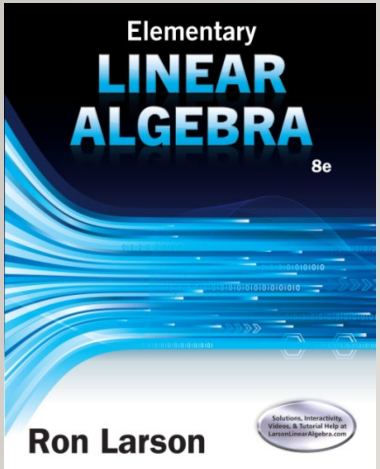
CENGAGE Learning



CHAPTER 3 DETERMINANTS

3.1 The Determinant of a Matrix
3.2 Determinant and Elementary Operations
3.3 Properties of Determinants
3.4 Application of Determinants

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3.3 Properties of Determinants

- Thm 3.5: (Determinant of a matrix product)

$$\det(AB) = \det(A)\det(B)$$

- Notes:

(1) $\det(EA) = \det(E)\det(A)$

(2) $\det(A + B) \neq \det(A) + \det(B)$

- Ex 1: (The determinant of a matrix product)

$$A=\begin{bmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{bmatrix} \qquad B=\begin{bmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{bmatrix}$$

Find $|A|$, $|B|$, and $|AB|$

Sol:

$$|A|=\begin{vmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{vmatrix}=-7 \qquad |B|=\begin{vmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{vmatrix}=11$$

$$AB=\begin{bmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{bmatrix}=\begin{bmatrix}8 & 4 & 1\\ 6 & -1 & -10\\ 5 & 1 & -1\end{bmatrix}$$

$$\Rightarrow |AB|=\begin{vmatrix}8 & 4 & 1\\ 6 & -1 & -10\\ 5 & 1 & -1\end{vmatrix}=-77$$

- Check:

$$|AB| = |A|\ |B|$$

- Thm 3.6: (Determinant of a scalar multiple of a matrix)

If A is an $n \times n$ matrix and c is a scalar, then

$$\det(cA) = c^n \det(A)$$

- Ex 2:

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}, \qquad \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5$$

Find $|A|$.

Sol:

$$A = 10\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \Rightarrow |A| = 10^3 \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = (1000)(5) = 5000$$

- Thm 3.7: (Determinant of an invertible matrix)

  A square matrix A is invertible (nonsingular) if and only if

  $$\det(A) \neq 0$$

- Ex 3: (Classifying square matrices as singular or nonsingular)

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Sol:

$|A| = 0 \quad \Rightarrow$ A has no inverse (it is singular).

$|B| = -12 \neq 0 \quad \Rightarrow$ B has an inverse (it is nonsingular).

- Thm 3.8: (Determinant of an inverse matrix)

$$\text{If } A \text{ is invertible, then } \det(A^{-1}) = \frac{1}{\det(A)}.$$

- Thm 3.9: (Determinant of a transpose)

$$\text{If } A \text{ is a square matrix, then } \det(A^{\mathrm{T}}) = \det(A).$$

- Ex 4:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \qquad \text{(a)} \ \left|A^{-1}\right| = ? \qquad \text{(b)} \ \left|A^{T}\right| = ?$$

Sol:

$$\because \ |A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 4$$

$$\therefore \ \left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{4}$$

$$\left|A^{T}\right| = |A| = 4$$

- Equivalent conditions for a nonsingular matrix:

If A is an $n \times n$ matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n

(5) A can be written as the product of elementary matrices.

(6) $\det(A) \neq 0$

- Ex 5: Which of the following system has a unique solution?

$$(a)\quad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

$$(b)\quad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= -4 \end{aligned}$$

Sol:

$(a)\quad A\mathbf{x}=\mathbf{b}$

$\because \qquad |A| = 0$

$\therefore$ This system does not have a unique solution.

$(b)\quad B\mathbf{x}=\mathbf{b}$

$\because \qquad |B| = -12 \neq 0$

$\therefore$ This system has a unique solution.

Key Learning in Section 3.3

- Find the determinant of a matrix product and a scalar multiple of a matrix.
- Find the determinant of an inverse matrix and recognize equivalent conditions for a nonsingular matrix.
- Find the determinant of the transpose of a matrix.

Keywords in Section 3.3

- determinant: 行列式
- matrix multiplication: 矩陣相乘
- scalar multiplication: 純量積
- invertible matrix: 可逆矩陣
- inverse matrix: 反矩陣
- nonsingular matrix: 非奇異矩陣
- transpose matrix: 轉置矩陣

3.4 Applications of Determinants

- Matrix of cofactors of A:

$$[C_{ij}] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

- Adjoint matrix of A:

$$adj(A) = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

- Thm 3.10: (The inverse of a matrix given by its adjoint)

If A is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} adj(A) \quad \Rightarrow \quad \det(A)I = A * adj(A)$$

$$A[\text{adj}(A)] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{j2} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{jn} & \cdots & C_{nn} \end{bmatrix}.$$

C= $a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}.$

⇒If i=j :

⇒If i≠j :

- Thm 3.10: (The inverse of a matrix given by its adjoint)

If A is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

- Ex:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \det(A) = ad - bc$$

$$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)} adj(A)$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Ex 1 & Ex 2:

$$A=\begin{bmatrix}-1 & 3 & 2\\ 0 & -2 & 1\\ 1 & 0 & -2\end{bmatrix}$$

(a) Find the adjoint of A.

(b) Use the adjoint of A to find A^{-1}

Sol: $\because \quad C_{ij}=(-1)^{i+j}M_{ij}$

$$\Rightarrow C_{11}=+\begin{vmatrix}-2 & 1\\ 0 & -2\end{vmatrix}=4,\ C_{12}=-\begin{vmatrix}0 & 1\\ 1 & -2\end{vmatrix}=1,\ C_{13}=+\begin{vmatrix}0 & -2\\ 1 & 0\end{vmatrix}=2$$

$$C_{21}=-\begin{vmatrix}3 & 2\\ 0 & 2\end{vmatrix}=6,\quad C_{22}=+\begin{vmatrix}-1 & 2\\ 1 & 2\end{vmatrix}=0,\ C_{23}=-\begin{vmatrix}-1 & 3\\ 1 & 0\end{vmatrix}=3$$

$$C_{31}=+\begin{vmatrix}3 & 2\\ -2 & 1\end{vmatrix}=7,\quad C_{32}=-\begin{vmatrix}-1 & 2\\ 0 & 1\end{vmatrix}=1,\ C_{33}=+\begin{vmatrix}-1 & 3\\ 0 & -2\end{vmatrix}=2$$

$\Rightarrow$ cofactor matrix of A

$$[C_{ij}] = \begin{bmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{bmatrix}$$

$\Rightarrow$ adjoint matrix of A

$$adj(A) = [C_{ij}]^T = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$\Rightarrow$ inverse matrix of A

$$A^{-1} = \frac{1}{\det(A)} adj(A) \qquad \because \quad \det(A) = 3$$

$$= \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}$$

- Check: $AA^{-1} = I$

Cramer's Rule

Cramer's Rule uses determinants to solve a system of linear equations in variables. This rule applies only to systems with unique solutions.

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \qquad x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{21}a_{12}} \qquad a_{11}a_{22} - a_{21}a_{12} \neq 0$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \qquad |A_1| = \begin{vmatrix} \boxed{\begin{matrix} b_1 \\ b_2 \end{matrix}} & \begin{matrix} a_{12} \\ a_{22} \end{matrix} \end{vmatrix} \qquad |A_2| = \begin{vmatrix} \begin{matrix} a_{11} \\ a_{21} \end{matrix} & \boxed{\begin{matrix} b_1 \\ b_2 \end{matrix}} \end{vmatrix}$$

$$x_1 = \frac{|A_1|}{|A|} \qquad x_2 = \frac{|A_2|}{|A|}$$

- Ex 3: (Using Cramer's Rule)

Use Cramer's Rule to solve the system of linear equations.

$$\begin{aligned} 4x_1 - 2x_2 &= \boxed{10} \\ 3x_1 - 5x_2 &= \boxed{11} \end{aligned}$$

Sol: Find the determinant of the coefficient matrix

$$|A| = \begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix} = -14$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} \boxed{\begin{matrix} 10 \\ 11 \end{matrix}} & \begin{matrix} -2 \\ -5 \end{matrix} \end{vmatrix}}{-14} = \frac{-28}{-14} = 2$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} \begin{matrix} 4 \\ 3 \end{matrix} & \boxed{\begin{matrix} 10 \\ 11 \end{matrix}} \end{vmatrix}}{-14} = \frac{14}{-14} = -1$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

$$x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

- Thm 3.11: (Cramer's Rule)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

$$A\mathbf{x} = \mathbf{b} \qquad A = \left[a_{ij}\right]_{n\times n} = \left[A^{(1)}, A^{(2)}, \cdots, A^{(n)}\right] \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

(this system has a unique solution)

$$A_j = \left[A^{(1)}, A^{(2)}, \cdots, A^{(j-1)}, b, A^{(j+1)}, \cdots, A^{(n)}\right]$$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2(j-1)} & b_2 & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & & & \ddots & & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}$$

$$(\text{ i.e. } \det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj})$$

$$\Rightarrow x_j = \frac{\det(A_j)}{\det(A)}, \qquad j = 1, 2, \cdots, n$$

Pf:

$$A\mathbf{x} = \mathbf{b}, \qquad \det(A) \neq 0$$

$$\Rightarrow \mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)} adj(A)\mathbf{b}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} b_1C_{11} & + & b_2C_{21} & + & \cdots & + & b_nC_{n1} \\ b_1C_{12} & + & b_2C_{22} & + & \cdots & + & b_nC_{n2} \\ & \vdots & & \vdots & & & \\ b_1C_{1n} & + & b_2C_{2n} & + & \cdots & + & b_nC_{nn} \end{bmatrix}$$

$$\Rightarrow x_j = \frac{1}{\det(A)}(b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj})$$

$$= \frac{\det(A_j)}{\det(A)} \qquad j = 1, 2, \cdots, n$$

- Ex 4: Use Cramer's rule to solve the system of linear equations.

$$\begin{aligned} -x + 2y - \boxed{3z} &= 1 \\ 2x \qquad\quad + \boxed{z} &= 0 \\ 3x - 4y + \boxed{4z} &= 2 \end{aligned}$$

Sol:

$$\det(A) = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10 \qquad \det(A_1) = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix} = 8$$

$$\det(A_2) = \begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -15, \qquad \det(A_3) = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix} = -16$$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{4}{5} \qquad y = \frac{\det(A_2)}{\det(A)} = \frac{-3}{2} \qquad z = \frac{\det(A_3)}{\det(A)} = \frac{-8}{5}$$

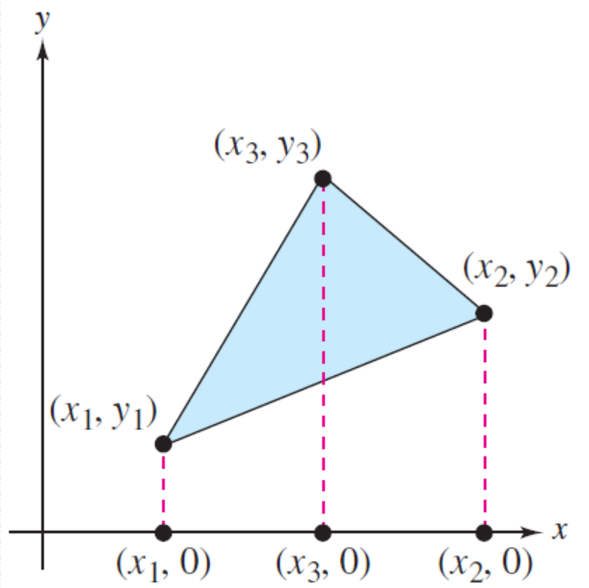
- Area of a triangle in the *xy*-plane:

  A triangle with vertices

  $$(x_1, y_1), (x_2, y_2), \text{and } (x_3, y_3)$$

  $$\text{Area} = \pm\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

  where <u>the sign (±)</u> is chosen to give <u>a positive area</u>.



Pf:

Consider the three trapezoid

Trapezoid 1: $(x_1, 0), (x_1, y_1), (x_3, y_3), (x_3, 0)$

Trapezoid 2: $(x_3, 0), (x_3, y_3), (x_2, y_2), (x_2, 0)$

Trapezoid 3: $(x_1, 0), (x_1, y_1), (x_2, y_2), (x_2, 0)$

$$\text{Area} = \frac{1}{2}(y_1 + y_3)(x_3 - x_1) + \frac{1}{2}(y_3 + y_2)(x_2 - x_3) - \frac{1}{2}(y_1 + y_2)(x_2 - x_1)$$

$$= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2)$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

If the vertices do not occur in the order $x_1 \leqq x_2 \leqq x_3$ or if the vertex (x_3, y_3) is not above the line segment connecting the other two vertices, then the formula above may yield the negative of the area. So, use ± and choose the correct sign to give a positive area.

- Ex 5: (Finding the Area of a Triangle)

Find the area of the triangle whose vertices are (1, 1), (2, 2), and (4, 3).

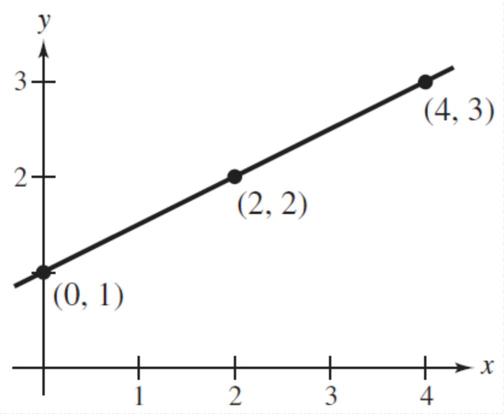
Sol:

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = -\frac{3}{2}$$

The area of the triangle is $\frac{3}{2}$ square units.

If three points in the *xy*-plane lie on the same line, then the determinant in the formula for the area of a triangle is zero.

$$\frac{1}{2}\begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = 0$$



- Test for collinear points in the *xy*-plane:

Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if and only if

$$\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$$

- Two-point form of the equation of a line:

An equation of the line passing through the distinct points (x_1, y_1) and (x_2, y_2) is given by

$$\det\begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0$$

- Ex 6: (Finding an Equation of the Line Passing Through Two Points)

  Find an equation of the line passing through the points (2, 4) and (-1, 3).

Sol:

$$\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0$$

$$x\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} - y\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} = 0$$

$$x(1) - y(3) + 1(10) = 0$$

$$x - 3y = -10$$

▪ Volume of a Tetrahedron:

The volume of a tetrahedron with vertices (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) is

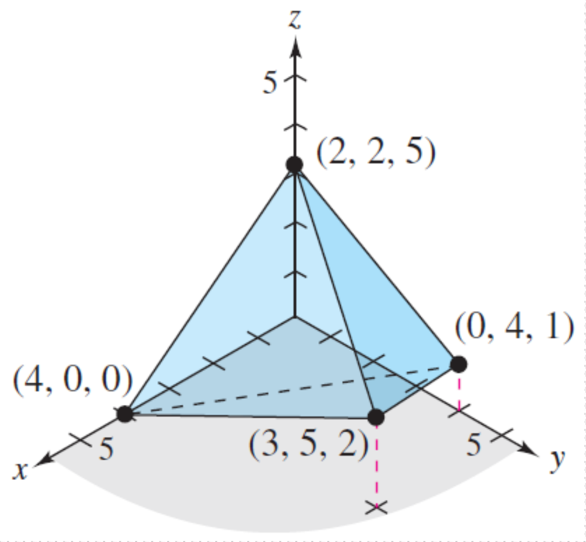
$$\text{Volume} = \pm\frac{1}{6}\det\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix}$$

where the sign $(\pm)$ is chosen to give a positive area.

- Ex 7: (Finding the Volume of a Tetrahedron)

  Find the volume of the tetrahedron shown in the following figure, whose vertices are (0, 4, 1), (4, 0, 0), (3, 5, 2), and (2, 2, 5).

Sol:



$$\frac{1}{6}\begin{vmatrix} 0 & 4 & 1 & 1 \\ 4 & 0 & 0 & 1 \\ 3 & 5 & 2 & 1 \\ 2 & 2 & 5 & 1 \end{vmatrix} = \frac{1}{6}(-72) = -12.$$

The volume of the tetrahedron is 12 cubic units.

- Test for coplanar points in space:

Four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) are coplanar if and only if

$$\det\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix} = 0$$

- Three-point form of the equation of a line:

An equation of the line passing through the distinct points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is given by

$$\det\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = 0$$

- Ex 8: (Finding an Equation of the Plane Passing Through Three Points)

  Find an equation of the plane passing through the points (0, 1, 0), (−1, 3, 2), and (-2, 0, 1).

Sol:

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 3 & 2 & 1 \\ -2 & 0 & 1 & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} x & y-1 & z & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & 2 & 1 \\ -2 & -1 & 1 & 1 \end{vmatrix} = 0$$

$$x\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} - (y-1)\begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} + z\begin{vmatrix} -1 & 2 \\ -2 & -1 \end{vmatrix} = 0$$

$$x(4) - (y-1)(3) + z(5) = 0$$

$$4x - 3y + 5z = -3$$

Key Learning in Section 3.4

- Find the adjoint of a matrix and use it to find the inverse of the matrix.
- Use Cramer's Rule to solve a system of n linear equations in n variables.
- Use determinants to find area, volume, and the equations of lines and planes.