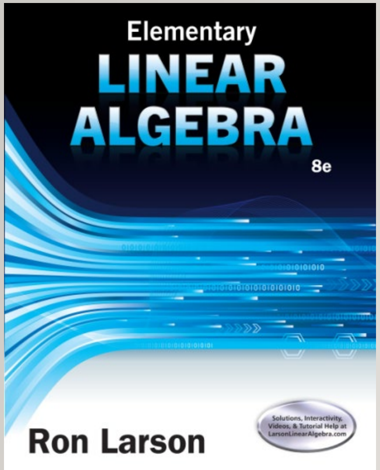

CHAPTER 5
INNER PRODUCT SPACES

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5.1 Length and Dot Product in R^n

- Length:

The length of a vector $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ in R^n is given by

$$\|\mathbf{v}\| = \sqrt{v_1^{\,2} + v_2^{\,2} + \cdots + v_n^{\,2}}$$

- Notes: The length of a vector is also called its **norm**.

- Notes: Properties of length

(1) $\|\mathbf{v}\| \ge 0$

(2) $\|\mathbf{v}\| = 1 \Rightarrow \mathbf{v}$ is called a **unit vector**.

(3) $\|\mathbf{v}\| = 0$ iff $\mathbf{v} = 0$

(4) $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$

- Ex 1:

(a) In R^5, the length of $\mathbf{v} = (0, -2, 1, 4, -2)$ is given by

$$\|\mathbf{v}\| = \sqrt{0^2 + (-2)^2 + 1^2 + 4^2 + (-2)^2} = \sqrt{25} = 5$$

(b) In R^3 the length of $\mathbf{v} = (\frac{2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}})$ is given by

$$\|\mathbf{v}\| = \sqrt{\left(\frac{2}{\sqrt{17}}\right)^2 + \left(\frac{-2}{\sqrt{17}}\right)^2 + \left(\frac{3}{\sqrt{17}}\right)^2} = \sqrt{\frac{17}{17}} = 1$$

($\mathbf{v}$ is a unit vector)

- A standard unit vector in R^n:

$$\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} = \{(1,0,\cdots,0), (0,1,\cdots,0), (0,0,\cdots,1)\}$$

- Ex:

the standard unit vector in R^2: $\{i, j\} = \{(1,0), (0,1)\}$

the standard unit vector in R^3: $\{i, j, k\} = \{(1,0,0), (0,1,0), (0,0,1)\}$

- Notes: (Two nonzero vectors are parallel)

$$\mathbf{u} = c\mathbf{v}$$

(1) $c > 0 \Rightarrow$ **u** and **v** have the same direction

(2) $c < 0 \Rightarrow$ **u** and **v** have the opposite direction

- Thm 5.1: (Length of a scalar multiple)

Let $\mathbf{v}$ be a vector in R^n and c be a scalar. Then

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$$

Pf:

$$\mathbf{v} = (v_1\,, v_2\,, \cdots, v_n)$$

$$\Rightarrow c\mathbf{v} = (cv_1\,, cv_2\,, \cdots, cv_n)$$

$$\|c\mathbf{v}\| = \|\,(cv_1\,, cv_2\,, \cdots, cv_n)\,\|$$

$$= \sqrt{(cv_1)^2 + (cv_2)^2 + \cdots + (cv_n)^2}$$

$$= \sqrt{c^2(v_1{}^2 + v_2{}^2 + \cdots + v_n{}^2)}$$

$$= |c|\sqrt{v_1{}^2 + v_2{}^2 + \cdots + v_n{}^2}$$

$$= |c|\,\|\mathbf{v}\|$$

- Thm 5.2: (Unit vector in the direction of **v**)

If **v** is a nonzero vector in R^n, then the vector $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$

has length 1 and has the same direction as **v**. This vector **u** is called the **unit vector in the direction of v**.

Pf:

$$\mathbf{v} \text{ is nonzero } \Rightarrow \|\mathbf{v}\| \neq 0 \Rightarrow \frac{1}{\|\mathbf{v}\|} > 0$$

$$\Rightarrow \mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} \qquad (\mathbf{u} \text{ has the same direction as } \mathbf{v})$$

$$\|\mathbf{u}\| = \left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1 \qquad (\text{u has length 1 })$$

- Notes:

  (1) The vector $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is called the unit vector in the direction of $\mathbf{v}$.

  (2) The process of finding the unit vector in the direction of $\mathbf{v}$ is called **normalizing** the vector $\mathbf{v}$.

- Ex 2: (Finding a unit vector)

Find the unit vector in the direction of $\mathbf{v}=(3,-1,2)$,

and verify that this vector has length 1.

Sol:

$$\mathbf{v}=(3,-1,2) \quad \Rightarrow \|\mathbf{v}\|=\sqrt{3^2+(-1)^2+2^2}=\sqrt{14}$$

$$\Rightarrow \frac{\mathbf{v}}{\|\mathbf{v}\|}=\frac{(3,-1,2)}{\sqrt{3^2+(-1)^2+2^2}}=\frac{1}{\sqrt{14}}(3,-1,2)=\left(\frac{3}{\sqrt{14}},\frac{-1}{\sqrt{14}},\frac{2}{\sqrt{14}}\right)$$

$$\because \quad \sqrt{\left(\frac{3}{\sqrt{14}}\right)^2+\left(\frac{-1}{\sqrt{14}}\right)^2+\left(\frac{2}{\sqrt{14}}\right)^2}=\sqrt{\frac{14}{14}}=1$$

$\therefore \quad \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector.

- Distance between two vectors:

  The **distance** between two vectors **u** and **v** in R^n is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

- Notes: (Properties of distance)

  (1) $d(\mathbf{u}, \mathbf{v}) \geq 0$

  (2) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

  (3) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

- Ex 3: (Finding the distance between two vectors)

The distance between **u** = (0, 2, 2) and **v** = (2, 0, 1) is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \|(0-2, 2-0, 2-1)\|$$

$$= \sqrt{(-2)^2 + 2^2 + 1^2} = 3$$

- Dot product in R^n:

  The **dot product** of $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ is the scalar quantity

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- Ex 4: (Finding the dot product of two vectors)

  The dot product of **u**=(1, 2, 0, -3) and **v**=(3, -2, 4, 2) is

$$\mathbf{u} \cdot \mathbf{v} = (1)(3) + (2)(-2) + (0)(4) + (-3)(2) = -7$$

- Thm 5.3: (Properties of the dot product)

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n and c is a scalar, then the following properties are true.

(1) $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$

(2) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$

(3) $c(\mathbf{u}\cdot\mathbf{v})=(c\mathbf{u})\cdot\mathbf{v}=\mathbf{u}\cdot(c\mathbf{v})$

(4) $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2$

(5) $\mathbf{v}\cdot\mathbf{v}\geq 0$, and $\mathbf{v}\cdot\mathbf{v}=0$ if and only if $\boldsymbol{v}=0$

- Euclidean *n*-space:

  R^n was defined to be the *set* of all order n-tuples of real numbers. When R^n is combined with the standard operations of vector addition, scalar multiplication, vector length, and the dot product, the resulting vector space is called **Euclidean *n*-space**.

- Ex 5: (Finding dot products)

$$\mathbf{u} = (2, -2), \mathbf{v} = (5, 8), \mathbf{w} = (-4, 3)$$

(a) $\mathbf{u} \cdot \mathbf{v}$ (b) $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ (c) $\mathbf{u} \cdot (2\mathbf{v})$ (d) $\|\mathbf{w}\|^2$ (e) $\mathbf{u} \cdot (\mathbf{v} - 2\mathbf{w})$

Sol:

(a) $\mathbf{u} \cdot \mathbf{v} = (2)(5) + (-2)(8) = -6$

(b) $(\mathbf{u} \cdot \mathbf{v})\mathbf{w} = -6\mathbf{w} = -6(-4, 3) = (24, -18)$

(c) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-6) = -12$

(d) $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w} = (-4)(-4) + (3)(3) = 25$

(e) $\mathbf{v} - 2\mathbf{w} = (5 - (-8), 8 - 6) = (13, 2)$

$\mathbf{u} \cdot (\mathbf{v} - 2\mathbf{w}) = (2)(13) + (-2)(2) = 26 - 4 = 22$

- Ex 6: (Using the properties of the dot product)

Given $\mathbf{u}\cdot\mathbf{u}=39 \quad \mathbf{u}\cdot\mathbf{v}=-3 \quad \mathbf{v}\cdot\mathbf{v}=79$

Find $(\mathbf{u}+2\mathbf{v})\cdot(3\mathbf{u}+\mathbf{v})$

Sol:

$$
\begin{aligned}
(\mathbf{u}+2\mathbf{v})\cdot(3\mathbf{u}+\mathbf{v}) &= \mathbf{u}\cdot(3\mathbf{u}+\mathbf{v})+2\mathbf{v}\cdot(3\mathbf{u}+\mathbf{v}) \\
&= \mathbf{u}\cdot(3\mathbf{u})+\mathbf{u}\cdot\mathbf{v}+(2\mathbf{v})\cdot(3\mathbf{u})+(2\mathbf{v})\cdot\mathbf{v} \\
&= 3(\mathbf{u}\cdot\mathbf{u})+\mathbf{u}\cdot\mathbf{v}+6(\mathbf{v}\cdot\mathbf{u})+2(\mathbf{v}\cdot\mathbf{v}) \\
&= 3(\mathbf{u}\cdot\mathbf{u})+7(\mathbf{u}\cdot\mathbf{v})+2(\mathbf{v}\cdot\mathbf{v}) \\
&= 3(39)+7(-3)+2(79)=254
\end{aligned}
$$

- Thm 5.4: (The Cauchy - Schwarz inequality)

  If **u** and **v** are vectors in R^n, then

  $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$ ($|\mathbf{u}\cdot\mathbf{v}|$ denotes the absolute value of $\mathbf{u}\cdot\mathbf{v}$)

- Ex 7: (An example of the Cauchy - Schwarz inequality)

  Verify the Cauchy - Schwarz inequality for **u**=(1, -1, 3) and **v**=(2, 0, -1)

Sol: $\mathbf{u}\cdot\mathbf{v} = -1, \quad \mathbf{u}\cdot\mathbf{u} = 11, \quad \mathbf{v}\cdot\mathbf{v} = 5$

$$\Rightarrow |\mathbf{u}\cdot\mathbf{v}| = |-1| = 1$$

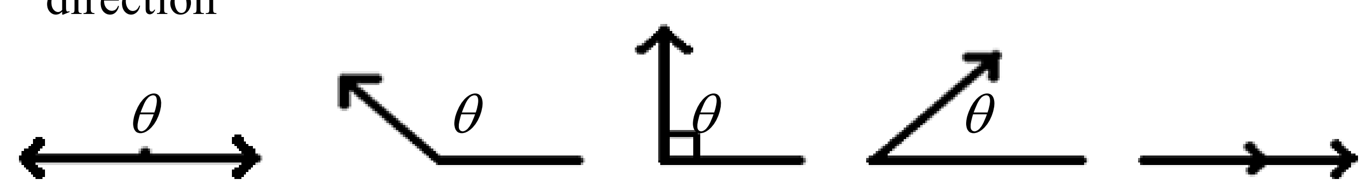
$$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}}\cdot\sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{11}\cdot\sqrt{5} = \sqrt{55}$$

$$\therefore |\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$$

- The angle between two vectors in R^n:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|},\ 0 \le \theta \le \pi$$

Opposite direction	$\mathbf{u}\cdot\mathbf{v} < 0$	$\mathbf{u}\cdot\mathbf{v} = 0$	$\mathbf{u}\cdot\mathbf{v} > 0$	Same direction



$\theta = \pi$	$\frac{\pi}{2} < \theta < \pi$	$\theta = \frac{\pi}{2}$	$0 < \theta < \frac{\pi}{2}$	$\theta = 0$
$\cos = -1$	$\cos < 0$	$\cos = 0$	$\cos > 0$	$\cos = 1$

- Note:

The angle between the zero vector and another vector is not defined.

- Ex 8: (Finding the angle between two vectors)

$$\mathbf{u} = (-4, 0, 2, -2) \quad \mathbf{v} = (2, 0, -1, 1)$$

Sol:

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{(-4)^2 + 0^2 + 2^2 + (-2)^2} = \sqrt{24}$$

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{2^2 + (0)^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$\mathbf{u}\cdot\mathbf{v} = (-4)(2) + (0)(0) + (2)(-1) + (-2)(1) = -12$$

$$\Rightarrow \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-12}{\sqrt{24}\sqrt{6}} = -\frac{12}{\sqrt{144}} = -1$$

$\Rightarrow \theta = \pi \qquad \therefore$ $\mathbf{u}$ and $\mathbf{v}$ have opposite directions. $(\mathbf{u} = -2\mathbf{v})$

- Orthogonal vectors:

  Two vectors **u** and **v** in R^n are orthogonal if

  $$\mathbf{u} \cdot \mathbf{v} = 0$$

- Note:

  The vector **0** is said to be orthogonal to every vector.

- Ex 10: (Finding orthogonal vectors)

Determine all vectors in R^n that are orthogonal to $\mathbf{u}$=(4, 2).

Sol:

$$\mathbf{u}=(4\,,2) \quad \text{Let} \quad \mathbf{v}=(v_1\,,v_2)$$

$$\Rightarrow \quad \mathbf{u}\cdot\mathbf{v}=(4\,,2)\cdot(v_1\,,v_2)$$
$$=4v_1+2v_2$$
$$=0$$

$$\begin{bmatrix}4 & 2 & 0\end{bmatrix} \to \begin{bmatrix}1 & \frac{1}{2} & 0\end{bmatrix}$$

$$\Rightarrow \quad v_1=\frac{-t}{2} \quad , \quad v_2=t$$

$$\therefore \quad \mathbf{v}=\left(\frac{-t}{2},t\right), \qquad t\in R$$

- Thm 5.5: (The triangle inequality)

If **u** and **v** are vectors in R^n, then $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$

Pf:

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$$

$$= \mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) + \mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}$$

$$= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 \quad \le \|\mathbf{u}\|^2 + 2\,|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2$$

$$\le \|\mathbf{u}\|^2 + 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2$$

$$= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$$

$$\therefore \|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$

- Note:

Equality occurs in the triangle inequality if and only if the vectors **u** and **v** have the same direction.

- Thm 5.6: (The Pythagorean theorem)

  If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

- Dot product and matrix multiplication:

$$\mathbf{u}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}\qquad \mathbf{v}=\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}$$

(A vector $\mathbf{u}=(u_1\ ,u_2\ ,\cdots,u_n)$ in R^n is represented as an $n\times 1$ column matrix)

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}=[u_1\quad u_2\quad\cdots\quad u_n]\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=[u_1v_1+u_2v_2+\cdots+u_nv_n]$$

Key Learning in Section 5.1

- Find the length of a vector and find a unit vector.
- Find the distance between two vectors.
- Find a dot product and the angle between two vectors, determine orthogonality, and verify the Cauchy-Schwarz Inequality, the triangle inequality, and the Pythagorean Theorem.
- Use a matrix product to represent a dot product.

Keywords in Section 5.1

- length: 長度
- norm: 範數
- unit vector: 單位向量
- standard unit vector : 標準單位向量
- normalizing: 單範化
- distance: 距離
- dot product: 點積
- Euclidean *n*-space: 歐基里德*n*維空間
- Cauchy-Schwarz inequality: 科西-舒瓦茲不等式
- angle: 夾角
- triangle inequality: 三角不等式
- Pythagorean theorem: 畢氏定理

5.2 Inner Product Spaces

- Inner product:

Let **u**, **v**, and **w** be vectors in a vector space V, and let c be any scalar. An inner product on V is <u>a function</u> that associates <u>a real number</u> $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors **u** and **v** and satisfies the following axioms.

(1) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

(2) $\langle \mathbf{u}, \mathbf{v}+\mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

(3) $c\langle \mathbf{u}, \mathbf{v} \rangle = \langle c\mathbf{u}, \mathbf{v} \rangle$

(4) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$

- Note:

$$\mathbf{u} \cdot \mathbf{v} = \text{dot product (Euclidean inner product for } R^n\text{)}$$

$$<\mathbf{u}\,,\,\mathbf{v}> = \text{general inner product for vector space } V$$

- Note:

A vector space V with an inner product is called an inner product space.

Vector space: $(V,\ +,\ \bullet)$

Inner product space: $(V,\ +,\ \bullet,\ <,>)$

- Ex 1: (The Euclidean inner product for R^n)

  Show that the dot product in R^n satisfies the four axioms of an inner product.

Sol:

$$\mathbf{u} = (u_1, u_2, \cdots, u_n) \quad , \quad \mathbf{v} = (v_1, v_2, \cdots, v_n)$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

By Theorem 5.3, this dot product satisfies the required four axioms. Thus it is an inner product on R^n.

Thm 5.3: (Properties of the dot product)

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n and c is a scalar, then the following properties are true.

(1) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(2) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(3) $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$

(4) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

(5) $\mathbf{v} \cdot \mathbf{v} \geq 0$, and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\boldsymbol{v} = 0$

- Ex 2: (A different inner product for R^n)

Show that the function defines an inner product on R^2, where $\mathbf{u}=(u_1\ ,u_2)$ and $\mathbf{v}=(v_1\ ,v_2)$.

$$\langle\,\mathbf{u}\ ,\mathbf{v}\,\rangle = u_1v_1 + 2u_2v_2$$

Sol:

(a) $\langle\,\mathbf{u}\ ,\mathbf{v}\,\rangle = u_1v_1 + 2u_2v_2 = v_1u_1 + 2v_2u_2 = \langle\,\mathbf{v}\ ,\mathbf{u}\,\rangle$

(b) $\mathbf{w}=(w_1\ ,w_2)$

$$\begin{aligned}\Rightarrow \langle\,\mathbf{u}\ ,\mathbf{v}+\mathbf{w}\,\rangle &= u_1(v_1+w_1)+2u_2(v_2+w_2)\\ &= u_1v_1+u_1w_1+2u_2v_2+2u_2w_2\\ &= (u_1v_1+2u_2v_2)+(u_1w_1+2u_2w_2)\\ &= \langle\,\mathbf{u}\ ,\mathbf{v}\,\rangle+\langle\,\mathbf{u}\ ,\mathbf{w}\,\rangle\end{aligned}$$

$$(c)\quad c\langle \mathbf{u}, \mathbf{v}\rangle = c(u_1v_1 + 2u_2v_2) = (cu_1)v_1 + 2(cu_2)v_2 = \langle c\mathbf{u}, \mathbf{v}\rangle$$

$$(d)\quad \langle \mathbf{v}, \mathbf{v}\rangle = v_1{}^2 + 2v_2{}^2 \geq 0$$

$$\langle \mathbf{v}, \mathbf{v}\rangle = 0 \Rightarrow v_1{}^2 + 2v_2{}^2 = 0 \qquad \Rightarrow \quad v_1 = v_2 = 0 \quad (\mathbf{v} = 0)$$

- Note: (An inner product on R^n)

$$\langle \mathbf{u}, \mathbf{v}\rangle = c_1u_1v_1 + c_2u_2v_2 + \cdots + c_nu_nv_n\,, \qquad c_i > 0$$

- Ex 3: (A function that is not an inner product)

Determine whether the following function is not an inner product on R^3 or not.

$$\langle \mathbf{u} \cdot \mathbf{v} \rangle = u_1 v_1 - 2u_2 v_2 + u_3 v_3$$

Sol:

Let $\mathbf{v} = (1, 2, 1)$

Then $< \mathbf{v}, \mathbf{v} >= (1)(1) - 2(2)(2) + (1)(1) = -6 < 0$

Axiom 4 is not satisfied.

Thus this function is not an inner product on R^3.

- Thm 5.7: (Properties of inner products)

  Let **u, v**, and **w** be vectors in an inner product space V, and let c be any real number.

  (1) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$

  (2) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

  (3) $\langle \mathbf{u}, c\mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$

- Norm (length) of **u**:

  $$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

- Note:

  $$\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$$

- Distance between **u** and **v**:

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

- Angle between two nonzero vectors **u** and **v**:

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|}, \quad 0 \le \theta \le \pi$$

- Orthogonal: $(\mathbf{u} \perp \mathbf{v})$

$\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

- Notes:

(1) If $\|\mathbf{v}\| = 1$, then v is called a unit vector.

(2) $\|\mathbf{v}\| \neq 1$, $\mathbf{v} \neq 0$ $\xrightarrow{\text{Normalizing}}$ $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ (the unit vector in the direction of $\mathbf{v}$)

not a unit vector

- Ex 6: (Finding inner product)

$$\langle p\, , q \rangle = a_0 b_0 + a_1 b_1 + \cdots + a_n b_n \quad \text{is an inner product}$$

Let $p(x) = 1 - 2x^2,\ q(x) = 4 - 2x + x^2$ be polynomials in $P_2(x)$

$(a)\ \langle p\, , q \rangle = ? \qquad (b)\ \ \|q\| = ? \qquad (c)\ \ d(p\, , q) = ?$

Sol:

$$\text{(a)}\ \langle p\, , q \rangle = (1)(4) + (0)(-2) + (-2)(1) = 2$$

$$\text{(b)}\ \ \|q\| = \sqrt{\langle q\, , q \rangle} = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{21}$$

$$\text{(c)}\ \ \because\ p - q = -3 + 2x - 3x^2$$

$$\therefore d(p\, , q) = \|p - q\| = \sqrt{\langle p - q, p - q \rangle}$$

$$= \sqrt{(-3)^2 + 2^2 + (-3)^2} = \sqrt{22}$$

- Properties of norm:

(1) $\|\mathbf{u}\| \geq 0$

(2) $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$

(3) $\|c\mathbf{u}\| = |c|\,\|\mathbf{u}\|$

- Properties of distance:

(1) $d(\mathbf{u}\,,\,\mathbf{v}) \geq 0$

(2) $d(\mathbf{u}\,,\,\mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

(3) $d(\mathbf{u}\,,\,\mathbf{v}) = d(\mathbf{v}\,,\,\mathbf{u})$

- Thm 5.8：

Let **u** and **v** be vectors in an inner product space *V*.

(1) Cauchy-Schwarz inequality:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{u}\| \, \|\mathbf{v}\| \qquad \text{Theorem 5.4}$$

(2) Triangle inequality:

$$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\| \qquad \text{Theorem 5.5}$$

(3) Pythagorean theorem :

u and **v** are orthogonal if and only if

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \qquad \text{Theorem 5.6}$$

- Orthogonal projections in inner product spaces:

Let **u** and **v** be two vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Then the **orthogonal projection of u onto v** is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}$$

- Note:

If **v** is a init vector, then $\langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{v}\|^2 = 1$.

The formula for the orthogonal projection of **u** onto **v** takes the following simpler form.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{v}$$

- Orthogonal projections in inner product spaces:

  Let **u** and **v** be two vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Then the **orthogonal projection of u onto v** is given by

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}\,,\,\mathbf{v}\rangle}{\langle \mathbf{v}\,,\,\mathbf{v}\rangle}\mathbf{v}$$

- proof:

- Ex 10: (Finding an orthogonal projection in R^3)

  Use the Euclidean inner product in R^3 to find the orthogonal projection of u=(6, 2, 4) onto v=(1, 2, 0).

  Sol:

$$\because \langle \mathbf{u}\,,\,\mathbf{v}\rangle = (6)(1)+(2)(2)+(4)(0)=10$$

$$\langle \mathbf{v}\,,\,\mathbf{v}\rangle = 1^2+2^2+0^2=5$$

$$\therefore \operatorname{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \tfrac{10}{5}(1\,,2\,,0)=(2\,,4\,,0)$$

- Note:

$$\boldsymbol{u}-\operatorname{proj}_{v}\boldsymbol{u}=(6,2,4)-(2,4,0)=(4,-2,4) \text{ is orthogonal to } v=(1,2,0).$$

- Thm 5.9: (Orthogonal projection and distance)

  Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Then

$$d(\mathbf{u}\,,\,\text{proj}_{\mathbf{v}}\mathbf{u}) < d(\mathbf{u}\,,\,c\mathbf{v})\,, \qquad c \neq \frac{\langle \mathbf{u}\,,\,\mathbf{v}\rangle}{\langle \mathbf{v}\,,\,\mathbf{v}\rangle}$$

Key Learning in Section 5.2

- Determine whether a function defines an inner product, and find the inner product of two vectors in R^n, $M_{m,n}$, P_n and C[a, b].
- Find an orthogonal projection of a vector onto another vector in an inner product space.

Keywords in Section 5.2

- inner product: 內積
- inner product space: 內積空間
- norm: 範數
- distance: 距離
- angle: 夾角
- orthogonal: 正交
- unit vector: 單位向量
- normalizing: 單範化
- Cauchy – Schwarz inequality: 科西 - 舒瓦茲不等式
- triangle inequality: 三角不等式
- Pythagorean theorem: 畢氏定理
- orthogonal projection: 正交投影

5.3 Orthonormal Bases: Gram-Schmidt Process

- A vector space can have many different bases:

  Ex: bases of R^3:

$$B_1 = \{(4,3,0), (3,4,0), (1,1,1)\}$$

$$B_2 = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$$B_2 = \{(\cos\theta, \sin\theta, 0), (-\sin\theta, \cos\theta, 0), (0,0,1)\}$$

- Features of vectors in B_2 and B_3
  - Mutually orthogonal
  - Unit vector
- B_2 is the standard basis for R^3
- Purpose of this section: find orthonormal bases of a vector space V

5.3 Orthonormal Bases: Gram-Schmidt Process

- Orthogonal:

  A set S of vectors in an inner product space V is called an **orthogonal set** if every pair of vectors in the set is orthogonal.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subseteq V$$

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \qquad i \neq j$$

- Orthonormal:

  An orthogonal set in which each vector is a unit vector is called **orthonormal**.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subseteq V$$

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- Note:

  If S is a basis, then it is called an orthogonal basis or an orthonormal basis.

- Ex 1: (A nonstandard orthonormal basis for R^3)

Show that the following set is an orthonormal basis.

$$S = \left\{ \overset{\mathbf{v}_1}{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}, \quad \overset{\mathbf{v}_2}{\left(-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right)}, \quad \overset{\mathbf{v}_3}{\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)} \right\}$$

Sol:

Show that the three vectors are mutually orthogonal.

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = -\tfrac{1}{6} + \tfrac{1}{6} + 0 = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0$$

$$\mathbf{v}_2 \cdot \mathbf{v}_3 = -\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0$$

Show that each vector is of length 1.

$$\|\mathbf{v}_1\| = \sqrt{\mathbf{v}_1 \cdot \mathbf{v}_1} = \sqrt{\tfrac{1}{2} + \tfrac{1}{2} + 0} = 1$$

$$\|\mathbf{v}_2\| = \sqrt{\mathbf{v}_2 \cdot \mathbf{v}_2} = \sqrt{\tfrac{2}{36} + \tfrac{2}{36} + \tfrac{8}{9}} = 1$$

$$\|\mathbf{v}_3\| = \sqrt{\mathbf{v}_3 \cdot \mathbf{v}_3} = \sqrt{\tfrac{4}{9} + \tfrac{4}{9} + \tfrac{1}{9}} = 1$$

Thus S is an orthonormal set.

- Ex 2: (An orthonormal basis for $P_3(x)$)

In $P_3(x)$, with the inner product

$$\langle p,q\rangle = a_0b_0 + a_1b_1 + a_2b_2$$

The standard basis $B = \{1, x, x^2, x^3\}$ is orthonormal.

Sol:

$\mathbf{v}_1 = 1 + 0x + 0x^2 + 0x^3,$ $\quad\quad$ $\mathbf{v}_2 = 0 + 1x + 0x^2 + 0x^3$

$\mathbf{v}_3 = 0 + 0x + 1x^2 + 0x^3$ $\quad\quad$ $\mathbf{v}_4 = 0 + 0x + 0x^2 + 1x^3$

Then

$$\langle \mathbf{v}_1, \mathbf{v}_2\rangle = (1)(0) + (0)(1) + (0)(0) + (0)(0) = 0,$$
$$\langle \mathbf{v}_1, \mathbf{v}_3\rangle = (1)(0) + (0)(0) + (0)(1) + (0)(0) = 0,$$
$$\langle \mathbf{v}_2, \mathbf{v}_3\rangle = (0)(0) + (1)(0) + (0)(1) + (0)(0) = 0,$$

....

$$\|\mathbf{v}_1\| = \sqrt{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} = \sqrt{(1)(1) + (0)(0) + (0)(0) + (0)(0)} = 1,$$
$$\|\mathbf{v}_2\| = \sqrt{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} = \sqrt{(0)(0) + (1)(1) + (0)(0) + (0)(0)} = 1,$$
$$\|\mathbf{v}_3\| = \sqrt{\langle \mathbf{v}_3, \mathbf{v}_3 \rangle} = \sqrt{(0)(0) + (0)(0) + (1)(1) + (0)(0)} = 1$$
$$\|\mathbf{v}_4\| = \sqrt{\langle \mathbf{v}_4, \mathbf{v}_4 \rangle} = \sqrt{(0)(0) + (0)(0) + (0)(0) + (1)(1)} = 1$$

- Thm 5.10: (Orthogonal sets are linearly independent)

  If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ is an orthogonal set of *nonzero* vectors in an inner product space *V*, then *S* is linearly independent.

Pf:

S is an orthogonal set of nonzero vectors

i.e. $\langle \mathbf{v}_i,\mathbf{v}_j\rangle=0\quad i\neq j$ and $\langle \mathbf{v}_i,\mathbf{v}_i\rangle>0$

Let $c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_n\mathbf{v}_n=0$

$$\Rightarrow \quad \langle c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_n\mathbf{v}_n,\mathbf{v}_i\rangle=\langle 0,\mathbf{v}_i\rangle=0 \qquad \forall i$$

$$\Rightarrow \quad c_1\langle\mathbf{v}_1,\mathbf{v}_i\rangle+c_2\langle\mathbf{v}_2,\mathbf{v}_i\rangle+\cdots+c_i\langle\mathbf{v}_i,\mathbf{v}_i\rangle+\cdots+c_n\langle\mathbf{v}_n,\mathbf{v}_i\rangle$$
$$=c_i\langle\mathbf{v}_i,\mathbf{v}_i\rangle$$

$\because \langle\mathbf{v}_i,\mathbf{v}_i\rangle\neq 0 \;\Rightarrow c_i=0 \quad \forall i \qquad \therefore S$ is linearly independent.

- Corollary to Thm 5.10:

  If V is an inner product space of dimension n, then any orthogonal set of n nonzero vectors is a basis for V.

- Ex 4: (Using orthogonality to test for a basis)

Show that the following set is a basis for R^4.

$$S = \{\underset{\mathbf{v}_1}{(2, 3, 2, -2)}, \underset{\mathbf{v}_2}{(1, 0, 0, 1)}, \underset{\mathbf{v}_3}{(-1, 0, 2, 1)}, \underset{\mathbf{v}_4}{(-1, 2, -1, 1)}\}$$

Sol:

$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$: nonzero vectors

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 2 + 0 + 0 - 2 = 0 \qquad \mathbf{v}_2 \cdot \mathbf{v}_3 = -1 + 0 + 0 + 1 = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = -2 + 0 + 4 - 2 = 0 \qquad \mathbf{v}_2 \cdot \mathbf{v}_4 = -1 + 0 + 0 + 1 = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_4 = -2 + 6 - 2 - 2 = 0 \qquad \mathbf{v}_3 \cdot \mathbf{v}_4 = 1 + 0 - 2 + 1 = 0$$

$\Rightarrow S$ is orthogonal.

$\Rightarrow S$ is a basis for R^4 (by Corollary to Theorem 5.10)

- Thm 5.11: (Coordinates relative to an orthonormal basis)

  If $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, then the coordinate representation of a vector $\mathbf{w}$ with respect to B is

$$\mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{w}, \mathbf{v}_n \rangle \mathbf{v}_n$$

Pf:

$B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for V

$\mathbf{w} \in V$

$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$ (unique representation)

$\because B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is orthonormal

$$\Rightarrow \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\begin{aligned}\langle \mathbf{w}\ ,\ \mathbf{v}_i\rangle &= \langle (k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ \cdots\ + k_n\mathbf{v}_n)\ ,\ \mathbf{v}_i\rangle \\ &= k_1\langle \mathbf{v}_1\ ,\ \mathbf{v}_i\rangle + \cdots + k_i\langle \mathbf{v}_i\ ,\ \mathbf{v}_i\rangle + \ \cdots \\ &= k_i \qquad \forall i\end{aligned}$$

$$\Rightarrow \mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1\rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2\rangle \mathbf{v}_2 + \cdots + \langle \mathbf{w}, \mathbf{v}_n\rangle \mathbf{v}_n$$

- Note:

  If $B = \{\mathbf{v}_1\ , \mathbf{v}_2\ , \cdots, \mathbf{v}_n\}$ is an orthonormal basis for V and $\mathbf{w} \in V$,

  Then the corresponding coordinate matrix of $\mathbf{w}$ relative to B is

$$[\mathbf{w}]_B = \begin{bmatrix} \langle \mathbf{w}, \mathbf{v}_1\rangle \\ \langle \mathbf{w}, \mathbf{v}_2\rangle \\ \vdots \\ \langle \mathbf{w}, \mathbf{v}_n\rangle \end{bmatrix}$$

- Ex 5: (Representing vectors relative to an orthonormal basis)

Find the coordinates of $\mathbf{w} = (5, -5, 2)$ relative to the following orthonormal basis for R^3.

$$B = \{(\tfrac{3}{5}, \tfrac{4}{5}, 0), (-\tfrac{4}{5}, \tfrac{3}{5}, 0), (0, 0, 1)\}$$

Sol:

$$\langle \mathbf{w}, \mathbf{v}_1 \rangle = \mathbf{w} \cdot \mathbf{v}_1 = (5, -5, 2) \cdot (\tfrac{3}{5}, \tfrac{4}{5}, 0) = -1$$

$$\langle \mathbf{w}, \mathbf{v}_2 \rangle = \mathbf{w} \cdot \mathbf{v}_2 = (5, -5, 2) \cdot (-\tfrac{4}{5}, \tfrac{3}{5}, 0) = -7$$

$$\langle \mathbf{w}, \mathbf{v}_3 \rangle = \mathbf{w} \cdot \mathbf{v}_3 = (5, -5, 2) \cdot (0, 0, 1) = 2$$

$$\Rightarrow [\mathbf{w}]_B = \begin{bmatrix} -1 \\ -7 \\ 2 \end{bmatrix}$$

- Gram-Schmidt orthonormalization process:

$B = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is a basis for an inner product space V

Let $\mathbf{v}_1 = \mathbf{u}_1$

$\mathbf{w}_1 = span(\{\mathbf{v}_1\})$

$$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{W}_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1$$

$\mathbf{w}_2 = span(\{\mathbf{v}_1, \mathbf{v}_2\})$

$$\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{\mathbf{W}_2}\mathbf{u}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2$$

$\vdots$

$$\mathbf{v}_n = \mathbf{u}_n - \text{proj}_{\mathbf{W}_{n-1}}\mathbf{u}_n = \mathbf{u}_n - \sum_{i=1}^{n-1}\frac{\langle \mathbf{v}_n, \mathbf{v}_i \rangle}{\langle \mathbf{v}_i, \mathbf{v}_i \rangle}\mathbf{v}_i$$

$\Rightarrow B' = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthogonal basis.

$\Rightarrow B'' = \{\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \cdots, \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}\}$ is an orthonormal basis.

- Ex 7: (Applying the Gram-Schmidt orthonormalization process)

Apply the Gram-Schmidt process to the following basis.

$$B = \{\underset{\mathbf{u}_1}{(1,1,0)},\quad \underset{\mathbf{u}_2}{(1,2,0)},\quad \underset{\mathbf{u}_3}{(0,1,2)}\}$$

Sol: $\mathbf{v}_1 = \mathbf{u}_1 = (1,1,0)$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (1,2,0) - \frac{3}{2}(1,1,0) = (-\frac{1}{2}, \frac{1}{2}, 0)$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

$$= (0,1,2) - \frac{1}{2}(1,1,0) - \frac{1/2}{1/2}(-\frac{1}{2}, \frac{1}{2}, 0) = (0,0,2)$$

Orthogonal basis

$$\Rightarrow B' = \{\mathbf{v}_1\ , \mathbf{v}_2\ , \mathbf{v}_3\} = \{(1, 1, 0), (\frac{-1}{2}, \frac{1}{2}, 0), (0, 0, 2)\}$$

Orthonormal basis

$$\Rightarrow B'' = \{\frac{\mathbf{v_1}}{\|\mathbf{v_1}\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|}\} = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (0, 0, 1)\}$$

- Ex 10: (Alternative form of Gram-Schmidt orthonormalization process)

Find an orthonormal basis for the solution space of the homogeneous system of linear equations.

$$\begin{aligned} x_1 + x_2 \qquad\quad &+ 7x_4 = 0 \\ 2x_1 + x_2 + 2x_3 &+ 6x_4 = 0 \end{aligned}$$

Sol:

$$\begin{bmatrix} 1 & 1 & 0 & 7 & 0 \\ 2 & 1 & 2 & 6 & 0 \end{bmatrix} \xrightarrow{G.J.E} \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -2 & 8 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s+t \\ 2s-8t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -8 \\ 0 \\ 1 \end{bmatrix}$$

Thus one basis for the solution space is

$$B=\{\mathbf{v}_1\ ,\ \mathbf{v}_2\}=\{(-2\ ,\ 2\ ,\ 1\ ,\ 0)\ ,\ (1\ ,\ -8\ ,\ 0\ ,\ 1)\}$$

$$\mathbf{v}_1=\mathbf{u}_1=(-2,2,1,0)$$

$$\mathbf{v}_2=\mathbf{u}_2-\frac{\langle\mathbf{u}_2,\mathbf{v}_1\rangle}{\langle\mathbf{v}_1,\mathbf{v}_1\rangle}\mathbf{v}_1=(1,-8,0,1)-\frac{-18}{9}(-2,2,1,0)$$

$$=(-3,-4,2,1)$$

$$\Rightarrow B'=\{(-2,2,1,0)(-3,-4,2,1)\}$$ (orthogonal basis)

$$\Rightarrow B''=\left\{\left(\frac{-2}{3},\frac{2}{3},\frac{1}{3},0\right),\left(\frac{-3}{\sqrt{30}},\frac{-4}{\sqrt{30}},\frac{2}{\sqrt{30}},\frac{1}{\sqrt{30}}\right)\right\}$$

(orthonormal basis)

Key Learning in Section 5.3

- Show that a set of vectors is orthogonal and forms an orthonormal basis, and represent a vector relative to an orthonormal basis.
- Apply the Gram-Schmidt orthonormalization process.

Keywords in Section 5.3

- orthogonal set: 正交集合
- orthonormal set: 單範正交集合
- orthogonal basis: 正交基底
- orthonormal basis: 單範正交基底
- linear independent: 線性獨立
- Gram-Schmidt Process: Gram-Schmidt過程

- Orthogonal subspaces:

The subspaces W_1 and W_2 of an inner product space V are orthogonal if $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = 0$ for all $\boldsymbol{v}_1$ in W_1 and all $\boldsymbol{v}_2$ in W_2.

- Ex 2: (Orthogonal subspaces)

The subspaces

$$W_1 = \text{span}(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}) \text{ and } W_2 = \text{span}(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix})$$

are orthogonal because $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = 0$ for any vector in W_1 and any vector in W_2 is zero.

- Orthogonal complement of W:

Let W be a subspace of an inner product space V.

(a) A vector $\mathbf{u}$ in V is said to **orthogonal to** W,

if $\mathbf{u}$ is orthogonal to every vector in W.

(b) The set of all vectors in V that are orthogonal to every

vector in W is called the **orthogonal complement of W**.

$$W^{\perp} = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0, \forall\, \mathbf{w} \in W\}$$

$W^{\perp}$ (read " W perp")

- Notes:

(1) $\left(\{0\}\right)^{\perp} = V$ (2) $V^{\perp} = \{0\}$

- Notes:

W is a subspace of V

(1) $W^{\perp}$ is a subspace of V

(2) $W \cap W^{\perp} = \{\mathbf{0}\}$

(3) $(W^{\perp})^{\perp} = W$

- Ex:

If $V = R^2, \quad W = x-\text{axis}$

Then (1) $W^{\perp} = y\text{-axis}$ is a subspace of R^2

(2) $W \cap W^{\perp} = \{(0,0)\}$

(3) $(W^{\perp})^{\perp} = W$

- Direct sum:

  Let W_1 and W_2 be two subspaces of R^n. If each vector $\mathbf{x} \in R^n$ can be uniquely written as a sum of a vector $\mathbf{w}_1$ from W_1 and a vector $\mathbf{w}_2$ from W_2, $\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2$, then R^n is the direct sum of W_1 and W_2, and you can write .

  $$R^n = W_1 \oplus W_2$$

- Thm 5.13: (Properties of orthogonal subspaces)

  Let W be a subspace of R^n. Then the following properties are true.

  (1) $\dim(W) + \dim(W^{\perp}) = n$

  (2) $R^n = W \oplus W^{\perp}$

  (3) $(W^{\perp})^{\perp} = W$

- Thm 5.14: (Projection onto a subspace)

If $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_t\}$ is an orthonormal basis for the subspace W of V, and $\mathbf{v} \in V$, then

$$\text{proj}_W \mathbf{v} = \langle \mathbf{v}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{v}, \mathbf{u}_2 \rangle \mathbf{u}_2 + \cdots + \langle \mathbf{v}, \mathbf{u}_t \rangle \mathbf{u}_t$$

Pf:

$\because \text{proj}_W \boldsymbol{v} \in \text{W}$ and $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_t\}$ is an orthonormal basis for W

$$\Rightarrow \text{proj}_W \boldsymbol{v} = \langle \text{proj}_W \boldsymbol{v}, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 + \cdots + \langle \text{proj}_W \boldsymbol{v}, \boldsymbol{u}_t \rangle \boldsymbol{u}_t$$

$$= \langle \boldsymbol{v} - \text{proj}_{W^\perp} \boldsymbol{v}, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 + \cdots + \langle \boldsymbol{v} - \text{proj}_{W^\perp} \boldsymbol{v}, \boldsymbol{u}_t \rangle \boldsymbol{u}_t$$

$$(\because \text{proj}_W \boldsymbol{v} = \boldsymbol{v} - \text{proj}_{W^\perp} \boldsymbol{v})$$

$$= \langle \boldsymbol{v}, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 + \cdots + \langle \boldsymbol{v}, \boldsymbol{u}_t \rangle \boldsymbol{u}_t \quad (\because \langle \text{proj}_{W^\perp} \boldsymbol{v}, \boldsymbol{u}_i \rangle = 0, \forall\, i)$$

- Ex 5: (Projection onto a subspace)

$$\mathbf{w}_1=(0,3,1),\ \mathbf{w}_2=(2,0,0),\ \mathbf{v}=(1,1,3)$$

Find the projection of the vector **v** onto the subspace W.

Sol:

$$W=span(\{\mathbf{w}_1,\mathbf{w}_2\})$$

$\{\mathbf{w}_1,\mathbf{w}_2\}$: an orthogonal basis for W

$$\Rightarrow\{\mathbf{u}_1,\mathbf{u}_2\}=\left\{\frac{\mathbf{w}_1}{\|\mathbf{w}_1\|},\frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}\right\}=\left\{(0,\frac{3}{\sqrt{10}},\frac{1}{\sqrt{10}}),(1,0,0)\right\}\ :$$

an orthonormal basis for W

$$\text{proj}_W\boldsymbol{v}=\langle\boldsymbol{v},\boldsymbol{u}_1\rangle\boldsymbol{u}_1+\langle\boldsymbol{v},\boldsymbol{u}_2\rangle\boldsymbol{u}_2$$

$$=\frac{6}{\sqrt{10}}(0,\frac{3}{\sqrt{10}},\frac{1}{\sqrt{10}})+(1,0,0)=(1,\frac{9}{5},\frac{3}{5})$$

- Thm 5.15: (Orthogonal projection and distance)

Let W be a subspace of an inner product space V, and $\mathbf{v} \in V$

Then for all $\mathbf{w} \in W,\ \mathbf{w} \neq \operatorname{proj}_W \mathbf{v}$

$$\|\mathbf{v} - \operatorname{proj}_W \mathbf{v}\| < \|\mathbf{v} - \mathbf{w}\|$$

or $$\|\boldsymbol{v} - \operatorname{proj}_W \boldsymbol{v}\| = \min_{\boldsymbol{w} \in \mathrm{W}} \|\boldsymbol{v} - \boldsymbol{w}\|$$

($\operatorname{proj}_W \mathbf{v}$ is the best approximation to $\mathbf{v}$ from W)

Pf:

$$\mathbf{v}-\mathbf{w}=(\mathbf{v}-\text{proj}_W\mathbf{v})+(\text{proj}_W\mathbf{v}-\mathbf{w})$$

$$(\mathbf{v}-\text{proj}_W\mathbf{v})\perp(\text{proj}_W\mathbf{v}-\mathbf{w})$$

By the Pythagorean theorem

$$\Rightarrow \|\mathbf{v}-\mathbf{w}\|^2=\|\mathbf{v}-\text{proj}_W\mathbf{v}\|^2+\|\text{proj}_W\mathbf{v}-\mathbf{w}\|^2$$

$$\mathbf{w}\neq\text{proj}_W\mathbf{v}\Rightarrow\|\text{proj}_W\mathbf{v}-\mathbf{w}\|>0$$

$$\Rightarrow \|\mathbf{v}-\mathbf{w}\|^2>\|\mathbf{v}-\text{proj}_W\mathbf{v}\|^2$$

$$\Rightarrow \|\mathbf{v}-\text{proj}_W\mathbf{v}\|<\|\mathbf{v}-\mathbf{w}\|$$

- Thm 5.16: (Fundamental subspaces of a matrix)

If A is an $m \times n$ matrix, then

(1) $(CS(A))^{\perp} = NS(A^{\mathrm{T}})$

$(NS(A^{\mathrm{T}}))^{\perp} = CS(A)$

(2) $(CS(A^{\mathrm{T}}))^{\perp} = NS(A)$

$(NS(A))^{\perp} = CS(A^{\mathrm{T}})$

(3) $CS(A) \oplus NS(A^T) = R^m$ $\quad CS(A) \oplus (CS(A))^{\perp} = R^m$

(4) $CS(A^T) \oplus NS(A) = R^n$ $\quad CS(A^T) \oplus (CS(A^{\mathrm{T}}))^{\perp} = R^n$

- Ex 6: (Fundamental subspaces)

Find the four fundamental subspaces of the matrix.

$$A=\begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\qquad \text{(reduced row-echelon form)}$$

Sol:

$$CS(A)=\text{span}(\{(1,0,0,0)\ \ (0,1,0,0)\})\ \text{ is a subspace of } R^4$$

$$CS(A^{\mathrm{T}})=RS(A)=\text{span}(\{(1,2,0)\ \ (0,0,1)\})\ \text{ is a subspace of } R^3$$

$$NS(A)=\text{span}(\{(-2,1,0)\})\ \text{ is a subspace of } R^3$$

$$A^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim R = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} & & s & t \end{array} \end{array}$$

$$NS(A^{\mathrm{T}}) = \operatorname{span}\left(\left\{(0,0,1,0)\quad(0,0,0,1)\right\}\right) \text{ is a subspace of } R^4$$

- Check:

$$(CS(A))^{\perp} = NS(A^{\mathrm{T}})$$

$$(CS(A^{\mathrm{T}}))^{\perp} = NS(A)$$

$$CS(A) \oplus NS(A^T) = R^4$$

$$CS(A^T) \oplus NS(A) = R^3$$

- Ex 3 & Ex 4:

$$W = \text{span}(\{\mathbf{w}_1, \mathbf{w}_2\})$$

Let W is a subspace of R^4 and $\mathbf{w}_1 = (1, 2, 1, 0), \quad \mathbf{w}_2 = (0, 0, 0, 1)$.

(a) Find a basis for W

(b) Find a basis for the orthogonal complement of W.

Sol:

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \mathbf{w}_1 \quad \mathbf{w}_2 \end{array} \sim R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{(reduced row-echelon form)}$$

$$(a)\quad W = CS(A)$$

$$\Rightarrow \quad \{(1,2,1,0),(0,0,0,1)\} \quad \text{is a basis for } W$$

$$(b)\quad W^{\perp} = (CS(A))^{\perp} = NS(A^{\mathrm{T}})$$

$$\because A^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s-t \\ s \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \{(-2,1,0,0)\ \ (-1,0,1,0)\} \text{ is a basis for } W^{\perp}$$

- Notes:

$$(1)\quad \dim(W) + \dim(W^{\perp}) = \dim(R^4)$$

$$(2)\quad W \oplus W^{\perp} = R^4$$

- Least squares problem:

$$\underset{m\times n}{A}\underset{n\times 1}{\mathbf{x}} = \underset{m\times 1}{\mathbf{b}} \qquad \text{(A system of linear equations)}$$

(1) When the system is consistent, we can use the Gaussian elimination with back-substitution to solve for $\mathbf{x}$

(2) When the system is inconsistent, how to find the "best possible" solution of the system. That is, the value of $\mathbf{x}$ for which the difference between $A\mathbf{x}$ and $\mathbf{b}$ is small.

- Least squares solution:

Given a system $A\mathbf{x} = \mathbf{b}$ of m linear equations in n unknowns, the least squares problem is to find a vector $\mathbf{x}$ in R^n that minimizes $\|A\mathbf{x} - \mathbf{b}\|$ with respect to the Euclidean inner product on R^n. Such a vector is called a least squares solution of $A\mathbf{x} = \mathbf{b}$.

- Notes:

The least square problem is to find a vector $\hat{\boldsymbol{x}}$ in R^n such that $A\hat{\boldsymbol{x}} = \text{proj}_{CS(A)}\boldsymbol{b}$ in the column space of A (i.e., $A\hat{\boldsymbol{x}} \in CS(A)$) is as close as possible to $\boldsymbol{b}$. That is,

$$\left\|\boldsymbol{b} - \text{proj}_{CS(A)}\boldsymbol{b}\right\| = \left\|\boldsymbol{b} - A\hat{\boldsymbol{x}}\right\| = \min_{\boldsymbol{x}\in R^n}\left\|\boldsymbol{b} - A\boldsymbol{x}\right\|$$

$A \in M_{m\times n}$

$\boldsymbol{x} \in R^n$

$A\boldsymbol{x} \in CS(A)$ $\quad$ ($CS(A)$ is a subspace of R^m)

$\because$ b $\notin CS(A)$ $\quad$ ($A\boldsymbol{x} =$ b is an inconsistent system)

Let $A\hat{\boldsymbol{x}} = proj_{CS(A)}\boldsymbol{b}$

$\Rightarrow (\boldsymbol{b} - A\hat{\boldsymbol{x}}) \perp CS(A)$

$\Rightarrow \boldsymbol{b} - A\hat{\boldsymbol{x}} \in (CS(A))^{\perp} = NS(A^{\mathrm{T}})$

$\Rightarrow A^{\mathrm{T}}(\boldsymbol{b} - A\hat{\boldsymbol{x}}) = 0$

i.e. $\quad A^{\mathrm{T}}A\hat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ (the normal system associated with $A\mathbf{x} = \mathbf{b}$)

- Note: ($A\mathbf{x} = \mathbf{b}$ is an inconsistent system)

The problem of finding the least squares solution of $A\mathbf{x} = \mathbf{b}$

is equal to he problem of finding an exact solution of the

associated normal system $A^{\mathrm{T}} A\hat{\mathbf{x}} = A^{\mathrm{T}}\mathbf{b}$.

- Ex 7: (Solving the normal equations)

Find the least squares solution of the following system

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$ (this system is inconsistent)

and find the orthogonal projection of $\mathbf{b}$ on the column space of A.

Sol:

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

the associated normal system

$$A^T A \hat{\boldsymbol{x}} = A^T \boldsymbol{b}$$

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

the least squares solution of $A\mathbf{x} = \mathbf{b}$

$$\hat{\boldsymbol{x}} = \begin{bmatrix} -\frac{5}{3} \\ \frac{3}{2} \end{bmatrix}$$

the orthogonal projection of $\mathbf{b}$ on the column space of A

$$\text{proj}_{CS(A)}\boldsymbol{b} = A\hat{\boldsymbol{x}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{-5}{3} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{-1}{6} \\ \frac{8}{6} \\ \frac{17}{6} \end{bmatrix}$$

Key Learning in Section 5.4

- Define the least squares problem.
- Find the orthogonal complement of a subspace and the projection of a vector onto a subspace.
- Find the four fundamental subspaces of a matrix.
- Solve a least squares problem.
- Use least squares for mathematical modeling.

Keywords in Section 5.4

- orthogonal to W: 正交於W
- orthogonal complement: 正交補集
- direct sum: 直和
- projection onto a subspace: 在子空間的投影
- fundamental subspaces: 基本子空間
- least squares problem: 最小平方問題
- normal equations: 一般方程式