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Assignment 2

1. (10%) Is stable matching unique (i.e., for any input, there is only one possible stable matching)? Please explain your answer.
2. (20%) Find a stable matching for the following input, and prove that your matching does not contain unstable pairs:

   Men: {w, x, y}, women: {a, b, c}

   w: c, b, a (c is w's favorite, and a is w's least favorite)

   x: c, b, a

   y: a, c, b

   a: w, x, y

   b: w, y, x

   c: y, w, x

stable matching

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3. (10%) Design a graph G such that:
   a. G has 8 vertices.
   b. Minimum degree = 3.
   c. Closure of G is a complete graph.

   Please explain why your graph satisfies the above constraints.
4. (10%) Prove that if $\alpha(G) \leq \kappa(G)$ and $G \neq K_2$, then the minimum degree of G is at least 2.
5. (10%) Design a graph G such that G has an Euler tour but G has no Hamiltonian cycle. Please explain why your graph satisfies the above constraints.
6. (10%) Design a graph G such that:
   a. G has 6 vertices.
   b. $\kappa(G) = \alpha(G)$.

   Clearly explain how you constructed your graph and justify why it meets the stated constraints.

Hamiltonian cycle

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7. (10%) Let G be a 2-connected graph. Let (u, v) be any edge of G. Construct a new graph G' by removing (u, v), adding a new vertex w, and adding two edges (u, w) and (w, v). Prove that G' is 2-connected.
8. (20%) Let G be a 3-connected graph, and let G' be a graph obtained from G by adding a new vertex y with at least 3 neighbors in G. Prove that G' is 3-connected.

   If your proof involves a case analysis (e.g., "if …; otherwise …"), please provide a concrete example graph for each case.

M-connected

1. Not always

e.g.

- stable matching 1:

$(f_2 > f_1 > f_3)\ w_1$ $\qquad$ $f_1\ (w_1 > w_2 > w_3)$

$(f_2 > f_3 > f_1)\ w_2$ $\qquad$ $f_2\ (w_1 > w_2 > w_3)$

$(f_1 > f_2 > f_3)\ w_3$ $\qquad$ $f_3\ (w_1 > w_3 > w_2)$

- stable matching 2:

$(f_2 > f_1 > f_3)\ w_1$ $\qquad$ $f_1\ (w_1 > w_2 > w_3)$

$(f_2 > f_3 > f_1)\ w_2$ $\qquad$ $f_2\ (w_1 > w_2 > w_3)$

$(f_1 > f_2 > f_3)\ w_3$ —— $f_3\ (w_1 > w_3 > w_2)$

2.

Gale-Shapely Algorithm:

w	c	b	a
x	~~c~~	b	a
y	a	c	b

$\xrightarrow{\text{Propose}}$

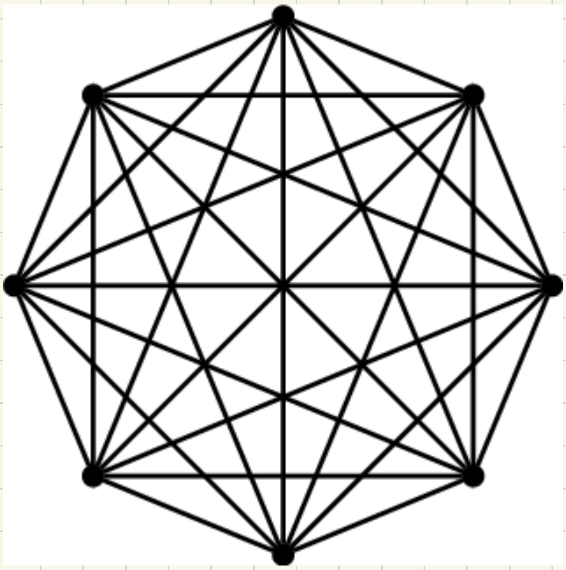
a	w	x	y
b	w	y	x
c	y	w	~~x~~

final matching : (w, c) , (x, b) , (y, a)

<pf>

is not unstable	reason
(w, a)	for w: $c > a$
(w, b)	for w: $c > b$
(x, a)	for x: $b > a$
(x, c)	for c: $w > x$
(y, c)	for y: $a > c$
(y, b)	for y: $a > b$

3. The graph I design is K_8.



a. $\#(\text{nodes in } K_8) = 8$

b. $\forall$ vertex v, $\deg(v) = 7$

$\Rightarrow$ min. deg $= 7$

c. $cl(K_8) = K_8$

4.

- $\alpha(G)$: independence number (size of the max. independent set)
- $\kappa(G)$: connectivity
- $\delta(G)$: min. degree

- The statement we'd like to prove is

$\alpha(G) \le \kappa(G)$ and $G \ne K_1 \Rightarrow \delta(G) \ge 2$

- The contrapositive is

$\delta(G) < 2 \Rightarrow \alpha(G) > \kappa(G)$ or $G = K_1$

$\searrow$ $\delta(G) = 0$ or 1

$\Rightarrow \exists v \in V(G)$, s.t. $\deg(v) = 0$ or 1

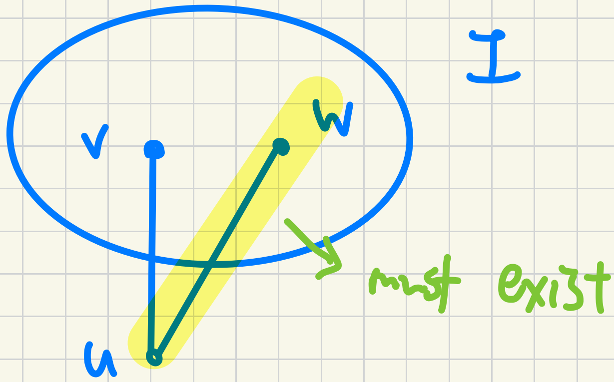
① case 1 : $\exists v \in V(G)$ s.t. $\deg(v) = 0$

- $\kappa(G) = 0$ as G is disconnected.
- $\alpha(G) \ge 1$ as v is included.

$\Rightarrow \alpha(G) > \kappa(G)$

② case 2 : $\exists v \in V(G)$ s.t. $\deg.(v) = 1$

- Let u be the only neighbor of v.

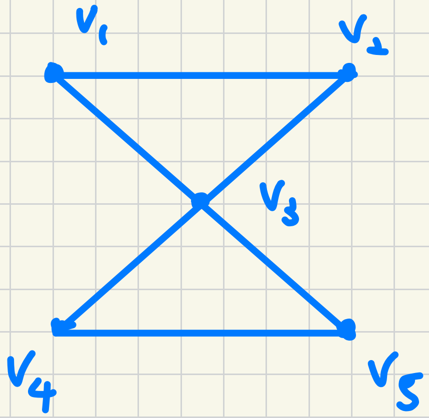


subcase I : $G = K_1$
(Covered in the R.H.S.)

subcase 2 : $G \neq K_1$
since u is v's only neighbor
v's neighbor(s) must be
included in the independent
set including v.

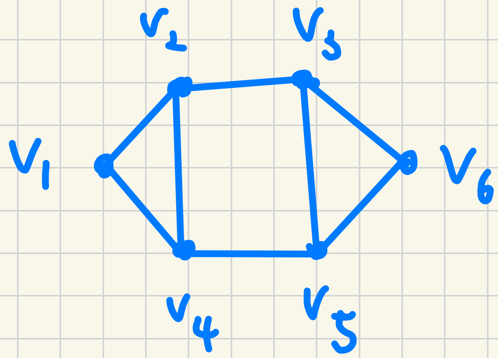
(Q.E.D.)

5.



- It is evident that we can find an Euler tour.
- Finding a Hamiltonian path is impossible as v_3 must be visited twice.

6.



- connectivity $\kappa(G) = 2$

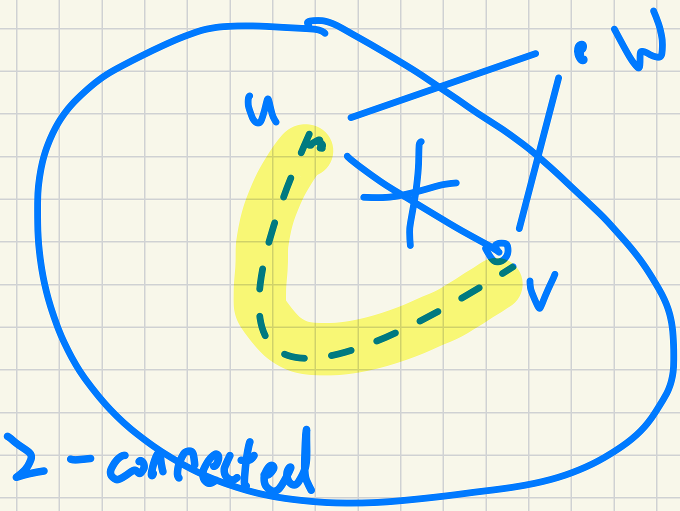
(We cannot disconnect the graph by removing any single vertex. A min. vertex cut is $\{v_2, v_5\}$)

- independence number $\alpha(G) =$ max. independent set $= 2$

$(\{v_1, v_6\}$ or $\{v_2, v_5\}$ or $\{v_3, v_4\})$

$$\Rightarrow \kappa(G) = \alpha(G) = 2$$

7.



(excluding the pair (u,v))

① For any 2 vertices in G, any path containing $\{u,v\}$ can be substituted by $\{u, w, v\}$ $\Rightarrow$ still 2-connected

② u and w : (wlog, v and w)

$\because$ G is 2-connected, a path must exist between u and v other than {u, v}

$\Rightarrow$ (a) A path between u and w is { u --- v , w }.

(b) Another path is { u, w }.

$\Rightarrow$ u and w are 2-connected.

③ u and v :

(a) { u, w, v }

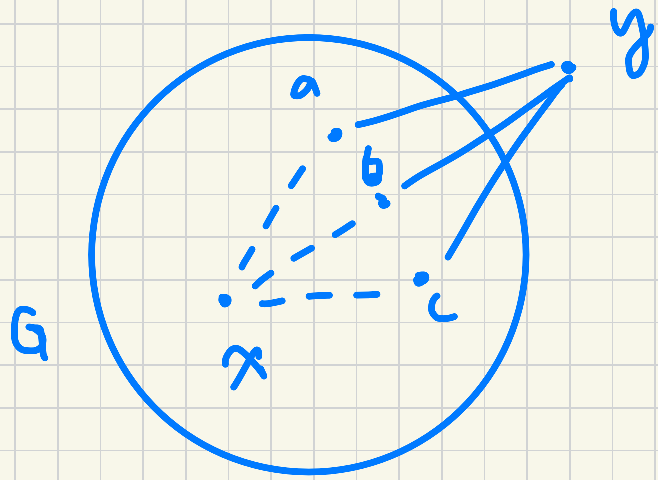
(b) { u ... v }

$\Rightarrow$ u and v are 2-connected

From ①, ②, ③, we know G' is 2-connected

(Q.E.D.)

8.



Say 3 of the neighbors of y are a, b, c

① Consider a point x other than a, b, c
Since G is 3-connected, an path must exist between x & a, b, c
$\Rightarrow$ x and y are 3-connected

② For any 2 vertices in G, they are 3-connected as G is 3-connected

③ y and a (wlog y & b, y & c)
$\{y, a\}$ $\{y, b \ldots a\}$, $\{y, c, a\}$

must exist as G is 3-connected

From ①, ②, ③, G' is 3-connected.
(Q.E.D.)