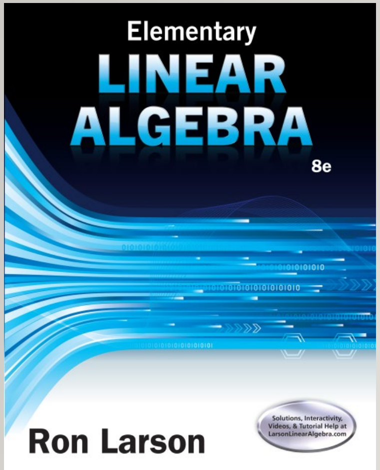


# CHAPTER 1 SYSTEMS OF LINEAR EQUATIONS





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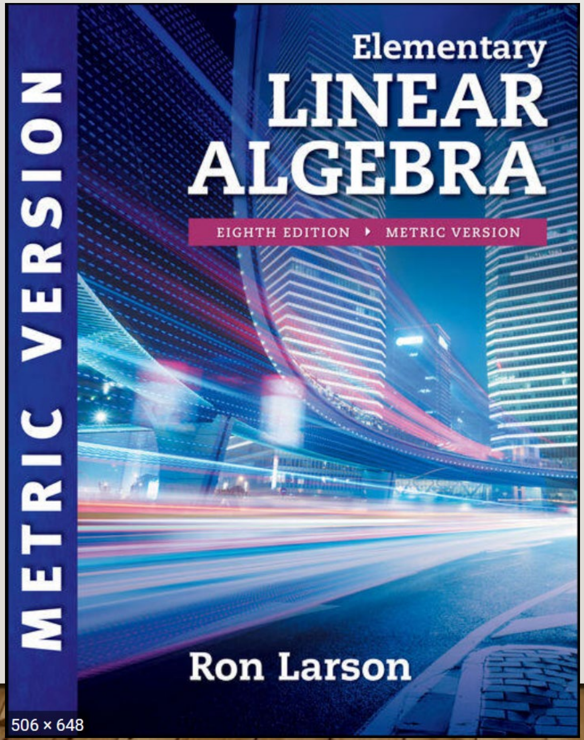
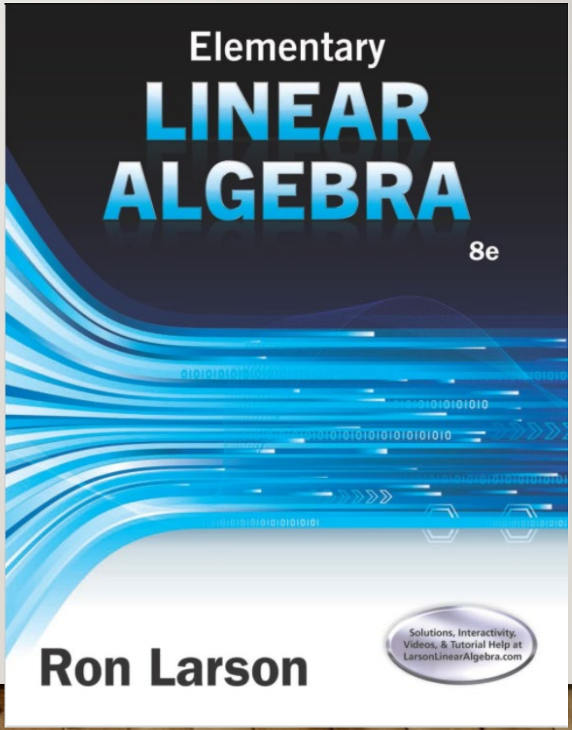
# 參考書目

- Elementary Linear Algebra 8E LARSON, Princeton
- 台灣代理：高立圖書 劉家宏
  - 0921-456018
  - 02-229-0318



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# 評分標準

- 課程參與：20%
- 作業：20%
- 期中考：30%
- 期末考：30%

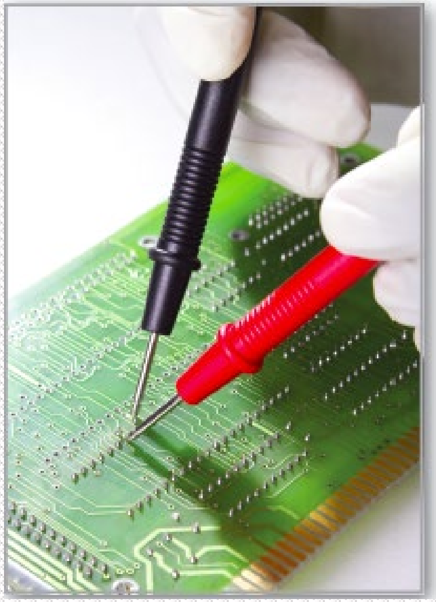
# CH 1 Linear Algebra Applied

Balancing Chemical Equations (p.4)

Airspeed of a Plane (p.11)

Global Positioning System (p.16)

Traffic Flow (p.28)



Electrical Network Analysis (p.30)

# 1.1 Introduction to Systems of Linear Equations

- a linear equation in $n$ variables:

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

$a_1, a_2, a_3, \ldots, a_n, b$: real number

$a_1$: leading coefficient

$x_1$: leading variable

- Notes:

(1) Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions.

(2) Variables appear only to the first power.

- Ex 1: (Linear or Nonlinear)

Linear $(a)\ 3x + 2y = 7$

$(b)\ \frac{1}{2}x + y - \pi z = \sqrt{2}$ Linear

Linear $(c)\ x_1 - 2x_2 + 10x_3 + x_4 = 0$

$(d)\ (\sin\frac{\pi}{2})x_1 - 4x_2 = e^2$ Linear

Nonlinear $(e)\ xy + z = 2$

Product of variables

Exponential

$(f)\ e^x - 2y = 4$ Nonlinear

Nonlinear $(g)\ \sin x_1 + 2x_2 - 3x_3 = 0$

trigonometric functions

$(h)\ \frac{1}{x} + \frac{1}{y} = 4$ Nonlinear

not the first power

- **a solution** of a linear equation in *n* variables:

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

$$x_1 = s_1,\ x_2 = s_2,\ x_3 = s_3,\ \cdots,\ x_n = s_n$$

such that $a_1s_1 + a_2s_2 + a_3s_3 + \cdots + a_ns_n = b$

- Solution set:

  the set of all solutions of a linear equation

- Ex 2： (Parametric representation of a solution set)

$$x_1 + 2x_2 = 4$$

a (special) solution: (2, 1), i.e. $x_1 = 2, x_2 = 1$

If you solve for $x_1$ in terms of $x_2$, you obtain

$$x_1 = 4 - 2x_2,$$

By letting $x_2 = t$ you can represent the solution set as

$$x_1 = 4 - 2t$$

And the solutions are $\{(4-2t, t) \mid t \in R\}$ or $\{(s, 2-\frac{1}{2}s) \mid s \in R\}$

- a system of $m$ linear equations in $n$ variables:

$$\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\ & & & & \vdots & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

- Consistent:

  A system of linear equations has at least one solution.

- Inconsistent:

  A system of linear equations has no solution.

- Notes:

Every system of linear equations has either

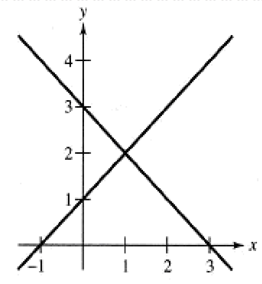
(1) **exactly one** solution,

(2) **infinitely many** solutions, or

(3) **no** solution.

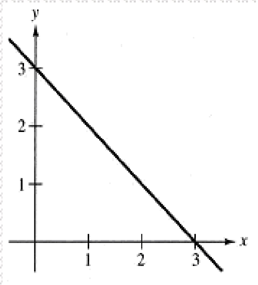
- Ex 4: (Solution of a system of linear equations)

(1)
$$\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$$
two intersecting lines



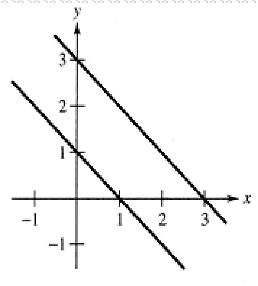

exactly one solution

(2)
$$\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$$
two coincident lines




infinitely many solutions

(3)
$$\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$$
two parallel lines




no solution

- Ex 5: (Using back substitution to solve a system in row echelon form)

$$\begin{aligned} x - 2y &= 5 \qquad (1) \\ y &= -2 \qquad (2) \end{aligned}$$

Sol: By substituting $y = -2$ into (1), you obtain

$$\begin{aligned} x - 2(-2) &= 5 \\ x &= 1 \end{aligned}$$

The system has exactly one solution: $x = 1,\ y = -2$

- Ex 6: (Using back substitution to solve a system in row echelon form)

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \quad (1) \\ & & y & + & 3z & = & 5 & \quad (2) \\ & & & & z & = & 2 & \quad (3) \end{array}$$

Sol: Substitute $z = 2$ into (2)

$$\begin{array}{rcrcr} y & + & 3(2) & = & 5 \\ & & y & = & -1 \end{array}$$

and substitute $y = -1$ and $z = 2$ into (1)

$$\begin{array}{rcrcrcr} x & - & 2(-1) & + & 3(2) & = & 9 \\ & & & & x & = & 1 \end{array}$$

The system has exactly one solution:

$x = 1, y = -1, z = 2$

- Equivalent:

  Two systems of linear equations are called **equivalent** if they have precisely the same solution set.

- Notes:

  Each of the following operations on a system of linear equations produces an equivalent system.

  (1) Interchange two equations.

  (2) Multiply an equation by a nonzero constant.

  (3) Add a multiple of an equation to another equation.

- Ex 7: Solve a system of linear equations (consistent system)

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ -x & + & 3y & & & = & -4 \\ 2x & - & 5y & + & 5z & = & 17 \end{array} \qquad \begin{array}{c}(1)\\(2)\\(3)\end{array}$$

Sol: $(1)+(2)\rightarrow(2)$

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ 2x & - & 5y & + & 5z & = & 17 \end{array} \qquad (4)$$

$(1)\times(-2)+(3)\rightarrow(3)$

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & -y & - & z & = & -1 \end{array} \qquad (5)$$

$(4)+(5)\rightarrow(5)$

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & & & 2z & = & 4 \end{array} \qquad (6)$$

$(6)\times\frac{1}{2}\rightarrow(6)$

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & & & z & = & 2 \end{array}$$

So the solution is $x=1,\ y=-1,\ z=2$ (<u>only one solution</u>)

- Ex 8: Solve a system of linear equations (inconsistent system)

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & x_3 & = & 1 \\ 2x_1 & - & x_2 & - & 2x_3 & = & 2 \\ x_1 & + & 2x_2 & - & 3x_3 & = & -1 \end{array} \qquad \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Sol: $(1)\times(-2)+(2)\rightarrow(2)$

$(1)\times(-1)+(3)\rightarrow(3)$

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & x_3 & = & 1 \\ & & 5x_2 & - & 4x_3 & = & 0 \\ & & 5x_2 & - & 4x_3 & = & -2 \end{array} \qquad \begin{array}{c} \\ (4) \\ (5) \end{array}$$

$$(4)\times(-1)+(5)\rightarrow(5)$$

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & x_3 & = & 1 \\ & & 5x_2 & - & 4x_3 & = & 0 \\ & & & & \boxed{0} & \boxed{=} & \boxed{-2} \end{array}$$

(a false statement)

So the system has no solution (an inconsistent system).

- Ex 9: Solve a system of linear equations (infinitely many solutions)

$$\begin{array}{rcrcrcrl} & & x_2 & - & x_3 & = & 0 & \quad (1) \\ x_1 & & & - & 3x_3 & = & -1 & \quad (2) \\ -x_1 & + & 3x_2 & & & = & 1 & \quad (3) \end{array}$$

Sol: $(1) \leftrightarrow (2)$

$$\begin{array}{rcrcrcrl} x_1 & & & - & 3x_3 & = & -1 & \quad (1) \\ & & x_2 & - & x_3 & = & 0 & \quad (2) \\ -x_1 & + & 3x_2 & & & = & 1 & \quad (3) \end{array}$$

$(1) + (3) \rightarrow (3)$

$$\begin{array}{rcrcrcrl} x_1 & & & - & 3x_3 & = & -1 & \\ & & x_2 & - & x_3 & = & 0 & \\ & & 3x_2 & - & 3x_3 & = & 0 & \quad (4) \end{array}$$

$$\begin{array}{rcrcrcr} x_1 & & & - & 3x_3 & = & -1 \\ & & x_2 & - & x_3 & = & 0 \end{array}$$

$$\Rightarrow x_2 = x_3, \quad x_1 = -1 + 3x_3$$

let $x_3 = t$

then $x_1 = 3t - 1,$

$x_2 = t, \qquad t \in R$

$x_3 = t,$

So this system has infinitely many solutions.

# Key Learning in Section 1.1

- Recognize a linear equation in *n* variables.
- Find a parametric representation of a solution set.
- Determine whether a system of linear equations is consistent or inconsistent.
- Use back-substitution and Gaussian elimination to solve a system of linear equations.

# Keywords in Section 1.1

- linear equation: 線性方程式
- system of linear equations: 線性方程式系統
- leading coefficient: 領先係數
- leading variable: 領先變數
- solution: 解
- solution set: 解集合
- parametric representation: 參數化表示
- consistent: 一致性(有解)
- inconsistent: 非一致性(無解、矛盾)
- equivalent: 等價

# 1.2 Gaussian Elimination and Gauss-Jordan Elimination

- $m \times n$ matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \quad m \text{ rows}$$

$n$ columns

- Notes:

(1) Every **entry** $a_{ij}$ in a matrix is a number.

(2) A matrix with $m$ rows and $n$ columns is said to be of **size** $m \times n$ .

(3) If $m = n$ , then the matrix is called **square of order *n***.

(4) For a square matrix, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called **the main diagonal entries**.

- Ex 1: Matrix Size

| Matrix | Size |
| --- | --- |
| $[2]$ | $1\times 1$ |
| $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ | $2\times 2$ |
| $\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$ | $1\times 4$ |
| $\begin{bmatrix} e & \pi \\ 2 & \sqrt{2} \\ -7 & 4 \end{bmatrix}$ | $3\times 2$ |

- Note:

  One very common use of **matrices** is to represent a system of linear equations.

- a system of $m$ equations in $n$ variables:

$$\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \dots & + & a_{3n}x_n & = & b_3 \\
 & & & & \vdots & & & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & = & b_m
\end{array}$$

Matrix form: $Ax = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- Augmented matrix:

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ & \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right] = [A \mid b]$$

- Coefficient matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = A$$

- Elementary row operation:

(1) Interchange two rows. $r_{ij} : R_i \leftrightarrow R_j$

(2) Multiply a row by a nonzero constant. $r_i^{(k)} : (k)R_i \rightarrow R_i,\ k \neq 0$

(3) Add a multiple of a row to another row. $r_{ij}^{(k)} : (k)R_i + R_j \rightarrow R_j$

- Row equivalent:

Two matrices are said to be **row equivalent** if one can be obtained from the other by a finite sequence of **elementary row operation**.

- Ex 2: (Elementary row operation)

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix} \xrightarrow{r_{12}} \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix} \xrightarrow{r_1^{(\frac{1}{2})}} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

- Ex 3: Using elementary row operations to solve a system

| Linear System | Associated Augmented Matrix | Elementary Row Operation |
| --- | --- | --- |
| $\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \phantom{+ 3z} &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$ | $\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$ | |
| $\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$ | $\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix}$ | $r_{12}^{(1)} : (1)R_1 + R_2 \rightarrow R_2$ |
| $\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$ | $\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix}$ | $r_{13}^{(-2)} : (-2)R_1 + R_3 \rightarrow R_3$ |

| Linear System | Associated Augmented Matrix | Elementary Row Operation |
|---|---|---|
| $\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$ | $\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$ | $r_{23}^{(1)} : (1)R_2 + R_3 \rightarrow R_3$ |
| $\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$ | $\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ | $r_3^{(\frac{1}{2})} : (\frac{1}{2})R_3 \rightarrow R_3$ |

$$\longrightarrow \quad \begin{aligned} x &= 1 \\ y &= -1 \\ z &= 2 \end{aligned}$$

- Row-echelon form: (1, 2, 3)
- Reduced row-echelon form: (1, 2, 3, 4)

(1) All row consisting entirely of zeros occur at the bottom of the matrix.

(2) For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called **a leading 1**).

(3) For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

(4) Every column that has a leading 1 has zeros in every position above and below its leading 1.

- Ex 4: (Row-echelon form or reduced row-echelon form)

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$ (row - echelon form)

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (reduced row - echelon form)

$$\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ (row - echelon form)

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (reduced row - echelon form)

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

- Gaussian elimination:

  The procedure for reducing a matrix to a row-echelon form.

- Gauss-Jordan elimination:

  The procedure for reducing a matrix to a reduced row-echelon form.

- Notes:

  (1) Every matrix has a **unique** reduced row echelon form.

  (2) A row-echelon form of a given matrix is not unique.

  (Different sequences of row operations can produce different row-echelon forms.)

- Ex: (Procedure of Gaussian elimination and Gauss-Jordan elimination)

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 8 & 12 \\ 2 & 8 & -6 & 4 & 12 & 28 \\ 2 & 8 & -1 & 4 & -5 & 4 \end{bmatrix} \xrightarrow{r_{12}} \begin{bmatrix} 2 & 8 & -6 & 4 & 12 & 28 \\ 0 & 0 & -2 & 0 & 8 & 12 \\ 2 & 8 & -1 & 4 & -5 & 4 \end{bmatrix}$$

The first nonzero column

Produce leading 1

$$\xrightarrow{r_1^{(\frac{1}{2})}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & -2 & 0 & 8 & 12 \\ 2 & 8 & -1 & 4 & -5 & 4 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & -2 & 0 & 8 & 12 \\ 0 & 0 & 5 & 0 & -17 & -24 \end{bmatrix}$$

leading 1

Zeros elements below leading 1

Produce leading 1

The first nonzero column

Submatrix

leading 1

$$\xrightarrow{r_2^{(-\frac{1}{2})}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 5 & 0 & -17 & -24 \end{bmatrix} \xrightarrow{r_{23}^{(-5)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 0 & 0 & \boxed{3} & \boxed{6} \end{bmatrix}$$

Zeros elements below leading 1

Submatrix

Produce leading 1

Zeros elsewhere

$$\xrightarrow{r_3^{(\frac{1}{3})}} \begin{bmatrix} 1 & 4 & \boxed{-3} & 2 & \boxed{6} & 14 \\ 0 & 0 & 1 & 0 & \boxed{-4} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_{31}^{(-6)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

leading 1

(row - echelon form) (row - echelon form)

$$\xrightarrow{r_{32}^{(4)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_{21}^{(3)}} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 & 8 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(row - echelon form) (reduced row - echelon form)

- Ex 7: Solve a system by Gauss-Jordan elimination method (only one solution)

$$\begin{aligned} x \; - \; 2y \; + \; 3z &= 9 \\ -x \; + \; 3y \qquad\quad &= -4 \\ 2x \; - \; 5y \; + \; 5z &= 17 \end{aligned}$$

Sol:

augmented matrix

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \xrightarrow{r_{12}^{(1)},\, r_{13}^{(-2)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \xrightarrow{r_{23}^{(1)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow{r_3^{(\frac{1}{2})}} \underset{\text{(row - echelon form)}}{\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}} \xrightarrow{r_{21}^{(2)},\, r_{32}^{(-3)},\, r_{31}^{(-9)}} \underset{\text{(reduced row - echelon form)}}{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}} \longrightarrow \begin{aligned} x \qquad\quad &= 1 \\ y \quad &= -1 \\ z &= 2 \end{aligned}$$

- Ex 8：Solve a system by Gauss-Jordan elimination method (infinitely many solutions)

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 \qquad &= 1 \end{aligned}$$

Sol: augmented matrix

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{r_1^{(\frac{1}{2})},\, r_{12}^{(-3)},\, r_2^{(-1)},\, r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$ (reduced row - echelon form)

the corresponding system of equations is

$$\begin{aligned} x_1 \qquad + 5x_3 &= 2 \\ x_2 - 3x_3 &= -1 \end{aligned}$$

leading variable：$x_1, x_2$

free variable：$x_3$

$$\begin{aligned} x_1 &= \phantom{-}2 - 5x_3 \\ x_2 &= -1 + 3x_3 \end{aligned}$$

Let $x_3 = t$

$$x_1 = 2 - 5t,$$

$$x_2 = -1 + 3t, \qquad t \in R$$

$$x_3 = t,$$

So this system has infinitely many solutions.

- Homogeneous systems of linear equations:

A system of linear equations is said to be **homogeneous** if **all the constant terms are zero**.

$$
\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & \boxed{0} \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & \boxed{0} \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & \boxed{0} \\
 & & & \vdots & & & & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & \boxed{0}
\end{array}
$$

Consistent or inconsistent?

- Trivial solution:

$$x_1 = x_2 = x_3 = \cdots = x_n = 0$$

- Nontrivial solution:

  other solutions

- Notes:

  (1) Every homogeneous system of linear equations is consistent.

  (2) If the homogenous system has fewer equations than variables, then it must have an infinite number of solutions.

  (3) For a homogeneous system, exactly one of the following is true.

  ($a$) The system has only the trivial solution.

  ($b$) The system has infinitely many nontrivial solutions in addition to the trivial solution.

- Ex 9: Solve the following homogeneous system

$$\begin{aligned} x_1 &- x_2 &+ 3x_3 &= 0 \\ 2x_1 &+ x_2 &+ 3x_3 &= 0 \end{aligned}$$

Sol: augmented matrix

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 2 & 1 & 3 & 0 \end{bmatrix} \xrightarrow{r_{12}^{(-2)},\, r_2^{(\frac{1}{3})},\, r_{21}^{(1)}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$ (reduced row - echelon form)

leading variable：$x_1, x_2$

free variable： $x_3$

Let $x_3 = t$

$x_1 = -2t, x_2 = t, x_3 = t, t \in R$

When $t = 0, x_1 = x_2 = x_3 = 0$ (trivial solution)

# Key Learning in Section 1.2

- Determine the size of a matrix .
- Write an augmented or coefficient matrix from a system of linear equations.
- Use matrices and Gaussian elimination with back-substitution to solve a system of linear equations.
- Use matrices and Gauss-Jordan elimination to solve a system of linear equations.
- Solve a homogeneous system of linear equations.

# Keywords in Section 1.2

- matrix: 矩陣
- row: 列
- column: 行
- entry: 元素
- size: 大小
- square matrix: 方陣
- order: 階
- main diagonal: 主對角線
- augmented matrix: 增廣矩陣
- coefficient matrix: 係數矩陣

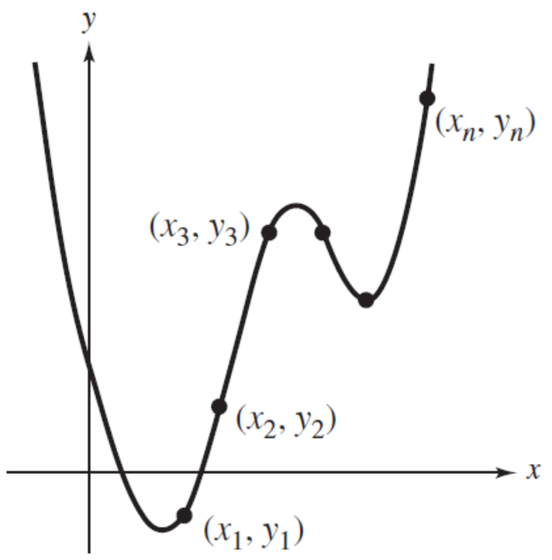
# Keywords in Section 1.2

- elementary row operation: 基本列運算
- row equivalent: 列等價
- row-echelon form: 列梯形形式
- reduced row-echelon form: 列簡梯形形式
- leading 1: 領先1
- Gaussian elimination: 高斯消去法
- Gauss-Jordan elimination: 高斯-喬登消去法
- free variable: 自由變數
- leading variable: 領先變數
- homogeneous system: 齊次系統
- trivial solution: 顯然解
- nontrivial solution: 非顯然解

# 1.3 Applications of Systems of Linear Equations

- Polynomial Curve Fitting:

  The procedure to fit a polynomial function to a set of data points in the plane is called polynomial curve fitting.




- $n$ points in the $xy$-plane:

$$(x_1, y_1), (x_1, y_1), \cdots, (x_n, y_n)$$

- a polynomial function of degree $n-1$:

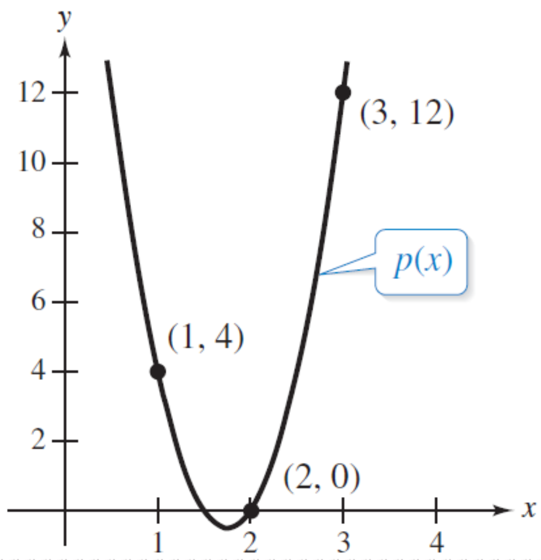
$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

- $n$ linear equations in $n$ variables $a_0$, $a_1$, $a_2$, …, and $a_{n-1}$:

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} &= y_1 \\ a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} &= y_2 \\ &\vdots \\ a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} &= y_n \end{aligned}$$

## Ex 1: (Polynomial Curve Fitting)

Determine the polynomial $p(x)= a_0+a_1x+a_2x^2$ whose graph passes through the points (1, 4), (2, 0), and (3, 12).




Sol: Substitute $x$ = 1, 2, and 3 into $p(x)$

$$p(1) = a_0 + a_1(1) + a_2(1)^2 = a_0 + a_1 + a_2 = 4$$

$$p(2) = a_0 + a_1(2) + a_2(2)^2 = a_0 + 2a_1 + 4a_2 = 0$$

$$p(3) = a_0 + a_1(3) + a_2(3)^2 = a_0 + 3a_1 + 9a_2 = 12$$

The solution of this system is

$$a_0 = 24, a_1 = -28, \text{and } a_2 = 8$$

So the polynomial function is

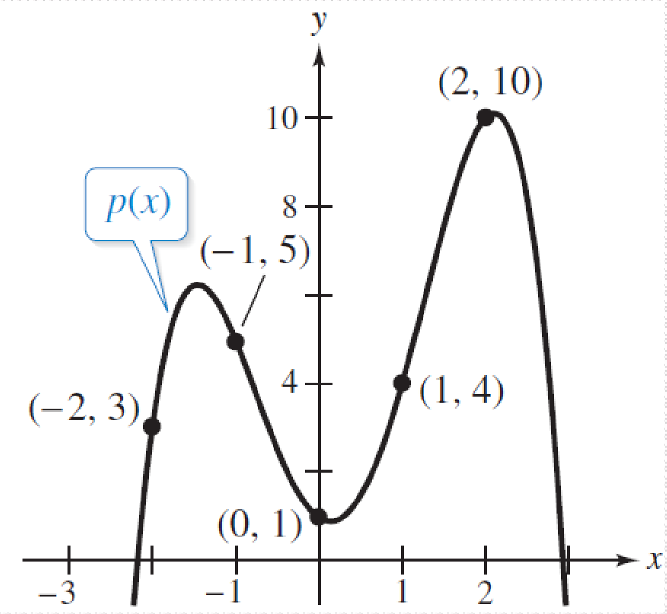
$$p(x) = 24 - 28x + 8x^2$$

- Ex 2: (Polynomial Curve Fitting)

Find a polynomial that fits the points (−2, 3), (−1, 5), (0, 1), (1, 4), and (2, 10).

Sol: Choose a fourth-degree polynomial function

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$




Substitute the given points into $p(x)$

$$\begin{aligned}
a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4 &= 3 \\
a_0 - a_1 + a_2 - a_3 + a_4 &= 5 \\
a_0 &= 1 \\
a_0 + a_1 + a_2 + a_3 + a_4 &= 4 \\
a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 &= 10
\end{aligned}$$

The solution is

$$a_0 = 1, a_1 = -\frac{5}{4}, a_2 = \frac{101}{24}, a_3 = \frac{3}{4}, \text{and } a_4 = -\frac{17}{24}$$

So the polynomial function is

$$p(x) = 1 - \frac{5}{4}x + \frac{101}{24}x^2 + \frac{3}{4}x^3 - \frac{17}{24}x^4$$

- Ex 3: (Translating Large $x$- Values Before Curve Fitting)

Find a polynomial that fits the points

$$\underbrace{(2011, 3)}_{(x_1, y_1)},\quad \underbrace{(2012, 5)}_{(x_2, y_2)},\quad \underbrace{(2013, 1)}_{(x_3, y_3)},\quad \underbrace{(2014, 4)}_{(x_4, y_4)},\quad \underbrace{(2015, 10)}_{(x_5, y_5)}.$$

Sol: Use the translation $z = x - 2013$ to obtain

$$\underbrace{(-2, 3)}_{(z_1, y_1)},\quad \underbrace{(-1, 5)}_{(z_2, y_2)},\quad \underbrace{(0, 1)}_{(z_3, y_3)},\quad \underbrace{(1, 4)}_{(z_4, y_4)},\quad \underbrace{(2, 10)}_{(z_5, y_5)}.$$ (the same as Ex.2)

So the polynomial function is

$$p(z) = 1 - \frac{5}{4}z + \frac{101}{24}z^2 + \frac{3}{4}z^3 - \frac{17}{24}z^4$$

Let $z = x - 2013$

$$p(x) = 1 - \frac{5}{4}(x-2013) + \frac{101}{24}(x-2013)^2 + \frac{3}{4}(x-2013)^3 - \frac{17}{24}(x-2013)^4$$

- Ex 4: (An Application of Curve Fitting)

Find a polynomial that relates the periods of the three planets that are closest to the Sun to their mean distances from the Sun, as shown in the table. Then use the polynomial to calculate the period of Mars and compare it to the value shown in the table.

| *Planet* | *Mercury* | *Venus* | *Earth* | *Mars* |
|---|---|---|---|---|
| ***Mean Distance*** | 0.387 | 0.723 | 1.000 | 1.524 |
| ***Period*** | 0.241 | 0.615 | 1.000 | 1.881 |

Sol: Choose a quadratic polynomial function

$$p(x) = a_0 + a_1 x + a_2 x^2$$

Substitute these points into $p(x)$

$$\begin{aligned} a_0 + (0.387)a_1 + (0.287)^2 a_2 &= 0.241 \\ a_0 + (0.723)a_1 + (0.723)^2 a_2 &= 0.615 \\ a_0 + \qquad a_1 + \qquad a_2 &= 1 \end{aligned}$$

The approximate solution of the system is

$$a_0 \approx -0.0634,\ a_1 \approx 0.6119,\ a_2 \approx 0.4515$$

An approximate of the polynomial function is

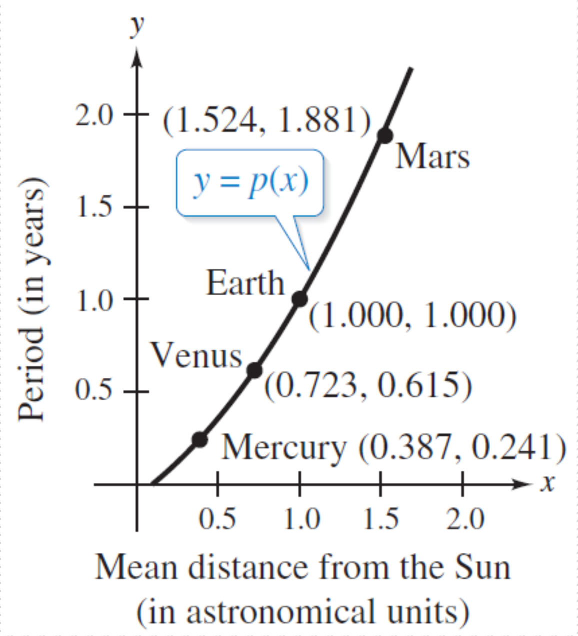
$$p(x) = -0.0634 + 0.6119x + 0.4515x^2$$

Let $x = 1.524$ (the mean distance of Mars) to produce $p(x)$ (the period of Mars)

$$p(1.524) \approx 1.918 \text{ years}$$

- Note:

The actual period of Mars is 1.881 years.



- Notes:

  (1) A polynomial that fits some of the points in a data set is not necessarily an accurate model for other points in the data set.

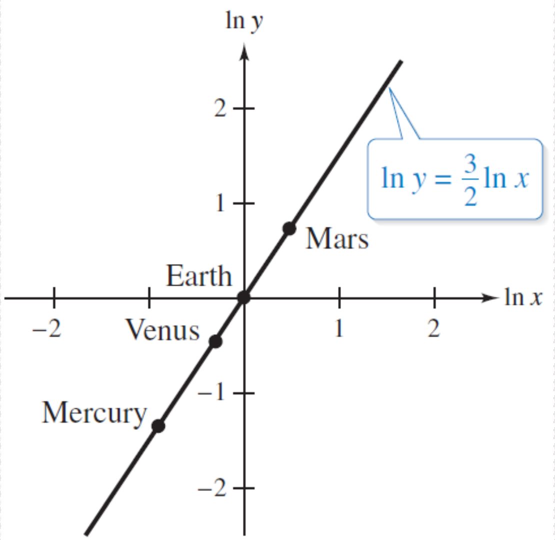
  (2) Generally, the farther the other points are from those used to fit the polynomial, the worse the fit.

- Note:

  Types of functions other than polynomial functions may provide better fits.

Taking the natural logarithms of the given distances and periods produces the following results.

| *Planet* | *Mercury* | *Venus* | *Earth* | *Mars* |
|---|---|---|---|---|
| *Mean Distance (x)* | 0.387 | 0.723 | 1.000 | 1.524 |
| **ln *x*** | −0.949 | −0.324 | 0.000 | 0.421 |
| *Period (y)* | 0.241 | 0.615 | 1.000 | 1.881 |
| **ln *y*** | −1.423 | −0.486 | 0.000 | 0.632 |




Fitting a polynomial to the logarithms of the distances and periods produces the linear relationship.

$$\ln y = \frac{3}{2}\ln x \quad (\text{i.e. } y = x^{3/2}, \text{or } y^2 = x^3 \ )$$