Sec. 4.4. Indeterminate forms and L'Hospital's Rule.

1. In before, if we want to solve $\lim_{x\to 1}\frac{x^2+2x-3}{x-1}$, we will cancel $x-1$ in 分子.分母 s.t. $\lim_{x\to 1}\frac{x^2+2x-3}{x-1}=\lim_{x\to 1}(x+3)=4$. But it's hard to solve $\lim_{x\to 1}\frac{\ln x}{x-1}$. So we will use l'Hospital's rule to solve it.

2. Indeterminate Forms (不定式): 牽扯到 infinite value 及0的四則運算.

ex.: $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, ∞^0, ...

3. L'Hospital's Rule:

① 關係式: $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(a)}{g'(a)}$

② 使用時机: $\boxed{\lim_{x\to a}f(x)=\lim_{x\to a} g(x)=0}$ or $\boxed{\lim_{x\to a}f(x)=\pm\infty,\ \lim_{x\to a}g(x)=\pm\infty}$

③ 限制: If $g'(x)=0$, and $f'(x)\neq 0$ when $x=c$, then we can't use l'Hospital's rule.

Eg1. $\lim_{x\to 1}\frac{\ln x}{x-1}=\boxed{\lim_{x\to 1}\frac{\frac{1}{x}}{1}=1}$

$\because \lim_{x\to 1}(x-1)=\lim_{x\to 1}\ln x=0$

Eg2. $\lim_{x\to\infty}\frac{e^x}{x^2}=\boxed{\lim_{x\to\infty}\frac{e^x}{2x}}=\lim_{x\to\infty}\frac{e^x}{2}=\infty$

$\because \lim_{x\to\infty}e^x=\lim_{x\to\infty}x^2=\infty$

Eg3. $\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}=\lim_{x\to\pi^-}\frac{\cos x}{\sin x}=-\infty$ (x) $\because \lim_{x\to\pi^-}\sin x=0$ but $\lim_{x\to\pi^-}1-\cos x=2\neq 0$, so we can't use l'Hospital rule's.

$=\frac{0}{2}=0$ (o)

4. Indeterminate products: Like $0\cdot\infty$. If we want to use l'Hospital rule, then we can change $0\cdot\infty$ to $\frac{0}{0}$.

Eg. $\lim_{x\to 0^+}(x\cdot\ln x)=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-1}{x^2}}=\lim_{x\to 0^+}(-x)=0$

5. Indeterminate differences: Like $\infty-\infty$. If we want to solve it, then we will change $\infty-\infty$ to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Eg. $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1^+}\frac{\boxed{(x-1)-\ln x}}{\boxed{(\ln x)\times(x-1)}}=\lim_{x\to 1^+}\frac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x}=\lim_{x\to 1^+}\frac{\frac{1}{x^2}}{\frac{1}{x^2}+\frac{1}{x}}=\frac{1}{2}$

$\lim_{x\to 1^+}[(x-1)-\ln x]=0$, $\lim_{x\to 1^+}[(\ln x)\times(x-1)]=0$

Eg. $\lim_{x\to\infty}(e^x-x)=\lim_{x\to\infty}[x\cdot(\frac{e^x}{x}-1)]=\infty$. It can't use l'Hospital's rule.

6. Indeterminate power: Like 0^0, ∞^0, 1^∞. We will use natural logarithm to solve problem.

$y=f(x)^{g(x)}\Rightarrow \ln y=g(x)\cdot\ln(f(x))$

Eg. $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$

Let $y=(1+\sin 4x)^{\cot x}$. Then $\ln y=(\cot x)\cdot\ln(1+\sin 4x)\Rightarrow \lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}[(\cot x)\cdot\ln(1+\sin 4x)]$

$=\lim_{x\to 0^+}\frac{\ln(1+\sin 4x)}{\tan x}=\lim_{x\to 0^+}\frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x}=4$

$\therefore \lim_{x\to 0^+}(1+\sin 4x)^{\cot x}=\lim_{x\to 0^+}y=\lim_{x\to 0^+}e^{\ln y}=e^{\lim_{x\to 0^+}\ln y}=e^4$

7. The proof of L'Hospital's rule:

Type 1: $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$. $g'(a) \neq 0$

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x \to a} \frac{g(x)-g(a)}{x-a}} = \frac{\lim_{x \to a}(f(x)-f(a))}{\lim_{x \to a}(g(x)-g(a))} = \frac{\lim_{x \to a} f(x) - \overset{0}{\overset{\shortparallel}{\lim_{x \to a} f(a)}}}{\lim_{x \to a} g(x) - \underset{0}{\underset{\shortparallel}{\lim_{x \to a} g(a)}}} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

Type 2: $\lim_{x \to a} f(x) = \pm\infty$, $\lim_{x \to a} g(x) = \pm\infty$. $g'(a) \neq 0$

Let $\frac{1}{f(x)} = h(x)$, $\frac{1}{g(x)} = u(x)$, 仿照 type 1 的証明

Sec. 4.5. Curve Sketching

1. 步驟: Find domain → Find asymptote → $\boxed{\text{Find critical points and the interval of increasing (decreasing)}}$ First-derivative

→ $\boxed{\text{Find the inflection points and concavity}}$ → graph

Second derivative