Instructor's Solutions Manuals for Calculus Early Transcendentals 9th Edition by James Stewart, Daniel K. Clegg, Saleem Watson

1 ☐ FUNCTIONS AND MODELS

1.1 Four Ways to Represent a Function

1. The functions $f(x) = x + \sqrt{2-x}$ and $g(u) = u + \sqrt{2-u}$ give exactly the same output values for every input value, so f and g are equal.

2. $f(x) = \dfrac{x^2 - x}{x-1} = \dfrac{x(x-1)}{x-1} = x$ for $x - 1 \neq 0$, so f and g [where $g(x) = x$] are not equal because $f(1)$ is undefined and $g(1) = 1$.

3. (a) The point $(-2, 2)$ lies on the graph of g, so $g(-2) = 2$. Similarly, $g\,(0) = -2$, $g\,(2) = 1$, and $g\,(3) \approx 2.5$.

(b) Only the point $(-4, 3)$ on the graph has a y-value of 3, so the only value of x for which $g(x) = 3$ is -4.

(c) The function outputs $g(x)$ are never greater than 3, so $g(x) \leq 3$ for the entire domain of the function. Thus, $g(x) \leq 3$ for $-4 \leq x \leq 4$ (or, equivalently, on the interval $[-4, 4]$).

(d) The domain consists of all x-values on the graph of g: $\{x \mid -4 \leq x \leq 4\} = [-4, 4]$. The range of g consists of all the y-values on the graph of g: $\{y \mid -2 \leq y \leq 3\} = [-2, 3]$.

(e) For any $x_1 < x_2$ in the interval $[0, 2]$, we have $g(x_1) < g(x_2)$. [The graph rises from $(0, -2)$ to $(2, 1)$.] Thus, $g(x)$ is increasing on $[0, 2]$.

4. (a) From the graph, we have $f(-4) = -2$ and $g(3) = 4$.

(b) Since $f(-3) = -1$ and $g(-3) = 2$, or by observing that the graph of g is above the graph of f at $x = -3$, $g(-3)$ is larger than $f(-3)$.

(c) The graphs of f and g intersect at $x = -2$ and $x = 2$, so $f(x) = g(x)$ at these two values of x.

(d) The graph of f lies below or on the graph of g for $-4 \leq x \leq -2$ and for $2 \leq x \leq 3$. Thus, the intervals on which $f(x) \leq g(x)$ are $[-4, -2]$ and $[2, 3]$.

(e) $f\,(x) = -1$ is equivalent to $y = -1$, and the points on the graph of f with y-values of -1 are $(-3, -1)$ and $(4, -1)$, so the solution of the equation $f(x) = -1$ is $x = -3$ or $x = 4$.

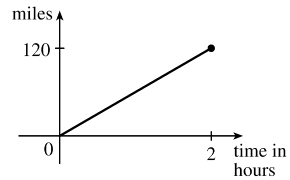
(f) For any $x_1 < x_2$ in the interval $[-4, 0]$, we have $g(x_1) > g(x_2)$. Thus, $g(x)$ is decreasing on $[-4, 0]$.

(g) The domain of f is $\{x \mid -4 \leq x \leq 4\} = [-4, 4]$. The range of f is $\{y \mid -2 \leq y \leq 3\} = [-2, 3]$.

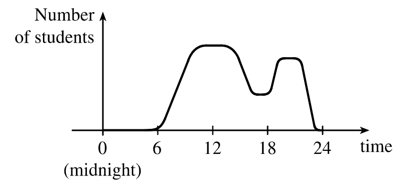
(h) The domain of g is $\{x \mid -4 \leq x \leq 3\} = [-4, 3]$. Estimating the lowest point of the graph of g as having coordinates $(0, 0.5)$, the range of g is approximately $\{y \mid 0.5 \leq y \leq 4\} = [0.5, 4]$.

5. From Figure 1 in the text, the lowest point occurs at about $(t, a) = (12, -85)$. The highest point occurs at about $(17, 115)$. Thus, the range of the vertical ground acceleration is $-85 \leq a \leq 115$. Written in interval notation, the range is $[-85, 115]$.

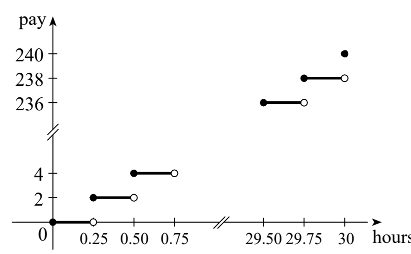
6. *Example 1:* A car is driven at 60 mi/h for 2 hours. The distance d traveled by the car is a function of the time t. The domain of the function is $\{t \mid 0 \leq t \leq 2\}$, where t is measured in hours. The range of the function is $\{d \mid 0 \leq d \leq 120\}$, where d is measured in miles.



Example 2: At a certain university, the number of students N on campus at any time on a particular day is a function of the time t after midnight. The domain of the function is $\{t \mid 0 \leq t \leq 24\}$, where t is measured in hours. The range of the function is $\{N \mid 0 \leq N \leq k\}$, where N is an integer and k is the largest number of students on campus at once.



Example 3: A certain employee is paid \$8.00 per hour and works a maximum of 30 hours per week. The number of hours worked is rounded down to the nearest quarter of an hour. This employee's gross weekly pay P is a function of the number of hours worked h. The domain of the function is $[0, 30]$ and the range of the function is $\{0, 2.00, 4.00, \ldots, 238.00, 240.00\}$.



7. We solve $3x - 5y = 7$ for y: $3x - 5y = 7 \quad \Leftrightarrow \quad -5y = -3x + 7 \quad \Leftrightarrow \quad y = \frac{3}{5}x - \frac{7}{5}$. Since the equation determines exactly one value of y for each value of x, the equation defines y as a function of x.

8. We solve $3x^2 - 2y = 5$ for y: $3x^2 - 2y = 5 \quad \Leftrightarrow \quad -2y = -3x^2 + 5 \quad \Leftrightarrow \quad y = \frac{3}{2}x^2 - \frac{5}{2}$. Since the equation determines exactly one value of y for each value of x, the equation defines y as a function of x.

9. We solve $x^2 + (y-3)^2 = 5$ for y: $x^2 + (y-3)^2 = 5 \quad \Leftrightarrow \quad (y-3)^2 = 5 - x^2 \quad \Leftrightarrow \quad y - 3 = \pm\sqrt{5 - x^2} \quad \Leftrightarrow$ $y = 3 \pm \sqrt{5 - x^2}$. Some input values x correspond to more than one output y. (For instance, $x = 1$ corresponds to $y = 1$ and to $y = 5$.) Thus, the equation does *not* define y as a function of x.

10. We solve $2xy + 5y^2 = 4$ for y: $2xy + 5y^2 = 4 \quad \Leftrightarrow \quad 5y^2 + (2x)\,y - 4 = 0 \quad \Leftrightarrow$

$$y = \frac{-2x \pm \sqrt{(2x)^2 - 4(5)(-4)}}{2(5)} = \frac{-2x \pm \sqrt{4x^2 + 80}}{10} = \frac{-x \pm \sqrt{x^2 + 20}}{5}$$

(using the quadratic formula). Some input values x correspond to more than one output y. (For instance, $x = 4$ corresponds to $y = -2$ and to $y = 2/5$.) Thus, the equation does *not* define y as a function of x.

11. We solve $(y+3)^3 + 1 = 2x$ for y: $(y+3)^3 + 1 = 2x \quad \Leftrightarrow \quad (y+3)^3 = 2x - 1 \quad \Leftrightarrow \quad y + 3 = \sqrt[3]{2x - 1} \quad \Leftrightarrow$ $y = -3 + \sqrt[3]{2x - 1}$. Since the equation determines exactly one value of y for each value of x, the equation defines y as a function of x.

12. We solve $2x - |y| = 0$ for y: $2x - |y| = 0 \Leftrightarrow |y| = 2x \Leftrightarrow y = \pm 2x$. Some input values x correspond to more than one output y. (For instance, $x = 1$ corresponds to $y = -2$ and to $y = 2$.) Thus, the equation does *not* define y as a function of x.

13. The height 60 in $(x = 60)$ corresponds to shoe sizes 7 and 8 $(y = 7$ and $y = 8)$. Since an input value x corresponds to more than output value y, the table does *not* define y as a function of x.

14. Each year x corresponds to exactly one tuition cost y. Thus, the table defines y as a function of x.

15. No, the curve is not the graph of a function because a vertical line intersects the curve more than once. Hence, the curve fails the Vertical Line Test.

16. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-2, 2]$ and the range is $[-1, 2]$.

17. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-3, 2]$ and the range is $[-3, -2) \cup [-1, 3]$.

18. No, the curve is not the graph of a function since for $x = 0$, ± 1, and ± 2, there are infinitely many points on the curve.

19. (a) When $t = 1950$, $T \approx 13.8^\circ\text{C}$, so the global average temperature in 1950 was about 13.8°C.

(b) When $T = 14.2^\circ\text{C}$, $t \approx 1990$.

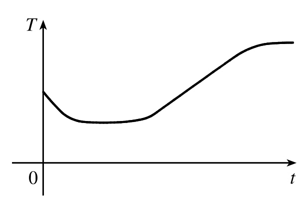
(c) The global average temperature was smallest in 1910 (the year corresponding to the lowest point on the graph) and largest in 2000 (the year corresponding to the highest point on the graph).

(d) When $t = 1910$, $T \approx 13.5^\circ\text{C}$, and when $t = 2000$, $T \approx 14.4^\circ\text{C}$. Thus, the range of T is about $[13.5, 14.4]$.

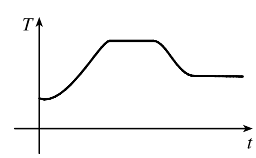
20. (a) The ring width varies from near 0 mm to about 1.6 mm, so the range of the ring width function is approximately $[0, 1.6]$.

(b) According to the graph, the earth gradually cooled from 1550 to 1700, warmed into the late 1700s, cooled again into the late 1800s, and has been steadily warming since then. In the mid-19th century, there was variation that could have been associated with volcanic eruptions.

21. The water will cool down almost to freezing as the ice melts. Then, when the ice has melted, the water will slowly warm up to room temperature.



22. The temperature of the pie would increase rapidly, level off to oven temperature, decrease rapidly, and then level off to room temperature.

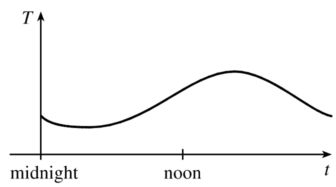


23. (a) The power consumption at 6 AM is 500 MW, which is obtained by reading the value of power P when $t = 6$ from the graph. At 6 PM we read the value of P when $t = 18$, obtaining approximately 730 MW.

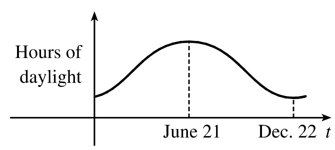
(b) The minimum power consumption is determined by finding the time for the lowest point on the graph, $t = 4$, or 4 AM. The maximum power consumption corresponds to the highest point on the graph, which occurs just before $t = 12$, or right before noon. These times are reasonable, considering the power consumption schedules of most individuals and businesses.

24. Runner A won the race, reaching the finish line at 100 meters in about 15 seconds, followed by runner B with a time of about 19 seconds, and then by runner C who finished in around 23 seconds. B initially led the race, followed by C, and then A. C then passed B to lead for a while. Then A passed first B, and then passed C to take the lead and finish first. Finally, B passed C to finish in second place. All three runners completed the race.

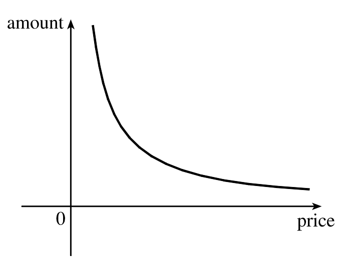
25. Of course, this graph depends strongly on the geographical location!



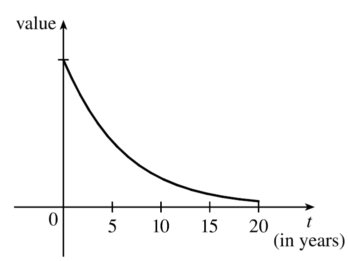
26. The summer solstice (the longest day of the year) is around June 21, and the winter solstice (the shortest day) is around December 22. (Exchange the dates for the southern hemisphere.)



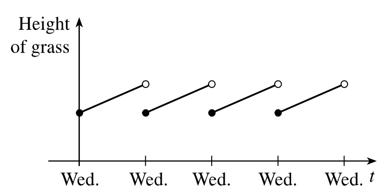
27. As the price increases, the amount sold decreases.



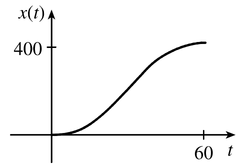
28. The value of the car decreases fairly rapidly initially, then somewhat less rapidly.



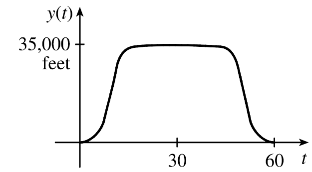
29.



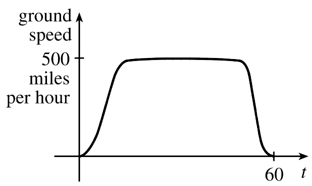
30. (a)



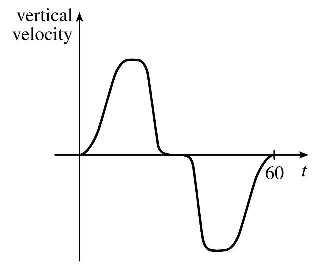
(b)



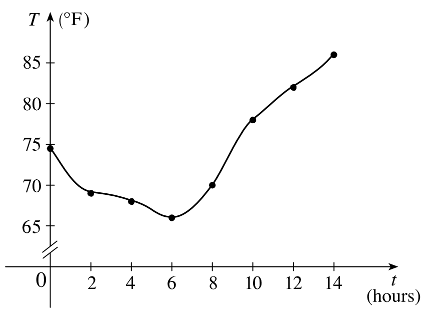
(c)



(d)

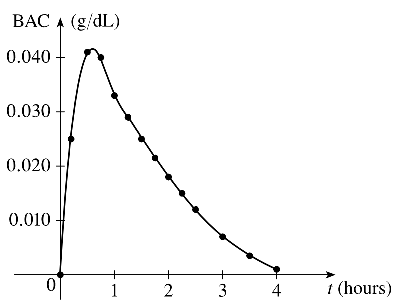


31. (a)



(b) 9:00 AM corresponds to $t = 9$. When $t = 9$, the temperature T is about $74°\text{F}$.

32. (a)



(b) The blood alcohol concentration rises rapidly, then slowly decreases to near zero.

33. $f(x) = 3x^2 - x + 2$.

$f(2) = 3(2)^2 - 2 + 2 = 12 - 2 + 2 = 12$.

$f(-2) = 3(-2)^2 - (-2) + 2 = 12 + 2 + 2 = 16$.

$f(a) = 3a^2 - a + 2$.

$f(-a) = 3(-a)^2 - (-a) + 2 = 3a^2 + a + 2$.

$f(a+1) = 3(a+1)^2 - (a+1) + 2 = 3(a^2 + 2a + 1) - a - 1 + 2 = 3a^2 + 6a + 3 - a + 1 = 3a^2 + 5a + 4$.

$2f(a) = 2 \cdot f(a) = 2(3a^2 - a + 2) = 6a^2 - 2a + 4$.

$f(2a) = 3(2a)^2 - (2a) + 2 = 3(4a^2) - 2a + 2 = 12a^2 - 2a + 2$.

[continued]

$f(a^2) = 3(a^2)^2 - (a^2) + 2 = 3(a^4) - a^2 + 2 = 3a^4 - a^2 + 2.$

$$\begin{aligned}[f(a)]^2 &= \left[3a^2 - a + 2\right]^2 = \left(3a^2 - a + 2\right)\left(3a^2 - a + 2\right) \\ &= 9a^4 - 3a^3 + 6a^2 - 3a^3 + a^2 - 2a + 6a^2 - 2a + 4 = 9a^4 - 6a^3 + 13a^2 - 4a + 4.\end{aligned}$$

$f(a+h) = 3(a+h)^2 - (a+h) + 2 = 3(a^2 + 2ah + h^2) - a - h + 2 = 3a^2 + 6ah + 3h^2 - a - h + 2.$

34. $g\,(x) = \dfrac{x}{\sqrt{x+1}}.$

$g(0) = \dfrac{0}{\sqrt{0+1}} = 0.$

$g\,(3) = \dfrac{3}{\sqrt{3+1}} = \dfrac{3}{2}.$

$5g(a) = 5 \cdot \dfrac{a}{\sqrt{a+1}} = \dfrac{5a}{\sqrt{a+1}}.$

$\dfrac{1}{2}g(4a) = \dfrac{1}{2} \cdot g(4a) = \dfrac{1}{2} \cdot \dfrac{4a}{\sqrt{4a+1}} = \dfrac{2a}{\sqrt{4a+1}}.$

$g(a^2) = \dfrac{a^2}{\sqrt{a^2+1}}$; $[g(a)]^2 = \left(\dfrac{a}{\sqrt{a+1}}\right)^2 = \dfrac{a^2}{a+1}.$

$g(a+h) = \dfrac{(a+h)}{\sqrt{(a+h)+1}} = \dfrac{a+h}{\sqrt{a+h+1}}.$

$g(x-a) = \dfrac{(x-a)}{\sqrt{(x-a)+1}} = \dfrac{x-a}{\sqrt{x-a+1}}.$

35. $f(x) = 4 + 3x - x^2$, so $f(3+h) = 4 + 3(3+h) - (3+h)^2 = 4 + 9 + 3h - (9 + 6h + h^2) = 4 - 3h - h^2$,

and $\dfrac{f(3+h) - f(3)}{h} = \dfrac{(4 - 3h - h^2) - 4}{h} = \dfrac{h(-3-h)}{h} = -3 - h.$

36. $f(x) = x^3$, so $f(a+h) = (a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$,

and $\dfrac{f(a+h) - f(a)}{h} = \dfrac{(a^3 + 3a^2h + 3ah^2 + h^3) - a^3}{h} = \dfrac{h(3a^2 + 3ah + h^2)}{h} = 3a^2 + 3ah + h^2.$

37. $f(x) = \dfrac{1}{x}$, so $\dfrac{f(x) - f(a)}{x - a} = \dfrac{\frac{1}{x} - \frac{1}{a}}{x - a} = \dfrac{\frac{a-x}{xa}}{x - a} = \dfrac{a-x}{xa(x-a)} = \dfrac{-1(x-a)}{xa(x-a)} = -\dfrac{1}{ax}.$

38. $f(x) = \sqrt{x+2}$, so $\dfrac{f(x) - f(1)}{x-1} = \dfrac{\sqrt{x+2} - \sqrt{3}}{x-1}$. Depending upon the context, this may be considered simplified.

Note: We may also rationalize the numerator:

$$\begin{aligned}\frac{\sqrt{x+2} - \sqrt{3}}{x-1} &= \frac{\sqrt{x+2} - \sqrt{3}}{x-1} \cdot \frac{\sqrt{x+2} + \sqrt{3}}{\sqrt{x+2} + \sqrt{3}} = \frac{(x+2) - 3}{(x-1)\left(\sqrt{x-2} + \sqrt{3}\right)} \\ &= \frac{x-1}{(x-1)\left(\sqrt{x-2} + \sqrt{3}\right)} = \frac{1}{\sqrt{x+2} + \sqrt{3}}\end{aligned}$$

39. $f(x) = (x+4)/(x^2 - 9)$ is defined for all x except when $0 = x^2 - 9 \quad \Leftrightarrow \quad 0 = (x+3)(x-3) \quad \Leftrightarrow \quad x = -3$ or 3, so the domain is $\{x \in \mathbb{R} \mid x \neq -3, 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

40. The function $f(x) = \dfrac{x^2+1}{x^2+4x-21}$ is defined for all values of x except those for which $x^2 + 4x - 21 = 0 \quad \Leftrightarrow$ $(x+7)(x-3) = 0 \quad \Leftrightarrow \quad x = -7$ or $x = 3$. Thus, the domain is $\{x \in \mathbb{R} \mid x \neq -7, 3\} = (-\infty, -7) \cup (-7, 3) \cup (3, \infty)$.

41. $f(t) = \sqrt[3]{2t-1}$ is defined for all real numbers. In fact $\sqrt[3]{p(t)}$, where $p(t)$ is a polynomial, is defined for all real numbers. Thus, the domain is $\mathbb{R}$, or $(-\infty, \infty)$.

42. $g(t) = \sqrt{3-t} - \sqrt{2+t}$ is defined when $3 - t \geq 0 \quad \Leftrightarrow \quad t \leq 3$ and $2 + t \geq 0 \quad \Leftrightarrow \quad t \geq -2$. Thus, the domain is $-2 \leq t \leq 3$, or $[-2, 3]$.

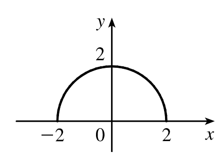
43. $h(x) = 1 \big/ \sqrt[4]{x^2-5x}$ is defined when $x^2 - 5x > 0 \quad \Leftrightarrow \quad x(x-5) > 0$. Note that $x^2 - 5x \neq 0$ since that would result in division by zero. The expression $x(x-5)$ is positive if $x < 0$ or $x > 5$. (See Appendix A for methods for solving inequalities.) Thus, the domain is $(-\infty, 0) \cup (5, \infty)$.

44. $f(u) = \dfrac{u+1}{1 + \dfrac{1}{u+1}}$ is defined when $u + 1 \neq 0$ $[u \neq -1]$ and $1 + \dfrac{1}{u+1} \neq 0$. Since $1 + \dfrac{1}{u+1} = 0 \quad \Leftrightarrow$ $\dfrac{1}{u+1} = -1 \quad \Leftrightarrow \quad 1 = -u - 1 \quad \Leftrightarrow \quad u = -2$, the domain is $\{u \mid u \neq -2, u \neq -1\} = (-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

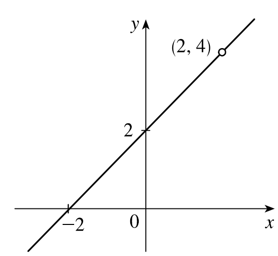
45. $F(p) = \sqrt{2 - \sqrt{p}}$ is defined when $p \geq 0$ and $2 - \sqrt{p} \geq 0$. Since $2 - \sqrt{p} \geq 0 \quad \Leftrightarrow \quad 2 \geq \sqrt{p} \quad \Leftrightarrow \quad \sqrt{p} \leq 2 \quad \Leftrightarrow$ $0 \leq p \leq 4$, the domain is $[0, 4]$.

46. The function $h(x) = \sqrt{x^2 - 4x - 5}$ is defined when $x^2 - 4x - 5 \geq 0 \quad \Leftrightarrow \quad (x+1)(x-5) \geq 0$. The polynomial $p(x) = x^2 - 4x - 5$ may change signs only at its zeros, so we test values of x on the intervals separated by $x = -1$ and $x = 5$: $p(-2) = 7 > 0$, $p(0) = -5 < 0$, and $p(6) = 7 > 0$. Thus, the domain of h, equivalent to the solution intervals of $p(x) \geq 0$, is $\{x \mid x \leq -1 \text{ or } x \geq 5\} = (-\infty, -1] \cup [5, \infty)$.

47. $h(x) = \sqrt{4 - x^2}$. Now $y = \sqrt{4 - x^2} \quad \Rightarrow \quad y^2 = 4 - x^2 \quad \Leftrightarrow \quad x^2 + y^2 = 4$, so the graph is the top half of a circle of radius 2 with center at the origin. The domain is $\left\{x \mid 4 - x^2 \geq 0\right\} = \left\{x \mid 4 \geq x^2\right\} = \{x \mid 2 \geq |x|\} = [-2, 2]$. From the graph, the range is $0 \leq y \leq 2$, or $[0, 2]$.

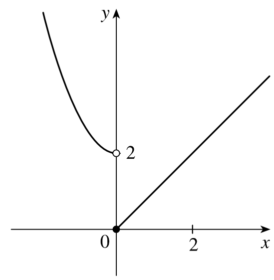


48. The function $f(x) = \dfrac{x^2-4}{x-2}$ is defined when $x - 2 \neq 0 \Leftrightarrow x \neq 2$, so the domain is $\{x \mid x \neq 2\} = (-\infty, 2) \cup (2, \infty)$. On its domain, $f(x) = \dfrac{x^2-4}{x-2} = \dfrac{(x-2)(x+2)}{x-2} = x + 2$. Thus, the graph of f is the line $y = x + 2$ with a hole at $(2, 4)$.



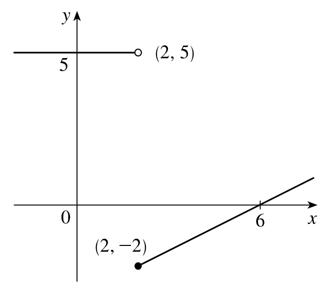
49. $f(x) = \begin{cases} x^2 + 2 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

$f(-3) = (-3)^2 + 2 = 11$, $f(0) = 0$, and $f(2) = 2$.



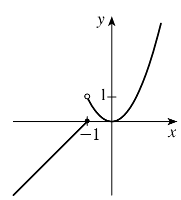
50. $f(x) = \begin{cases} 5 & \text{if } x < 2 \\ \frac{1}{2}x - 3 & \text{if } x \geq 2 \end{cases}$

$f(-3) = 5$, $f(0) = 5$, and $f(2) = \frac{1}{2}(2) - 3 = -2$.



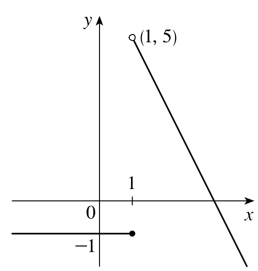
51. $f(x) = \begin{cases} x + 1 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

$f(-3) = -3 + 1 = -2$, $f(0) = 0^2 = 0$, and $f(2) = 2^2 = 4$.



52. $f(x) = \begin{cases} -1 & \text{if } x \leq 1 \\ 7 - 2x & \text{if } x > 1 \end{cases}$

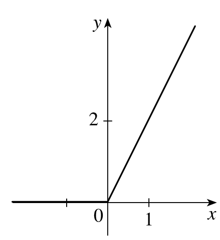
$f(-3) = -1$, $f(0) = -1$, and $f(2) = 7 - 2(2) = 3$.



53. $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

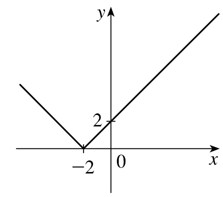
so $f(x) = x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

Graph the line $y = 2x$ for $x \geq 0$ and graph $y = 0$ (the x-axis) for $x < 0$.



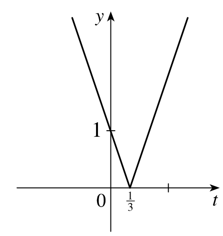
54. $f(x) = |x+2| = \begin{cases} x+2 & \text{if } x+2 \geq 0 \\ -(x+2) & \text{if } x+2 < 0 \end{cases}$

$$= \begin{cases} x+2 & \text{if } x \geq -2 \\ -x-2 & \text{if } x < -2 \end{cases}$$



55. $g(t) = |1-3t| = \begin{cases} 1-3t & \text{if } 1-3t \geq 0 \\ -(1-3t) & \text{if } 1-3t < 0 \end{cases}$

$$= \begin{cases} 1-3t & \text{if } t \leq \frac{1}{3} \\ 3t-1 & \text{if } t > \frac{1}{3} \end{cases}$$

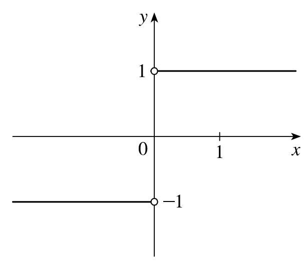


56. $f(x) = \dfrac{|x|}{x}$

The domain of f is $\{x \mid x \neq 0\}$ and $|x| = x$ if $x > 0$, $|x| = -x$ if $x < 0$.

So we can write

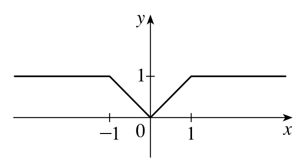
$$f(x) = \begin{cases} \dfrac{-x}{x} = -1 & \text{if } x < 0 \\ \dfrac{x}{x} = 1 & \text{if } x > 0 \end{cases}$$



57. To graph $f(x) = \begin{cases} |x| & \text{if } |x| \leq 1 \\ 1 & \text{if } |x| > 1 \end{cases}$, graph $y = |x|$ [Figure 16]

for $-1 \leq x \leq 1$ and graph $y = 1$ for $x > 1$ and for $x < -1$.

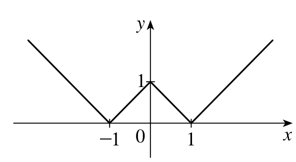
We could rewrite f as $f(x) = \begin{cases} 1 & \text{if } x < -1 \\ -x & \text{if } -1 \leq x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$.



58. $g(x) = \big||x|-1\big| = \begin{cases} |x|-1 & \text{if } |x|-1 \geq 0 \\ -(|x|-1) & \text{if } |x|-1 < 0 \end{cases}$

$$= \begin{cases} |x|-1 & \text{if } |x| \geq 1 \\ -|x|+1 & \text{if } |x| < 1 \end{cases}$$

$$= \begin{cases} x-1 & \text{if } |x| \geq 1 \text{ and } x \geq 0 \\ -x-1 & \text{if } |x| \geq 1 \text{ and } x < 0 \\ -x+1 & \text{if } |x| < 1 \text{ and } x \geq 0 \\ -(-x)+1 & \text{if } |x| < 1 \text{ and } x < 0 \end{cases} = \begin{cases} x-1 & \text{if } x \geq 1 \\ -x-1 & \text{if } x \leq -1 \\ -x+1 & \text{if } 0 \leq x < 1 \\ x+1 & \text{if } -1 < x < 0 \end{cases}$$



59. Recall that the slope m of a line between the two points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and an equation of the line connecting those two points is $y - y_1 = m(x - x_1)$. The slope of the line segment joining the points $(1, -3)$ and $(5, 7)$ is $\dfrac{7 - (-3)}{5 - 1} = \dfrac{5}{2}$, so an equation is $y - (-3) = \frac{5}{2}(x - 1)$. The function is $f(x) = \frac{5}{2}x - \frac{11}{2}$, $1 \le x \le 5$.

60. The slope of the line segment joining the points $(-5, 10)$ and $(7, -10)$ is $\dfrac{-10 - 10}{7 - (-5)} = -\dfrac{5}{3}$, so an equation is $y - 10 = -\frac{5}{3}[x - (-5)]$. The function is $f(x) = -\frac{5}{3}x + \frac{5}{3}$, $-5 \le x \le 7$.

61. We need to solve the given equation for y. $x + (y-1)^2 = 0 \quad\Leftrightarrow\quad (y-1)^2 = -x \quad\Leftrightarrow\quad y - 1 = \pm\sqrt{-x} \quad\Leftrightarrow$ $y = 1 \pm \sqrt{-x}$. The expression with the positive radical represents the top half of the parabola, and the one with the negative radical represents the bottom half. Hence, we want $f(x) = 1 - \sqrt{-x}$. Note that the domain is $x \le 0$.

62. $x^2 + (y-2)^2 = 4 \quad\Leftrightarrow\quad (y-2)^2 = 4 - x^2 \quad\Leftrightarrow\quad y - 2 = \pm\sqrt{4 - x^2} \quad\Leftrightarrow\quad y = 2 \pm \sqrt{4 - x^2}$. The top half is given by the function $f(x) = 2 + \sqrt{4 - x^2}$, $-2 \le x \le 2$.

63. For $0 \le x \le 3$, the graph is the line with slope -1 and y-intercept 3, that is, $y = -x + 3$. For $3 < x \le 5$, the graph is the line with slope 2 passing through $(3, 0)$; that is, $y - 0 = 2(x - 3)$, or $y = 2x - 6$. So the function is

$$f(x) = \begin{cases} -x + 3 & \text{if } 0 \le x \le 3 \\ 2x - 6 & \text{if } 3 < x \le 5 \end{cases}$$

64. For $-4 \le x \le -2$, the graph is the line with slope $-\frac{3}{2}$ passing through $(-2, 0)$; that is, $y - 0 = -\frac{3}{2}[x - (-2)]$, or $y = -\frac{3}{2}x - 3$. For $-2 < x < 2$, the graph is the top half of the circle with center $(0, 0)$ and radius 2. An equation of the circle is $x^2 + y^2 = 4$, so an equation of the top half is $y = \sqrt{4 - x^2}$. For $2 \le x \le 4$, the graph is the line with slope $\frac{3}{2}$ passing through $(2, 0)$; that is, $y - 0 = \frac{3}{2}(x - 2)$, or $y = \frac{3}{2}x - 3$. So the function is

$$f(x) = \begin{cases} -\frac{3}{2}x - 3 & \text{if } -4 \le x \le -2 \\ \sqrt{4 - x^2} & \text{if } -2 < x < 2 \\ \frac{3}{2}x - 3 & \text{if } \;\; 2 \le x \le 4 \end{cases}$$

65. Let the length and width of the rectangle be L and W. Then the perimeter is $2L + 2W = 20$ and the area is $A = LW$. Solving the first equation for W in terms of L gives $W = \dfrac{20 - 2L}{2} = 10 - L$. Thus, $A(L) = L(10 - L) = 10L - L^2$. Since lengths are positive, the domain of A is $0 < L < 10$. If we further restrict L to be larger than W, then $5 < L < 10$ would be the domain.

66. Let the length and width of the rectangle be L and W. Then the area is $LW = 16$, so that $W = 16/L$. The perimeter is $P = 2L + 2W$, so $P(L) = 2L + 2(16/L) = 2L + 32/L$, and the domain of P is $L > 0$, since lengths must be positive quantities. If we further restrict L to be larger than W, then $L > 4$ would be the domain.

67. Let the length of a side of the equilateral triangle be x. Then by the Pythagorean Theorem, the height y of the triangle satisfies $y^2 + \left(\frac{1}{2}x\right)^2 = x^2$, so that $y^2 = x^2 - \frac{1}{4}x^2 = \frac{3}{4}x^2$ and $y = \frac{\sqrt{3}}{2}x$. Using the formula for the area A of a triangle, $A = \frac{1}{2}(\text{base})(\text{height})$, we obtain $A(x) = \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2$, with domain $x > 0$.

68. Let the length, width, and height of the closed rectangular box be denoted by L, W, and H, respectively. The length is twice the width, so $L = 2W$. The volume V of the box is given by $V = LWH$. Since $V = 8$, we have $8 = (2W)WH \quad \Rightarrow$ $8 = 2W^2H \quad \Rightarrow \quad H = \dfrac{8}{2W^2} = \dfrac{4}{W^2}$, and so $H = f(W) = \dfrac{4}{W^2}$.

69. Let each side of the base of the box have length x, and let the height of the box be h. Since the volume is 2, we know that $2 = hx^2$, so that $h = 2/x^2$, and the surface area is $S = x^2 + 4xh$. Thus, $S(x) = x^2 + 4x(2/x^2) = x^2 + (8/x)$, with domain $x > 0$.

70. Let r and h denote the radius and the height of the right circular cylinder, respectively. Then the volume V is given by $V = \pi r^2 h$, and for this particular cylinder we have $\pi r^2 h = 25 \quad \Leftrightarrow \quad r^2 = \dfrac{25}{\pi h}$. Solving for r and rejecting the negative solution gives $r = \dfrac{5}{\sqrt{\pi h}}$, so $r = f(h) = \dfrac{5}{\sqrt{\pi h}}$ in.

71. The height of the box is x and the length and width are $L = 20 - 2x$, $W = 12 - 2x$. Then $V = LWx$ and so

$V(x) = (20 - 2x)(12 - 2x)(x) = 4(10 - x)(6 - x)(x) = 4x(60 - 16x + x^2) = 4x^3 - 64x^2 + 240x$.

The sides L, W, and x must be positive. Thus, $L > 0 \quad \Leftrightarrow \quad 20 - 2x > 0 \quad \Leftrightarrow \quad x < 10$;

$W > 0 \quad \Leftrightarrow \quad 12 - 2x > 0 \quad \Leftrightarrow \quad x < 6$; and $x > 0$. Combining these restrictions gives us the domain $0 < x < 6$.

72. The area of the window is $A = xh + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = xh + \dfrac{\pi x^2}{8}$, where h is the height of the rectangular portion of the window.

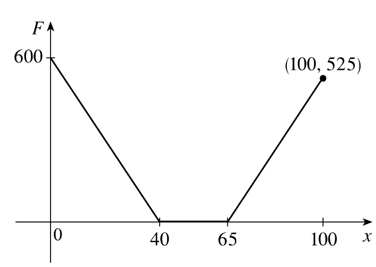
The perimeter is $P = 2h + x + \frac{1}{2}\pi x = 30 \quad \Leftrightarrow \quad 2h = 30 - x - \frac{1}{2}\pi x \quad \Leftrightarrow \quad h = \frac{1}{4}(60 - 2x - \pi x)$. Thus,

$$A(x) = x\,\frac{60 - 2x - \pi x}{4} + \frac{\pi x^2}{8} = 15x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2 = 15x - \frac{4}{8}x^2 - \frac{\pi}{8}x^2 = 15x - x^2\left(\frac{\pi + 4}{8}\right).$$

Since the lengths x and h must be positive quantities, we have $x > 0$ and $h > 0$. For $h > 0$, we have $2h > 0 \quad \Leftrightarrow$ $30 - x - \frac{1}{2}\pi x > 0 \quad \Leftrightarrow \quad 60 > 2x + \pi x \quad \Leftrightarrow \quad x < \dfrac{60}{2 + \pi}$. Hence, the domain of A is $0 < x < \dfrac{60}{2 + \pi}$.

73. We can summarize the amount of the fine with a piecewise defined function.

$$F(x) = \begin{cases} 15(40 - x) & \text{if } 0 \le x < 40 \\ 0 & \text{if } 40 \le x \le 65 \\ 15(x - 65) & \text{if } x > 65 \end{cases}$$



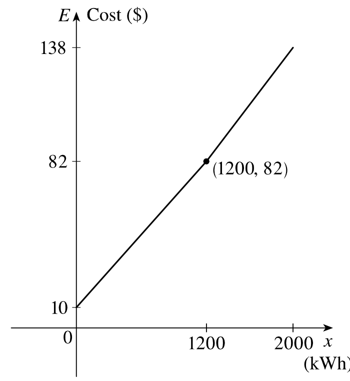
74. For the first 1200 kWh, $E(x) = 10 + 0.06x$.

For usage over 1200 kWh, the cost is

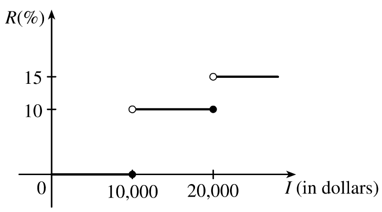
$E(x) = 10 + 0.06(1200) + 0.07(x - 1200) = 82 + 0.07(x - 1200)$.

Thus,

$$E(x) = \begin{cases} 10 + 0.06x & \text{if } 0 \le x \le 1200 \\ 82 + 0.07(x - 1200) & \text{if } x > 1200 \end{cases}$$



75. (a)

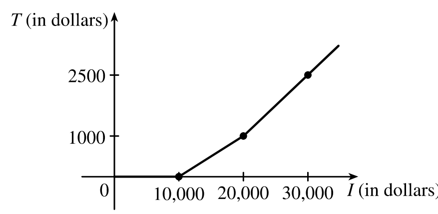


(b) On $14,000, tax is assessed on $4000, and $10\%(\$4000) = \$400$.

On $26,000, tax is assessed on $16,000, and

$10\%(\$10{,}000) + 15\%(\$6000) = \$1000 + \$900 = \$1900$.

(c) As in part (b), there is $1000 tax assessed on $20,000 of income, so the graph of T is a line segment from $(10{,}000, 0)$ to $(20{,}000, 1000)$. The tax on $30,000 is $2500, so the graph of T for $x > 20{,}000$ is the ray with initial point $(20{,}000, 1000)$ that passes through $(30{,}000, 2500)$.



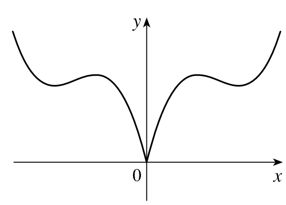
76. (a) Because an even function is symmetric with respect to the y-axis, and the point $(5, 3)$ is on the graph of this even function, the point $(-5, 3)$ must also be on its graph.

(b) Because an odd function is symmetric with respect to the origin, and the point $(5, 3)$ is on the graph of this odd function, the point $(-5, -3)$ must also be on its graph.

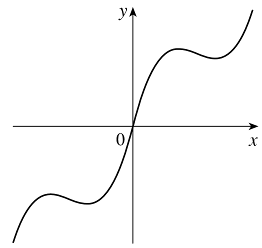
77. f is an odd function because its graph is symmetric about the origin. g is an even function because its graph is symmetric with respect to the y-axis.

78. f is not an even function since it is not symmetric with respect to the y-axis. f is not an odd function since it is not symmetric about the origin. Hence, f is *neither* even nor odd. g is an even function because its graph is symmetric with respect to the y-axis.

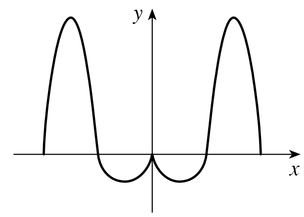
79. (a) The graph of an even function is symmetric about the y-axis. We reflect the given portion of the graph of f about the y-axis in order to complete it.



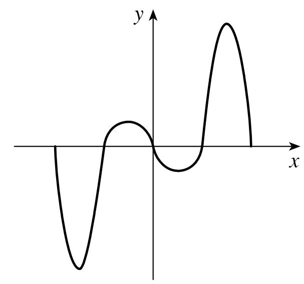
(b) For an odd function, $f(-x) = -f(x)$. The graph of an odd function is symmetric about the origin. We rotate the given portion of the graph of f through 180° about the origin in order to complete it.



80. (a) The graph of an even function is symmetric about the y-axis. We reflect the given portion of the graph of f about the y-axis in order to complete it.



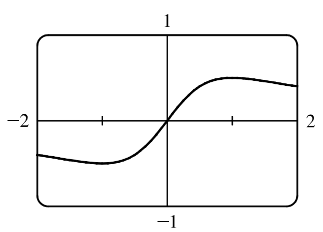
(b) The graph of an odd function is symmetric about the origin. We rotate the given portion of the graph of f through 180° about the origin in order to complete it.



81. $f(x) = \dfrac{x}{x^2+1}$.

$$f(-x) = \frac{-x}{(-x)^2+1} = \frac{-x}{x^2+1} = -\frac{x}{x^2+1} = -f(x).$$

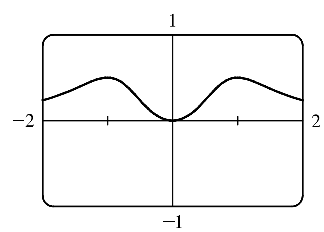
Since $f(-x) = -f(x)$, f is an odd function.



82. $f(x) = \dfrac{x^2}{x^4+1}$.

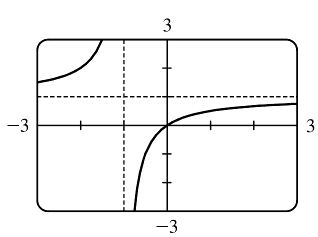
$$f(-x) = \frac{(-x)^2}{(-x)^4+1} = \frac{x^2}{x^4+1} = f(x).$$

Since $f(-x) = f(x)$, f is an even function.



83. $f(x) = \dfrac{x}{x+1}$, so $f(-x) = \dfrac{-x}{-x+1} = \dfrac{x}{x-1}$.

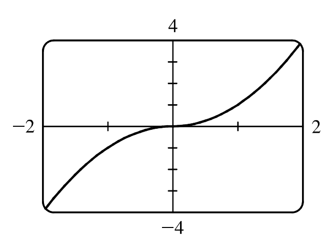
Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.



84. $f(x) = x\,|x|$.

$$f(-x) = (-x)\,|-x| = (-x)\,|x| = -(x\,|x|)$$
$$= -f(x)$$

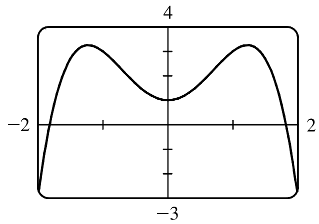
Since $f(-x) = -f(x)$, f is an odd function.



85. $f(x) = 1 + 3x^2 - x^4$.

$f(-x) = 1+3(-x)^2-(-x)^4 = 1+3x^2-x^4 = f(x)$.

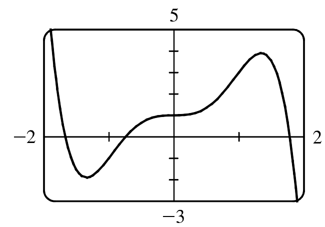
Since $f(-x) = f(x)$, f is an even function.



86. $f(x) = 1 + 3x^3 - x^5$, so

$$f(-x) = 1 + 3(-x)^3 - (-x)^5 = 1 + 3(-x^3) - (-x^5)$$
$$= 1 - 3x^3 + x^5$$

Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.



87. (i) If f and g are both even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$. Now $(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$, so $f + g$ is an *even* function.

(ii) If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now $(f+g)(-x) = f(-x) + g(-x) = -f(x) + [-g(x)] = -[f(x) + g(x)] = -(f+g)(x)$, so $f + g$ is an *odd* function.

(iii) If f is an even function and g is an odd function, then $(f+g)(-x) = f(-x)+g(-x) = f(x)+[-g(x)] = f(x)-g(x)$, which is not $(f+g)(x)$ nor $-(f+g)(x)$, so $f+g$ is *neither* even nor odd. (Exception: if f is the zero function, then $f + g$ will be *odd*. If g is the zero function, then $f + g$ will be *even*.)

88. (i) If f and g are both even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$. Now $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$, so fg is an *even* function.

(ii) If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now $(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = (fg)(x)$, so fg is an *even* function.

(iii) If f is an even function and g is an odd function, then $(fg)(-x) = f(-x)g(-x) = f(x)[-g(x)] = -[f(x)g(x)] = -(fg)(x)$, so fg is an *odd* function.

1.2 Mathematical Models: A Catalog of Essential Functions

1. (a) $f(x) = x^3 + 3x^2$ is a polynomial function of degree 3. (This function is also an algebraic function.)

(b) $g(t) = \cos^2 t - \sin t$ is a trigonometric function.

(c) $r(t) = t^{\sqrt{3}}$ is a power function.

(d) $v(t) = 8^t$ is an exponential function.

(e) $y = \dfrac{\sqrt{x}}{x^2+1}$ is an algebraic function. It is the quotient of a root of a polynomial and a polynomial of degree 2.

(f) $g(u) = \log_{10} u$ is a logarithmic function.

2. (a) $f(t) = \dfrac{3t^2 + 2}{t}$ is a rational function. (This function is also an algebraic function.)

(b) $h(r) = 2.3^r$ is an exponential function.

(c) $s(t) = \sqrt{t + 4}$ is an algebraic function. It is a root of a polynomial.

(d) $y = x^4 + 5$ is a polynomial function of degree 4.

(e) $g(x) = \sqrt[3]{x}$ is a root function. Rewriting $g(x)$ as $x^{1/3}$, we recognize the function also as a power function. (This function is, further, an algebraic function because it is a root of a polynomial.)

(f) $y = \dfrac{1}{x^2}$ is a rational function. Rewriting y as x^{-2}, we recognize the function also as a power function. (This function is, further, an algebraic function because it is the quotient of two polynomials.)

3. We notice from the figure that g and h are even functions (symmetric with respect to the y-axis) and that f is an odd function (symmetric with respect to the origin). So (b) $\left[y = x^5\right]$ must be f. Since g is flatter than h near the origin, we must have (c) $\left[y = x^8\right]$ matched with g and (a) $\left[y = x^2\right]$ matched with h.

4. (a) The graph of $y = 3x$ is a line (choice G).

(b) $y = 3^x$ is an exponential function (choice f).

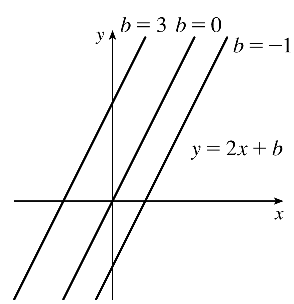
(c) $y = x^3$ is an odd polynomial function or power function (choice F).

(d) $y = \sqrt[3]{x} = x^{1/3}$ is a root function (choice g).

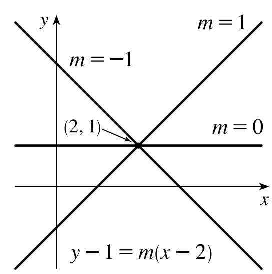
5. The denominator cannot equal 0, so $1 - \sin x \neq 0 \quad \Leftrightarrow \quad \sin x \neq 1 \quad \Leftrightarrow \quad x \neq \frac{\pi}{2} + 2n\pi$. Thus, the domain of $f(x) = \dfrac{\cos x}{1 - \sin x}$ is $\left\{x \mid x \neq \frac{\pi}{2} + 2n\pi, n \text{ an integer}\right\}$.

6. The denominator cannot equal 0, so $1 - \tan x \neq 0 \quad \Leftrightarrow \quad \tan x \neq 1 \quad \Leftrightarrow \quad x \neq \frac{\pi}{4} + n\pi$. The tangent function is not defined if $x \neq \frac{\pi}{2} + n\pi$. Thus, the domain of $g(x) = \dfrac{1}{1 - \tan x}$ is $\left\{x \mid x \neq \frac{\pi}{4} + n\pi, x \neq \frac{\pi}{2} + n\pi, n \text{ an integer}\right\}$.

7. (a) An equation for the family of linear functions with slope 2 is $y = f(x) = 2x + b$, where b is the y-intercept.

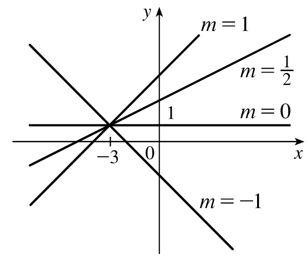


(b) $f(2) = 1$ means that the point $(2, 1)$ is on the graph of f. We can use the point-slope form of a line to obtain an equation for the family of linear functions through the point $(2, 1)$. $\quad y - 1 = m(x - 2)$, which is equivalent to $y = mx + (1 - 2m)$ in slope-intercept form.

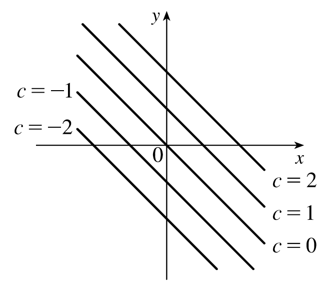


(c) To belong to both families, an equation must have slope $m = 2$, so the equation in part (b), $y = mx + (1 - 2m)$, becomes $y = 2x - 3$. It is the *only* function that belongs to both families.

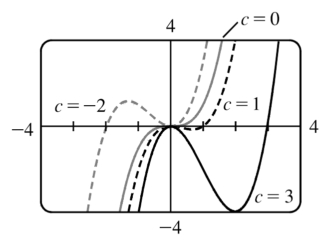
8. All members of the family of linear functions $f(x) = 1 + m(x + 3)$ have graphs that are lines passing through the point $(-3, 1)$.



9. All members of the family of linear functions $f(x) = c - x$ have graphs that are lines with slope -1. The y-intercept is c.



10. We graph $P(x) = x^3 - cx^2$ for $c = -2$, 0, 1, and 3. For $c \neq 0$, $P(x) = x^3 - cx^2 = x^2(x - c)$ has two x-intercepts, 0 and c. The curve has one decreasing portion that begins or ends at the origin and increases in length as $|c|$ increases; the decreasing portion is in quadrant II for $c < 0$ and in quadrant IV for $c > 0$.



11. Because f is a quadratic function, we know it is of the form $f(x) = ax^2 + bx + c$. The y-intercept is 18, so $f(0) = 18 \Rightarrow c = 18$ and $f(x) = ax^2 + bx + 18$. Since the points $(3, 0)$ and $(4, 2)$ lie on the graph of f, we have

$$f(3) = 0 \quad\Rightarrow\quad 9a + 3b + 18 = 0 \quad\Rightarrow\quad 3a + b = -6 \qquad \textbf{(1)}$$
$$f(4) = 2 \quad\Rightarrow\quad 16a + 4b + 18 = 2 \quad\Rightarrow\quad 4a + b = -4 \qquad \textbf{(2)}$$

This is a system of two equations in the unknowns a and b, and subtracting **(1)** from **(2)** gives $a = 2$. From **(1)**, $3(2) + b = -6 \Rightarrow b = -12$, so a formula for f is $f(x) = 2x^2 - 12x + 18$.

12. g is a quadratic function so $g(x) = ax^2 + bx + c$. The y-intercept is 1, so $g(0) = 1 \Rightarrow c = 1$ and $g(x) = ax^2 + bx + 1$. Since the points $(-2, 2)$ and $(1, -2.5)$ lie on the graph of g, we have

$$g(-2) = 2 \quad\Rightarrow\quad 4a - 2b + 1 = 2 \quad\Rightarrow\quad 4a - 2b = 1 \qquad \textbf{(1)}$$
$$g(1) = -2.5 \quad\Rightarrow\quad a + b + 1 = -2.5 \quad\Rightarrow\quad a + b = -3.5 \qquad \textbf{(2)}$$

Then **(1)** $+ 2 \cdot$ **(2)** gives us $6a = -6 \Rightarrow a = -1$ and from **(2)**, we have $-1 + b = -3.5 \Rightarrow b = -2.5$, so a formula for g is $g(x) = -x^2 - 2.5x + 1$.

13. Since $f(-1) = f(0) = f(2) = 0$, f has zeros of -1, 0, and 2, so an equation for f is $f(x) = a[x-(-1)](x-0)(x-2)$, or $f(x) = ax(x+1)(x-2)$. Because $f(1) = 6$, we'll substitute 1 for x and 6 for $f(x)$.

$6 = a(1)(2)(-1) \quad \Rightarrow \quad -2a = 6 \quad \Rightarrow \quad a = -3$, so an equation for f is $f(x) = -3x(x+1)(x-2)$.

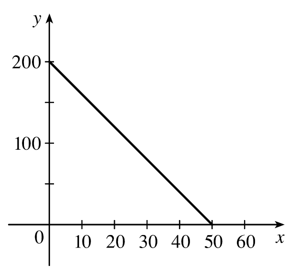
14. (a) For $T = 0.02t + 8.50$, the slope is 0.02, which means that the average surface temperature of the world is increasing at a rate of $0.02\ ^\circ\mathrm{C}$ per year. The T-intercept is 8.50, which represents the average surface temperature in $^\circ\mathrm{C}$ in the year 1900.

(b) $t = 2100 - 1900 = 200 \quad \Rightarrow \quad T = 0.02(200) + 8.50 = 12.50\,^\circ\mathrm{C}$

15. (a) $D = 200$, so $c = 0.0417D(a+1) = 0.0417(200)(a+1) = 8.34a + 8.34$. The slope is 8.34, which represents the change in mg of the dosage for a child for each change of 1 year in age.

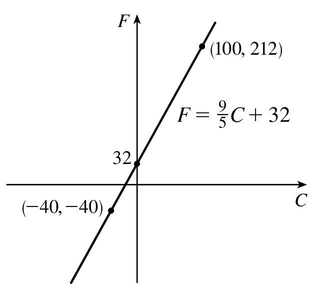
(b) For a newborn, $a = 0$, so $c = 8.34$ mg.

16. (a)



(b) The slope of -4 means that for each increase of 1 dollar for a rental space, the number of spaces rented *decreases* by 4. The y-intercept of 200 is the number of spaces that would be occupied if there were no charge for each space. The x-intercept of 50 is the smallest rental fee that results in no spaces rented.

17. (a)



(b) The slope of $\frac{9}{5}$ means that F increases $\frac{9}{5}$ degrees for each increase of $1^\circ\mathrm{C}$. (Equivalently, F increases by 9 when C increases by 5 and F decreases by 9 when C decreases by 5.) The F-intercept of 32 is the Fahrenheit temperature corresponding to a Celsius temperature of 0.

18. (a) Jari is traveling faster since the line representing her distance versus time is steeper than the corresponding line for Jade.

(b) At $t = 0$, Jade has traveled 10 miles. At $t = 6$, Jade has traveled 16 miles. Thus, Jade's speed is

$$\frac{16 \text{ miles} - 10 \text{ miles}}{6 \text{ minutes} - 0 \text{ minutes}} = 1 \text{ mi/min. This is } \frac{1 \text{ mile}}{1 \text{ minute}} \times \frac{60 \text{ minutes}}{1 \text{ hour}} = 60 \text{ mi/h}$$

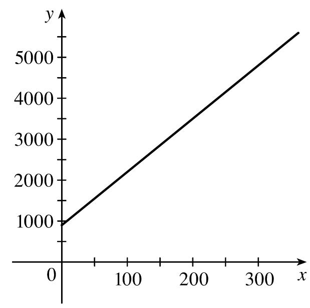
At $t = 0$, Jari has traveled 0 miles. At $t = 6$, Jari has traveled 7 miles. Thus, Jari's speed is

$$\frac{7 \text{ miles} - 0 \text{ miles}}{6 \text{ minutes} - 0 \text{ minutes}} = \frac{7}{6} \text{ mi/min or } \frac{7 \text{ miles}}{6 \text{ minutes}} \times \frac{60 \text{ minutes}}{1 \text{ hour}} = 70 \text{ mi/h}$$

(c) From part (b), we have a slope of 1 (mile/minute) for the linear function f modeling the distance traveled by Jade and from the graph the y-intercept is 10. Thus, $f(t) = 1t + 10 = t + 10$. Similarly, we have a slope of $\frac{7}{6}$ miles/minute for

Jari and a y-intercept of 0. Thus, the distance traveled by Jari as a function of time t (in minutes) is modeled by $g(t) = \frac{7}{6}t + 0 = \frac{7}{6}t$.

19. (a) Let x denote the number of chairs produced in one day and y the associated cost. Using the points $(100, 2200)$ and $(300, 4800)$, we get the slope $\frac{4800-2200}{300-100} = \frac{2600}{200} = 13$. So $y - 2200 = 13(x - 100) \quad \Leftrightarrow \quad y = 13x + 900$.



(b) The slope of the line in part (a) is 13 and it represents the cost (in dollars) of producing each additional chair.

(c) The y-intercept is 900 and it represents the fixed daily costs of operating the factory.

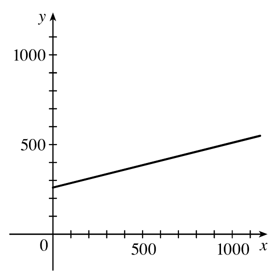
20. (a) Using d in place of x and C in place of y, we find the slope to be $\dfrac{C_2 - C_1}{d_2 - d_1} = \dfrac{460 - 380}{800 - 480} = \dfrac{80}{320} = \dfrac{1}{4}$.

So a linear equation is $C - 460 = \frac{1}{4}(d - 800) \quad \Leftrightarrow \quad C - 460 = \frac{1}{4}d - 200 \quad \Leftrightarrow \quad C = \frac{1}{4}d + 260$.

(b) Letting $d = 1500$ we get $C = \frac{1}{4}(1500) + 260 = 635$.

The cost of driving 1500 miles is \$635.

(c)



The slope of the line represents the cost per mile, \$0.25.

(d) The y-intercept represents the fixed cost, \$260.

(e) A linear function gives a suitable model in this situation because you have fixed monthly costs such as insurance and car payments, as well as costs that increase as you drive, such as gasoline, oil, and tires, and the cost of these for each additional mile driven is a constant.

21. (a) We are given $\dfrac{\text{change in pressure}}{\text{10 feet change in depth}} = \dfrac{4.34}{10} = 0.434$. Using P for pressure and d for depth with the point $(d, P) = (0, 15)$, we have the slope-intercept form of the line, $P = 0.434d + 15$.

(b) When $P = 100$, then $100 = 0.434d + 15 \quad \Leftrightarrow \quad 0.434d = 85 \quad \Leftrightarrow \quad d = \frac{85}{0.434} \approx 195.85$ feet. Thus, the pressure is $100\ \text{lb/in}^2$ at a depth of approximately 196 feet.

22. (a) $R(x) = kx^{-2}$ and $R\,(0.005) = 140$, so $140 = k(0.005)^{-2} \quad \Leftrightarrow \quad k = 140(0.005)^2 = 0.0035$.

(b) $R(x) = 0.0035x^{-2}$, so for a diameter of 0.008 m the resistance is $R(0.008) = 0.0035(0.008)^{-2} \approx 54.7$ ohms.

23. If x is the original distance from the source, then the illumination is $f(x) = kx^{-2} = k/x^2$. Moving halfway to the lamp gives an illumination of $f\left(\frac{1}{2}x\right) = k\left(\frac{1}{2}x\right)^{-2} = k(2/x)^2 = 4(k/x^2)$, so the light is four times as bright.

24. (a) $P = k/V$ and $P = 39$ kPa when $V = 0.671\text{ m}^3$, so $39 = k/0.671 \quad \Leftrightarrow \quad k = 39(0.671) = 26.169$.

(b) When $V = 0.916$, $P = 26.169/V = 26.169/0.916 \approx 28.6$, so the pressure is reduced to approximately 28.6 kPa.

25. (a) $P = kAv^3$ so doubling the windspeed v gives $P = kA(2v)^3 = 8(kAv^3)$. Thus, the power output is increased by a factor of eight.

(b) The area swept out by the blades is given by $A = \pi l^2$, where l is the blade length, so the power output is $P = kAv^3 = k\pi l^2 v^3$. Doubling the blade length gives $P = k\pi(2l)^2 v^3 = 4(k\pi l^2 v^3)$. Thus, the power output is increased by a factor of four.

(c) From part (b) we have $P = k\pi l^2 v^3$, and $k = 0.214\text{ kg/m}^3$, $l = 30$ m gives

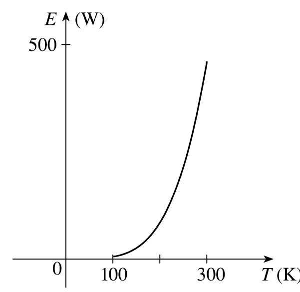
$$P = 0.214\,\frac{\text{kg}}{\text{m}^3} \cdot 900\pi\text{ m}^2 \cdot v^3 = 192.6\pi v^3\,\frac{\text{kg}}{\text{m}}$$

For $v = 10\text{ m/s}$, we have

$$P = 192.6\pi\left(10\,\frac{\text{m}}{\text{s}}\right)^3\,\frac{\text{kg}}{\text{m}} = 192{,}600\pi\,\frac{\text{m}^2\cdot\text{kg}}{\text{s}^3} \approx 605{,}000\text{ W}$$

Similarly, $v = 15\text{ m/s}$ gives $P = 650{,}025\pi \approx 2{,}042{,}000$ W and $v = 25\text{ m/s}$ gives $P = 3{,}009{,}375\pi \approx 9{,}454{,}000$ W.

26. (a) We graph $E(T) = (5.67 \times 10^{-8})T^4$ for $100 \leq T \leq 300$:



(b) From the graph, we see that as temperature increases, energy increases—slowly at first, but then at an increasing rate.

27. (a) The data appear to be periodic and a sine or cosine function would make the best model. A model of the form $f(x) = a\cos(bx) + c$ seems appropriate.

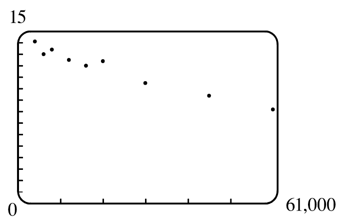
(b) The data appear to be decreasing in a linear fashion. A model of the form $f(x) = mx + b$ seems appropriate.

28. (a) The data appear to be increasing exponentially. A model of the form $f(x) = a \cdot b^x$ or $f(x) = a \cdot b^x + c$ seems appropriate.

(b) The data appear to be decreasing similarly to the values of the reciprocal function. A model of the form $f(x) = a/x$ seems appropriate.

Exercises 29–33: Some values are given to many decimal places. The results may depend on the technology used—rounding is left to the reader.

29. (a)

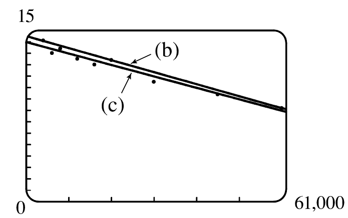


A linear model does seem appropriate.

(b) Using the points $(4000, 14.1)$ and $(60{,}000, 8.2)$, we obtain

$$y - 14.1 = \frac{8.2 - 14.1}{60{,}000 - 4000}\,(x - 4000) \text{ or, equivalently,}$$

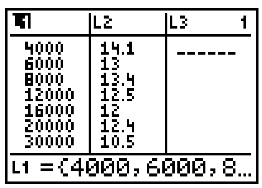
$y \approx -0.000105357x + 14.521429$.

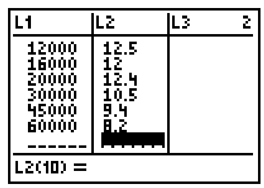


(c) Using a computing device, we obtain the regression line $y = -0.0000997855x + 13.950764$.

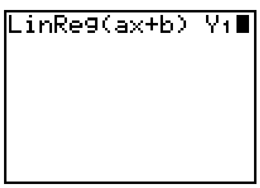
The following commands and screens illustrate how to find the regression line on a TI-84 Plus calculator.

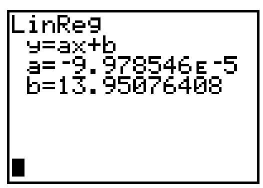
Enter the data into list one (L1) and list two (L2). Press STAT 1 to enter the editor.

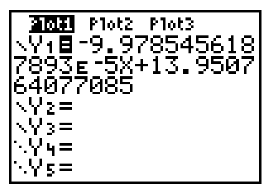




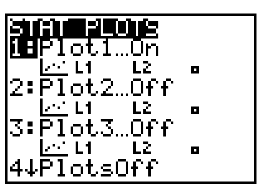
Find the regession line and store it in Y_1. Press 2nd QUIT STAT ▸ 4 VARS ▸ 1 1 ENTER.

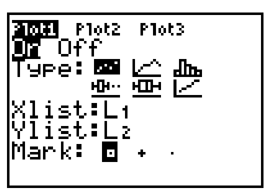




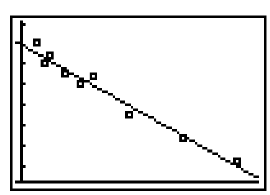


Note from the last figure that the regression line has been stored in Y_1 and that Plot1 has been turned on (Plot1 is highlighted). You can turn on Plot1 from the Y= menu by placing the cursor on Plot1 and pressing ENTER or by pressing 2nd STAT PLOT 1 ENTER.





Now press ZOOM 9 to produce a graph of the data and the regression line. Note that choice 9 of the ZOOM menu automatically selects a window that displays all of the data.



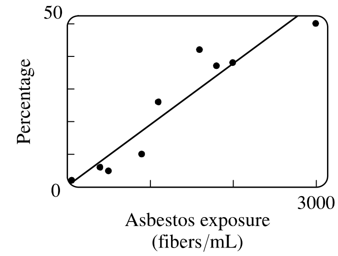
(d) When $x = 25{,}000$, $y \approx 11.456$; or about 11.5 per 100 population.

(e) When $x = 80{,}000$, $y \approx 5.968$; or about a 6% chance.

(f) When $x = 200{,}000$, y is negative, so the model does not apply.

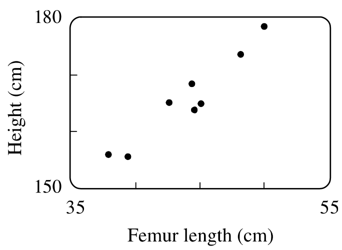
30. (a) Using a computing device, we obtain the regression line $y = 0.01879x + 0.30480$.

(b) The regression line appears to be a suitable model for the data.

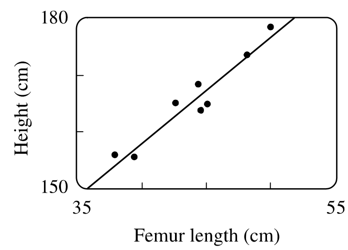


(c) The y-intercept represents the percentage of laboratory rats that develop lung tumors when *not* exposed to asbestos fibers.

31. (a)



(b) Using a computing device, we obtain the regression line $y = 1.88074x + 82.64974$.

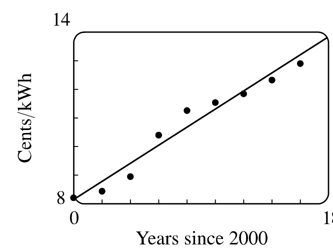


(c) When $x = 53$ cm, $y \approx 182.3$ cm.

32. (a) See the scatter plot in part (b). A linear model seems appropriate.

(b) Using a computing device, we obtain the regression line

$y = 0.31567x + 8.15578$.



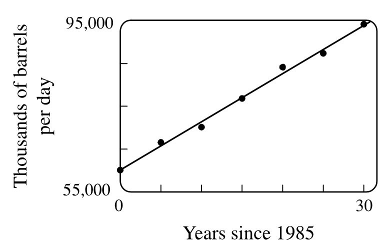
(c) For 2005, $x = 5$ and $y \approx 9.73$ cents/kWh. For 2017, $x = 17$ and

$y \approx 13.52$ cents/kWh.

33. (a) See the scatter plot in part (b). A linear model seems appropriate.

(b) Using a computing device, we obtain the regression line

$y = 1124.86x + 60{,}119.86$.



(c) For 2002, $x = 17$ and $y \approx 79{,}242$ thousands of barrels per day.

For 2017, $x = 32$ and $y \approx 96{,}115$ thousands of barrels per day.

34. (a) $T = 1.000\,431\,227 d^{1.499\,528\,750}$

(b) The power model in part (a) is approximately $T = d^{1.5}$. Squaring both sides gives us $T^2 = d^3$, so the model matches Kepler's Third Law, $T^2 = kd^3$.

35. (a) If $A = 60$, then $S = 0.7A^{0.3} \approx 2.39$, so you would expect to find 2 species of bats in that cave.

(b) $S = 4 \quad \Rightarrow \quad 4 = 0.7A^{0.3} \quad \Rightarrow \quad \frac{40}{7} = A^{3/10} \quad \Rightarrow \quad A = \left(\frac{40}{7}\right)^{10/3} \approx 333.6$, so we estimate the surface area of the cave to be 334 m^2.

36. (a) Using a computing device, we obtain a power function $N = cA^b$, where $c \approx 3.1046$ and $b \approx 0.308$.

(b) If $A = 291$, then $N = cA^b \approx 17.8$, so you would expect to find 18 species of reptiles and amphibians on Dominica.

37. We have $I = \dfrac{S}{4\pi r^2} = \left(\dfrac{S}{4\pi}\right)\left(\dfrac{1}{r^2}\right) = \dfrac{S/(4\pi)}{r^2}$. Thus, $I = \dfrac{k}{r^2}$ with $k = \dfrac{S}{4\pi}$.

1.3 New Functions from Old Functions

1. (a) If the graph of f is shifted 3 units upward, its equation becomes $y = f(x) + 3$.

(b) If the graph of f is shifted 3 units downward, its equation becomes $y = f(x) - 3$.

(c) If the graph of f is shifted 3 units to the right, its equation becomes $y = f(x - 3)$.

(d) If the graph of f is shifted 3 units to the left, its equation becomes $y = f(x + 3)$.

(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 3, its equation becomes $y = 3f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 3, its equation becomes $y = \frac{1}{3}f(x)$.

2. (a) To obtain the graph of $y = f(x) + 8$ from the graph of $y = f(x)$, shift the graph 8 units upward.

(b) To obtain the graph of $y = f(x + 8)$ from the graph of $y = f(x)$, shift the graph 8 units to the left.

(c) To obtain the graph of $y = 8f(x)$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 8.

(d) To obtain the graph of $y = f(8x)$ from the graph of $y = f(x)$, shrink the graph horizontally by a factor of 8.

(e) To obtain the graph of $y = -f(x) - 1$ from the graph of $y = f(x)$, first reflect the graph about the x-axis, and then shift it 1 unit downward.

(f) To obtain the graph of $y = 8f(\frac{1}{8}x)$ from the graph of $y = f(x)$, stretch the graph horizontally and vertically by a factor of 8.

3. (a) *Graph 3:* The graph of f is shifted 4 units to the right and has equation $y = f(x - 4)$.

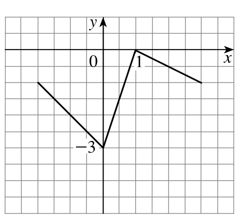
(b) *Graph 1:* The graph of f is shifted 3 units upward and has equation $y = f(x) + 3$.

(c) *Graph 4:* The graph of f is shrunk vertically by a factor of 3 and has equation $y = \frac{1}{3}f(x)$.

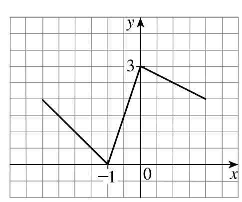
(d) *Graph 5:* The graph of f is shifted 4 units to the left and reflected about the x-axis. Its equation is $y = -f(x + 4)$.

(e) *Graph 2:* The graph of f is shifted 6 units to the left and stretched vertically by a factor of 2. Its equation is $y = 2f(x + 6)$.

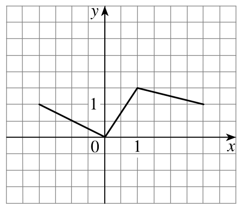
4. (a) $y = f(x) - 3$: Shift the graph of f 3 units down.



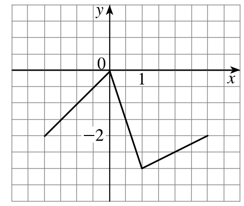
(b) $y = f(x + 1)$: Shift the graph of f 1 unit to the left.



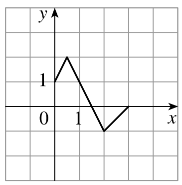
(c) $y = \frac{1}{2}f(x)$: Shrink the graph of f vertically by a factor of 2.



(d) $y = -f(x)$: Reflect the graph of f about the x-axis.

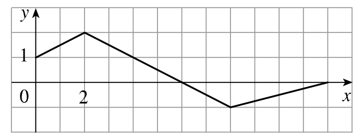


5. (a) To graph $y = f(2x)$ we shrink the graph of f horizontally by a factor of 2.



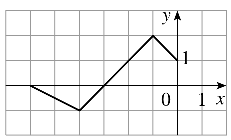
The point $(4, -1)$ on the graph of f corresponds to the point $\left(\frac{1}{2} \cdot 4, -1\right) = (2, -1)$.

(b) To graph $y = f\left(\frac{1}{2}x\right)$ we stretch the graph of f horizontally by a factor of 2.



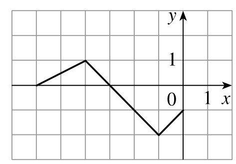
The point $(4, -1)$ on the graph of f corresponds to the point $(2 \cdot 4, -1) = (8, -1)$.

(c) To graph $y = f(-x)$ we reflect the graph of f about the y-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1) = (-4, -1)$.

(d) To graph $y = -f(-x)$ we reflect the graph of f about the y-axis, then about the x-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1 \cdot -1) = (-4, 1)$.

6. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 2 units to the right and stretched vertically by a factor of 2. Thus, a function describing the graph is

$$y = 2f(x-2) = 2\sqrt{3(x-2) - (x-2)^2} = 2\sqrt{3x - 6 - (x^2 - 4x + 4)} = 2\sqrt{-x^2 + 7x - 10}$$

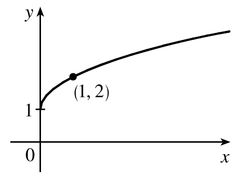
7. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 4 units to the left, reflected about the x-axis, and shifted downward 1 unit. Thus, a function describing the graph is

$$y = \underbrace{-1\cdot}_{\substack{\text{reflect} \\ \text{about } x\text{-axis}}} \quad f\underbrace{(x+4)}_{\substack{\text{shift} \\ \text{4 units left}}} \quad \underbrace{-\ 1}_{\substack{\text{shift} \\ \text{1 unit left}}}$$

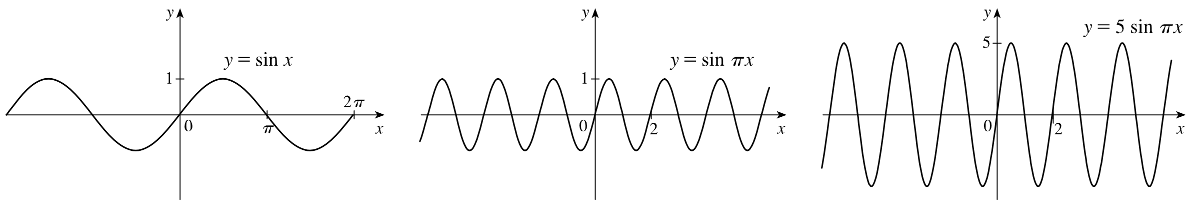
This function can be written as

$$\begin{aligned} y &= -f(x+4) - 1 = -\sqrt{3(x+4) - (x+4)^2} - 1 \\ &= -\sqrt{3x + 12 - (x^2 + 8x + 16)} - 1 = -\sqrt{-x^2 - 5x - 4} - 1 \end{aligned}$$

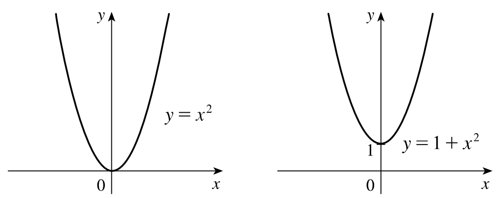
8. (a) The graph of $y = 1 + \sqrt{x}$ can be obtained from the graph of $y = \sqrt{x}$ by shifting it upward 1 unit.



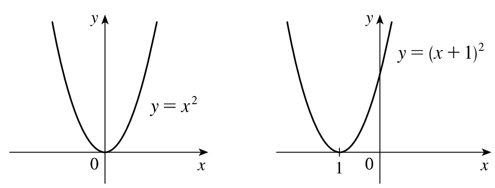
(b) The graph of $y = \sin \pi x$ can be obtained from the graph of $y = \sin x$ by compressing horizontally by a factor of π, giving a period of $2\pi/\pi = 2$. The graph of $y = 5 \sin \pi x$ is then obtained by stretching vertically by a factor of 5.



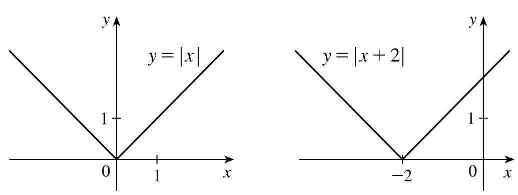
9. $y = 1 + x^2$. Start with the graph of $y = x^2$ and shift 1 unit upward



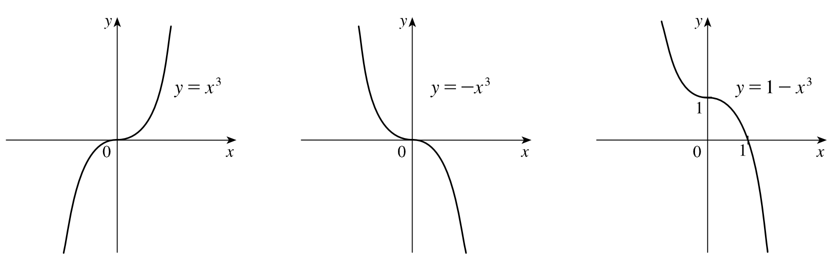
10. $y = (x + 1)^2$. Start with the graph of $y = x^2$ and shift 1 unit to the left.



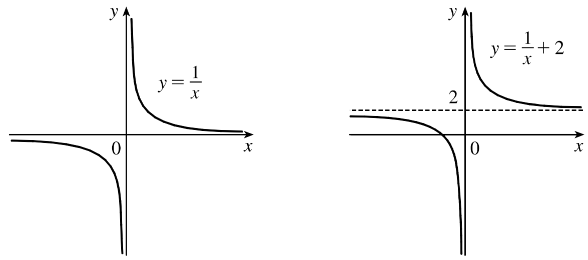
11. $y = |x + 2|$. Start with the graph of $y = |x|$ and shift 2 units to the left.



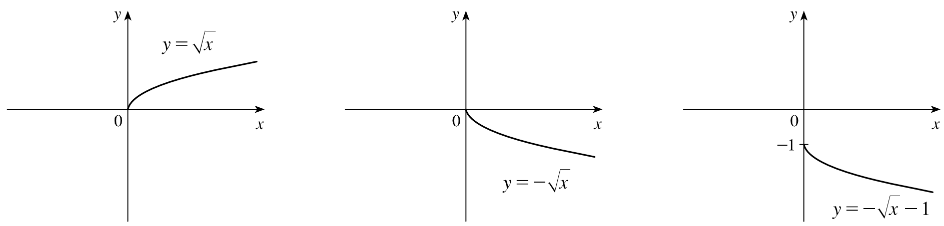
12. $y = 1 - x^3$. Start with the graph of $y = x^3$, reflect about the x-axis, and then shift 1 unit upward.



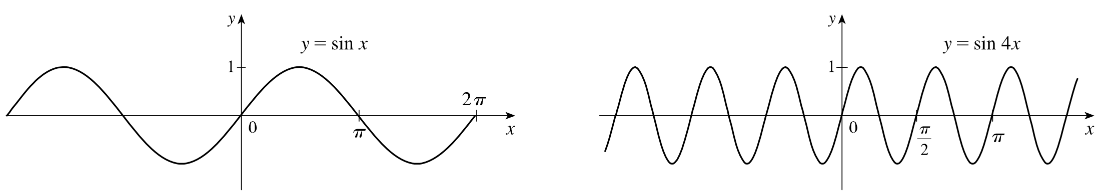
13. $y = \dfrac{1}{x} + 2$. Start with the graph of $y = \dfrac{1}{x}$ and shift 2 units upward.



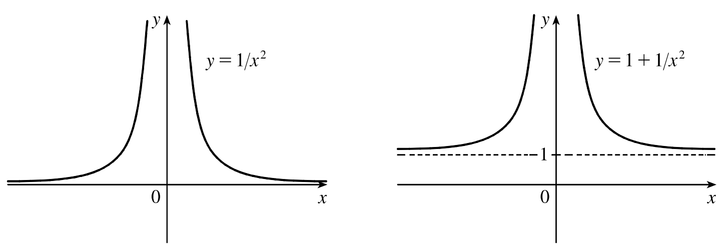
14. $y = -\sqrt{x} - 1$. Start with the graph of $y = \sqrt{x}$, reflect about the x-axis, and then shift 1 unit downward.



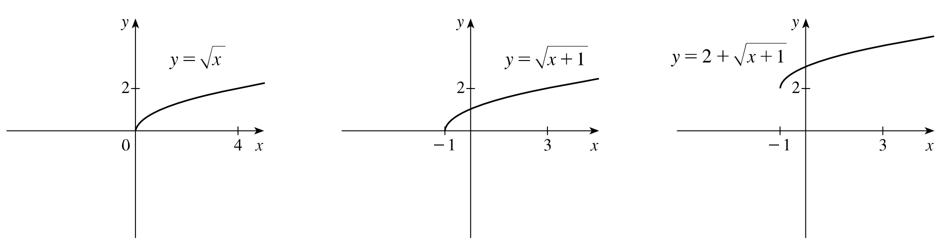
15. $y = \sin 4x$. Start with the graph of $y = \sin x$ and compress horizontally by a factor of 4. The period becomes $2\pi/4 = \pi/2$.



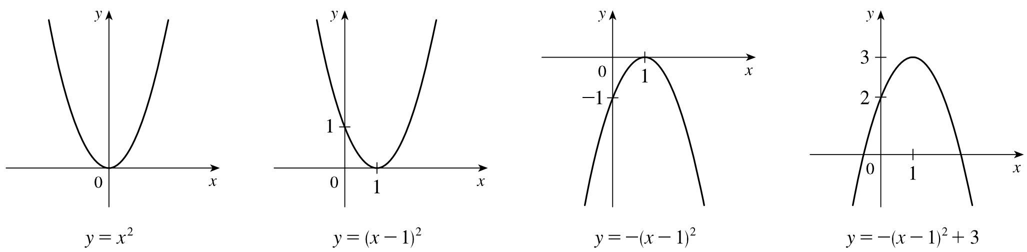
16. $y = 1 + \dfrac{1}{x^2}$. Start with the graph of $y = \dfrac{1}{x^2}$ and shift 1 unit upward.



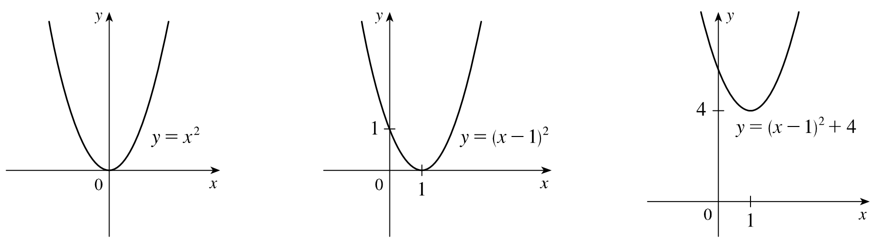
17. $y = 2 + \sqrt{x+1}$. Start with the graph of $y = \sqrt{x}$, shift 1 unit to the left, and then shift 2 units upward.



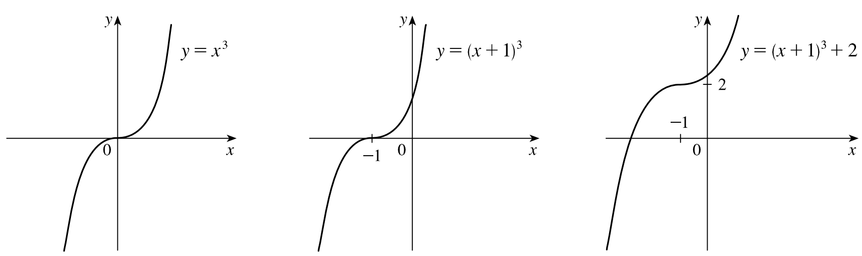
18. $y = -(x-1)^2 + 3$. Start with the graph of $y = x^2$, shift 1 unit to the right, reflect about the x-axis, and then shift 3 units upward.



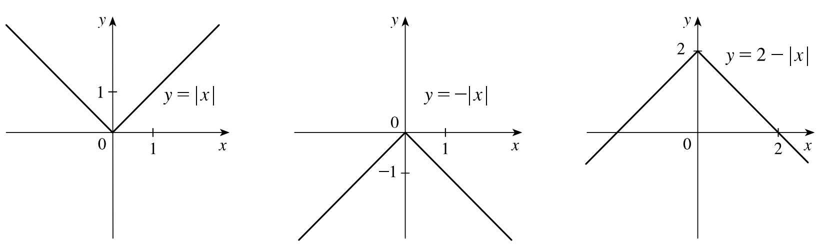
19. $y = x^2 - 2x + 5 = (x^2 - 2x + 1) + 4 = (x-1)^2 + 4$. Start with the graph of $y = x^2$, shift 1 unit to the right, and then shift 4 units upward.



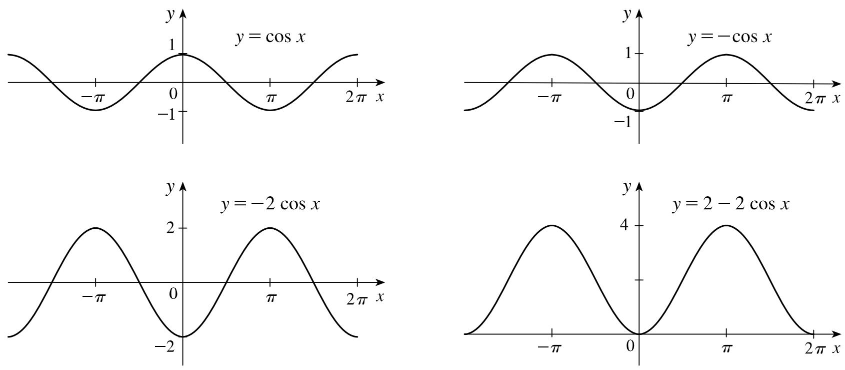
20. $y = (x+1)^3 + 2$. Start with the graph of $y = x^3$, shift 1 unit to the left, and then shift 2 units upward.



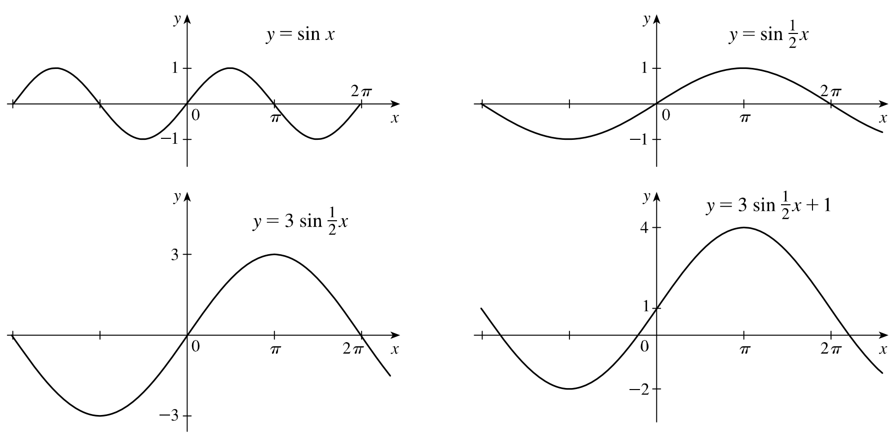
21. $y = 2 - |x|$. Start with the graph of $y = |x|$, reflect about the x-axis, and then shift 2 units upward.



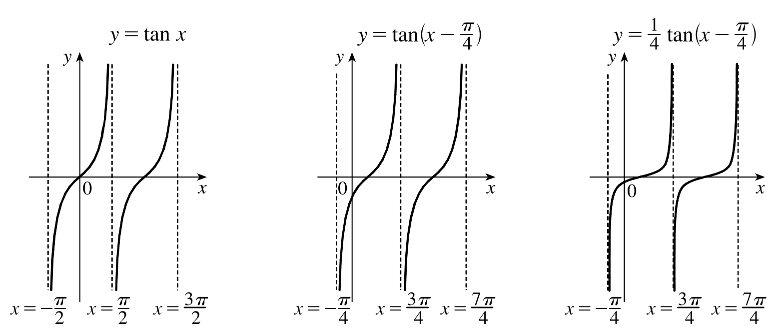
22. $y = 2 - 2\cos x$. Start with the graph of $y = \cos x$, reflect about the x-axis, stretch vertically by a factor of 2, and then shift 2 units upward.



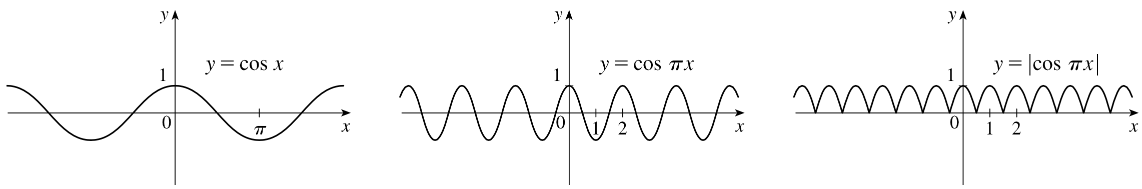
23. $y = 3\sin\frac{1}{2}x + 1$. Start with the graph of $y = \sin x$, stretch horizontally by a factor of 2, stretch vertically by a factor of 3, and then shift 1 unit upward.



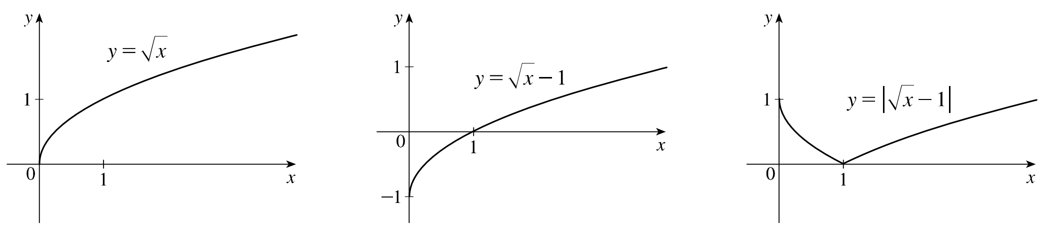
24. $y = \frac{1}{4}\tan\left(x - \frac{\pi}{4}\right)$. Start with the graph of $y = \tan x$, shift $\frac{\pi}{4}$ units to the right, and then compress vertically by a factor of 4.



25. $y = |\cos \pi x|$. Start with the graph of $y = \cos x$, shrink horizontally by a factor of π, and reflect all the parts of the graph below the x-axis about the x-axis.



26. $y = \left|\sqrt{x} - 1\right|$. Start with the graph of $y = \sqrt{x}$, shift 1 unit downward, and then reflect the portion of the graph below the x-axis about the x-axis.



27. This is just like the solution to Example 4 except the amplitude of the curve (the 30°N curve in Figure 9 on June 21) is $14 - 12 = 2$. So the function is $L(t) = 12 + 2\sin\left[\frac{2\pi}{365}(t - 80)\right]$. March 31 is the 90th day of the year, so the model gives $L(90) \approx 12.34$ h. The daylight time (5:51 AM to 6:18 PM) is 12 hours and 27 minutes, or 12.45 h. The model value differs from the actual value by $\frac{12.45 - 12.34}{12.45} \approx 0.009$, less than 1%.

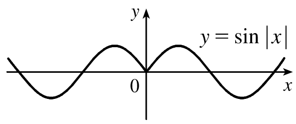
28. Using a sine function to model the brightness of Delta Cephei as a function of time, we take its period to be 5.4 days, its amplitude to be 0.35 (on the scale of magnitude), and its average magnitude to be 4.0. If we take $t = 0$ at a time of average brightness, then the magnitude (brightness) as a function of time t in days can be modeled by the formula $M(t) = 4.0 + 0.35\sin\left(\frac{2\pi}{5.4}t\right)$.

29. The water depth $D(t)$ can be modeled by a cosine function with amplitude $\dfrac{12 - 2}{2} = 5$ m, average magnitude $\dfrac{12 + 2}{2} = 7$ m, and period 12 hours. High tide occurred at time 6:45 AM ($t = 6.75$ h), so the curve begins a cycle at time $t = 6.75$ h (shift 6.75 units to the right). Thus, $D(t) = 5\cos\left[\frac{2\pi}{12}(t - 6.75)\right] + 7 = 5\cos\left[\frac{\pi}{6}(t - 6.75)\right] + 7$, where D is in meters and t is the number of hours after midnight.

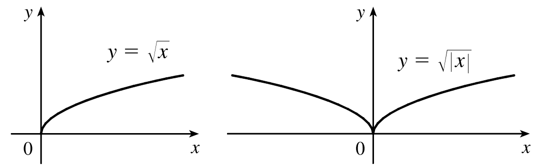
30. The total volume of air $V(t)$ in the lungs can be modeled by a sine function with amplitude $\dfrac{2500 - 2000}{2} = 250$ mL, average volume $\dfrac{2500 + 2000}{2} = 2250$ mL, and period 4 seconds. Thus, $V(t) = 250\sin\frac{2\pi}{4}t + 2250 = 250\sin\frac{\pi}{2}t + 2250$, where V is in mL and t is in seconds.

31. (a) To obtain $y = f(|x|)$, the portion of the graph of $y = f(x)$ to the right of the y-axis is reflected about the y-axis.

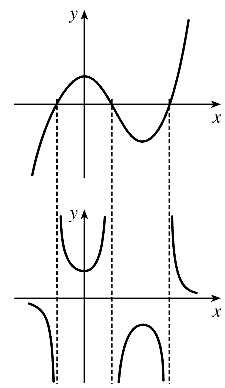
(b) $y = \sin|x|$



(c) $y = \sqrt{|x|}$



32. The most important features of the given graph are the x-intercepts and the maximum and minimum points. The graph of $y = 1/f(x)$ has vertical asymptotes at the x-values where there are x-intercepts on the graph of $y = f(x)$. The maximum of 1 on the graph of $y = f(x)$ corresponds to a minimum of $1/1 = 1$ on $y = 1/f(x)$. Similarly, the minimum on the graph of $y = f(x)$ corresponds to a maximum on the graph of $y = 1/f(x)$. As the values of y get large (positively or negatively) on the graph of $y = f(x)$, the values of y get close to zero on the graph of $y = 1/f(x)$.



33. $f(x) = \sqrt{25 - x^2}$ is defined only when $25 - x^2 \geq 0 \quad \Leftrightarrow \quad x^2 \leq 25 \quad \Leftrightarrow \quad -5 \leq x \leq 5$, so the domain of f is $[-5, 5]$. For $g(x) = \sqrt{x+1}$, we must have $x + 1 \geq 0 \quad \Leftrightarrow \quad x \geq -1$, so the domain of g is $[-1, \infty)$.

(a) $(f+g)(x) = \sqrt{25 - x^2} + \sqrt{x+1}$. The domain of $f + g$ is found by intersecting the domains of f and g: $[-1, 5]$.

(b) $(f-g)(x) = \sqrt{25 - x^2} - \sqrt{x+1}$. The domain of $f - g$ is found by intersecting the domains of f and g: $[-1, 5]$.

(c) $(fg)(x) = \sqrt{25 - x^2} \cdot \sqrt{x+1} = \sqrt{-x^3 - x^2 + 25x + 25}$. The domain of fg is found by intersecting the domains of f and g: $[-1, 5]$.

(d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{\sqrt{25 - x^2}}{\sqrt{x+1}} = \sqrt{\dfrac{25 - x^2}{x+1}}$. Notice that we must have $x + 1 \neq 0$ in addition to any previous restrictions. Thus, the domain of f/g is $(-1, 5]$.

34. For $f(x) = \dfrac{1}{x-1}$, we must have $x - 1 \neq 0 \quad \Leftrightarrow \quad x \neq 1$. For $g(x) = \dfrac{1}{x} - 2$, we must have $x \neq 0$.

(a) $(f+g)(x) = \dfrac{1}{x-1} + \dfrac{1}{x} - 2 = \dfrac{x + x - 1 - 2x(x-1)}{x(x-1)} = \dfrac{2x - 1 - 2x^2 + 2x}{x^2 - x} = -\dfrac{2x^2 - 4x + 1}{x^2 - x}, \quad \{x \mid x \neq 0, 1\}$

(b) $(f-g)(x) = \dfrac{1}{x-1} - \left(\dfrac{1}{x} - 2\right) = \dfrac{x - (x-1) + 2x(x-1)}{x(x-1)} = \dfrac{1 + 2x^2 - 2x}{x^2 - x} = \dfrac{2x^2 - 2x + 1}{x^2 - x}, \quad \{x \mid x \neq 0, 1\}$

(c) $(fg)(x) = \dfrac{1}{x-1}\left(\dfrac{1}{x} - 2\right) = \dfrac{1}{x^2 - x} - \dfrac{2}{x-1} = \dfrac{1 - 2x}{x^2 - x}, \quad \{x \mid x \neq 0, 1\}$

(d)
$$\left(\frac{f}{g}\right)(x) = \frac{\dfrac{1}{x-1}}{\dfrac{1}{x} - 2} = \frac{\dfrac{1}{x-1}}{\dfrac{1 - 2x}{x}} = \frac{1}{x-1} \cdot \frac{x}{1 - 2x} = \frac{x}{(x-1)(1-2x)} = -\frac{x}{(x-1)(2x-1)}$$
$$= -\frac{x}{2x^2 - 3x + 1}, \quad \{x \mid x \neq 0, \tfrac{1}{2}, 1\}$$

[Note the additional domain restriction $g(x) \neq 0 \quad \Rightarrow \quad x \neq \frac{1}{2}$.]

35. $f(x) = x^3 + 5$ and $g(x) = \sqrt[3]{x}$. The domain of each function is $(-\infty, \infty)$.

(a) $(f \circ g)(x) = f(g(x)) = f\left(\sqrt[3]{x}\right) = \left(\sqrt[3]{x}\right)^3 + 5 = x + 5$. The domain is $(-\infty, \infty)$.

(b) $(g \circ f)(x) = g(f(x)) = g(x^3 + 5) = \sqrt[3]{x^3 + 5}$. The domain is $(-\infty, \infty)$.

(c) $(f \circ f)(x) = f(f(x)) = f(x^3 + 5) = (x^3 + 5)^3 + 5$. The domain is $(-\infty, \infty)$.

(d) $(g \circ g)(x) = g(g(x)) = g\left(\sqrt[3]{x}\right) = \sqrt[3]{\sqrt[3]{x}} = \sqrt[9]{x}$. The domain is $(-\infty, \infty)$.

36. $f(x) = 1/x$ and $g(x) = 2x + 1$. The domain of f is $(-\infty, 0) \cup (0, \infty)$. The domain of g is $(-\infty, \infty)$.

(a) $(f \circ g)(x) = f(g(x)) = f(2x + 1) = \dfrac{1}{2x + 1}$. The domain is $\{x \mid 2x + 1 \neq 0\} = \{x \mid x \neq -\frac{1}{2}\} = \left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\dfrac{1}{x}\right) = 2\left(\dfrac{1}{x}\right) + 1 = \dfrac{2}{x} + 1$. We must have $x \neq 0$, so the domain is $(-\infty, 0) \cup (0, \infty)$.

(c) $(f \circ f)(x) = f(f(x)) = f\left(\dfrac{1}{x}\right) = \dfrac{1}{1/x} = x$. Since f requires $x \neq 0$, the domain is $(-\infty, 0) \cup (0, \infty)$.

(d) $(g \circ g)(x) = g(g(x)) = g(2x + 1) = 2(2x + 1) + 1 = 4x + 3$. The domain is $(-\infty, \infty)$.

37. $f(x) = \dfrac{1}{\sqrt{x}}$ and $g(x) = x + 1$. The domain of f is $(0, \infty)$. The domain of g is $(-\infty, \infty)$.

(a) $(f \circ g)(x) = f(g(x)) = f(x + 1) = \dfrac{1}{\sqrt{x + 1}}$. We must have $x + 1 > 0$, or $x > -1$, so the domain is $(-1, \infty)$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\dfrac{1}{\sqrt{x}}\right) = \dfrac{1}{\sqrt{x}} + 1$. We must have $x > 0$, so the domain is $(0, \infty)$.

(c) $(f \circ f)(x) = f(f(x)) = f\left(\dfrac{1}{\sqrt{x}}\right) = \dfrac{1}{\sqrt{1/\sqrt{x}}} = \dfrac{1}{1/\sqrt{\sqrt{x}}} = \sqrt{\sqrt{x}} = \sqrt[4]{x}$. We must have $x > 0$, so the domain is $(0, \infty)$.

(d) $(g \circ g)(x) = g(g(x)) = g(x + 1) = (x + 1) + 1 = x + 2$. The domain is $(-\infty, \infty)$.

38. $f(x) = \dfrac{x}{x + 1}$ and $g(x) = 2x - 1$. The domain of f is $(-\infty, -1) \cup (-1, \infty)$. The domain of g is $(-\infty, \infty)$.

(a) $(f \circ g)(x) = f(g(x)) = f(2x - 1) = \dfrac{2x - 1}{(2x - 1) + 1} = \dfrac{2x - 1}{2x}$. We must have $2x \neq 0 \iff x \neq 0$. Thus, the domain is $(-\infty, 0) \cup (0, \infty)$.

(b) $(g \circ f)(x) = g(f(x)) = 2\left(\dfrac{x}{x + 1}\right) - 1 = \dfrac{2x}{x + 1} - 1 = \dfrac{2x - 1(x + 1)}{x + 1} = \dfrac{x - 1}{x + 1}$. We must have $x + 1 \neq 0 \iff x \neq -1$. Thus, the domain is $(-\infty, -1) \cup (-1, \infty)$.

(c) $(f \circ f)(x) = f(f(x)) = \dfrac{\dfrac{x}{x + 1}}{\dfrac{x}{x + 1} + 1} = \dfrac{\dfrac{x}{x + 1}}{\dfrac{x}{x + 1} + 1} \cdot \dfrac{x + 1}{x + 1} = \dfrac{x}{x + (x + 1)} = \dfrac{x}{2x + 1}$. We must have both $x + 1 \neq 0$ and $2x + 1 \neq 0$, so the domain excludes both -1 and $-\frac{1}{2}$. Thus, the domain is $(-\infty, -1) \cup \left(-1, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$.

(d) $(g \circ g)(x) = g(g(x)) = g(2x - 1) = 2(2x - 1) - 1 = 4x - 3$. The domain is $(-\infty, \infty)$.

39. $f(x) = \dfrac{2}{x}$ and $g(x) = \sin x$. The domain of f is $(-\infty, 0) \cup (0, \infty)$. The domain of g is $(-\infty, \infty)$.

(a) $(f \circ g)(x) = f(g(x)) = f(\sin x) = \dfrac{2}{\sin x} = 2\csc x$. We must have $\sin x \neq 0$, so the domain is $\{x \mid x \neq k\pi,\ k \text{ an integer}\}$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\dfrac{2}{x}\right) = \sin\left(\dfrac{2}{x}\right)$. We must have $x \neq 0$, so the domain is $(-\infty, 0) \cup (0, \infty)$.

(c) $(f \circ f)(x) = f(f(x)) = f\left(\dfrac{2}{x}\right) = \dfrac{2}{\frac{2}{x}} = x$. Since f requires $x \neq 0$, the domain is $(-\infty, 0) \cup (0, \infty)$.

(d) $(g \circ g)(x) = g(g(x)) = g(\sin x) = \sin(\sin x)$. The domain is $(-\infty, \infty)$.

40. $f(x) = \sqrt{5 - x}$ and $g(x) = \sqrt{x - 1}$. The domain of f is $(-\infty, 5]$ and the domain of g is $[1, \infty)$.

(a) $(f \circ g)(x) = f(g(x)) = f\left(\sqrt{x-1}\right) = \sqrt{5 - \sqrt{x-1}}$. We must have $x - 1 \geq 0 \quad \Leftrightarrow \quad x \geq 1$ and $5 - \sqrt{x-1} \geq 0 \quad \Leftrightarrow \quad \sqrt{x-1} \leq 5 \quad \Leftrightarrow \quad 0 \leq x - 1 \leq 25 \quad \Leftrightarrow \quad 1 \leq x \leq 26$. Thus, the domain is $[1, 26]$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\sqrt{5-x}\right) = \sqrt{\sqrt{5-x} - 1}$. We must have $5 - x \geq 0 \quad \Leftrightarrow \quad x \leq 5$ and $\sqrt{5-x} - 1 \geq 0 \quad \Leftrightarrow \quad \sqrt{5-x} \geq 1 \quad \Leftrightarrow \quad 5 - x \geq 1 \quad \Leftrightarrow \quad x \leq 4$. Intersecting the restrictions on x gives a domain of $(-\infty, 4]$.

(c) $(f \circ f)(x) = f(f(x)) = f\left(\sqrt{5-x}\right) = \sqrt{5 - \sqrt{5-x}}$. We must have $5 - x \geq 0 \quad \Leftrightarrow \quad x \leq 5$ and $5 - \sqrt{5-x} \geq 0 \quad \Leftrightarrow \quad \sqrt{5-x} \leq 5 \quad \Leftrightarrow \quad 0 \leq 5 - x \leq 25 \quad \Leftrightarrow \quad -5 \leq -x \leq 20 \quad \Leftrightarrow \quad -20 \leq x \leq 5$. Intersecting the restrictions on x gives a domain of $[-20, 5]$.

(d) $(g \circ g)(x) = g(g(x)) = g\left(\sqrt{x-1}\right) = \sqrt{\sqrt{x-1} - 1}$. We must have $x - 1 \geq 0 \quad \Leftrightarrow \quad x \geq 1$ and $\sqrt{x-1} - 1 \geq 0 \quad \Leftrightarrow \quad \sqrt{x-1} \geq 1 \quad \Leftrightarrow \quad x - 1 \geq 1 \quad \Leftrightarrow \quad x \geq 2$. Intersecting the restrictons on x gives a domain of $[2, \infty)$.

41. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^2)) = f(\sin(x^2)) = 3\sin(x^2) - 2$

42. $(f \circ g \circ h)(x) = f(g(h(x))) = f\left(g\left(\sqrt{x}\right)\right) = f\left(2^{\sqrt{x}}\right) = \left|2^{\sqrt{x}} - 4\right|$

43.
$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x^3 + 2)) = f[(x^3 + 2)^2] = f(x^6 + 4x^3 + 4) \\ &= \sqrt{(x^6 + 4x^3 + 4) - 3} = \sqrt{x^6 + 4x^3 + 1}\end{aligned}$$

44. $(f \circ g \circ h)(x) = f(g(h(x))) = f\left(g\left(\sqrt[3]{x}\right)\right) = f\left(\dfrac{\sqrt[3]{x}}{\sqrt[3]{x} - 1}\right) = \tan\left(\dfrac{\sqrt[3]{x}}{\sqrt[3]{x} - 1}\right)$

45. Let $g(x) = 2x + x^2$ and $f(x) = x^4$. Then $(f \circ g)(x) = f(g(x)) = f(2x + x^2) = (2x + x^2)^4 = F(x)$.

46. Let $g(x) = \cos x$ and $f(x) = x^2$. Then $(f \circ g)(x) = f(g(x)) = f(\cos x) = (\cos x)^2 = \cos^2 x = F(x)$.

47. Let $g(x) = \sqrt[3]{x}$ and $f(x) = \dfrac{x}{1 + x}$. Then $(f \circ g)(x) = f(g(x)) = f\left(\sqrt[3]{x}\right) = \dfrac{\sqrt[3]{x}}{1 + \sqrt[3]{x}} = F(x)$.

48. Let $g(x) = \dfrac{x}{1+x}$ and $f(x) = \sqrt[3]{x}$. Then $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{x}{1+x}\right) = \sqrt[3]{\dfrac{x}{1+x}} = G(x)$.

49. Let $g(t) = t^2$ and $f(t) = \sec t\, \tan t$. Then $(f \circ g)(t) = f(g(t)) = f(t^2) = \sec(t^2)\, \tan(t^2) = v(t)$.

50. Let $g(x) = \sqrt{x}$ and $f(x) = \sqrt{1+x}$. Then $(f \circ g)(x) = f(g(x)) = f\left(\sqrt{x}\right) = \sqrt{1+\sqrt{x}} = H(x)$.

51. Let $h(x) = \sqrt{x}$, $g(x) = x - 1$, and $f(x) = \sqrt{x}$. Then

$(f \circ g \circ h)(x) = f(g(h(x))) = f\left(g\left(\sqrt{x}\right)\right) = f\left(\sqrt{x} - 1\right) = \sqrt{\sqrt{x} - 1} = R(x)$.

52. Let $h(x) = |x|$, $g(x) = 2 + x$, and $f(x) = \sqrt[8]{x}$. Then

$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(|x|)) = f(2 + |x|) = \sqrt[8]{2 + |x|} = H(x)$.

53. Let $h(t) = \cos t$, $g(t) = \sin t$, and $f(t) = t^2$. Then

$(f \circ g \circ h)(t) = f(g(h(t))) = f(g(\cos t)) = f(\sin(\cos t)) = [\sin(\cos t)]^2 = \sin^2(\cos t) = S(t)$.

54. Let $h(t) = \tan t$, $g(t) = \sqrt{t} + 1$, and $f(t) = \cos t$. Then

$(f \circ g \circ h)(t) = f(g(h(t))) = f(g(\tan t)) = f\left(\sqrt{\tan t} + 1\right) = \cos\left(\sqrt{\tan t} + 1\right) = H(t)$.

55. (a) $f(g(3)) = f(4) = 6$.
(b) $g(f(2)) = g(1) = 5$.
(c) $(f \circ g)(5) = f(g(5)) = f(3) = 5$.
(d) $(g \circ f)(5) = g(f(5)) = g(2) = 3$.

56. (a) $g(g(g(2))) = g(g(3)) = g(4) = 1$.
(b) $(f \circ f \circ f)(1) = f(f(f(1))) = f(f(3)) = f(5) = 2$.
(c) $(f \circ f \circ g)(1) = f(f(g(1))) = f(f(5)) = f(2) = 1$.
(d) $(g \circ f \circ g)(3) = g(f(g(3))) = g(f(4)) = g(6) = 2$.

57. (a) $g(2) = 5$, because the point $(2, 5)$ is on the graph of g. Thus, $f(g(2)) = f(5) = 4$, because the point $(5, 4)$ is on the graph of f.

(b) $g(f(0)) = g(0) = 3$

(c) $(f \circ g)(0) = f(g(0)) = f(3) = 0$

(d) $(g \circ f)(6) = g(f(6)) = g(6)$. This value is not defined, because there is no point on the graph of g that has x-coordinate 6.

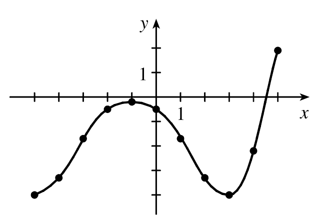
(e) $(g \circ g)(-2) = g(g(-2)) = g(1) = 4$

(f) $(f \circ f)(4) = f(f(4)) = f(2) = -2$

58. To find a particular value of $f(g(x))$, say for $x = 0$, we note from the graph that $g(0) \approx 2.8$ and $f(2.8) \approx -0.5$. Thus, $f(g(0)) \approx f(2.8) \approx -0.5$. The other values listed in the table were obtained in a similar fashion.

x	$g(x)$	$f(g(x))$
-5	-0.2	-4
-4	1.2	-3.3
-3	2.2	-1.7
-2	2.8	-0.5
-1	3	-0.2

x	$g(x)$	$f(g(x))$
0	2.8	-0.5
1	2.2	-1.7
2	1.2	-3.3
3	-0.2	-4
4	-1.9	-2.2
5	-4.1	1.9



59. (a) Using the relationship *distance* $=$ *rate* $\cdot$ *time* with the radius r as the distance, we have $r(t) = 60t$.

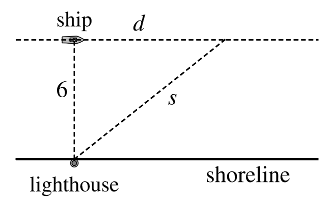
(b) $A = \pi r^2 \quad \Rightarrow \quad (A \circ r)(t) = A(r(t)) = \pi(60t)^2 = 3600\pi t^2$. This formula gives us the extent of the rippled area (in cm^2) at any time t.

60. (a) The radius r of the balloon is increasing at a rate of 2 cm/s, so $r(t) = (2\text{ cm/s})(t\text{ s}) = 2t$ (in cm).

(b) Using $V = \frac{4}{3}\pi r^3$, we get $(V \circ r)(t) = V(r(t)) = V(2t) = \frac{4}{3}\pi(2t)^3 = \frac{32}{3}\pi t^3$.

The result, $V = \frac{32}{3}\pi t^3$, gives the volume of the balloon (in cm^3) as a function of time (in s).

61. (a) From the figure, we have a right triangle with legs 6 and d, and hypotenuse s. By the Pythagorean Theorem, $d^2 + 6^2 = s^2 \quad \Rightarrow \quad s = f(d) = \sqrt{d^2 + 36}$.



(b) Using $d = rt$, we get $d = (30\text{ km/h})(t\text{ hours}) = 30t$ (in km). Thus, $d = g(t) = 30t$.

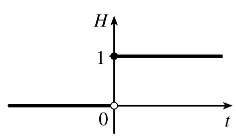
(c) $(f \circ g)(t) = f(g(t)) = f(30t) = \sqrt{(30t)^2 + 36} = \sqrt{900t^2 + 36}$. This function represents the distance between the lighthouse and the ship as a function of the time elapsed since noon.

62. (a) $d = rt \quad \Rightarrow \quad d(t) = 350t$

(b) There is a Pythagorean relationship involving the legs with lengths d and 1 and the hypotenuse with length s: $d^2 + 1^2 = s^2$. Thus, $s(d) = \sqrt{d^2 + 1}$.

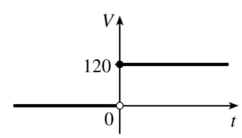
(c) $(s \circ d)(t) = s(d(t)) = s(350t) = \sqrt{(350t)^2 + 1}$

63. (a)



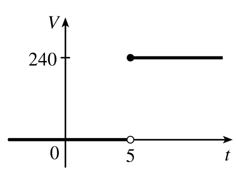
$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

(b)



$$V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 120 & \text{if } t \geq 0 \end{cases} \quad \text{so } V(t) = 120H(t).$$

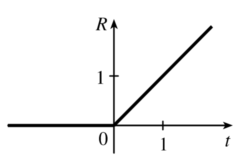
(c)



Starting with the formula in part (b), we replace 120 with 240 to reflect the different voltage. Also, because we are starting 5 units to the right of $t = 0$, we replace t with $t - 5$. Thus, the formula is $V(t) = 240H(t - 5)$.

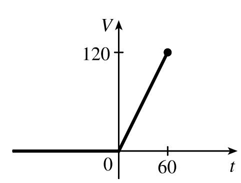
64. (a) $R(t) = tH(t)$

$$= \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases}$$



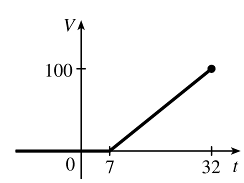
(b) $V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 2t & \text{if } 0 \leq t \leq 60 \end{cases}$

so $V(t) = 2tH(t)$, $t \leq 60$.



(c) $V(t) = \begin{cases} 0 & \text{if } t < 7 \\ 4(t - 7) & \text{if } 7 \leq t \leq 32 \end{cases}$

so $V(t) = 4(t - 7)H(t - 7)$, $t \leq 32$.



65. If $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$, then

$(f \circ g)(x) = f(g(x)) = f(m_2x + b_2) = m_1(m_2x + b_2) + b_1 = m_1m_2x + m_1b_2 + b_1$.

So $f \circ g$ is a linear function with slope m_1m_2.

66. If $A(x) = 1.04x$, then

$(A \circ A)(x) = A(A(x)) = A(1.04x) = 1.04(1.04x) = (1.04)^2x$,

$(A \circ A \circ A)(x) = A((A \circ A)(x)) = A((1.04)^2x) = 1.04(1.04)^2x = (1.04)^3x$, and

$(A \circ A \circ A \circ A)(x) = A((A \circ A \circ A)(x)) = A((1.04)^3x) = 1.04(1.04)^3x, = (1.04)^4x$.

These compositions represent the amount of the investment after 2, 3, and 4 years.

Based on this pattern, when we compose n copies of A, we get the formula $\underbrace{(A \circ A \circ \cdots \circ A)}_{n\ A\text{'s}}(x) = (1.04)^n x$.

67. (a) By examining the variable terms in g and h, we deduce that we must square g to get the terms $4x^2$ and $4x$ in h. If we let $f(x) = x^2 + c$, then $(f \circ g)(x) = f(g(x)) = f(2x + 1) = (2x + 1)^2 + c = 4x^2 + 4x + (1 + c)$. Since $h(x) = 4x^2 + 4x + 7$, we must have $1 + c = 7$. So $c = 6$ and $f(x) = x^2 + 6$.

(b) We need a function g so that $f(g(x)) = 3(g(x)) + 5 = h(x)$. But $h(x) = 3x^2 + 3x + 2 = 3(x^2 + x) + 2 = 3(x^2 + x - 1) + 5$, so we see that $g(x) = x^2 + x - 1$.

68. We need a function g so that $g(f(x)) = g(x + 4) = h(x) = 4x - 1 = 4(x + 4) - 17$. So we see that the function g must be $g(x) = 4x - 17$.

69. We need to examine $h(-x)$.

$$h(-x) = (f \circ g)(-x) = f(g(-x)) = f(g(x)) \quad [\text{because } g \text{ is even}] \quad = h(x)$$

Because $h(-x) = h(x)$, h is an even function.

70. $h(-x) = f(g(-x)) = f(-g(x))$. At this point, we can't simplify the expression, so we might try to find a counterexample to show that h is not an odd function. Let $g(x) = x$, an odd function, and $f(x) = x^2 + x$. Then $h(x) = x^2 + x$, which is neither even nor odd.

Now suppose f is an odd function. Then $f(-g(x)) = -f(g(x)) = -h(x)$. Hence, $h(-x) = -h(x)$, and so h is odd if both f and g are odd.

Now suppose f is an even function. Then $f(-g(x)) = f(g(x)) = h(x)$. Hence, $h(-x) = h(x)$, and so h is even if g is odd and f is even.

71. (a) $E(x) = f(x) + f(-x) \quad \Rightarrow \quad E(-x) = f(-x) + f(-(-x)) = f(-x) + f(x) = E(x)$. Since $E(-x) = E(x)$, E is an even function.

(b) $O(x) = f(x) - f(-x) \quad \Rightarrow \quad O(-x) = f(-x) - f(-(-x)) = f(-x) - f(x) = -[f(x) - f(-x)] = -O(x)$.

Since $O(-x) = -O(x)$, O is an odd function.

(c) For any function f with domain $\mathbb{R}$, define functions E and O as in parts (a) and (b). Then $\frac{1}{2}E$ is even, $\frac{1}{2}O$ is odd, and we show that $f(x) = \frac{1}{2}E(x) + \frac{1}{2}O(x)$:

$$\begin{aligned}\tfrac{1}{2}E(x) + \tfrac{1}{2}O(x) &= \tfrac{1}{2}\left[f(x) + f(-x)\right] + \tfrac{1}{2}\left[f(x) - f(-x)\right] \\ &= \tfrac{1}{2}[f(x) + f(-x) + f(x) - f(-x)] \\ &= \tfrac{1}{2}[2f(x)] = f(x)\end{aligned}$$

as desired.

(d) $f(x) = 2^x + (x-3)^2$ has domain $\mathbb{R}$, so we know from part (c) that $f(x) = \frac{1}{2}E\,(x) + \frac{1}{2}O\,(x)$, where

$$\begin{aligned}E(x) = f(x) + f\,(-x) &= 2^x + (x-3)^2 + 2^{-x} + (-x-3)^2 \\ &= 2^x + 2^{-x} + (x-3)^2 + (x+3)^2\end{aligned}$$

and

$$\begin{aligned}O(x) = f(x) - f\,(-x) &= 2^x + (x-3)^2 - [2^{-x} + (-x-3)^2] \\ &= 2^x - 2^{-x} + (x-3)^2 - (x+3)^2\end{aligned}$$

1.4 Exponential Functions

1. (a) $\dfrac{-2^6}{4^3} = \dfrac{-2^6}{(2^2)^3} = -\dfrac{2^6}{2^6} = -1$

(b) $\dfrac{(-3)^6}{9^6} = \left(\dfrac{-3}{9}\right)^6 = \left(-\dfrac{1}{3}\right)^6 = \dfrac{1}{3^6}$

(c) $\dfrac{1}{\sqrt[4]{x^5}} = \dfrac{1}{\sqrt[4]{x^4 \cdot x}} = \dfrac{1}{x\sqrt[4]{x}}$

(d) $\dfrac{x^3 \cdot x^n}{x^{n+1}} = \dfrac{x^{3+n}}{x^{n+1}} = x^{(3+n)-(n+1)} = x^2$

(e) $b^3(3b^{-1})^{-2} = b^3 3^{-2}(b^{-1})^{-2} = \dfrac{b^3 \cdot b^2}{3^2} = \dfrac{b^5}{9}$

(f) $\dfrac{2x^2y}{(3x^{-2}y)^2} = \dfrac{2x^2y}{3^2(x^{-2})^2y^2} = \dfrac{2x^2y}{9x^{-4}y^2} = \dfrac{2}{9}x^{2-(-4)}y^{1-2} = \dfrac{2}{9}x^6y^{-1} = \dfrac{2x^6}{9y}$

2. (a) $\dfrac{\sqrt[3]{4}}{\sqrt[3]{108}} = \dfrac{\sqrt[3]{4}}{\sqrt[3]{4 \cdot 27}} = \dfrac{\sqrt[3]{4}}{\sqrt[3]{4} \cdot \sqrt[3]{27}} = \dfrac{1}{\sqrt[3]{27}} = \dfrac{1}{3}$

(b) $27^{2/3} = (27^{1/3})^2 = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$

(c) $2x^2(3x^5)^2 = 2x^2 \cdot 3^2(x^5)^2 = 2x^2 \cdot 9x^{10} = 2 \cdot 9x^{2+10} = 18x^{12}$

(d) $(2x^{-2})^{-3}x^{-3} = 2^{-3}(x^{-2})^{-3}x^{-3} = \dfrac{x^6 \cdot x^{-3}}{2^3} = \dfrac{x^{6+(-3)}}{8} = \dfrac{x^3}{8}$

(e) $\dfrac{3a^{3/2} \cdot a^{1/2}}{a^{-1}} = 3a^{3/2+1/2} \cdot a^1 = 3a^2 \cdot a = 3a^3$

(f) $\dfrac{\sqrt{a\sqrt{b}}}{\sqrt[3]{ab}} = \dfrac{(ab^{1/2})^{1/2}}{(ab)^{1/3}} = \dfrac{a^{1/2}\,(b^{1/2})^{1/2}}{a^{1/3}\,b^{1/3}} = \dfrac{a^{1/2}\,b^{1/4}}{a^{1/3}\,b^{1/3}} = a^{1/2-1/3}\,b^{1/4-1/3} = a^{1/6}\,b^{-1/12} = \dfrac{a^{1/6}}{b^{1/12}} = \dfrac{\sqrt[6]{a}}{\sqrt[12]{b}}$

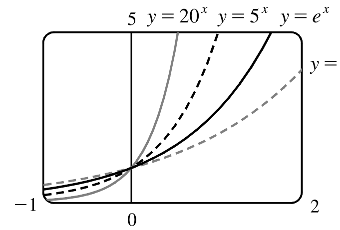
3. (a) $f(x) = b^x,\ \ b > 0$ (b) $\mathbb{R}$ (c) $(0, \infty)$ (d) See Figures 4(c), 4(b), and 4(a), respectively.

4. (a) The number e is the value of a such that the slope of the tangent line at $x = 0$ on the graph of $y = a^x$ is exactly 1.

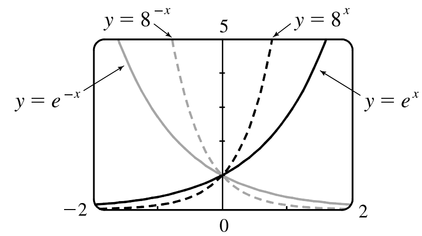
(b) $e \approx 2.71828$ (c) $f(x) = e^x$

5. All of these graphs approach 0 as $x \to -\infty$, all of them pass through the point $(0,1)$, and all of them are increasing and approach ∞ as $x \to \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \to -\infty$.

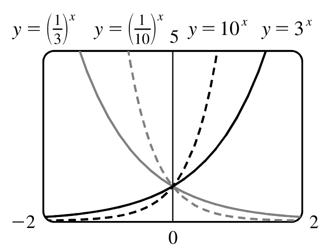
Note: The notation "$x \to \infty$" can be thought of as "x becomes large" at this point. More details on this notation are given in Chapter 2.



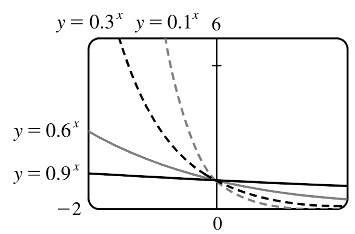
6. The graph of e^{-x} is the reflection of the graph of e^x about the y-axis, and the graph of 8^{-x} is the reflection of that of 8^x about the y-axis. The graph of 8^x increases more quickly than that of e^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



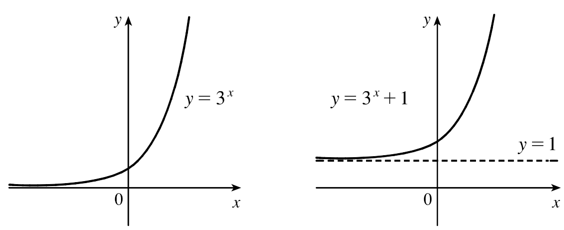
7. The functions with base greater than 1 (3^x and 10^x) are increasing, while those with base less than 1 $\left[\left(\frac{1}{3}\right)^x \text{ and } \left(\frac{1}{10}\right)^x\right]$ are decreasing. The graph of $\left(\frac{1}{3}\right)^x$ is the reflection of that of 3^x about the y-axis, and the graph of $\left(\frac{1}{10}\right)^x$ is the reflection of that of 10^x about the y-axis. The graph of 10^x increases more quickly than that of 3^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



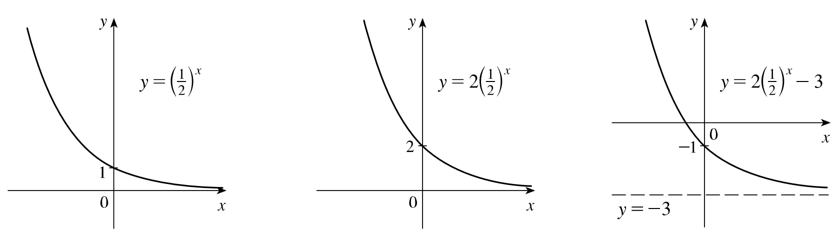
8. Each of the graphs approaches ∞ as $x \to -\infty$, and each approaches 0 as $x \to \infty$. The smaller the base, the faster the function grows as $x \to -\infty$, and the faster it approaches 0 as $x \to \infty$.



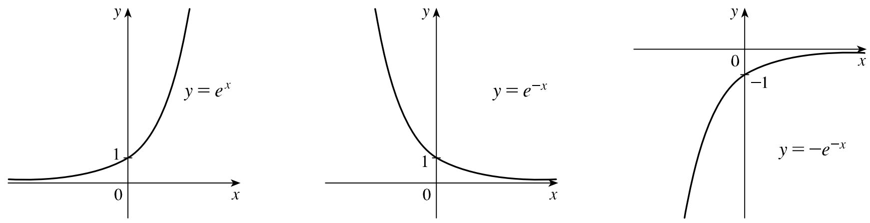
9. We start with the graph of $y = 3^x$ (Figure 15) and shift 1 unit upward to get the graph of $g(x) = 3^x + 1$.



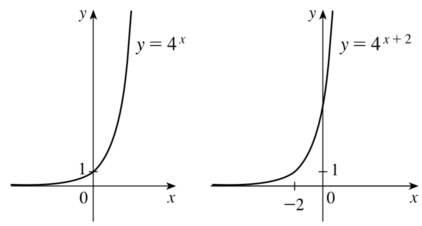
10. We start with the graph of $y = \left(\frac{1}{2}\right)^x$ (Figure 3) and stretch vertically by a factor of 2 to obtain the graph of $y = 2\left(\frac{1}{2}\right)^x$. Then we shift the graph 3 units downward to get the graph of $h(x) = 2\left(\frac{1}{2}\right)^x - 3$.



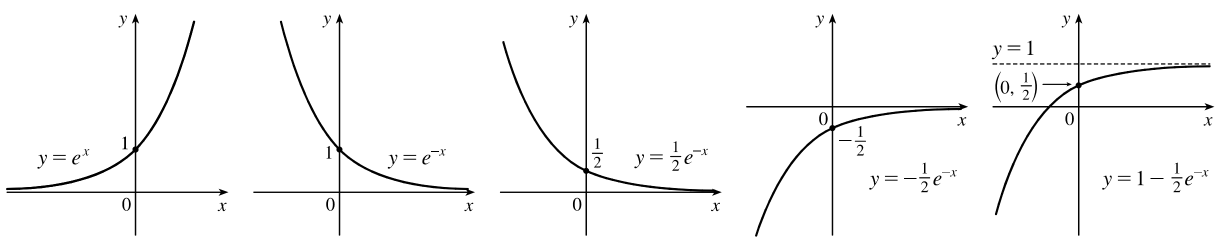
11. We start with the graph of $y = e^x$ (Figure 15) and reflect about the y-axis to get the graph of $y = e^{-x}$. Then we reflect the graph about the x-axis to get the graph of $y = -e^{-x}$.



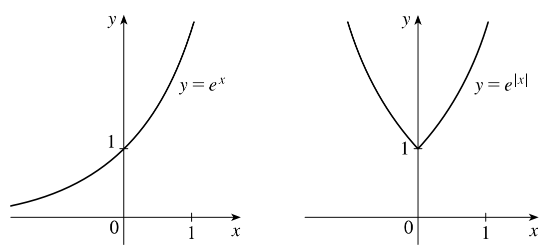
12. We start with the graph of $y = 4^x$ (Figure 3) and shift 2 units to the left to get the graph of $y = 4^{x+2}$.



13. We start with the graph of $y = e^x$ (Figure 15) and reflect about the y-axis to get the graph of $y = e^{-x}$. Then we compress the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$ and then reflect about the x-axis to get the graph of $y = -\frac{1}{2}e^{-x}$. Finally, we shift the graph one unit upward to get the graph of $y = 1 - \frac{1}{2}e^{-x}$.



14. We start with the graph of $y = e^x$ (Figure 15) and reflect the portion of the graph in the first quadrant about the y-axis to obtain the graph of $y = e^{|x|}$.



15. (a) To find the equation of the graph that results from shifting the graph of $y = e^x$ two units downward, we subtract 2 from the original function to get $y = e^x - 2$.

(b) To find the equation of the graph that results from shifting the graph of $y = e^x$ two units to the right, we replace x with $x - 2$ in the original function to get $y = e^{x-2}$.

(c) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis, we multiply the original function by -1 to get $y = -e^x$.

(d) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the y-axis, we replace x with $-x$ in the original function to get $y = e^{-x}$.

(e) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis and then about the y-axis, we first multiply the original function by -1 (to get $y = -e^x$) and then replace x with $-x$ in this equation to get $y = -e^{-x}$.

16. (a) This reflection consists of first reflecting the graph about the x-axis (giving the graph with equation $y = -e^x$) and then shifting this graph $2 \cdot 4 = 8$ units upward. So the equation is $y = -e^x + 8$.

(b) This reflection consists of first reflecting the graph about the y-axis (giving the graph with equation $y = e^{-x}$) and then shifting this graph $2 \cdot 2 = 4$ units to the right. So the equation is $y = e^{-(x-4)}$.

17. (a) The denominator is zero when $1 - e^{1-x^2} = 0 \quad \Leftrightarrow \quad e^{1-x^2} = 1 \quad \Leftrightarrow \quad 1 - x^2 = 0 \quad \Leftrightarrow \quad x = \pm 1$. Thus, the function $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{1-x^2}}$ has domain $\{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

(b) The denominator is never equal to zero, so the function $f(x) = \dfrac{1 + x}{e^{\cos x}}$ has domain $\mathbb{R}$, or $(-\infty, \infty)$.

18. (a) The function $g(t) = \sqrt{10^t - 100}$ has domain $\{t \mid 10^t - 100 \geq 0\} = \{t \mid 10^t \geq 10^2\} = \{t \mid t \geq 2\} = [2, \infty)$.

(b) The sine and exponential functions have domain $\mathbb{R}$, so $g(t) = \sin(e^t - 1)$ also has domain $\mathbb{R}$.

19. Use $y = Cb^x$ with the points $(1, 6)$ and $(3, 24)$. $6 = Cb^1 \quad \left[C = \frac{6}{b}\right]$ and $24 = Cb^3 \quad \Rightarrow \quad 24 = \left(\dfrac{6}{b}\right)b^3 \quad \Rightarrow$ $4 = b^2 \quad \Rightarrow \quad b = 2 \quad [\text{since } b > 0]$ and $C = \frac{6}{2} = 3$. The function is $f(x) = 3 \cdot 2^x$.

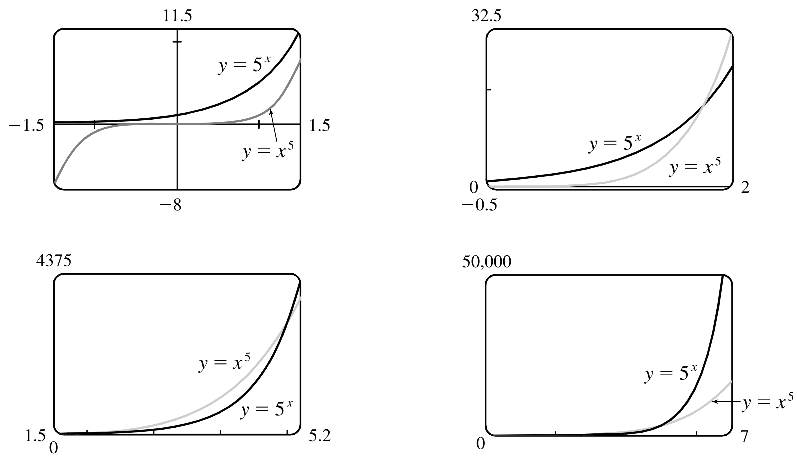
20. Use $y = Cb^x$ with the points $(-1, 3)$ and $\left(1, \frac{4}{3}\right)$. From the point $(-1, 3)$, we have $3 = Cb^{-1}$, hence $C = 3b$. Using this and the point $\left(1, \frac{4}{3}\right)$, we get $\frac{4}{3} = Cb^1 \quad \Rightarrow \quad \frac{4}{3} = (3b)b \quad \Rightarrow \quad \frac{4}{9} = b^2 \quad \Rightarrow \quad b = \frac{2}{3} \quad [\text{since } b > 0]$ and $C = 3\left(\frac{2}{3}\right) = 2$. The function is $f(x) = 2\left(\frac{2}{3}\right)^x$.

21. If $f(x) = 5^x$, then $\dfrac{f(x+h) - f(x)}{h} = \dfrac{5^{x+h} - 5^x}{h} = \dfrac{5^x 5^h - 5^x}{h} = \dfrac{5^x\left(5^h - 1\right)}{h} = 5^x\left(\dfrac{5^h - 1}{h}\right)$.

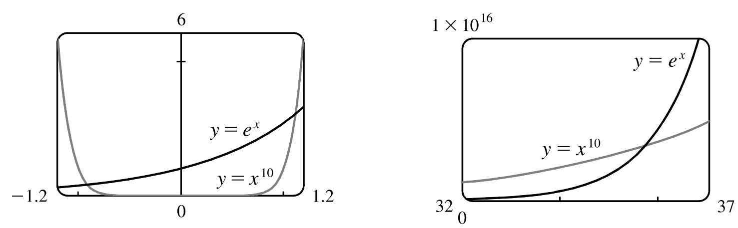
22. Suppose the month is February. Your payment on the 28th day would be $2^{28-1} = 2^{27} = 134{,}217{,}728$ cents, or \$1,342,177.28. Clearly, the second method of payment results in a larger amount for any month.

23. 2 ft $= 24$ in, $f(24) = 24^2$ in $= 576$ in $= 48$ ft. $\quad g(24) = 2^{24}$ in $= 2^{24}/(12 \cdot 5280)$ mi ≈ 265 mi

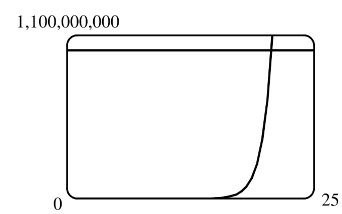
24. We see from the graphs that for x less than about 1.8, $g(x) = 5^x > f(x) = x^5$, and then near the point $(1.8, 17.1)$ the curves intersect. Then $f(x) > g(x)$ from $x \approx 1.8$ until $x = 5$. At $(5, 3125)$ there is another point of intersection, and for $x > 5$ we see that $g(x) > f(x)$. In fact, g increases much more rapidly than f beyond that point.



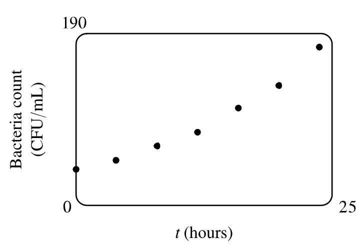
25. The graph of g finally surpasses that of f at $x \approx 35.8$.



26. We graph $y = e^x$ and $y = 1{,}000{,}000{,}000$ and determine where $e^x = 1 \times 10^9$. This seems to be true at $x \approx 20.723$, so $e^x > 1 \times 10^9$ for $x > 20.723$.

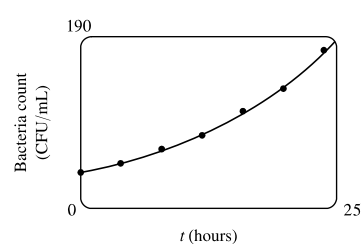


27. (a)



(b) Using a graphing calculator, we obtain the exponential curve $f(t) = 36.89301(1.06614)^t$.

(c) Using the TRACE and zooming in, we find that the bacteria count doubles from 37 to 74 in about 10.87 hours.



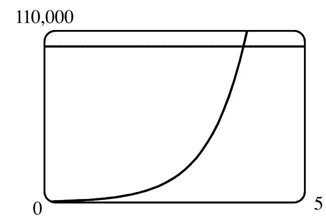
28. Let $t = 0$ correspond to 1900 to get the model $P = ab^t$, where $a \approx 80.8498$ and $b \approx 1.01269$. To estimate the population in 1925, let $t = 25$ to obtain $P \approx 111$ million. To predict the population in 2020, let $t = 120$ to obtain $P \approx 367$ million.

29. (a) Three hours represents 6 doubling periods (one doubling period is 30 minutes). Thus, $500 \cdot 2^6 = 32{,}000$.

(b) In t hours, there will be $2t$ doubling periods. The initial population is 500, so the population y at time t is $y = 500 \cdot 2^{2t}$.

(c) $t = \frac{40}{60} = \frac{2}{3} \quad \Rightarrow \quad y = 500 \cdot 2^{2(2/3)} \approx 1260$

(d) We graph $y_1 = 500 \cdot 2^{2t}$ and $y_2 = 100{,}000$. The two curves intersect at $t \approx 3.82$, so the population reaches 100,000 in about 3.82 hours.



30. (a) Let a be the initial population. Since 18 years is 3 doubling periods, $a \cdot 2^3 = 600 \quad \Rightarrow \quad a = \dfrac{600}{8} = 75$. The initial squirrel popluation was 75.

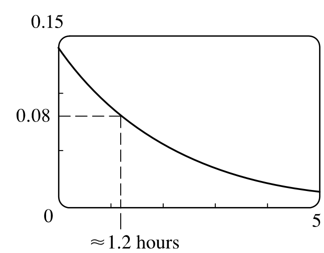
(b) A period of t years corresponds to $t/6$ doubling periods, so the expected squirrel population t years after introduction is $P = 75 \cdot 2^{t/6}$.

(c) Ten years from now will be $18 + 10 = 28$ years from introduction. The population is estimated to be $P = 75 \cdot 2^{28/6} \approx 1905$ squirrels.

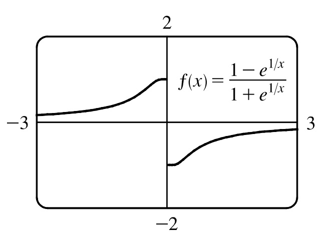
31. Half of 76.0 RNA copies per mL, corresponding to $t = 1$, is 38.0 RNA copies per mL. Using the graph of V in Figure 11, we estimate that it takes about 3.5 additional days for the patient's viral load to decrease to 38 RNA copies per mL.

32. (a) The exponential decay model has the form $C(t) = a\left(\frac{1}{2}\right)^{t/1.5}$, where t is the number of hours after midnight and $C(t)$ is the BAC. We are given that $C(0) = 0.14$, so $a = 0.14$, and the model is $C(t) = 0.14\left(\frac{1}{2}\right)^{t/1.5}$.

(b) From the graph, we estimate that the BAC is 0.08 g/dL when $t \approx 1.2$ hours.



33.



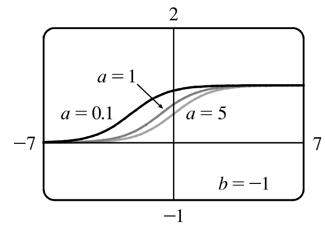
From the graph, it appears that f is an odd function (f is undefined for $x = 0$). To prove this, we must show that $f(-x) = -f(x)$.

$$f(-x) = \frac{1 - e^{1/(-x)}}{1 + e^{1/(-x)}} = \frac{1 - e^{(-1/x)}}{1 + e^{(-1/x)}} = \frac{1 - \dfrac{1}{e^{1/x}}}{1 + \dfrac{1}{e^{1/x}}} \cdot \frac{e^{1/x}}{e^{1/x}} = \frac{e^{1/x} - 1}{e^{1/x} + 1}$$

$$= -\frac{1 - e^{1/x}}{1 + e^{1/x}} = -f(x)$$

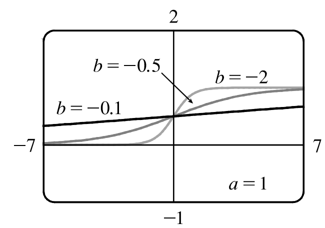
so f is an odd function.

34. We'll start with $b = -1$ and graph $f(x) = \dfrac{1}{1 + ae^{bx}}$ for $a = 0.1$, 1, and 5.



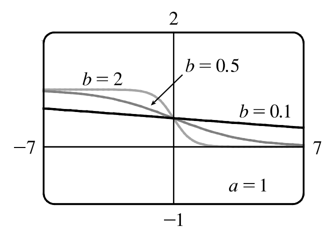
From the graph, we see that there is a horizontal asymptote $y = 0$ as $x \to -\infty$ and a horizontal asymptote $y = 1$ as $x \to \infty$. If $a = 1$, the y-intercept is $\left(0, \frac{1}{2}\right)$. As a gets smaller (close to 0), the graph of f moves left. As a gets larger, the graph of f moves right.

As b changes from -1 to 0, the graph of f is stretched horizontally. As b changes through large negative values, the graph of f is compressed horizontally. (This takes care of negatives values of b.)

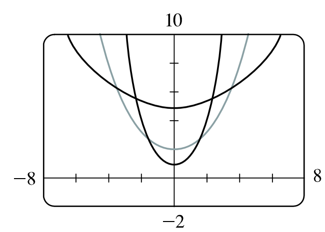


If b is positive, the graph of f is reflected through the y-axis.

Last, if $b = 0$, the graph of f is the horizontal line $y = 1/(1 + a)$.



35. We graph the function $f(x) = \dfrac{a}{2}(e^{x/a} + e^{-x/a})$ for $a = 1$, 2, and 5. Because $f(0) = a$, the y-intercept is a, so the y-intercept moves upward as a increases. Notice that the graph also widens, becoming flatter near the y-axis as a increases.



1.5 Inverse Functions and Logarithms

1. (a) See Definition 1.

(b) It must pass the Horizontal Line Test.

2. (a) $f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$ for any y in B. The domain of f^{-1} is B and the range of f^{-1} is A.

(b) See the steps in Box 5.

(c) Reflect the graph of f about the line $y = x$.

3. f is not one-to-one because $2 \neq 6$, but $f(2) = 2.0 = f(6)$.

4. f is one-to-one because it never takes on the same value twice.

5. We could draw a horizontal line that intersects the graph in more than one point. Thus, by the Horizontal Line Test, the function is not one-to-one.

6. No horizontal line intersects the graph more than once. Thus, by the Horizontal Line Test, the function is one-to-one.

7. No horizontal line intersects the graph more than once. Thus, by the Horizontal Line Test, the function is one-to-one.

8. We could draw a horizontal line that intersects the graph in more than one point. Thus, by the Horizontal Line Test, the function is not one-to-one.

9. The graph of $f(x) = 2x - 3$ is a line with slope 2. It passes the Horizontal Line Test, so f is one-to-one.

Algebraic solution: If $x_1 \neq x_2$, then $2x_1 \neq 2x_2 \;\Rightarrow\; 2x_1 - 3 \neq 2x_2 - 3 \;\Rightarrow\; f(x_1) \neq f(x_2)$, so f is one-to-one.

10. The graph of $f(x) = x^4 - 16$ is symmetric with respect to the y-axis. Pick any x-values equidistant from 0 to find two equal function values. For example, $f(-1) = -15$ and $f(1) = -15$, so f is not one-to-one.

11. No horizontal line intersects the graph of $r(t) = t^3 + 4$ more than once. Thus, by the Horizontal Line Test, the function is one-to-one.

Algebraic solution: If $t_1 \neq t_2$, then $t_1^3 \neq t_2^3 \;\Rightarrow\; t_1^3 + 4 \neq t_2^3 + 4 \;\Rightarrow\; r(t_1) \neq r(t_2)$, so r is one-to-one.

12. The graph of $g(x) = \sqrt[3]{x}$ passes the Horizontal Line Test, so g is one-to-one.

13. $g(x) = 1 - \sin x$. $\;g(0) = 1$ and $g(\pi) = 1$, so g is not one-to-one.

14. The graph of $f(x) = x^4 - 1$ passes the Horizontal Line Test when x is restricted to the interval $[0,10]$, so f is one-to-one.

15. A football will attain every height h up to its maximum height twice: once on the way up, and again on the way down. Thus, even if t_1 does not equal t_2, $f(t_1)$ may equal $f(t_2)$, so f is not 1-1.

16. f is not 1-1 because eventually we all stop growing and therefore, there are two times at which we have the same height.

17. (a) Since f is 1-1, $f(6) = 17 \;\Leftrightarrow\; f^{-1}(17) = 6$.

(b) Since f is 1-1, $f^{-1}(3) = 2 \;\Leftrightarrow\; f(2) = 3$.

18. First, we must determine x such that $f(x) = 3$. By inspection, we see that if $x = 1$, then $f(1) = 3$. Since f is 1-1 (f is an increasing function), it has an inverse, and $f^{-1}(3) = 1$. If f is a 1-1 function, then $f(f^{-1}(a)) = a$, so $f(f^{-1}(2)) = 2$.

19. First, we must determine x such that $g(x) = 4$. By inspection, we see that if $x = 0$, then $g(x) = 4$. Since g is 1-1 (g is an increasing function), it has an inverse, and $g^{-1}(4) = 0$.

20. (a) f is 1-1 because it passes the Horizontal Line Test.

(b) Domain of $f = [-3, 3] =$ Range of f^{-1}. Range of $f = [-1, 3] =$ Domain of f^{-1}.

(c) Since $f(0) = 2$, $f^{-1}(2) = 0$.

(d) Since $f(-1.7) \approx 0$, $f^{-1}(0) \approx -1.7$.

21. We solve $C = \frac{5}{9}(F - 32)$ for F: $\frac{9}{5}C = F - 32 \;\Rightarrow\; F = \frac{9}{5}C + 32$. This gives us a formula for the inverse function, that is, the Fahrenheit temperature F as a function of the Celsius temperature C. $F \geq -459.67 \;\Rightarrow\; \frac{9}{5}C + 32 \geq -459.67 \;\Rightarrow\; \frac{9}{5}C \geq -491.67 \;\Rightarrow\; C \geq -273.15$, the domain of the inverse function.

22. $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}} \;\Rightarrow\; 1 - \dfrac{v^2}{c^2} = \dfrac{m_0^2}{m^2} \;\Rightarrow\; \dfrac{v^2}{c^2} = 1 - \dfrac{m_0^2}{m^2} \;\Rightarrow\; v^2 = c^2\left(1 - \dfrac{m_0^2}{m^2}\right) \;\Rightarrow\; v = c\sqrt{1 - \dfrac{m_0^2}{m^2}}.$

This formula gives us the speed v of the particle in terms of its mass m, that is, $v = f^{-1}(m)$.

23. First note that $f(x) = 1 - x^2$, $x \geq 0$, is one-to-one. We first write $y = 1 - x^2$, $x \geq 0$, and solve for x: $x^2 = 1 - y \;\Rightarrow\; x = \sqrt{1 - y}$ (since $x \geq 0$). Interchanging x and y gives $y = \sqrt{1 - x}$, so the inverse function is $f^{-1}(x) = \sqrt{1 - x}$.

24. Completing the square, we have $g(x) = x^2 - 2x = (x^2 - 2x + 1) - 1 = (x - 1)^2 - 1$ and, with the restriction $x \geq 1$, g is one-to-one. We write $y = (x - 1)^2 - 1$, $x \geq 1$, and solve for x: $x - 1 = \sqrt{y + 1}$ (since $x \geq 1 \;\Leftrightarrow\; x - 1 \geq 0$), so $x = 1 + \sqrt{y + 1}$. Interchanging x and y gives $y = 1 + \sqrt{x + 1}$, so $g^{-1}(x) = 1 + \sqrt{x + 1}$.

25. First write $y = g(x) = 2 + \sqrt{x + 1}$ and note that $y \geq 2$. Solve for x: $y - 2 = \sqrt{x + 1} \;\Rightarrow\; (y - 2)^2 = x + 1 \;\Rightarrow\; x = (y - 2)^2 - 1$ $(y \geq 2)$. Interchanging x and y gives $y = (x - 2)^2 - 1$, so $g^{-1}(x) = (x - 2)^2 - 1$ with domain $x \geq 2$.

26. We write $y = h(x) = \dfrac{6 - 3x}{5x + 7}$ and solve for x: $y(5x + 7) = 6 - 3x \;\Rightarrow\; 5xy + 7y = 6 - 3x \;\Rightarrow\; 5xy + 3x = 6 - 7y \;\Rightarrow\; x(5y + 3) = 6 - 7y \;\Rightarrow\; x = \dfrac{6 - 7y}{5y + 3}$. Interchanging x and y gives $y = \dfrac{6 - 7x}{5x + 3}$, so $h^{-1}(x) = \dfrac{6 - 7x}{5x + 3}$.

27. We solve $y = e^{1-x}$ for x: $\ln y = \ln e^{1-x} \;\Rightarrow\; \ln y = 1 - x \;\Rightarrow\; x = 1 - \ln y$. Interchanging x and y gives the inverse function $y = 1 - \ln x$.

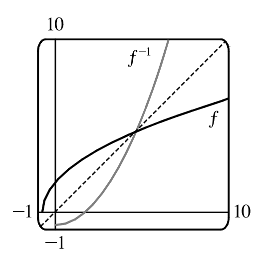
28. We solve $y = 3\ln(x - 2)$ for x: $y/3 = \ln(x - 2) \;\Rightarrow\; e^{y/3} = x - 2 \;\Rightarrow\; x = 2 + e^{y/3}$. Interchanging x and y gives the inverse function $y = 2 + e^{x/3}$.

29. We solve $y = \left(2 + \sqrt[3]{x}\right)^5$ for x: $\sqrt[5]{y} = 2 + \sqrt[3]{x} \;\Rightarrow\; \sqrt[3]{x} = \sqrt[5]{y} - 2 \;\Rightarrow\; x = \left(\sqrt[5]{y} - 2\right)^3$. Interchanging x and y gives the inverse function $y = \left(\sqrt[5]{x} - 2\right)^3$.

30. We solve $y = \dfrac{1 - e^{-x}}{1 + e^{-x}}$ for x: $y(1 + e^{-x}) = 1 - e^{-x} \;\Rightarrow\; y + ye^{-x} = 1 - e^{-x} \;\Rightarrow\; e^{-x} + ye^{-x} = 1 - y \;\Rightarrow\; e^{-x}(1 + y) = 1 - y \;\Rightarrow\; e^{-x} = \dfrac{1 - y}{1 + y} \;\Rightarrow\; -x = \ln\dfrac{1 - y}{1 + y} \;\Rightarrow\; x = -\ln\dfrac{1 - y}{1 + y}$ or, equivalently, $x = \ln\left(\dfrac{1 - y}{1 + y}\right)^{-1} = \ln\dfrac{1 + y}{1 - y}$. Interchanging x and y gives the inverse function $y = \ln\dfrac{1 + x}{1 - x}$.

31. $y = f(x) = \sqrt{4x + 3}\;\; (y \geq 0) \;\Rightarrow\; y^2 = 4x + 3 \;\Rightarrow\; x = \dfrac{y^2 - 3}{4}$.

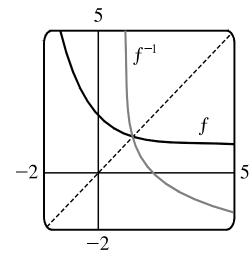
Interchange x and y: $y = \dfrac{x^2 - 3}{4}$. So $f^{-1}(x) = \dfrac{x^2 - 3}{4}$ $(x \geq 0)$. From the graph, we see that f and f^{-1} are reflections about the line $y = x$.



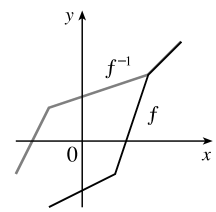
32. $y = f(x) = 1 + e^{-x} \quad \Rightarrow \quad e^{-x} = y - 1 \quad \Rightarrow \quad -x = \ln(y-1) \quad \Rightarrow$

$x = -\ln(y-1)$. Interchange x and y: $y = -\ln(x-1)$.

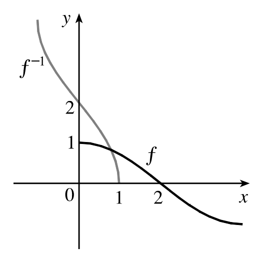
So $f^{-1}(x) = -\ln(x-1)$. From the graph, we see that f and f^{-1} are reflections about the line $y = x$.



33. Reflect the graph of f about the line $y = x$. The points $(-1,-2)$, $(1,-1)$, $(2,2)$, and $(3,3)$ on f are reflected to $(-2,-1)$, $(-1,1)$, $(2,2)$, and $(3,3)$ on f^{-1}.



34. Reflect the graph of f about the line $y = x$.

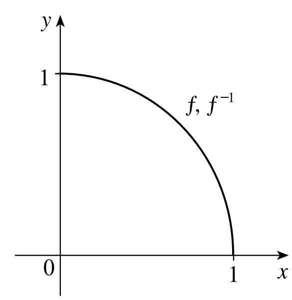


35. (a) $y = f(x) = \sqrt{1-x^2} \quad (0 \le x \le 1$ and note that $y \ge 0) \quad \Rightarrow$

$y^2 = 1 - x^2 \quad \Rightarrow \quad x^2 = 1 - y^2 \quad \Rightarrow \quad x = \sqrt{1-y^2}$. So

$f^{-1}(x) = \sqrt{1-x^2}$, $0 \le x \le 1$. We see that $f^{-1}$and f are the same function.

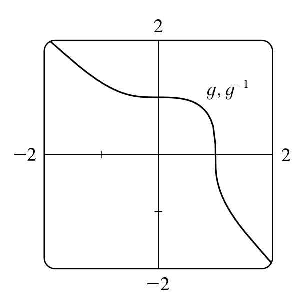
(b) The graph of f is the portion of the circle $x^2 + y^2 = 1$ with $0 \le x \le 1$ and $0 \le y \le 1$ (quarter-circle in the first quadrant). The graph of f is symmetric with respect to the line $y = x$, so its reflection about $y = x$ is itself, that is, $f^{-1} = f$.



36. (a) $y = g(x) = \sqrt[3]{1-x^3} \quad \Rightarrow \quad y^3 = 1 - x^3 \quad \Rightarrow \quad x^3 = 1 - y^3 \quad \Rightarrow$

$x = \sqrt[3]{1-y^3}$. So $g^{-1}(x) = \sqrt[3]{1-x^3}$. We see that g and g^{-1} are the same function.

(b) The graph of g is symmetric with respect to the line $y = x$, so its reflection about $y = x$ is itself, that is, $g^{-1} = g$.



37. (a) It is defined as the inverse of the exponential function with base b, that is, $\log_b x = y \quad \Leftrightarrow \quad b^y = x$.

(b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 11.

38. (a) The natural logarithm is the logarithm with base e, denoted $\ln x$.

(b) The common logarithm is the logarithm with base 10, denoted $\log x$.

(c) See Figure 13.

39. (a) $\log_3 81 = \log_3 3^4 = 4$ (b) $\log_3\left(\frac{1}{81}\right) = \log_3 3^{-4} = -4$ (c) $\log_9 3 = \log_9 9^{1/2} = \frac{1}{2}$

40. (a) $\ln \dfrac{1}{e^2} = \ln e^{-2} = -2$ (b) $\ln \sqrt{e} = \ln e^{1/2} = \frac{1}{2}$ (c) $\ln\left(\ln e^{e^{50}}\right) = \ln(e^{50}) = 50$

41. (a) $\log_2 30 - \log_2 15 = \log_2\left(\dfrac{30}{15}\right) = \log_2 2 = 1$

(b) $$\log_3 10 - \log_3 5 - \log_3 18 = \log_3\left(\frac{10}{5}\right) - \log_3 18 = \log_3 2 - \log_3 18 = \log_3\left(\frac{2}{18}\right) = \log_3\left(\frac{1}{9}\right)$$
$$= \log_3 3^{-2} = -2$$

(c) $$2\log_5 100 - 4\log_5 50 = \log_5 100^2 - \log_5 50^4 = \log_5\left(\frac{100^2}{50^4}\right) = \log_5\left(\frac{10^4}{5^4 \cdot 10^4}\right) = \log_5 5^{-4} = -4$$

42. (a) $e^{3\ln 2} = e^{\ln 2^3} = 2^3 = 8$ (b) $e^{-2\ln 5} = e^{\ln 5^{-2}} = 5^{-2} = \frac{1}{25}$ (c) $e^{\ln(\ln e^3)} = e^{\ln(3)} = 3$

43. (a) $$\log_{10}\left(x^2y^3z\right) = \log_{10} x^2 + \log_{10} y^3 + \log_{10} z \qquad \text{[Law 1]}$$
$$= 2\log_{10} x + 3\log_{10} y + \log_{10} z \qquad \text{[Law 3]}$$

(b) $$\ln\left(\frac{x^4}{\sqrt{x^2-4}}\right) = \ln x^4 - \ln(x^2-4)^{1/2} \qquad \text{[Law 2]}$$
$$= 4\ln x - \tfrac{1}{2}\ln[(x+2)(x-2)] \qquad \text{[Law 3]}$$
$$= 4\ln x - \tfrac{1}{2}[\ln(x+2) + \ln(x-2)] \qquad \text{[Law 1]}$$
$$= 4\ln x - \tfrac{1}{2}\ln(x+2) - \tfrac{1}{2}\ln(x-2)$$

44. (a) $$\ln\sqrt{\frac{3x}{x-3}} = \ln\left(\frac{3x}{x-3}\right)^{1/2} = \frac{1}{2}\ln\left(\frac{3x}{x-3}\right) \qquad \text{[Law 3]}$$
$$= \tfrac{1}{2}[\ln 3 + \ln x - \ln(x-3)] \qquad \text{[Laws 1 and 2]}$$
$$= \tfrac{1}{2}\ln 3 + \tfrac{1}{2}\ln x - \tfrac{1}{2}\ln(x-3)$$

(b) $$\log_2\left[(x^3+1)\sqrt[3]{(x-3)^2}\right] = \log_2(x^3+1) + \log_2\sqrt[3]{(x-3)^2} \qquad \text{[Law 1]}$$
$$= \log_2(x^3+1) + \log_2(x-3)^{2/3}$$
$$= \log_2(x^3+1) + \tfrac{2}{3}\log_2(x-3) \qquad \text{[Law 3]}$$

45. (a) $$\log_{10} 20 - \tfrac{1}{3}\log_{10} 1000 = \log_{10} 20 - \log_{10} 1000^{1/3} = \log_{10} 20 - \log_{10}\sqrt[3]{1000}$$
$$= \log_{10} 20 - \log_{10} 10 = \log_{10}\left(\tfrac{20}{10}\right) = \log_{10} 2$$

(b) $$\ln a - 2\ln b + 3\ln c = \ln a - \ln b^2 + \ln c^3 = \ln\frac{a}{b^2} + \ln c^3 = \ln\frac{ac^3}{b^2}$$

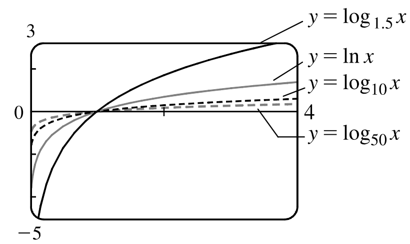
46. (a) $$3\ln(x-2) - \ln(x^2-5x+6) + 2\ln(x-3) = \ln(x-2)^3 - \ln[(x-2)(x-3)] + \ln(x-3)^2$$
$$= \ln\left[\frac{(x-2)^3(x-3)^2}{(x-2)(x-3)}\right] = \ln[(x-2)^2(x-3)]$$

(b) $$c\log_a x - d\log_a y + \log_a z = \log_a x^c - \log_a y^d + \log_a z = \log_a\left(\frac{x^cz}{y^d}\right)$$

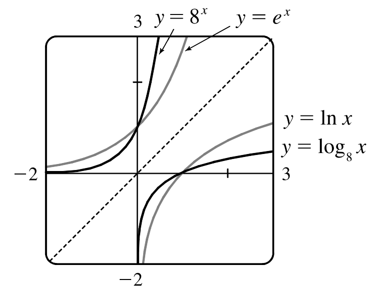
47. (a) $\log_5 10 = \dfrac{\ln 10}{\ln 5} \approx 1.430677$ (b) $\log_{15} 12 = \dfrac{\ln 12}{\ln 15} \approx 0.917600$

48. (a) $\log_3 12 = \dfrac{\ln 12}{\ln 3} \approx 2.261860$ (b) $\log_{12} 6 = \dfrac{\ln 6}{\ln 12} \approx 0.721057$

49. To graph these functions, we use $\log_{1.5} x = \dfrac{\ln x}{\ln 1.5}$ and $\log_{50} x = \dfrac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y-axis more closely as $x \to 0^+$.

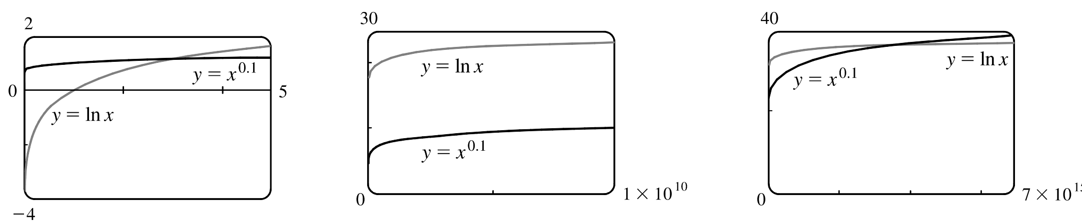


50. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y = x$, and that the graph of $\log_8 x$ is the reflection of the graph of 8^x about the same line. The graph of 8^x increases more quickly than that of e^x. Also note that $\log_8 x \to \infty$ as $x \to \infty$ more slowly than $\ln x$.



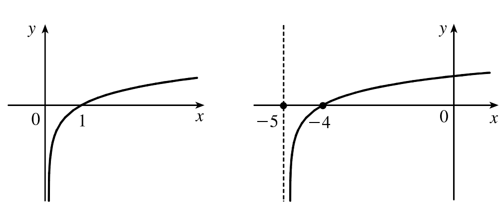
51. $3 \text{ ft} = 36 \text{ in}$, so we need x such that $\log_2 x = 36 \quad \Leftrightarrow \quad x = 2^{36} = 68{,}719{,}476{,}736$. In miles, this is $68{,}719{,}476{,}736 \text{ in} \cdot \dfrac{1 \text{ ft}}{12 \text{ in}} \cdot \dfrac{1 \text{ mi}}{5280 \text{ ft}} \approx 1{,}084{,}587.7 \text{ mi}$.

52.



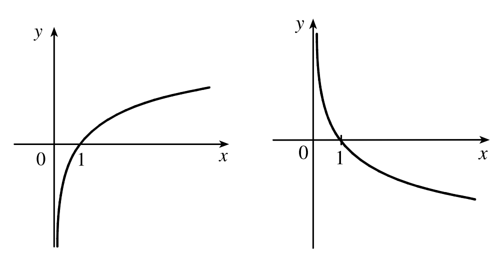
From the graphs, we see that $f(x) = x^{0.1} > g(x) = \ln x$ for approximately $0 < x < 3.06$, and then $g(x) > f(x)$ for $3.06 < x < 3.43 \times 10^{15}$ (approximately). At that point, the graph of f finally surpasses the graph of g for good.

53. (a) Shift the graph of $y = \log_{10} x$ five units to the left to obtain the graph of $y = \log_{10}(x + 5)$. Note the vertical asymptote of $x = -5$.



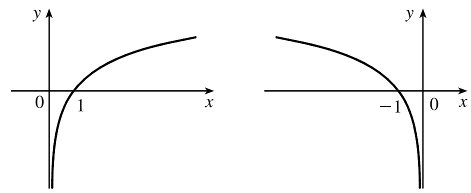
$y = \log_{10} x$ $y = \log_{10}(x + 5)$

(b) Reflect the graph of $y = \ln x$ about the x-axis to obtain the graph of $y = -\ln x$.



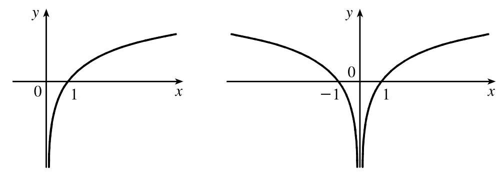
$y = \ln x$ $y = -\ln x$

54. (a) Reflect the graph of $y = \ln x$ about the y-axis to obtain the graph of $y = \ln(-x)$.



$y = \ln x$ $\qquad$ $y = \ln(-x)$

(b) Reflect the portion of the graph of $y = \ln x$ to the right of the y-axis about the y-axis. The graph of $y = \ln|x|$ is that reflection in addition to the original portion.

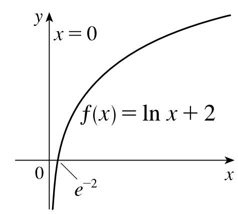


$y = \ln x$ $\qquad$ $y = \ln|x|$

55. (a) The domain of $f(x) = \ln x + 2$ is $x > 0$ and the range is $\mathbb{R}$.

(b) $y = 0 \quad\Rightarrow\quad \ln x + 2 = 0 \quad\Rightarrow\quad \ln x = -2 \quad\Rightarrow\quad x = e^{-2}$

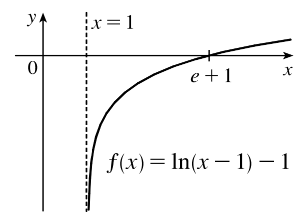
(c) We shift the graph of $y = \ln x$ two units upward.



56. (a) The domain of $f(x) = \ln(x-1) - 1$ is $x > 1$ and the range is $\mathbb{R}$.

(b) $y = 0 \quad\Rightarrow\quad \ln(x-1) - 1 = 0 \quad\Rightarrow\quad \ln(x-1) = 1 \quad\Rightarrow$
$x - 1 = e^1 \quad\Rightarrow\quad x = e + 1$

(c) We shift the graph of $y = \ln x$ one unit to the right and one unit downward.



57. (a) $\ln(4x+2) = 3 \quad\Rightarrow\quad e^{\ln(4x+2)} = e^3 \quad\Rightarrow\quad 4x + 2 = e^3 \quad\Rightarrow\quad 4x = e^3 - 2 \quad\Rightarrow\quad x = \frac{1}{4}(e^3 - 2) \approx 4.521$

(b) $e^{2x-3} = 12 \quad\Rightarrow\quad \ln e^{2x-3} = \ln 12 \quad\Rightarrow\quad 2x - 3 = \ln 12 \quad\Rightarrow\quad 2x = 3 + \ln 12 \quad\Rightarrow\quad x = \frac{1}{2}(3 + \ln 12) \approx 2.742$

58. (a) $\log_2(x^2 - x - 1) = 2 \quad\Rightarrow\quad x^2 - x - 1 = 2^2 = 4 \quad\Rightarrow\quad x^2 - x - 5 = 0 \quad\Rightarrow$

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-5)}}{2(1)} = \frac{1 \pm \sqrt{21}}{2}.$$

Solutions are $x_1 = \dfrac{1 - \sqrt{21}}{2} \approx -1.791$ and $x_2 = \dfrac{1 + \sqrt{21}}{2} \approx 2.791$.

(b) $1 + e^{4x+1} = 20 \quad\Rightarrow\quad e^{4x+1} = 19 \quad\Rightarrow\quad \ln e^{4x+1} = \ln 19 \quad\Rightarrow\quad 4x + 1 = \ln 19 \quad\Rightarrow\quad 4x = -1 + \ln 19 \quad\Rightarrow$
$x = \frac{1}{4}(-1 + \ln 19) \approx 0.486$

59. (a) $\ln x + \ln(x-1) = 0 \quad\Rightarrow\quad \ln[x(x-1)] = 0 \quad\Rightarrow\quad e^{\ln[x^2 - x]} = e^0 \quad\Rightarrow\quad x^2 - x = 1 \quad\Rightarrow\quad x^2 - x - 1 = 0$. The quadratic formula gives $x = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \dfrac{1 \pm \sqrt{5}}{2}$, but we note that $\ln \dfrac{1 - \sqrt{5}}{2}$ is undefined because $\dfrac{1 - \sqrt{5}}{2} < 0$. Thus, $x = \dfrac{1 + \sqrt{5}}{2} \approx 1.618$.

(b) $5^{1-2x} = 9 \quad\Rightarrow\quad \ln 5^{1-2x} = \ln 9 \quad\Rightarrow\quad (1 - 2x)\ln 5 = \ln 9 \quad\Rightarrow\quad 1 - 2x = \dfrac{\ln 9}{\ln 5} \quad\Rightarrow\quad x = \dfrac{1}{2} - \dfrac{\ln 9}{2\ln 5} \approx -0.183$

60. (a) $\ln(\ln x) = 0 \quad\Rightarrow\quad e^{\ln(\ln x)} = e^0 \quad\Rightarrow\quad \ln x = 1 \quad\Rightarrow\quad x = e \approx 2.718$

(b) $\dfrac{60}{1+e^{-x}} = 4 \quad\Rightarrow\quad 60 = 4(1+e^{-x}) \quad\Rightarrow\quad 15 = 1+e^{-x} \quad\Rightarrow\quad 14 = e^{-x} \quad\Rightarrow\quad \ln 14 = \ln e^{-x} \quad\Rightarrow$

$\ln 14 = -x \quad\Rightarrow\quad x = -\ln 14 \approx -2.639$

61. (a) $\ln x < 0 \quad\Rightarrow\quad x < e^0 \quad\Rightarrow\quad x < 1$. Since the domain of $f(x) = \ln x$ is $x > 0$, the solution of the original inequality is $0 < x < 1$.

(b) $e^x > 5 \quad\Rightarrow\quad \ln e^x > \ln 5 \quad\Rightarrow\quad x > \ln 5$

62. (a) $1 < e^{3x-1} < 2 \quad\Rightarrow\quad \ln 1 < 3x-1 < \ln 2 \quad\Rightarrow\quad 0 < 3x-1 < \ln 2 \quad\Rightarrow\quad 1 < 3x < 1+\ln 2 \quad\Rightarrow$

$\frac{1}{3} < x < \frac{1}{3}(1+\ln 2)$

(b) $1 - 2\ln x < 3 \quad\Rightarrow\quad -2\ln x < 2 \quad\Rightarrow\quad \ln x > -1 \quad\Rightarrow\quad x > e^{-1}$

63. (a) We must have $e^x - 3 > 0 \quad\Leftrightarrow\quad e^x > 3 \quad\Leftrightarrow\quad x > \ln 3$. Thus, the domain of $f(x) = \ln(e^x - 3)$ is $(\ln 3, \infty)$.

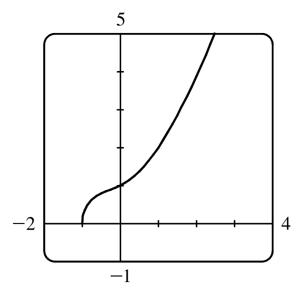
(b) $y = \ln(e^x - 3) \quad\Rightarrow\quad e^y = e^x - 3 \quad\Rightarrow\quad e^x = e^y + 3 \quad\Rightarrow\quad x = \ln(e^y + 3)$, so $f^{-1}(x) = \ln(e^x + 3)$.

Now $e^x + 3 > 0 \quad\Rightarrow\quad e^x > -3$, which is true for any real x, so the domain of f^{-1} is $\mathbb{R}$.

64. (a) By (9), $e^{\ln 300} = 300$ and $\ln(e^{300}) = 300$.

(b) A calculator gives $e^{\ln 300} = 300$ and an error message for $\ln(e^{300})$ because e^{300} is larger than most calculators can evaluate.

65. We see that the graph of $y = f(x) = \sqrt{x^3 + x^2 + x + 1}$ is increasing, so f is 1-1.



Enter $x = \sqrt{y^3 + y^2 + y + 1}$ and use your CAS to solve the equation for y. You will likely get two (irrelevant) solutions involving imaginary expressions, as well as one which can be simplified to

$y = f^{-1}(x) = -\frac{\sqrt[3]{4}}{6}\left(\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2}\right)$

where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$ or, equivalently, $\dfrac{1}{6}\dfrac{M^{2/3} - 8 - 2M^{1/3}}{2M^{1/3}}$,

where $M = 108x^2 + 12\sqrt{48 - 120x^2 + 81x^4} - 80$.

66. (a) Depending on the software used, solving $x = y^6 + y^4$ for y may give six solutions of the form $y = \pm\frac{\sqrt{3}}{3}\sqrt{B-1}$, where

$B \in \left\{-2\sin\dfrac{A}{3}, 2\sin\left(\dfrac{A}{3} + \dfrac{\pi}{3}\right), -2\cos\left(\dfrac{A}{3} + \dfrac{\pi}{6}\right)\right\}$ and $A = \sin^{-1}\left(\dfrac{27x-2}{2}\right)$. The inverse for $y = x^6 + x^4$

$(x \geq 0)$ is $y = \frac{\sqrt{3}}{3}\sqrt{B-1}$ with $B = 2\sin\left(\dfrac{A}{3} + \dfrac{\pi}{3}\right)$, but because the domain of A is $\left[0, \frac{4}{27}\right]$, this expression is only valid for $x \in \left[0, \frac{4}{27}\right]$.

If we solve $x = y^6 + y^4$ for y using Maple, we get the two real solutions $\pm\dfrac{\sqrt{6}}{6}\dfrac{\sqrt{C^{1/3}\left(C^{2/3} - 2C^{1/3} + 4\right)}}{C^{1/3}}$,

where $C = 108x + 12\sqrt{3}\sqrt{x(27x-4)}$, and the inverse for $y = x^6 + x^4$ $(x \geq 0)$ is the positive solution, whose domain is $\left[\frac{4}{27}, \infty\right)$.

[continued]

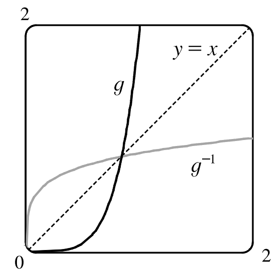
Mathematica also gives two real solutions, equivalent to those of Maple.

The positive one is $\dfrac{\sqrt{6}}{6}\left(\sqrt[3]{4}D^{1/3}+2\sqrt[3]{2}D^{-1/3}-2\right)$, where

$D=-2+27x+3\sqrt{3}\sqrt{x}\sqrt{27x-4}$. Although this expression also has domain

$\left[\frac{4}{27},\infty\right)$, Mathematica is mysteriously able to plot the solution for all $x\geq 0$.

(b)



67. (a) $n=f(t)=100\cdot 2^{t/3} \quad\Rightarrow\quad \dfrac{n}{100}=2^{t/3} \quad\Rightarrow\quad \log_2\left(\dfrac{n}{100}\right)=\dfrac{t}{3} \quad\Rightarrow\quad t=3\log_2\left(\dfrac{n}{100}\right)$. Using the Change of Base Formula, we can write this as $t=f^{-1}(n)=3\cdot\dfrac{\ln(n/100)}{\ln 2}$. This function tells us how long it will take to obtain n bacteria (given the number n).

(b) $n=50{,}000 \quad\Rightarrow\quad t=f^{-1}(50{,}000)=3\cdot\dfrac{\ln\left(\frac{50{,}000}{100}\right)}{\ln 2}=3\left(\dfrac{\ln 500}{\ln 2}\right)\approx 26.9$ hours

68. (a) We write $Q=Q_0(1-e^{-t/a})$ and solve for t: $\dfrac{Q}{Q_0}=1-e^{-t/a} \quad\Rightarrow\quad e^{-t/a}=1-\dfrac{Q}{Q_0} \quad\Rightarrow$

$-\dfrac{t}{a}=\ln\left(1-\dfrac{Q}{Q_0}\right) \quad\Rightarrow\quad t=-a\ln\left(1-\dfrac{Q}{Q_0}\right)$. This formula gives the time (in seconds) needed after a discharge to obtain a given charge Q.

(b) We set $Q=0.9Q_0$ and $a=50$ to get $t=-50\ln\left(1-\dfrac{0.9Q_0}{Q_0}\right)=-50\ln(0.1)\approx 115.1$ seconds. It will take approximately 115 seconds—just shy of two minutes—to recharge the capacitors to 90% of capacity.

69. (a) $\cos^{-1}(-1)=\pi$ because $\cos\pi=-1$ and π is in the interval $[0,\pi]$ (the range of $\cos^{-1}$).

(b) $\sin^{-1}(0.5)=\frac{\pi}{6}$ because $\sin\frac{\pi}{6}=0.5$ and $\frac{\pi}{6}$ is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (the range of $\sin^{-1}$).

70. (a) $\tan^{-1}\sqrt{3}=\frac{\pi}{3}$ because $\tan\frac{\pi}{3}=\sqrt{3}$ and $\frac{\pi}{3}$ is in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (the range of $\tan^{-1}$).

(b) $\arctan(-1)=-\frac{\pi}{4}$ because $\tan\left(-\frac{\pi}{4}\right)=-1$ and $-\frac{\pi}{4}$ is in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (the range of arctan).

71. (a) $\csc^{-1}\sqrt{2}=\frac{\pi}{4}$ because $\csc\frac{\pi}{4}=\sqrt{2}$ and $\frac{\pi}{4}$ is in $\left(0,\frac{\pi}{2}\right]\cup\left(\pi,\frac{3\pi}{2}\right]$ (the range of $\csc^{-1}$).

(b) $\arcsin 1=\frac{\pi}{2}$ because $\sin\frac{\pi}{2}=1$ and $\frac{\pi}{2}$ is in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (the range of arcsin).

72. (a) $\sin^{-1}(-1/\sqrt{2})=-\frac{\pi}{4}$ because $\sin\left(-\frac{\pi}{4}\right)=-1/\sqrt{2}$ and $-\frac{\pi}{4}$ is in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

(b) $\cos^{-1}\left(\sqrt{3}/2\right)=\frac{\pi}{6}$ because $\cos\frac{\pi}{6}=\sqrt{3}/2$ and $\frac{\pi}{6}$ is in $[0,\pi]$.

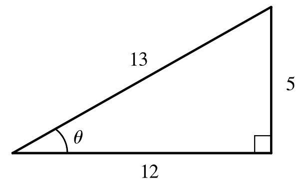
73. (a) $\cot^{-1}\left(-\sqrt{3}\right)=\frac{5\pi}{6}$ because $\cot\frac{5\pi}{6}=-\sqrt{3}$ and $\frac{5\pi}{6}$ is in $(0,\pi)$ (the range of $\cot^{-1}$).

(b) $\sec^{-1}2=\frac{\pi}{3}$ because $\sec\frac{\pi}{3}=2$ and $\frac{\pi}{3}$ is in $\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$ (the range of $\sec^{-1}$).

74. (a) $\arcsin(\sin(5\pi/4))=\arcsin\left(-1/\sqrt{2}\right)=-\frac{\pi}{4}$ because $\sin\left(-\frac{\pi}{4}\right)=-1/\sqrt{2}$ and $-\frac{\pi}{4}$ is in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

(b) Let $\theta = \sin^{-1}\left(\frac{5}{13}\right)$ [see the figure].

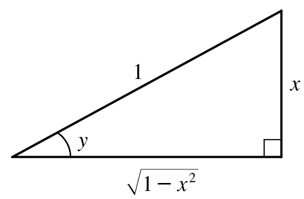
$$\cos\left(2\sin^{-1}\left(\tfrac{5}{13}\right)\right) = \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \left(\tfrac{12}{13}\right)^2 - \left(\tfrac{5}{13}\right)^2 = \tfrac{144}{169} - \tfrac{25}{169} = \tfrac{119}{169}$$



75. Let $y = \sin^{-1} x$. Then $-\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow \cos y \ge 0$, so $\cos(\sin^{-1} x) = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$.

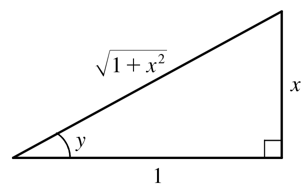
76. Let $y = \sin^{-1} x$. Then $\sin y = x$, so from the triangle (which illustrates the case $y > 0$), we see that

$$\tan(\sin^{-1} x) = \tan y = \frac{x}{\sqrt{1-x^2}}.$$



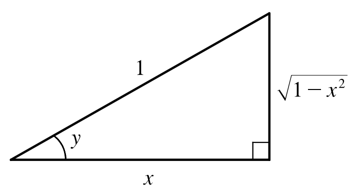
77. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle (which illustrates the case $y > 0$), we see that

$$\sin(\tan^{-1} x) = \sin y = \frac{x}{\sqrt{1+x^2}}.$$

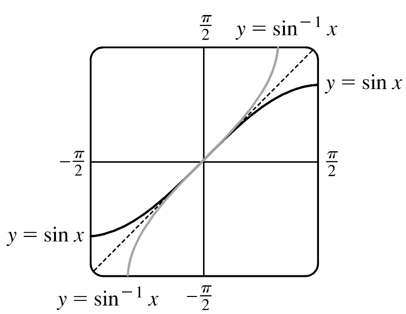


78. Let $y = \arccos x$. Then $\cos y = x$, so from the triangle (which illustrates the case $y > 0$), we see that

$$\sin(2\arccos x) = \sin 2y = 2\sin y\,\cos y$$
$$= 2\left(\sqrt{1-x^2}\right)(x) = 2x\sqrt{1-x^2}$$

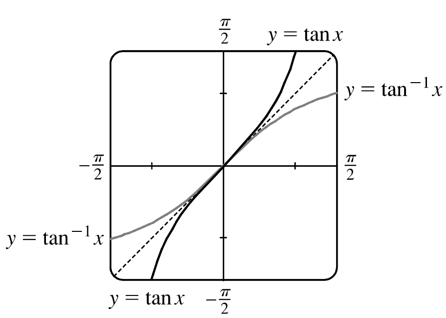


79.



The graph of $\sin^{-1} x$ is the reflection of the graph of $\sin x$ about the line $y = x$.

80.



The graph of $\tan^{-1} x$ is the reflection of the graph of $\tan x$ about the line $y = x$.

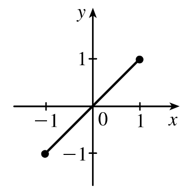
81. $g(x) = \sin^{-1}(3x+1)$.

Domain $(g) = \{x \mid -1 \le 3x+1 \le 1\} = \{x \mid -2 \le 3x \le 0\} = \{x \mid -\frac{2}{3} \le x \le 0\} = [-\frac{2}{3}, 0]$.

Range $(g) = \{y \mid -\frac{\pi}{2} \le y \le \frac{\pi}{2}\} = [-\frac{\pi}{2}, \frac{\pi}{2}]$.

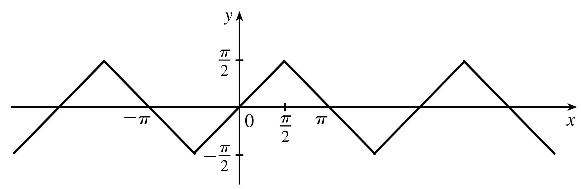
82. (a) $f(x) = \sin(\sin^{-1} x)$

Since one function undoes what the other one does, we get the identity function, $y = x$, on the restricted domain $-1 \le x \le 1$.



(b) $g(x) = \sin^{-1}(\sin x)$

This is similar to part (a), but with domain $\mathbb{R}$. Equations for g on intervals of the form $\left(-\frac{\pi}{2} + \pi n, \frac{\pi}{2} + \pi n\right)$, for any integer n, can be found using $g(x) = (-1)^n x + (-1)^{n+1} n\pi$.



The sine function is monotonic on each of these intervals, and hence, so is g (but in a linear fashion).

83. (a) If the point (x, y) is on the graph of $y = f(x)$, then the point $(x-c, y)$ is that point shifted c units to the left. Since f is 1-1, the point (y, x) is on the graph of $y = f^{-1}(x)$ and the point corresponding to $(x-c, y)$ on the graph of f is $(y, x-c)$ on the graph of f^{-1}. Thus, the curve's reflection is shifted *down* the same number of units as the curve itself is shifted to the left. So an expression for the inverse function is $g^{-1}(x) = f^{-1}(x) - c$.

(b) If we compress (or stretch) a curve horizontally, the curve's reflection in the line $y = x$ is compressed (or stretched) *vertically* by the same factor. Using this geometric principle, we see that the inverse of $h(x) = f(cx)$ can be expressed as $h^{-1}(x) = (1/c)\, f^{-1}(x)$.

1 Review

TRUE-FALSE QUIZ

1. False. Let $f(x) = x^2$, $s = -1$, and $t = 1$. Then $f(s+t) = (-1+1)^2 = 0^2 = 0$, but $f(s) + f(t) = (-1)^2 + 1^2 = 2 \ne 0 = f(s+t)$.

2. False. Let $f(x) = x^2$. Then $f(-2) = 4 = f(2)$, but $-2 \ne 2$.

3. False. Let $f(x) = x^2$. Then $f(3x) = (3x)^2 = 9x^2$ and $3f(x) = 3x^2$. So $f(3x) \ne 3f(x)$.

4. True. The inverse function f^{-1} of a one-to-one function f is *defined* by $f^{-1}(y) = x \Leftrightarrow f(x) = y$.

5. True. See the Vertical Line Test.

6. False. Let $f(x) = x^2$ and $g(x) = 2x$. Then $(f \circ g)(x) = f(g(x)) = f(2x) = (2x)^2 = 4x^2$ and $(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2$. So $f \circ g \neq g \circ f$.

7. False. Let $f(x) = x^3$. Then f is one-to-one and $f^{-1}(x) = \sqrt[3]{x}$. But $1/f(x) = 1/x^3$, which is not equal to $f^{-1}(x)$.

8. True. We can divide by e^x since $e^x \neq 0$ for every x.

9. True. The function $\ln x$ is an increasing function on $(0, \infty)$.

10. False. Let $x = e$. Then $(\ln x)^6 = (\ln e)^6 = 1^6 = 1$, but $6 \ln x = 6 \ln e = 6 \cdot 1 = 6 \neq 1 = (\ln x)^6$. What *is* true, however, is that $\ln(x^6) = 6 \ln x$ for $x > 0$.

11. False. Let $x = e^2$ and $a = e$. Then $\dfrac{\ln x}{\ln a} = \dfrac{\ln e^2}{\ln e} = \dfrac{2 \ln e}{\ln e} = 2$ and $\ln \dfrac{x}{a} = \ln \dfrac{e^2}{e} = \ln e = 1$, so in general the statement is false. What *is* true, however, is that $\ln \dfrac{x}{a} = \ln x - \ln a$.

12. False. It is true that $\tan \frac{3\pi}{4} = -1$, but since the range of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we must have $\tan^{-1}(-1) = -\frac{\pi}{4}$.

13. False. For example, $\tan^{-1} 20$ is defined; $\sin^{-1} 20$ and $\cos^{-1} 20$ are not.

14. False. For example, if $x = -3$, then $\sqrt{(-3)^2} = \sqrt{9} = 3$, not -3.

EXERCISES

1. (a) When $x = 2$, $y \approx 2.7$. Thus, $f(2) \approx 2.7$.

(b) $f(x) = 3 \quad \Rightarrow \quad x \approx 2.3, 5.6$

(c) The domain of f is $-6 \leq x \leq 6$, or $[-6, 6]$.

(d) The range of f is $-4 \leq y \leq 4$, or $[-4, 4]$.

(e) f is increasing on $[-4, 4]$, that is, on $-4 \leq x \leq 4$.

(f) f is not one-to-one because it fails the Horizontal Line Test.

(g) f is odd because its graph is symmetric about the origin.

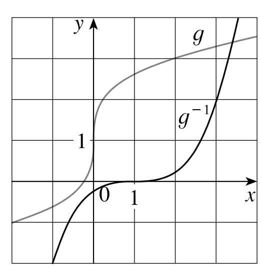
2. (a) When $x = 2$, $y = 3$. Thus, $g(2) = 3$.

(b) g is one-to-one because it passes the Horizontal Line Test.

(c) When $y = 2$, $x \approx 0.2$. So $g^{-1}(2) \approx 0.2$.

(d) The range of g is $[-1, 3.5]$, which is the same as the domain of g^{-1}.

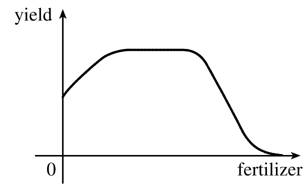
(e) We reflect the graph of g through the line $y = x$ to obtain the graph of g^{-1}.



3. $f(x) = x^2 - 2x + 3$, so $f(a + h) = (a + h)^2 - 2(a + h) + 3 = a^2 + 2ah + h^2 - 2a - 2h + 3$, and

$$\frac{f(a + h) - f(a)}{h} = \frac{(a^2 + 2ah + h^2 - 2a - 2h + 3) - (a^2 - 2a + 3)}{h} = \frac{h(2a + h - 2)}{h} = 2a + h - 2.$$

4. There will be some yield with no fertilizer, increasing yields with increasing fertilizer use, a leveling-off of yields at some point, and disaster with too much fertilizer use.



5. $f(x) = 2/(3x-1)$. Domain: $3x - 1 \neq 0 \quad \Rightarrow \quad 3x \neq 1 \quad \Rightarrow \quad x \neq \frac{1}{3}$. $\quad D = \left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

Range: all reals except 0 ($y = 0$ is the horizontal asymptote for f.)

$R = (-\infty, 0) \cup (0, \infty)$

6. $g(x) = \sqrt{16 - x^4}$. Domain: $16 - x^4 \geq 0 \quad \Rightarrow \quad x^4 \leq 16 \quad \Rightarrow \quad |x| \leq \sqrt[4]{16} \quad \Rightarrow \quad |x| \leq 2$. $\quad D = [-2, 2]$

Range: $y \geq 0$ and $y \leq \sqrt{16} \quad \Rightarrow \quad 0 \leq y \leq 4$.

$R = [0, 4]$

7. $h(x) = \ln(x+6)$. Domain: $x + 6 > 0 \quad \Rightarrow \quad x > -6$. $\quad D = (-6, \infty)$

Range: $x + 6 > 0$, so $\ln(x+6)$ takes on all real numbers and, hence, the range is $\mathbb{R}$.

$R = (-\infty, \infty)$

8. $y = F(t) = 3 + \cos 2t$. Domain: $\mathbb{R}$. $\quad D = (-\infty, \infty)$

Range: $-1 \leq \cos 2t \leq 1 \quad \Rightarrow \quad 2 \leq 3 + \cos 2t \leq 4 \quad \Rightarrow \quad 2 \leq y \leq 4$.

$R = [2, 4]$

9. (a) To obtain the graph of $y = f(x) + 5$, we shift the graph of $y = f(x)$ 5 units upward.

(b) To obtain the graph of $y = f(x+5)$, we shift the graph of $y = f(x)$ 5 units to the left.

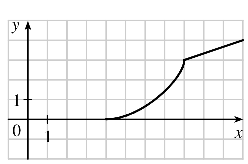
(c) To obtain the graph of $y = 1 + 2f(x)$, we stretch the graph of $y = f(x)$ vertically by a factor of 2, and then shift the resulting graph 1 unit upward.

(d) To obtain the graph of $y = f(x-2) - 2$, we shift the graph of $y = f(x)$ 2 units to the right (for the "-2" inside the parentheses), and then shift the resulting graph 2 units downward.

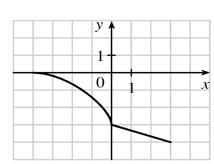
(e) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.

(f) To obtain the graph of $y = f^{-1}(x)$, we reflect the graph of $y = f(x)$ about the line $y = x$ (assuming f is one–to-one).

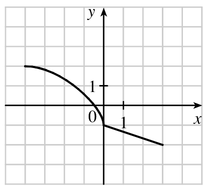
10. (a) To obtain the graph of $y = f(x-8)$, we shift the graph of $y = f(x)$ right 8 units.



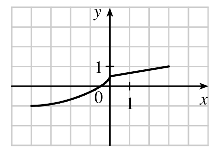
(b) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.



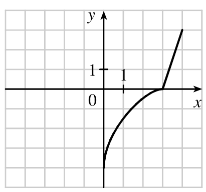
(c) To obtain the graph of $y = 2 - f(x)$, we reflect the graph of $y = f(x)$ about the x-axis, and then shift the resulting graph 2 units upward.



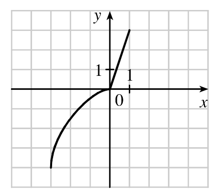
(d) To obtain the graph of $y = \frac{1}{2}f(x) - 1$, we shrink the graph of $y = f(x)$ by a factor of 2, and then shift the resulting graph 1 unit downward.



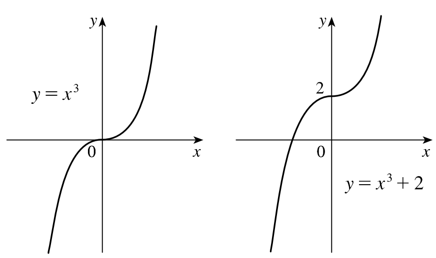
(e) To obtain the graph of $y = f^{-1}(x)$, we reflect the graph of $y = f(x)$ about the line $y = x$.



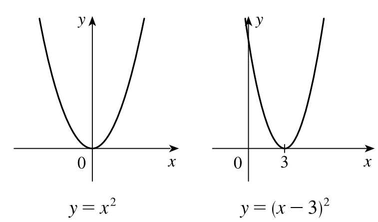
(f) To obtain the graph of $y = f^{-1}(x+3)$, we reflect the graph of $y = f(x)$ about the line $y = x$ [see part (e)], and then shift the resulting graph left 3 units.



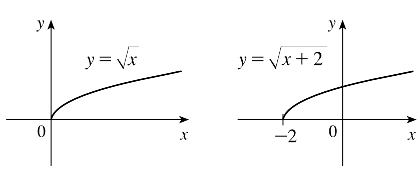
11. $f(x) = x^3 + 2$. Start with the graph of $y = x^3$ and shift 2 units upward.



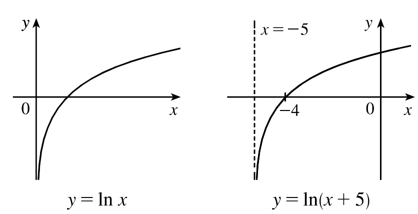
12. $f(x) = (x-3)^2$. Start with the graph of $y = x^2$ and shift 3 units to the right.



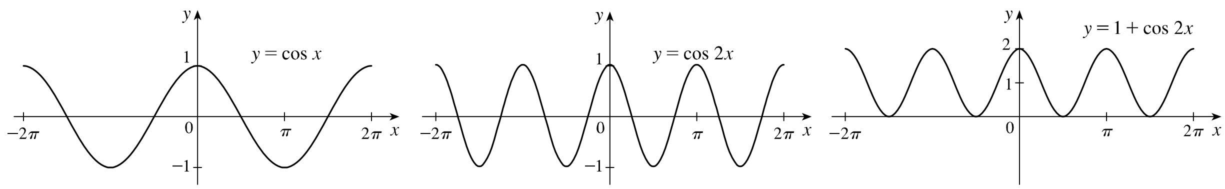
13. $y = \sqrt{x+2}$. Start with the graph of $y = \sqrt{x}$ and shift 2 units to the left.



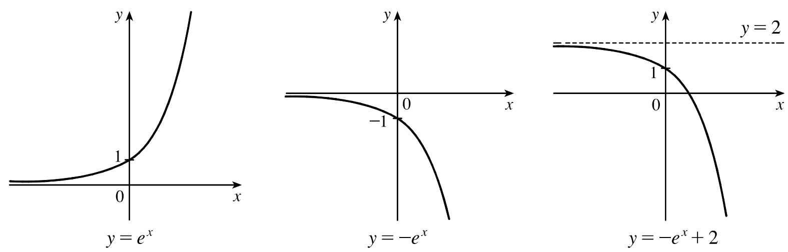
14. $y = \ln(x+5)$. Start with the graph of $y = \ln x$ and shift 5 units to the left.



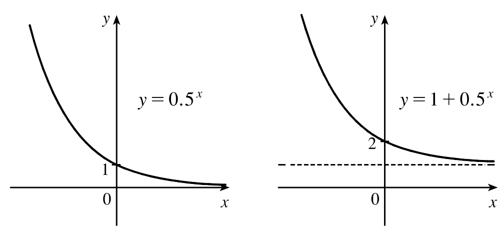
15. $g(x) = 1 + \cos 2x$. Start with the graph of $y = \cos x$, compress horizontally by a factor of 2, and then shift 1 unit upward.



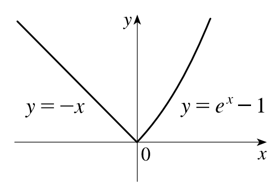
16. $h(x) = -e^x + 2$. Start with the graph of $y = e^x$, reflect about the x-axis, and then shift 2 units upward.



17. $s(x) = 1 + 0.5^x$. Start with the graph of $y = 0.5^x = \left(\frac{1}{2}\right)^x$ and shift 1 unit upward.



18. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ e^x - 1 & \text{if } x \geq 0 \end{cases}$



On $(-\infty, 0)$, graph $y = -x$ (the line with slope -1 and y-intercept 0) with open endpoint $(0, 0)$.

On $[0, \infty)$, graph $y = e^x - 1$ (the graph of $y = e^x$ shifted 1 unit downward) with closed endpoint $(0, 0)$.

19. (a) $f(x) = 2x^5 - 3x^2 + 2 \quad \Rightarrow \quad f(-x) = 2(-x)^5 - 3(-x)^2 + 2 = -2x^5 - 3x^2 + 2$. Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, f is neither even nor odd.

(b) $f(x) = x^3 - x^7 \quad \Rightarrow \quad f(-x) = (-x)^3 - (-x)^7 = -x^3 + x^7 = -(x^3 - x^7) = -f(x)$, so f is odd.

(c) $f(x) = e^{-x^2} \quad \Rightarrow \quad f(-x) = e^{-(-x)^2} = e^{-x^2} = f(x)$, so f is even.

(d) $f(x) = 1 + \sin x \quad \Rightarrow \quad f(-x) = 1 + \sin(-x) = 1 - \sin x$. Now $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd.

(e) $f(x) = 1 - \cos 2x \quad \Rightarrow \quad f(-x) = 1 - \cos[2(-x)] = 1 - \cos(-2x) = 1 - \cos 2x = f(x)$, so f is even.

(f) $f(x) = (x+1)^2 = x^2 + 2x + 1$. Now $f(-x) = (-x)^2 + 2(-x) + 1 = x^2 - 2x + 1$. Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, f is neither even nor odd.

20. For the line segment from $(-2, 2)$ to $(-1, 0)$, the slope is $\dfrac{0-2}{-1+2} = -2$, and an equation is $y - 0 = -2(x+1)$ or, equivalently, $y = -2x - 2$. The circle has equation $x^2 + y^2 = 1$; the top half has equation $y = \sqrt{1 - x^2}$ (we have solved for positive y). Thus, $f(x) = \begin{cases} -2x - 2 & \text{if } -2 \le x \le -1 \\ \sqrt{1-x^2} & \text{if } -1 < x \le 1 \end{cases}$.

21. $f(x) = \ln x, \quad D = (0, \infty); \quad g(x) = x^2 - 9, \quad D = \mathbb{R}$.

(a) $(f \circ g)(x) = f(g(x)) = f(x^2 - 9) = \ln(x^2 - 9)$.

Domain: $x^2 - 9 > 0 \quad \Rightarrow \quad x^2 > 9 \quad \Rightarrow \quad |x| > 3 \quad \Rightarrow \quad x \in (-\infty, -3) \cup (3, \infty)$

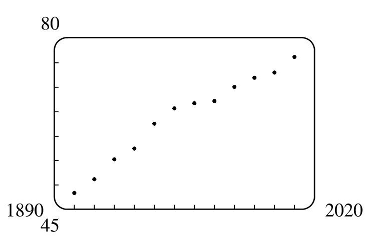
(b) $(g \circ f)(x) = g(f(x)) = g(\ln x) = (\ln x)^2 - 9$. Domain: $x > 0$, or $(0, \infty)$

(c) $(f \circ f)(x) = f(f(x)) = f(\ln x) = \ln \ln x$. Domain: $\ln x > 0 \quad \Rightarrow \quad x > e^0 = 1$, or $(1, \infty)$

(d) $(g \circ g)(x) = g(g(x)) = g(x^2 - 9) = (x^2 - 9)^2 - 9$. Domain: $x \in \mathbb{R}$, or $(-\infty, \infty)$

22. Let $h(x) = x + \sqrt{x}$, $g(x) = \sqrt{x}$, and $f(x) = 1/x$. Then $(f \circ g \circ h)(x) = \dfrac{1}{\sqrt{x + \sqrt{x}}} = F(x)$.

23.

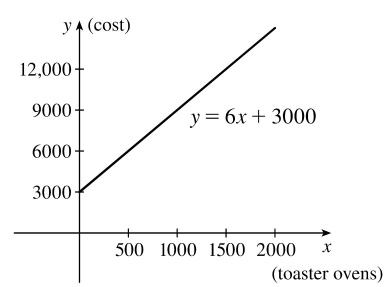


More than one model appears to be plausible. Your choice of model depends on whether you think medical advances will keep increasing life expectancy, or if there is bound to be a natural leveling-off of life expectancy. A linear model, $y = 0.2441x - 413.3960$, gives us an estimate of 82.1 years for the year 2030.

24. (a) Let x denote the number of toaster ovens produced in one week and y the associated cost. Using the points $(1000, 9000)$ and $(1500, 12{,}000)$, we get an equation of a line:

$$y - 9000 = \frac{12{,}000 - 9000}{1500 - 1000}(x - 1000) \quad \Rightarrow$$

$$y = 6(x - 1000) + 9000 \quad \Rightarrow \quad y = 6x + 3000.$$



(b) The slope of 6 means that each additional toaster oven produced adds \$6 to the weekly production cost.

(c) The y-intercept of 3000 represents the overhead cost—the cost incurred without producing anything.

25. The value of x for which $f(x) = 2x + 4^x$ equals 6 will be $f^{-1}(6)$. To solve $2x + 4^x = 6$, we either observe that letting $x = 1$ gives us equality, or we graph $y_1 = 2x + 4^x$ and $y_2 = 6$ to find the intersection at $x = 1$. Since $f(1) = 6$, $f^{-1}(6) = 1$.

26. We write $y = \dfrac{2x+3}{1-5x}$ and solve for x: $y(1-5x) = 2x+3 \;\Rightarrow\; y - 5xy = 2x+3 \;\Rightarrow\; y - 3 = 2x + 5xy \;\Rightarrow$ $y - 3 = x(2+5y) \;\Rightarrow\; x = \dfrac{y-3}{2+5y}$. Interchanging x and y gives $y = \dfrac{x-3}{2+5x}$, so $f^{-1}(x) = \dfrac{x-3}{2+5x}$.

27. (a) $\ln x\sqrt{x+1} = \ln x + \ln\sqrt{x+1}$ [Law 1]

$= \ln x + \ln(x+1)^{1/2} = \ln x + \frac{1}{2}\ln(x+1)$ [Law 3]

(b) $\log_2\sqrt{\dfrac{x^2+1}{x-1}} = \log_2\left(\dfrac{x^2+1}{x-1}\right)^{1/2}$

$= \dfrac{1}{2}\log_2\left(\dfrac{x^2+1}{x-1}\right)$ [Law 3]

$= \frac{1}{2}\left[\log_2(x^2+1) - \log_2(x-1)\right]$ [Law 2]

$= \frac{1}{2}\log_2(x^2+1) - \frac{1}{2}\log_2(x-1)$

28. (a) $\frac{1}{2}\ln x - 2\ln(x^2+1) = \ln x^{1/2} - \ln(x^2+1)^2 = \ln\dfrac{\sqrt{x}}{(x^2+1)^2}$

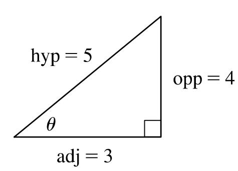
(b) $\ln(x-3) + \ln(x+3) - 2\ln(x^2-9) = \ln[(x-3)(x+3)] - \ln(x^2-9)^2$

$= \ln\dfrac{(x-3)(x+3)}{(x^2-9)^2} = \ln\dfrac{x^2-9}{(x^2-9)^2} = \ln\dfrac{1}{x^2-9}$

29. (a) $e^{2\ln 5} = e^{\ln 5^2} = 5^2 = 25$

(b) $\log_6 4 + \log_6 54 = \log_6(4\cdot 54) = \log_6 216 = \log_6 6^3 = 3$

(c) Let $\theta = \arcsin\frac{4}{5}$, so $\sin\theta = \frac{4}{5}$. Draw a right triangle with angle θ as shown in the figure. By the Pythagorean Theorem, the adjacent side has length 3, and $\tan\left(\arcsin\dfrac{4}{5}\right) = \tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{4}{3}$.



30. (a) $\ln\dfrac{1}{e^3} = \ln e^{-3} = -3$

(b) $\sin(\tan^{-1}1) = \sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$

(c) $10^{-3\log 4} = 10^{\log 4^{-3}} = 4^{-3} = \dfrac{1}{4^3} = \dfrac{1}{64}$

31. $e^{2x} = 3 \;\Rightarrow\; \ln(e^{2x}) = \ln 3 \;\Rightarrow\; 2x = \ln 3 \;\Rightarrow\; x = \frac{1}{2}\ln 3 \approx 0.549$

32. $\ln x^2 = 5 \;\Rightarrow\; e^{\ln x^2} = e^5 \;\Rightarrow\; x^2 = e^5 \;\Rightarrow\; x = \pm\sqrt{e^5} \approx \pm 12.182$

33. $e^{e^x} = 10 \;\Rightarrow\; \ln\left(e^{e^x}\right) = \ln 10 \;\Rightarrow\; e^x = \ln 10 \;\Rightarrow\; \ln e^x = \ln(\ln 10) \;\Rightarrow\; x = \ln(\ln 10) \approx 0.834$

34. $\cos^{-1}x = 2 \;\Rightarrow\; \cos(\cos^{-1}x) = \cos 2 \;\Rightarrow\; x = \cos 2 \approx -0.416$

35. $\tan^{-1}(3x^2) = \dfrac{\pi}{4} \;\Rightarrow\; \tan(\tan^{-1}(3x^2)) = \tan\dfrac{\pi}{4} \;\Rightarrow\; 3x^2 = 1 \;\Rightarrow\; x^2 = \dfrac{1}{3} \;\Rightarrow\; x = \pm\dfrac{1}{\sqrt{3}} \approx \pm 0.577$

36. $\ln x - 1 = \ln(5+x) - 4 \quad\Rightarrow\quad \ln x - \ln(5+x) = -4+1 \quad\Rightarrow\quad \ln\dfrac{x}{5+x} = -3 \quad\Rightarrow\quad e^{\ln(x/(5+x))} = e^{-3} \quad\Rightarrow$

$\dfrac{x}{5+x} = e^{-3} \quad\Rightarrow\quad x = 5e^{-3} + xe^{-3} \quad\Rightarrow\quad x - xe^{-3} = 5e^{-3} \quad\Rightarrow\quad x(1-e^{-3}) = 5e^{-3} \quad\Rightarrow\quad x = \dfrac{5e^{-3}}{1-e^{-3}}$

or, multiplying by $\dfrac{e^3}{e^3}$, we have $x = \dfrac{5}{e^3-1} \approx 0.262$.

37. (a) The half-life of the virus with this treatment is eight days and 24 days is 3 half-lives, so the viral load after 24 days is

$52.0\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 52.0\left(\frac{1}{2}\right)^3 = 6.5$ RNA copies/mL.

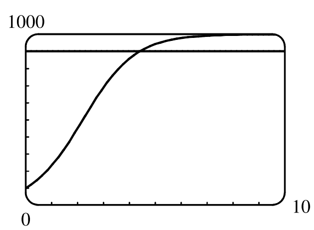
(b) The viral load is halved every $t/8$ days, so $V(t) = 52.0\left(\frac{1}{2}\right)^{t/8}$.

(c) $V = V(t) = 52.0\left(\frac{1}{2}\right)^{t/8} \quad\Rightarrow\quad \dfrac{V}{52.0} = \left(\frac{1}{2}\right)^{t/8} = 2^{-t/8} \quad\Rightarrow\quad \log_2\left(\dfrac{V}{52.0}\right) = \log_2\left(2^{-t/8}\right) = -\dfrac{t}{8} \quad\Rightarrow$

$t = t(V) - 8\log_2\left(\dfrac{V}{52.0}\right)$. This gives the number of days t needed after treatment begins for the viral load to be reduced to V RNA copies/mL.

(d) Using the function from part (c), we have $t(2.0) = -8\log_2\left(\dfrac{2.0}{52.0}\right) = -8 \cdot \dfrac{\ln\frac{1}{26}}{\ln 2} \approx 37.6$ days.

38. (a)



The population would reach 900 in about 4.4 years.

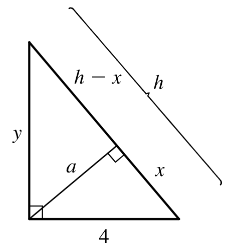
(b) $P = \dfrac{100{,}000}{100 + 900e^{-t}} \Rightarrow\quad 100P + 900Pe^{-t} = 100{,}000 \Rightarrow\quad 900Pe^{-t} = 100{,}000 - 100P \quad\Rightarrow$

$e^{-t} = \dfrac{100{,}000 - 100P}{900P} \quad\Rightarrow\quad -t = \ln\left(\dfrac{1000-P}{9P}\right) \quad\Rightarrow\quad t = -\ln\left(\dfrac{1000-P}{9P}\right)$, or $\ln\left(\dfrac{9P}{1000-P}\right)$;

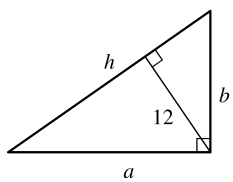
this is the time required for the population to reach a given number P.

(c) $P = 900 \quad\Rightarrow\quad t = \ln\left(\dfrac{9 \cdot 900}{1000-900}\right) = \ln 81 \approx 4.4$ years, as in part (a).

PRINCIPLES OF PROBLEM SOLVING

1. 

By using the area formula for a triangle, $\frac{1}{2}$ (base) (height), in two ways, we see that $\frac{1}{2}(4)(y) = \frac{1}{2}(h)(a)$, so $a = \dfrac{4y}{h}$. Since $4^2 + y^2 = h^2$, $y = \sqrt{h^2 - 16}$, and $a = \dfrac{4\sqrt{h^2-16}}{h}$.

2. 

Refer to Example 1, where we obtained $h = \dfrac{P^2 - 100}{2P}$. The 100 came from 4 times the area of the triangle. In this case, the area of the triangle is $\frac{1}{2}(h)(12) = 6h$. Thus, $h = \dfrac{P^2 - 4\,(6h)}{2P} \;\Rightarrow\; 2Ph = P^2 - 24h \;\Rightarrow\; 2Ph + 24h = P^2 \;\Rightarrow\; h\,(2P + 24) = P^2 \;\Rightarrow\; h = \dfrac{P^2}{2P+24}$.

3. $\big|4x - |x+1|\,\big| = 3 \;\Rightarrow\; 4x - |x+1| = -3$ (Equation 1) or $4x - |x+1| = 3$ (Equation 2).

If $x + 1 < 0$, or $x < -1$, then $|x+1| = -(x+1) = -x - 1$. If $x + 1 \geq 0$, or $x \geq -1$, then $|x+1| = x + 1$.

We thus consider two cases, $x < -1$ (Case 1) and $x \geq -1$ (Case 2), for each of Equations 1 and 2.

Equation 1, Case 1: $4x - |x+1| = -3 \;\Rightarrow\; 4x - (-x-1) = -3 \;\Rightarrow\; 5x + 1 = -3 \;\Rightarrow\; 5x = -4 \;\Rightarrow\; x = -\frac{4}{5}$ which is invalid since $x < -1$.

Equation 1, Case 2: $4x - |x+1| = -3 \;\Rightarrow\; 4x - (x-1) = -3 \;\Rightarrow\; 3x - 1 = -3 \;\Rightarrow\; 3x = -2 \;\Rightarrow\; x = -\frac{2}{3}$, which is valid since $x \geq -1$.

Equation 2, Case 1: $4x - |x+1| = 3 \;\Rightarrow\; 4x - (-x-1) = 3 \;\Rightarrow\; 5x + 1 = 3 \;\Rightarrow\; 5x = 2 \;\Rightarrow\; x = \frac{2}{5}$, which is invalid since $x < -1$.

Equation 2, Case 2: $4x - |x+1| = 3 \;\Rightarrow\; 4x - (x+1) = 3 \;\Rightarrow\; 3x - 1 = 3 \;\Rightarrow\; 3x = 4 \;\Rightarrow\; x = \frac{4}{3}$, which is valid since $x \geq -1$.

Thus, the solution set is $\left\{-\frac{2}{3}, \frac{4}{3}\right\}$.

4. $|x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}$ and $|x-3| = \begin{cases} x-3 & \text{if } x \geq 3 \\ 3-x & \text{if } x < 3 \end{cases}$

Therefore, we consider the three cases $x < 1$, $1 \leq x < 3$, and $x \geq 3$.

If $x < 1$, we must have $1 - x - (3 - x) \geq 5 \;\Leftrightarrow\; 0 \geq 7$, which is false.

If $1 \leq x < 3$, we must have $x - 1 - (3 - x) \geq 5 \;\Leftrightarrow\; x \geq \frac{9}{2}$, which is false because $x < 3$.

If $x \geq 3$, we must have $x - 1 - (x - 3) \geq 5 \;\Leftrightarrow\; 2 \geq 5$, which is false.

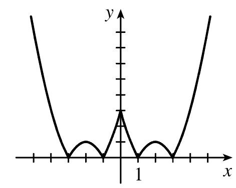
All three cases lead to falsehoods, so the inequality has no solution.

5. $f(x) = \left|x^2 - 4\,|x| + 3\right|$. If $x \geq 0$, then $f(x) = \left|x^2 - 4x + 3\right| = |(x-1)(x-3)|$.

Case (i): If $0 < x \leq 1$, then $f(x) = x^2 - 4x + 3$.

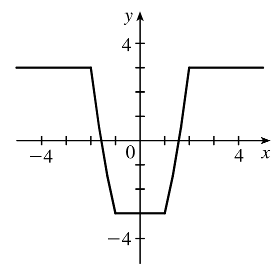
Case (ii): If $1 < x \leq 3$, then $f(x) = -(x^2 - 4x + 3) = -x^2 + 4x - 3$.

Case (iii): If $x > 3$, then $f(x) = x^2 - 4x + 3$.



This enables us to sketch the graph for $x \geq 0$. Then we use the fact that f is an even function to reflect this part of the graph about the y-axis to obtain the entire graph. Or, we could consider also the cases $x < -3$, $-3 \leq x < -1$, and $-1 \leq x < 0$.

6. $g(x) = \left|x^2 - 1\right| - \left|x^2 - 4\right|$.



$$\left|x^2 - 1\right| = \begin{cases} x^2 - 1 & \text{if } |x| \geq 1 \\ 1 - x^2 & \text{if } |x| < 1 \end{cases} \quad \text{and} \quad \left|x^2 - 4\right| = \begin{cases} x^2 - 4 & \text{if } |x| \geq 2 \\ 4 - x^2 & \text{if } |x| < 2 \end{cases}$$

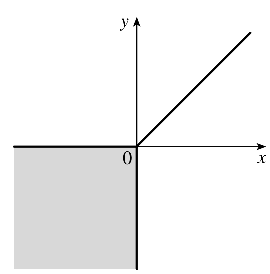
So for $0 \leq |x| < 1$, $g(x) = 1 - x^2 - (4 - x^2) = -3$, for $1 \leq |x| < 2$, $g(x) = x^2 - 1 - (4 - x^2) = 2\,x^2 - 5$, and for $|x| \geq 2$, $g(x) = x^2 - 1 - (x^2 - 4) = 3$.

7. Remember that $|a| = a$ if $a \geq 0$ and that $|a| = -a$ if $a < 0$. Thus,

$$x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad \text{and} \qquad y + |y| = \begin{cases} 2y & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

We will consider the equation $x + |x| = y + |y|$ in four cases.

(1) $x \geq 0, y \geq 0$	(2) $x \geq 0, y < 0$	(3) $x < 0, y \geq 0$	(4) $x < 0, y < 0$
$2x = 2y$	$2x = 0$	$0 = 2y$	$0 = 0$
$x = y$	$x = 0$	$0 = y$	



Case 1 gives us the line $y = x$ with nonnegative x and y.

Case 2 gives us the portion of the y-axis with y negative.

Case 3 gives us the portion of the x-axis with x negative.

Case 4 gives us the entire third quadrant.

8. $|x - y| + |x| - |y| \leq 2$ [call this inequality ($\star$)]

Case (i): $x \geq y \geq 0$. Then ($\star$) $\Leftrightarrow$ $x - y + x - y \leq 2$ $\Leftrightarrow$ $x - y \leq 1$ $\Leftrightarrow$ $y \geq x - 1$.

Case (ii): $y \geq x \geq 0$. Then ($\star$) $\Leftrightarrow$ $y - x + x - y \leq 2$ $\Leftrightarrow$ $0 \leq 2$ (true).

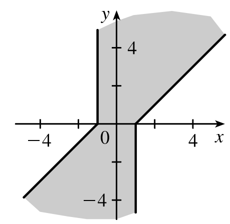
Case (iii): $x \geq 0$ and $y \leq 0$. Then ($\star$) $\Leftrightarrow$ $x - y + x + y \leq 2$ $\Leftrightarrow$ $2x \leq 2$ $\Leftrightarrow$ $x \leq 1$.

Case (iv): $x \leq 0$ and $y \geq 0$. Then ($\star$) $\Leftrightarrow$ $y - x - x - y \leq 2$ $\Leftrightarrow$ $-2x \leq 2$ $\Leftrightarrow$ $x \geq -1$.

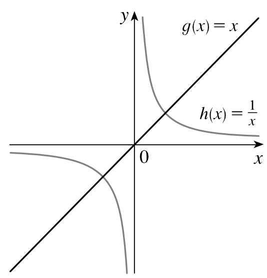
Case (v): $y \leq x \leq 0$. Then ($\star$) $\Leftrightarrow$ $x - y - x + y \leq 2$ $\Leftrightarrow$ $0 \leq 2$ (true).

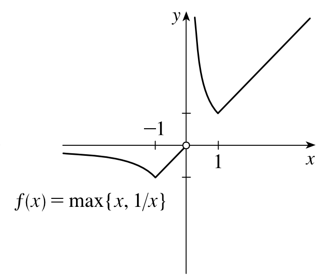
Case (vi): $x \leq y \leq 0$. Then ($\star$) $\Leftrightarrow$ $y - x - x + y \leq 2$ $\Leftrightarrow$ $y - x \leq 1$ $\Leftrightarrow$ $y \leq x + 1$.

Note: Instead of considering cases (iv), (v), and (vi), we could have noted that the region is unchanged if x and y are replaced by $-x$ and $-y$, so the region is symmetric about the origin. Therefore, we need only draw cases (i), (ii), and (iii), and rotate through $180°$ about the origin.

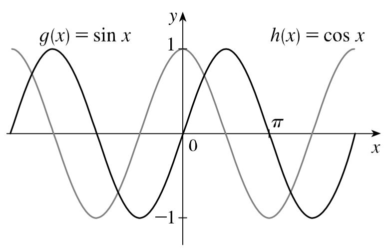


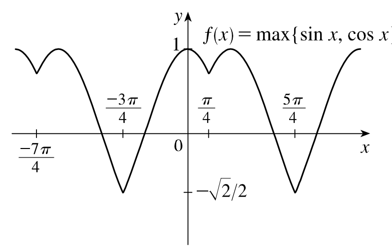
9. (a) To sketch the graph of $f(x) = \max\{x, 1/x\}$, we first graph $g(x) = x$ and $h(x) = 1/x$ on the same coordinate axes. Then create the graph of f by plotting the largest y-value of g and h for every value of x.



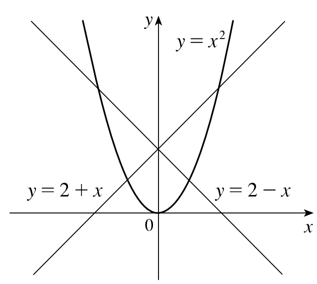


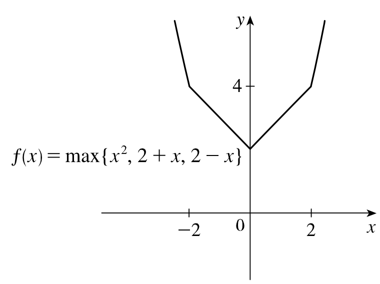
(b)





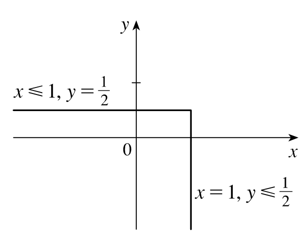
(c)



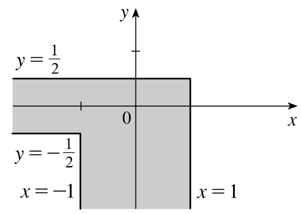


On the TI-84 Plus, max is found under `LIST`, then under `MATH`. To graph $f(x) = \max\left\{x^2, 2+x, 2-x\right\}$, use `Y` $= \max(x^2, \max(2+x, 2-x))$.

10. (a) If $\max\{x, 2y\} = 1$, then either $x = 1$ *and* $2y \le 1$ ***or*** $x \le 1$ *and* $2y = 1$. Thus, we obtain the set of points such that $x = 1$ *and* $y \le \frac{1}{2}$ [a vertical line with highest point $(1, \frac{1}{2})$] ***or*** $x \le 1$ *and* $y = \frac{1}{2}$ [a horizontal line with rightmost point $(1, \frac{1}{2})$].

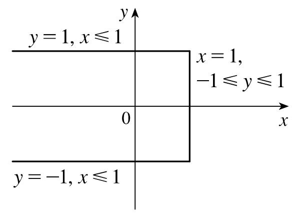


(b) The graph of $\max\{x, 2y\} = 1$ is shown in part (a), and the graph of $\max\{x, 2y\} = -1$ can be found in a similar manner. The inequalities in $-1 \le \max\{x, 2y\} \le 1$ give us all the points on or inside the boundaries.



(c) $\max\{x, y^2\} = 1 \quad \Leftrightarrow \quad x = 1 \ \textit{and} \ y^2 \le 1 \ [-1 \le y \le 1]$

$\textbf{\textit{or}} \ \ x \le 1 \ \textit{and} \ y^2 = 1 \ [y = \pm 1]$.



11. $$\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_5 x} = \frac{1}{\dfrac{\log x}{\log 2}} + \frac{1}{\dfrac{\log x}{\log 3}} + \frac{1}{\dfrac{\log x}{\log 5}} \qquad \text{[Change of Base formula]}$$

$$= \frac{\log 2}{\log x} + \frac{\log 3}{\log x} + \frac{\log 5}{\log x}$$

$$= \frac{\log 2 + \log 3 + \log 5}{\log x} = \frac{\log(2 \cdot 3 \cdot 5)}{\log x} \qquad \text{[Law 1 of Lograithms]}$$

$$= \frac{\log 30}{\log x} = \frac{1}{\dfrac{\log x}{\log 30}} = \frac{1}{\log_{30} x} \qquad \text{[Change of Base formula]}$$

12. We note that $-1 \le \sin x \le 1$ for all x. Thus, any solution of $\sin x = x/100$ will have $-1 \le x/100 \le 1$, or $-100 \le x \le 100$. We next observe that the period of $\sin x$ is 2π, and $\sin x$ takes on each value in its range, except for -1 and 1, twice each cycle. We observe that $x = 0$ is a solution. Finally, we note that because $\sin x$ and $x/100$ are both odd functions, every solution on $0 \le x \le 100$ gives us a corresponding solution on $-100 \le x \le 0$.

$100/2\pi \approx 15.9$, so there 15 full cycles of $\sin x$ on $[0, 100]$ .Each of the 15 intervals $[0, 2\pi]$, $[2\pi, 4\pi]$, $\ldots$, $[28\pi, 30\pi]$ must contain two solutions of $\sin x = x/100$, as the graph of $\sin x$ will intersect the graph of $x/100$ twice each cycle. We must be careful with the next (16th) interval $[30\pi, 32\pi]$, because 100 is contained in the interval. A graph of $y_1 = \sin x$ and $y_2 = x/100$ over this interval reveals that two intersections occur within the interval with $x \le 100$.

Thus, there are $16 \cdot 2 = 32$ solutions of $\sin x = x/100$ on $[0, 100]$. There are also 32 solutions of the equation on $[-100, 0]$. Being careful to not count the solution $x = 0$ twice, we find that there are $32 + 32 - 1 = 63$ solutions of the equation $\sin x = x/100$.

13. By rearranging terms, we write the given expression as

$$\left(\sin\frac{\pi}{100}+\sin\frac{199\pi}{100}\right)+\left(\sin\frac{2\pi}{100}+\sin\frac{198\pi}{100}\right)+\cdots+\left(\sin\frac{99\pi}{100}+\sin\frac{101\pi}{100}\right)+\sin\frac{100\pi}{100}+\sin\frac{200\pi}{100}$$

Each grouped sum is of the form $\sin x+\sin y$ with $x+y=2\pi$ so that $\dfrac{x+y}{2}=\dfrac{2\pi}{2}=\pi$. We now derive a useful identity from the product-to-sum identity $\sin x\,\cos y=\frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$. If in this identity we replace x with $\dfrac{x+y}{2}$ and y with $\dfrac{x-y}{2}$, we have

$$\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)=\frac{1}{2}\left[\sin\left(\frac{x+y}{2}+\frac{x-y}{2}\right)+\sin\left(\frac{x+y}{2}-\frac{x-y}{2}\right)\right]=\frac{1}{2}(\sin x+\sin y)$$

Multiplication of the left and right members of this equality by 2 gives the sum-to-product identity $\sin x+\sin y=2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x+y}{2}\right)$. Using this sum-to-product identity, we have each grouped sum equal to 0, since $\sin\left(\dfrac{x+y}{2}\right)=\sin\pi=0$ is always a factor of the right side. Since $\sin\dfrac{100\pi}{100}=\sin\pi=0$ and $\sin\dfrac{200\pi}{100}=\sin 2\pi=0$, the sum of the given expression is 0.

Another approach: Since the sine function is odd, $\sin(-x)=-\sin x$. Because the period of the sine function is 2π, we have $\sin(-x+2\pi)=-\sin x$. Multiplying each side by -1 and rearranging, we have $\sin x=-\sin(2\pi-x)$. This means that $\sin\dfrac{\pi}{100}=-\sin\left(2\pi-\dfrac{\pi}{100}\right)=-\sin\dfrac{199\pi}{100}$, $\sin\dfrac{2\pi}{100}=-\sin\left(2\pi-\dfrac{2\pi}{100}\right)=\sin\dfrac{198\pi}{100}$, and so on, until we have $\sin\dfrac{99\pi}{100}=-\sin\left(2\pi-\dfrac{99\pi}{100}\right)=-\sin\dfrac{101\pi}{100}$. As before we rearrange terms to write the given expression as

$$\left(\sin\frac{\pi}{100}+\sin\frac{199\pi}{100}\right)+\left(\sin\frac{2\pi}{100}+\sin\frac{198\pi}{100}\right)+\cdots+\left(\sin\frac{99\pi}{100}+\sin\frac{101\pi}{100}\right)+\sin\frac{100\pi}{100}+\sin\frac{200\pi}{100}$$

Each sum in parentheses is 0 since the two terms are opposites, and the last two terms again reduce to $\sin\pi$ and $\sin 2\pi$, respectively, each also 0. Thus, the value of the original expression is 0.

14. (a) $$f(-x)=\ln\left(-x+\sqrt{(-x)^2+1}\right)=\ln\left(-x+\sqrt{x^2+1}\cdot\frac{-x-\sqrt{x^2+1}}{-x-\sqrt{x^2+1}}\right)$$
$$=\ln\left(\frac{x^2-(x^2+1)}{-x-\sqrt{x^2+1}}\right)=\ln\left(\frac{-1}{-x-\sqrt{x^2+1}}\right)=\ln\left(\frac{1}{x+\sqrt{x^2+1}}\right)$$
$$=\ln 1-\ln\left(x+\sqrt{x^2+1}\right)=-\ln\left(x+\sqrt{x^2-1}\right)=-f(x)$$

(b) $y=\ln\left(x+\sqrt{x^2+1}\right)$. Interchanging x and y, we get $x=\ln\left(y+\sqrt{y^2+1}\right)\ \Rightarrow\ e^x=y+\sqrt{y^2+1}\ \Rightarrow$

$$e^x-y=\sqrt{y^2+1}\ \Rightarrow\ e^{2x}-2ye^x+y^2=y^2+1\ \Rightarrow\ e^{2x}-1=2ye^x\ \Rightarrow\ y=\frac{e^{2x}-1}{2e^x}=f^{-1}(x).$$

15. $\ln\left(x^2 - 2x - 2\right) \le 0 \quad\Rightarrow\quad x^2 - 2x - 2 \le e^0 = 1 \quad\Rightarrow\quad x^2 - 2x - 3 \le 0 \quad\Rightarrow\quad (x-3)(x+1) \le 0 \quad\Rightarrow\quad x \in [-1, 3]$.

Since the argument must be positive, $x^2 - 2x - 2 > 0 \quad\Rightarrow\quad \left[x - \left(1 - \sqrt{3}\right)\right]\left[x - \left(1 + \sqrt{3}\right)\right] > 0 \quad\Rightarrow$

$x \in \left(-\infty, 1 - \sqrt{3}\right) \cup \left(1 + \sqrt{3}, \infty\right)$. The intersection of these intervals is $\left[-1, 1 - \sqrt{3}\right) \cup \left(1 + \sqrt{3}, 3\right]$.

16. Assume that $\log_2 5$ is rational. Then $\log_2 5 = m/n$ for natural numbers m and n. Changing to exponential form gives us $2^{m/n} = 5$ and then raising both sides to the nth power gives $2^m = 5^n$. But 2^m is even and 5^n is odd. We have arrived at a contradiction, so we conclude that our hypothesis, that $\log_2 5$ is rational, is false. Thus, $\log_2 5$ is irrational.

17. Let d be the distance traveled on each half of the trip. Let t_1 and t_2 be the times taken for the first and second halves of the trip. For the first half of the trip we have $t_1 = d/30$ and for the second half we have $t_2 = d/60$. Thus, the average speed for the entire trip is $\dfrac{\text{total distance}}{\text{total time}} = \dfrac{2d}{t_1 + t_2} = \dfrac{2d}{\dfrac{d}{30} + \dfrac{d}{60}} \cdot \dfrac{60}{60} = \dfrac{120d}{2d + d} = \dfrac{120d}{3d} = 40$. The average speed for the entire trip is 40 mi/h.

18. Let $f(x) = \sin x$, $g(x) = x$, and $h(x) = x$. Then the left-hand side of the equation is

$[f \circ (g + h)](x) = \sin(x + x) = \sin 2x = 2 \sin x \, \cos x$; and the right-hand side is

$(f \circ g)(x) + (f \circ h)(x) = \sin x + \sin x = 2 \sin x$. The two sides are not equal, so the given statement is false.

19. Let S_n be the statement that $7^n - 1$ is divisible by 6.

- S_1 is true because $7^1 - 1 = 6$ is divisible by 6.
- Assume S_k is true, that is, $7^k - 1$ is divisible by 6. In other words, $7^k - 1 = 6m$ for some positive integer m. Then $7^{k+1} - 1 = 7^k \cdot 7 - 1 = (6m + 1) \cdot 7 - 1 = 42m + 6 = 6(7m + 1)$, which is divisible by 6, so S_{k+1} is true.
- Therefore, by mathematical induction, $7^n - 1$ is divisible by 6 for every positive integer n.

20. Let S_n be the statement that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

- S_1 is true because $[2(1) - 1] = 1 = 1^2$.
- Assume S_k is true, that is, $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. Then

  $1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] = 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2$

  which shows that S_{k+1} is true.
- Therefore, by mathematical induction, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for every positive integer n.

21. $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$.

$f_1(x) = f_0(f_0(x)) = f_0\left(x^2\right) = \left(x^2\right)^2 = x^4$, $f_2(x) = f_0(f_1(x)) = f_0(x^4) = (x^4)^2 = x^8$,

$f_3(x) = f_0(f_2(x)) = f_0(x^8) = (x^8)^2 = x^{16}, \ldots$. Thus, a general formula is $f_n(x) = x^{2^{n+1}}$.

22. (a) $f_0(x) = 1/(2-x)$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$.

$$f_1(x) = f_0\left(\frac{1}{2-x}\right) = \frac{1}{2 - \dfrac{1}{2-x}} = \frac{2-x}{2(2-x)-1} = \frac{2-x}{3-2x},$$

$$f_2(x) = f_0\left(\frac{2-x}{3-2x}\right) = \frac{1}{2 - \dfrac{2-x}{3-2x}} = \frac{3-2x}{2(3-2x)-(2-x)} = \frac{3-2x}{4-3x},$$

$$f_3(x) = f_0\left(\frac{3-2x}{4-3x}\right) = \frac{1}{2 - \dfrac{3-2x}{4-3x}} = \frac{4-3x}{2(4-3x)-(3-2x)} = \frac{4-3x}{5-4x}, \ldots$$

Thus, we conjecture that the general formula is $f_n(x) = \dfrac{n+1-nx}{n+2-(n+1)x}$.

To prove this, we use the Principle of Mathematical Induction. We have already verified that f_n is true for $n = 1$.

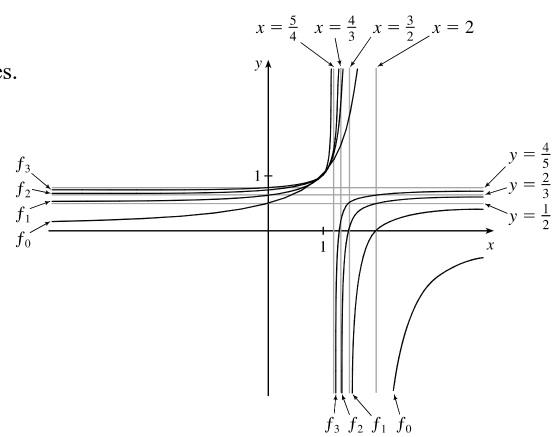
Assume that the formula is true for $n = k$; that is, $f_k(x) = \dfrac{k+1-kx}{k+2-(k+1)x}$. Then

$$\begin{aligned} f_{k+1}(x) = (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{k+1-kx}{k+2-(k+1)x}\right) &= \frac{1}{2 - \dfrac{k+1-kx}{k+2-(k+1)x}} \\ = \frac{k+2-(k+1)x}{2\,[k+2-(k+1)x] - (k+1-kx)} &= \frac{k+2-(k+1)x}{k+3-(k+2)x} \end{aligned}$$

This shows that the formula for f_n is true for $n = k+1$. Therefore, by mathematical induction, the formula is true for all positive integers n.

(b) From the graph, we can make several observations:

- The values at each fixed $x = a$ keep increasing as n increases.
- The vertical asymptote gets closer to $x = 1$ as n increases.
- The horizontal asymptote gets closer to $y = 1$ as n increases.
- The x-intercept for f_{n+1} is the value of the vertical asymptote for f_n.
- The y-intercept for f_n is the value of the horizontal asymptote for f_{n+1}.



2 ☐ LIMITS AND DERIVATIVES

2.1 The Tangent and Velocity Problems

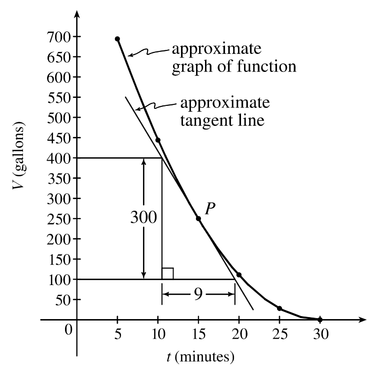
1. (a) Using $P(15, 250)$, we construct the following table:

t	Q	slope $= m_{PQ}$
5	$(5, 694)$	$\frac{694-250}{5-15} = -\frac{444}{10} = -44.4$
10	$(10, 444)$	$\frac{444-250}{10-15} = -\frac{194}{5} = -38.8$
20	$(20, 111)$	$\frac{111-250}{20-15} = -\frac{139}{5} = -27.8$
25	$(25, 28)$	$\frac{28-250}{25-15} = -\frac{222}{10} = -22.2$
30	$(30, 0)$	$\frac{0-250}{30-15} = -\frac{250}{15} = -16.\overline{6}$

(b) Using the values of t that correspond to the points closest to P ($t = 10$ and $t = 20$), we have

$$\frac{-38.8 + (-27.8)}{2} = -33.3$$

(c) From the graph, we can estimate the slope of the tangent line at P to be $\frac{-300}{9} = -33.\overline{3}$.



2. (a) (i) On the interval $[0, 40]$, slope $= \dfrac{7398 - 3438}{40 - 0} = 99$.

(ii) On the interval $[10, 20]$, slope $= \dfrac{5622 - 4559}{20 - 10} = 106.3$.

(iii) On the interval $[20, 30]$, slope $= \dfrac{6536 - 5622}{30 - 20} = 91.4$.

The slopes represent the average number of steps per minute the student walked during the respective time intervals.

(b) Averaging the slopes of the secant lines corresponding to the intervals immediately before and after $t = 20$, we have

$$\frac{106.3 + 91.4}{2} = 98.85$$

The student's walking pace is approximately 99 steps per minute at 3:20 PM.

3. (a) $y = \dfrac{1}{1-x}$, $P(2,-1)$

	x	$Q(x, 1/(1-x))$	m_{PQ}
(i)	1.5	$(1.5, -2)$	2
(ii)	1.9	$(1.9, -1.111\,111)$	1.111 111
(iii)	1.99	$(1.99, -1.010\,101)$	1.010 101
(iv)	1.999	$(1.999, -1.001\,001)$	1.001 001
(v)	2.5	$(2.5, -0.666\,667)$	0.666 667
(vi)	2.1	$(2.1, -0.909\,091)$	0.909 091
(vii)	2.01	$(2.01, -0.990\,099)$	0.990 099
(viii)	2.001	$(2.001, -0.999\,001)$	0.999 001

(b) The slope appears to be 1.

(c) Using $m = 1$, an equation of the tangent line to the curve at $P(2, -1)$ is $y - (-1) = 1(x - 2)$, or $y = x - 3$.

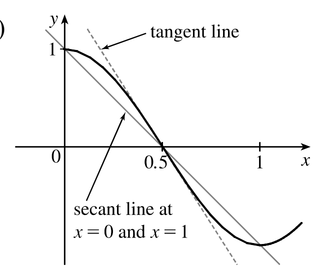
4. (a) $y = \cos \pi x$, $P(0.5, 0)$

	x	Q	m_{PQ}
(i)	0	$(0, 1)$	-2
(ii)	0.4	$(0.4, 0.309017)$	-3.090170
(iii)	0.49	$(0.49, 0.031411)$	-3.141076
(iv)	0.499	$(0.499, 0.003142)$	-3.141587
(v)	1	$(1, -1)$	-2
(vi)	0.6	$(0.6, -0.309017)$	-3.090170
(vii)	0.51	$(0.51, -0.031411)$	-3.141076
(viii)	0.501	$(0.501, -0.003142)$	-3.141587

(b) The slope appears to be $-\pi$.

(c) $y - 0 = -\pi(x - 0.5)$ or $y = -\pi x + \frac{1}{2}\pi$.

(d)



5. (a) $y = y(t) = 275 - 16t^2$. At $t = 4$, $y = 275 - 16(4)^2 = 19$. The average velocity between times 4 and $4 + h$ is

$$v_{\text{avg}} = \frac{y(4+h) - y(4)}{(4+h) - 4} = \frac{\left[275 - 16(4+h)^2\right] - 19}{h} = \frac{-128h - 16h^2}{h} = -128 - 16h \quad \text{if } h \neq 0$$

(i) 0.1 seconds: $h = 0.1$, $v_{\text{avg}} = -129.6$ ft/s

(ii) 0.05 seconds: $h = 0.05$, $v_{\text{avg}} = -128.8$ ft/s

(iii) 0.01 seconds: $h = 0.01$, $v_{\text{avg}} = -128.16$ ft/s

(b) The instantaneous velocity when $t = 4$ (h approaches 0) is -128 ft/s.

6. (a) $y = y(t) = 10t - 1.86t^2$. At $t = 1$, $y = 10(1) - 1.86(1)^2 = 8.14$. The average velocity between times 1 and $1 + h$ is

$$v_{\text{avg}} = \frac{y(1+h) - y(1)}{(1+h) - 1} = \frac{\left[10(1+h) - 1.86(1+h)^2\right] - 8.14}{h} = \frac{6.28h - 1.86h^2}{h} = 6.28 - 1.86h, \text{ if } h \neq 0.$$

(i) $[1, 2]$: $h = 1$, $v_{\text{avg}} = 4.42$ m/s

(ii) $[1, 1.5]$: $h = 0.5$, $v_{\text{avg}} = 5.35$ m/s

(iii) $[1, 1.1]$: $h = 0.1$, $v_{\text{avg}} = 6.094$ m/s

(iv) $[1, 1.01]$: $h = 0.01$, $v_{\text{avg}} = 6.2614$ m/s

(v) $[1, 1.001]$: $h = 0.001$, $v_{\text{avg}} = 6.27814$ m/s

(b) The instantaneous velocity when $t = 1$ (h approaches 0) is 6.28 m/s.

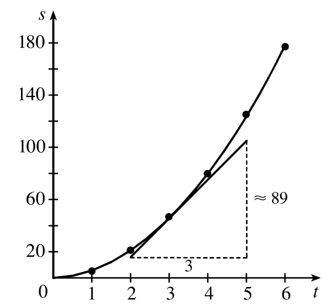
7. (a) (i) On the interval $[2,4]$, $v_{\text{avg}} = \dfrac{s(4)-s(2)}{4-2} = \dfrac{79.2-20.6}{2} = 29.3$ ft/s.

(ii) On the interval $[3,4]$, $v_{\text{avg}} = \dfrac{s(4)-s(3)}{4-3} = \dfrac{79.2-46.5}{1} = 32.7$ ft/s.

(iii) On the interval $[4,5]$, $v_{\text{avg}} = \dfrac{s(5)-s(4)}{5-4} = \dfrac{124.8-79.2}{1} = 45.6$ ft/s.

(iv) On the interval $[4,6]$, $v_{\text{avg}} = \dfrac{s(6)-s(4)}{6-4} = \dfrac{176.7-79.2}{2} = 48.75$ ft/s.

(b) Using the points $(2,16)$ and $(5,105)$ from the approximate tangent line, the instantaneous velocity at $t=3$ is about $\dfrac{105-16}{5-2} = \dfrac{89}{3} \approx 29.7$ ft/s.



8. (a) (i) $s = s(t) = 2\sin\pi t + 3\cos\pi t$. On the interval $[1,2]$, $v_{\text{avg}} = \dfrac{s(2)-s(1)}{2-1} = \dfrac{3-(-3)}{1} = 6$ cm/s.

(ii) On the interval $[1,1.1]$, $v_{\text{avg}} = \dfrac{s(1.1)-s(1)}{1.1-1} \approx \dfrac{-3.471-(-3)}{0.1} = -4.71$ cm/s.

(iii) On the interval $[1,1.01]$, $v_{\text{avg}} = \dfrac{s(1.01)-s(1)}{1.01-1} \approx \dfrac{-3.0613-(-3)}{0.01} = -6.13$ cm/s.

(iv) On the interval $[1,1.001]$, $v_{\text{avg}} = \dfrac{s(1.001)-s(1)}{1.001-1} \approx \dfrac{-3.00627-(-3)}{0.001} = -6.27$ cm/s.

(b) The instantaneous velocity of the particle when $t=1$ appears to be about -6.3 cm/s.

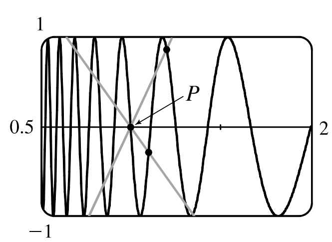
9. (a) For the curve $y = \sin(10\pi/x)$ and the point $P(1,0)$:

x	Q	m_{PQ}
2	$(2,0)$	0
1.5	$(1.5, 0.8660)$	1.7321
1.4	$(1.4, -0.4339)$	-1.0847
1.3	$(1.3, -0.8230)$	-2.7433
1.2	$(1.2, 0.8660)$	4.3301
1.1	$(1.1, -0.2817)$	-2.8173

x	Q	m_{PQ}
0.5	$(0.5, 0)$	0
0.6	$(0.6, 0.8660)$	-2.1651
0.7	$(0.7, 0.7818)$	-2.6061
0.8	$(0.8, 1)$	-5
0.9	$(0.9, -0.3420)$	3.4202

As x approaches 1, the slopes do not appear to be approaching any particular value.

(b)



We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at P that we need to take x-values much closer to 1 in order to get accurate estimates of its slope.

(c) If we choose $x = 1.001$, then the point Q is $(1.001, -0.0314)$ and $m_{PQ} \approx -31.3794$. If $x = 0.999$, then Q is $(0.999, 0.0314)$ and $m_{PQ} = -31.4422$. The average of these slopes is -31.4108. So we estimate that the slope of the tangent line at P is about -31.4.

2.2 The Limit of a Function

1. As x approaches 2, $f(x)$ approaches 5. [Or, the values of $f(x)$ can be made as close to 5 as we like by taking x sufficiently close to 2 (but $x \neq 2$).] Yes, the graph could have a hole at $(2, 5)$ and be defined such that $f(2) = 3$.

2. As x approaches 1 from the left, $f(x)$ approaches 3; and as x approaches 1 from the right, $f(x)$ approaches 7. No, the limit does not exist because the left- and right-hand limits are different.

3. (a) $\lim\limits_{x \to -3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).

(b) $\lim\limits_{x \to 4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.

4. (a) As x approaches 2 from the left, the values of $f(x)$ approach 3, so $\lim\limits_{x \to 2^-} f(x) = 3$.

(b) As x approaches 2 from the right, the values of $f(x)$ approach 1, so $\lim\limits_{x \to 2^+} f(x) = 1$.

(c) $\lim\limits_{x \to 2} f(x)$ does not exist since the left-hand limit does not equal the right-hand limit.

(d) When $x = 2$, $y = 3$, so $f(2) = 3$.

(e) As x approaches 4, the values of $f(x)$ approach 4, so $\lim\limits_{x \to 4} f(x) = 4$.

(f) There is no value of $f(x)$ when $x = 4$, so $f(4)$ does not exist.

5. (a) As x approaches 1, the values of $f(x)$ approach 2, so $\lim\limits_{x \to 1} f(x) = 2$.

(b) As x approaches 3 from the left, the values of $f(x)$ approach 1, so $\lim\limits_{x \to 3^-} f(x) = 1$.

(c) As x approaches 3 from the right, the values of $f(x)$ approach 4, so $\lim\limits_{x \to 3^+} f(x) = 4$.

(d) $\lim\limits_{x \to 3} f(x)$ does not exist since the left-hand limit does not equal the right-hand limit.

(e) When $x = 3$, $y = 3$, so $f(3) = 3$.

6. (a) $h(x)$ approaches 4 as x approaches -3 from the left, so $\lim\limits_{x \to -3^-} h(x) = 4$.

(b) $h(x)$ approaches 4 as x approaches -3 from the right, so $\lim\limits_{x \to -3^+} h(x) = 4$.

(c) $\lim_{x\to -3} h(x) = 4$ because the limits in part (a) and part (b) are equal.

(d) $h(-3)$ is not defined, so it doesn't exist.

(e) $h(x)$ approaches 1 as x approaches 0 from the left, so $\lim_{x\to 0^-} h(x) = 1$.

(f) $h(x)$ approaches -1 as x approaches 0 from the right, so $\lim_{x\to 0^+} h(x) = -1$.

(g) $\lim_{x\to 0} h(x)$ does not exist because the limits in part (e) and part (f) are not equal.

(h) $h(0) = 1$ since the point $(0, 1)$ is on the graph of h.

(i) Since $\lim_{x\to 2^-} h(x) = 2$ and $\lim_{x\to 2^+} h(x) = 2$, we have $\lim_{x\to 2} h(x) = 2$.

(j) $h(2)$ is not defined, so it doesn't exist.

(k) $h(x)$ approaches 3 as x approaches 5 from the right, so $\lim_{x\to 5^+} h(x) = 3$.

(l) $h(x)$ does not approach any one number as x approaches 5 from the left, so $\lim_{x\to 5^-} h(x)$ does not exist.

7. (a) $\lim_{x\to 4^-} g(x) \neq \lim_{x\to 4^+} g(x)$, so $\lim_{x\to 4} g(x)$ does not exist. However, there is a point on the graph representing $g(4)$. Thus, $a = 4$ satisfies the given description.

(b) $\lim_{x\to 5^-} g(x) = \lim_{x\to 5^+} g(x)$, so $\lim_{x\to 5} g(x)$ exists. However, $g(5)$ is not defined. Thus, $a = 5$ satisfies the given description.

(c) From part (a), $a = 4$ satisfies the given description. Also, $\lim_{x\to 2^-} g(x)$ and $\lim_{x\to 2^+} g(x)$ exist, but $\lim_{x\to 2^-} g(x) \neq \lim_{x\to 2^+} g(x)$. Thus, $\lim_{x\to 2} g(x)$ does not exist, and $a = 2$ also satisfies the given description.

(d) $\lim_{x\to 4^+} g(x) = g(4)$, but $\lim_{x\to 4^-} g(x) \neq g(4)$. Thus, $a = 4$ satisfies the given description.

8. (a) $\lim_{x\to -3} A(x) = \infty$ (b) $\lim_{x\to 2^-} A(x) = -\infty$

(c) $\lim_{x\to 2^+} A(x) = \infty$ (d) $\lim_{x\to -1} A(x) = -\infty$

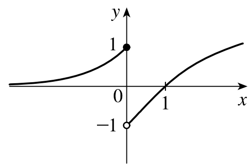
(e) The equations of the vertical asymptotes are $x = -3$, $x = -1$ and $x = 2$.

9. (a) $\lim_{x\to -7} f(x) = -\infty$ (b) $\lim_{x\to -3} f(x) = \infty$ (c) $\lim_{x\to 0} f(x) = \infty$

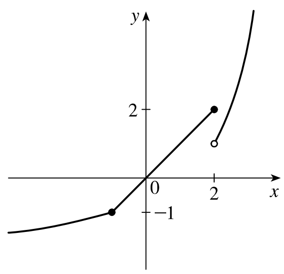
(d) $\lim_{x\to 6^-} f(x) = -\infty$ (e) $\lim_{x\to 6^+} f(x) = \infty$

(f) The equations of the vertical asymptotes are $x = -7$, $x = -3$, $x = 0$, and $x = 6$.

10. $\lim_{t\to 12^-} f(t) = 150$ mg and $\lim_{t\to 12^+} f(t) = 300$ mg. These limits show that there is an abrupt change in the amount of drug in the patient's bloodstream at $t = 12$ h. The left-hand limit represents the amount of the drug just before the fourth injection. The right-hand limit represents the amount of the drug just after the fourth injection.

11. 

From the graph of f we see that $\lim_{x\to 0^-} f(x) = 1$, but $\lim_{x\to 0^+} f(x) = -1$, so $\lim_{x\to a} f(x)$ does not exist for $a = 0$. However, $\lim_{x\to a} f(x)$ exists for all other values of a. Thus, $\lim_{x\to a} f(x)$ exists for all a in $(-\infty, 0) \cup (0, \infty)$.

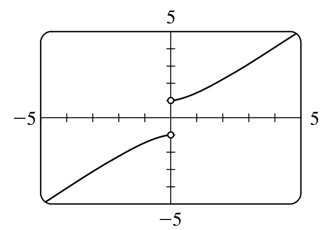
12. 

From the graph of f we see that $\lim_{x\to 2^-} f(x) = 2$, but $\lim_{x\to 2^+} f(x) = 1$, so $\lim_{x\to a} f(x)$ does not exist for $a = 2$. However, $\lim_{x\to a} f(x)$ exists for all other values of a. Thus, $\lim_{x\to a} f(x)$ exists for all a in $(-\infty, 2) \cup (2, \infty)$.

13. (a) From the graph, $\lim_{x\to 0^-} f(x) = -1$.

(b) From the graph, $\lim_{x\to 0^+} f(x) = 1$.

(c) Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, $\lim_{x\to 0} f(x)$ does not exist.

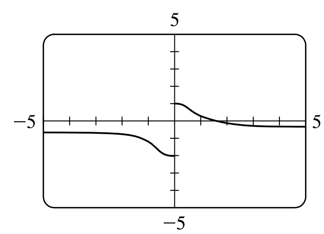


$f(x) = x\sqrt{1 + x^{-2}}$

14. (a) From the graph, $\lim_{x\to 0^-} f(x) = -2$.

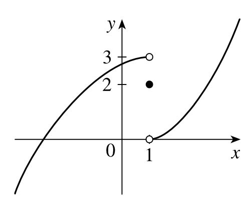
(b) From the graph, $\lim_{x\to 0^+} f(x) = 1$.

(c) Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, $\lim_{x\to 0} f(x)$ does not exist.



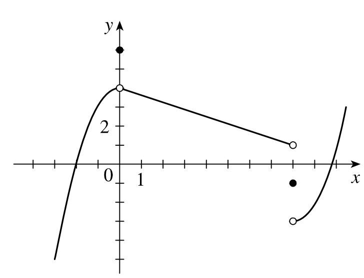
$f(x) = \dfrac{e^{1/x} - 2}{e^{1/x} + 1}$

15. $\lim_{x\to 1^-} f(x) = 3$, $\lim_{x\to 1^+} f(x) = 0$, $f(1) = 2$



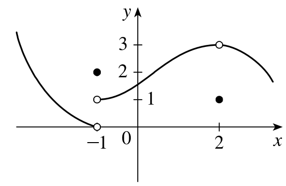
16. $\lim_{x\to 0} f(x) = 4$, $\lim_{x\to 8^-} f(x) = 1$, $\lim_{x\to 8^+} f(x) = -3$,

$f(0) = 6$, $f(8) = -1$



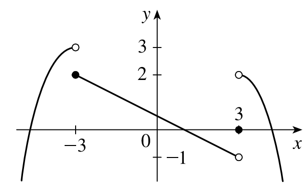
17. $\lim\limits_{x \to -1^-} f(x) = 0$, $\lim\limits_{x \to -1^+} f(x) = 1$, $\lim\limits_{x \to 2} f(x) = 3$,

$f(-1) = 2$, $f(2) = 1$



18. $\lim\limits_{x \to -3^-} f(x) = 3$, $\lim\limits_{x \to -3^+} f(x) = 2$, $\lim\limits_{x \to 3^-} f(x) = -1$,

$\lim\limits_{x \to 3^+} f(x) = 2$, $f(-3) = 2$, $f(3) = 0$



19. For $f(x) = \dfrac{x^2 - 3x}{x^2 - 9}$:

x	$f(x)$
3.1	0.508 197
3.05	0.504 132
3.01	0.500 832
3.001	0.500 083
3.0001	0.500 008

x	$f(x)$
2.9	0.491 525
2.95	0.495 798
2.99	0.499 165
2.999	0.499 917
2.9999	0.499 992

It appears that $\lim\limits_{x \to 3} \dfrac{x^2 - 3x}{x^2 - 9} = \dfrac{1}{2}$.

20. For $f(x) = \dfrac{x^2 - 3x}{x^2 - 9}$:

x	$f(x)$
−2.5	−5
−2.9	−29
−2.95	−59
−2.99	−299
−2.999	−2999
−2.9999	−29,999

x	$f(x)$
−3.5	7
−3.1	31
−3.05	61
−3.01	301
−3.001	3001
−3.0001	30,001

It appears that $\lim\limits_{x \to -3^+} f(x) = -\infty$ and that

$\lim\limits_{x \to -3^-} f(x) = \infty$, so $\lim\limits_{x \to -3} \dfrac{x^2 - 3x}{x^2 - 9}$ does not exist.

21. For $f(t) = \dfrac{e^{5t} - 1}{t}$:

t	$f(t)$
0.5	22.364 988
0.1	6.487 213
0.01	5.127 110
0.001	5.012 521
0.0001	5.001 250

t	$f(t)$
−0.5	1.835 830
−0.1	3.934 693
−0.01	4.877 058
−0.001	4.987 521
−0.0001	4.998 750

It appears that $\lim\limits_{t \to 0} \dfrac{e^{5t} - 1}{t} = 5$.

22. For $f(h) = \dfrac{(2+h)^5 - 32}{h}$:

h	$f(h)$
0.5	131.312 500
0.1	88.410 100
0.01	80.804 010
0.001	80.080 040
0.0001	80.008 000

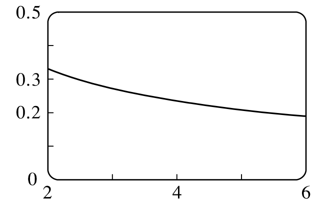
h	$f(h)$
−0.5	48.812 500
−0.1	72.390 100
−0.01	79.203 990
−0.001	79.920 040
−0.0001	79.992 000

It appears that $\lim\limits_{h \to 0} \dfrac{(2+h)^5 - 32}{h} = 80$.

23. For $f(x) = \dfrac{\ln x - \ln 4}{x - 4}$:

x	$f(x)$
3.9	0.253 178
3.99	0.250 313
3.999	0.250 031
3.9999	0.250 003

x	$f(x)$
4.1	0.246 926
4.01	0.249 688
4.001	0.249 969
4.0001	0.249 997

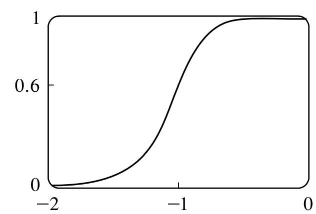


It appears that $\lim\limits_{x \to 4} f(x) = 0.25$. The graph confirms that result.

24. For $f(p) = \dfrac{1 + p^9}{1 + p^{15}}$:

p	$f(p)$
-1.1	0.427 397
-1.01	0.582 008
-1.001	0.598 200
-1.0001	0.599 820

p	$f(p)$
-0.9	0.771 405
-0.99	0.617 992
-0.999	0.601 800
-0.9999	0.600 180



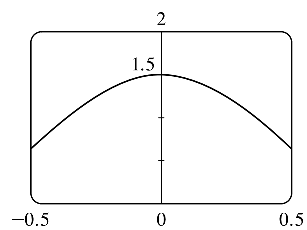
It appears that $\lim\limits_{p \to -1} f(p) = 0.6$. The graph confirms that result.

25. For $f(\theta) = \dfrac{\sin 3\theta}{\tan 2\theta}$:

θ	$f(\theta)$
± 0.1	1.457 847
± 0.01	1.499 575
± 0.001	1.499 996
± 0.0001	1.500 000

It appears that $\lim\limits_{\theta \to 0} \dfrac{\sin 3\theta}{\tan 2\theta} = 1.5$.

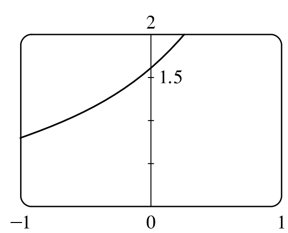
The graph confirms that result.



26. For $f(t) = \dfrac{5^t - 1}{t}$:

t	$f(t)$
0.1	1.746 189
0.01	1.622 459
0.001	1.610 734
0.0001	1.609 567

t	$f(t)$
-0.1	1.486 601
-0.01	1.596 556
-0.001	1.608 143
-0.0001	1.609 308



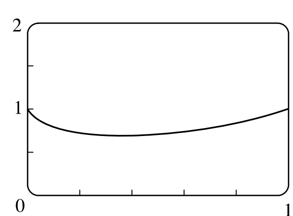
It appears that $\lim\limits_{t \to 0} f(t) \approx 1.6094$. The graph confirms that result.

27. For $f(x) = x^x$:

x	$f(x)$
0.1	0.794 328
0.01	0.954 993
0.001	0.993 116
0.0001	0.999 079

It appears that $\lim\limits_{x \to 0^+} f(x) = 1$.

The graph confirms that result.

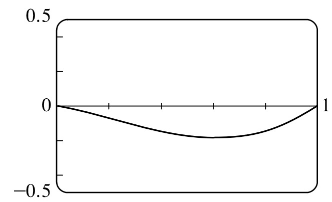


28. For $f(x) = x^2 \ln x$:

x	$f(x)$
0.1	$-0.023\,026$
0.01	$-0.000\,461$
0.001	$-0.000\,007$
0.0001	$-0.000\,000$

It appears that $\lim\limits_{x \to 0^+} f(x) = 0$.

The graph confirms that result.



29. $\lim\limits_{x \to 5^+} \dfrac{x+1}{x-5} = \infty$ since the numerator is positive and the denominator approaches 0 from the positive side as $x \to 5^+$.

30. $\lim\limits_{x \to 5^-} \dfrac{x+1}{x-5} = -\infty$ since the numerator is positive and the denominator approaches 0 from the negative side as $x \to 5^-$.

31. $\lim\limits_{x \to 2} \dfrac{x^2}{(x-2)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \to 2$.

32. $\lim\limits_{x \to 3^-} \dfrac{\sqrt{x}}{(x-3)^5} = -\infty$ since the numerator is positive and the denominator approaches 0 from the negative side as $x \to 3^-$.

33. $\lim\limits_{x \to 1^+} \ln(\sqrt{x} - 1) = -\infty$ since $\sqrt{x} - 1 \to 0^+$ as $x \to 1^+$.

34. $\lim\limits_{x \to 0^+} \ln(\sin x) = -\infty$ since $\sin x \to 0^+$ as $x \to 0^+$.

35. $\lim\limits_{x \to (\pi/2)^+} \dfrac{1}{x} \sec x = -\infty$ since $\dfrac{1}{x}$ is positive and $\sec x \to -\infty$ as $x \to (\pi/2)^+$.

36. $\lim\limits_{x \to \pi^-} x \cot x = -\infty$ since x is positive and $\cot x \to -\infty$ as $x \to \pi^-$.

37. $\lim\limits_{x \to 1} \dfrac{x^2 + 2x}{x^2 - 2x + 1} = \lim\limits_{x \to 1} \dfrac{x^2 + 2x}{(x-1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \to 1$.

38. $\lim\limits_{x \to 3^-} \dfrac{x^2 + 4x}{x^2 - 2x - 3} = \lim\limits_{x \to 3^-} \dfrac{x^2 + 4x}{(x-3)(x+1)} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $x \to 3^-$.

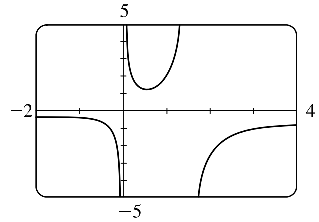
39. $\lim\limits_{x \to 0} (\ln x^2 - x^{-2}) = -\infty$ since $\ln x^2 \to -\infty$ and $x^{-2} \to \infty$ as $x \to 0$.

40. $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \ln x \right) = \infty$ since $\dfrac{1}{x} \to \infty$ and $\ln x \to -\infty$ as $x \to 0^+$.

41. The denominator of $f(x) = \dfrac{x-1}{2x+4}$ is equal to 0 when $x = -2$ (and the numerator is not), so $x = -2$ is the vertical asymptote of the function.

42. (a) The denominator of $y = \dfrac{x^2+1}{3x-2x^2} = \dfrac{x^2+1}{x(3-2x)}$ is equal to zero when $x = 0$ and $x = \frac{3}{2}$ (and the numerator is not), so $x = 0$ and $x = 1.5$ are vertical asymptotes of the function.

(b)



43. (a) $f(x) = \dfrac{1}{x^3 - 1}$.

x	$f(x)$
0.5	-1.14
0.9	-3.69
0.99	-33.7
0.999	-333.7
0.9999	-3333.7
0.99999	$-33{,}333.7$

x	$f(x)$
1.5	0.42
1.1	3.02
1.01	33.0
1.001	333.0
1.0001	3333.0
1.00001	33,333.3

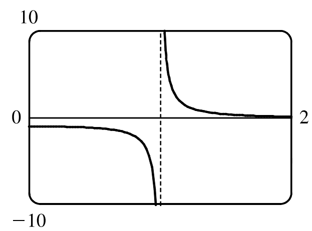
From these calculations, it seems that $\lim\limits_{x\to1^-} f(x) = -\infty$ and $\lim\limits_{x\to1^+} f(x) = \infty$.

(b) If x is slightly smaller than 1, then $x^3 - 1$ will be a negative number close to 0, and the reciprocal of $x^3 - 1$, that is, $f(x)$, will be a negative number with large absolute value. So $\lim\limits_{x\to1^-} f(x) = -\infty$.

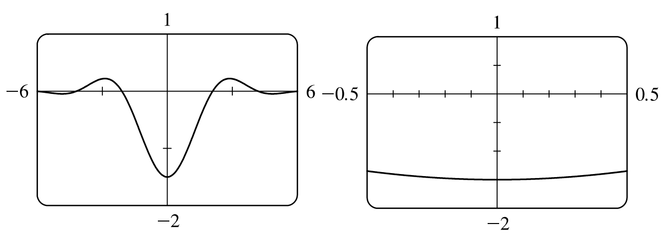
If x is slightly larger than 1, then $x^3 - 1$ will be a small positive number, and its reciprocal, $f(x)$, will be a large positive number. So $\lim\limits_{x\to1^+} f(x) = \infty$.

(c) It appears from the graph of f that

$\lim\limits_{x\to1^-} f(x) = -\infty$ and $\lim\limits_{x\to1^+} f(x) = \infty$.



44. (a) From the graphs, it seems that $\lim\limits_{x\to0} \dfrac{\cos 2x - \cos x}{x^2} = -1.5$.



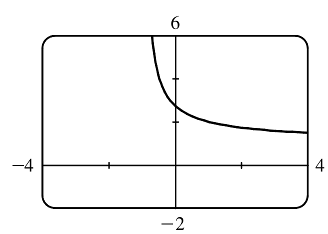
(b)

x	$f(x)$
±0.1	$-1.493\,759$
±0.01	$-1.499\,938$
±0.001	$-1.499\,999$
±0.0001	$-1.500\,000$

45. (a) Let $h(x) = (1+x)^{1/x}$.

x	$h(x)$
-0.001	2.71964
-0.0001	2.71842
-0.00001	2.71830
-0.000001	2.71828
0.000001	2.71828
0.00001	2.71827
0.0001	2.71815
0.001	2.71692

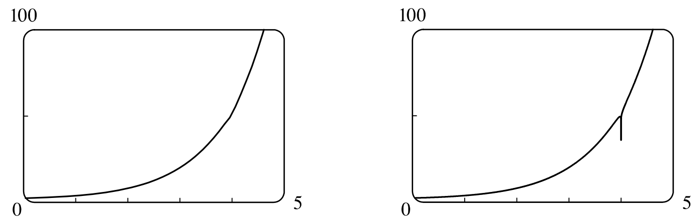
(b)



It appears that $\lim\limits_{x\to0} (1+x)^{1/x} \approx 2.71828$, which is approximately e.

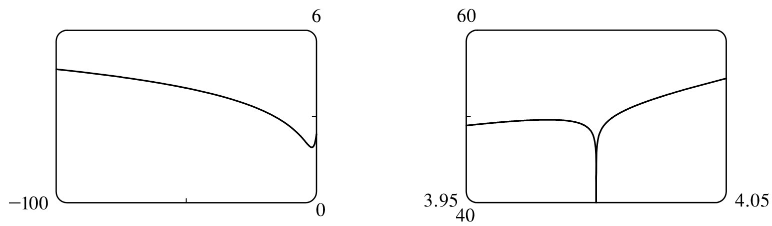
In Section 3.6 we will see that the value of the limit is exactly e.

46. (a)

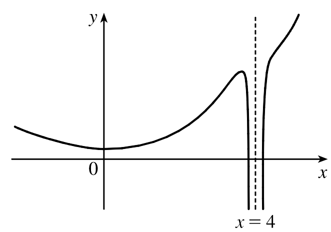


No, because the calculator-produced graph of $f(x) = e^x + \ln|x - 4|$ looks like an exponential function, but the graph of f has an infinite discontinuity at $x = 4$. A second graph, obtained by increasing the `numpoints` option in Maple, begins to reveal the discontinuity at $x = 4$.

(b) There isn't a single graph that shows all the features of f. Several graphs are needed since f looks like $\ln|x - 4|$ for large negative values of x and like e^x for $x > 5$, but yet has the infinite discontiuity at $x = 4$.



A hand-drawn graph, though distorted, might be better at revealing the main features of this function.



47. For $f(x) = x^2 - (2^x/1000)$:

(a)

x	$f(x)$
1	0.998 000
0.8	0.638 259
0.6	0.358 484
0.4	0.158 680
0.2	0.038 851
0.1	0.008 928
0.05	0.001 465

It appears that $\lim\limits_{x\to 0} f(x) = 0$.

(b)

x	$f(x)$
0.04	0.000 572
0.02	−0.000 614
0.01	−0.000 907
0.005	−0.000 978
0.003	−0.000 993
0.001	−0.001 000

It appears that $\lim\limits_{x\to 0} f(x) = -0.001$.

48. For $h(x) = \dfrac{\tan x - x}{x^3}$:

(a)

x	$h(x)$
1.0	0.557 407 73
0.5	0.370 419 92
0.1	0.334 672 09
0.05	0.333 667 00
0.01	0.333 346 67
0.005	0.333 336 67

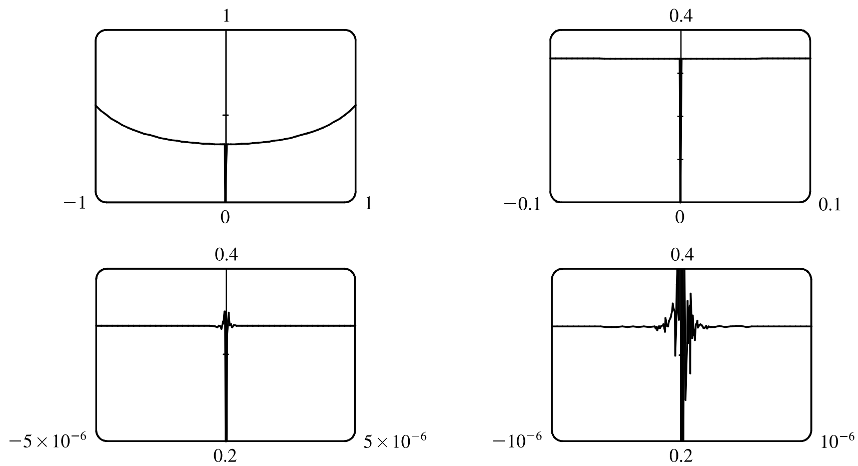
(b) It seems that $\lim\limits_{x\to 0} h(x) = \frac{1}{3}$.

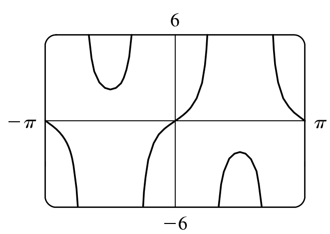
(c)

x	$h(x)$
0.001	0.333 333 50
0.0005	0.333 333 44
0.0001	0.333 330 00
0.00005	0.333 336 00
0.00001	0.333 000 00
0.000001	0.000 000 00

Here the values will vary from one calculator to another. Every calculator will eventually give *false values*.

(d) As in part (c), when we take a small enough viewing rectangle we get incorrect output.



49. 

There appear to be vertical asymptotes of the curve $y = \tan(2\sin x)$ at $x \approx \pm 0.90$ and $x \approx \pm 2.24$. To find the exact equations of these asymptotes, we note that the graph of the tangent function has vertical asymptotes at $x = \frac{\pi}{2} + \pi n$. Thus, we must have $2\sin x = \frac{\pi}{2} + \pi n$, or equivalently, $\sin x = \frac{\pi}{4} + \frac{\pi}{2}n$. Since $-1 \le \sin x \le 1$, we must have $\sin x = \pm\frac{\pi}{4}$ and so $x = \pm\sin^{-1}\frac{\pi}{4}$ (corresponding to $x \approx \pm 0.90$). Just as 150° is the reference angle for 30°, $\pi - \sin^{-1}\frac{\pi}{4}$ is the reference angle for $\sin^{-1}\frac{\pi}{4}$. So $x = \pm\left(\pi - \sin^{-1}\frac{\pi}{4}\right)$ are also equations of vertical asymptotes (corresponding to $x \approx \pm 2.24$).

50. (a) For any positive integer n, if $x = \dfrac{1}{n\pi}$, then $f(x) = \tan\dfrac{1}{x} = \tan(n\pi) = 0$. (Remember that the tangent function has period π.)

(b) For any nonnegative number n, if $x = \dfrac{4}{(4n+1)\pi}$, then

$$f(x) = \tan\frac{1}{x} = \tan\frac{(4n+1)\pi}{4} = \tan\left(\frac{4n\pi}{4} + \frac{\pi}{4}\right) = \tan\left(n\pi + \frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$$

(c) From part (a), $f(x) = 0$ infinitely often as $x \to 0$. From part (b), $f(x) = 1$ infinitely often as $x \to 0$. Thus, $\lim\limits_{x\to 0}\tan\dfrac{1}{x}$ does not exist since $f(x)$ does not get close to a fixed number as $x \to 0$.

51. $\lim\limits_{v\to c^-} m = \lim\limits_{v\to c^-}\dfrac{m_0}{\sqrt{1 - v^2/c^2}}$. As $v \to c^-$, $\sqrt{1 - v^2/c^2} \to 0^+$, and $m \to \infty$.

2.3 Calculating Limits Using the Limit Laws

1. (a) $\lim_{x\to 2}[f(x)+5g(x)] = \lim_{x\to 2} f(x) + \lim_{x\to 2}[5g(x)]$ [Limit Law 1]

$= \lim_{x\to 2} f(x) + 5\lim_{x\to 2} g(x)$ [Limit Law 3]

$= 4 + 5(-2) = -6$

(b) $\lim_{x\to 2}[g(x)]^3 = \left[\lim_{x\to 2} g(x)\right]^3$ [Limit Law 6]

$= (-2)^3 = -8$

(c) $\lim_{x\to 2}\sqrt{f(x)} = \sqrt{\lim_{x\to 2} f(x)}$ [Limit Law 7]

$= \sqrt{4} = 2$

(d) $\lim_{x\to 2}\dfrac{3f(x)}{g(x)} = \dfrac{\lim_{x\to 2}[3f(x)]}{\lim_{x\to 2} g(x)}$ [Limit Law 5]

$= \dfrac{3\lim_{x\to 2} f(x)}{\lim_{x\to 2} g(x)}$ [Limit Law 3]

$= \dfrac{3(4)}{-2} = -6$

(e) Because the limit of the denominator is 0, we can't use Limit Law 5. The given limit, $\lim_{x\to 2}\dfrac{g(x)}{h(x)}$, does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

(f) $\lim_{x\to 2}\dfrac{g(x)\,h(x)}{f(x)} = \dfrac{\lim_{x\to 2}[g(x)\,h(x)]}{\lim_{x\to 2} f(x)}$ [Limit Law 5]

$= \dfrac{\lim_{x\to 2} g(x)\cdot \lim_{x\to 2} h(x)}{\lim_{x\to 2} f(x)}$ [Limit Law 4]

$= \dfrac{-2\cdot 0}{4} = 0$

2. (a) $\lim_{x\to 2}[f(x)+g(x)] = \lim_{x\to 2} f(x) + \lim_{x\to 2} g(x)$ [Limit Law 1]

$= -1 + 2$

$= 1$

(b) $\lim_{x\to 0} f(x)$ exists, but $\lim_{x\to 0} g(x)$ does not exist, so we cannot apply Limit Law 2 to $\lim_{x\to 0}[f(x)-g(x)]$. The limit does not exist.

(c) $\lim_{x\to -1}[f(x)\,g(x)] = \lim_{x\to -1} f(x)\cdot \lim_{x\to -1} g(x)$ [Limit Law 4]

$= 1\cdot 2$

$= 2$

(d) $\lim_{x\to 3} f(x) = 1$, but $\lim_{x\to 3} g(x) = 0$, so we cannot apply Limit Law 5 to $\lim_{x\to 3}\dfrac{f(x)}{g(x)}$. The limit does not exist.

Note: $\lim_{x\to 3^-}\dfrac{f(x)}{g(x)} = \infty$ since $g(x)\to 0^+$ as $x\to 3^-$ and $\lim_{x\to 3^+}\dfrac{f(x)}{g(x)} = -\infty$ since $g(x)\to 0^-$ as $x\to 3^+$. Therefore, the limit does not exist, even as an infinite limit.

(e) $\lim_{x\to 2}\left[x^2 f(x)\right] = \lim_{x\to 2} x^2\cdot \lim_{x\to 2} f(x)$ [Limit Law 4]

$= 2^2\cdot(-1)$

$= -4$

(f) $f(-1) + \lim_{x\to -1} g(x)$ is undefined since $f(-1)$ is not defined.

3. $$\begin{aligned}\lim_{x\to5}(4x^2-5x) &= \lim_{x\to5}(4x^2)-\lim_{x\to5}(5x) && \text{[Limit Law 2]}\\ &= 4\lim_{x\to5}x^2-5\lim_{x\to5}x && [3]\\ &= 4(5^2)-5(5) && [10, 9]\\ &= 75\end{aligned}$$

4. $$\begin{aligned}\lim_{x\to-3}(2x^3+6x^2-9) &= \lim_{x\to-3}(2x^3)+\lim_{x\to-3}(6x^2)-\lim_{x\to-3}9 && \text{[Limits Laws 1 and 2]}\\ &= 2\lim_{x\to-3}x^3+6\lim_{x\to-3}x^2-\lim_{x\to-3}9 && [3]\\ &= 2(-3)^3+6(-3)^2-9 && [10, 8]\\ &= -9\end{aligned}$$

5. $$\begin{aligned}\lim_{v\to2}(v^2+2v)(2v^3-5) &= \lim_{v\to2}(v^2+2v)\cdot\lim_{v\to2}(2v^3-5) && \text{[Limit Law 4]}\\ &= \left(\lim_{v\to2}v^2+\lim_{v\to2}2v\right)\left(\lim_{v\to2}2v^3-\lim_{v\to2}5\right) && \text{[1 and 2]}\\ &= \left(\lim_{v\to2}v^2+2\lim_{v\to2}v\right)\left(2\lim_{v\to2}v^3-\lim_{v\to2}5\right) && [3]\\ &= \left[2^2+2(2)\right]\left[2(2)^3-5\right] && \text{[10, 9, and 8]}\\ &= (8)(11)=88\end{aligned}$$

6. $$\begin{aligned}\lim_{t\to7}\frac{3t^2+1}{t^2-5t+2} &= \frac{\lim_{t\to7}(3t^2+1)}{\lim_{t\to7}(t^2-5t+2)} && \text{[Limit Law 5]}\\ &= \frac{\lim_{t\to7}3t^2+\lim_{t\to7}1}{\lim_{t\to7}t^2-\lim_{t\to7}5t+\lim_{t\to7}2} && \text{[1 and 2]}\\ &= \frac{3\lim_{t\to7}t^2+\lim_{t\to7}1}{\lim_{t\to7}t^2-5\lim_{t\to7}t+\lim_{t\to7}2} && [3]\\ &= \frac{3(7^2)+1}{7^2-5(7)+2} && \text{[10, 9, and 8]}\\ &= \frac{148}{16}=\frac{37}{4}\end{aligned}$$

7. $$\begin{aligned}\lim_{u\to-2}\sqrt{9-u^3+2u^2} &= \sqrt{\lim_{u\to-2}(9-u^3+2u^2)} && \text{[Limit Law 7]}\\ &= \sqrt{\lim_{u\to-2}9-\lim_{u\to-2}u^3+\lim_{u\to-2}2u^2} && \text{[2 and 1]}\\ &= \sqrt{\lim_{u\to-2}9-\lim_{u\to-2}u^3+2\lim_{u\to-2}u^2} && [3]\\ &= \sqrt{9-(-2)^3+2(-2)^2} && \text{[8 and 10]}\\ &= \sqrt{25}=5\end{aligned}$$

8. $\lim\limits_{x\to 3} \sqrt[3]{x+5}\,(2x^2-3x) = \lim\limits_{x\to 3} \sqrt[3]{x+5} \cdot \lim\limits_{x\to 3} \left(2x^2-3x\right)$ [Limit Law 4]

$= \sqrt[3]{\lim\limits_{x\to 3}(x+5)} \cdot \lim\limits_{x\to 3}(2x^2-3x)$ [7]

$= \sqrt[3]{\lim\limits_{x\to 3} x + \lim\limits_{x\to 3} 5} \cdot \left(\lim\limits_{x\to 3} 2x^2 - \lim\limits_{x\to 3} 3x\right)$ [1 and 2]

$= \sqrt[3]{\lim\limits_{x\to 3} x + \lim\limits_{x\to 3} 5} \cdot \left(2\lim\limits_{x\to 3} x^2 - 3\lim\limits_{x\to 3} x\right)$ [3]

$= \sqrt[3]{3+5} \cdot \left[2(3^2) - 3(3)\right]$ [9, 8, and 10]

$= 2 \cdot (18-9) = 18$

9. $\lim\limits_{t\to -1} \left(\dfrac{2t^5-t^4}{5t^2+4}\right)^3 = \left(\lim\limits_{t\to -1} \dfrac{2t^5-t^4}{5t^2+4}\right)^3$ [Limit Law 6]

$= \left(\dfrac{\lim\limits_{t\to -1}(2t^5-t^4)}{\lim\limits_{t\to -1}(5t^2+4)}\right)^3$ [5]

$= \left(\dfrac{2\lim\limits_{t\to -1} t^5 - \lim\limits_{t\to -1} t^4}{5\lim\limits_{t\to -1} t^2 + \lim\limits_{t\to -1} 4}\right)^3$ [3, 2, and 1]

$= \left(\dfrac{2(-1)^5-(-1)^4}{5(-1)^2+4}\right)^3$ [10 and 8]

$= \left(-\dfrac{3}{9}\right)^3 = -\dfrac{1}{27}$

10. (a) The left-hand side of the equation is not defined for $x = 2$, but the right-hand side is.

(b) Since the equation holds for all $x \neq 2$, it follows that both sides of the equation approach the same limit as $x \to 2$, just as in Example 3. Remember that in finding $\lim\limits_{x\to a} f(x)$, we never consider $x = a$.

11. $\lim\limits_{x\to -2}(3x-7) = 3(-2) - 7 = -13$

12. $\lim\limits_{x\to 6}\left(8 - \frac{1}{2}x\right) = 8 - \frac{1}{2}(6) = 5$

13. $\lim\limits_{t\to 4} \dfrac{t^2-2t-8}{t-4} = \lim\limits_{t\to 4} \dfrac{(t-4)(t+2)}{t-4} = \lim\limits_{t\to 4}(t+2) = 4+2 = 6$

14. $\lim\limits_{x\to -3} \dfrac{x^2+3x}{x^2-x-12} = \lim\limits_{x\to -3} \dfrac{x(x+3)}{(x-4)(x+3)} = \lim\limits_{x\to -3} \dfrac{x}{x-4} = \dfrac{-3}{-3-4} = \dfrac{3}{7}$

15. $\lim\limits_{x\to 2} \dfrac{x^2+5x+4}{x-2}$ does not exist since $x - 2 \to 0$, but $x^2 + 5x + 4 \to 18$ as $x \to 2$.

16. $\lim\limits_{x\to 4} \dfrac{x^2+3x}{x^2-x-12} = \lim\limits_{x\to 4} \dfrac{x(x+3)}{(x-4)(x+3)} = \lim\limits_{x\to 4} \dfrac{x}{x-4}$. The last limit does not exist since $\lim\limits_{x\to 4^-} \dfrac{x}{x-4} = -\infty$ and $\lim\limits_{x\to 4^+} \dfrac{x}{x-4} = \infty$.

17. $\displaystyle\lim_{x\to-2}\frac{x^2-x-6}{3x^2+5x-2}=\lim_{x\to-2}\frac{(x-3)(x+2)}{(3x-1)(x+2)}=\lim_{x\to-2}\frac{x-3}{3x-1}=\frac{-2-3}{3(-2)-1}=\frac{-5}{-7}=\frac{5}{7}$

18. $\displaystyle\lim_{x\to-5}\frac{2x^2+9x-5}{x^2-25}=\lim_{x\to-5}\frac{(2x-1)(x+5)}{(x-5)(x+5)}=\lim_{x\to-5}\frac{2x-1}{x-5}=\frac{2(-5)-1}{-5-5}=\frac{-11}{-10}=\frac{11}{10}$

19. Factoring t^3-27 as the difference of two cubes, we have

$$\lim_{t\to3}\frac{t^3-27}{t^2-9}=\lim_{t\to3}\frac{(t-3)(t^2+3t+9)}{(t-3)(t+3)}=\lim_{t\to3}\frac{t^2+3t+9}{t+3}=\frac{3^2+3(3)+9}{3+3}=\frac{27}{6}=\frac{9}{2}.$$

20. Factoring u^3+1 as the sum of two cubes, we have

$$\lim_{u\to-1}\frac{u+1}{u^3+1}=\lim_{u\to-1}\frac{u+1}{(u+1)(u^2-u+1)}=\lim_{u\to-1}\frac{1}{u^2-u+1}=\frac{1}{(-1)^2-(-1)+1}=\frac{1}{3}.$$

21. $\displaystyle\lim_{h\to0}\frac{(h-3)^2-9}{h}=\lim_{h\to0}\frac{h^2-6h+9-9}{h}=\lim_{h\to0}\frac{h^2-6h}{h}=\lim_{h\to0}\frac{h(h-6)}{h}=\lim_{h\to0}(h-6)=0-6=-6$

22. $\displaystyle\lim_{x\to9}\frac{9-x}{3-\sqrt{x}}=\lim_{x\to9}\frac{9-x}{3-\sqrt{x}}\cdot\frac{3+\sqrt{x}}{3+\sqrt{x}}=\lim_{x\to9}\frac{(9-x)(3+\sqrt{x})}{9-x}=\lim_{x\to9}(3+\sqrt{x})=3+\sqrt{9}=6$

23. $\displaystyle\lim_{h\to0}\frac{\sqrt{9+h}-3}{h}=\lim_{h\to0}\frac{\sqrt{9+h}-3}{h}\cdot\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}=\lim_{h\to0}\frac{\left(\sqrt{9+h}\right)^2-3^2}{h\left(\sqrt{9+h}+3\right)}=\lim_{h\to0}\frac{(9+h)-9}{h\left(\sqrt{9+h}+3\right)}$

$$=\lim_{h\to0}\frac{h}{h\left(\sqrt{9+h}+3\right)}=\lim_{h\to0}\frac{1}{\sqrt{9+h}+3}=\frac{1}{3+3}=\frac{1}{6}$$

24. $\displaystyle\lim_{x\to2}\frac{2-x}{\sqrt{x+2}-2}=\lim_{x\to2}\frac{2-x}{\sqrt{x+2}-2}\cdot\frac{\sqrt{x+2}+2}{\sqrt{x+2}+2}=\lim_{x\to2}\frac{(2-x)\left(\sqrt{x+2}+2\right)}{\left(\sqrt{x+2}\right)^2-4}=\lim_{x\to2}\frac{-(x-2)\left(\sqrt{x+2}+2\right)}{x-2}$

$$=\lim_{x\to2}\left[-(\sqrt{x+2}+2)\right]=-\left(\sqrt{4}+2\right)=-4$$

25. $\displaystyle\lim_{x\to3}\frac{\dfrac{1}{x}-\dfrac{1}{3}}{x-3}=\lim_{x\to3}\frac{\dfrac{1}{x}-\dfrac{1}{3}}{x-3}\cdot\frac{3x}{3x}=\lim_{x\to3}\frac{3-x}{3x(x-3)}=\lim_{x\to3}\frac{-1}{3x}=-\frac{1}{9}$

26. $\displaystyle\lim_{h\to0}\frac{(-2+h)^{-1}+2^{-1}}{h}=\lim_{h\to0}\frac{\dfrac{1}{h-2}+\dfrac{1}{2}}{h}=\lim_{h\to0}\frac{\dfrac{2+(h-2)}{2(h-2)}}{h}=\lim_{h\to0}\frac{2+(h-2)}{2h(h-2)}$

$$=\lim_{h\to0}\frac{h}{2h(h-2)}=\lim_{h\to0}\frac{1}{2(h-2)}=\frac{1}{2(0-2)}=-\frac{1}{4}$$

27. $\displaystyle\lim_{t\to0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t}=\lim_{t\to0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t}\cdot\frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}}=\lim_{t\to0}\frac{\left(\sqrt{1+t}\right)^2-\left(\sqrt{1-t}\right)^2}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}$

$$=\lim_{t\to0}\frac{(1+t)-(1-t)}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}=\lim_{t\to0}\frac{2t}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}=\lim_{t\to0}\frac{2}{\sqrt{1+t}+\sqrt{1-t}}$$

$$=\frac{2}{\sqrt{1}+\sqrt{1}}=\frac{2}{2}=1$$

28. $\lim\limits_{t\to 0}\left(\dfrac{1}{t}-\dfrac{1}{t^2+t}\right)=\lim\limits_{t\to 0}\left(\dfrac{1}{t}-\dfrac{1}{t(t+1)}\right)=\lim\limits_{t\to 0}\dfrac{t+1-1}{t(t+1)}=\lim\limits_{t\to 0}\dfrac{1}{t+1}=\dfrac{1}{0+1}=1$

29. $$\lim_{x\to 16}\frac{4-\sqrt{x}}{16x-x^2}=\lim_{x\to 16}\frac{(4-\sqrt{x})(4+\sqrt{x})}{(16x-x^2)(4+\sqrt{x})}=\lim_{x\to 16}\frac{16-x}{x(16-x)(4+\sqrt{x})}$$
$$=\lim_{x\to 16}\frac{1}{x(4+\sqrt{x})}=\frac{1}{16\left(4+\sqrt{16}\right)}=\frac{1}{16(8)}=\frac{1}{128}$$

30. $$\lim_{x\to 2}\frac{x^2-4x+4}{x^4-3x^2-4}=\lim_{x\to 2}\frac{(x-2)^2}{(x^2-4)(x^2+1)}=\lim_{x\to 2}\frac{(x-2)^2}{(x+2)(x-2)(x^2+1)}$$
$$=\lim_{x\to 2}\frac{x-2}{(x+2)(x^2+1)}=\frac{0}{4\cdot 5}=0$$

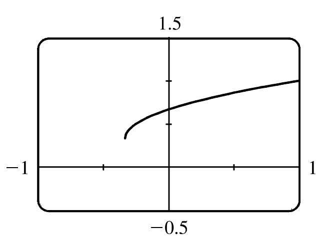
31. $$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)=\lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}=\lim_{t\to 0}\frac{\left(1-\sqrt{1+t}\right)\left(1+\sqrt{1+t}\right)}{t\sqrt{t+1}\left(1+\sqrt{1+t}\right)}=\lim_{t\to 0}\frac{-t}{t\sqrt{1+t}\left(1+\sqrt{1+t}\right)}$$
$$=\lim_{t\to 0}\frac{-1}{\sqrt{1+t}\left(1+\sqrt{1+t}\right)}=\frac{-1}{\sqrt{1+0}\left(1+\sqrt{1+0}\right)}=-\frac{1}{2}$$

32. $$\lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4}=\lim_{x\to -4}\frac{\left(\sqrt{x^2+9}-5\right)\left(\sqrt{x^2+9}+5\right)}{(x+4)\left(\sqrt{x^2+9}+5\right)}=\lim_{x\to -4}\frac{(x^2+9)-25}{(x+4)\left(\sqrt{x^2+9}+5\right)}$$
$$=\lim_{x\to -4}\frac{x^2-16}{(x+4)\left(\sqrt{x^2+9}+5\right)}=\lim_{x\to -4}\frac{(x+4)(x-4)}{(x+4)\left(\sqrt{x^2+9}+5\right)}$$
$$=\lim_{x\to -4}\frac{x-4}{\sqrt{x^2+9}+5}=\frac{-4-4}{\sqrt{16+9}+5}=\frac{-8}{5+5}=-\frac{4}{5}$$

33. $$\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}=\lim_{h\to 0}\frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$$
$$=\lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}=\lim_{h\to 0}(3x^2+3xh+h^2)=3x^2$$

34. $$\lim_{h\to 0}\frac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}=\lim_{h\to 0}\frac{\dfrac{x^2-(x+h)^2}{(x+h)^2x^2}}{h}=\lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}=\lim_{h\to 0}\frac{-h(2x+h)}{hx^2(x+h)^2}$$
$$=\lim_{h\to 0}\frac{-(2x+h)}{x^2(x+h)^2}=\frac{-2x}{x^2\cdot x^2}=-\frac{2}{x^3}$$

35. (a)



$\lim\limits_{x\to 0}\dfrac{x}{\sqrt{1+3x}-1}\approx\dfrac{2}{3}$

(b)

x	$f(x)$
−0.001	0.666 166 3
−0.000 1	0.666 616 7
−0.000 01	0.666 661 7
−0.000 001	0.666 666 2
0.000 001	0.666 667 2
0.000 01	0.666 671 7
0.000 1	0.666 716 7
0.001	0.667 166 3

The limit appears to be $\dfrac{2}{3}$.

(c) $\lim\limits_{x\to 0}\left(\dfrac{x}{\sqrt{1+3x}-1}\cdot\dfrac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}\right)=\lim\limits_{x\to 0}\dfrac{x\left(\sqrt{1+3x}+1\right)}{(1+3x)-1}=\lim\limits_{x\to 0}\dfrac{x\left(\sqrt{1+3x}+1\right)}{3x}$

$=\dfrac{1}{3}\lim\limits_{x\to 0}\left(\sqrt{1+3x}+1\right)$ [Limit Law 3]

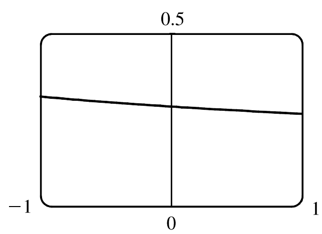
$=\dfrac{1}{3}\left[\sqrt{\lim\limits_{x\to 0}(1+3x)}+\lim\limits_{x\to 0}1\right]$ [1 and 7]

$=\dfrac{1}{3}\left(\sqrt{\lim\limits_{x\to 0}1+3\lim\limits_{x\to 0}x}+1\right)$ [1, 3, and 8]

$=\dfrac{1}{3}\left(\sqrt{1+3\cdot 0}+1\right)$ [8 and 9]

$=\dfrac{1}{3}(1+1)=\dfrac{2}{3}$

36. (a)



$\lim\limits_{x\to 0}\dfrac{\sqrt{3+x}-\sqrt{3}}{x}\approx 0.29$

(b)

x	$f(x)$
-0.001	$0.288\,699\,2$
$-0.000\,1$	$0.288\,677\,5$
$-0.000\,01$	$0.288\,675\,4$
$-0.000\,001$	$0.288\,675\,2$
$0.000\,001$	$0.288\,675\,1$
$0.000\,01$	$0.288\,674\,9$
$0.000\,1$	$0.288\,672\,7$
0.001	$0.288\,651\,1$

The limit appears to be approximately 0.2887.

(c) $\lim\limits_{x\to 0}\left(\dfrac{\sqrt{3+x}-\sqrt{3}}{x}\cdot\dfrac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}\right)=\lim\limits_{x\to 0}\dfrac{(3+x)-3}{x\left(\sqrt{3+x}+\sqrt{3}\right)}=\lim\limits_{x\to 0}\dfrac{1}{\sqrt{3+x}+\sqrt{3}}$

$=\dfrac{\lim\limits_{x\to 0}1}{\lim\limits_{x\to 0}\sqrt{3+x}+\lim\limits_{x\to 0}\sqrt{3}}$ [Limit Laws 5 and 1]

$=\dfrac{1}{\sqrt{\lim\limits_{x\to 0}(3+x)}+\sqrt{3}}$ [7 and 8]

$=\dfrac{1}{\sqrt{3+0}+\sqrt{3}}$ [1, 8, and 9]

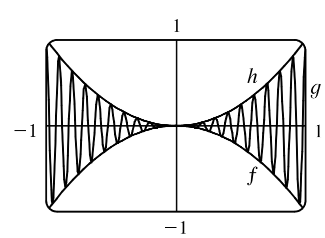
$=\dfrac{1}{2\sqrt{3}}$

37. Let $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$ and $h(x)=x^2$. Then

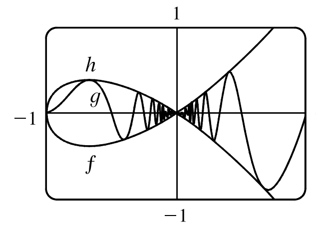
$-1\leq\cos 20\pi x\leq 1\quad\Rightarrow\quad -x^2\leq x^2\cos 20\pi x\leq x^2\quad\Rightarrow\quad f(x)\leq g(x)\leq h(x)$.

So since $\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}h(x)=0$, by the Squeeze Theorem we have

$\lim\limits_{x\to 0}g(x)=0$.



38. Let $f(x) = -\sqrt{x^3 + x^2}$, $g(x) = \sqrt{x^3 + x^2}\,\sin(\pi/x)$, and $h(x) = \sqrt{x^3 + x^2}$. Then $-1 \le \sin(\pi/x) \le 1 \quad\Rightarrow\quad -\sqrt{x^3 + x^2} \le \sqrt{x^3 + x^2}\,\sin(\pi/x) \le \sqrt{x^3 + x^2} \quad\Rightarrow$ $f(x) \le g(x) \le h(x)$. So since $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} h(x) = 0$, by the Squeeze Theorem we have $\lim\limits_{x\to 0} g(x) = 0$.



39. We have $\lim\limits_{x\to 4}(4x - 9) = 4(4) - 9 = 7$ and $\lim\limits_{x\to 4}\left(x^2 - 4x + 7\right) = 4^2 - 4(4) + 7 = 7$. Since $4x - 9 \le f(x) \le x^2 - 4x + 7$ for $x \ge 0$, $\lim\limits_{x\to 4} f(x) = 7$ by the Squeeze Theorem.

40. We have $\lim\limits_{x\to 1}(2x) = 2(1) = 2$ and $\lim\limits_{x\to 1}(x^4 - x^2 + 2) = 1^4 - 1^2 + 2 = 2$. Since $2x \le g(x) \le x^4 - x^2 + 2$ for all x, $\lim\limits_{x\to 1} g(x) = 2$ by the Squeeze Theorem.

41. $-1 \le \cos(2/x) \le 1 \quad\Rightarrow\quad -x^4 \le x^4\cos(2/x) \le x^4$. Since $\lim\limits_{x\to 0}\left(-x^4\right) = 0$ and $\lim\limits_{x\to 0} x^4 = 0$, we have $\lim\limits_{x\to 0}\left[x^4\cos(2/x)\right] = 0$ by the Squeeze Theorem.

42. $-1 \le \sin(\pi/x) \le 1 \quad\Rightarrow\quad e^{-1} \le e^{\sin(\pi/x)} \le e^1 \quad\Rightarrow\quad \sqrt{x}/e \le \sqrt{x}\,e^{\sin(\pi/x)} \le \sqrt{x}\,e$. Since $\lim\limits_{x\to 0^+}\left(\sqrt{x}/e\right) = 0$ and $\lim\limits_{x\to 0^+}\left(\sqrt{x}\,e\right) = 0$, we have $\lim\limits_{x\to 0^+}\left[\sqrt{x}\,e^{\sin(\pi/x)}\right] = 0$ by the Squeeze Theorem.

43. $|x+4| = \begin{cases} x+4 & \text{if } x+4 \ge 0 \\ -(x+4) & \text{if } x+4 < 0 \end{cases} = \begin{cases} x+4 & \text{if } x \ge -4 \\ -(x+4) & \text{if } x < -4 \end{cases}$

Thus, $\lim\limits_{x\to -4^+}(|x+4| - 2x) = \lim\limits_{x\to -4^+}(x + 4 - 2x) = \lim\limits_{x\to -4^+}(-x + 4) = 4 + 4 = 8$ and

$\lim\limits_{x\to -4^-}(|x+4| - 2x) = \lim\limits_{x\to -4^-}(-(x+4) - 2x) = \lim\limits_{x\to -4^-}(-3x - 4) = 12 - 4 = 8$.

The left and right limits are equal, so $\lim\limits_{x\to -4}(|x+4| - 2x) = 8$.

44. $|x+4| = \begin{cases} x+4 & \text{if } x+4 \ge 0 \\ -(x+4) & \text{if } x+4 < 0 \end{cases} = \begin{cases} x+4 & \text{if } x \ge -4 \\ -(x+4) & \text{if } x < -4 \end{cases}$

Thus, $\lim\limits_{x\to -4^+}\dfrac{|x+4|}{2x+8} = \lim\limits_{x\to -4^+}\dfrac{x+4}{2x+8} = \lim\limits_{x\to -4^+}\dfrac{x+4}{2(x+4)} = \lim\limits_{x\to -4^+}\dfrac{1}{2} = \dfrac{1}{2}$ and

$\lim\limits_{x\to -4^-}\dfrac{|x+4|}{2x+8} = \lim\limits_{x\to -4^-}\dfrac{-(x+4)}{2x+8} = \lim\limits_{x\to -4^-}\dfrac{-(x+4)}{2(x+4)} = \lim\limits_{x\to -4^-}\dfrac{-1}{2} = -\dfrac{1}{2}$.

The left and right limits are different, so $\lim\limits_{x\to -4}\dfrac{|x+4|}{2x+8}$ does not exist.

45. $\left|2x^3 - x^2\right| = \left|x^2(2x-1)\right| = \left|x^2\right| \cdot |2x-1| = x^2\,|2x-1|$

$|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \ge 0 \\ -(2x-1) & \text{if } 2x-1 < 0 \end{cases} = \begin{cases} 2x-1 & \text{if } x \ge 0.5 \\ -(2x-1) & \text{if } x < 0.5 \end{cases}$

So $\left|2x^3 - x^2\right| = x^2[-(2x-1)]$ for $x < 0.5$.

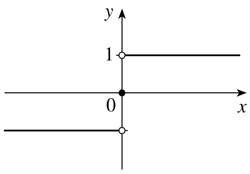
Thus, $\lim\limits_{x\to 0.5^-}\dfrac{2x-1}{\left|2x^3 - x^2\right|} = \lim\limits_{x\to 0.5^-}\dfrac{2x-1}{x^2[-(2x-1)]} = \lim\limits_{x\to 0.5^-}\dfrac{-1}{x^2} = \dfrac{-1}{(0.5)^2} = \dfrac{-1}{0.25} = -4$.

46. Since $|x| = -x$ for $x < 0$, we have $\lim\limits_{x\to-2} \dfrac{2-|x|}{2+x} = \lim\limits_{x\to-2} \dfrac{2-(-x)}{2+x} = \lim\limits_{x\to-2} \dfrac{2+x}{2+x} = \lim\limits_{x\to-2} 1 = 1$.

47. Since $|x| = -x$ for $x < 0$, we have $\lim\limits_{x\to0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim\limits_{x\to0^-} \left(\dfrac{1}{x} - \dfrac{1}{-x}\right) = \lim\limits_{x\to0^-} \dfrac{2}{x}$, which does not exist since the denominator approaches 0 and the numerator does not.

48. Since $|x| = x$ for $x > 0$, we have $\lim\limits_{x\to0^+} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim\limits_{x\to0^+} \left(\dfrac{1}{x} - \dfrac{1}{x}\right) = \lim\limits_{x\to0^+} 0 = 0$.

49. (a)



(b) (i) Since $\operatorname{sgn} x = 1$ for $x > 0$, $\lim\limits_{x\to0^+} \operatorname{sgn} x = \lim\limits_{x\to0^+} 1 = 1$.

(ii) Since $\operatorname{sgn} x = -1$ for $x < 0$, $\lim\limits_{x\to0^-} \operatorname{sgn} x = \lim\limits_{x\to0^-} -1 = -1$.

(iii) Since $\lim\limits_{x\to0^-} \operatorname{sgn} x \neq \lim\limits_{x\to0^+} \operatorname{sgn} x$, $\lim\limits_{x\to0} \operatorname{sgn} x$ does not exist.

(iv) Since $|\operatorname{sgn} x| = 1$ for $x \neq 0$, $\lim\limits_{x\to0} |\operatorname{sgn} x| = \lim\limits_{x\to0} 1 = 1$.

50. (a) $g(x) = \operatorname{sgn}(\sin x) = \begin{cases} -1 & \text{if } \sin x < 0 \\ 0 & \text{if } \sin x = 0 \\ 1 & \text{if } \sin x > 0 \end{cases}$

(i) $\lim\limits_{x\to0^+} g(x) = \lim\limits_{x\to0^+} \operatorname{sgn}(\sin x) = 1$ since $\sin x$ is positive for small positive values of x.

(ii) $\lim\limits_{x\to0^-} g(x) = \lim\limits_{x\to0^-} \operatorname{sgn}(\sin x) = -1$ since $\sin x$ is negative for small negative values of x.

(iii) $\lim\limits_{x\to0} g(x)$ does not exist since $\lim\limits_{x\to0^+} g(x) \neq \lim\limits_{x\to0^-} g(x)$.

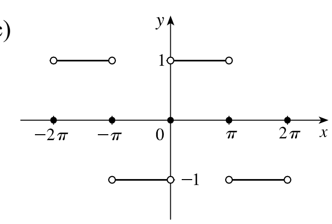
(iv) $\lim\limits_{x\to\pi^+} g(x) = \lim\limits_{x\to\pi^+} \operatorname{sgn}(\sin x) = -1$ since $\sin x$ is negative for values of x slightly greater than π.

(v) $\lim\limits_{x\to\pi^-} g(x) = \lim\limits_{x\to\pi^-} \operatorname{sgn}(\sin x) = 1$ since $\sin x$ is positive for values of x slightly less than π.

(vi) $\lim\limits_{x\to\pi} g(x)$ does not exist since $\lim\limits_{x\to\pi^+} g(x) \neq \lim\limits_{x\to\pi^-} g(x)$.

(b) The sine function changes sign at every integer multiple of π, so the signum function equals 1 on one side and -1 on the other side of $n\pi$, n an integer. Thus, $\lim\limits_{x\to a} g(x)$ does not exist for $a = n\pi$, n an integer.

(c)

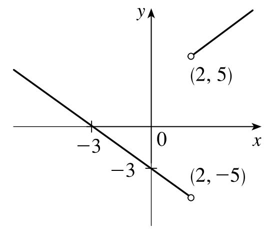


51. (a) (i)
$$\begin{aligned}\lim_{x\to2^+} g(x) = \lim_{x\to2^+} \frac{x^2+x-6}{|x-2|} &= \lim_{x\to2^+} \frac{(x+3)(x-2)}{|x-2|} \\ &= \lim_{x\to2^+} \frac{(x+3)(x-2)}{x-2} \qquad [\text{since } x-2>0 \text{ if } x\to2^+] \\ &= \lim_{x\to2^+} (x+3) = 5\end{aligned}$$

(ii) The solution is similar to the solution in part (i), but now $|x-2| = 2-x$ since $x-2<0$ if $x\to2^-$. Thus, $\lim\limits_{x\to2^-} g(x) = \lim\limits_{x\to2^-} -(x+3) = -5$.

(b) Since the right-hand and left-hand limits of g at $x = 2$ are not equal, $\lim_{x \to 2} g(x)$ does not exist.

(c)

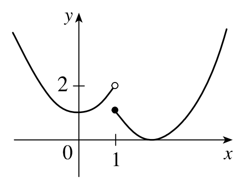


52. (a) $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x-2)^2 & \text{if } x \geq 1 \end{cases}$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1) = 1^2 + 1 = 2, \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-2)^2 = (-1)^2 = 1$

(b) Since the right-hand and left-hand limits of f at $x = 1$ are not equal, $\lim_{x \to 1} f(x)$ does not exist.

(c)



53. For the $\lim_{t \to 2} B(t)$ to exist, the one-sided limits at $t = 2$ must be equal. $\lim_{t \to 2^-} B(t) = \lim_{t \to 2^-} \left(4 - \frac{1}{2}t\right) = 4 - 1 = 3$ and $\lim_{t \to 2^+} B(t) = \lim_{t \to 2^+} \sqrt{t + c} = \sqrt{2 + c}$. Now $3 = \sqrt{2 + c} \;\Rightarrow\; 9 = 2 + c \;\Leftrightarrow\; c = 7$.

54. (a) (i) $\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} x = 1$

(ii) $\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (2 - x^2) = 2 - 1^2 = 1$. Since $\lim_{x \to 1^-} g(x) = 1$ and $\lim_{x \to 1^+} g(x) = 1$, we have $\lim_{x \to 1} g(x) = 1$.

Note that the fact $g(1) = 3$ does not affect the value of the limit.

(iii) When $x = 1$, $g(x) = 3$, so $g(1) = 3$.

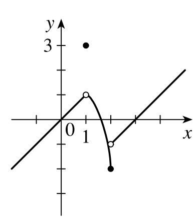
(iv) $\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} (2 - x^2) = 2 - 2^2 = 2 - 4 = -2$

(v) $\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (x - 3) = 2 - 3 = -1$

(vi) $\lim_{x \to 2} g(x)$ does not exist since $\lim_{x \to 2^-} g(x) \neq \lim_{x \to 2^+} g(x)$.

(b)

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$



55. (a) (i) $[\![x]\!] = -2$ for $-2 \leq x < -1$, so $\lim_{x \to -2^+} [\![x]\!] = \lim_{x \to -2^+} (-2) = -2$

(ii) $[\![x]\!] = -3$ for $-3 \leq x < -2$, so $\lim_{x \to -2^-} [\![x]\!] = \lim_{x \to -2^-} (-3) = -3$.

The right and left limits are different, so $\lim_{x \to -2} [\![x]\!]$ does not exist.

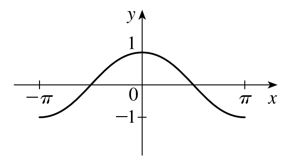
(iii) $[\![x]\!] = -3$ for $-3 \leq x < -2$, so $\lim_{x \to -2.4} [\![x]\!] = \lim_{x \to -2.4} (-3) = -3$.

(b) (i) $[\![x]\!] = n - 1$ for $n - 1 \le x < n$, so $\lim\limits_{x \to n^-} [\![x]\!] = \lim\limits_{x \to n^-} (n - 1) = n - 1$.

(ii) $[\![x]\!] = n$ for $n \le x < n + 1$, so $\lim\limits_{x \to n^+} [\![x]\!] = \lim\limits_{x \to n^+} n = n$.

(c) $\lim\limits_{x \to a} [\![x]\!]$ exists $\Leftrightarrow$ a is not an integer.

56. (a) See the graph of $y = \cos x$.

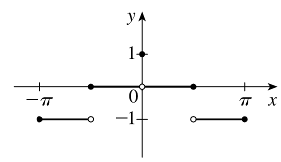


Since $-1 \le \cos x < 0$ on $[-\pi, -\pi/2)$, we have $y = f(x) = [\![\cos x]\!] = -1$ on $[-\pi, -\pi/2)$.

Since $0 \le \cos x < 1$ on $[-\pi/2, 0) \cup (0, \pi/2]$, we have $f(x) = 0$ on $[-\pi/2, 0) \cup (0, \pi/2]$.

Since $-1 \le \cos x < 0$ on $(\pi/2, \pi]$, we have $f(x) = -1$ on $(\pi/2, \pi]$.

Note that $f(0) = 1$.



(b) (i) $\lim\limits_{x \to 0^-} f(x) = 0$ and $\lim\limits_{x \to 0^+} f(x) = 0$, so $\lim\limits_{x \to 0} f(x) = 0$.

(ii) As $x \to (\pi/2)^-$, $f(x) \to 0$, so $\lim\limits_{x \to (\pi/2)^-} f(x) = 0$.

(iii) As $x \to (\pi/2)^+$, $f(x) \to -1$, so $\lim\limits_{x \to (\pi/2)^+} f(x) = -1$.

(iv) Since the answers in parts (ii) and (iii) are not equal, $\lim\limits_{x \to \pi/2} f(x)$ does not exist.

(c) $\lim\limits_{x \to a} f(x)$ exists for all a in the open interval $(-\pi, \pi)$ except $a = -\pi/2$ and $a = \pi/2$.

57. The graph of $f(x) = [\![x]\!] + [\![-x]\!]$ is the same as the graph of $g(x) = -1$ with holes at each integer, since $f(a) = 0$ for any integer a. Thus, $\lim\limits_{x \to 2^-} f(x) = -1$ and $\lim\limits_{x \to 2^+} f(x) = -1$, so $\lim\limits_{x \to 2} f(x) = -1$. However,

$f(2) = [\![2]\!] + [\![-2]\!] = 2 + (-2) = 0$, so $\lim\limits_{x \to 2} f(x) \ne f(2)$.

58. $\lim\limits_{v \to c^-} \left(L_0 \sqrt{1 - \dfrac{v^2}{c^2}} \right) = L_0 \sqrt{1 - 1} = 0$. As the velocity approaches the speed of light, the length approaches 0.

A left-hand limit is necessary since L is not defined for $v > c$.

59. Since $p(x)$ is a polynomial, $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. Thus, by the Limit Laws,

$$\begin{aligned} \lim_{x \to a} p(x) &= \lim_{x \to a} \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\right) = a_0 + a_1 \lim_{x \to a} x + a_2 \lim_{x \to a} x^2 + \cdots + a_n \lim_{x \to a} x^n \\ &= a_0 + a_1 a + a_2 a^2 + \cdots + a_n a^n = p(a) \end{aligned}$$

Thus, for any polynomial p, $\lim\limits_{x \to a} p(x) = p(a)$.

60. Let $r(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are any polynomials, and suppose that $q(a) \ne 0$. Then

$$\lim_{x \to a} r(x) = \lim_{x \to a} \frac{p(x)}{q(x)} = \frac{\lim\limits_{x \to a} p(x)}{\lim\limits_{x \to a} q(x)} \quad \text{[Limit Law 5]} \quad = \frac{p(a)}{q(a)} \quad \text{[Exercise 59]} \quad = r(a).$$

61. $\lim_{x\to 1}[f(x)-8] = \lim_{x\to 1}\left[\dfrac{f(x)-8}{x-1}\cdot(x-1)\right] = \lim_{x\to 1}\dfrac{f(x)-8}{x-1}\cdot\lim_{x\to 1}(x-1) = 10\cdot 0 = 0.$

Thus, $\lim_{x\to 1} f(x) = \lim_{x\to 1}\{[f(x)-8]+8\} = \lim_{x\to 1}[f(x)-8] + \lim_{x\to 1} 8 = 0+8 = 8.$

Note: The value of $\lim_{x\to 1}\dfrac{f(x)-8}{x-1}$ does not affect the answer since it's multiplied by 0. What's important is that $\lim_{x\to 1}\dfrac{f(x)-8}{x-1}$ exists.

62. (a) $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\dfrac{f(x)}{x^2}\cdot x^2\right] = \lim_{x\to 0}\dfrac{f(x)}{x^2}\cdot\lim_{x\to 0}x^2 = 5\cdot 0 = 0$

(b) $\lim_{x\to 0}\dfrac{f(x)}{x} = \lim_{x\to 0}\left[\dfrac{f(x)}{x^2}\cdot x\right] = \lim_{x\to 0}\dfrac{f(x)}{x^2}\cdot\lim_{x\to 0}x = 5\cdot 0 = 0$

63. Observe that $0 \le f(x) \le x^2$ for all x, and $\lim_{x\to 0} 0 = 0 = \lim_{x\to 0} x^2$. So, by the Squeeze Theorem, $\lim_{x\to 0} f(x) = 0$.

64. Let $f(x) = [\![x]\!]$ and $g(x) = -[\![x]\!]$. Then $\lim_{x\to 3} f(x)$ and $\lim_{x\to 3} g(x)$ do not exist [Example 10] but $\lim_{x\to 3}[f(x)+g(x)] = \lim_{x\to 3}([\![x]\!]-[\![x]\!]) = \lim_{x\to 3} 0 = 0.$

65. Let $f(x) = H(x)$ and $g(x) = 1 - H(x)$, where H is the Heaviside function defined in Exercise 1.3.63.

Thus, either f or g is 0 for any value of x. Then $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist, but $\lim_{x\to 0}[f(x)g(x)] = \lim_{x\to 0} 0 = 0.$

66.
$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} &= \lim_{x\to 2}\left(\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}\cdot\frac{\sqrt{6-x}+2}{\sqrt{6-x}+2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}\right)\\ &= \lim_{x\to 2}\left[\frac{\left(\sqrt{6-x}\right)^2-2^2}{\left(\sqrt{3-x}\right)^2-1^2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right] = \lim_{x\to 2}\left(\frac{6-x-4}{3-x-1}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right)\\ &= \lim_{x\to 2}\frac{(2-x)\left(\sqrt{3-x}+1\right)}{(2-x)\left(\sqrt{6-x}+2\right)} = \lim_{x\to 2}\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{1}{2}\end{aligned}$$

67. Since the denominator approaches 0 as $x \to -2$, the limit will exist only if the numerator also approaches 0 as $x \to -2$. In order for this to happen, we need $\lim_{x\to -2}\left(3x^2+ax+a+3\right) = 0 \quad\Leftrightarrow$

$3(-2)^2 + a(-2) + a + 3 = 0 \quad\Leftrightarrow\quad 12 - 2a + a + 3 = 0 \quad\Leftrightarrow\quad a = 15$. With $a = 15$, the limit becomes

$$\lim_{x\to -2}\frac{3x^2+15x+18}{x^2+x-2} = \lim_{x\to -2}\frac{3(x+2)(x+3)}{(x-1)(x+2)} = \lim_{x\to -2}\frac{3(x+3)}{x-1} = \frac{3(-2+3)}{-2-1} = \frac{3}{-3} = -1.$$

68. *Solution 1:* First, we find the coordinates of P and Q as functions of r. Then we can find the equation of the line determined by these two points, and thus find the x-intercept (the point R), and take the limit as $r \to 0$. The coordinates of P are $(0, r)$. The point Q is the point of intersection of the two circles $x^2 + y^2 = r^2$ and $(x-1)^2 + y^2 = 1$. Eliminating y from these equations, we get $r^2 - x^2 = 1 - (x-1)^2 \quad\Leftrightarrow\quad r^2 = 1 + 2x - 1 \quad\Leftrightarrow\quad x = \frac{1}{2}r^2$. Substituting back into the equation of the shrinking circle to find the y-coordinate, we get $\left(\frac{1}{2}r^2\right)^2 + y^2 = r^2 \quad\Leftrightarrow\quad y^2 = r^2\left(1 - \frac{1}{4}r^2\right) \quad\Leftrightarrow\quad y = r\sqrt{1 - \frac{1}{4}r^2}$

(the positive y-value). So the coordinates of Q are $\left(\frac{1}{2}r^2, r\sqrt{1 - \frac{1}{4}r^2}\right)$. The equation of the line joining P and Q is thus

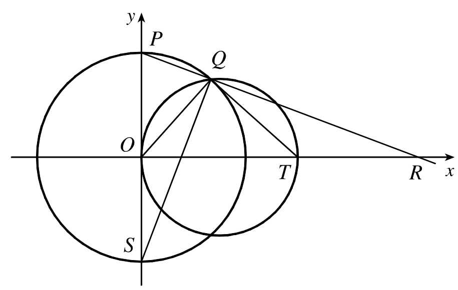
$y - r = \dfrac{r\sqrt{1 - \frac{1}{4}r^2} - r}{\frac{1}{2}r^2 - 0}(x - 0)$. We set $y = 0$ in order to find the x-intercept, and get

$$x = -r\frac{\frac{1}{2}r^2}{r\left(\sqrt{1 - \frac{1}{4}r^2} - 1\right)} = \frac{-\frac{1}{2}r^2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right)}{1 - \frac{1}{4}r^2 - 1} = 2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right)$$

Now we take the limit as $r \to 0^+$: $\lim\limits_{r\to 0^+} x = \lim\limits_{r\to 0^+} 2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right) = \lim\limits_{r\to 0^+} 2\left(\sqrt{1} + 1\right) = 4$.

So the limiting position of R is the point $(4, 0)$.

Solution 2: We add a few lines to the diagram, as shown. Note that $\angle PQS = 90^\circ$ (subtended by diameter PS). So $\angle SQR = 90^\circ = \angle OQT$ (subtended by diameter OT). It follows that $\angle OQS = \angle TQR$. Also $\angle PSQ = 90^\circ - \angle SPQ = \angle ORP$. Since $\triangle QOS$ is isosceles, so is $\triangle QTR$, implying that $QT = TR$. As the circle C_2 shrinks, the point Q plainly approaches the origin, so the point R must approach a point twice as far from the origin as T, that is, the point $(4, 0)$, as above.



2.4 The Precise Definition of a Limit

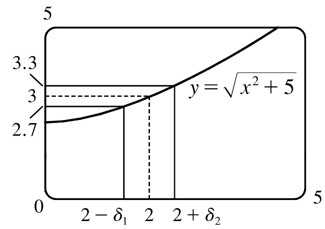
1. If $|f(x) - 1| < 0.2$, then $-0.2 < f(x) - 1 < 0.2 \quad\Rightarrow\quad 0.8 < f(x) < 1.2$. From the graph, we see that the last inequality is true if $0.7 < x < 1.1$, so we can choose $\delta = \min\{1 - 0.7, 1.1 - 1\} = \min\{0.3, 0.1\} = 0.1$ (or any smaller positive number).

2. If $|f(x) - 2| < 0.5$, then $-0.5 < f(x) - 2 < 0.5 \quad\Rightarrow\quad 1.5 < f(x) < 2.5$. From the graph, we see that the last inequality is true if $2.6 < x < 3.8$, so we can take $\delta = \min\{3 - 2.6, 3.8 - 3\} = \min\{0.4, 0.8\} = 0.4$ (or any smaller positive number). Note that $x \neq 3$.

3. The leftmost question mark is the solution of $\sqrt{x} = 1.6$ and the rightmost, $\sqrt{x} = 2.4$. So the values are $1.6^2 = 2.56$ and $2.4^2 = 5.76$. On the left side, we need $|x - 4| < |2.56 - 4| = 1.44$. On the right side, we need $|x - 4| < |5.76 - 4| = 1.76$. To satisfy both conditions, we need the more restrictive condition to hold—namely, $|x - 4| < 1.44$. Thus, we can choose $\delta = 1.44$, or any smaller positive number.

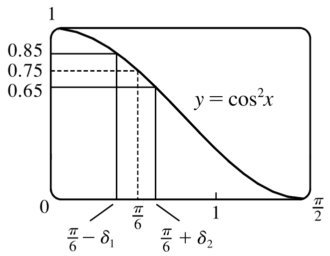
4. The leftmost question mark is the positive solution of $x^2 = \frac{1}{2}$, that is, $x = \frac{1}{\sqrt{2}}$, and the rightmost question mark is the positive solution of $x^2 = \frac{3}{2}$, that is, $x = \sqrt{\frac{3}{2}}$. On the left side, we need $|x - 1| < \left|\frac{1}{\sqrt{2}} - 1\right| \approx 0.292$ (rounding down to be safe). On the right side, we need $|x - 1| < \left|\sqrt{\frac{3}{2}} - 1\right| \approx 0.224$. The more restrictive of these two conditions must apply, so we choose $\delta = 0.224$ (or any smaller positive number).

5.



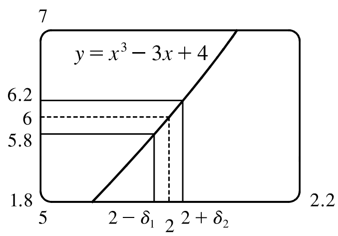
From the graph, we find that $y = \sqrt{x^2 + 5} = 2.7$ $[3 - 0.3]$ when $x \approx 1.513$, so $2 - \delta_1 \approx 1.513 \;\Rightarrow\; \delta_1 \approx 2 - 1.513 = 0.487$. Also, $y = \sqrt{x^2 + 5} = 3.3$ $[3 + 0.3]$ when $x \approx 2.426$, so $2 + \delta_2 \approx 2.426 \;\Rightarrow\; \delta_2 \approx 2.426 - 2 = 0.426$. Thus, we choose $\delta = 0.426$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

6.



From the graph, we find that $y = \cos^2 x = 0.85$ $[0.75 + 0.10]$ when $x \approx 0.398$, so $\frac{\pi}{6} - \delta_1 \approx 0.398 \;\Rightarrow\; \delta_1 \approx \frac{\pi}{6} - 0.398 \approx 0.126$. Also, $y = \cos^2 x = 0.65$ $[0.75 - 0.10]$ when $x \approx 0.633$, so $\frac{\pi}{6} + \delta_2 \approx 0.633 \;\Rightarrow\; \delta_2 \approx 0.633 - \frac{\pi}{6} \approx 0.109$. Thus, we choose $\delta = 0.109$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

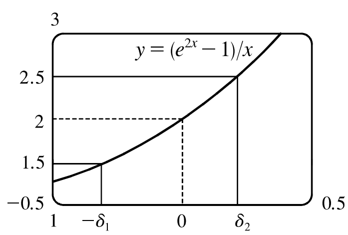
7.



From the graph with $\varepsilon = 0.2$, we find that $y = x^3 - 3x + 4 = 5.8$ $[6 - \varepsilon]$ when $x \approx 1.9774$, so $2 - \delta_1 \approx 1.9774 \;\Rightarrow\; \delta_1 \approx 0.0226$. Also, $y = x^3 - 3x + 4 = 6.2$ $[6 + \varepsilon]$ when $x \approx 2.022$, so $2 + \delta_2 \approx 2.0219 \;\Rightarrow\; \delta_2 \approx 0.0219$. Thus, we choose $\delta = 0.0219$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

For $\varepsilon = 0.1$, we get $\delta_1 \approx 0.0112$ and $\delta_2 \approx 0.0110$, so we choose $\delta = 0.011$ (or any smaller positive number).

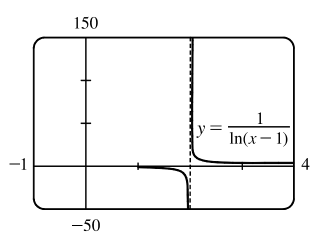
8.

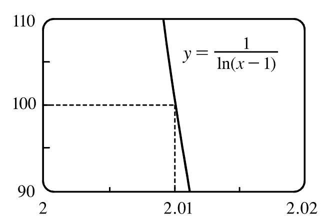


From the graph with $\varepsilon = 0.5$, we find that $y = (e^{2x} - 1)/x = 1.5$ $[2 - \varepsilon]$ when $x \approx -0.303$, so $\delta_1 \approx 0.303$. Also, $y = (e^{2x} - 1)/x = 2.5$ $[2 + \varepsilon]$ when $x \approx 0.215$, so $\delta_2 \approx 0.215$. Thus, we choose $\delta = 0.215$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

For $\varepsilon = 0.1$, we get $\delta_1 \approx 0.052$ and $\delta_2 \approx 0.048$, so we choose $\delta = 0.048$ (or any smaller positive number).

9. (a)



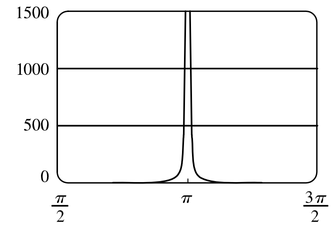


The first graph of $y = \dfrac{1}{\ln(x - 1)}$ shows a vertical asymptote at $x = 2$. The second graph shows that $y = 100$ when $x \approx 2.01$ (more accurately, 2.01005). Thus, we choose $\delta = 0.01$ (or any smaller positive number).

(b) From part (a), we see that as x gets closer to 2 from the right, y increases without bound. In symbols,

$$\lim_{x \to 2^+} \frac{1}{\ln(x - 1)} = \infty.$$

10. We graph $y = \csc^2 x$ and $y = 500$. The graphs intersect at $x \approx 3.186$, so we choose $\delta = 3.186 - \pi \approx 0.044$. Thus, if $0 < |x - \pi| < 0.044$, then $\csc^2 x > 500$. Similarly, for $M = 1000$, we get $\delta = 3.173 - \pi \approx 0.031$.



11. (a) $A = \pi r^2$ and $A = 1000\text{ cm}^2 \quad \Rightarrow \quad \pi r^2 = 1000 \quad \Rightarrow \quad r^2 = \frac{1000}{\pi} \quad \Rightarrow \quad r = \sqrt{\frac{1000}{\pi}} \quad (r > 0) \quad \approx 17.8412$ cm.

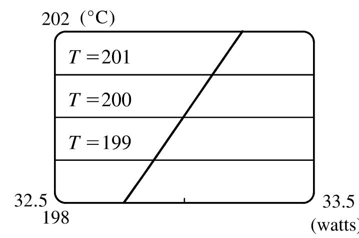
(b) $|A - 1000| \le 5 \quad \Rightarrow \quad -5 \le \pi r^2 - 1000 \le 5 \quad \Rightarrow \quad 1000 - 5 \le \pi r^2 \le 1000 + 5 \quad \Rightarrow$

$\sqrt{\frac{995}{\pi}} \le r \le \sqrt{\frac{1005}{\pi}} \quad \Rightarrow \quad 17.7966 \le r \le 17.8858$. $\sqrt{\frac{1000}{\pi}} - \sqrt{\frac{995}{\pi}} \approx 0.04466$ and $\sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}} \approx 0.04455$. So if the machinist gets the radius within 0.0445 cm of 17.8412, the area will be within 5 cm^2 of 1000.

(c) x is the radius, $f(x)$ is the area, a is the target radius given in part (a), L is the target area (1000 cm^2), ε is the magnitude of the error tolerance in the area (5 cm^2), and δ is the tolerance in the radius given in part (b).

12. (a) $T = 0.1w^2 + 2.155w + 20$ and $T = 200 \quad \Rightarrow$

$0.1w^2 + 2.155w + 20 = 200 \quad \Rightarrow$ [by the quadratic formula or from the graph] $\quad w \approx 33.0$ watts $(w > 0)$



(b) From the graph, $199 \le T \le 201 \quad \Rightarrow \quad 32.89 < w < 33.11$.

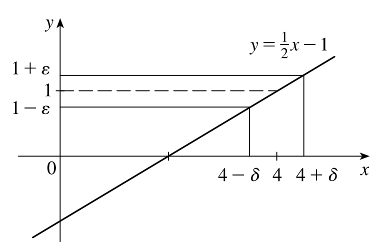
(c) x is the input power, $f(x)$ is the temperature, a is the target input power given in part (a), L is the target temperature (200), ε is the tolerance in the temperature (1), and δ is the tolerance in the power input in watts indicated in part (b) (0.11 watts).

13. (a) $|4x - 8| = 4\,|x - 2| < 0.1 \quad \Leftrightarrow \quad |x - 2| < \dfrac{0.1}{4}$, so $\delta = \dfrac{0.1}{4} = 0.025$.

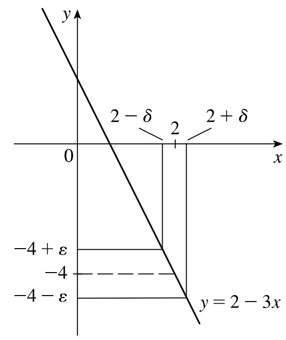
(b) $|4x - 8| = 4\,|x - 2| < 0.01 \quad \Leftrightarrow \quad |x - 2| < \dfrac{0.01}{4}$, so $\delta = \dfrac{0.01}{4} = 0.0025$.

14. $|(5x - 7) - 3| = |5x - 10| = |5(x - 2)| = 5\,|x - 2|$. We must have $|f(x) - L| < \varepsilon$, so $5\,|x - 2| < \varepsilon \quad \Leftrightarrow$ $|x - 2| < \varepsilon/5$. Thus, choose $\delta = \varepsilon/5$. For $\varepsilon = 0.1$, $\delta = 0.02$; for $\varepsilon = 0.05$, $\delta = 0.01$; for $\varepsilon = 0.01$, $\delta = 0.002$.

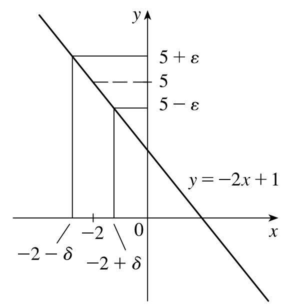
15. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 4| < \delta$, then $\left|\left(\frac{1}{2}x - 1\right) - 1\right| < \varepsilon$. But $\left|\left(\frac{1}{2}x - 1\right) - 1\right| < \varepsilon \quad \Leftrightarrow \quad \left|\frac{1}{2}x - 2\right| < \varepsilon \quad \Leftrightarrow$ $\left|\frac{1}{2}\right|\,|x - 4| < \varepsilon \quad \Leftrightarrow \quad |x - 4| < 2\varepsilon$. So if we choose $\delta = 2\varepsilon$, then $0 < |x - 4| < \delta \quad \Rightarrow \quad \left|\left(\frac{1}{2}x - 1\right) - 1\right| < \varepsilon$. Thus, $\lim\limits_{x \to 4} \left(\frac{1}{2}x - 1\right) = 1$ by the definition of a limit.



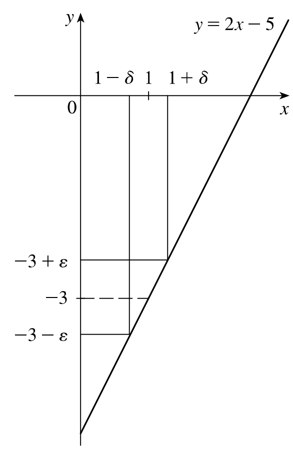
16. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|(2 - 3x) - (-4)| < \varepsilon$. But $|(2 - 3x) - (-4)| < \varepsilon \quad \Leftrightarrow$ $|6 - 3x| < \varepsilon \quad \Leftrightarrow \quad |-3|\,|x - 2| < \varepsilon \quad \Leftrightarrow \quad |x - 2| < \frac{1}{3}\varepsilon$. So if we choose $\delta = \frac{1}{3}\varepsilon$, then $0 < |x - 2| < \delta \quad \Rightarrow \quad |(2 - 3x) - (-4)| < \varepsilon$. Thus, $\lim\limits_{x \to 2} (2 - 3x) = -4$ by the definition of a limit.



17. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $|(-2x + 1) - 5| < \varepsilon$. But $|(-2x + 1) - 5| < \varepsilon \quad \Leftrightarrow$ $|-2x - 4| < \varepsilon \quad \Leftrightarrow \quad |-2|\,|x - (-2)| < \varepsilon \quad \Leftrightarrow \quad |x - (-2)| < \frac{1}{2}\varepsilon$. So if we choose $\delta = \frac{1}{2}\varepsilon$, then $0 < |x - (-2)| < \delta \quad \Rightarrow$ $|(-2x + 1) - 5| < \varepsilon$. Thus, $\lim\limits_{x \to -2} (-2x + 1) = 5$ by the definition of a limit.



18. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 1| < \delta$, then $|(2x - 5) - (-3)| < \varepsilon$. But $|(2x - 5) - (-3)| < \varepsilon \quad \Leftrightarrow$ $|2x - 2| < \varepsilon \quad \Leftrightarrow \quad |2|\,|x - 1| < \varepsilon \quad \Leftrightarrow \quad |x - 1| < \frac{1}{2}\varepsilon$. So if we choose $\delta = \frac{1}{2}\varepsilon$, then $0 < |x - 1| < \delta \quad \Rightarrow \quad |(2x - 5) - (-3)| < \varepsilon$. Thus, $\lim\limits_{x \to 1} (2x - 5) = -3$ by the definition of a limit.



19. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 9| < \delta$, then $\left|\left(1 - \frac{1}{3}x\right) - (-2)\right| < \varepsilon$. But $\left|\left(1 - \frac{1}{3}x\right) - (-2)\right| < \varepsilon \quad \Leftrightarrow$ $\left|3 - \frac{1}{3}x\right| < \varepsilon \quad \Leftrightarrow \quad \left|-\frac{1}{3}\right| |x - 9| < \varepsilon \quad \Leftrightarrow \quad |x - 9| < 3\varepsilon$. So if we choose $\delta = 3\varepsilon$, then $0 < |x - 9| < \delta \quad \Rightarrow$ $\left|\left(1 - \frac{1}{3}x\right) - (-2)\right| < \varepsilon$. Thus, $\lim\limits_{x \to 9} \left(1 - \frac{1}{3}x\right) = -2$ by the definition of a limit.

20. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 5| < \delta$, then $\left|\left(\frac{3}{2}x - \frac{1}{2}\right) - 7\right| < \varepsilon$. But $\left|\left(\frac{3}{2}x - \frac{1}{2}\right) - 7\right| < \varepsilon \quad \Leftrightarrow$ $\left|\frac{3}{2}x - \frac{15}{2}\right| < \varepsilon \quad \Leftrightarrow \quad \left|\frac{3}{2}\right| |x - 5| < \varepsilon \quad \Leftrightarrow \quad |x - 5| < \frac{2}{3}\varepsilon$. So if we choose $\delta = \frac{2}{3}\varepsilon$, then $0 < |x - 5| < \delta \quad \Rightarrow$ $\left|\left(\frac{3}{2}x - \frac{1}{2}\right) - 7\right| < \varepsilon$. Thus, $\lim\limits_{x \to 5} \left(\frac{3}{2}x - \frac{1}{2}\right) = 7$ by the definition of a limit.

21. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-4| < \delta$, then $\left|\dfrac{x^2-2x-8}{x-4} - 6\right| < \varepsilon \iff$

$\left|\dfrac{(x-4)(x+2)}{x-4} - 6\right| < \varepsilon \iff |x+2-6| < \varepsilon \quad [x \neq 4] \iff |x-4| < \varepsilon$. So choose $\delta = \varepsilon$. Then

$0 < |x-4| < \delta \Rightarrow |x-4| < \varepsilon \Rightarrow |x+2-6| < \varepsilon \Rightarrow \left|\dfrac{(x-4)(x+2)}{x-4} - 6\right| < \varepsilon \quad [x \neq 4] \Rightarrow$

$\left|\dfrac{x^2-2x-8}{x-4} - 6\right| < \varepsilon$. By the definition of a limit, $\lim\limits_{x\to 4} \dfrac{x^2-2x-8}{x-4} = 6$.

22. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x+1.5| < \delta$, then $\left|\dfrac{9-4x^2}{3+2x} - 6\right| < \varepsilon \iff$

$\left|\dfrac{(3+2x)(3-2x)}{3+2x} - 6\right| < \varepsilon \iff |3-2x-6| < \varepsilon \quad [x \neq -1.5] \iff |-2x-3| < \varepsilon \iff |-2|\,|x+1.5| < \varepsilon \iff$

$|x+1.5| < \varepsilon/2$. So choose $\delta = \varepsilon/2$. Then $0 < |x+1.5| < \delta \Rightarrow |x+1.5| < \varepsilon/2 \Rightarrow |-2|\,|x+1.5| < \varepsilon \Rightarrow$

$|-2x-3| < \varepsilon \Rightarrow |3-2x-6| < \varepsilon \Rightarrow \left|\dfrac{(3+2x)(3-2x)}{3+2x} - 6\right| < \varepsilon \quad [x \neq -1.5] \Rightarrow \left|\dfrac{9-4x^2}{3+2x} - 6\right| < \varepsilon$.

By the definition of a limit, $\lim\limits_{x\to -1.5} \dfrac{9-4x^2}{3+2x} = 6$.

23. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|x-a| < \varepsilon$. So $\delta = \varepsilon$ will work.

24. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|c-c| < \varepsilon$. But $|c-c| = 0$, so this will be true no matter what δ we pick.

25. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-0| < \delta$, then $\left|x^2-0\right| < \varepsilon \iff x^2 < \varepsilon \iff |x| < \sqrt{\varepsilon}$. Take $\delta = \sqrt{\varepsilon}$.

Then $0 < |x-0| < \delta \Rightarrow \left|x^2-0\right| < \varepsilon$. Thus, $\lim\limits_{x\to 0} x^2 = 0$ by the definition of a limit.

26. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-0| < \delta$, then $\left|x^3-0\right| < \varepsilon \iff |x|^3 < \varepsilon \iff |x| < \sqrt[3]{\varepsilon}$. Take $\delta = \sqrt[3]{\varepsilon}$.

Then $0 < |x-0| < \delta \Rightarrow \left|x^3-0\right| < \delta^3 = \varepsilon$. Thus, $\lim\limits_{x\to 0} x^3 = 0$ by the definition of a limit.

27. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-0| < \delta$, then $\big||x|-0\big| < \varepsilon$. But $\big||x|\big| = |x|$. So this is true if we pick $\delta = \varepsilon$.

Thus, $\lim\limits_{x\to 0} |x| = 0$ by the definition of a limit.

28. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < x-(-6) < \delta$, then $\left|\sqrt[8]{6+x} - 0\right| < \varepsilon$. But $\left|\sqrt[8]{6+x} - 0\right| < \varepsilon \iff$

$\sqrt[8]{6+x} < \varepsilon \iff 6+x < \varepsilon^8 \iff x-(-6) < \varepsilon^8$. So if we choose $\delta = \varepsilon^8$, then $0 < x-(-6) < \delta \Rightarrow$

$\left|\sqrt[8]{6+x} - 0\right| < \varepsilon$. Thus, $\lim\limits_{x\to -6^+} \sqrt[8]{6+x} = 0$ by the definition of a right-hand limit.

29. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-2| < \delta$, then $\left|\left(x^2-4x+5\right) - 1\right| < \varepsilon \iff \left|x^2-4x+4\right| < \varepsilon \iff$

$\left|(x-2)^2\right| < \varepsilon$. So take $\delta = \sqrt{\varepsilon}$. Then $0 < |x-2| < \delta \iff |x-2| < \sqrt{\varepsilon} \iff \left|(x-2)^2\right| < \varepsilon$. Thus,

$\lim\limits_{x\to 2} \left(x^2-4x+5\right) = 1$ by the definition of a limit.

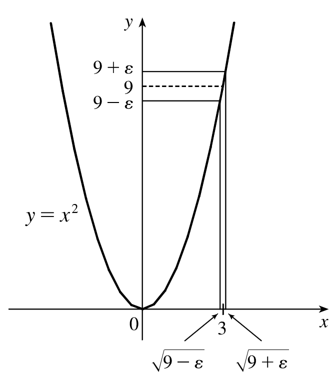
30. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-2| < \delta$, then $\left|(x^2+2x-7)-1\right| < \varepsilon$. But $\left|(x^2+2x-7)-1\right| < \varepsilon \quad \Leftrightarrow$ $\left|x^2+2x-8\right| < \varepsilon \quad \Leftrightarrow \quad |x+4|\,|x-2| < \varepsilon$. Thus our goal is to make $|x-2|$ small enough so that its product with $|x+4|$ is less than ε. Suppose we first require that $|x-2| < 1$. Then $-1 < x-2 < 1 \quad \Rightarrow \quad 1 < x < 3 \quad \Rightarrow \quad 5 < x+4 < 7 \quad \Rightarrow$ $|x+4| < 7$, and this gives us $7\,|x-2| < \varepsilon \quad \Rightarrow \quad |x-2| < \varepsilon/7$. Choose $\delta = \min\{1, \varepsilon/7\}$. Then if $0 < |x-2| < \delta$, we have $|x-2| < \varepsilon/7$ and $|x+4| < 7$, so $\left|(x^2+2x-7)-1\right| = |(x+4)(x-2)| = |x+4|\,|x-2| < 7(\varepsilon/7) = \varepsilon$, as desired. Thus, $\lim\limits_{x\to 2}(x^2+2x-7) = 1$ by the definition of a limit.

31. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-(-2)| < \delta$, then $\left|\left(x^2-1\right)-3\right| < \varepsilon$ or upon simplifying we need $\left|x^2-4\right| < \varepsilon$ whenever $0 < |x+2| < \delta$. Notice that if $|x+2| < 1$, then $-1 < x+2 < 1 \quad \Rightarrow \quad -5 < x-2 < -3 \quad \Rightarrow$ $|x-2| < 5$. So take $\delta = \min\{\varepsilon/5, 1\}$. Then $0 < |x+2| < \delta \quad \Rightarrow \quad |x-2| < 5$ and $|x+2| < \varepsilon/5$, so $\left|\left(x^2-1\right)-3\right| = |(x+2)(x-2)| = |x+2|\,|x-2| < (\varepsilon/5)(5) = \varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x\to -2}(x^2-1) = 3$.

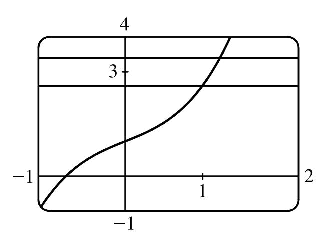
32. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-2| < \delta$, then $\left|x^3-8\right| < \varepsilon$. Now $\left|x^3-8\right| = \left|(x-2)\left(x^2+2x+4\right)\right|$. If $|x-2| < 1$, that is, $1 < x < 3$, then $x^2+2x+4 \; < \; 3^2+2(3)+4 = 19$ and so $\left|x^3-8\right| = |x-2|\left(x^2+2x+4\right) < 19\,|x-2|$. So if we take $\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$, then $0 < |x-2| < \delta \quad \Rightarrow$ $\left|x^3-8\right| = |x-2|\left(x^2+2x+4\right) < \frac{\varepsilon}{19}\cdot 19 = \varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x\to 2} x^3 = 8$.

33. Given $\varepsilon > 0$, we let $\delta = \min\left\{2, \frac{\varepsilon}{8}\right\}$. If $0 < |x-3| < \delta$, then $|x-3| < 2 \quad \Rightarrow \quad -2 < x-3 < 2 \quad \Rightarrow$ $4 < x+3 < 8 \quad \Rightarrow \quad |x+3| < 8$. Also $|x-3| < \frac{\varepsilon}{8}$, so $\left|x^2-9\right| = |x+3|\,|x-3| < 8\cdot\frac{\varepsilon}{8} = \varepsilon$. Thus, $\lim\limits_{x\to 3} x^2 = 9$.

34. From the figure, our choices for δ are $\delta_1 = 3 - \sqrt{9-\varepsilon}$ and $\delta_2 = \sqrt{9+\varepsilon} - 3$. The *largest* possible choice for δ is the minimum value of $\{\delta_1, \delta_2\}$; that is, $\delta = \min\{\delta_1, \delta_2\} = \delta_2 = \sqrt{9+\varepsilon} - 3$.



35. (a) The points of intersection in the graph are $(x_1, 2.6)$ and $(x_2, 3.4)$ with $x_1 \approx 0.891$ and $x_2 \approx 1.093$. Thus, we can take δ to be the smaller of $1 - x_1$ and $x_2 - 1$. So $\delta = x_2 - 1 \approx 0.093$.



(b) Solving $x^3 + x + 1 = 3 + \varepsilon$ with a CAS gives us two nonreal complex solutions and one real solution, which is

$$x(\varepsilon) = \frac{\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\right)^{2/3} - 12}{6\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\right)^{1/3}}. \text{ Thus, } \delta = x(\varepsilon) - 1.$$

(c) If $\varepsilon = 0.4$, then $x(\varepsilon) \approx 1.093\,272\,342$ and $\delta = x(\varepsilon) - 1 \approx 0.093$, which agrees with our answer in part (a).

36. *1. Guessing a value for* δ Let $\varepsilon > 0$ be given. We have to find a number $\delta > 0$ such that $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$ whenever $0 < |x - 2| < \delta$. But $\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2 - x}{2x}\right| = \frac{|x - 2|}{|2x|} < \varepsilon$. We find a positive constant C such that $\frac{1}{|2x|} < C \Rightarrow \frac{|x-2|}{|2x|} < C\,|x - 2|$ and we can make $C\,|x - 2| < \varepsilon$ by taking $|x - 2| < \frac{\varepsilon}{C} = \delta$. We restrict x to lie in the interval $|x - 2| < 1 \Rightarrow 1 < x < 3$ so $1 > \frac{1}{x} > \frac{1}{3} \Rightarrow \frac{1}{6} < \frac{1}{2x} < \frac{1}{2} \Rightarrow \frac{1}{|2x|} < \frac{1}{2}$. So $C = \frac{1}{2}$ is suitable. Thus, we should choose $\delta = \min\{1, 2\varepsilon\}$.

2. Showing that δ *works* Given $\varepsilon > 0$ we let $\delta = \min\{1, 2\varepsilon\}$. If $0 < |x - 2| < \delta$, then $|x - 2| < 1 \Rightarrow 1 < x < 3 \Rightarrow \frac{1}{|2x|} < \frac{1}{2}$ (as in part 1). Also $|x - 2| < 2\varepsilon$, so $\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x - 2|}{|2x|} < \frac{1}{2} \cdot 2\varepsilon = \varepsilon$. This shows that $\lim_{x \to 2} (1/x) = \frac{1}{2}$.

37. *1. Guessing a value for* δ Given $\varepsilon > 0$, we must find $\delta > 0$ such that $|\sqrt{x} - \sqrt{a}| < \varepsilon$ whenever $0 < |x - a| < \delta$. But $|\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \varepsilon$ (from the hint). Now if we can find a positive constant C such that $\sqrt{x} + \sqrt{a} > C$ then $\frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \frac{|x - a|}{C} < \varepsilon$, and we take $|x - a| < C\varepsilon$. We can find this number by restricting x to lie in some interval centered at a. If $|x - a| < \frac{1}{2}a$, then $-\frac{1}{2}a < x - a < \frac{1}{2}a \Rightarrow \frac{1}{2}a < x < \frac{3}{2}a \Rightarrow \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$, and so $C = \sqrt{\frac{1}{2}a} + \sqrt{a}$ is a suitable choice for the constant. So $|x - a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon$. This suggests that we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon\right\}$.

2. Showing that δ *works* Given $\varepsilon > 0$, we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon\right\}$. If $0 < |x - a| < \delta$, then $|x - a| < \frac{1}{2}a \Rightarrow \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$ (as in part 1). Also $|x - a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon$, so

$$|\sqrt{x} - \sqrt{a}\,| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \frac{\left(\sqrt{a/2} + \sqrt{a}\right)\varepsilon}{\left(\sqrt{a/2} + \sqrt{a}\right)} = \varepsilon. \text{ Therefore, } \lim_{x \to a} \sqrt{x} = \sqrt{a} \text{ by the definition of a limit.}$$

38. Suppose that $\lim_{t \to 0} H(t) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |t| < \delta \Rightarrow |H(t) - L| < \frac{1}{2} \Leftrightarrow L - \frac{1}{2} < H(t) < L + \frac{1}{2}$. For $0 < t < \delta$, $H(t) = 1$, so $1 < L + \frac{1}{2} \Rightarrow L > \frac{1}{2}$. For $-\delta < t < 0$, $H(t) = 0$, so $L - \frac{1}{2} < 0 \Rightarrow L < \frac{1}{2}$. This contradicts $L > \frac{1}{2}$. Therefore, $\lim_{t \to 0} H(t)$ does not exist.

39. Suppose that $\lim_{x \to 0} f(x) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |x| < \delta \Rightarrow |f(x) - L| < \frac{1}{2}$. Take any rational number r with $0 < |r| < \delta$. Then $f(r) = 0$, so $|0 - L| < \frac{1}{2}$, so $L \leq |L| < \frac{1}{2}$. Now take any irrational number s with

$0 < |s| < \delta$. Then $f(s) = 1$, so $|1 - L| < \frac{1}{2}$. Hence, $1 - L < \frac{1}{2}$, so $L > \frac{1}{2}$. This contradicts $L < \frac{1}{2}$, so $\lim\limits_{x\to 0} f(x)$ does not exist.

40. First suppose that $\lim\limits_{x\to a} f(x) = L$. Then, given $\varepsilon > 0$ there exists $\delta > 0$ so that $0 < |x - a| < \delta \quad\Rightarrow\quad |f(x) - L| < \varepsilon$. Then $a - \delta < x < a \quad\Rightarrow\quad 0 < |x - a| < \delta$ so $|f(x) - L| < \varepsilon$. Thus, $\lim\limits_{x\to a^-} f(x) = L$. Also $a < x < a + \delta \quad\Rightarrow$ $0 < |x - a| < \delta$ so $|f(x) - L| < \varepsilon$. Hence, $\lim\limits_{x\to a^+} f(x) = L$.

Now suppose $\lim\limits_{x\to a^-} f(x) = L = \lim\limits_{x\to a^+} f(x)$. Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to a^-} f(x) = L$, there exists $\delta_1 > 0$ so that $a - \delta_1 < x < a \quad\Rightarrow\quad |f(x) - L| < \varepsilon$. Since $\lim\limits_{x\to a^+} f(x) = L$, there exists $\delta_2 > 0$ so that $a < x < a + \delta_2 \quad\Rightarrow$ $|f(x) - L| < \varepsilon$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x - a| < \delta \quad\Rightarrow\quad a - \delta_1 < x < a$ or $a < x < a + \delta_2$ so $|f(x) - L| < \varepsilon$. Hence, $\lim\limits_{x\to a} f(x) = L$. So we have proved that $\lim\limits_{x\to a} f(x) = L \quad\Leftrightarrow\quad \lim\limits_{x\to a^-} f(x) = L = \lim\limits_{x\to a^+} f(x)$.

41. $\dfrac{1}{(x+3)^4} > 10{,}000 \quad\Leftrightarrow\quad (x+3)^4 < \dfrac{1}{10{,}000} \quad\Leftrightarrow\quad |x+3| < \dfrac{1}{\sqrt[4]{10{,}000}} \quad\Leftrightarrow\quad |x - (-3)| < \dfrac{1}{10}$

42. Given $M > 0$, we need $\delta > 0$ such that $0 < |x+3| < \delta \quad\Rightarrow\quad 1/(x+3)^4 > M$. Now $\dfrac{1}{(x+3)^4} > M \quad\Leftrightarrow$ $(x+3)^4 < \dfrac{1}{M} \quad\Leftrightarrow\quad |x+3| < \dfrac{1}{\sqrt[4]{M}}$. So take $\delta = \dfrac{1}{\sqrt[4]{M}}$. Then $0 < |x+3| < \delta = \dfrac{1}{\sqrt[4]{M}} \quad\Rightarrow\quad \dfrac{1}{(x+3)^4} > M$, so $\lim\limits_{x\to -3} \dfrac{1}{(x+3)^4} = \infty$.

43. Given $M < 0$ we need $\delta > 0$ so that $\ln x < M$ whenever $0 < x < \delta$; that is, $x = e^{\ln x} < e^M$ whenever $0 < x < \delta$. This suggests that we take $\delta = e^M$. If $0 < x < e^M$, then $\ln x < \ln e^M = M$. By the definition of a limit, $\lim\limits_{x\to 0^+} \ln x = -\infty$.

44. (a) Let M be given. Since $\lim\limits_{x\to a} f(x) = \infty$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \quad\Rightarrow\quad f(x) > M + 1 - c$. Since $\lim\limits_{x\to a} g(x) = c$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \quad\Rightarrow\quad |g(x) - c| < 1 \quad\Rightarrow\quad g(x) > c - 1$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x - a| < \delta \quad\Rightarrow\quad f(x) + g(x) > (M + 1 - c) + (c - 1) = M$. Thus, $\lim\limits_{x\to a} [f(x) + g(x)] = \infty$.

(b) Let $M > 0$ be given. Since $\lim\limits_{x\to a} g(x) = c > 0$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \quad\Rightarrow$ $|g(x) - c| < c/2 \quad\Rightarrow\quad g(x) > c/2$. Since $\lim\limits_{x\to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \quad\Rightarrow$ $f(x) > 2M/c$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - a| < \delta \quad\Rightarrow\quad f(x)\,g(x) > \dfrac{2M}{c}\,\dfrac{c}{2} = M$, so $\lim\limits_{x\to a} f(x)\,g(x) = \infty$.

(c) Let $N < 0$ be given. Since $\lim\limits_{x\to a} g(x) = c < 0$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \quad\Rightarrow$ $|g(x) - c| < -c/2 \quad\Rightarrow\quad g(x) < c/2$. Since $\lim\limits_{x\to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \quad\Rightarrow$ $f(x) > 2N/c$. (Note that $c < 0$ and $N < 0 \quad\Rightarrow\quad 2N/c > 0$.) Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - a| < \delta \quad\Rightarrow$ $f(x) > 2N/c \quad\Rightarrow\quad f(x)\,g(x) < \dfrac{2N}{c} \cdot \dfrac{c}{2} = N$, so $\lim\limits_{x\to a} f(x)\,g(x) = -\infty$.

2.5 Continuity

1. From Definition 1, $\lim_{x \to 4} f(x) = f(4)$.

2. The graph of f has no hole, jump, or vertical asymptote.

3. (a) f is discontinuous at -4 since $f(-4)$ is not defined and at -2, 2, and 4 since the limit does not exist (the left and right limits are not the same).

(b) f is continuous from the left at -2 since $\lim_{x \to -2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since $\lim_{x \to 2^+} f(x) = f(2)$ and $\lim_{x \to 4^+} f(x) = f(4)$. The function is not continuous from either side at -4 since $f(-4)$ is undefined.

4. g is not continuous at -2 since $g(-2)$ is not defined. g is not continuous at $a = -1$ since the limit does not exist (the left and right limits are $-\infty$). g is not continuous at $a = 0$ and $a = 1$ since the limit does not exist (the left and right limits are not equal).

5. (a) From the graph we see that $\lim_{x \to a} f(x)$ does not exist at $a = 1$ since the left and right limits are not the same.

(b) f is not continuous at $a = 1$ since $\lim_{x \to 1} f(x)$ does not exist by part (a). Also, f is not continuous at $a = 3$ since $\lim_{x \to 3} f(x) \neq f(3)$.

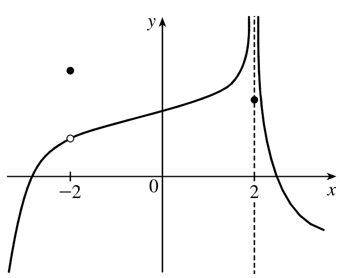
(c) From the graph we see that $\lim_{x \to 3} f(x) = 3$, but $f(3) = 2$. Since the limit is not equal to $f(3)$, f is not continuous at $a = 3$.

6. (a) From the graph we see that $\lim_{x \to a} f(x)$ does not exist at $a = 1$ since the function increases without bound from the left and from the right. Also, $\lim_{x \to a} f(x)$ does not exist at $a = 5$ since the left and right limits are not the same.

(b) f is not continuous at $a = 1$ and at $a = 5$ since the limits do not exist by part (a). Also, f is not continuous at $a = 3$ since $\lim_{x \to 3} f(x) \neq f(3)$.

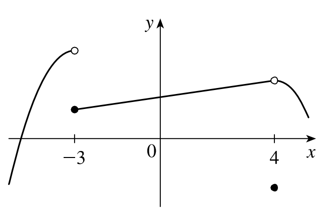
(c) From the graph we see that $\lim_{x \to 3} f(x)$ exists, but the limit is not equal to $f(3)$, so f is not continuous at $a = 3$.

7.

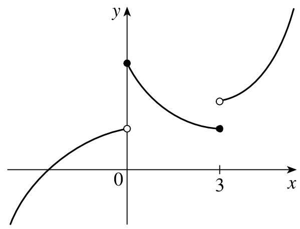


The graph of $y = f(x)$ must have a removable discontinuity (a hole) at $x = -2$ and an infinite discontinuity (a vertical asymptote) at $x = 2$.

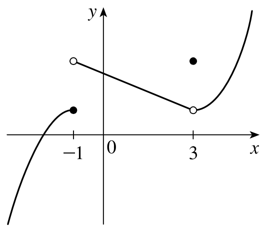
8.



The graph of $y = f(x)$ must have a jump discontinuity at $x = -3$ and a removable discontinuity (a hole) at $x = 4$.

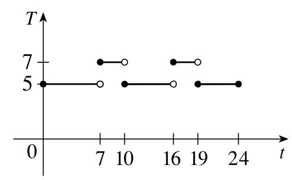
9. 

The graph of $y = f(x)$ must have discontinuities at $x = 0$ and $x = 3$. It must show that $\lim_{x \to 0^+} f(x) = f(0)$ and $\lim_{x \to 3^-} f(x) = f(3)$.

10. 

The graph of $y = f(x)$ must have a discontinuity at $x = -1$ with $\lim_{x \to -1^-} f(x) = f(-1)$ and $\lim_{x \to -1^+} f(x) \neq f(-1)$. The graph must also show that $\lim_{x \to 3^-} f(x) \neq f(3)$ and $\lim_{x \to 3^+} f(x) \neq f(3)$.

11. (a) The toll is \$5 except between 7:00 AM and 10:00 AM and between 4:00 PM and 7:00 PM, when the toll is \$7.



(b) The function T has jump discontinuities at $t = 7$, 10, 16, and 19. Their significance to someone who uses the road is that, because of the sudden jumps in the toll, they may want to avoid the higher rates between $t = 7$ and $t = 10$ and between $t = 16$ and $t = 19$ if feasible.

12. (a) Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.

(b) Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.

(c) Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values — at a cliff, for example.

(d) Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.

(e) Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.

13. $$\lim_{x \to -1} f(x) = \lim_{x \to -1} \left[3x^2 + (x+2)^5\right] = \lim_{x \to -1} 3x^2 + \lim_{x \to -1} (x+2)^5 = 3 \lim_{x \to -1} x^2 + \lim_{x \to -1} (x+2)^5$$
$$= 3(-1)^2 + (-1+2)^5 = 4 = f(-1)$$

By the definition of continuity, f is continuous at $a = -1$.

14. $$\lim_{t \to 2} g(t) = \lim_{t \to 2} \frac{t^2 + 5t}{2t + 1} = \frac{\lim\limits_{t \to 2}(t^2 + 5t)}{\lim\limits_{t \to 2}(2t + 1)} = \frac{\lim\limits_{t \to 2} t^2 + 5 \lim\limits_{t \to 2} t}{2 \lim\limits_{t \to 2} t + \lim\limits_{t \to 2} 1} = \frac{2^2 + 5(2)}{2(2) + 1} = \frac{14}{5} = g(2).$$

By the definition of continuity, g is continuous at $a = 2$.

15. $\lim\limits_{v\to 1} p(v) = \lim\limits_{v\to 1} 2\sqrt{3v^2+1} = 2\lim\limits_{v\to 1}\sqrt{3v^2+1} = 2\sqrt{\lim\limits_{v\to 1}(3v^2+1)} = 2\sqrt{3\lim\limits_{v\to 1} v^2 + \lim\limits_{v\to 1} 1}$

$= 2\sqrt{3(1)^2+1} = 2\sqrt{4} = 4 = p(1)$

By the definition of continuity, p is continuous at $a = 1$.

16. $\lim\limits_{r\to -2} f(r) = \lim\limits_{r\to -2} \sqrt[3]{4r^2-2r+7} = \sqrt[3]{\lim\limits_{r\to -2}(4r^2-2r+7)} = \sqrt[3]{4(-2)^2-2(-2)+7} = \sqrt[3]{27} = 3 = f(-2)$

By the definition of continuity, f is continuous at $a = -2$.

17. For $a > 4$, we have

$$\begin{aligned} \lim_{x\to a} f(x) &= \lim_{x\to a}(x+\sqrt{x-4}) = \lim_{x\to a} x + \lim_{x\to a}\sqrt{x-4} && \text{[Limit Law 1]} \\ &= a + \sqrt{\lim_{x\to a} x - \lim_{x\to a} 4} && \text{[8, 11, and 2]} \\ &= a + \sqrt{a-4} && \text{[8 and 7]} \\ &= f(a) \end{aligned}$$

So f is continuous at $x = a$ for every a in $(4,\infty)$. Also, $\lim\limits_{x\to 4^+} f(x) = 4 = f(4)$, so f is continuous from the right at 4.

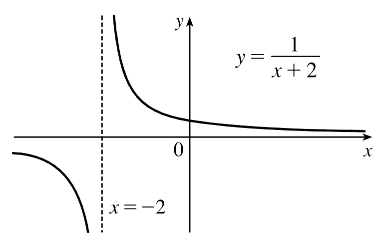
Thus, f is continuous on $[4,\infty)$.

18. For $a < -2$, we have

$$\begin{aligned} \lim_{x\to a} g(x) = \lim_{x\to a}\frac{x-1}{3x+6} &= \frac{\lim\limits_{x\to a}(x-1)}{\lim\limits_{x\to a}(3x+6)} && \text{[Limit Law 5]} \\ &= \frac{\lim\limits_{x\to a} x - \lim\limits_{x\to a} 1}{3\lim\limits_{x\to a} x + \lim\limits_{x\to a} 6} && \text{[2, 1, and 3]} \\ &= \frac{a-1}{3a+6} && \text{[8 and 7]} \end{aligned}$$

Thus, g is continuous at $x = a$ for every a in $(-\infty,-2)$; that is, g is continuous on $(-\infty,-2)$.

19. $f(x) = \dfrac{1}{x+2}$ is discontinuous at $a = -2$ because $f(-2)$ is undefined.



20. $f(x) = \begin{cases} \dfrac{1}{x+2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$

Here $f(-2) = 1$, but $\lim\limits_{x\to -2^-} f(x) = -\infty$ and $\lim\limits_{x\to -2^+} f(x) = \infty$,

so $\lim\limits_{x\to -2} f(x)$ does not exist and f is discontinuous at -2.