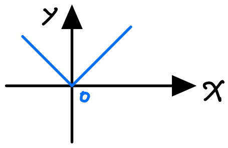
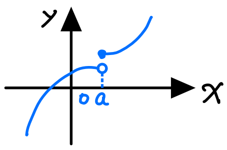
How can a function fail to be differentiable?

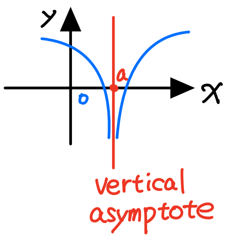


Not be differentiable at $x=0$

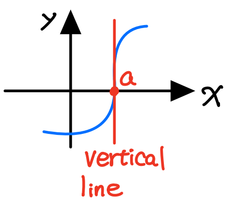
$\because$ slope of tangent lines from right $\neq$ from left



discontinuous, so not be differentiable at $x=a$



Not be differentiable at $x=a$



Not be differentiable at a

slope D.N.E.

Higer derivatives

$f'(x) \to f''(x) \to f'''(x) \to f^{(4)}(x) \to \cdots$ $\quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$y' \to y'' \to y''' \to y^{(4)} \to \cdots$

3.1. Derivatives of Polynomials and Exponential Functions

① $\frac{d}{dx}c = 0$ Constant rule

② $\frac{d}{dx}cf(x) = c\cdot\frac{d}{dx}f(x)$ constant multiple rule

③ $\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$ Power rule

④ $\frac{d}{dx}x^n = nx^{n-1}$ Power rule

pf.: $\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$

$= \lim_{h\to 0}\frac{x^n + \binom{n}{1}x^{n-1}\cdot h + \cdots + h^n - x^n}{h} = nx^{n-1}$

Eg. $y = x^4 - 6x^2 + 4$

$y' = 4x^3 - 12x$

Eg. $f(x) = \frac{1}{x^2} = x^{-2}$

$f'(x) = -2x^{-3} = -\frac{2}{x^3}$

Eg. $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

Exponential function

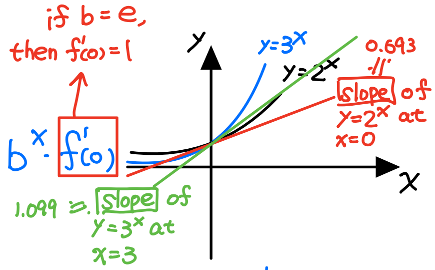
Let $f(x) = b^x$

$f'(x) = \lim_{h\to 0}\frac{b^{x+h} - b^x}{h} = \lim_{h\to 0}\frac{b^x \cdot b^h - b^x}{h} = b^x \cdot \boxed{\lim_{h\to 0}\frac{b^h - 1}{h}} = b^x \cdot \boxed{f'(0)}$

$f'(0)$ ॥

if $b=e$, then $f'(0)=1$

$f'(0) = \lim_{h\to 0}\frac{b^{0+h} - b^0}{h} = \lim_{h\to 0}\frac{b^h - 1}{h}$



※ If $b=e$, then we call b^x is a natural exponential function.

let $2<e<3$ s.t. $f'(0) = \lim_{h\to 0}\frac{e^h - 1}{h} = 1$

Eg. $f(x)=5e^x-x^3+\frac{1}{x}$

$f'(x)=5e^x-3x^2-\frac{1}{x^2}$

3.2. The product and quotient rules

① $$\frac{d}{dx}(f(x)\cdot g(x))=\lim_{h\to0}\frac{f(x+h)\cdot g(x+h)-f(x)\cdot g(x)+f(x+h)\cdot g(x)-f(x+h)\cdot g(x)}{h}$$

$$=\lim_{h\to0}\frac{f(x+h)\cdot(g(x+h)-g(x))+g(x)\cdot(f(x+h)-f(x))}{h}$$

$$=\lim_{h\to0}\frac{f(x+h)\cdot(g(x+h)-g(x))}{h}+\lim_{h\to0}\frac{g(x)\cdot(f(x+h)-f(x))}{h}$$

$$=\lim_{h\to0}f(x+h)\cdot\lim_{h\to0}\frac{g(x+h)-g(x)}{h}+g(x)\cdot\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$=f'(x)\cdot g(x)+g'(x)\cdot f(x)$$

② $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{[g(x)]^2}$$

Eg. $f(x)=\dfrac{xe^x}{x^2+2}$

$$f'(x)=\frac{(xe^x)'\cdot(x^2+2)-xe^x\cdot(x^2+2)'}{(x^2+2)^2}$$

$$=\frac{(e^x+xe^x)\cdot(x^2+2)-xe^x\cdot 2x}{(x^2+2)^2}$$

Sec 3.3. Derivative of trigonometric Functions

1. Let $f(x)=\sin x$

$$f'(x)=\lim_{h\to0}\frac{\sin(x+h)-\sin x}{h}=\lim_{h\to0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$

$$=\lim_{h\to0}\left[\sin x\cdot\frac{\cos h-1}{h}+\cos x\cdot\frac{\sin h}{h}\right]$$

$$=\sin x\cdot\overset{0}{\boxed{\lim_{h\to0}\frac{\cos h-1}{h}}}+\cos x\cdot\boxed{\lim_{h\to0}\frac{\sin h}{h}}^{=1}=\cos x$$

2. $\frac{d}{dx}\cos x=-\sin x$

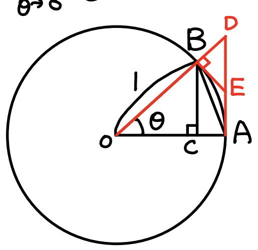
3. $\frac{d}{dx}\tan x=\frac{d}{dx}\frac{\sin x}{\cos x}=\frac{\cos^2x+\sin^2x}{(\cos x)^2}=\frac{1}{(\cos x)^2}=\sec^2x$

4. $\frac{d}{dx}\cot x=-\csc^2x$

5. $\frac{d}{dx}\sec x=\sec x\tan x$

6. $\frac{d}{dx}\csc x=-\csc x\cot x$

7. $\lim_{\theta\to0}\frac{\sin\theta}{\theta}=1$



pf.: ① $\overset{\sin\theta}{\boxed{\overline{BC}}}<\overline{AB}<\overset{\theta}{\boxed{\text{arc}\,\overset{\frown}{AB}}}$

$\Rightarrow\frac{\sin\theta}{\theta}<\frac{\theta}{\theta}=1$

② $\theta=\text{arc}\,\overset{\frown}{AB}<\overline{AE}+\overline{BE}<\overline{AE}+\overline{ED}=\overline{AD}=\tan\theta$

$\Rightarrow\theta<\frac{\sin\theta}{\cos\theta}\Rightarrow\frac{\sin\theta}{\theta}>\cos\theta$

$\therefore\lim_{\theta\to0}\cos\theta<\lim_{\theta\to0}\frac{\sin\theta}{\theta}<\lim_{\theta\to0}1\Rightarrow\lim_{\theta\to0}\frac{\sin\theta}{\theta}=1$

8. $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta}$

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta}$$

$$= \lim_{\theta\to 0}\left(\frac{\cos\theta-1}{\theta}\cdot\frac{\cos\theta+1}{\cos\theta+1}\right)$$

$$= -\lim_{\theta\to 0}\frac{-\cos^2\theta+1}{\theta(\cos\theta+1)} = -\lim_{\theta\to 0}\frac{\sin\theta\cdot\sin\theta}{\theta(\cos\theta+1)} = 0$$

Eg. $\lim_{x\to 0}\frac{\sin 7x}{4x}$

$$= \lim_{x\to 0}\frac{7}{4}\cdot\frac{\sin 7x}{7x} = \frac{7}{4}$$

3.4. Chain Rule

$$\frac{d}{dx}f(g(x)) = f'(g(x))\cdot g'(x)$$

pf.: $\frac{d}{dx}f(g(x)) = \lim_{h\to 0}\left(\frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)}\right)\left(\frac{g(x+h)-g(x)}{h}\right)$

$$= \lim_{h\to 0}\left(\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\right)\boxed{\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}}$$

$$= f'(g(x))\cdot g'(x) \qquad \text{''} \; g'(x)$$

Let $u = g(x)$, $u+k = g(x+h)$

$$\lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)} = \lim_{k\to 0}\frac{f(u+k)-f(u)}{k} = f'(u)$$

Eg. $f(x) = \sin(x^2+e^x)$

$$f'(x) = [\cos(x^2+e^x)]\cdot\frac{d}{dx}(x^2+e^x) = [\cos(x^2+e^x)]\cdot(2x+e^x)$$

Eg. $f(x) = \sin(e^{x^2+9})$

$$f'(x) = [\cos(e^{x^2+9})]\cdot\frac{d}{dx}(e^{x^2+9})$$

$$= [\cos(e^{x^2+9})]\cdot e^{x^2+9}\cdot\frac{d}{dx}(x^2+9)$$

$$= [\cos(e^{x^2+9})]\cdot e^{x^2+9}\cdot 2x$$

Eg. $g(t) = \left(\frac{t-2}{2t+1}\right)^9$

$$g'(t) = 9\cdot\left(\frac{t-2}{2t+1}\right)^8\cdot\left(\frac{t-2}{2t+1}\right)'$$

$b^x = b^x\cdot \ln b$

pf.: $b^x = (e^{\ln b})^x = e^{x\ln b}$

$$\Rightarrow \frac{d}{dx}b^x = \frac{d}{dx}(e^{x\ln b}) = e^{x\ln b}\cdot\frac{d}{dx}(x\ln b) = b^x\cdot\ln b$$

3.5. Implicit Differentiation

If a function like $y = f(x)$, is called explicitly. Otherwise, we call it implicitly

Eg. $x^2+y^2=9$

$\Rightarrow 2x+2y\cdot y'=0$

$\Rightarrow y' = -\frac{x}{y} = \pm\frac{x}{\sqrt{9-x^2}}$

Eg. $y^5+3x^2y^2+5x^4=12$

$\Rightarrow 5y^4y'+6xy^2+6x^2yy'+20x^3=0$

$\Rightarrow y'(5y^4+6x^2y) = -20x^3-6xy^2$

$\Rightarrow y' = \frac{-20x^3-6xy^2}{5y^4+6x^2y}$

Eg. $\sin(x+y) = y^2\cos x$, find y'

$\Rightarrow [\cos(x+y)](1+y') = 2yy'\cos x - y^2\sin x$

$\Rightarrow y' = \frac{-y^2\sin x-\cos(x+y)}{\cos(x+y)-2y\cos x}$

3.6. Derivatives of Log and Inverse Trigonometric Functions

1. $\frac{d}{dx}(\log_b x) = \frac{1}{x\ln b}$

pf.: Let $y=\log_b x \Rightarrow b^y = x \Rightarrow b^y\cdot\ln b\cdot y' = 1 \Rightarrow y' = \frac{1}{x\cdot\ln b}$

2. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Eg. $\frac{d}{dx}\ln(\sin x)$

$= \frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)$

$= \frac{\cos x}{\sin x} = \cot x$

Eg. $f(x) = \log_{10}(2+\sin x)$

$f'(x) = \frac{1}{(2+\sin x)\ln 10}\cdot\frac{d}{dx}(2+\sin x) = \frac{\cos x}{(2+\sin x)\cdot\ln 10}$

Eg. $f(x) = \ln|x|$

$\Rightarrow f(x) = \begin{cases} \ln x, & \text{if } x>0 \longrightarrow f'(x) = \frac{1}{x} \\ \ln(-x), & \text{if } x<0 \longrightarrow f'(x) = \frac{1}{x} \end{cases}$