \dfrac{d}{dx}(e^{x/2}) = \dfrac{1}{2}e^{x/2}$.

55. $\dfrac{d}{dx}\arctan(\tanh x) = \dfrac{1}{1+(\tanh x)^2}\dfrac{d}{dx}(\tanh x) = \dfrac{\operatorname{sech}^2 x}{1+\tanh^2 x} = \dfrac{1/\cosh^2 x}{1+(\sinh^2 x)/\cosh^2 x}$

$$= \frac{1}{\cosh^2 x + \sinh^2 x} = \frac{1}{\cosh 2x} \quad [\text{by Exercise 20}] \quad = \operatorname{sech} 2x$$

56. (a) Let $a = 0.03291765$. A graph of the central curve,

$y = f(x) = 211.49 - 20.96\cosh ax$, is shown.



(b) $f(0) = 211.49 - 20.96\cosh 0 = 211.49 - 20.96(1) = 190.53$ m.

(c) $y = 100 \quad\Rightarrow\quad 100 = 211.49 - 20.96\cosh ax \quad\Rightarrow$

$20.96\cosh ax = 111.49 \quad\Rightarrow\quad \cosh ax = \dfrac{111.49}{20.96} \quad\Rightarrow$

$ax = \pm\cosh^{-1}\dfrac{111.49}{20.96} \quad\Rightarrow\quad x = \pm\dfrac{1}{a}\cosh^{-1}\dfrac{111.49}{20.96} \approx \pm 71.56$ m. The points are approximately $(\pm 71.56, 100)$.

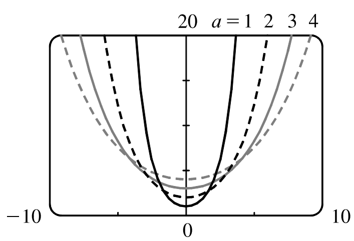
(d) $f(x) = 211.49 - 20.96\cosh ax \quad\Rightarrow\quad f'(x) = -20.96\sinh ax \cdot a$.

$f'\left(\pm\dfrac{1}{a}\cosh^{-1}\dfrac{111.49}{20.96}\right) = -20.96a\sinh\left[a\left(\pm\dfrac{1}{a}\cosh^{-1}\dfrac{111.49}{20.96}\right)\right] = -20.96a\sinh\left(\pm\cosh^{-1}\dfrac{111.49}{20.96}\right) \approx \mp 3.6.$

So the slope at $(71.56, 100)$ is about -3.6 and the slope at $(-71.56, 100)$ is about 3.6.

57. As the depth d of the water gets large, the fraction $\dfrac{2\pi d}{L}$ gets large, and from Figure 5 or Exercise 27(a), $\tanh\left(\dfrac{2\pi d}{L}\right)$ approaches 1. Thus, $v = \sqrt{\dfrac{gL}{2\pi}\tanh\left(\dfrac{2\pi d}{L}\right)} \approx \sqrt{\dfrac{gL}{2\pi}(1)} = \sqrt{\dfrac{gL}{2\pi}}$.

58.



For $y = a\cosh(x/a)$ with $a > 0$, we have the y-intercept equal to a. As a increases, the graph flattens.

59. (a) $y = 20\cosh(x/20) - 15 \quad\Rightarrow\quad y' = 20\sinh(x/20)\cdot\frac{1}{20} = \sinh(x/20)$. Since the right pole is positioned at $x = 7$, we have $y'(7) = \sinh\frac{7}{20} \approx 0.3572$.

(b) If α is the angle between the tangent line and the x-axis, then $\tan\alpha =$ slope of the line $= \sinh\frac{7}{20}$, so $\alpha = \tan^{-1}\left(\sinh\frac{7}{20}\right) \approx 0.343$ rad $\approx 19.66^\circ$. Thus, the angle between the line and the pole is $\theta = 90^\circ - \alpha \approx 70.34^\circ$.

60. We differentiate the function twice, then substitute into the differential equation: $y = \dfrac{T}{\rho g}\cosh\dfrac{\rho g x}{T} \quad\Rightarrow$

$\dfrac{dy}{dx} = \dfrac{T}{\rho g}\sinh\left(\dfrac{\rho g x}{T}\right)\dfrac{\rho g}{T} = \sinh\dfrac{\rho g x}{T} \quad\Rightarrow\quad \dfrac{d^2y}{dx^2} = \cosh\left(\dfrac{\rho g x}{T}\right)\dfrac{\rho g}{T} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$. We evaluate the two sides separately: LHS $= \dfrac{d^2y}{dx^2} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$ and RHS $= \dfrac{\rho g}{T}\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \dfrac{\rho g}{T}\sqrt{1 + \sinh^2\dfrac{\rho g x}{T}} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$, by the identity proved in Example 1(a).

61. (a) From Exercise 60, the shape of the cable is given by $y = f(x) = \dfrac{T}{\rho g}\cosh\left(\dfrac{\rho g x}{T}\right)$. The shape is symmetric about the y-axis, so the lowest point is $(0, f(0)) = \left(0, \dfrac{T}{\rho g}\right)$ and the poles are at $x = \pm 100$. We want to find T when the lowest point is 60 m, so $\dfrac{T}{\rho g} = 60 \;\Rightarrow\; T = 60\rho g = (60\text{ m})(2\text{ kg/m})(9.8\text{ m/s}^2) = 1176\,\dfrac{\text{kg-m}}{\text{s}^2}$, or 1176 N (newtons).

The height of each pole is $f(100) = \dfrac{T}{\rho g}\cosh\left(\dfrac{\rho g \cdot 100}{T}\right) = 60\cosh\left(\dfrac{100}{60}\right) \approx 164.50\text{ m}.$

(b) If the tension is doubled from T to $2T$, then the low point is doubled since $\dfrac{T}{\rho g} = 60 \;\Rightarrow\; \dfrac{2T}{\rho g} = 120$. The height of the poles is now $f(100) = \dfrac{2T}{\rho g}\cosh\left(\dfrac{\rho g \cdot 100}{2T}\right) = 120\cosh\left(\dfrac{100}{120}\right) \approx 164.13$ m, just a slight decrease.

62. (a) $\displaystyle\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\sqrt{\frac{mg}{k}}\tanh\left(t\sqrt{\frac{gk}{m}}\right) = \sqrt{\frac{mg}{k}}\lim_{t\to\infty}\tanh\left(t\sqrt{\frac{gk}{m}}\right) = \sqrt{\frac{mg}{k}}\cdot 1 \quad \left[\begin{smallmatrix}\text{as } t\to\infty,\\ t\sqrt{gk/m}\to\infty\end{smallmatrix}\right] \quad = \sqrt{\frac{mg}{k}}$

(b) Belly-to-earth: $g = 9.8$, $k = 0.515$, $m = 60$, so the terminal velocity is $\sqrt{\dfrac{60(9.8)}{0.515}} \approx 33.79$ m/s.

Feet-first: $g = 9.8$, $k = 0.067$, $m = 60$, so the terminal velocity is $\sqrt{\dfrac{60(9.8)}{0.067}} \approx 93.68$ m/s.

63. (a) $y = A\sinh mx + B\cosh mx \;\Rightarrow\; y' = mA\cosh mx + mB\sinh mx \;\Rightarrow$

$y'' = m^2 A\sinh mx + m^2 B\cosh mx = m^2(A\sinh mx + B\cosh mx) = m^2 y$

(b) From part (a), a solution of $y'' = 9y$ is $y(x) = A\sinh 3x + B\cosh 3x$. Now $-4 = y(0) = A\sinh 0 + B\cosh 0 = B$, so $B = -4$. Also, $y'(x) = 3A\cosh 3x - 12\sinh 3x$, so $6 = y'(0) = 3A \;\Rightarrow\; A = 2$. Thus, $y = 2\sinh 3x - 4\cosh 3x$.

64. $\cosh x = \cosh[\ln(\sec\theta + \tan\theta)] = \dfrac{1}{2}\left[e^{\ln(\sec\theta+\tan\theta)} + e^{-\ln(\sec\theta+\tan\theta)}\right] = \dfrac{1}{2}\left[\sec\theta + \tan\theta + \dfrac{1}{\sec\theta+\tan\theta}\right]$

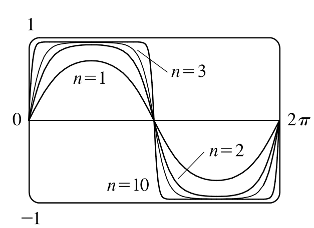
$= \dfrac{1}{2}\left[\sec\theta + \tan\theta + \dfrac{\sec\theta - \tan\theta}{(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}\right] = \dfrac{1}{2}\left[\sec\theta + \tan\theta + \dfrac{\sec\theta-\tan\theta}{\sec^2\theta - \tan^2\theta}\right]$

$= \frac{1}{2}(\sec\theta + \tan\theta + \sec\theta - \tan\theta) = \sec\theta$

65. The tangent to $y = \cosh x$ has slope 1 when $y' = \sinh x = 1 \;\Rightarrow\; x = \sinh^{-1} 1 = \ln\left(1+\sqrt{2}\right)$, by Equation 3.

Since $\sinh x = 1$ and $y = \cosh x = \sqrt{1+\sinh^2 x}$, we have $\cosh x = \sqrt{2}$. The point is $\left(\ln\left(1+\sqrt{2}\right), \sqrt{2}\right)$.

66. $f_n(x) = \tanh(n\sin x)$, where n is a positive integer. Note that $f_n(x+2\pi) = f_n(x)$; that is, f_n is periodic with period 2π. Also, from Figure 3, $-1 < \tanh x < 1$, so we can choose a viewing rectangle of $[0, 2\pi]\times[-1, 1]$. From the graph, we see that $f_n(x)$ becomes more rectangular looking as n increases. As n becomes large, the graph of f_n approaches the graph of $y = 1$ on the intervals $(2k\pi, (2k+1)\pi)$ and $y = -1$ on the intervals $((2k-1)\pi, 2k\pi)$.



67. If $ae^x + be^{-x} = \alpha\cosh(x+\beta)$ [or $\alpha\sinh(x+\beta)$], then

$ae^x + be^{-x} = \frac{\alpha}{2}\left(e^{x+\beta} \pm e^{-x-\beta}\right) = \frac{\alpha}{2}\left(e^x e^\beta \pm e^{-x}e^{-\beta}\right) = \left(\frac{\alpha}{2}e^\beta\right)e^x \pm \left(\frac{\alpha}{2}e^{-\beta}\right)e^{-x}$. Comparing coefficients of e^x and e^{-x}, we have $a = \frac{\alpha}{2}e^\beta$ **(1)** and $b = \pm\frac{\alpha}{2}e^{-\beta}$ **(2)**. We need to find α and β. Dividing equation **(1)** by equation **(2)** gives us $\frac{a}{b} = \pm e^{2\beta} \quad \Rightarrow \quad (\star) \quad 2\beta = \ln\left(\pm\frac{a}{b}\right) \quad \Rightarrow \quad \beta = \frac{1}{2}\ln\left(\pm\frac{a}{b}\right)$. Solving equations **(1)** and **(2)** for e^{β} gives us

$e^{\beta} = \dfrac{2a}{\alpha}$ and $e^{\beta} = \pm\dfrac{\alpha}{2b}$, so $\dfrac{2a}{\alpha} = \pm\dfrac{\alpha}{2b} \quad \Rightarrow \quad \alpha^2 = \pm 4ab \quad \Rightarrow \quad \alpha = 2\sqrt{\pm ab}$.

$(\star)$ If $\frac{a}{b} > 0$, we use the $+$ sign and obtain a cosh function, whereas if $\frac{a}{b} < 0$, we use the $-$ sign and obtain a sinh function.

In summary, if a and b have the same sign, we have $ae^{x} + be^{-x} = 2\sqrt{ab}\cosh\left(x + \frac{1}{2}\ln\frac{a}{b}\right)$, whereas, if a and b have the opposite sign, then $ae^{x} + be^{-x} = 2\sqrt{-ab}\sinh\left(x + \frac{1}{2}\ln\left(-\frac{a}{b}\right)\right)$.

3 Review

TRUE-FALSE QUIZ

1. True. This is the Sum Rule.

2. False. See the warning before the Product Rule.

3. True. This is the Chain Rule.

4. True. $\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{d}{dx}[f(x)]^{1/2} = \dfrac{1}{2}[f(x)]^{-1/2} f'(x) = \dfrac{f'(x)}{2\sqrt{f(x)}}$

5. False. $\dfrac{d}{dx} f(\sqrt{x}) = f'(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2} = \dfrac{f'(\sqrt{x})}{2\sqrt{x}}$, which is not $\dfrac{f'(x)}{2\sqrt{x}}$.

6. False. $y = e^2$ is a constant, so $y' = 0$, not $2e$.

7. False. $\dfrac{d}{dx}(10^x) = 10^x \ln 10$, which is not equal to $x10^{x-1}$.

8. False. $\ln 10$ is a constant, so its derivative, $\dfrac{d}{dx}(\ln 10)$, is 0, not $\frac{1}{10}$.

9. True. $\dfrac{d}{dx}(\tan^2 x) = 2\tan x \sec^2 x$, and $\dfrac{d}{dx}(\sec^2 x) = 2\sec x(\sec x \tan x) = 2\tan x \sec^2 x$.

Or: $\dfrac{d}{dx}(\sec^2 x) = \dfrac{d}{dx}(1 + \tan^2 x) = \dfrac{d}{dx}(\tan^2 x)$.

10. False. $f(x) = \left|x^2 + x\right| = x^2 + x$ for $x \geq 0$ or $x \leq -1$ and $\left|x^2 + x\right| = -(x^2 + x)$ for $-1 < x < 0$.
So $f'(x) = 2x + 1$ for $x > 0$ or $x < -1$ and $f'(x) = -(2x + 1)$ for $-1 < x < 0$. But $|2x + 1| = 2x + 1$ for $x \geq -\frac{1}{2}$ and $|2x + 1| = -2x - 1$ for $x < -\frac{1}{2}$.

11. True. If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$, which is a polynomial.

12. True. $f(x) = (x^6 - x^4)^5$ is a polynomial of degree 30, so its 31st derivative, $f^{(31)}(x)$, is 0.

13. True. If $r(x) = \dfrac{p(x)}{q(x)}$, then $r'(x) = \dfrac{q(x)p'(x) - p(x)q'(x)}{[q(x)]^2}$, which is a quotient of polynomials, that is, a rational function.

14. False. A tangent line to the parabola $y = x^2$ has slope $dy/dx = 2x$, so at $(-2, 4)$ the slope of the tangent is $2(-2) = -4$ and an equation of the tangent line is $y - 4 = -4(x + 2)$. [The given equation, $y - 4 = 2x(x + 2)$, is not even linear!]

15. True. $g(x) = x^5 \;\Rightarrow\; g'(x) = 5x^4 \;\Rightarrow\; g'(2) = 5(2)^4 = 80$, and by the definition of the derivative, $\displaystyle\lim_{x\to 2} \frac{g(x) - g(2)}{x - 2} = g'(2) = 5(2)^4 = 80$.

EXERCISES

1. $y = (x^2 + x^3)^4 \;\Rightarrow\; y' = 4(x^2 + x^3)^3(2x + 3x^2) = 4(x^2)^3(1 + x)^3 x(2 + 3x) = 4x^7(x + 1)^3(3x + 2)$

2. $y = \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt[5]{x^3}} = x^{-1/2} - x^{-3/5} \;\Rightarrow\; y' = -\dfrac{1}{2}x^{-3/2} + \dfrac{3}{5}x^{-8/5}$ or $\dfrac{3}{5x\sqrt[5]{x^3}} - \dfrac{1}{2x\sqrt{x}}$ or $\dfrac{1}{10}x^{-8/5}(-5x^{1/10} + 6)$

3. $y = \dfrac{x^2 - x + 2}{\sqrt{x}} = x^{3/2} - x^{1/2} + 2x^{-1/2} \;\Rightarrow\; y' = \dfrac{3}{2}x^{1/2} - \dfrac{1}{2}x^{-1/2} - x^{-3/2} = \dfrac{3}{2}\sqrt{x} - \dfrac{1}{2\sqrt{x}} - \dfrac{1}{\sqrt{x^3}}$

4. $y = \dfrac{\tan x}{1 + \cos x} \;\Rightarrow\; y' = \dfrac{(1 + \cos x)\sec^2 x - \tan x(-\sin x)}{(1 + \cos x)^2} = \dfrac{(1 + \cos x)\sec^2 x + \tan x\,\sin x}{(1 + \cos x)^2}$

5. $y = x^2 \sin \pi x \;\Rightarrow\; y' = x^2(\cos \pi x)\pi + (\sin \pi x)(2x) = x(\pi x \cos \pi x + 2 \sin \pi x)$

6. $y = x \cos^{-1} x \;\Rightarrow\; y' = x\left(-\dfrac{1}{\sqrt{1 - x^2}}\right) + (\cos^{-1} x)(1) = \cos^{-1} x - \dfrac{x}{\sqrt{1 - x^2}}$

7. $y = \dfrac{t^4 - 1}{t^4 + 1} \;\Rightarrow\; y' = \dfrac{(t^4 + 1)4t^3 - (t^4 - 1)4t^3}{(t^4 + 1)^2} = \dfrac{4t^3[(t^4 + 1) - (t^4 - 1)]}{(t^4 + 1)^2} = \dfrac{8t^3}{(t^4 + 1)^2}$

8. $\dfrac{d}{dx}(xe^y) = \dfrac{d}{dx}(y \sin x) \;\Rightarrow\; xe^y y' + e^y \cdot 1 = y \cos x + \sin x \cdot y' \;\Rightarrow\; xe^y y' - \sin x \cdot y' = y \cos x - e^y \;\Rightarrow$

$(xe^y - \sin x)y' = y \cos x - e^y \;\Rightarrow\; y' = \dfrac{y \cos x - e^y}{xe^y - \sin x}$

9. $y = \ln(x \ln x) \;\Rightarrow\; y' = \dfrac{1}{x \ln x}(x \ln x)' = \dfrac{1}{x \ln x}\left(x \cdot \dfrac{1}{x} + \ln x \cdot 1\right) = \dfrac{1 + \ln x}{x \ln x}$

Another method: $y = \ln(x \ln x) = \ln x + \ln \ln x \;\Rightarrow\; y' = \dfrac{1}{x} + \dfrac{1}{\ln x} \cdot \dfrac{1}{x} = \dfrac{\ln x + 1}{x \ln x}$

10. $y = e^{mx} \cos nx \;\Rightarrow$

$y' = e^{mx}(\cos nx)' + \cos nx\,(e^{mx})' = e^{mx}(-\sin nx \cdot n) + \cos nx\,(e^{mx} \cdot m) = e^{mx}(m \cos nx - n \sin nx)$

11. $y = \sqrt{x}\cos\sqrt{x} \quad \Rightarrow$

$$y' = \sqrt{x}\,\left(\cos\sqrt{x}\right)' + \cos\sqrt{x}\,\left(\sqrt{x}\right)' = \sqrt{x}\left[-\sin\sqrt{x}\left(\tfrac{1}{2}x^{-1/2}\right)\right] + \cos\sqrt{x}\left(\tfrac{1}{2}x^{-1/2}\right)$$

$$= \tfrac{1}{2}x^{-1/2}\left(-\sqrt{x}\,\sin\sqrt{x} + \cos\sqrt{x}\right) = \frac{\cos\sqrt{x} - \sqrt{x}\,\sin\sqrt{x}}{2\sqrt{x}}$$

12. $y = (\arcsin 2x)^2 \quad \Rightarrow \quad y' = 2(\arcsin 2x)\cdot(\arcsin 2x)' = 2\arcsin 2x \cdot \dfrac{1}{\sqrt{1-(2x)^2}}\cdot 2 = \dfrac{4\arcsin 2x}{\sqrt{1-4x^2}}$

13. $y = \dfrac{e^{1/x}}{x^2} \quad \Rightarrow \quad y' = \dfrac{x^2(e^{1/x})' - e^{1/x}\left(x^2\right)'}{(x^2)^2} = \dfrac{x^2(e^{1/x})(-1/x^2) - e^{1/x}(2x)}{x^4} = \dfrac{-e^{1/x}(1+2x)}{x^4}$

14. $y = \ln\sec x \quad \Rightarrow \quad y' = \dfrac{1}{\sec x}\dfrac{d}{dx}(\sec x) = \dfrac{1}{\sec x}(\sec x\,\tan x) = \tan x$

15. $\dfrac{d}{dx}(y + x\cos y) = \dfrac{d}{dx}(x^2 y) \quad \Rightarrow \quad y' + x(-\sin y\cdot y') + \cos y\cdot 1 = x^2y' + y\cdot 2x \quad \Rightarrow$

$$y' - x\sin y\cdot y' - x^2y' = 2xy - \cos y \quad \Rightarrow \quad (1 - x\sin y - x^2)y' = 2xy - \cos y \quad \Rightarrow \quad y' = \frac{2xy - \cos y}{1 - x\sin y - x^2}$$

16. $y = \left(\dfrac{u-1}{u^2+u+1}\right)^4 \quad \Rightarrow$

$$y' = 4\left(\frac{u-1}{u^2+u+1}\right)^3\frac{d}{du}\left(\frac{u-1}{u^2+u+1}\right) = 4\left(\frac{u-1}{u^2+u+1}\right)^3\frac{(u^2+u+1)(1)-(u-1)(2u+1)}{(u^2+u+1)^2}$$

$$= \frac{4(u-1)^3}{(u^2+u+1)^3}\frac{u^2+u+1-2u^2+u+1}{(u^2+u+1)^2} = \frac{4(u-1)^3(-u^2+2u+2)}{(u^2+u+1)^5}$$

17. $y = \sqrt{\arctan x} \quad \Rightarrow \quad y' = \dfrac{1}{2}(\arctan x)^{-1/2}\dfrac{d}{dx}(\arctan x) = \dfrac{1}{2\sqrt{\arctan x}\,(1+x^2)}$

18. $y = \cot(\csc x) \quad \Rightarrow \quad y' = -\csc^2(\csc x)\dfrac{d}{dx}(\csc x) = -\csc^2(\csc x)\cdot(-\csc x\,\cot x) = \csc^2(\csc x)\,\csc x\,\cot x$

19. $y = \tan\left(\dfrac{t}{1+t^2}\right) \quad \Rightarrow$

$$y' = \sec^2\left(\frac{t}{1+t^2}\right)\frac{d}{dt}\left(\frac{t}{1+t^2}\right) = \sec^2\left(\frac{t}{1+t^2}\right)\cdot\frac{(1+t^2)(1)-t(2t)}{(1+t^2)^2} = \frac{1-t^2}{(1+t^2)^2}\sec^2\left(\frac{t}{1+t^2}\right)$$

20. $y = e^{x\sec x} \quad \Rightarrow \quad y' = e^{x\sec x}\dfrac{d}{dx}(x\sec x) = e^{x\sec x}(x\sec x\,\tan x + \sec x\cdot 1) = \sec x\,e^{x\sec x}(x\tan x + 1)$

21. $y = 3^{x\ln x} \quad \Rightarrow \quad y' = 3^{x\ln x}(\ln 3)\dfrac{d}{dx}(x\ln x) = 3^{x\ln x}(\ln 3)\left(x\cdot\dfrac{1}{x} + \ln x\cdot 1\right) = 3^{x\ln x}(\ln 3)(1+\ln x)$

22. $y = \sec(1+x^2) \quad \Rightarrow \quad y' = 2x\sec(1+x^2)\,\tan(1+x^2)$

23. $y = (1-x^{-1})^{-1} \quad \Rightarrow$

$$y' = -1(1-x^{-1})^{-2}[-(-1x^{-2})] = -(1-1/x)^{-2}x^{-2} = -((x-1)/x)^{-2}x^{-2} = -(x-1)^{-2}$$

24. $y = \dfrac{1}{\sqrt[3]{x + \sqrt{x}}} = \left(x + \sqrt{x}\right)^{-1/3} \quad \Rightarrow \quad y' = -\frac{1}{3}\left(x + \sqrt{x}\right)^{-4/3}\left(1 + \dfrac{1}{2\sqrt{x}}\right)$

25. $\sin(xy) = x^2 - y \quad \Rightarrow \quad \cos(xy)(xy' + y \cdot 1) = 2x - y' \quad \Rightarrow \quad x\cos(xy)y' + y' = 2x - y\cos(xy) \quad \Rightarrow$

$y'[x\cos(xy) + 1] = 2x - y\cos(xy) \quad \Rightarrow \quad y' = \dfrac{2x - y\cos(xy)}{x\cos(xy) + 1}$

26. $y = \sqrt{\sin\sqrt{x}} \quad \Rightarrow \quad y' = \frac{1}{2}\left(\sin\sqrt{x}\right)^{-1/2}\left(\cos\sqrt{x}\right)\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$

27. $y = \log_5(1 + 2x) \quad \Rightarrow \quad y' = \dfrac{1}{(1 + 2x)\ln 5}\dfrac{d}{dx}(1 + 2x) = \dfrac{2}{(1 + 2x)\ln 5}$

28. $y = (\cos x)^x \quad \Rightarrow \quad \ln y = \ln(\cos x)^x = x\ln\cos x \quad \Rightarrow \quad \dfrac{y'}{y} = x \cdot \dfrac{1}{\cos x} \cdot (-\sin x) + \ln\cos x \cdot 1 \quad \Rightarrow$

$y' = (\cos x)^x(\ln\cos x - x\tan x)$

29. $y = \ln\sin x - \frac{1}{2}\sin^2 x \quad \Rightarrow \quad y' = \dfrac{1}{\sin x} \cdot \cos x - \frac{1}{2} \cdot 2\sin x \cdot \cos x = \cot x - \sin x\,\cos x$

30. $y = \dfrac{(x^2 + 1)^4}{(2x + 1)^3(3x - 1)^5} \quad \Rightarrow$

$\ln y = \ln\dfrac{(x^2 + 1)^4}{(2x + 1)^3(3x - 1)^5} = \ln(x^2 + 1)^4 - \ln[(2x + 1)^3(3x - 1)^5] = 4\ln(x^2 + 1) - [\ln(2x + 1)^3 + \ln(3x - 1)^5]$

$= 4\ln(x^2 + 1) - 3\ln(2x + 1) - 5\ln(3x - 1) \quad \Rightarrow$

$\dfrac{y'}{y} = 4 \cdot \dfrac{1}{x^2 + 1} \cdot 2x - 3 \cdot \dfrac{1}{2x + 1} \cdot 2 - 5 \cdot \dfrac{1}{3x - 1} \cdot 3 \quad \Rightarrow \quad y' = \dfrac{(x^2 + 1)^4}{(2x + 1)^3(3x - 1)^5}\left(\dfrac{8x}{x^2 + 1} - \dfrac{6}{2x + 1} - \dfrac{15}{3x - 1}\right).$

[The answer could be simplified to $y' = -\dfrac{(x^2 + 56x + 9)(x^2 + 1)^3}{(2x + 1)^4(3x - 1)^6}$, but this is unnecessary.]

31. $y = x\tan^{-1}(4x) \quad \Rightarrow \quad y' = x \cdot \dfrac{1}{1 + (4x)^2} \cdot 4 + \tan^{-1}(4x) \cdot 1 = \dfrac{4x}{1 + 16x^2} + \tan^{-1}(4x)$

32. $y = e^{\cos x} + \cos(e^x) \quad \Rightarrow \quad y' = e^{\cos x}(-\sin x) + [-\sin(e^x) \cdot e^x] = -\sin x\, e^{\cos x} - e^x\sin(e^x)$

33. $y = \ln|\sec 5x + \tan 5x| \quad \Rightarrow$

$y' = \dfrac{1}{\sec 5x + \tan 5x}(\sec 5x\,\tan 5x \cdot 5 + \sec^2 5x \cdot 5) = \dfrac{5\sec 5x\,(\tan 5x + \sec 5x)}{\sec 5x + \tan 5x} = 5\sec 5x$

34. $y = 10^{\tan \pi\theta} \quad \Rightarrow \quad y' = 10^{\tan \pi\theta} \cdot \ln 10 \cdot \sec^2 \pi\theta \cdot \pi = \pi(\ln 10)10^{\tan \pi\theta}\sec^2 \pi\theta$

35. $y = \cot(3x^2 + 5) \quad \Rightarrow \quad y' = -\csc^2(3x^2 + 5)(6x) = -6x\csc^2(3x^2 + 5)$

36. $y = \sqrt{t\ln(t^4)} \quad \Rightarrow$

$$y' = \frac{1}{2}[t\ln(t^4)]^{-1/2}\,\frac{d}{dt}\,[t\ln(t^4)] = \frac{1}{2\,\sqrt{t\ln(t^4)}}\cdot\left[1\cdot\ln(t^4) + t\cdot\frac{1}{t^4}\cdot 4t^3\right] = \frac{1}{2\,\sqrt{t\ln(t^4)}}\cdot[\ln(t^4)+4] = \frac{\ln(t^4)+4}{2\,\sqrt{t\ln(t^4)}}$$

Or: Since y is only defined for $t > 0$, we can write $y = \sqrt{t\cdot 4\ln t} = 2\,\sqrt{t\ln t}$. Then

$y' = 2\cdot\dfrac{1}{2\,\sqrt{t\ln t}}\cdot\left(1\cdot\ln t + t\cdot\dfrac{1}{t}\right) = \dfrac{\ln t + 1}{\sqrt{t\ln t}}$. This agrees with our first answer since

$$\frac{\ln(t^4)+4}{2\,\sqrt{t\ln(t^4)}} = \frac{4\ln t + 4}{2\,\sqrt{t\cdot 4\ln t}} = \frac{4(\ln t + 1)}{2\cdot 2\,\sqrt{t\ln t}} = \frac{\ln t + 1}{\sqrt{t\ln t}}.$$

37. $y = \sin\left(\tan\sqrt{1+x^3}\,\right) \quad\Rightarrow\quad y' = \cos\left(\tan\sqrt{1+x^3}\,\right)\left(\sec^2\sqrt{1+x^3}\,\right)\left[3x^2/\left(2\,\sqrt{1+x^3}\,\right)\right]$

38. $y = x\sec^{-1}x \quad\Rightarrow\quad y' = x\cdot\dfrac{1}{x\sqrt{x^2-1}} + \left(\sec^{-1}x\right)\cdot 1 = \dfrac{1}{\sqrt{x^2-1}} + \sec^{-1}x$

39. $y = 5\arctan\dfrac{1}{x} \quad\Rightarrow\quad y' = 5\cdot\dfrac{1}{1+\left(\dfrac{1}{x}\right)^2}\cdot\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{5}{1+\dfrac{1}{x^2}}\left(-\dfrac{1}{x^2}\right) = -\dfrac{5}{x^2+1}$

40. $y = \sin^{-1}(\cos\theta) \quad\Rightarrow$

$$\begin{aligned} y' &= \frac{1}{\sqrt{1-\cos^2\theta}}\cdot\frac{d}{d\theta}(\cos\theta) = \frac{1}{\sqrt{1-\cos^2\theta}}\cdot(-\sin\theta) = -\frac{\sin\theta}{\sqrt{1-\cos^2\theta}} \\ &= -\frac{\sin\theta}{\sin\theta} \qquad [\sin^2\theta + \cos^2\theta = 1 \quad\Rightarrow\quad \sin\theta = \sqrt{1-\cos^2\theta} \text{ for } 0 < \theta < \pi] \\ &= -1 \end{aligned}$$

41. $y = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) \quad\Rightarrow$

$$y' = x\cdot\frac{1}{1+x^2} + \left(\tan^{-1}x\right)\cdot 1 - \frac{1}{2}\left(\frac{2x}{1+x^2}\right) = \frac{x}{1+x^2} + \tan^{-1}x - \frac{x}{1+x^2} = \tan^{-1}x$$

42. $y = \ln(\arcsin x^2) \quad\Rightarrow\quad y' = \dfrac{1}{\arcsin x^2}\cdot\dfrac{d}{dx}\,(\arcsin x^2) = \dfrac{1}{\arcsin x^2}\cdot\dfrac{1}{\sqrt{1-(x^2)^2}}\cdot 2x = \dfrac{2x}{(\arcsin x^2)\sqrt{1-x^4}}$

43. $y = \tan^2(\sin\theta) = [\tan(\sin\theta)]^2 \quad\Rightarrow\quad y' = 2[\tan(\sin\theta)]\cdot\sec^2(\sin\theta)\cdot\cos\theta$

44. $y + \ln y = xy^2 \quad\Rightarrow\quad y' + (1/y)\,y' = x\cdot 2y\,y' + y^2\cdot 1 \quad\Rightarrow\quad y' + (1/y)\,y' - 2xy\,y' = y^2 \quad\Rightarrow$

$$[1 + (1/y) - 2xy]\,y' = y^2 \quad\Rightarrow\quad y' = \frac{y^2}{1+(1/y)-2xy}\cdot\frac{y}{y} = \frac{y^3}{y+1-2xy^2}$$

45. $y = \dfrac{\sqrt{x+1}\,(2-x)^5}{(x+3)^7} \quad\Rightarrow\quad \ln y = \frac{1}{2}\ln(x+1) + 5\ln(2-x) - 7\ln(x+3) \quad\Rightarrow\quad \dfrac{y'}{y} = \dfrac{1}{2(x+1)} + \dfrac{-5}{2-x} - \dfrac{7}{x+3} \quad\Rightarrow$

$$y' = \frac{\sqrt{x+1}\,(2-x)^5}{(x+3)^7}\left[\frac{1}{2(x+1)} - \frac{5}{2-x} - \frac{7}{x+3}\right] \quad\text{or}\quad y' = \frac{(2-x)^4(3x^2-55x-52)}{2\,\sqrt{x+1}\,(x+3)^8}.$$

46. $y = \dfrac{(x+\lambda)^4}{x^4+\lambda^4} \quad\Rightarrow\quad y' = \dfrac{(x^4+\lambda^4)(4)(x+\lambda)^3 - (x+\lambda)^4(4x^3)}{(x^4+\lambda^4)^2} = \dfrac{4(x+\lambda)^3(\lambda^4-\lambda x^3)}{(x^4+\lambda^4)^2}$

47. $y = x\sinh(x^2) \quad\Rightarrow\quad y' = x\cosh(x^2)\cdot 2x + \sinh(x^2)\cdot 1 = 2x^2\cosh(x^2) + \sinh(x^2)$

48. $y = \dfrac{\sin mx}{x} \quad\Rightarrow\quad y' = \dfrac{x\cdot\cos mx\cdot m - \sin mx\cdot 1}{(x)^2} = \dfrac{mx\cos mx - \sin mx}{x^2}$

49. $y = \ln(\cosh 3x) \quad\Rightarrow\quad y' = (1/\cosh 3x)(\sinh 3x)(3) = 3\tanh 3x$

50. $y = \ln\left|\dfrac{x^2-4}{2x+5}\right| = \ln\left|x^2-4\right| - \ln|2x+5| \quad\Rightarrow\quad y' = \dfrac{2x}{x^2-4} - \dfrac{2}{2x+5}$ or $\dfrac{2(x+1)(x+4)}{(x+2)(x-2)(2x+5)}$

51. $y = \cosh^{-1}(\sinh x) \quad\Rightarrow\quad y' = \dfrac{1}{\sqrt{(\sinh x)^2-1}}\cdot\cosh x = \dfrac{\cosh x}{\sqrt{\sinh^2 x - 1}}$

52. $y = x\tanh^{-1}\sqrt{x} \quad\Rightarrow\quad y' = \tanh^{-1}\sqrt{x} + x\,\dfrac{1}{1-\left(\sqrt{x}\right)^2}\,\dfrac{1}{2\sqrt{x}} = \tanh^{-1}\sqrt{x} + \dfrac{\sqrt{x}}{2(1-x)}$

53. $y = \cos\left(e^{\sqrt{\tan 3x}}\right) \quad\Rightarrow$

$$y' = -\sin\left(e^{\sqrt{\tan 3x}}\right)\cdot\left(e^{\sqrt{\tan 3x}}\right)' = -\sin\left(e^{\sqrt{\tan 3x}}\right)\,e^{\sqrt{\tan 3x}}\cdot\tfrac{1}{2}(\tan 3x)^{-1/2}\cdot\sec^2(3x)\cdot 3$$
$$= \frac{-3\sin\left(e^{\sqrt{\tan 3x}}\right)\,e^{\sqrt{\tan 3x}}\,\sec^2(3x)}{2\sqrt{\tan 3x}}$$

54. $y = \sin^2\left(\cos\sqrt{\sin\pi x}\right) = \left[\sin\left(\cos\sqrt{\sin\pi x}\right)\right]^2 \quad\Rightarrow$

$$y' = 2\left[\sin\left(\cos\sqrt{\sin\pi x}\right)\right]\left[\sin\left(\cos\sqrt{\sin\pi x}\right)\right]' = 2\sin\left(\cos\sqrt{\sin\pi x}\right)\cos\left(\cos\sqrt{\sin\pi x}\right)\left(\cos\sqrt{\sin\pi x}\right)'$$
$$= 2\sin\left(\cos\sqrt{\sin\pi x}\right)\cos\left(\cos\sqrt{\sin\pi x}\right)\left(-\sin\sqrt{\sin\pi x}\right)\left(\sqrt{\sin\pi x}\right)'$$
$$= -2\sin\left(\cos\sqrt{\sin\pi x}\right)\cos\left(\cos\sqrt{\sin\pi x}\right)\sin\sqrt{\sin\pi x}\cdot\tfrac{1}{2}(\sin\pi x)^{-1/2}(\sin\pi x)'$$
$$= \frac{-\sin\left(\cos\sqrt{\sin\pi x}\right)\cos\left(\cos\sqrt{\sin\pi x}\right)\sin\sqrt{\sin\pi x}}{\sqrt{\sin\pi x}}\cdot\cos\pi x\cdot\pi$$
$$= \frac{-\pi\sin\left(\cos\sqrt{\sin\pi x}\right)\cos\left(\cos\sqrt{\sin\pi x}\right)\sin\sqrt{\sin\pi x}\,\cos\pi x}{\sqrt{\sin\pi x}}$$

55. $f(t) = \sqrt{4t+1} \quad\Rightarrow\quad f'(t) = \tfrac{1}{2}(4t+1)^{-1/2}\cdot 4 = 2(4t+1)^{-1/2} \quad\Rightarrow$

$f''(t) = 2(-\tfrac{1}{2})(4t+1)^{-3/2}\cdot 4 = -4/(4t+1)^{3/2}$, so $f''(2) = -4/9^{3/2} = -\tfrac{4}{27}$.

56. $g(\theta) = \theta\sin\theta \quad\Rightarrow\quad g'(\theta) = \theta\cos\theta + \sin\theta\cdot 1 \quad\Rightarrow\quad g''(\theta) = \theta(-\sin\theta) + \cos\theta\cdot 1 + \cos\theta = 2\cos\theta - \theta\sin\theta$,

so $g''(\pi/6) = 2\cos(\pi/6) - (\pi/6)\sin(\pi/6) = 2\left(\sqrt{3}/2\right) - (\pi/6)(1/2) = \sqrt{3} - \pi/12$.

57. $x^6 + y^6 = 1 \quad \Rightarrow \quad 6x^5 + 6y^5y' = 0 \quad \Rightarrow \quad y' = -x^5/y^5 \quad \Rightarrow$

$$y'' = -\frac{y^5(5x^4) - x^5(5y^4y')}{(y^5)^2} = -\frac{5x^4y^4\left[y - x(-x^5/y^5)\right]}{y^{10}} = -\frac{5x^4\left[(y^6 + x^6)/y^5\right]}{y^6} = -\frac{5x^4}{y^{11}}$$

58. $f(x) = (2-x)^{-1} \quad \Rightarrow \quad f'(x) = (2-x)^{-2} \quad \Rightarrow \quad f''(x) = 2(2-x)^{-3} \quad \Rightarrow \quad f'''(x) = 2 \cdot 3(2-x)^{-4} \quad \Rightarrow$

$f^{(4)}(x) = 2 \cdot 3 \cdot 4(2-x)^{-5}$. In general, $f^{(n)}(x) = 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n(2-x)^{-(n+1)} = \dfrac{n!}{(2-x)^{(n+1)}}$.

59. We first show it is true for $n = 1$: $f(x) = xe^x \quad \Rightarrow \quad f'(x) = xe^x + e^x = (x+1)e^x$. We now assume it is true for $n = k$: $f^{(k)}(x) = (x+k)e^x$. With this assumption, we must show it is true for $n = k+1$:

$$f^{(k+1)}(x) = \frac{d}{dx}\left[f^{(k)}(x)\right] = \frac{d}{dx}\left[(x+k)e^x\right] = (x+k)e^x + e^x = [(x+k)+1]e^x = [x+(k+1)]e^x.$$

Therefore, $f^{(n)}(x) = (x+n)e^x$ by mathematical induction.

60. $$\lim_{t\to 0}\frac{t^3}{\tan^3(2t)} = \lim_{t\to 0}\frac{t^3\cos^3(2t)}{\sin^3(2t)} = \lim_{t\to 0}\cos^3(2t)\cdot\frac{1}{8\dfrac{\sin^3(2t)}{(2t)^3}} = \lim_{t\to 0}\frac{\cos^3(2t)}{8\left(\displaystyle\lim_{t\to 0}\frac{\sin(2t)}{2t}\right)^3} = \frac{1}{8\cdot 1^3} = \frac{1}{8}$$

61. $y = 4\sin^2 x \quad \Rightarrow \quad y' = 4 \cdot 2\sin x\cos x$. At $\left(\frac{\pi}{6}, 1\right)$, $y' = 8 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$, so an equation of the tangent line is $y - 1 = 2\sqrt{3}\left(x - \frac{\pi}{6}\right)$, or $y = 2\sqrt{3}\,x + 1 - \pi\sqrt{3}/3$.

62. $y = \dfrac{x^2-1}{x^2+1} \quad \Rightarrow \quad y' = \dfrac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$.

At $(0, -1)$, $y' = 0$, so an equation of the tangent line is $y + 1 = 0(x - 0)$, or $y = -1$.

63. $y = \sqrt{1 + 4\sin x} \quad \Rightarrow \quad y' = \frac{1}{2}(1 + 4\sin x)^{-1/2} \cdot 4\cos x = \dfrac{2\cos x}{\sqrt{1 + 4\sin x}}$.

At $(0, 1)$, $y' = \dfrac{2}{\sqrt{1}} = 2$, so an equation of the tangent line is $y - 1 = 2(x - 0)$, or $y = 2x + 1$.

64. $x^2 + 4xy + y^2 = 13 \quad \Rightarrow \quad 2x + 4(xy' + y \cdot 1) + 2yy' = 0 \quad \Rightarrow \quad x + 2xy' + 2y + yy' = 0 \quad \Rightarrow$

$2xy' + yy' = -x - 2y \quad \Rightarrow \quad y'(2x + y) = -x - 2y \quad \Rightarrow \quad y' = \dfrac{-x - 2y}{2x + y}$.

At $(2, 1)$, $y' = \dfrac{-2-2}{4+1} = -\dfrac{4}{5}$, so an equation of the tangent line is $y - 1 = -\frac{4}{5}(x - 2)$, or $y = -\frac{4}{5}x + \frac{13}{5}$.

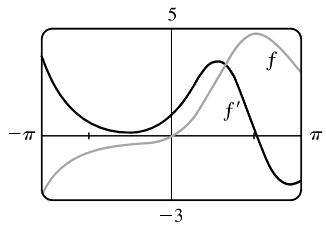
The slope of the normal line is $\frac{5}{4}$, so an equation of the normal line is $y - 1 = \frac{5}{4}(x - 2)$, or $y = \frac{5}{4}x - \frac{3}{2}$.

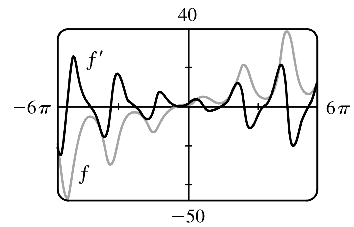
65. $y = (2 + x)e^{-x} \quad \Rightarrow \quad y' = (2 + x)(-e^{-x}) + e^{-x} \cdot 1 = e^{-x}[-(2 + x) + 1] = e^{-x}(-x - 1)$.

At $(0, 2)$, $y' = 1(-1) = -1$, so an equation of the tangent line is $y - 2 = -1(x - 0)$, or $y = -x + 2$.

The slope of the normal line is 1, so an equation of the normal line is $y - 2 = 1(x - 0)$, or $y = x + 2$.

66. $f(x) = xe^{\sin x} \quad \Rightarrow \quad f'(x) = x[e^{\sin x}(\cos x)] + e^{\sin x}(1) = e^{\sin x}(x\cos x + 1)$. As a check on our work, we notice from the graphs that $f'(x) > 0$ when f is increasing. Also, we see in the larger viewing rectangle a certain similarity in the graphs of f and f': the sizes of the oscillations of f and f' are linked.





67. (a) $f(x) = x\sqrt{5-x} \quad \Rightarrow$

$$f'(x) = x\left[\frac{1}{2}(5-x)^{-1/2}(-1)\right] + \sqrt{5-x} = \frac{-x}{2\sqrt{5-x}} + \sqrt{5-x}\cdot\frac{2\sqrt{5-x}}{2\sqrt{5-x}} = \frac{-x}{2\sqrt{5-x}} + \frac{2(5-x)}{2\sqrt{5-x}}$$
$$= \frac{-x+10-2x}{2\sqrt{5-x}} = \frac{10-3x}{2\sqrt{5-x}}$$

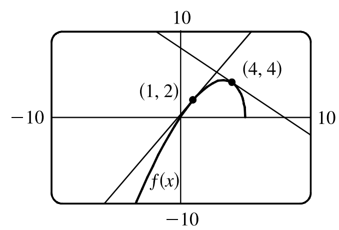
(b) At $(1,2)$: $f'(1) = \frac{7}{4}$.

So an equation of the tangent line is $y - 2 = \frac{7}{4}(x-1)$ or $y = \frac{7}{4}x + \frac{1}{4}$.

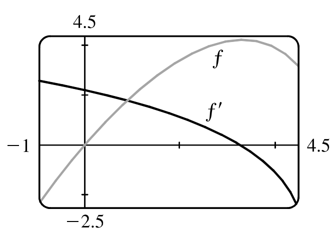
At $(4,4)$: $f'(4) = -\frac{2}{2} = -1$.

So an equation of the tangent line is $y - 4 = -1(x-4)$ or $y = -x + 8$.

(c)



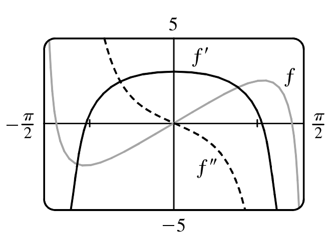
(d)



The graphs look reasonable, since f' is positive where f has tangents with positive slope, and f' is negative where f has tangents with negative slope.

68. (a) $f(x) = 4x - \tan x \quad \Rightarrow \quad f'(x) = 4 - \sec^2 x \quad \Rightarrow \quad f''(x) = -2\sec x\,(\sec x\,\tan x) = -2\sec^2 x\,\tan x$.

(b)



We can see that our answers are reasonable, since the graph of f' is 0 where f has a horizontal tangent, and the graph of f' is positive where f has tangents with positive slope and negative where f has tangents with negative slope. The same correspondence holds between the graphs of f' and f''.

69. $y = \sin x + \cos x \quad \Rightarrow \quad y' = \cos x - \sin x = 0 \quad \Leftrightarrow \quad \cos x = \sin x$ and $0 \le x \le 2\pi \quad \Leftrightarrow \quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$, so the points are $\left(\frac{\pi}{4}, \sqrt{2}\right)$ and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$.

70. $x^2 + 2y^2 = 1 \Rightarrow 2x + 4yy' = 0 \Rightarrow y' = -x/(2y) = 1 \Leftrightarrow x = -2y$. Since the points lie on the ellipse, we have $(-2y)^2 + 2y^2 = 1 \Rightarrow 6y^2 = 1 \Rightarrow y = \pm\frac{1}{\sqrt{6}}$. The points are $\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$.

71. $f(x) = (x-a)(x-b)(x-c) \Rightarrow f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$.

So $\dfrac{f'(x)}{f(x)} = \dfrac{(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)}{(x-a)(x-b)(x-c)} = \dfrac{1}{x-a} + \dfrac{1}{x-b} + \dfrac{1}{x-c}$.

Or: $f(x) = (x-a)(x-b)(x-c) \Rightarrow \ln|f(x)| = \ln|x-a| + \ln|x-b| + \ln|x-c| \Rightarrow$

$\dfrac{f'(x)}{f(x)} = \dfrac{1}{x-a} + \dfrac{1}{x-b} + \dfrac{1}{x-c}$

72. (a) $\cos 2x = \cos^2 x - \sin^2 x \Rightarrow -2\sin 2x = -2\cos x \sin x - 2\sin x \cos x \Leftrightarrow \sin 2x = 2\sin x \cos x$

(b) $\sin(x+a) = \sin x\, \cos a + \cos x\, \sin a \Rightarrow \cos(x+a) = \cos x\, \cos a - \sin x\, \sin a$.

73. (a) $S(x) = f(x) + g(x) \Rightarrow S'(x) = f'(x) + g'(x) \Rightarrow S'(1) = f'(1) + g'(1) = 3 + 1 = 4$

(b) $P(x) = f(x)\,g(x) \Rightarrow P'(x) = f(x)\,g'(x) + g(x)\,f'(x) \Rightarrow$

$P'(2) = f(2)\,g'(2) + g(2)f'(2) = 1(4) + 1(2) = 4 + 2 = 6$

(c) $Q(x) = \dfrac{f(x)}{g(x)} \Rightarrow Q'(x) = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2} \Rightarrow$

$Q'(1) = \dfrac{g(1)\,f'(1) - f(1)\,g'(1)}{[g(1)]^2} = \dfrac{3(3) - 2(1)}{3^2} = \dfrac{9-2}{9} = \dfrac{7}{9}$

(d) $C(x) = f(g(x)) \Rightarrow C'(x) = f'(g(x))\,g'(x) \Rightarrow C'(2) = f'(g(2))\,g'(2) = f'(1)\cdot 4 = 3\cdot 4 = 12$

74. (a) $P(x) = f(x)\,g(x) \Rightarrow P'(x) = f(x)\,g'(x) + g(x)\,f'(x) \Rightarrow$

$P'(2) = f(2)\,g'(2) + g(2)\,f'(2) = (1)\left(\frac{6-0}{3-0}\right) + (4)\left(\frac{0-3}{3-0}\right) = (1)(2) + (4)(-1) = 2 - 4 = -2$

(b) $Q(x) = \dfrac{f(x)}{g(x)} \Rightarrow Q'(x) = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2} \Rightarrow$

$Q'(2) = \dfrac{g(2)\,f'(2) - f(2)\,g'(2)}{[g(2)]^2} = \dfrac{(4)(-1) - (1)(2)}{4^2} = \dfrac{-6}{16} = -\dfrac{3}{8}$

(c) $C(x) = f(g(x)) \Rightarrow C'(x) = f'(g(x))g'(x) \Rightarrow$

$C'(2) = f'(g(2))g'(2) = f'(4)g'(2) = \left(\frac{6-0}{5-3}\right)(2) = (3)(2) = 6$

75. $f(x) = x^2 g(x) \Rightarrow f'(x) = x^2 g'(x) + g(x)(2x)$ or $x[xg'(x) + 2g(x)]$

76. $f(x) = g(x^2) \Rightarrow f'(x) = g'(x^2)(2x) = 2xg'(x^2)$

77. $f(x) = [\,g(x)]^2 \Rightarrow f'(x) = 2[\,g(x)]\cdot g'(x) = 2g(x)\,g'(x)$

78. $f(x) = g(g(x)) \Rightarrow f'(x) = g'(g(x))\,g'(x)$

79. $f(x) = g(e^x) \quad \Rightarrow \quad f'(x) = g'(e^x)\, e^x$

80. $f(x) = e^{g(x)} \quad \Rightarrow \quad f'(x) = e^{g(x)} g'(x)$

81. $f(x) = \ln|g(x)| \quad \Rightarrow \quad f'(x) = \dfrac{1}{g(x)} g'(x) = \dfrac{g'(x)}{g(x)}$

82. $f(x) = g(\ln x) \quad \Rightarrow \quad f'(x) = g'(\ln x) \cdot \dfrac{1}{x} = \dfrac{g'(\ln x)}{x}$

83. $h(x) = \dfrac{f(x)\, g(x)}{f(x) + g(x)} \quad \Rightarrow$

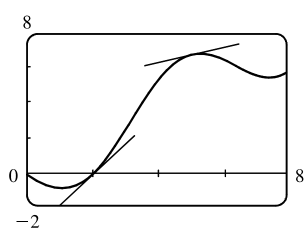
$$\begin{aligned} h'(x) &= \frac{[f(x) + g(x)]\,[f(x)\,g'(x) + g(x)\,f'(x)] - f(x)\,g(x)\,[f'(x) + g'(x)]}{[f(x) + g(x)]^2} \\ &= \frac{[f(x)]^2\, g'(x) + f(x)\,g(x)\,f'(x) + f(x)\,g(x)\,g'(x) + [\,g(x)]^2\, f'(x) - f(x)\,g(x)\,f'(x) - f(x)\,g(x)\,g'(x)}{[f(x) + g(x)]^2} \\ &= \frac{f'(x)\,[\,g(x)]^2 + g'(x)\,[f(x)]^2}{[f(x) + g(x)]^2} \end{aligned}$$

84. $h(x) = \sqrt{\dfrac{f(x)}{g(x)}} \quad \Rightarrow \quad h'(x) = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{2\,\sqrt{f(x)/g(x)}\,[g(x)]^2} = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{2[g(x)]^{3/2}\sqrt{f(x)}}$

85. Using the Chain Rule repeatedly, $h(x) = f(g(\sin 4x)) \quad \Rightarrow$

$h'(x) = f'(g(\sin 4x)) \cdot \dfrac{d}{dx}\,(g(\sin 4x)) = f'(g(\sin 4x)) \cdot g'(\sin 4x) \cdot \dfrac{d}{dx}\,(\sin 4x) = f'(g(\sin 4x))g'(\sin 4x)(\cos 4x)(4).$

86. (a)



(b) The average rate of change is larger on $[2, 3]$.

(c) The instantaneous rate of change (the slope of the tangent) is larger at $x = 2$.

(d) $f(x) = x - 2\sin x \quad \Rightarrow \quad f'(x) = 1 - 2\cos x$,

so $f'(2) = 1 - 2\cos 2 \approx 1.8323$ and $f'(5) = 1 - 2\cos 5 \approx 0.4327$.

So $f'(2) > f'(5)$, as predicted in part (c).

87. $y = [\ln(x+4)]^2 \quad \Rightarrow \quad y' = 2[\ln(x+4)]^1 \cdot \dfrac{1}{x+4} \cdot 1 = 2\,\dfrac{\ln(x+4)}{x+4}$ and $y' = 0 \quad \Leftrightarrow \quad \ln(x+4) = 0 \quad \Leftrightarrow$

$x + 4 = e^0 \quad \Rightarrow \quad x + 4 = 1 \quad \Leftrightarrow \quad x = -3$, so the tangent is horizontal at the point $(-3, 0)$.

88. (a) The line $x - 4y = 1$ has slope $\frac{1}{4}$. A tangent to $y = e^x$ has slope $\frac{1}{4}$ when $y' = e^x = \frac{1}{4} \quad \Rightarrow \quad x = \ln\frac{1}{4} = -\ln 4$.

Since $y = e^x$, the y-coordinate is $\frac{1}{4}$ and the point of tangency is $\left(-\ln 4, \frac{1}{4}\right)$. Thus, an equation of the tangent line is $y - \frac{1}{4} = \frac{1}{4}(x + \ln 4)$ or $y = \frac{1}{4}x + \frac{1}{4}(\ln 4 + 1)$.

(b) The slope of the tangent at the point (a, e^a) is $\left.\dfrac{d}{dx}\, e^x\right|_{x=a} = e^a$. Thus, an equation of the tangent line is

$y - e^a = e^a(x - a)$. We substitute $x = 0$, $y = 0$ into this equation, since we want the line to pass through the origin:

$0 - e^a = e^a(0-a) \quad \Leftrightarrow \quad -e^a = e^a(-a) \quad \Leftrightarrow \quad a = 1$. So an equation of the tangent line at the point $(a, e^a) = (1, e)$ is $y - e = e(x-1)$ or $y = ex$.

89. $y = f(x) = ax^2 + bx + c \quad \Rightarrow \quad f'(x) = 2ax + b$. We know that $f'(-1) = 6$ and $f'(5) = -2$, so $-2a + b = 6$ and $10a + b = -2$. Subtracting the first equation from the second gives $12a = -8 \quad \Rightarrow \quad a = -\frac{2}{3}$. Substituting $-\frac{2}{3}$ for a in the first equation gives $b = \frac{14}{3}$. Now $f(1) = 4 \quad \Rightarrow \quad 4 = a + b + c$, so $c = 4 + \frac{2}{3} - \frac{14}{3} = 0$ and hence, $f(x) = -\frac{2}{3}x^2 + \frac{14}{3}x$.

90. (a) $\lim\limits_{t\to\infty} C(t) = \lim\limits_{t\to\infty} [K(e^{-at} - e^{-bt})] = K \lim\limits_{t\to\infty} (e^{-at} - e^{-bt}) = K(0 - 0) = 0$ because $-at \to -\infty$ and $-bt \to -\infty$ as $t \to \infty$.

(b) $C(t) = K(e^{-at} - e^{-bt}) \quad \Rightarrow \quad C'(t) = K(e^{-at}(-a) - e^{-bt}(-b)) = K(-ae^{-at} + be^{-bt})$

(c) $C'(t) = 0 \quad \Leftrightarrow \quad be^{-bt} = ae^{-at} \quad \Leftrightarrow \quad \dfrac{b}{a} = e^{(-a+b)t} \quad \Leftrightarrow \quad \ln \dfrac{b}{a} = (b-a)t \quad \Leftrightarrow \quad t = \dfrac{\ln(b/a)}{b-a}$

91. $s(t) = Ae^{-ct}\cos(\omega t + \delta) \quad \Rightarrow$

$$v(t) = s'(t) = A\{e^{-ct}\,[-\omega\sin(\omega t+\delta)] + \cos(\omega t+\delta)(-ce^{-ct})\} = -Ae^{-ct}\,[\omega\sin(\omega t+\delta) + c\cos(\omega t+\delta)] \quad \Rightarrow$$

$$\begin{aligned} a(t) = v'(t) &= -A\{e^{-ct}[\omega^2\cos(\omega t+\delta) - c\omega\sin(\omega t+\delta)] + [\omega\sin(\omega t+\delta) + c\cos(\omega t+\delta)](-ce^{-ct})\} \\ &= -Ae^{-ct}[\omega^2\cos(\omega t+\delta) - c\omega\sin(\omega t+\delta) - c\omega\sin(\omega t+\delta) - c^2\cos(\omega t+\delta)] \\ &= -Ae^{-ct}[(\omega^2 - c^2)\cos(\omega t+\delta) - 2c\omega\sin(\omega t+\delta)] = Ae^{-ct}[(c^2-\omega^2)\cos(\omega t+\delta) + 2c\omega\sin(\omega t+\delta)] \end{aligned}$$

92. (a) $x = \sqrt{b^2 + c^2t^2} \quad \Rightarrow \quad v(t) = x' = \left[1/\left(2\sqrt{b^2+c^2t^2}\right)\right] 2c^2t = c^2t/\sqrt{b^2+c^2t^2} \quad \Rightarrow$

$$a(t) = v'(t) = \frac{c^2\sqrt{b^2+c^2t^2} - c^2t\left(c^2t/\sqrt{b^2+c^2t^2}\right)}{b^2+c^2t^2} = \frac{b^2c^2}{(b^2+c^2t^2)^{3/2}}$$

(b) $v(t) > 0$ for $t > 0$, so the particle always moves in the positive direction.

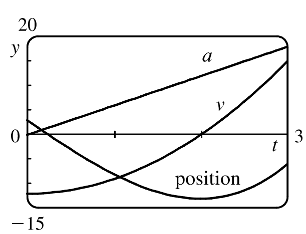
93. (a) $y = t^3 - 12t + 3 \quad \Rightarrow \quad v(t) = y' = 3t^2 - 12 \quad \Rightarrow \quad a(t) = v'(t) = 6t$

(b) $v(t) = 3(t^2 - 4) > 0$ when $t > 2$, so it moves upward when $t > 2$ and downward when $0 \le t < 2$.

(c) Distance upward $= y(3) - y(2) = -6 - (-13) = 7$,

Distance downward $= y(0) - y(2) = 3 - (-13) = 16$. Total distance $= 7 + 16 = 23$.

(d)



(e) The particle is speeding up when v and a have the same sign, that is, when $t > 2$. The particle is slowing down when v and a have opposite signs; that is, when $0 < t < 2$.

94. (a) $V = \frac{1}{3}\pi r^2 h \quad \Rightarrow \quad dV/dh = \frac{1}{3}\pi r^2 \quad [r \text{ constant}]$

(b) $V = \frac{1}{3}\pi r^2 h \quad \Rightarrow \quad dV/dr = \frac{2}{3}\pi rh \quad [h \text{ constant}]$

95. The linear density ρ is the rate of change of mass m with respect to length x.

$m = x\left(1+\sqrt{x}\right) = x + x^{3/2} \quad \Rightarrow \quad \rho = dm/dx = 1 + \frac{3}{2}\sqrt{x}$, so the linear density when $x = 4$ is $1 + \frac{3}{2}\sqrt{4} = 4$ kg/m.

96. (a) $C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3 \quad \Rightarrow \quad C'(x) = 2 - 0.04x + 0.00021x^2$

(b) $C'(100) = 2 - 4 + 2.1 = \$0.10/\text{unit}$. This value represents the rate at which costs are increasing as the hundredth unit is produced, and is the approximate cost of producing the 101st unit.

(c) The cost of producing the 101st item is $C(101) - C(100) = 990.10107 - 990 = \$0.10107$, slightly larger than $C'(100)$.

97. (a) $y(t) = y(0)e^{kt} = 200e^{kt} \quad \Rightarrow \quad y(0.5) = 200e^{0.5k} = 360 \quad \Rightarrow \quad e^{0.5k} = 1.8 \quad \Rightarrow \quad 0.5k = \ln 1.8 \quad \Rightarrow$

$k = 2\ln 1.8 = \ln(1.8)^2 = \ln 3.24 \quad \Rightarrow \quad y(t) = 200e^{(\ln 3.24)t} = 200(3.24)^t$

(b) $y(4) = 200(3.24)^4 \approx 22{,}040$ cells

(c) $y'(t) = 200(3.24)^t \cdot \ln 3.24$, so $y'(4) = 200(3.24)^4 \cdot \ln 3.24 \approx 25{,}910$ cells per hour

(d) $200(3.24)^t = 10{,}000 \quad \Rightarrow \quad (3.24)^t = 50 \quad \Rightarrow \quad t\ln 3.24 = \ln 50 \quad \Rightarrow \quad t = \ln 50/\ln 3.24 \approx 3.33$ hours

98. (a) If $y(t)$ is the mass remaining after t years, then $y(t) = y(0)e^{kt} = 100e^{kt}$. $y(5.24) = 100e^{5.24k} = \frac{1}{2} \cdot 100 \quad \Rightarrow$

$e^{5.24k} = \frac{1}{2} \quad \Rightarrow \quad 5.24k = -\ln 2 \quad \Rightarrow \quad k = -\frac{1}{5.24}\ln 2 \quad \Rightarrow \quad y(t) = 100e^{-(\ln 2)t/5.24} = 100 \cdot 2^{-t/5.24}$. Thus,

$y(20) = 100 \cdot 2^{-20/5.24} \approx 7.1$ mg.

(b) $100 \cdot 2^{-t/5.24} = 1 \quad \Rightarrow \quad 2^{-t/5.24} = \dfrac{1}{100} \quad \Rightarrow \quad -\dfrac{t}{5.24}\ln 2 = \ln\dfrac{1}{100} \quad \Rightarrow \quad t = 5.24\,\dfrac{\ln 100}{\ln 2} \approx 34.8$ years

99. (a) $C'(t) = -kC(t) \quad \Rightarrow \quad C(t) = C(0)e^{-kt}$ by Theorem 3.8.2. But $C(0) = C_0$, so $C(t) = C_0e^{-kt}$.

(b) $C(30) = \frac{1}{2}C_0$ since the concentration is reduced by half. Thus, $\frac{1}{2}C_0 = C_0e^{-30k} \quad \Rightarrow \ln\frac{1}{2} = -30k \quad \Rightarrow$

$k = -\frac{1}{30}\ln\frac{1}{2} = \frac{1}{30}\ln 2$. Since 10% of the original concentration remains if 90% is eliminated, we want the value of t such that $C(t) = \frac{1}{10}C_0$. Therefore, $\frac{1}{10}C_0 = C_0e^{-t(\ln 2)/30} \quad \Rightarrow \quad \ln 0.1 = -t(\ln 2)/30 \quad \Rightarrow \quad t = -\frac{30}{\ln 2}\ln 0.1 \approx 100$ h.

100. (a) If $y = u - 20$, $u(0) = 80 \quad \Rightarrow \quad y(0) = 80 - 20 = 60$, and the initial-value problem is $dy/dt = ky$ with $y(0) = 60$.

So the solution is $y(t) = 60e^{kt}$. Now $y(0.5) = 60e^{k(0.5)} = 60 - 20 \quad \Rightarrow \quad e^{0.5k} = \frac{40}{60} = \frac{2}{3} \quad \Rightarrow \quad k = 2\ln\frac{2}{3} = \ln\frac{4}{9}$,

so $y(t) = 60e^{(\ln 4/9)t} = 60(\frac{4}{9})^t$. Thus, $y(1) = 60(\frac{4}{9})^1 = \frac{80}{3} = 26\frac{2}{3}$ °C and $u(1) = 46\frac{2}{3}$ °C.

(b) $u(t) = 40 \quad \Rightarrow \quad y(t) = 20. \quad y(t) = 60\left(\dfrac{4}{9}\right)^t = 20 \quad \Rightarrow \quad \left(\dfrac{4}{9}\right)^t = \dfrac{1}{3} \quad \Rightarrow \quad t\ln\dfrac{4}{9} = \ln\dfrac{1}{3} \quad \Rightarrow \quad t = \dfrac{\ln\frac{1}{3}}{\ln\frac{4}{9}} \approx 1.35$ h

or 81.3 min.

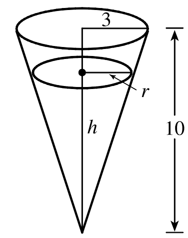
101. If $x =$ edge length, then $V = x^3 \quad \Rightarrow \quad dV/dt = 3x^2\,dx/dt = 10 \quad \Rightarrow \quad dx/dt = 10/(3x^2)$ and $S = 6x^2 \quad \Rightarrow$

$dS/dt = (12x)\,dx/dt = 12x[10/(3x^2)] = 40/x$. When $x = 30$, $dS/dt = \frac{40}{30} = \frac{4}{3}$ cm^2/min.

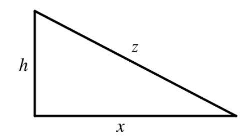
102. Given $dV/dt = 2$, find dh/dt when $h = 5$. $V = \frac{1}{3}\pi r^2 h$ and, from similar triangles, $\dfrac{r}{h} = \dfrac{3}{10} \quad \Rightarrow \quad V = \dfrac{\pi}{3}\left(\dfrac{3h}{10}\right)^2 h = \dfrac{3\pi}{100}h^3$, so

$$2 = \frac{dV}{dt} = \frac{9\pi}{100}h^2\frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{200}{9\pi h^2} = \frac{200}{9\pi\,(5)^2} = \frac{8}{9\pi}\ \text{cm/s}$$

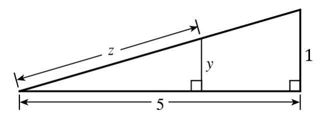
when $h = 5$.



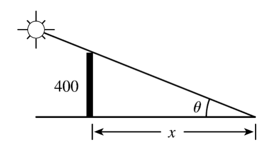
103. Given $dh/dt = 2$ and $dx/dt = 5$, find dz/dt. $z^2 = x^2 + h^2 \Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2h\frac{dh}{dt} - x\frac{dh}{dt} - h\frac{dx}{dt} \Rightarrow \frac{dz}{dt} = \frac{1}{z}(5x + 2h)$. When $t = 3$, $h = 15 + 3(2) = 21$ and $x = 5(3) = 15 \Rightarrow z = \sqrt{21^2 + 15^2} = \sqrt{666}$, so $\frac{dz}{dt} = \frac{1}{\sqrt{666}}[5(15) + 2(21)] = \frac{117}{\sqrt{666}} \approx 4.5$ m/s.



104. We are given $dz/dt = 10$ m/s. By similar triangles, $\frac{y}{z} = \frac{1}{\sqrt{26}} \Rightarrow y = \frac{1}{\sqrt{26}}z$, so $\frac{dy}{dt} = \frac{1}{\sqrt{26}}\frac{dz}{dt} = \frac{10}{\sqrt{26}} \approx 2.0$ m/s.



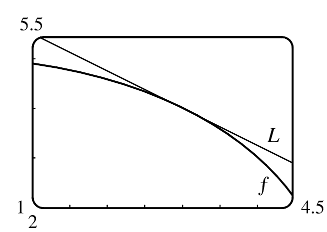
105. We are given $d\theta/dt = -0.25$ rad/h. $\tan\theta = 400/x \Rightarrow x = 400\cot\theta \Rightarrow \frac{dx}{dt} = -400\csc^2\theta\,\frac{d\theta}{dt}$. When $\theta = \frac{\pi}{6}$, $\frac{dx}{dt} = -400(2)^2(-0.25) = 400$ m/h.



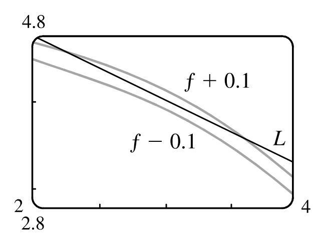
106. (a) $f(x) = \sqrt{25 - x^2} \quad \Rightarrow \quad f'(x) = \dfrac{-2x}{2\sqrt{25 - x^2}} = -x(25 - x^2)^{-1/2}$.

So the linear approximation to $f(x)$ near 3 is $f(x) \approx f(3) + f'(3)(x - 3) = 4 - \frac{3}{4}(x - 3)$.

(b)

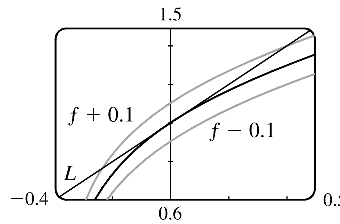


(c) For the required accuracy, we want $\sqrt{25 - x^2} - 0.1 < 4 - \frac{3}{4}(x - 3)$ and $4 - \frac{3}{4}(x - 3) < \sqrt{25 - x^2} + 0.1$. From the graph, it appears that these both hold for $2.24 < x < 3.66$.



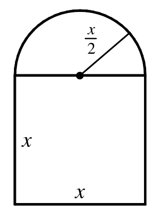
107. (a) $f(x) = \sqrt[3]{1 + 3x} = (1 + 3x)^{1/3} \quad \Rightarrow \quad f'(x) = (1 + 3x)^{-2/3}$, so the linearization of f at $a = 0$ is $L(x) = f(0) + f'(0)(x - 0) = 1^{1/3} + 1^{-2/3}x = 1 + x$. Thus, $\sqrt[3]{1 + 3x} \approx 1 + x \quad \Rightarrow$ $\sqrt[3]{1.03} = \sqrt[3]{1 + 3(0.01)} \approx 1 + (0.01) = 1.01$.

(b) The linear approximation is $\sqrt[3]{1+3x} \approx 1+x$, so for the required accuracy we want $\sqrt[3]{1+3x} - 0.1 < 1+x < \sqrt[3]{1+3x} + 0.1$. From the graph, it appears that this is true when $-0.235 < x < 0.401$.



108. $y = x^3 - 2x^2 + 1 \quad \Rightarrow \quad dy = (3x^2 - 4x)\,dx$. When $x = 2$ and $dx = 0.2$, $dy = \left[3(2)^2 - 4(2)\right](0.2) = 0.8$.

109. $A = x^2 + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = \left(1 + \frac{\pi}{8}\right)x^2 \quad \Rightarrow \quad dA = \left(2 + \frac{\pi}{4}\right)x\,dx$. When $x = 60$ and $dx = 0.1$, $dA = \left(2 + \frac{\pi}{4}\right)60(0.1) = 12 + \frac{3\pi}{2}$, so the maximum error is approximately $12 + \frac{3\pi}{2} \approx 16.7\ \text{cm}^2$.



110. $\lim\limits_{x\to 1} \dfrac{x^{17} - 1}{x - 1} = \left[\dfrac{d}{dx}\, x^{17}\right]_{x=1} = 17(1)^{16} = 17$

111. $\lim\limits_{h\to 0} \dfrac{\sqrt[4]{16+h} - 2}{h} = \left[\dfrac{d}{dx}\,\sqrt[4]{x}\right]_{x=16} = \dfrac{1}{4}x^{-3/4}\Big|_{x=16} = \dfrac{1}{4\left(\sqrt[4]{16}\right)^3} = \dfrac{1}{32}$

112. $\lim\limits_{\theta\to\pi/3} \dfrac{\cos\theta - 0.5}{\theta - \pi/3} = \left[\dfrac{d}{d\theta}\cos\theta\right]_{\theta=\pi/3} = -\sin\dfrac{\pi}{3} = -\dfrac{\sqrt{3}}{2}$

113.
$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3} &= \lim_{x\to 0} \frac{\left(\sqrt{1+\tan x} - \sqrt{1+\sin x}\right)\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)} \\
&= \lim_{x\to 0} \frac{(1+\tan x) - (1+\sin x)}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)} = \lim_{x\to 0} \frac{\sin x\,(1/\cos x - 1)}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)} \cdot \frac{\cos x}{\cos x} \\
&= \lim_{x\to 0} \frac{\sin x\,(1 - \cos x)}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)\cos x} \cdot \frac{1+\cos x}{1+\cos x} \\
&= \lim_{x\to 0} \frac{\sin x \cdot \sin^2 x}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)\cos x\,(1+\cos x)} \\
&= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^3 \lim_{x\to 0} \frac{1}{\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)\cos x\,(1+\cos x)} \\
&= 1^3 \cdot \frac{1}{\left(\sqrt{1} + \sqrt{1}\right)\cdot 1 \cdot (1+1)} = \frac{1}{4}
\end{aligned}$$

114. Differentiating the first given equation implicitly with respect to x and using the Chain Rule, we obtain $f(g(x)) = x \quad \Rightarrow$ $f'(g(x))\,g'(x) = 1 \quad \Rightarrow \quad g'(x) = \dfrac{1}{f'(g(x))}$. Using the second given equation to expand the denominator of this expression gives $g'(x) = \dfrac{1}{1 + [f(g(x))]^2}$. But the first given equation states that $f(g(x)) = x$, so $g'(x) = \dfrac{1}{1+x^2}$.

115. $\dfrac{d}{dx}\left[f(2x)\right] = x^2 \quad\Rightarrow\quad f'(2x)\cdot 2 = x^2 \quad\Rightarrow\quad f'(2x) = \frac{1}{2}x^2$. Let $t = 2x$. Then $f'(t) = \frac{1}{2}\left(\frac{1}{2}t\right)^2 = \frac{1}{8}t^2$, so $f'(x) = \frac{1}{8}x^2$.

116. Let (b, c) be on the curve, that is, $b^{2/3} + c^{2/3} = a^{2/3}$. Now $x^{2/3} + y^{2/3} = a^{2/3} \quad\Rightarrow\quad \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{y^{1/3}}{x^{1/3}} = -\left(\dfrac{y}{x}\right)^{1/3}$, so at (b, c) the slope of the tangent line is $-(c/b)^{1/3}$ and an equation of the tangent line is $y - c = -(c/b)^{1/3}(x - b)$ or $y = -(c/b)^{1/3}x + (c + b^{2/3}c^{1/3})$. Setting $y = 0$, we find that the x-intercept is $b^{1/3}c^{2/3} + b = b^{1/3}(c^{2/3} + b^{2/3}) = b^{1/3}a^{2/3}$ and setting $x = 0$ we find that the y-intercept is $c + b^{2/3}c^{1/3} = c^{1/3}(c^{2/3} + b^{2/3}) = c^{1/3}a^{2/3}$. So the length of the tangent line between these two points is

$$\begin{aligned}\sqrt{(b^{1/3}a^{2/3})^2 + (c^{1/3}a^{2/3})^2} &= \sqrt{b^{2/3}a^{4/3} + c^{2/3}a^{4/3}} = \sqrt{(b^{2/3} + c^{2/3})a^{4/3}} \\ &= \sqrt{a^{2/3}a^{4/3}} = \sqrt{a^2} = a = \text{constant}\end{aligned}$$