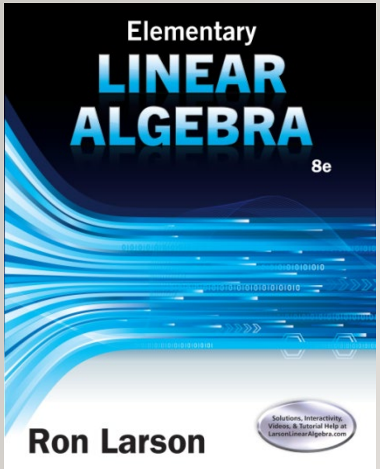


CHAPTER 4
VECTOR SPACES

4.4 Spanning Sets and Linear Independence

- Linear combination:

A vector $\mathbf{v}$ in a vector space V is called a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k \qquad c_1, c_2, \cdots, c_k : \text{scalars}$$

- Ex 2-3: (Finding a linear combination)

$$\mathbf{v}_1 = (1,2,3) \quad \mathbf{v}_2 = (0,1,2) \quad \mathbf{v}_3 = (-1,0,1)$$

Prove (a) $\mathbf{w} = (1,1,1)$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

(b) $\mathbf{w} = (1,-2,2)$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Sol:

(a) $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

$$\begin{aligned}(1,1,1) &= c_1(1,2,3) + c_2(0,1,2) + c_3(-1,0,1)\\ &= (c_1 - c_3,\ 2c_1 + c_2,\ 3c_1 + 2c_2 + c_3)\end{aligned}$$

$$\Rightarrow \begin{array}{rcl} c_1 \qquad\quad - c_3 & = & 1 \\ 2c_1 + \quad c_2 \qquad\quad & = & 1 \\ 3c_1 + 2c_2 + c_3 & = & 1 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{Guass}-\text{Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = 1 + t\ ,\ c_2 = -1 - 2t\ ,\ c_3 = t$$

(this system has infinitely many solutions)

$$\overset{t=1}{\Rightarrow} \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$$

(b) $\mathbf{w} = (1,-2,2)$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{Guass-Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

$\Rightarrow$ this system has no solution $(\because 0 \neq 7)$

$$\Rightarrow \mathbf{w} \neq c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

- the span of a set: span (S)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then **the span of S** is the set of all linear combinations of the vectors in S,

$$\text{span}(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \mid \forall c_i \in R\}$$

(the set of all linear combinations of vectors in S)

- a spanning set of a vector space:

If every vector in a given vector space can be written as a linear combination of vectors in a given set S, then S is called **a spanning set** of the vector space.

$$\text{span}(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \mid \forall c_i \in R\} = V$$

- Notes:

$$\text{span}(S) = V$$

$$\Rightarrow \; S \text{ spans (generates) } V$$

$$V \text{ is spanned (generated) by } S$$

$$S \text{ is a spanning set of } V$$

- Ex 5: (A spanning set for R^3)

Show that the set $S=\{(1,2,3),(0,1,2),(-2,0,1)\}$ spans R^3

Sol:

We must determine whether an arbitrary vector $\mathbf{u}=(u_1,u_2,u_3)$ in R^3 can be as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.

$$\mathbf{u}\in R^3 \Rightarrow \mathbf{u}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3$$

$$\begin{aligned} \Rightarrow\ c_1 - 2c_3 &= u_1 \\ 2c_1 + c_2 &= u_2 \\ 3c_1 + 2c_2 + c_3 &= u_3 \end{aligned}$$

The problem thus reduces to determining whether this system is consistent for all values of u_1, u_2, and u_3.

$$\because |A| = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \neq 0$$

$\Rightarrow A\mathbf{x} = \mathbf{b}$ has exactly one solution for every u.

$$\Rightarrow span(S) = R^3$$

- Thm 4.7: (Span(S) is a subspace of V)

If $S=\{\mathbf{v}_1, \mathbf{v}_2,\ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V,

then

(a) span (S) is a subspace of V.

Prove：

(b) span (S) is the smallest subspace of V that contains S.

(Every other subspace of V that contains S must contain span (S).)

- Linear Independent (L.I.) and Linear Dependent (L.D.):

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\} \quad \text{: a set of vectors in a vector space V}$$

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

(1) If the equation has only the trivial solution $(c_1 = c_2 = \cdots = c_k = 0)$

then S is called linearly independent.

(2) If the equation has a nontrivial solution (i.e., not all zeros),

then S is called linearly dependent.

- Ex 8: (Testing for linearly independent)

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{\underset{\mathbf{v}_1}{(1,2,3)}, \underset{\mathbf{v}_2}{(0,1,2)}, \underset{\mathbf{v}_3}{(-2,0,1)}\}$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \Rightarrow \begin{aligned} c_1 \qquad\quad - 2c_3 &= 0 \\ 2c_1 + \ c_2 + \qquad\quad &= 0 \\ 3c_1 + 2c_2 + \ c_3 &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{Gauss - Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = c_2 = c_3 = 0 \quad (\text{only the trivial solution})$$

$$\Rightarrow S \text{ is linearly independent}$$

- Ex 9: (Testing for linearly independent)

Determine whether the following set of vectors in P_2 is L.I. or L.D.

$$S = \{\underbrace{1+x-2x^2}_{\mathbf{v}_1},\ \underbrace{2+5x-x^2}_{\mathbf{v}_2},\ \underbrace{x+x^2}_{\mathbf{v}_3}\}$$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}$$

i.e. $c_1(1+x-2x^2)+c_2(2+5x-x^2)+c_3(x+x^2)=0+0x+0x^2$

$$\Rightarrow \begin{aligned} c_1+2c_2 \quad\quad &= 0 \\ c_1+5c_2+c_3 &= 0 \\ -2c_1-c_2+c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{G.J.}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ This system has infinitely many solutions.
(i.e., This system has nontrivial solutions.)

$\Rightarrow$ S is linearly dependent. (Ex: $c_1=2$, $c_2=-1$, $c_3=3$)

- Ex 10: (Testing for linearly independent)

Determine whether the following set of vectors in 2×2 matrix space is L.I. or L.D.

$$S = \left\{ \underset{\mathbf{v}_1}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}, \underset{\mathbf{v}_2}{\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}}, \underset{\mathbf{v}_3}{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}} \right\}$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$$c_1\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2c_1+3c_2+\ c_3 &= 0 \\ c_1 \qquad\qquad &= 0 \\ 2c_2+2c_3 &= 0 \\ c_1+ c_2 \qquad &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{Gauss - Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ $c_1 = c_2 = c_3 = 0$ (This system has only the trivial solution.)

$\Rightarrow$ S is linearly independent.

- Thm 4.8: (A property of linearly dependent sets)

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\boldsymbol{v}_j$ in S can be written as a linear combination of the other vectors in S.

Pf:

$(\Rightarrow)$ $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$

$\because S$ is linearly dependent

$\Rightarrow c_i \neq 0$ for some i

$$\Rightarrow \mathbf{v}_i = \frac{c_1}{-c_\mathrm{i}}\mathbf{v}_1 + \cdots + \frac{c_{i-1}}{-c_i}\mathbf{v}_{i-1} + \frac{c_{i+1}}{-c_i}\mathbf{v}_{i+1} + \cdots + \frac{c_k}{-c_i}\mathbf{v}_k$$

($\Leftarrow$)

Let $\mathbf{v}_i = d_1\mathbf{v}_1+\ldots+d_{i\text{-}1}\mathbf{v}_{i\text{-}1}+d_{i+1}\mathbf{v}_{i+1}+\ldots+d_k\mathbf{v}_k$

$\Rightarrow d_1\mathbf{v}_1+\ldots+d_{i\text{-}1}\mathbf{v}_{i\text{-}1}\text{-}\mathbf{v}_i+d_{i+1}\mathbf{v}_{i+1}+\ldots+d_k\mathbf{v}_k = \mathbf{0}$

$\Rightarrow c_1=d_1, \ldots, c_{i\text{-}1}=d_{i\text{-}1}, c_i=\text{-}1, c_{i+1}=d_{i+1}, \ldots, c_k=d_k$ (nontrivial solution)

$\Rightarrow S$ is linearly dependent

Key Learning in Section 4.4

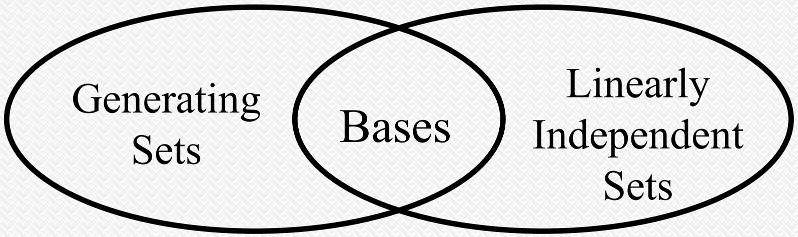
- Write a linear combination of a set of vectors in a vector space *V*.
- Determine whether a set *S* of vectors in a vector space *V* is a spanning set of *V*.
- Determine whether a set of vectors in a vector space *V* is linearly independent.

Keywords in Section 4.4:

- linear combination：線性組合
- spanning set：生成集合
- trivial solution：顯然解
- linear independent：線性獨立
- linear dependent：線性相依

4.5 Basis and Dimension

- Basis:



V : a vector space

$S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq V$

$$\begin{cases} (a) & S \text{ spans } V \text{ (i.e., } span(S) = V\text{)} \\ (b) & S \text{ is linearly independent} \end{cases}$$

$\Rightarrow$ S is called a **basis** for V

- Notes:

(1) Ø is a basis for $\{\mathbf{0}\}$

(2) the standard basis for R^3:

$\{i, j, k\}$ $\quad i = (1, 0, 0),\ j = (0, 1, 0),\ k = (0, 0, 1)$

(3) the standard basis for R^n :

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ $\quad \mathbf{e}_1 = (1,0,\ldots,0),\ \mathbf{e}_2 = (0,1,\ldots,0),\ \mathbf{e}_n = (0,0,\ldots,1)$

Ex: R^4 $\quad \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

(4) the standard basis for $m\times n$ matrix space:

$\{ E_{ij} \mid 1 \le i \le m ,\ 1 \le j \le n \}$

Ex: 2×2 matrix space:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

(5) the standard basis for $P_n(x)$:

$\{1, x, x^2, \ldots, x^n\}$

Ex: $P_3(x)$ $\quad \{1, x, x^2, x^3\}$

- Thm 4.9: (Uniqueness of basis representation)

If $S=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

Pf:

$$\because S \text{ is a basis } \Rightarrow \begin{cases} 1. & span(S)=V \\ 2. & S \text{ is linearly independent} \end{cases}$$

$\because span(S)=V$ Let $\mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n$

$\mathbf{v}=b_1\mathbf{v}_1+b_2\mathbf{v}_2+\ldots+b_n\mathbf{v}_n$

$$\Rightarrow \mathbf{0}=(c_1-b_1)\mathbf{v}_1+(c_2-b_2)\mathbf{v}_2+\ldots+(c_n-b_n)\mathbf{v}_n$$

$\because S$ is linearly independent

$\Rightarrow c_1=b_1,\ c_2=b_2, \ldots, c_n=b_n$ (i.e., uniqueness)

- Thm 4.10: (Bases and linear dependence)

If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent.

Pf:

Let $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$, $m > n$

$\because \text{span}(S)=V$

$$\mathbf{u}_i \in V \quad \Rightarrow \quad \begin{aligned} \mathbf{u}_1 &= c_{11}\mathbf{v}_1 + c_{21}\mathbf{v}_2 + \cdots + c_{n1}\mathbf{v}_n \\ \mathbf{u}_2 &= c_{12}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \cdots + c_{n2}\mathbf{v}_n \\ &\vdots \\ \mathbf{u}_m &= c_{1m}\mathbf{v}_1 + c_{2m}\mathbf{v}_2 + \cdots + c_{nm}\mathbf{v}_n \end{aligned}$$

Let $k_1\mathbf{u}_1+k_2\mathbf{u}_2+\ldots+k_m\mathbf{u}_m=\mathbf{0}$

$\Rightarrow d_1\mathbf{v}_1+d_2\mathbf{v}_2+\ldots+d_n\mathbf{v}_n=\mathbf{0}$ (where $d_i=c_{i1}k_1+c_{i2}k_2+\ldots+c_{im}k_m$)

$\because S$ is L.I.

$\Rightarrow d_i=0 \quad \forall i$ i.e.

$$\begin{aligned} c_{11}k_1+c_{12}k_2+\cdots+c_{1m}k_m &= 0 \\ c_{21}k_1+c_{22}k_2+\cdots+c_{2m}k_m &= 0 \\ &\vdots \\ c_{n1}k_1+c_{n2}k_2+\cdots+c_{nm}k_m &= 0 \end{aligned}$$

$\because$ Thm 1.1: If the homogeneous system has fewer equations than variables, then it must have infinitely many solution.

$m>n \Rightarrow k_1\mathbf{u}_1+k_2\mathbf{u}_2+\ldots+k_m\mathbf{u}_m=\mathbf{0}$ has nontrivial solution

$\Rightarrow S_1$ is linearly dependent

- Thm 4.11: (Number of vectors in a basis)

If a vector space V has one basis with n vectors, then every basis for V has n vectors. (All bases for a finite-dimensional vector space has the same number of vectors.)

Pf:

$S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$
$S'=\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ two bases for a vector space

$$\left.\begin{array}{l}\left.\begin{array}{r} S \text{ is a basis} \\ S' \text{ is L.I.}\end{array}\right\} \overset{Thm.4.10}{\Longrightarrow} n \geq m \\ \left.\begin{array}{r} S \text{ is L.I.} \\ S' \text{ is a basis}\end{array}\right\} \overset{Thm.4.10}{\Longrightarrow} n \leq m\end{array}\right\} \Rightarrow n=m$$

- Finite dimensional:

  A vector space V is called **finite dimensional**,

  if it has a basis consisting of a finite number of elements.

- Infinite dimensional:

  If a vector space V is not finite dimensional,

  then it is called **infinite dimensional**.

- Dimension:

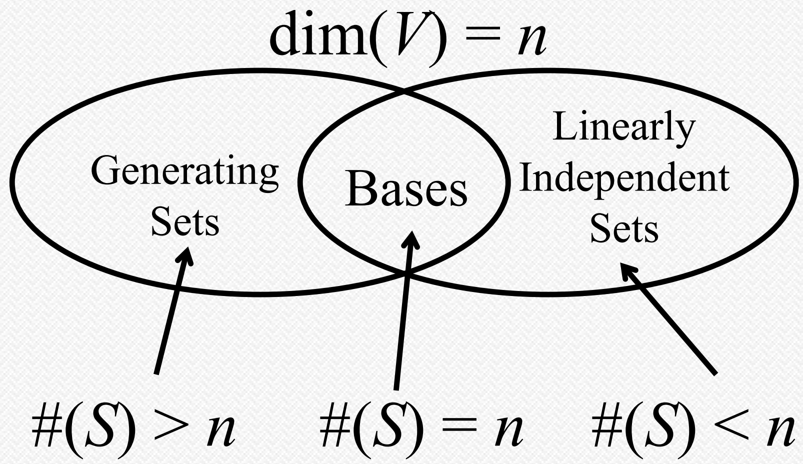
  The **dimension** of a finite dimensional vector space V is defined to be <u>the number of vectors in a basis</u> for V.

  V: a vector space $\qquad$ S: a basis for V

  $\Rightarrow \dim(V) = \#(S)$ $\qquad$ (the number of vectors in S)

- Notes:

(1) $\dim(\{\mathbf{0}\}) = 0 = \#(\varnothing)$



(2) $\dim(V) = n$, $S \subseteq V$

S : a generating set $\Rightarrow \#(S) \geq n$

S : a L.I. set $\Rightarrow \#(S) \leq n$

S : a basis $\Rightarrow \#(S) = n$

(3) $\dim(V) = n$, W is a subspace of V $\Rightarrow$ $\dim(W) \leq n$

- Ex:

(1) Vector space R^n $\Rightarrow$ basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

$\Rightarrow \dim(R^n) = n$

(2) Vector space $M_{m\times n}$ $\Rightarrow$ basis $\{E_{ij} \mid 1\le i\le m,\ 1\le j\le n\}$

$\Rightarrow \dim(M_{m\times n}) = mn$

(3) Vector space $P_n(x)$ $\Rightarrow$ basis $\{1, x, x^2, \ldots, x^n\}$

$\Rightarrow \dim(P_n(x)) = n+1$

(4) Vector space $P(x)$ $\Rightarrow$ basis $\{1, x, x^2, \ldots\}$

$\Rightarrow \dim(P(x)) = \infty$

- Ex 9: (Finding the dimension of a subspace)

  (a) $W=\{(d, c-d, c)\text{: } c \text{ and } d \text{ are real numbers}\}$

  (b) $W=\{(2b, b, 0)\text{: } b \text{ is a real number}\}$

Sol: (Note: Find a set of L.I. vectors that spans the subspace)

(a) $(d, c-d, c) = c(0, 1, 1) + d(1, -1, 0)$

$\Rightarrow S = \{(0, 1, 1), (1, -1, 0)\}$ (S is L.I. and S *spans* W)

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 2$

(b) $\because (2b, b, 0) = b(2, 1, 0)$

$\Rightarrow S = \{(2, 1, 0)\}$ spans W and S is L.I.

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 1$

- Ex 11: (Finding the dimension of a subspace)

  Let W be the subspace of all symmetric matrices in $M_{2\times 2}$.

  What is the dimension of W?

  Sol:

$$W=\left\{\begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a,b,c\in R\right\}$$

$$\because \begin{bmatrix} a & b \\ b & c \end{bmatrix}=a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}+b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}+c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow S=\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\} \text{ spans } W \text{ and } S \text{ is L.I.}$$

$\Rightarrow S$ is a basis for W $\quad \Rightarrow \dim(W)=\#(S)=3$

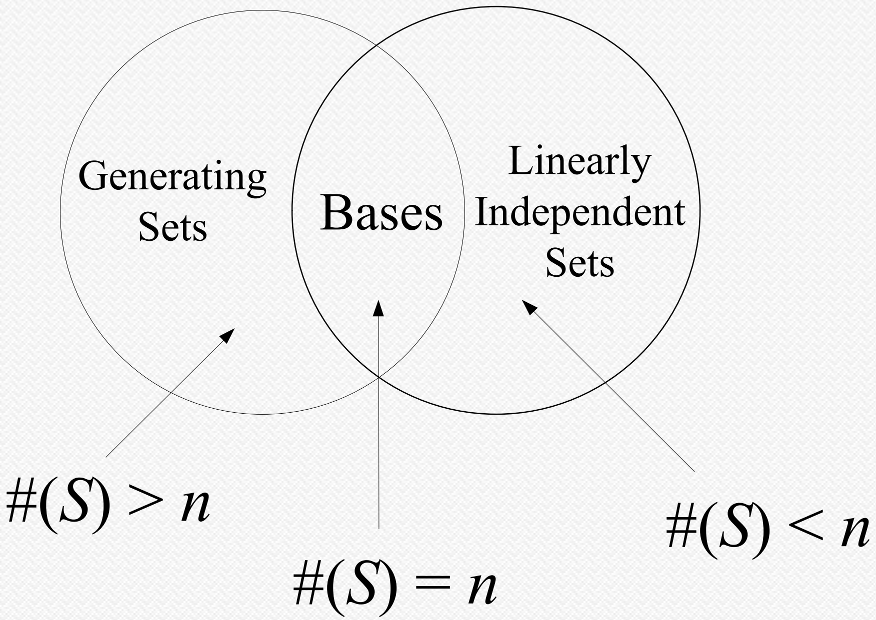
- Thm 4.12: (Basis tests in an n-dimensional space)

Let V be a vector space of dimension n.

(1) If $S=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.

(2) If $S=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ spans V, then S is a basis for V.

$$\dim(V)=n$$



Key Learning in Section 4.5

- Recognize bases in the vector spaces R^n, P_n and $M_{m,n}$
- Find the dimension of a vector space.

Keywords in Section 4.5

- basis：基底
- dimension：維度
- finite dimension：有限維度
- infinite dimension：無限維度

4.6 Rank of a Matrix and Systems of Linear Equations

- row vectors:

Row vectors of A

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}=\begin{bmatrix} A_{(1)} \\ A_{(2)} \\ \vdots \\ A_{(m)} \end{bmatrix}$$

$$\begin{gathered}[a_{11}, a_{12}, \ldots, a_{1n}] = A_{(1)} \\ [a_{21}, a_{22}, \ldots, a_{2n}] = A_{(2)} \\ \vdots \\ [a_{m1}, a_{m2}, \ldots, a_{mn}] = A_{(n)}\end{gathered}$$

- column vectors:

Column vectors of A

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}=\left[A^{(1)} \vdots A^{(2)} \vdots \cdots \vdots A^{(n)}\right]$$

$$\underset{\displaystyle A^{(1)}}{\overset{\|}{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}}} \underset{\displaystyle A^{(2)}}{\overset{\|}{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}}} \cdots \underset{\displaystyle A^{(n)}}{\overset{\|}{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}}}$$

Let A be an $m\times n$ matrix.

- Row space:

  The **row space** of A is the subspace of R^n spanned by the row vectors of A.

$$RS(A)=\{\alpha_1 A_{(1)}+\alpha_2 A_{(2)}+\ldots+\alpha_m A_{(m)} \mid \alpha_1,\alpha_2,\ldots,\alpha_m\in R\}$$

- Column space:

  The **column space** of A is the subspace of R^m spanned by the column vectors of A.

$$CS(A)=\{\beta_1 A^{(1)}+\beta_2 A^{(2)}+\cdots+\beta_n A^{(n)} \mid \beta_1,\beta_2,\cdots\beta_n\in R\}$$

- Null space (零空間):

  The **null space** of A is the set of all solutions of $A\mathbf{x}=\mathbf{0}$ and it is a subspace of R^n.

$$NS(A)=\{\mathbf{x}\in R^n \mid A\mathbf{x}=\mathbf{0}\}$$

- Thm 4.13: (Row-equivalent matrices have the same row space)

  If an $m \times n$ matrix A is row equivalent to an $m \times n$ matrix B,

  then the row space of A is equal to the row space of B.

- Proof:

- Ex 2: (Finding a basis for a row space)

Find a basis of row space of $A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix}$

Sol:

$$A = \begin{array}{c} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix} \\ \begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{array} \end{array} \xrightarrow{\text{G.E.}} B = \begin{array}{cc} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \\ \\ \end{array} \\ \begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{array} & \end{array}$$

a basis for $RS(A)$ = {the nonzero row vectors of B}

$= \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \{(1, 3, 1, 3), (0, 1, 1, 0), (0, 0, 0, 1)\}$

- Thm 4.14: (Basis for the row space of a matrix)

  If a matrix A is row equivalent to a matrix B in row-echelon form, then the nonzero row vectors of B form a basis for the row space of A.

- Ex 3: (Finding a basis for a subspace)

Find a basis for the subspace of R^3 spanned by

$$S = \{(\overset{\mathbf{v}_1}{-1, 2, 5}), (\overset{\mathbf{v}_2}{3, 0, 3}), (\overset{\mathbf{v}_3}{5, 1, 8})\}$$

Sol:

$$A = \begin{array}{c} \begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{array} \xrightarrow{\text{G.E.}} B = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \mathrm{w}_1 \\ \mathrm{w}_2 \\ \end{array}$$

a basis for $span(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$

= a basis for $RS(A)$

= {the nonzero row vectors of B} (Thm 4.14)

= $\{\mathbf{w}_1, \mathbf{w}_2\}$

= {(1, –2, – 5) , (0, 1, 3)}

- Ex 4-5: (Finding a basis for the column space of a matrix)

Find a basis for the column space of the matrix A given in Ex 2.

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

Sol. (Method 1):

$$A^T = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & -4 \\ 3 & 0 & -1 & 1 & -2 \end{bmatrix} \xrightarrow{G.E.} B = \begin{array}{l} \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \boldsymbol{w}_3 \\ \\ \end{array} \end{array}$$

$\because \quad CS(A) = RS(A^T)$

$\therefore$ a basis for $CS(A)$

$=$ a basis for $RS(A^T)$

$=$ {the nonzero vectors of B}

$= \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$

$$= \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 9 \\ -5 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$ (a basis for the column space of A)

- Note: This basis is not a subset of $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$.

- Sol. (Method 2):

$$A = \begin{array}{c} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix} \\ \begin{matrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_4 \end{matrix} \end{array} \xrightarrow{G.E.} B = \begin{array}{c} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{matrix} \end{array}$$

Leading 1 => $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis for $CS(B)$

$\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4\}$ is a basis for $CS(A)$

- Notes:

(1) This basis is a subset of $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$.

(2) $\mathbf{v}_3 = -2\mathbf{v}_1 + \mathbf{v}_2$, thus $\mathbf{c}_3 = -2\mathbf{c}_1 + \mathbf{c}_2$.

$$A=\begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

a basis for $RS(A)$ = {the nonzero row vectors of B}

$= \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$

$= \{(1, 3, 1, 3), (0, 1, 1, 0), (0, 0, 0, 1)\}$

a basis for $CS(A)$

= a basis for $RS(A^T)$

= {the nonzero vectors of B}

$= \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$

$$= \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 9 \\ -5 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

dim(RS(A))=3=dim(CS(A))

- Thm 4.15: (Row and column space have equal dimensions)

  If A is an $m \times n$ matrix, then the row space and the column space of A have the same dimension.

$$\dim(RS(A)) = \dim(CS(A))$$

- Rank(矩陣的秩):

  The dimension of the row (or column) space of a matrix A is called the **rank** of A and is denoted by rank(A).

$$\operatorname{rank}(A) = \dim(RS(A)) = \dim(CS(A))$$

- Thm 4.16: (Solutions of a homogeneous system)

  If A is an $m \times n$ matrix, then the set of all solutions of the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$ is a subspace of R^n called the nullspace (零空間) of A.

$$NS(A) = \{x \in R^n \mid Ax = 0\}$$

Pf:

$$NS(A) \in R^n$$

$$NS(A) \neq \phi \quad (\because A\mathbf{0} = \mathbf{0})$$

Let $\mathbf{x}_1, \mathbf{x}_2 \in NS(A)$ (i.e. $A\mathbf{x}_1 = \mathbf{0}$, $A\mathbf{x}_2 = \mathbf{0}$)

Then (1) $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$ Addition

(2) $A(c\mathbf{x}_1) = c(A\mathbf{x}_1) = c(\mathbf{0}) = \mathbf{0}$ Scalar multiplication

Thus $NS(A)$ is a subspace of R^n

- Notes: The nullspace of A is also called the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

- Ex 7: (Finding the solution space of a homogeneous system)

Find the nullspace of the matrix A.

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$$

Sol: The nullspace of A is the solution space of $A\mathbf{x} = \mathbf{0}$.

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix} \xrightarrow{G.J.E} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x_1 = -2s - 3t,\ x_2 = s,\ x_3 = -t,\ x_4 = t$$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s-3t \\ s \\ -t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix} = s\mathbf{v}_1 + t\mathbf{v}_2$$

$$\Rightarrow NS(A) = \{s\mathbf{v}_1 + t\mathbf{v}_2 \mid s, t \in R\}$$

- Nullity:

The dimension of the nullspace of A is called the nullity of A.

$$\text{nullity}(A) = \dim(NS(A))$$

- Note: $\text{rank}(A^T) = \text{rank}(A)$

Pf: $\text{rank}(A^T) = \dim(RS(A^T)) = \dim(CS(A)) = \text{rank}(A)$

- Thm 4.17: (Dimension of the solution space)

  If A is an $m \times n$ matrix of rank r, then the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is $n - r$. That is

  $$n = \text{rank}(A) + \text{nullity}(A)$$

- Notes:

(1) rank(A): The number of nonzero rows in the row-echelon form of A

(2) nullity (A): The number of free variables in the solution of A$\mathbf{x} = \mathbf{0}$.

- Notes:

  If A is an $m \times n$ matrix and rank(A) = r, then

Fundamental Space	Dimension
$RS(A)=CS(A^T)$	r
$CS(A)=RS(A^T)$	r
$NS(A)$	$n-r$
$NS(A^T)$	$m-r$

- Ex 8: (Rank and nullity of a matrix)

  Let the column vectors of the matrix A be denoted by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, and $\mathbf{a}_5$.

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \\ \begin{array}{ccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{array} \end{array}$$

(a) Find the rank and nullity of A.

(b) Find a subset of the column vectors of A that forms a basis for the column space of A.

Sol: Let B be the reduced row-echelon form of A.

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{matrix} \qquad\qquad \begin{matrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \end{matrix}$$

(a) $\text{rank}(A) = 3$ (the number of nonzero rows in B)

$\text{nuillity}(A) = n - \text{rank}(A) = 5 - 3 = 2$

(b) Leading 1

$\Rightarrow \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4\}$ is a basis for $CS(B)$

$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is a basis for $CS(A)$

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \; \mathbf{a}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \text{ and } \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

- Thm 4.18: (Solutions of a nonhomogeneous linear system)

If $\mathbf{x}_p$ is a particular solution of the nonhomogeneous system $A\mathbf{x} = \boldsymbol{b}$, then every solution of this system can be written in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, wher $\mathbf{x}_h$ is a solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$.

Pf: Let $\mathbf{x}$ be any solution of $A\mathbf{x} = \boldsymbol{b}$.

$$\Rightarrow A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

$$\Rightarrow (\mathbf{x} - \mathbf{x}_p) \quad \text{is a solution of } A\mathbf{x} = \mathbf{0}$$

Let $\mathbf{x}_h = \mathbf{x} - \mathbf{x}_p$

$$\Rightarrow \mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$$

- Ex 9: (Finding the solution set of a nonhomogeneous system)

Find the set of all solution vectors of the system of linear equations.

$$\begin{array}{rcrcrcrcr} x_1 & & & - & 2x_3 & + & x_4 & = & 5 \\ 3x_1 & + & x_2 & - & 5x_3 & & & = & 8 \\ x_1 & + & 2x_2 & & & - & 5x_4 & = & -9 \end{array}$$

Sol:

$$\left[\begin{array}{rrrr:r} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{array}\right] \xrightarrow{G.J.E} \begin{array}{c} \left[\begin{array}{rrrr:r} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{rrrrr} & & s & t & \end{array} \end{array}$$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s & - & t & + & 5 \\ -s & + & 3t & - & 7 \\ s & + & 0t & + & 0 \\ 0s & + & t & + & 0 \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$$

$$= s\mathbf{u}_1 + t\mathbf{u}_2 + \mathbf{x}_p$$

i.e. $\mathbf{x}_p = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution vector of $A\mathbf{x}=\mathbf{b}$.

$\mathbf{x}_h = s\mathbf{u}_1 + t\mathbf{u}_2$ is a solution of $A\mathbf{x} = \mathbf{0}$

- Thm 4.19: (Solution of a system of linear equations)

  The system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Pf:

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

be the coefficient matrix, the column matrix of unknowns, and the right-hand side, respectively, of the system $A\mathbf{x} = \mathbf{b}$.

Then

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n \\ \vdots & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Hence, $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A. That is, the system is consistent if and only if $\mathbf{b}$ is in the subspace of R^m spanned by the columns of A.

- Note:

If rank$([A|\mathbf{b}])$=rank(A)

Then the system $A\mathbf{x}=\mathbf{b}$ is consistent.

- Ex 10: (Consistency of a system of linear equations)

$$
\begin{array}{rcrcrcr}
x_1 & + & x_2 & - & x_3 & = & -1 \\
x_1 & & & + & x_3 & = & 3 \\
3x_1 & + & 2x_2 & - & x_3 & = & 1
\end{array}
$$

Sol:

$$
A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix} \xrightarrow{G.J.E.} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$[A \vdots \mathbf{b}] = \begin{array}{c} \begin{bmatrix} 1 & 1 & -1 & \vdots & -1 \\ 1 & 0 & 1 & \vdots & 3 \\ 3 & 2 & -1 & \vdots & 1 \end{bmatrix} \\ \begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{b} \end{array} \end{array} \xrightarrow{G.J.E.} \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & \vdots & 3 \\ 0 & 1 & -2 & \vdots & -4 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \\ \begin{array}{cccc} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{v} \end{array} \end{array}$$

$$\because \mathbf{v} = 3\mathbf{w}_1 - 4\mathbf{w}_2$$

$$\Rightarrow \mathbf{b} = 3\mathbf{c}_1 - 4\mathbf{c}_2 + 0\mathbf{c}_3 \quad (\mathbf{b} \text{ is in the column space of } A)$$

$\Rightarrow$ The system of linear equations is consistent.

- Check:

$$\text{rank}(A) = \text{rank}([A \vdots \mathbf{b}]) = 2$$

- Summary of equivalent conditions for square matrices:

If A is an $n \times n$ matrix, then the following conditions are equivalent.

(1) A is invertible

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $n \times 1$ matrix $\mathbf{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(4) A is row-equivalent to I_n

(5) $|A| \neq 0$

(6) $\text{rank}(A) = n$

(7) The n row vectors of A are linearly independent.

(8) The n column vectors of A are linearly independent.

Key Learning in Section 4.6

- Find a basis for the row space, a basis for the column space, and the rank of a matrix.
- Find the nullspace of a matrix.
- Find the solution of a consistent system $A\mathbf{x} = \mathbf{b}$ in the form $\mathbf{x}_p + \mathbf{x}_h$.

Keywords in Section 4.6:

- row space : 列空間
- column space : 行空間
- null space: 零空間
- solution space : 解空間
- rank: 秩
- nullity : 核次數

4.7 Coordinates and Change of Basis

- Coordinate representation relative to a basis

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered basis for a vector space V and let $\mathbf{x}$ be a vector in V such that

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

The scalars $c_1, c_2, \ldots, c_n$ are called the **coordinates of x relative to the basis *B***. The **coordinate matrix** (or **coordinate vector**) of $\mathbf{x}$ relative to B is the column matrix in R^n whose components are the coordinates of $\mathbf{x}$.

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

- Ex 1: (Coordinates and components in R^n)

  Find the coordinate matrix of $\mathbf{x} = (-2, 1, 3)$ in R^3

  relative to the standard basis

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

Sol:

$$\because \mathbf{x} = (-2, 1, 3) = -2(1, 0, 0) + 1(0, 1, 0) + 3(0, 0, 1),$$

$$\therefore [\mathbf{x}]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}.$$

- Ex 3: (Finding a coordinate matrix relative to a nonstandard basis)

Find the coordinate matrix of $\mathbf{x} = (1, 2, -1)$ in R^3

relative to the (nonstandard) basis

$$B' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$$

Sol: $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 \Rightarrow (1, 2, -1) = c_1(1, 0, 1) + c_2(0, -1, 2) + c_3(2, 3, -5)$

$$\Rightarrow \begin{array}{rcrcrcr} c_1 & & & + & 2c_3 & = & 1 \\ & & -c_2 & + & 3c_3 & = & 2 \\ c_1 & + & 2c_2 & - & 5c_3 & = & -1 \end{array} \quad \text{i.e.} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{rrrr} 1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 2 \\ 1 & 2 & -5 & -1 \end{array}\right] \xrightarrow{\text{G.J.E.}} \left[\begin{array}{rrrr} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -2 \end{array}\right] \Rightarrow [\boldsymbol{x}]_{B'} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

- Change of basis problem:

You were given the coordinates of a vector relative to one basis B and were asked to find the coordinates relative to another basis B'.

- Ex: (Change of basis)

Consider two bases for a vector space V

$$B=\{\mathbf{u}_1,\mathbf{u}_2\},\ B'=\{\mathbf{u}_1',\mathbf{u}_2'\}$$

$$\text{If } [\mathbf{u}_1']_B=\begin{bmatrix} a \\ b \end{bmatrix},\ [\mathbf{u}_2']_B=\begin{bmatrix} c \\ d \end{bmatrix}$$

$$\text{i.e., } \mathbf{u}_1'=a\mathbf{u}_1+b\mathbf{u}_2,\quad \mathbf{u}_2'=c\mathbf{u}_1+d\mathbf{u}_2$$

$$\text{Let } \mathbf{v} \in V,\ [\mathbf{v}]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \mathbf{v} &= k_1\mathbf{u}_1' + k_2\mathbf{u}_2' \\ &= k_1(a\mathbf{u}_1 + b\mathbf{u}_2) + k_2(c\mathbf{u}_1 + d\mathbf{u}_2) \\ &= (k_1a + k_2c)\mathbf{u}_1 + (k_1b + k_2d)\mathbf{u}_2 \end{aligned}$$

$$\begin{aligned} \Rightarrow [\mathbf{v}]_B &= \begin{bmatrix} k_1a + k_2c \\ k_1b + k_2d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \\ &= \left[[\mathbf{u}_1']_B \ [\mathbf{u}_2']_B \right] [\mathbf{v}]_{B'} \end{aligned}$$

- Transition matrix from *B*' to *B*:

Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2 \ldots, \mathbf{u}'_n\}$ be two bases for a vector space V

If $[\mathbf{v}]_B$ is the coordinate matrix of **v** relative to *B*

$[\mathbf{v}]_{B'}$ is the coordinate matrix of **v** relative to *B*'

then $[\mathbf{v}]_B = P[\mathbf{v}]_{B'}$

$$= \left[[\mathbf{u}'_1]_B, [\mathbf{u}'_2]_B, \ldots, [\mathbf{u}'_n]_B\right] [v]_{B'}$$

where

$$P = \left[[\mathbf{u}'_1]_B, [\mathbf{u}'_2]_B, \ldots, [\mathbf{u}'_n]_B\right]$$

is called **the transition matrix from *B*' to *B***

- Thm 4.20: (The inverse of a transition matrix)

If P is the transition matrix from a basis B' to a basis B in R^n, then

(1) P is invertible

(2) The transition matrix from B to B' is P^{-1}

- Notes:

$$B = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}, \quad B' = \{\mathbf{u}'_1, \mathbf{u}'_2, ..., \mathbf{u}'_n\}$$

$$\left[\mathbf{v}\right]_B = \left[[\mathbf{u}'_1]_B, [\mathbf{u}'_2]_B, ..., [\mathbf{u}'_n]_B\right] \left[\mathbf{v}\right]_{B'} = P\left[\mathbf{v}\right]_{B'}$$

$$\left[\mathbf{v}\right]_{B'} = \left[[\mathbf{u}_1]_{B'}, [\mathbf{u}_2]_{B'}, ..., [\mathbf{u}_n]_{B'}\right] \left[\mathbf{v}\right]_B = P^{-1}\left[\mathbf{v}\right]_B$$

- Thm 4.21: (Transition matrix from B to B')

  Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be two bases for R^n. Then the transition matrix P^{-1} from B to B' can be found by using Gauss-Jordan elimination on the $n \times 2n$ matrix $[B' \vdots B]$ as follows.

$$[B' \vdots B] \longrightarrow [I_n \vdots P^{-1}]$$

- Ex 5: (Finding a transition matrix)

$B=\{(-3, 2), (4,-2)\}$ and $B'=\{(-1, 2), (2,-2)\}$ are two bases for R^2

(a) Find the transition matrix from B' to B.

(b) Let $[\mathbf{v}]_{B'}=\begin{bmatrix}1\\2\end{bmatrix}$, find $[\mathbf{v}]_B$

(c) Find the transition matrix from B to B' .

Sol:

(a) $$\underset{B \qquad\qquad\quad B'}{\left[\begin{array}{cc:cc} -3 & 4 & -1 & 2 \\ 2 & -2 & 2 & -2 \end{array}\right]} \xrightarrow{\text{G.J.E.}} \underset{I \qquad\quad P}{\left[\begin{array}{cc:cc} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & -1 \end{array}\right]}$$

$$\therefore P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \quad \text{(the transition matrix from } B' \text{ to } B\text{)}$$

(b)

Check :

$$[\boldsymbol{v}]_{B'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow [\boldsymbol{v}]_B = P[\boldsymbol{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$[\boldsymbol{v}]_{B'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \boldsymbol{v} = (1)(-1,2) + (2)(2,-2) = (3,-2)$$

$$[\boldsymbol{v}]_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{v} = (-1)(3,-2) + (0)(4,-2) = (3,-2)$$

(c)

$$\left[\begin{array}{cc:cc} -1 & 2 & -3 & 4 \\ 2 & -2 & 2 & -2 \end{array}\right] \xrightarrow{\text{G.J.E.}} \left[\begin{array}{cc:cc} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \end{array}\right]$$

$$\qquad B' \qquad\qquad B \qquad\qquad\qquad\qquad I \qquad\qquad P^{-1}$$

$$\therefore P^{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \quad \text{(the transition matrix from } B \textit{ to } B'\text{)}$$

- Check:

$$PP^{-1} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

- Ex 6: (Coordinate representation in $P_3(x)$)

  (a) Find the coordinate matrix of $p = 3x^3 - 2x^2 + 4$ relative to the standard basis $S = \{1, x, x^2, x^3\}$ in $P_3(x)$.

  (b) Find the coordinate matrix of $p = 3x^3 - 2x^2 + 4$ relative to the basis $S = \{1, 1+x, 1+x^2, 1+x^3\}$ in $P_3(x)$.

Sol:

$$\text{(a) } p = (4)(1) + (0)(x) + (\text{-}2)(x^2) + (3)(x^3) \quad \Rightarrow [p]_B = \begin{bmatrix} 4 \\ 0 \\ -2 \\ 3 \end{bmatrix}$$

$$\text{(b) } p = (3)(1) + (0)(1+x) + (\text{-}2)(1+x^2) + (3)(1+x^3) \quad \Rightarrow [p]_B = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 3 \end{bmatrix}$$

- Ex: (Coordinate representation in M_{2x2})

Find the coordinate matrix of $x = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ relative to the standard basis in $M_{2\times 2}$.

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Sol:

$$x = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = 5\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 7\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 8\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow [x]_B = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

Key Learning in Section 4.7

- Find a coordinate matrix relative to a basis in R^n
- Find the transition matrix from the basis to the basis B' in R^n.
- Represent coordinates in general n-dimensional spaces.

Keywords in Section 4.7

- coordinates of **x** relative to B: **x**相對於B的座標
- coordinate matrix: 座標矩陣
- coordinate vector: 座標向量
- change of basis problem: 基底變換問題
- transition matrix from B' to B: 從 B' 到 B的轉移矩陣