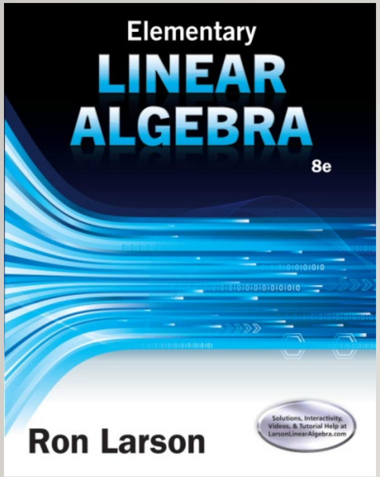

CHAPTER 6 LINEAR TRANSFORMATIONS

6.3 Matrices for Linear Transformations

6.4 Transition Matrices and Similarity

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6.3 Matrices for Linear Transformations

- Two representations of the linear transformation $T: R^3 \rightarrow R^3$:

$$(1)\ T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

$$(2)\ T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Five reasons for matrix representation of a linear transformation:
  - Simpler to write.
  - Simpler to read.
  - More easily adapted for computer use.
  - Easy to represent using a basis representation of a matrix
  - Easy to represent the inverse of a linear transformation

- Thm 6.10: (Standard matrix for a linear transformation)

$$T(\mathbf{v})=A\mathbf{v}$$
$$A=?$$

Pf:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n$$

$$\begin{aligned} T \text{ is a L.T.} \Rightarrow T(\boldsymbol{v}) &= T(v_1 e_1 + v_2 e_2 + \cdots + v_n e_n) \\ &= T(v_1 e_1) + T(v_2 e_2) + \cdots + T(v_n e_n) \\ &= v_1 T(e_1) + v_2 T(e_2) + \cdots + v_n T(e_n) \end{aligned}$$

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

$$= v_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= v_1T(e_1) + v_2T(e_2) + \cdots + v_nT(e_n)$$

Therefore, $T(\mathbf{v}) = A\mathbf{v}$ for each $\mathbf{v}$ in R^n

- Ex 1: (Finding the standard matrix of a linear transformation)

Find the standard matrix for the L.T. $T: R^3 \rightarrow R^2$ define by

$$T(x, y, z) = (x - 2y, 2x + y)$$

Sol:

Vector Notation	Matrix Notation
$T(e_1) = T(1, 0, 0) = (1, 2)$	$T(e_1) = T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
$T(e_2) = T(0, 1, 0) = (-2, 1)$	$T(e_2) = T(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$
$T(e_3) = T(0, 0, 1) = (0, 0)$	$T(e_3) = T(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$A = \left[T(e_1) \mid T(e_2) \mid T(e_3) \right]$$
$$= \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

- Check:

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-2y \\ 2x+y \end{bmatrix}$$

$$\text{i.e.}\, T(x, y, z) = (x-2y, 2x+y)$$

- Ex 2: (Finding the standard matrix of a linear transformation)

  The linear transformation $T : R^2 \rightarrow R^2$ is given by projecting each point in R^2 onto the x-axis. Find the standard matrix for T.

  Sol:

$$T(x, y) = (x, 0)$$

$$A = \left[T(e_1) \mid T(e_2)\right] = \left[T(1, 0) \mid T(0, 1)\right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- Notes:

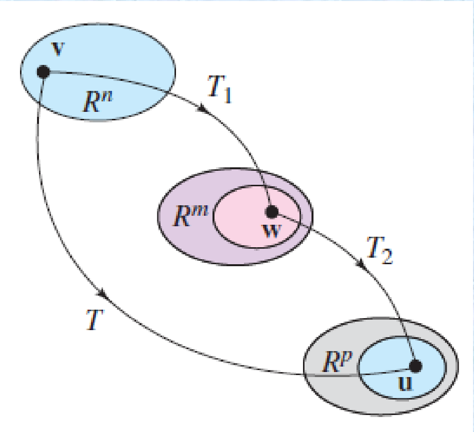
  (1) The standard matrix for the zero transformation from R^n into R^m is the $m \times n$ zero matrix.

  (2) The standard matrix for the identity transformation from R^n into R^n is the $n \times n$ identity matrix I_n.

- Composition of $T_1:R^n \to R^m$ with $T_2:R^m \to R^p$:

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})), \quad \mathbf{v} \in R^n$$

$$T = T_2 \circ T_1, \quad \text{domain of } T = \text{domain of } T_1$$



- Thm 6.11: (Composition of linear transformations)

Let $T_1 : R^n \to R^m$ and $T_2 : R^m \to R^p$ be L.T.

with standard matrices A_1 and A_2 , then

(1)The composition $T : R^n \to R^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is a L.T.

(2) The standard matrix A for T is given by the matrix product $A = \boxed{?}$

Pf:

(1)(T is a L.T.)

Let **u** and **v** be vectors in R^n and let c be any scalar then

$$T(\mathbf{u}+\mathbf{v}) = T_2(T_1(\mathbf{u}+\mathbf{v})) = T_2(T_1(\mathbf{u})+T_1(\mathbf{v}))$$
$$= T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) = T(\mathbf{u}) + T(\mathbf{v})$$

$$T(c\mathbf{v}) = T_2(T_1(c\mathbf{v})) = T_2(cT_1(\mathbf{v})) = cT_2(T_1(\mathbf{v})) = cT(\mathbf{v})$$

(2)(A_2A_1 is the standard matrix for T)

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})) = T_2(A_1\mathbf{v}) = A_2A_1\mathbf{v} = (A_2A_1)\mathbf{v}$$

- Note: (1) $T_1 \circ T_2 \neq T_2 \circ T_1$

  (2) $\mathrm{T(v)} = \mathrm{T_n}(T_{\mathrm{n}-1} \cdots (T_2(T_1(\mathrm{v}))) \cdots)$

  $\mathrm{A} = \mathrm{A_n A_{n-1}} \cdots \mathrm{A_2 A_1}$

- Ex 3: (The standard matrix of a composition)

Let T_1 and T_2 be L.T. from R^3 into R^3 s.t.

$$T_1(x, y, z) = (2x + y, 0, x + z)$$

$$T_2(x, y, z) = (x - y, z, y)$$

Find the standard matrices for the compositions

$T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$,

Sol:

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{(standard matrix for } T_1\text{)}$$

$$A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{(standard matrix for } T_2\text{)}$$

The standard matrix for $T = T_2 \circ T_1$

$$A = A_2 A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The standard matrix for $T' = T_1 \circ T_2$

$$A' = A_1 A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- Inverse linear transformation:

If $T_1 : R^n \to R^n$ and $T_2 : R^n \to R^n$ are L.T. s.t. for every $\mathbf{v}$ in R^n

$$T_2(T_1(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T_1(T_2(\mathbf{v})) = \mathbf{v}$$

Then T_2 is called the inverse of T_1 and T_1 is said to be invertible

- Note:

If the transformation T is invertible, then the inverse is unique and denoted by T^{-1} .

- Thm 6.12: (Existence of an inverse transformation)

Let $T : R^n \rightarrow R^n$ be a L.T. with standard matrix A,

Then the following condition are equivalent.

(1) T is invertible.

(2) T is an isomorphism.

(3) A is invertible.

- Note:

If T is invertible with standard matrix A, then the standard matrix for T^{-1} is A^{-1}.

- Ex 4: (Finding the inverse of a linear transformation)

The L.T. $T: R^3 \rightarrow R^3$ is defined by

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3, 3x_1 + 3x_2 + x_3, 2x_1 + 4x_2 + x_3)$$

Show that T is invertible, and find its inverse.

Sol:

The standard matrix for T

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \quad \begin{matrix} \leftarrow 2x_1 + 3x_2 + x_3 \\ \leftarrow 3x_1 + 3x_2 + x_3 \\ \leftarrow 2x_1 + 4x_2 + x_3 \end{matrix}$$

$$\left[A \mid I_3\right] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{G.J.E} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right] = \left[I \mid A^{-1}\right]$$

Therefore T is invertible and the standard matrix for T^{-1} is A^{-1}

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$T^{-1}(\mathbf{v}) = A^{-1}\mathbf{v} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ 6x_1 - 2x_2 - 3x_3 \end{bmatrix}$$

In other words,

$$T^{-1}(x_1, x_2, x_3) = (-x_1 + x_2, -x_1 + x_3, 6x_1 - 2x_2 - 3x_3)$$

- the matrix of T relative to the bases B *and* B':

$$T : V \rightarrow W \qquad \text{(a L.T.)}$$

$$B = \{v_1, v_2, \cdots, v_n\} \qquad \text{(a basis for } V\text{)}$$

$$B' = \{w_1, w_2, \cdots, w_m\} \qquad \text{(a basis for } W\text{)}$$

Thus, the matrix of T relative to the bases B *and* B' is

$$A = \left[[T(v_1)]_{B'}, [T(v_2)]_{B'}, \cdots, [T(v_n)]_{B'} \right] \in M_{m \times n}$$

- Transformation matrix for nonstandard bases:

Let V and W be finite - dimensional vector spaces with basis B and B', respectively, where $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$

If $T: V \rightarrow W$ is a L.T. s.t.

$$[T(\mathbf{v}_1)]_{B'} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad [T(\mathbf{v}_2)]_{B'} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, \quad [T(\mathbf{v}_n)]_{B'} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

then the $m \times n$ matrix whose n columns correspond to $[T(\mathbf{v}_i)]_{B'}$

$$A = \left[T(v_1)\ \middle|\ T(v_2)\ \middle|\ \cdots\ \middle|\ T(v_n)\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $\left[T(\mathbf{v})\right]_{B'} = A[\mathbf{v}]_B$ for every $\mathbf{v}$ in V.

- Ex 5: (Finding a matrix relative to nonstandard bases)

Let $T: R^2 \rightarrow R^2$ be a L.T. defined by

$$T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$$

Find the matrix of T relative to the basis

$B = \{(1, 2), (-1, 1)\}$ and $B' = \{(1, 0), (0, 1)\}$

Sol:

$$T(1, 2) = (3, 0) = 3(1, 0) + 0(0, 1)$$

$$T(-1, 1) = (0, -3) = 0(1, 0) - 3(0, 1)$$

$$\left[T(1, 2)\right]_{B'} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \left[T(-1, 1)\right]_{B'} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

the matrix for T relative to B and B'

$$A = \left[\left[T(1, 2)\right]_{B'} \quad \left[T(1, 2)\right]_{B'}\right] = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

- Ex 6:

For the L.T. $T: R^2 \to R^2$ given in Example 5, use the matrix A to find $T(\mathbf{v})$, where $\mathbf{v} = (2, 1)$

Sol:

$$\mathbf{v} = (2,1) = 1(1,2) - 1(-1,1) \qquad B = \{(1, 2), (-1, 1)\}$$

$$\Rightarrow [\mathbf{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow [T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow T(\mathbf{v}) = 3(1,0) + 3(0,1) = (3,3) \qquad B' = \{(1, 0), (0, 1)\}$$

- Check:

$$T(2,1) = (2+1, 2(2)-1) = (3,3)$$

- Notes:

(1)In the special case where $V = W$ and $B = B'$,

the matrix A is called the matrix of T relative to the basis B

(2)$T: V \rightarrow V$: the identity transformation

$B = \{v_1, v_2, \cdots, v_n\}$: a basis for V

$\Rightarrow$ the matrix of T relative to the basis B

$$A = \left[[T(v_1)]_B, [T(v_2)]_B, \cdots, [T(v_n)]_B \right] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n$$

6.4 Transition Matrices and Similarity

$$T : V \rightarrow V \qquad \text{(a L.T.)}$$

$$B = \{v_1, v_2, \cdots, v_n\} \qquad \text{(a basis of } V\text{)}$$

$$B' = \{w_1, w_2, \cdots, w_n\} \qquad \text{(a basis of } V\text{)}$$

$$A = \left[[T(v_1)]_B, [T(v_2)]_B, \cdots, [T(v_n)]_B\right] \qquad \text{(matrix of } T \text{ relative to } B\text{)}$$

$$A' = \left[[T(w_1)]_{B'}, [T(w_2)]_{B'}, \cdots, [T(w_n)]_{B'}\right] \qquad \text{(matrix of } T \text{ relative to } B'\text{)}$$

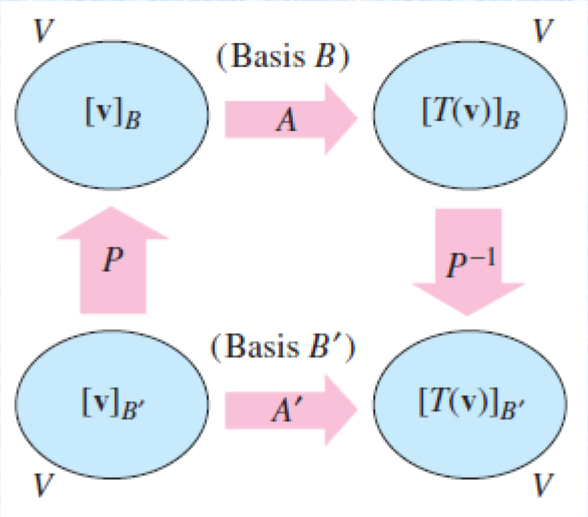
$$P = \left[[w_1]_B, [w_2]_B, \cdots, [w_n]_B\right] \qquad \text{(transition matrix from } B' \text{ to } B\text{)}$$

$$P^{-1} = \left[[v_1]_{B'}, [v_2]_{B'}, \cdots, [v_n]_{B'}\right] \qquad \text{(transition matrix from } B \text{ to } B'\text{)}$$

$$\therefore [\mathbf{v}]_B = P[\mathbf{v}]_{B'}, \ [\mathbf{v}]_{B'} = P^{-1}[\mathbf{v}]_B$$

$$[T(\mathbf{v})]_B = A[\mathbf{v}]_B$$

$$[T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'}$$



- Two ways to get from $[\mathbf{v}]_{B'}$ to $[T(\mathbf{v})]_{B'}$:

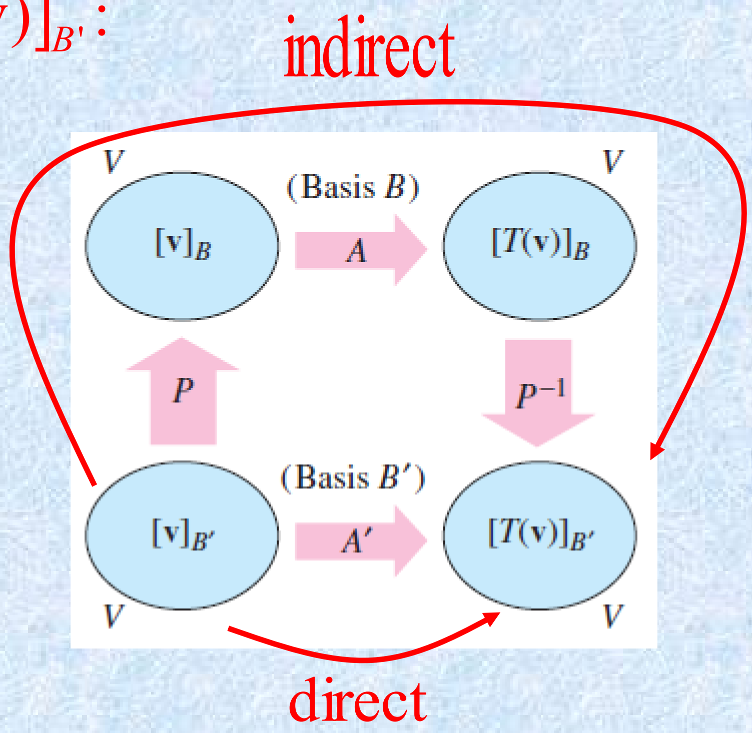
(1)(direct)

$$A'[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$$

(2)(indirect)

$$P^{-1}AP[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$$

$$\Rightarrow A' = P^{-1}AP$$

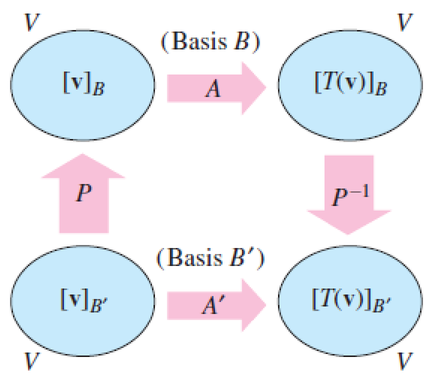


- Ex 1: (Finding a matrix for a linear transformation)

Find the matrix A' for $T: R^2 \rightarrow R^2$

$$T(x_1, x_2) = (2x_1 - 2x_2, -x_1 + 3x_2)$$

relative to the basis $B' = \{(1, 0), (1, 1)\}$



Sol:

(I) $A' = \left[[T(1,0)]_{B'} \quad [T(1,1)]_{B'} \right]$

$$T(1,0) = (2,-1) = 3(1,0) - 1(1,1) \;\Rightarrow\; [T(1,0)]_{B'} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$T(1,1) = (0,2) = -2(1,0) + 2(1,1) \;\Rightarrow\; [T(1,1)]_{B'} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\Rightarrow A' = \left[[T(1,0)]_{B'} \quad [T(1,1)]_{B'} \right] = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

(II) standard matrix for T (matrix of T relative to $B = \{(1,0),(0,1)\}$)

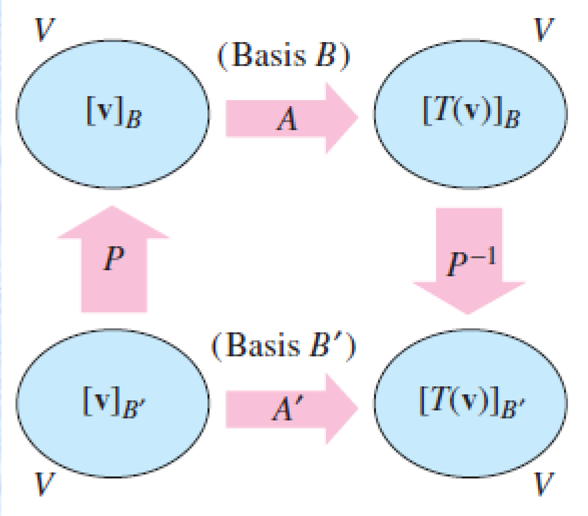
$$A = \begin{bmatrix} T(1,0) & T(0,1) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

transition matrix from B' to B

$$P = \begin{bmatrix} [(1,0)]_B & [(1,1)]_B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

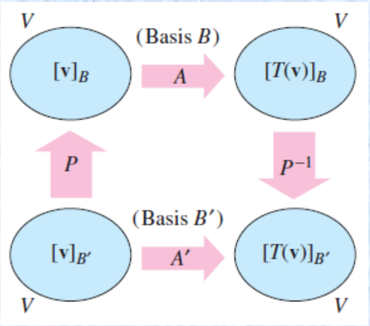
transition matrix from B to B'

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$



matrix of T relative B'

$$A' = P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$



- Ex 2: (Finding a matrix for a linear transformation)

Let $B=\{(-3,2),(4,-2)\}$ and $B'=\{(-1,2),(2,-2)\}$ be basis for R^2,

and let $A=\begin{bmatrix}-2 & 7\\ -3 & 7\end{bmatrix}$ be the matrix for $T: R^2 \rightarrow R^2$ relative to B.

Find the matrix of T relative to B'.

Sol:

transition matrix from B' to B: $P=\begin{bmatrix}[(-1,2)]_B & [(2,-2)]_B\end{bmatrix}=\begin{bmatrix}3 & -2\\ 2 & -1\end{bmatrix}$

transition matrix from B to B': $P^{-1}=\begin{bmatrix}[(-3,2)]_{B'} & [(4,-2)]_{B'}\end{bmatrix}=\begin{bmatrix}-1 & 2\\ -2 & 3\end{bmatrix}$

matrix of T relative to B':

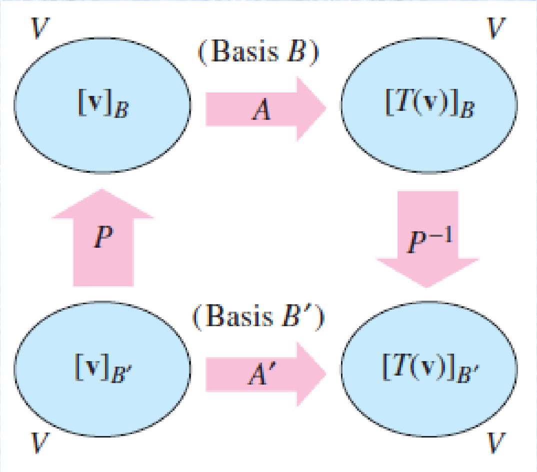
$$A'=P^{-1}AP=\begin{bmatrix}-1 & 2\\ -2 & 3\end{bmatrix}\begin{bmatrix}-2 & 7\\ -3 & 7\end{bmatrix}\begin{bmatrix}3 & -2\\ 2 & -1\end{bmatrix}=\begin{bmatrix}2 & 1\\ -1 & 3\end{bmatrix}$$

$B = \{(-3, 2), (4, -2)\}$ and $B' = \{(-1, 2), (2, -2)\}$

- Ex 3: (Finding a matrix for a linear transformation)

For the linear transformation $T : R^2 \to R^2$ given in Ex.2, find $[\mathbf{v}]_B$ $[T(\mathbf{v})]_B$ and $[T(\mathbf{v})]_{B'}$, for the vector $\mathbf{v}$ whose coordinate matrix is

$$[\mathbf{v}]_{B'} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$



Sol:

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \end{bmatrix}$$

$$[T(\mathbf{v})]_B = A[\mathbf{v}]_B = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} -7 \\ -5 \end{bmatrix} = \begin{bmatrix} -21 \\ -14 \end{bmatrix}$$

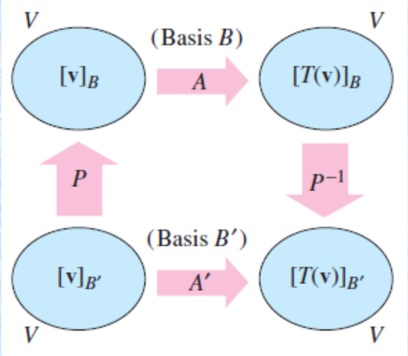
$$[T(\mathbf{v})]_{B'} = P^{-1}[T(\mathbf{v})]_B = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -21 \\ -14 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}$$

$$\text{or } [T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}$$

- Similar matrix:

For square matrices A and A' of order n, A' is said to be similar to A if there exist an invertible matrix P *s.t.* $A' = P^{-1}AP$

- Thm 6.13: (Properties of similar matrices)



Let A, B, and C be square matrices of order n. Then the following properties are true.

(1) A is similar to A.

(2) If A is similar to B, then B is similar to A.

(3) If A is similar to B and B is similar to C, then A is similar to C.

Pf:

$$(1)\ A = I_n A I_n$$

$$(2)\ A = P^{-1}BP \quad \Rightarrow \quad PAP^{-1} = P(P^{-1}BP)P^{-1}$$

$$PAP^{-1} = B \quad \Rightarrow \quad Q^{-1}AQ = B \ \ (Q = P^{-1})$$

- Ex 4: (Similar matrices)

$$(a)\ A=\begin{bmatrix}2 & -2\\ -1 & 3\end{bmatrix} \text{ and } A'=\begin{bmatrix}3 & -2\\ -1 & 2\end{bmatrix} \text{ are similar}$$

$$\text{because } A'=P^{-1}AP, \text{ where } P=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

$$(b)\ A=\begin{bmatrix}-2 & 7\\ -3 & 7\end{bmatrix} \text{ and } A'=\begin{bmatrix}2 & 1\\ -1 & 3\end{bmatrix} \text{ are similar}$$

$$\text{because } A'=P^{-1}AP, \text{ where } P=\begin{bmatrix}3 & -2\\ 2 & -1\end{bmatrix}$$

- Ex 5: (A comparison of two matrices for a linear transformation)

Suppose $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ is the matrix for $T: R^3 \rightarrow R^3$ relative to the standard basis. Find the matrix for T relative to the basis

$$B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$$

Sol:

The transition matrix from B' to the standard matrix

$$P = \big[[(1, 1, 0)]_B \quad [(1, -1, 0)]_B \quad [(0, 0, 1)]_B \big] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

matrix of T relative to B':

$$A' = P^{-1}AP = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Notes: Computational advantages of diagonal matrices:

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

$$(1) D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

$$(2) D^T = D$$

$$(3) D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}, \quad d_i \neq 0$$