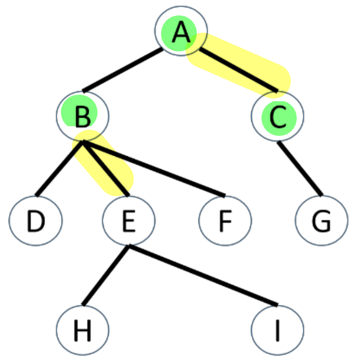
Assignment 1

Due: 4/8 12:00

1. (10%) Is the degree sequence 4, 3, 3, 3, 3, 2, 1, 1 graphical? Please explain your answer.
2. (40%) Consider the following graph.
    a. (5%) Prove that the graph is bipartite by specifying the partite sets X and Y.
    b. (5%) Prove that $M = \{(A,C), (B,E)\}$ is a maximal matching.
    c. (10%) Find an M-augmenting path P and specify the sets S and T.
    d. (10%) Find a larger matching M^* by P.
    e. (10%) Prove that M^* is the largest matching by finding a corresponding vertex cover. König's thm



3. (20%) Find the maximum weighted matching and the minimum weighted vertex cover of the following weighted bipartite graph in the matrix form.

$$\begin{bmatrix} 6 & 0 & 3 & 6 & 8 \\ 1 & 8 & 5 & 5 & 3 \\ 1 & 9 & 4 & 7 & 5 \\ 6 & 5 & 8 & 6 & 5 \\ 0 & 6 & 5 & 4 & 3 \end{bmatrix}$$

4. (20%) Find the dual problem of the following optimization problem.

$$\max 3x_1 - 5x_2 + 6x_3$$

such that:

$$x_1 - 3x_2 \geq 5$$
$$2x_1 + 5x_3 \leq 3$$
$$2x_1 - 3x_2 + x_3 \geq 3$$
$$2x_2 + 3x_3 \leq -10$$
$$x_1, x_2, x_3 \geq 0$$

: impossible to satisfy, no feasible solution

5. (10%) Prove that the randomized algorithm for the vertex-weighted vertex cover problem in page 120 is a 2-approximation algorithm.

1. Harel - Hakimi Algorithm

1° [4, 3, 3, 3, 3, 2, 2, 1, 1] : sorted

2° Remove 4, subtract 1 from the next 4 entries

[0, 2, 2, 2, 2, 2, 2, 1, 1]

3° Sort descending

[2, 2, 2, 2, 2, 2, 1, 1, 0]

4° Remove 2, subtract 1 from the next 2 entries

[0, 1, 1, 2, 2, 2, 1, 1, 0]

5° Sort descending

[2, 2, 2, 1, 1, 1, 1, 0, 0]

6° Remove 2, subtract 1 from the next 2 entries

[0, 1, 1, 1, 1, 1, 1, 0, 0]

7° Sort descending

[1, 1, 1, 1, 1, 1, 0, 0, 0]

8° Remove 1, subtract 1 from the next 1 entry

[0, 0, 1, 1, 1, 1, 0, 0, 0]

9° Sort descending

[1, 1, 1, 1, 0, 0, 0, 0, 0]

⋮

$[0, 0, 0, 0, 0, 0, 0, 0, 0]$: all zeros

$\Rightarrow$ the given degree sequence is graphical #

2.

(a) G is a tree $\Rightarrow$ G has no (odd-length cycle)

$\Rightarrow$ G is bipartite #

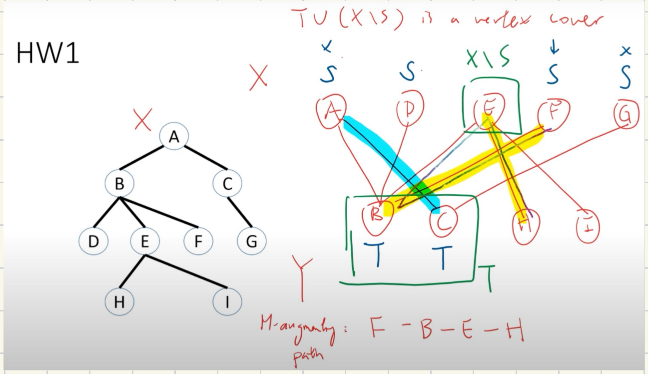
$X = \{A, D, E, F, G\}$, $Y = \{B, C, H, I\}$ #

(b) no edges can be added to $M = \{(A,C), (B,E)\}$ w/o sharing an endpoint $\Rightarrow$ it is maximal #

(c)

$X = \{ A \quad D \quad E \quad F \quad G \}$

$Y = \{ B \quad C \quad H \quad I \}$



$$\begin{cases} P = D - B - E - H \\ S = \{D, E\} \\ T = \{B, H\} \end{cases}$$ #

(d) $M^* = \{(A,C), (D,B), (E,H)\}$

maximum matching minimum vertex cover

(e) $|M^*| = 3 = |C^*| = |\{B, C, E\}|$ (König's theorem) #

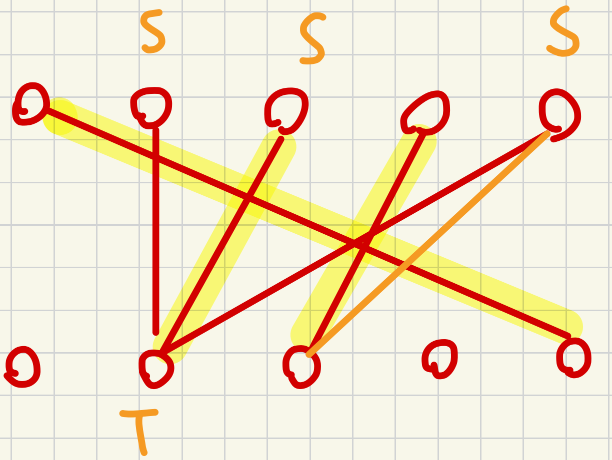
3.

①

$$
\begin{array}{c|ccccc|}
 & 0 & \cancel{0}\ 1 & 0 & 0 & 0 \\
\hline
8 & 6 & 0 & 3 & 6 & 8 \\
7\ \cancel{8} & 1 & 8 & 5 & 5 & 3 \\
8\ \cancel{9} & 1 & 9 & 4 & 7 & 5 \\
8 & 6 & 5 & 8 & 6 & 5 \\
5\ \cancel{6} & 0 & 6 & 5 & 4 & 3 \\
\end{array}
$$

Y

X



: maximal matching

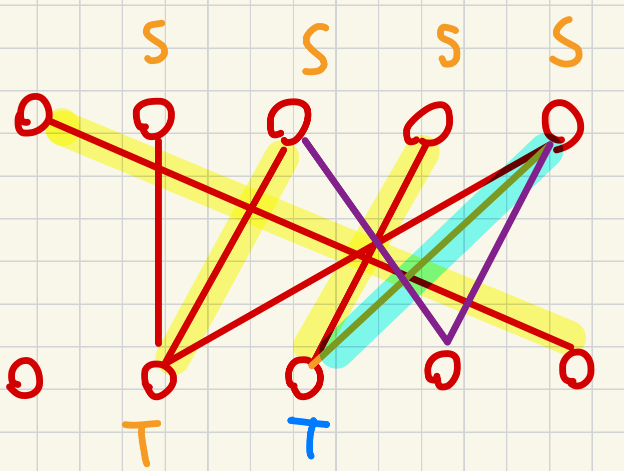
mingap $\varepsilon = 1$

②

$$
\begin{array}{c|ccccc|}
 & 0 & \cancel{0}\ \cancel{1}\ 2 & \cancel{0}\ 1 & 0 & 0 \\
\hline
8 & 6 & 0 & 3 & 6 & 8 \\
6\ \cancel{7}\ \cancel{8} & 1 & 8 & 5 & 5 & 3 \\
7\ \cancel{8}\ \cancel{9} & 1 & 9 & 4 & 7 & 5 \\
7\ \cancel{8} & 6 & 5 & 8 & 6 & 5 \\
4\ \cancel{5}\ \cancel{6} & 0 & 6 & 5 & 4 & 3 \\
\end{array}
$$

Y

X

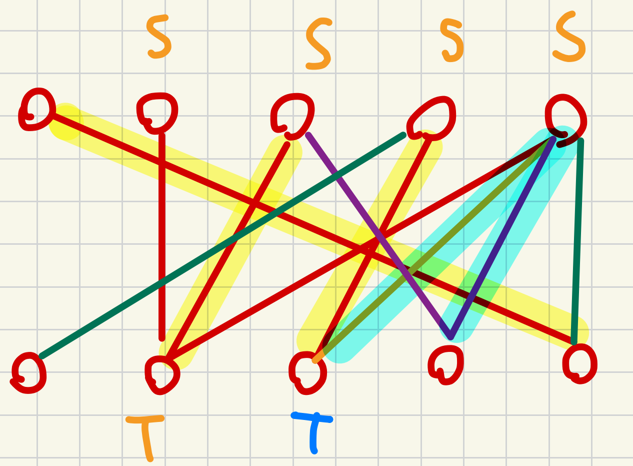


: maximal matching

mingap $\varepsilon = 1$

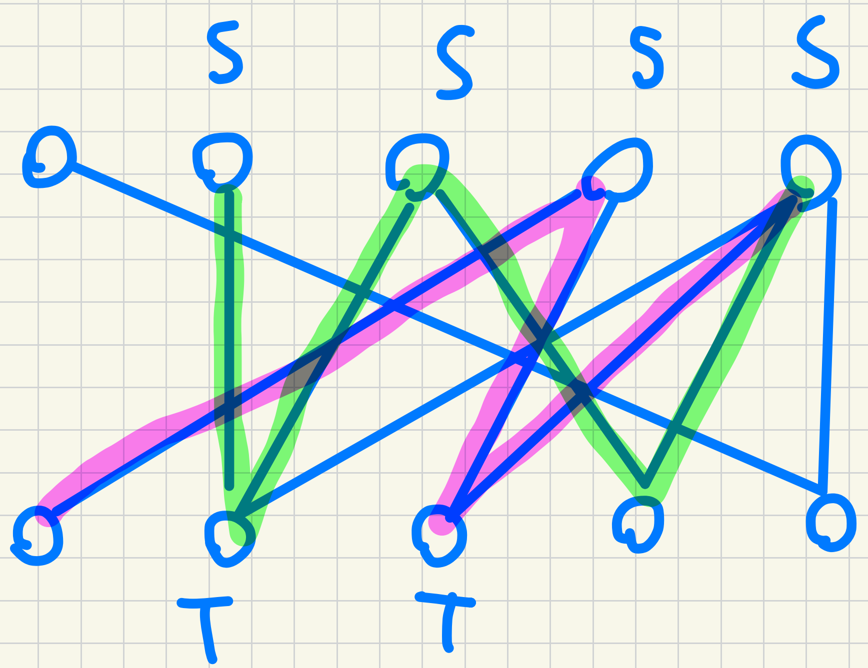
③

$$
\begin{array}{c|c}
 & \begin{matrix} 0 & \cancel{0}\,\cancel{1}\,2 & \cancel{0}\,1 & 0 & 0 \end{matrix} \quad Y \\
\hline
\begin{matrix} 8 \\ 6\,\cancel{7}\,\cancel{8} \\ 7\,\cancel{8}\,\cancel{9} \\ 7\,\cancel{8} \\ 4\,\cancel{5}\,\cancel{6} \\ X \end{matrix} &
\begin{bmatrix} 6 & 0 & 3 & 6 & 8 \\ 1 & 8 & 5 & 5 & 3 \\ 1 & 9 & 4 & 7 & 5 \\ 6 & 5 & 8 & 6 & 5 \\ 0 & 6 & 5 & 4 & 3 \end{bmatrix}
\end{array}
$$



: maximal matching

min gap $\varepsilon = 1$



2 augmenting paths

$x_2 \to y_2 \to x_3 \to y_4 \to x_5 \to y_3 \to x_4 \to y_1$

weighted matching = 34, vertex cover weight = 34 maximum weighted matching

$= 0 + 3 + 4 + 6 + 0 = 13$

minimum weighted vertex cover = mwm $= 13$

4. Dual LP

Minimize $5y_1 + 3y_2 + 3y_3 - 10y_4$

Subject to:

$$\begin{cases} y_1 + 2y_2 + 2y_3 \leq 3 \\ -3y_1 - 3y_3 + 2y_4 \leq -5 \\ 5y_2 + y_3 + 3y_4 \leq 6 \\ y_1, y_3 \geq 0 \\ y_2, y_4 \leq 0 \end{cases}$$

5. 1° Minimize $\sum_{v \in V} w(v) \cdot x_v$

subject to $x_u + x_v \geq 1 \quad \forall (u, v) \in E$

$0 \leq x_v \leq 1 \quad \forall v \in V$

Let x^* be the optimal solution w/ cost

$OPTLP = \sum v \in V_w(v) \cdot x_v^*$

2° Vertex cover $C = \{ v \in V \mid x_v^* \geq \frac{1}{2} \}$

3° $\forall (u, v) \in E$, the LP constraint guarantees

$x_u^* + x_v^* \geq 1 \Rightarrow$ at least one of

u or $v \in C$

So, every edge is covered

$\rightarrow C$ is a valid vertex cover

4° $\sum_{v \in C} w(v) \leq \sum_{v \in V} w(v) \cdot 2x_v^* = 2 \cdot OPT_{LP}$

Since $OPT_{LP} \leq OPT_{IP}$ (Relaxation bound),

we get $\sum_{v \in V} w(v) \leq 2 \cdot OPT_{IP}$

5° The algorithm produces a valid vertex cover

w/ total weight at most twice the optimal.

4.

Primal

$$\max \quad 3x_1 - 5x_2 + 6x_3$$

subject to

$$\begin{array}{rrrcrl}
-x_1 & +3x_2 & & \le & -5 & \quad y_1 \\
2x_1 & & +5x_3 & \le & 3 & \quad y_2 \\
-2x_1 & +3x_2 & -x_3 & \le & -3 & \quad y_3 \\
 & 2x_2 & +3x_3 & \le & -10 & \quad y_4
\end{array}$$

$$x_1, x_2, x_3 \ge 0$$

Dual

$$\min \quad -5y_1 + 3y_2 - 3y_3 - 10y_4$$

subject to

$$\begin{array}{rrrrcr}
-y_1 & +2y_2 & -2y_3 & & \ge & 3 \\
3y_1 & & +3y_3 & +2y_4 & \ge & -5 \\
 & 5y_2 & -y_3 & +3y_4 & \ge & 6
\end{array}$$

$$y_1, y_2, y_3, y_4 \ge 0$$

p.s. $y_3, y_2, y_4 = 0$

$y_1 = A$

As $A \uparrow$, objective value $\downarrow$

5. See Vertex Cover Algorithms / Proof for General Graph