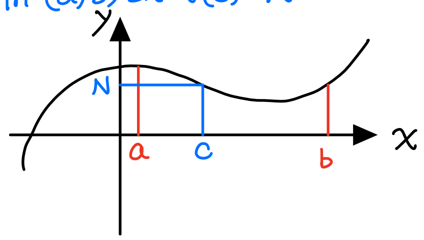
Theorem: Polynomials, rational, root, triagonometric, investing, exponential, log

Theorem: If f is continuous at $x=b$ and $\lim_{x\to a} f(x)=b$, then $\lim_{x\to a} f(g(x))=f(b)$ and

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$$

The Intermediate Value Theorem:

Suppose that f is countinuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a)\neq f(b)$. Then there exists a number c in (a,b) s.t. $f(c)=N$



$f(c)=N$

2.6. Limits at infinity; Horizontal Asymptotes

$\begin{cases} \text{infinite limits} \Rightarrow \text{vertical asymptote} \\ \text{limits at infinite} \Rightarrow \text{Horizontal asymptote} \end{cases}$

Thm. If $r>0$, then $\begin{cases} \lim_{x\to\infty} \frac{1}{x^r}=0 \\ \lim_{x\to-\infty} \frac{1}{x^r}=0 \end{cases}$

Evaluating the limits at infinity

Eg3. $\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2-4x+1} = \lim_{x\to\infty} \frac{3-\frac{1}{x}-\frac{2}{x}}{5-\frac{4}{x}+\frac{1}{x^2}} = \frac{3}{5}$. $y=\frac{3}{5}$ is a horizontal asymptote

Eg4. $f(x)=\frac{\sqrt{2x^2+1}}{3x-5}$

$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{2+1/x^2}}{3-5/x} = \frac{\sqrt{2}}{3}$. $y=\frac{\sqrt{2}}{3}$ is a horizontal asymptote

$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{-\sqrt{2+1/x^2}}{3-5/x} = -\frac{\sqrt{2}}{3}$. $y=-\frac{\sqrt{2}}{3}$ is a horizontal asymptote

Eg5. $\lim_{x\to\infty} \frac{3x^2+\cdots}{5x^2+\cdots} = \frac{3}{5}$, $\lim_{x\to-\infty} \frac{3x^2+\cdots}{5x^2+\cdots} = \frac{3}{5}$.

There is only one horizontal asymptote

图示:

Eg5. $\lim_{x\to\infty} (\sqrt{x^2+1}-x)(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x})$

$= \lim_{x\to\infty} \frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = 0$

Eg8. $\lim_{x\to\infty} \sin x$ D.N.E, so it doesn't has horizontal

Infinite Limits at infinity

Eg 10.

$$\lim_{x\to\infty}(x^2-x)=\lim_{x\to\infty}(x-1)x=\infty$$

Eg 11. $\lim_{x\to\infty}\frac{x^2+x}{3-x}$

$$=\lim_{x\to\infty}\frac{1+\frac{1}{x}}{\frac{3}{x^2}-\frac{1}{x}}=\infty$$

2.7. Derivatives and Rates of Change

Instantaneous Velocity

$$\lim_{h\to 0}\underbrace{\boxed{\frac{f(a+h)-f(a)}{h}}}_{\text{Average velocity}} \Rightarrow$$

The derivative of a function f at a number a is $\boxed{f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}}$

a點切線之slope

$\Rightarrow f'(a)$ is $\underbrace{\text{instantaneous rate of change}}_{\text{How fast (slow) at } x=a}$ or slope of the tangent line.

Eg. 4 Use the definition to find the derivative of $f(x)=x^2-8x+9$ at $x=2$

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{(2+h)^2-8\cdot(2+h)+9-(-3)}{h}=-4$$

Eg. 5 $f(x)=\frac{1}{\sqrt{x}}$

$$f'(x)=\lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}=\lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\cdot\sqrt{x+h}\cdot\sqrt{x}}=\lim_{h\to 0}\frac{(\sqrt{x}-\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}{h\cdot\sqrt{x(x+h)}\cdot(\sqrt{x}+\sqrt{x+h})}$$

$$=\lim_{h\to 0}\frac{-h}{h\cdot\sqrt{x(x+h)}\cdot(\sqrt{x}+\sqrt{x+h})}=\frac{-1}{x\cdot 2\sqrt{x}}=-\frac{1}{2}x^{-\frac{3}{2}}$$

2.8. Other Notations

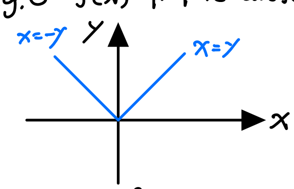
$$\boxed{f'(x)}=\boxed{y'}$$

$$f'(x)=\frac{d}{dx}f(x)=Df(x)=D_xf(x)$$

$$y'=\frac{dy}{dx}$$

1. derivative 微分的
2. differentiation 微分形式
3. differentiable 可被微分
4. differentiate 進行微分

Eg. 5 $f(x)=|x|$ is differentiable?



$$\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}=1$$

$$\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h}=-1$$

$f(x)$ can not be differentiated at $x=0$

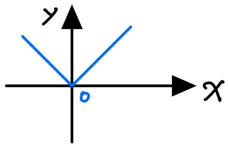
Thm. If f is differentiated at a, then f is continuous at a

pf.: $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$. $f(x)-f(a)=\frac{f(x)-f(a)}{x-a}\cdot(x-a)$

$$\lim_{x\to a}[f(x)-f(a)]=\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)\cdot\lim_{x\to a}(x-a)=f'(a)\cdot 0=0$$

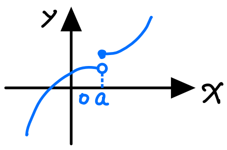
Hence, $\lim_{x\to a}f(x)=f(a)\Rightarrow f(x)$ is continuity at a.

How can a function fail to be differentiable?

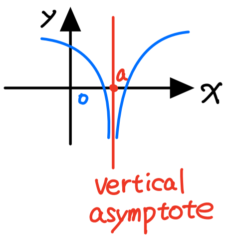


Not be differentiable at $x=0$

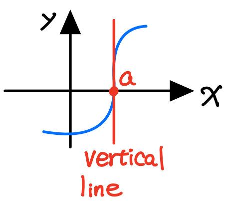
$\because$ slope of tangent lines from right $\neq$ from left



discontinuous, so not be differentiable at $x=a$



Not be differentiable at $x=a$



Not be differentiable at a

Slope D.N.E.

Higer derivatives

$f'(x) \to f''(x) \to f'''(x) \to f^{(4)}(x) \to \cdots$

$y' \to y'' \to y''' \to y^{(4)} \to \cdots$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$