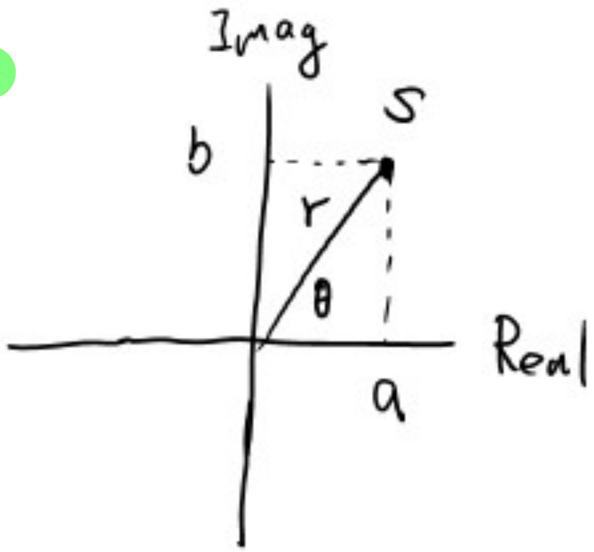
△ Sinusoids, phasor, complex exponential, impedance

⊙ Complex number and Euler's equation

$$S = a + bj \qquad j = \sqrt{-1}$$



$$= r \cdot \cos\theta + r \cdot \sin\theta \cdot j$$

$$r = \sqrt{a^2+b^2} \qquad \theta = \tan^{-1}\frac{b}{a}$$

Euler's equation : $e^{j\theta} = \cos\theta + j\sin\theta$ complex exponential

$S = r \cdot e^{j\theta}$, $e = 2.71828\ldots\ldots$ mathematic constant Euler's number.

$a = \mathrm{Re}\{s\}$ base of natural logarithm 自然對數的底數

$b = \mathrm{Im}\{s\}$ exponential function 指數函數的底數.

$$\frac{10}{a+bj} = 2.36 \cdot e^{j(45^\circ)}$$

sol ①.

$$\frac{10\, e^{j0^\circ}}{r \cdot e^{j\theta}} = 2.36\, e^{j(45^\circ)}$$

$$\frac{10}{r} = 2.36 \Rightarrow r = 4.25$$

$$e^{j(0^\circ - \theta)} = e^{j(45^\circ)} \Rightarrow \theta = -45^\circ$$

$$a = r \cdot \cos\theta = 4.25 \cos(-45^\circ) = 3, \quad b = r \cdot \sin\theta = -3$$

sol ②.

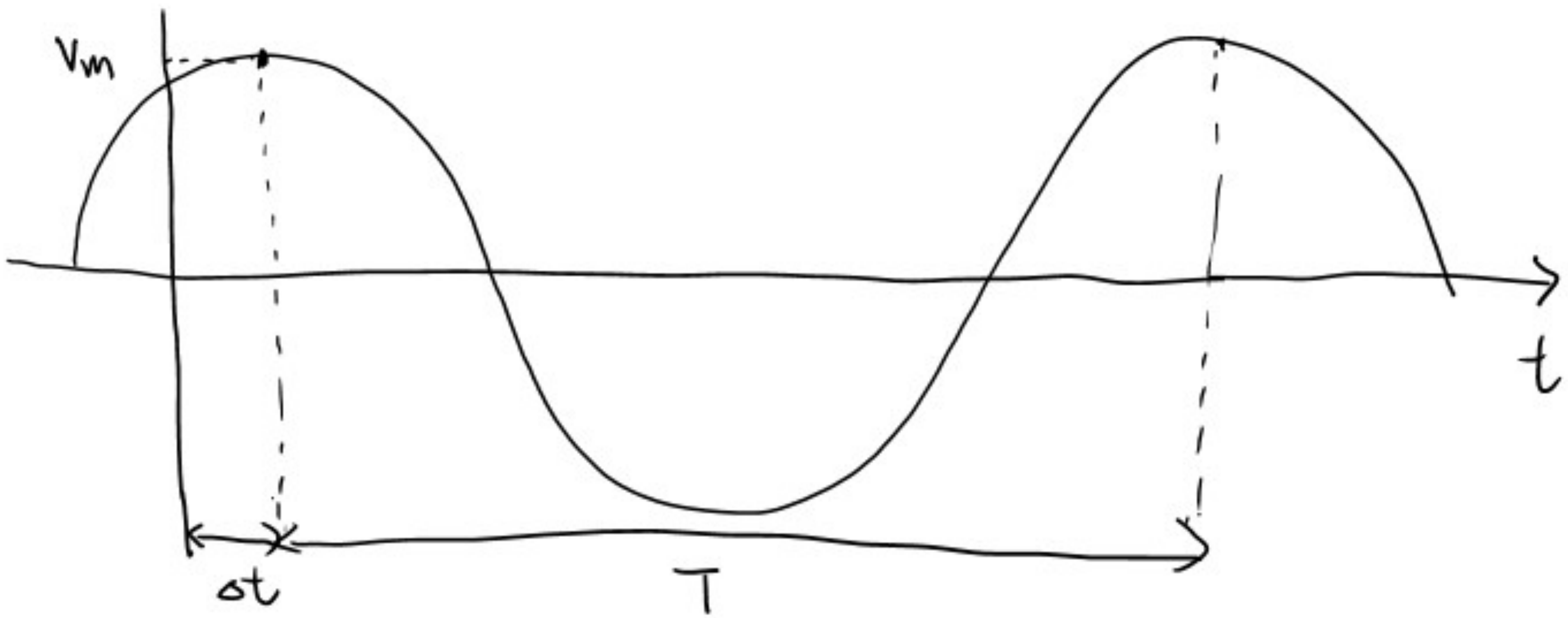
$$\frac{10}{(a+bj)} \cdot \frac{(a-bj)}{(a-bj)} = \frac{10(a-bj)}{a^2+b^2} = \frac{10}{\sqrt{a^2+b^2}}\left[\frac{a}{\sqrt{a^2+b^2}} + \frac{-b}{\sqrt{a^2+b^2}} j\right] = 2.36 \cdot e^{j(45^\circ)}$$

$$(bj)(bj) = b^2 \cdot j^2 = -b^2$$

$$\frac{10}{\sqrt{a^2+b^2}} = 2.36 \qquad \frac{-b}{a} = \tan(45^\circ) = 1 \Rightarrow a = -b \qquad a = 3, \quad b = -3$$

⓪ AC signal model : Sinusoids

$V_s(t) = V_m \cos(\omega t + \phi)$



V_m = amplitude (V, A)

ω = angular frequency (rad/s)

$\omega = 2\pi \cdot f = \frac{2\pi}{T}$

ϕ = phase (rad) $\phi = \omega \cdot \Delta t = \frac{2\pi}{T} \cdot \Delta t$

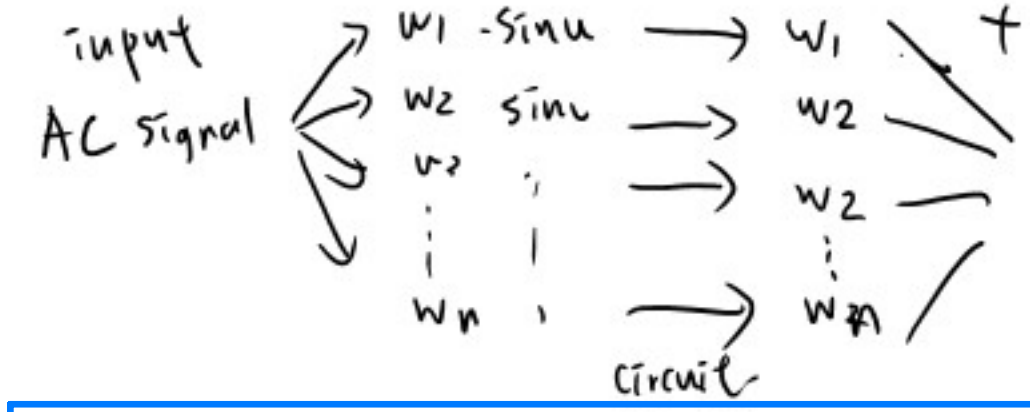
- its natural, most oscillations comes in the form of sinusoids
  For example. mechanical vibrations, propagation of EM waves
- In electric system, power and communications all come in sinusoids.
- In signal processing theory, all AC signals can be expressed as linear combinations of series fundemental Sinusoids

note. $V_m(t) = \mathrm{Re}\{V_m \cdot e^{j(\omega t + \phi)}\} = \mathrm{Re}\{V_m \cdot e^{j\omega t} \cdot e^{j\phi}\}$

① Sinusoids in phasor and complex exponential

- when all independent sources are sinusoides with the same frequency.

  phasor and complex exponential can be used to seek for steady state response of the linear circuit



$V_m \cos(\omega t + \phi)$

$$\boxed{\begin{aligned} V_s(t) &= \mathrm{Re}\{V_m \cdot e^{j\omega t} \cdot e^{j\phi}\} \dashrightarrow \text{time domain} \\ V_s &= V_m \angle\phi = V_m e^{j\phi} \dashrightarrow \text{frequency domain} \end{aligned}}$$

- analyze the response of the circuit by frequency.
- simplify the calculation, avoid differential equations
- easier to conceptualize circuit design.

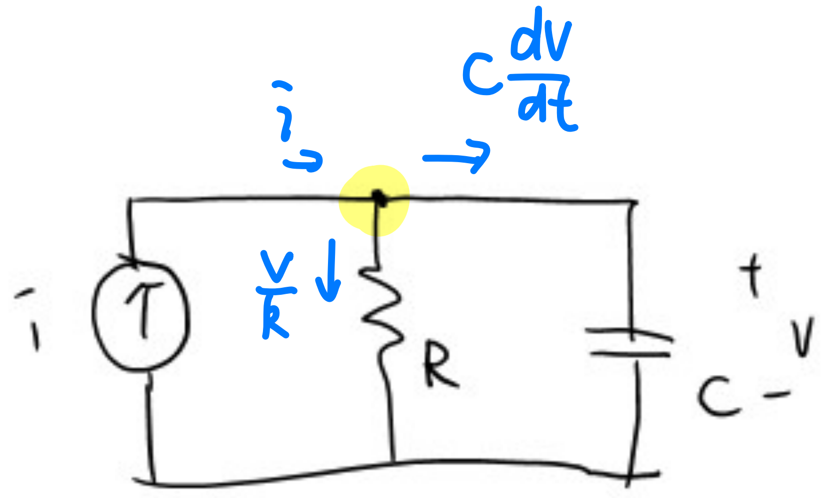
① $t \rightarrow f$

- $i(t) = 4 \cdot \cos(\omega t - 80^\circ) \longrightarrow I = 4\angle(-80^\circ) = 4 \cdot e^{j(-80^\circ)}$

② $f \rightarrow t$

- $V = 10\angle(-140^\circ) \longrightarrow V(t) = 10 \cos(\omega t - 140^\circ)$
- $V = 80 + 75j = \sqrt{80^2 + 75^2} \angle\left(\tan^{-1}\frac{75}{80}\right) = 109.7\angle(43.2^\circ) = 109.7 e^{j(43.2^\circ)}$

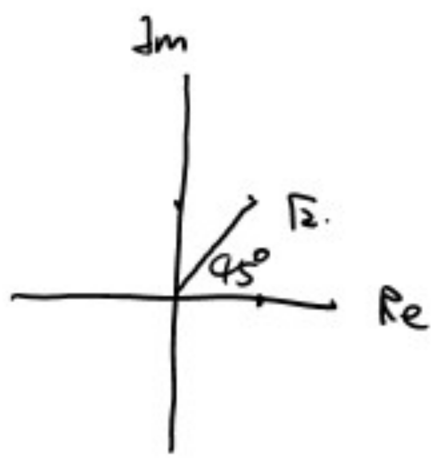
$\longrightarrow v(t) = 109.7 \cos(\omega t + 43.2^\circ)$



$R = 1\Omega, \quad C = 10\,mF, \quad \omega = 100\ \text{rad}/s$

$i(t) = 10 \cdot \cos(\omega t) = Re\{10 \cdot e^{j\omega t}\}$

$v(t) = A \cdot \cos(\omega t + \phi) = A\, Re\{e^{j\omega t} \cdot e^{j\phi}\}$



< Sol 1 > time domain

$i = \frac{v}{R} + C \cdot \frac{dv}{dt}$ (KCL)

$$10\, e^{j\omega t} = \frac{A}{R} e^{j\omega t} \cdot e^{j\phi} + C \frac{d(A \cdot e^{j\omega t} \cdot e^{j\phi})}{dt} \rightarrow 10 \cdot [e^{j\omega t}] = \frac{A}{R} e^{j\phi} [e^{j\omega t}] + C \cdot A \cdot e^{j\phi} (j\omega) [e^{j\omega t}]$$

$$10 = \frac{A}{R} e^{j\phi} + C \cdot A\, e^{j\phi} (j\omega) \rightarrow 10 \cdot e^{j0^\circ} = A\left(\frac{1}{R} + C \cdot j \cdot \omega\right) e^{j\phi} = A(1+j)e^{j\phi} = A\,(\sqrt{2} e^{45^\circ j})\, e^{j\phi}$$

$$= \sqrt{2} A\, e^{j(\phi + 45^\circ)}$$

$10 = \sqrt{2} A \rightarrow A = \frac{10}{\sqrt{2}}. \quad \phi = -45^\circ$

$$v(t) = \frac{10}{\sqrt{2}} \cos(100t - 45^\circ)$$

#

<sol 2> frequency domain

$$i = V\left(\frac{1}{R} + j\frac{1}{\frac{1}{\omega C}}\right)$$

ωC → capacitive reactance X_c

$$\Rightarrow i = V\left(\frac{1}{R} + j\omega C\right)$$

$j\omega C$ → admittance of the capacitor

$$\Rightarrow V = \frac{i}{\frac{1}{R} + j\omega C} \qquad \left(\begin{array}{l} i = 10\angle 0^\circ \\ V = A\angle\phi \end{array}\right)$$

$$= \frac{10\angle 0^\circ}{\frac{1}{1} + j(100)(10^{-2})}$$

$$= \frac{10\angle 0^\circ}{1+j} = \frac{10\angle 0^\circ}{1+j} = \frac{10\angle 0^\circ}{\sqrt{2}\angle 45^\circ} = \frac{10}{\sqrt{2}}\angle -45^\circ$$

$$\left(= \frac{10}{\sqrt{2}}\cos(100t - 45^\circ)\right)_{\#}$$

⑤ circuit element in frequency domain

max. voltage

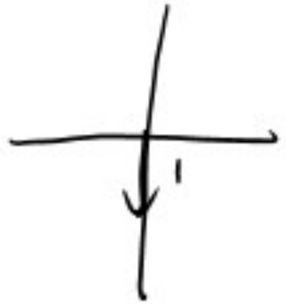
Impedance. Z .. (阻抗)

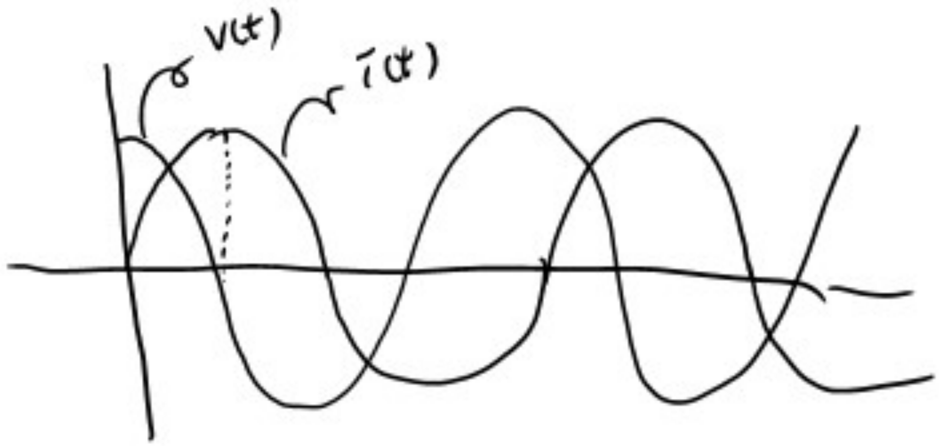
$$Z = \frac{V}{I} = \frac{V_m \angle \phi}{I_m \angle \theta} = \frac{V_m e^{j\phi}}{I_m e^{j\theta}} = \frac{V_m}{I_m} e^{j(\phi-\theta)}$$

Z is a complex number, act as Resistor

Resistor $v(t) = i(t) \cdot R \longrightarrow Z_R = R \angle 0^\circ$

capacitor $i(t) = C \frac{dv(t)}{dt} \longrightarrow V_c = V_m e^{j(wt+\phi)}$ $\quad I_c = C \cdot V_m (jw) e^{j(wt+\phi)}$





$$Z_c = \frac{V_c}{I_c} = \frac{V_m e^{j(wt+\phi)}}{C \cdot V_m (jw) e^{j(wt+\phi)}} = \frac{1}{jwC} = \frac{1}{wC}(-j) = \frac{1}{wC} \angle(-90^\circ) = \frac{1}{wC} e^{j(-90^\circ)}$$

$f \uparrow \quad \frac{1}{wC} \downarrow \quad Z_c \downarrow$ (low), $\quad f \downarrow \quad \frac{1}{wC} \uparrow \quad Z_c \uparrow$ (high) current source.

$i(t) = I_m \cos(wt+\theta)$

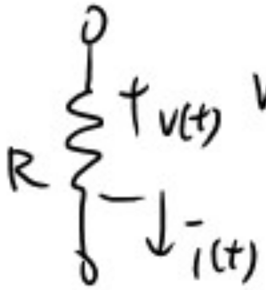
$I = I_m \cdot e^{j\theta}$

voltage source

$V(t) = V_m \cos(wt+\theta)$

$V = V_m e^{j\theta}$

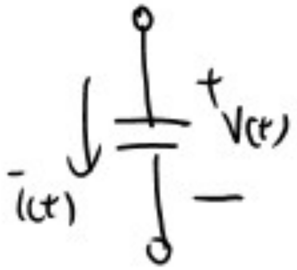
Resistor



$v(t) = R\, i(t)$

$V(w) = \left[R e^{j0^\circ}\right] I(w)$

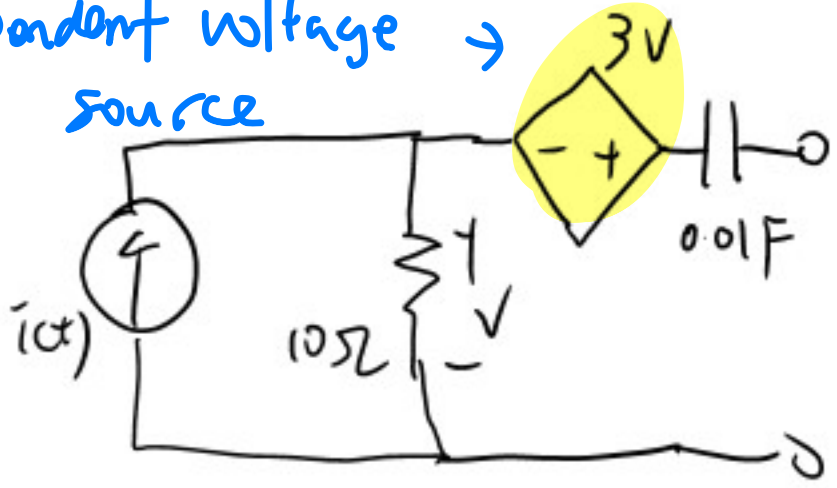
Capacitor



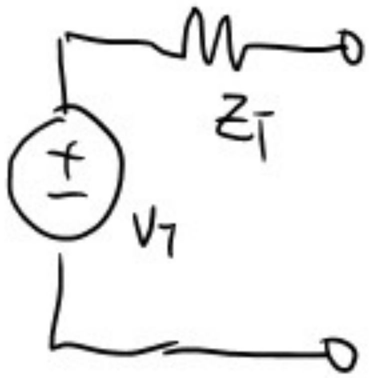
$i(t) = C \cdot \frac{dv(t)}{dt}$

$V(w) = \frac{1}{jwC} \cdot I(w) = \left[\frac{1}{wC} e^{j(-90^\circ)}\right] \cdot I(w)$

dependent voltage source →



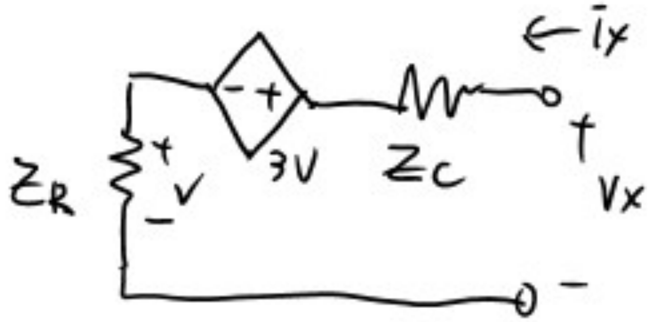
→



$i(t) = 2\cos(10t) \rightarrow I = 2e^{j0^\circ}, \ w = 10$

$Z_C = \frac{1}{j(10)(0.01)} = -10j = 10 e^{(-90^\circ)j}$

$V_T = V_{oc} = 10 \cdot I + 3(10I) = 40I = 80 \cdot e^{j(0^\circ)}$



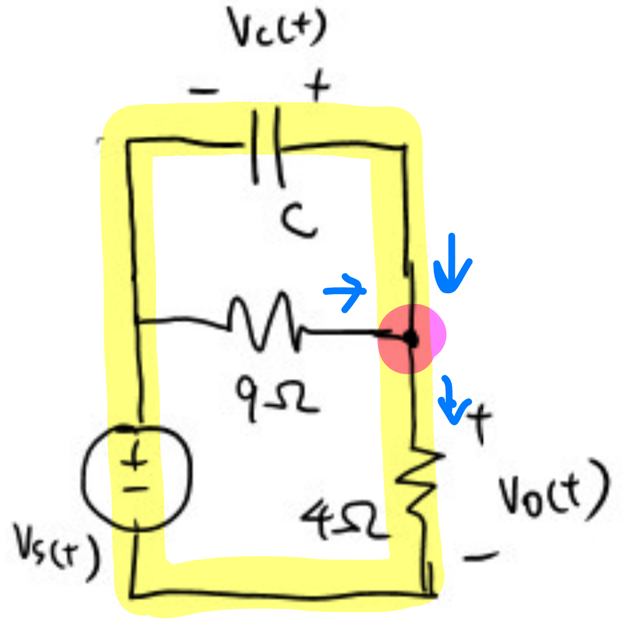
$V_x = (Z_R + Z_C) i_x + 3(Z_R i_x)$

$Z_T = \frac{V_x}{i_x} = 4Z_R + Z_C$

$= 40(e^{j0^\circ}) + 10 \cdot e^{(-90^\circ)j}$

$= 40 - 10j$

$= 41.23\, e^{(-14^\circ)j}$



$\omega = 2\ \mathrm{rad/s}$, $Z_c = \frac{1}{j\omega C} = \frac{1}{2Cj}$, $V_s = 8.93\,e^{j(54^\circ)} = 5.25 + 7.22j$

$V_o = 3.83\,e^{j83^\circ} = 0.47 + 3.8j$

$V_s(t) = 8.93\cos(2t + 54^\circ)$

$V_o(t) = 3.83\cos(2t + 83^\circ)$

$C = ?$

sol ①

$$\begin{cases} V_o = V_c + V_s \quad \text{(KVL)} \\ \frac{V_o}{4} + \frac{V_c}{9} + \frac{V_c}{Z_c} = 0 \quad \text{(KCL)} \end{cases}$$

$V_c = (0.47 + 3.8j) - (5.25 + 7.22j) = -4.78 - 3.42j = 5.88\,e^{j(216^\circ)}$

$-\frac{1}{Z_c} = \frac{9V_o + 4V_c}{36V_c} = \frac{9(0.47+3.8j) + 4(-4.78-3.42j)}{36(5.88\,e^{j216^\circ})} = \frac{-14.89 + 20.52j}{36(5.88\,e^{j216^\circ})}$

$-\frac{1}{Z_c} = -2Cj = 2C\,e^{j(-90^\circ)} = \frac{25.35\,e^{j126^\circ}}{36 \cdot 5.88\,e^{j216^\circ}} = 0.12\,e^{j(-90^\circ)} \rightarrow 2C = 0.12 \rightarrow C = 0.06\,F$ #

sol ②

$V_o = V_s \frac{4}{Z_c \parallel 9 + 4} \rightarrow \frac{V_o}{V_s} = \frac{4}{\frac{1}{2Cj + 1/9} + 4} = \frac{4(18Cj + 1)}{9 + 4(18Cj + 1)} = \frac{4 + 72Cj}{13 + 72Cj}$

in parallel

$\frac{3.83}{8.93} = \frac{\sqrt{4^2 + 72^2C^2}}{\sqrt{13^2 + 72^2C^2}} \rightarrow 0.184 = \frac{16 + (72C)^2}{169 + (72C)^2} \rightarrow (72C)^2(0.816) = 15.1 \rightarrow C = \frac{4.3}{72} = 0.06\,F$ #