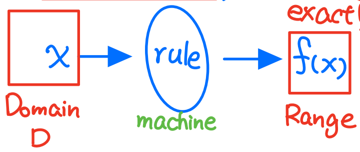
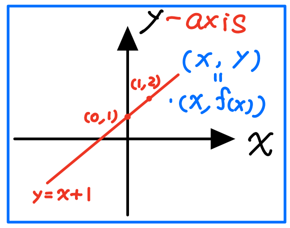
Cha 1. Fuction of Methods

Def.: A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, is a set E.

independent variable

exactly one → dependent variable, marked as y.





x-y plane

$\boxed{f(x)}$: value of f at x

f of x

Our goal of $f(x)$:

— Extrema of $f(x)$

— Sketch the graph of $f(x)$

— predict

Eg.: $f(x)=\sqrt{x+2}$

domain (of $f(x)$ in $\mathbb{R}$): $\boxed{\{x \mid x \geq -2,\ x \in \mathbb{R}\}}$, $x \geq -2$, $[-2, \infty)$

教授寫的

range (of $f(x)$ in $\mathbb{R}$): $\{f(x) \mid f(x) \geq 0,\ f(x) \in \mathbb{R}\}$

Eg: $g(x) = \dfrac{1}{x^2-x}$ ← 1 numerate, x^2-x denominator

domain (of $g(x)$ in $\mathbb{R}$): $\{x \mid x \neq 1 \text{ and } x \neq 0\}$

range (of $g(x)$ in $\mathbb{R}$): $\{g(x) \mid g(x) \neq 0\}$

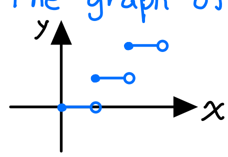
Piecewise Defined Fuctions:

$$f(x) = \begin{cases} 1-x, & \text{if } x \leq -1 \\ x^2, & \text{if } x > -1 \\ \vdots \end{cases}$$

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Step function

The graph of $\boxed{f(x) = [x]}$ is

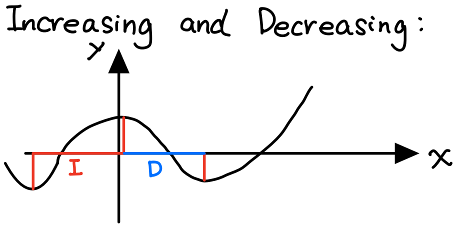


Even and Odd Fuctions.

$$\begin{cases} f(-x) = f(x) \Rightarrow \text{Even function} \\ f(-x) = -f(x) \Rightarrow \text{Odd function} \end{cases}$$

The graph of an
- even function is symmetric with respect to the y-axis.
- odd function isn't symmetric with respect to the y-axis.
- odd function is symmetric with respect to the original point.

Increasing and Decreasing:



f is increasing on I if $f(x_1) < f(x_2)$ as $x_1 < x_2$

decreasing on D if $f(x_1) > f(x_2)$ as $x_1 < x_2$

Polynomials:

terms $\sqrt{x}+(x-2)+(x^2-9)$

factor $\frac{(x-2)\cdot\sqrt{x}}{x^2-9}$

$P(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_1x^1+a_0x^0$

$a_n \neq 0$, n: nonnegative integer, a_i: coefficient, n: degree

a_n → leading coefficient

Power Function:

$f(x)=x^a$, a: constant

Rational Function:

$\frac{P(x)}{Q(x)}$ → polynomial

Trigonometric Function:

$f(x)=\sin x$

Exponential Function:

$f(x)=b^x$, b: constant

logarithmic Function:

$f(x)=\log_a x$, $a>0$, $a\neq 1$

1.3 New Functions from Old Functions

Translation: Vertical and Horizontal Shifts $(c>0)$

$y=f(x)+c$ ↑

$y=f(x)-c$ ↓

$y=f(x-c)$ →

$y=f(x+c)$ ←

Stretch and Shrink

$y=cf(x)$ stretched a factor of c 高

$y=\frac{1}{c}f(x)$ shrink a factor of c 矮

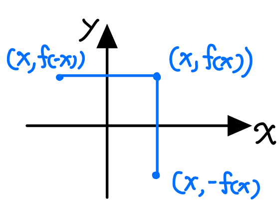
$y=f(cx)$ shrink 瘦

$y=f(\frac{x}{c})$ Strecthed 胖

Reflexion

$y=-f(x)$

$y=f(-x)$



addition $f \overset{plus}{\oplus} g$ ⇒ sum

substraction $f \overset{minus}{\ominus} g$ ⇒ difference

multiplication $f \overset{times}{\odot} g$ ⇒ product

division $f \underset{over}{\oslash} g$ ⇒ quotient $(g\neq 0)$

1.4 Exponential Fuctions

Law of Exponents

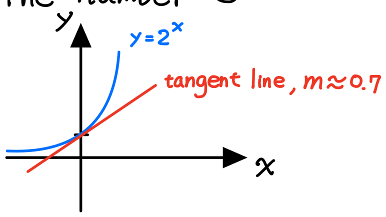
(1) $b^{x+y} = b^x \cdot b^y$

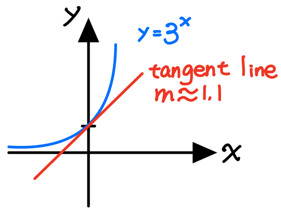
(2) $b^{x-y} = b^x / b^y$

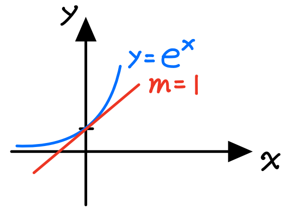
(3) $(b^x)^y = b^{xy}$

(4) $(a \cdot b)^x = a^x \cdot b^x$

The number e

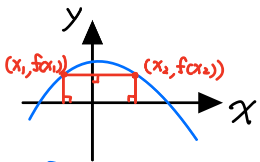






1.5 Inverse Function

Def.: f is one-to-one, when $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$ ($x_1, x_2 \in$ domain of f)



f is not one-to-one by horizontal of line test.

Def.: $f: A \to B$

If f is invertible function (marked as f^{-1}) has domain B and range of A

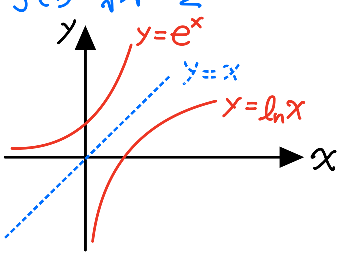
$f^{-1}(y) = x \Leftrightarrow f(x) = y \ \forall y \in B$

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$

$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y$

Find the inverse function of $f(x) = x^3 + 2$

$f^{-1}(x) = \sqrt[3]{x-2}$



f is symmetric to f^{-1} with respective to the line $y = x$

Laws of Logs: $b > 0$, $b \neq 1$, $r \in \mathbb{R}$

(1) $\log_b(xy) = \log_b x + \log_b y$

(2) $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

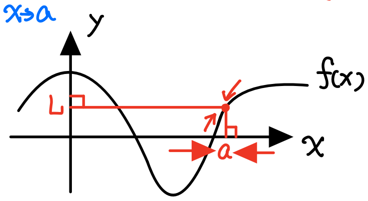
(3) $\log_b(x^r) = r \log_b x$

(4) $\log_a b = \dfrac{\ln b}{\ln a}$

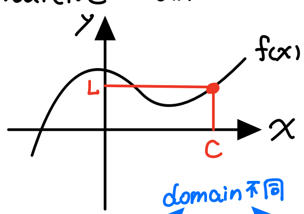
2.2 Limits

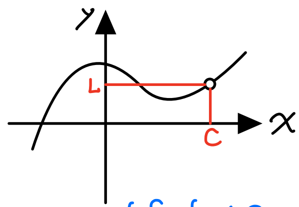
Intuitive definition of a limit.

$\lim_{x\to a} f(x) = L$ is the limit of $f(x)$ as x approaches a, equals L.

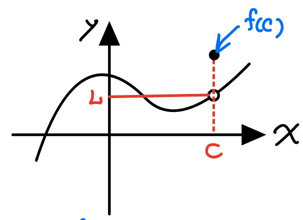


Intuitive Definition of One-sided Limit





not defined at c



$\lim_{x\to c} f(x) = L$

domain不同

Eg. $\lim_{x\to 1} \boxed{\frac{x-1}{x^2-1}} = \lim_{x\to 1} \boxed{\frac{1}{x+1}} = \frac{1}{2}$

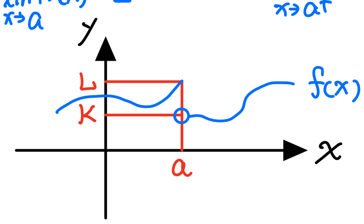
Eg1. $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t\to 0} \frac{(\sqrt{t^2+9}-3)(\sqrt{t^2+9}+3)}{t^2\cdot(\sqrt{t^2+9}+3)} = \frac{t^2+9-9}{t^2\cdot(\sqrt{t^2+9}+3)} = \frac{1}{6}$

$\lim_{x\to a^-} f(x) = L \Rightarrow$ left-hand side limit

$\lim_{x\to a^+} f(x) = L \Rightarrow$ right-hand side limit

$\lim_{x\to a} f(x) = L$ exists if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$

$\sin\frac{\pi}{a}$ $\quad$ $\sin(-\frac{\pi}{a})$

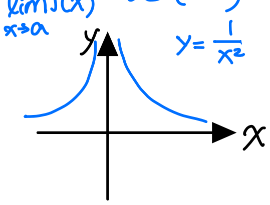


$\underset{L}{\boxed{\lim_{x\to a^-} f(x)}} \neq \underset{K}{\boxed{\lim_{x\to a^+} f(x)}}$

$\therefore \lim_{x\to a} f(x)$ D.N.E.

Eg. $\lim_{x\to 0} \frac{1}{x^2} = \infty$, D.N.E.

$\lim_{x\to a} f(x) = \infty \ (-\infty)$



vertical asymptote of $f(x) = \frac{1}{x^2}$ is $x=0$

Def.: The vertical line $x=a$ is called a vertical asymptote of the curve $y=f(x)$ if at least one of the following statement is true:

$\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to a^-} f(x) = \infty$, $\lim_{x\to a} f(x) = \infty$

$\lim_{x\to a^+} f(x) = -\infty$, $\lim_{x\to a^-} f(x) = -\infty$, $\lim_{x\to a} f(x) = -\infty$

Eg.: $y = \frac{2x}{x-3}$

$$\left.\begin{aligned} \lim_{x\to 3^+} \frac{2x}{x-3} &= \infty \\ \lim_{x\to 3^-} \frac{2x}{x-3} &= -\infty \end{aligned}\right\} x=3 \text{ is an vertical asymptote}$$

Eg.: $f(x) = \tan x = \frac{\sin x}{\cos x}$

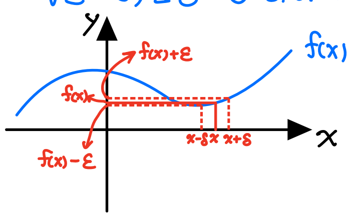
$\cos x = 0,\ x = \frac{2n+1}{2}\pi\ \forall n \in \mathbb{Z}$

$\sin(\frac{1}{2}\pi) = 1$

$\lim_{x\to \frac{\pi}{2}^+} \frac{\sin x}{\cos x} = -\infty,\ \lim_{x\to \frac{\pi}{2}^-} \frac{\sin x}{\cos x} = \infty,\ \therefore \lim_{x\to\frac{\pi}{2}} \frac{\sin x}{\cos x}$ D.N.E

2.4 The Precise Definition of a Limit

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ if $0 < |x - a| < \delta$, then $\lim_{x\to a} f(x) = L$



2.3

Sum Law of Limits: If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a}(f(x)+g(x)) = L+M$

pf.: Given any $\varepsilon > 0$, $\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$

$\exists \delta_2 > 0$ s.t. $0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon$

$\delta = \min\{\delta_1, \delta_2\}$, $0 < |x-a| < \delta \Rightarrow |f(x) + g(x) - (L+M)|$

$= |(f(x)-L) + (g(x)-M)| \le |f(x)-L| + |g(x)-M| < \varepsilon + \varepsilon = 2\varepsilon$

Limit Laws (Rules): Suppose that c is a constant and the limits $\lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ exists. Then

1. $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) \Rightarrow$ sum(difference) rule
2. $\lim_{x\to a}[cf(x)] = c \lim_{x\to a} f(x) \Rightarrow$ constant multiple rule
3. $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \Rightarrow$ product rule
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0 \Rightarrow$ quotient rule
5. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$, $n \in \mathbb{N} \Rightarrow$ power rule
6. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ [if n is even, we assume that $\lim_{x\to a} f(x) > 0$]
7. $\lim_{x\to a} C = C \Rightarrow$ constant rule
8. $\lim_{x\to a} x = a$

$p(x) = \sum_{i=0}^{n} a_i x^i \Rightarrow$ Polynomial function

$\lim_{x\to a} p(x) = \sum_{i=0}^{n} \lim_{x\to a} a_i x^i = \sum_{i=0}^{n} a_i \lim_{x\to a} x^i = \sum_{i=0}^{n} a_i \cdot a^i = p(a)$

$\lim_{x\to a} \frac{p(x)}{g(x)} = \frac{p(a)}{g(a)},\ g(a) \neq 0$

Eg:

(a) $\lim_{x\to5}(2x^2-3x+4)=39$

(b) $\lim_{x\to-2}\frac{x^3+2x^2-1}{5-3x}=-\frac{1}{11}$

Eg.: $\lim_{x\to0}|x|=0$

$|x|=\begin{cases}x, & x>0\\ -x, & x\le0\end{cases}$, $\lim_{x\to0^+}|x|=\lim_{x\to0^+}x=0$, $\lim_{x\to0^-}|x|=\lim_{x\to0^-}(-x)=0$

Eg.: $\lim_{x\to0}\frac{|x|}{x}$ D.N.E.

$\lim_{x\to0^+}\frac{|x|}{x}=\lim_{x\to0^+}\frac{x}{x}=1$

$\lim_{x\to0^-}\frac{|x|}{x}=\lim_{x\to0^-}\frac{-x}{x}=-1$

Eg.: $[x]$: largest integer which is less than or equal to x

$\lim_{x\to5^+}[x]=5$, $\lim_{x\to5^-}[x]=4$, $\therefore \lim_{x\to5}[x]$ D.N.E.

Eg.: $\lim_{x\to0}x^2\sin\frac{1}{x}=0$ (Can't use product rule)

$\lim_{x\to0}\sin\frac{1}{x}$ D.N.E.

$-1\le\sin\frac{1}{x}\le1$

$\Rightarrow -x^2\le x^2\sin\frac{1}{x}\le x^2$

$\Rightarrow \lim_{x\to0}(-x^2)\le\lim_{x\to0}x^2\sin\frac{1}{x}\le\lim_{x\to0}x^2$

$\Rightarrow 0\le\lim_{x\to0}x^2\frac{1}{x}\le0$

Theorem: $f(x)\le g(x)$ when x is near a, $\lim_{x\to a}f(x)\le\lim_{x\to a}g(x)$

Squeeze theorem: If $f(x)\le g(x)\le h(x)$ when x is near and $\lim_{x\to a}f(x)=\lim_{x\to a}h(x)=L$, then $\lim_{x\to a}g(x)=L$

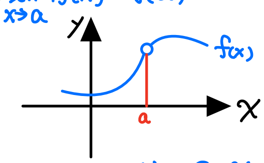
2.5 Continuity

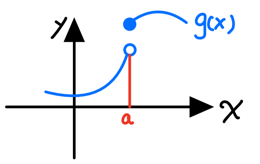
Def.: A function of f is continuity at a number a if $\lim_{x\to a}f(x)=f(a)$

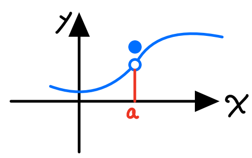
① $f(x)$ is defined at a

② $\lim_{x\to a}f(x)$ exists

③ $\lim_{x\to a}f(x)=f(a)$







not continue at $x=a$ not continue at $x=a$ not continue at $x=a$

Eg. $f(x)=\frac{x^2-x-2}{x-2}$

$\because f(x)$ is not defined at $x=2$, $\therefore f(x)$ isn't continuous at $x=2$

Eg. $f(x)=\begin{cases}\frac{x^2-x-2}{x-2}, & x\ne2\\ 3, & x=2\end{cases}$

$\lim_{x\to2}\frac{x^2-x-2}{x-2}=\lim_{x\to2}\frac{(x-2)(x+1)}{x-2}=3=f(2)$

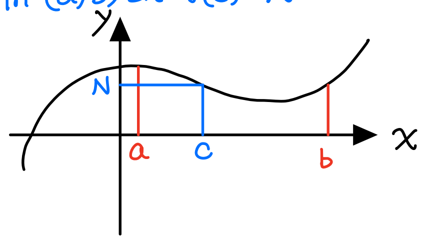
Theorem: Polynomials, rational, root, triagonometric, investing, exponential, log

Theorem: If f is continuous at $x=b$ and $\lim_{x\to a} f(x)=b$, then $\lim_{x\to a} f(g(x))=f(b)$ and

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$$

The Intermediate Value Theorem:

Suppose that f is countinuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number C in (a,b) s.t. $f(c)=N$



$f(c)=N$