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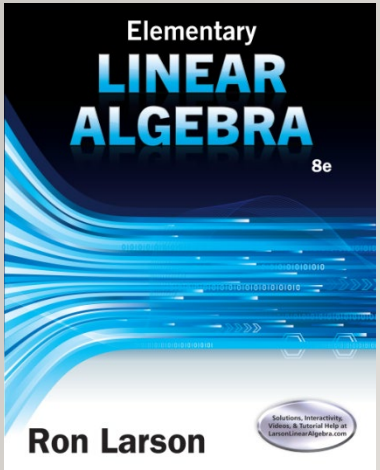

CHAPTER 6 LINEAR TRANSFORMATIONS

6.1 Introduction to Linear Transformations

6.2 The Kernel and Range of a Linear Transformation

6.3 Matrices for Linear Transformations

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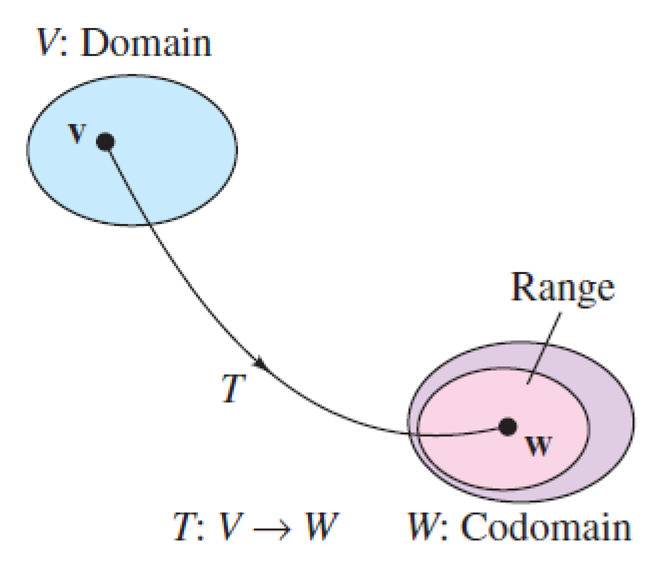
6.1 Introduction to Linear Transformations

- Function T that maps a vector space V into a vector space W:

$$T : V \xrightarrow{\text{mapping}} W, \quad V, W : \text{vector space}$$

V: the domain of T

W: the codomain of T



- Image of **v** under T:

If **v** is in V and **w** is in W such that

$$T(\mathbf{v}) = \mathbf{w}$$

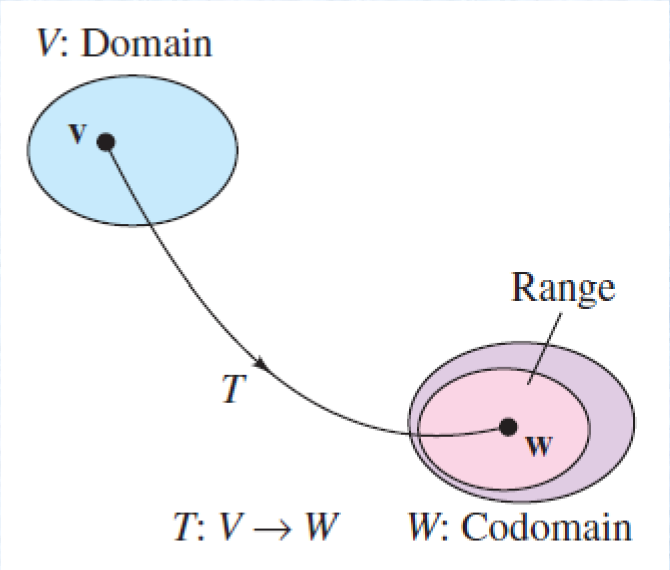
Then **w** is called the image of **v** under T.

- the range of T:

The set of all images of vectors in V.

- the preimage of w:

The set of all **v** in V such that $T(\mathbf{v})=\mathbf{w}$.



- Ex 1: (A function from R^2 into R^2)

$$T: R^2 \to R^2 \qquad \mathbf{v} = (v_1, v_2) \in R^2$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

(a) Find the image of **v**=(-1,2). (b) Find the preimage of **w**=(-1,11)

Sol:

$$(a)\ \ \mathbf{v} = (-1, 2)$$

$$\Rightarrow T(\mathbf{v}) = T(-1, 2) = (-1-2, -1+2(2)) = (-3, 3)$$

$$(b)\ \ T(\mathbf{v}) = \mathbf{w} = (-1, 11)$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$$

$$\Rightarrow v_1 - v_2 = -1$$

$$v_1 + 2v_2 = 11$$

$\Rightarrow v_1 = 3,\ v_2 = 4$ Thus {(3, 4)} is the preimage of **w**=(-1, 11).

- Linear Transformation (L.T.):
- T is a linear transformation of V into W if (1) (2) are true for all u and v in V and any scalar c.

V, W: vector space

$T: V \rightarrow W$: V to W linear transformation

$$(1)\ \ T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V$$

$$(2)\ \ T(c\mathbf{u}) = cT(\mathbf{u}), \quad \forall c \in R$$

- Notes:

(1) A linear transformation is said to be operation preserving.

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \qquad T(c\mathbf{u}) = cT(\mathbf{u})$$

Addition in V

Addition in W

Scalar multiplication in V

Scalar multiplication in W

(2) A linear transformation $T: V \rightarrow V$ from a vector space into itself is called a **linear operator**.

- Ex 2: (Verifying a linear transformation T from R^2 into R^2)

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

Pf:

$\mathbf{u} = (u_1, u_2),\ \ \mathbf{v} = (v_1, v_2)$: vector in R^2, c : any real number

(1) Vector addition :

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(u_1 + v_1, u_2 + v_2) \\ &= ((u_1 + v_1) - (u_2 + v_2), (u_1 + v_1) + 2(u_2 + v_2)) \\ &= ((u_1 - u_2) + (v_1 - v_2), (u_1 + 2u_2) + (v_1 + 2v_2)) \\ &= (u_1 - u_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

(2) Scalar multiplication

$$c\mathbf{u} = c(u_1, u_2) = (cu_1, cu_2)$$

$$\begin{aligned} T(c\mathbf{u}) &= T(cu_1, cu_2) = (cu_1 - cu_2, cu_1 + 2cu_2) \\ &= c(u_1 - u_2, u_1 + 2u_2) \\ &= cT(\mathbf{u}) \end{aligned}$$

Therefore, T is a linear transformation.

- Ex 3: (Linear transformation or not?)

$$f(x)=x+1$$

$$f(x_1+x_2)=x_1+x_2+1$$

$$f(x_1)+f(x_2)=(x_1+1)+(x_2+1)=x_1+x_2+2$$

$$f(x_1+x_2)\neq f(x_1)+f(x_2)$$

$\Leftarrow f(x)=x+1$ is not linear transformation

- Notes: Two uses of the term "linear".

(1) $f(x) = x + 1$ is called a linear function because its graph is a line.

(2) $f(x) = x + 1$ is not a linear transformation from a vector space R into R because it preserves neither vector addition nor scalar multiplication.

- Ex 3: (Functions that are not linear transformations)

(a) $f(x) = \sin x$

$\sin(x_1 + x_2) \neq \sin(x_1) + \sin(x_2)$ $\Leftarrow f(x) = \sin x$ is not linear transformation

$\sin(\frac{\pi}{2} + \frac{\pi}{3}) \neq \sin(\frac{\pi}{2}) + \sin(\frac{\pi}{3})$

(b) $f(x) = x^2$

$(x_1 + x_2)^2 \neq x_1^2 + x_2^2$ $\Leftarrow f(x) = x^2$ is not linear transformation

$(1+2)^2 \neq 1^2 + 2^2$

(c) $f(x) = x + 1$

$f(x_1 + x_2) = x_1 + x_2 + 1$

$f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2$

$f(x_1 + x_2) \neq f(x_1) + f(x_2)$ $\Leftarrow f(x) = x + 1$ is not linear transformation

Two simple linear transformations

- Zero transformation:

$$T : V \to W \qquad\qquad T(\mathbf{v}) = 0,\ \forall \mathbf{v} \in V$$

- Identity transformation:

$$T : V \to V \qquad\qquad T(\mathbf{v}) = \mathbf{v},\ \forall \mathbf{v} \in V$$

- Thm 6.1: (Properties of linear transformations)

$$T : V \to W,\quad \mathbf{u}, \mathbf{v} \in V$$

$$(1)\ T(\mathbf{0}) = \mathbf{0}$$

$$(2)\ T(-\mathbf{v}) = -T(\mathbf{v})$$

$$(3)\ T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$$

(4) If $\mathbf{v} = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

Then $T(\mathbf{v}) = T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$

$$= c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$$

- Ex 4: (Linear transformations and bases)

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that

$$T(1,0,0) = (2,-1,4)$$
$$T(0,1,0) = (1,5,-2)$$
$$T(0,0,1) = (0,3,1)$$

Find $T(2, 3, -2)$.

Sol:

$$(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$$

$$T(2,3,-2) = 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1) \qquad (T \text{ is a L.T.})$$
$$= 2(2,-1,4) + 3(1,5,-2) - 2T(0,3,1)$$
$$= (7,7,0)$$

- Ex 5: (A linear transformation defined by a matrix)

The function $T: R^2 \rightarrow R^3$ is defined as $T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(a) Find $T(\mathbf{v})$, where $\mathbf{v} = (2,-1)$

(b) Show that T is a linear transformation form R^2 into R^3

Sol: (a) $\mathbf{v} = (2,-1)$

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{R^2 \text{ vector}} = \underbrace{\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}}_{R^3 \text{ vector}}$$

$\therefore T(2,-1) = (6,3,0)$

(b) $T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$ (vector addition)

$T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u})$ (scalar multiplication)

- Thm 6.2: (The linear transformation given by a matrix)

  Let A be an $m \times n$ matrix. The function T defined by

  $$T(\mathbf{v}) = A\mathbf{v}$$

  is a linear transformation from R^n into R^m.

- Note:

$$\begin{array}{ccccc} & & R^n \text{ vector} & & R^m \text{ vector} \\ & & \downarrow & & \downarrow \\ A\mathbf{v} = & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} & \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} & = & \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix} \end{array}$$

$$T(\mathbf{v}) = A\mathbf{v}$$

$$T : R^n \longrightarrow R^m$$

- Ex 7: (Rotation in the plane)

Show that the L.T. $T: R^2 \rightarrow R^2$ given by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

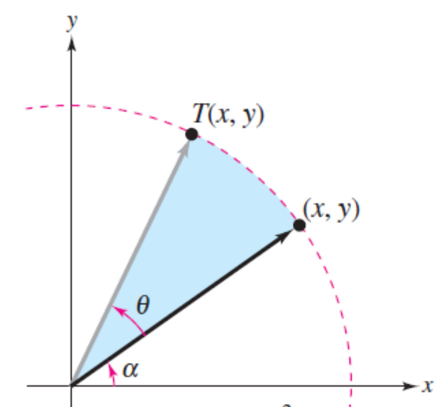
has the property that it rotates every vector in R^2 counterclockwise about the origin through the angle θ.

Sol:

$v = (x, y) = (r\cos\alpha, r\sin\alpha)$ (polar coordinates)

r : the length of v

α : the angle from the positive x-axis counterclockwise to the vector v



Rotation in R^2

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix}$$

r ： the length of $T(\mathbf{v})$

$\theta+\alpha$ ： the angle from the positive x-axis counterclockwise to the vector $T(\mathbf{v})$

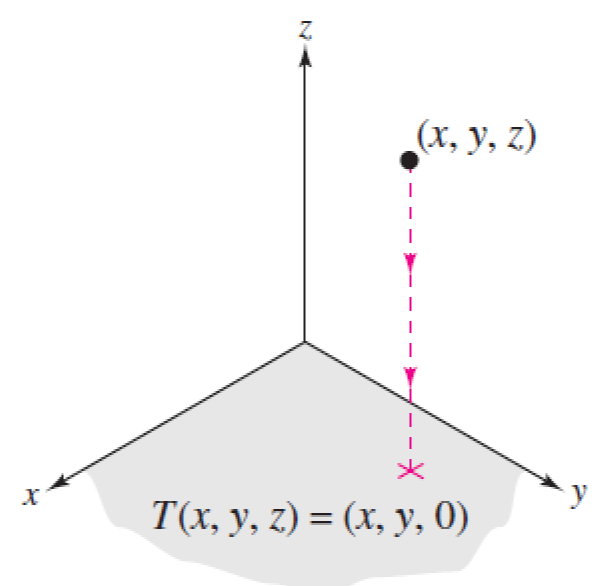
Thus, $T(\mathbf{v})$ is the vector that results from rotating the vector $\mathbf{v}$ counterclockwise through the angle θ.

▪ Ex 8: (A projection in R^3)

The linear transformation $T: R^3 \to R^3$ is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is called a projection in R^3.



Projection onto xy-plane

- Ex 9: (A linear transformation from $M_{m\times n}$ into $M_{n\times m}$)

$$T(A) = A^T \quad (T : M_{m\times n} \rightarrow M_{n\times m})$$

Show that T is a linear transformation.

Sol:

$$A, B \in M_{m\times n}$$

$$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$$

$$T(cA) = (cA)^T = cA^T = cT(A)$$

Therefore, T is a linear transformation from $M_{m\times n}$ into $M_{n\times m}$.

6.2 The Kernel and Range of a Linear Transformation

- Kernel of a linear transformation T:

Let $T: V \to W$ be a linear transformation

Then the set of all vectors **v** in V that satisfy $T(\mathbf{v}) = 0$ is called the kernel of T and is denoted by ker(T).

$$\ker(T) = \{\mathbf{v} \mid T(\mathbf{v}) = 0, \forall \mathbf{v} \in V\}$$

- Ex 1: (Find the kernel of a linear transformation)

$$T(A) = A^T \quad (T: M_{3\times 2} \to M_{2\times 3})$$

Sol:

$$\ker(T) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

- Ex 2: (The kernel of the zero and identity transformations)

(a) $T(\mathbf{v}) = \mathbf{0}$ (the zero transformation $T: V \to W$)

$$\ker(T) = V$$

(b) $T(\mathbf{v}) = \mathbf{v}$ (the identity transformation $T: V \to V$)

$$\ker(T) = \{\mathbf{0}\}$$

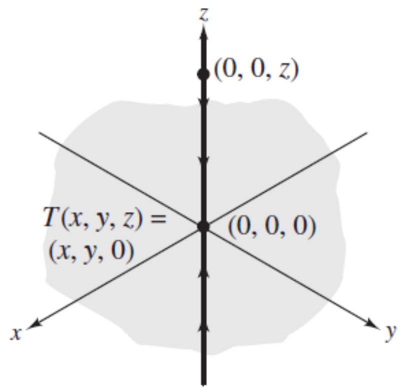
- Ex 3: (Find the kernel of a linear transformation)

$$T(x, y, z) = (x, y, 0) \qquad (T: R^3 \to R^3)$$

$$\ker(T) = ?$$

Sol:

$$\ker(T) = \{(0,0,z) \mid z \text{ is a real number}\}$$



The kernel of T is the set of all vectors on the z-axis.

- Ex 5: (Find the kernel of a linear transformation)

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (T : R^3 \rightarrow R^2)$$

$$\ker(T) = ?$$

Sol:

$$\ker(T) = \{(x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = (0,0),\, x = (x_1, x_2, x_3) \in R^3\}$$

$$T(x_1, x_2, x_3) = (0,0)$$

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{G.J.E} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \ker(T) = \{t(1,-1,1) \mid t \text{ is a real number}\}$$
$$= \text{span}\{(1,-1,1)\}$$

- **Thm 6.3: (The kernel is a subspace of *V*)**

The kernel of a linear transformation $T : V \to W$ is a subspace of the domain *V*.

Pf: $\because T(0) = 0$ (Theorem 6.1)

$\therefore \ker(T)$ is a nonempty subset of V

(sufficient by showing the closure property)

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in the kernel of T. then

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = 0 + 0 = 0 \qquad \Rightarrow \mathbf{u}+\mathbf{v} \in \ker(T)$$

$$T(c\mathbf{u}) = cT(\mathbf{u}) = c0 = 0 \qquad \Rightarrow c\mathbf{u} \in \ker(T)$$

Thus, $\ker(T)$ is a subspace of V.

- Note:

The kernel of *T* is sometimes called the **nullspace** of *T*.

- Ex 6: (Find a basis for the kernel)

Let $T: R^5 \rightarrow R^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is in R^5 and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for ker(T) as a subspace of R^5.

Sol:

$$[A \mid 0] =$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 8 & 0 \end{bmatrix} \xrightarrow{G.J.E} \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} & & s & & t & \end{matrix} \end{array}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s+t \\ s+2t \\ s \\ -4t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

$B = \{(-2, 1, 1, 0, 0), (1, 2, 0, -4, 1)\}$: one basis for the kernel of T

- Corollary to Thm 6.3:

Let $T: R^n \to R^m$ be the L.T given by $T(\mathbf{x}) = A\mathbf{x}$

Then the kernel of T is equal to the solution space of $A\mathbf{x} = 0$

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{(a linear transformation } T: R^n \to R^m\text{)}$$

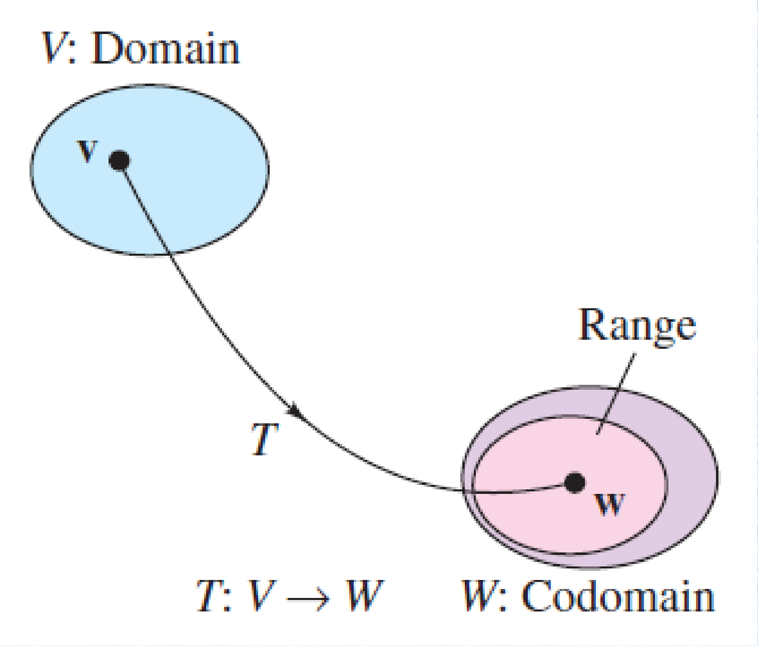
$$\Rightarrow Ker(T) = NS(A) = \left\{\mathbf{x} \mid A\mathbf{x} = 0, \forall \mathbf{x} \in R^m\right\} \text{ (subspace of } R^m\text{)}$$

- Range of a linear transformation T:

Let $T : V \rightarrow W$ be a L.T.

Then the set of all vectors w in W that are images of vector in V is called the range of T and is denoted by $range(T)$

$$range(T) = \{T(\mathbf{v}) \mid \forall \mathbf{v} \in V\}$$



- Thm 6.4: (The range of T is a subspace of W)

The range of a linear transformation $\mathrm{T}: V \rightarrow W$ is a subspace of W.

Pf:

$$\because T(0) = 0 \quad (\text{Thm.6.1})$$

$$\therefore range(T) \text{ is a nonempty subset of } W$$

Let $T(\mathbf{u})$ and $T(\mathbf{v})$ be vector in the range of T

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \in range(T) \quad (\mathbf{u} \in V, \mathbf{v} \in V \Rightarrow \mathbf{u}+\mathbf{v} \in V)$$

$$T(c\mathbf{u}) = cT(\mathbf{u}) \in range(T) \quad (\mathbf{u} \in V \Rightarrow c\mathbf{u} \in V)$$

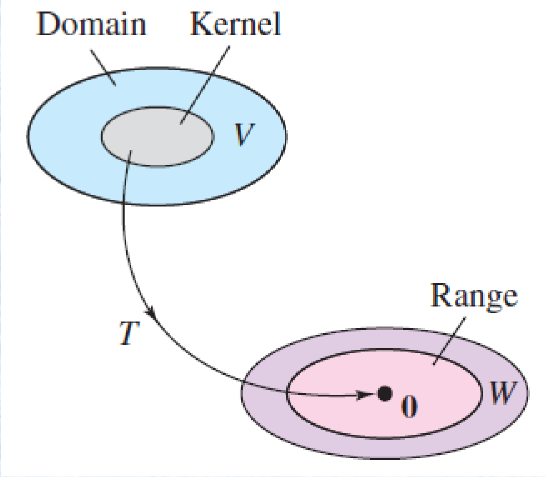
Therefore, $range(T)$ is W subspace.

- Notes:

$T: V \to W$ is a L.T.

(1) $Ker(T)$ is subspace of V

(2) $range(T)$ is subspace of W



- Corollary to Thm 6.4:

Let $T: R^n \to R^m$ be the L.T. given by $T(\mathbf{x}) = A\mathbf{x}$

Then the range of T is equal to the column space of A

$\Rightarrow \text{range}(T) = CS(A)$

Since Ax=b. Detail see page 312

- Ex 7: (Find a basis for the range of a linear transformation)

Let $T: R^5 \to R^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is R^5 and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for the range of T.

Sol:

$$A=\begin{array}{c}\begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \\ \begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & c_5 \end{array}\end{array} \xrightarrow{G.J.E} \begin{array}{c}\begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & w_5 \end{array}\end{array} = B$$

$\Rightarrow \{w_1, w_2, w_4\}$ is a basis for $CS(B)$

$\{c_1, c_2, c_4\}$ is a basis for $CS(A)$

$\Rightarrow \{(1, 2, -1, 0), (2, 1, 0, 0), (1, 1, 0, 2)\}$ is a basis for the range of T

Def. : Rank and Nullity of a L. T

- Rank of a linear transformation $T:V\rightarrow W$:

$$rank(T) = \text{the dimension of the range of } T$$

- Nullity of a linear transformation $T:V\rightarrow W$:

$$nullity(T) = \text{the dimension of the kernel of } T$$

- Note:

Let $T: R^n \rightarrow R^m$ be the L.T. given by $T(\mathbf{x}) = A\mathbf{x}$, then

$$rank(T) = rank(A)$$

$$nullity(T) = nullity(A)$$

- Thm 6.5: (Sum of rank and nullity)

Let $T: V \to W$ be a L. T. from an n − dimensional vector space V into a vector space W. then

$$\text{rank}(T) + \text{nullity}(T) = n$$

$$\dim(\text{range of } T) + \dim(\text{kernel of } T) = \dim(\text{domain of } T)$$

Pf:

Let T is represented by an $m \times n$ *matrix* A

Assume $rank(A) = r$

$$(1)\ \text{rank}(T) = \dim(\text{range of } T) = \dim(\text{column space of } A)$$
$$= \text{rank}(A) = r$$

$$(2)\ \text{nullity}(T) = \dim(\text{kernel of } T) = \dim(\text{solution space of } A)$$
$$= n - r$$

$$\Rightarrow rank(T) + nullity(T) = r + (n - r) = n$$

- Ex 8: (Find the rank and nullity of a linear transformation)

Find the rank and nullity of the L.T. $T: R^3 \rightarrow R^3$ define by

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol:

$$\text{rank}(T) = \text{rank}(A) = 2$$

$$\text{nullity}(T) = \dim(\text{domain of } T) - \text{rank}(T) = 3 - 2 = 1$$

- Ex 9: (Find the rank and nullity of a linear transformation)

  Let $T : R^5 \to R^7$ be a linear transformation.

  (a) Find the dimension of the kernel of T if the dimension of the range is 2

  (b) Find the rank of T if the nullity of T is 4

  (c) Find the rank of T if $\ker(T) = \{0\}$

Sol:

(a) $\dim(\text{domain of } T) = 5$

$\dim(\text{kernel of } T) = n - \dim(\text{range of } T) = 5 - 2 = 3$

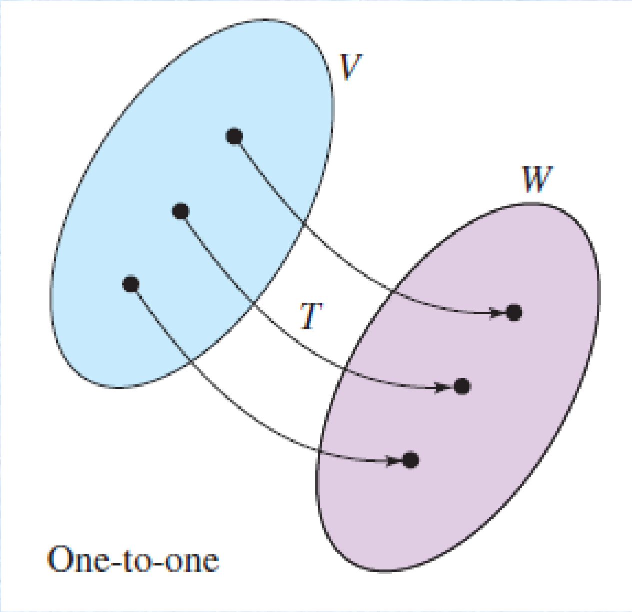
(b) $\text{rank}(T) = n - \text{nullity}(T) = 5 - 4 = 1$

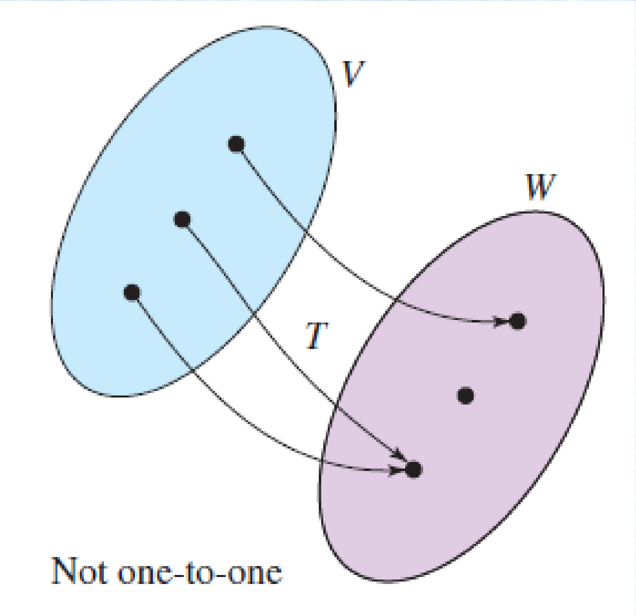
(c) $\text{rank}(T) = n - \text{nullity}(T) = 5 - 0 = 5$

- One-to-one:

A function $T: V \rightarrow W$ is called one-to-one if the preimage of every w in the range consists of a single vector.

T is one-to-one iff for all u and v in V, $T(\mathbf{u}) = T(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$.

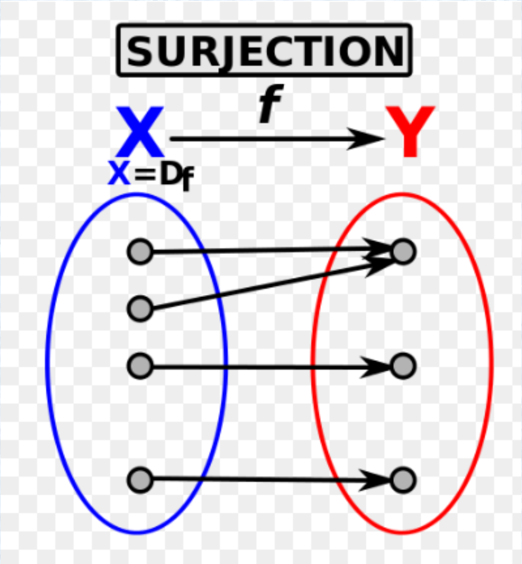


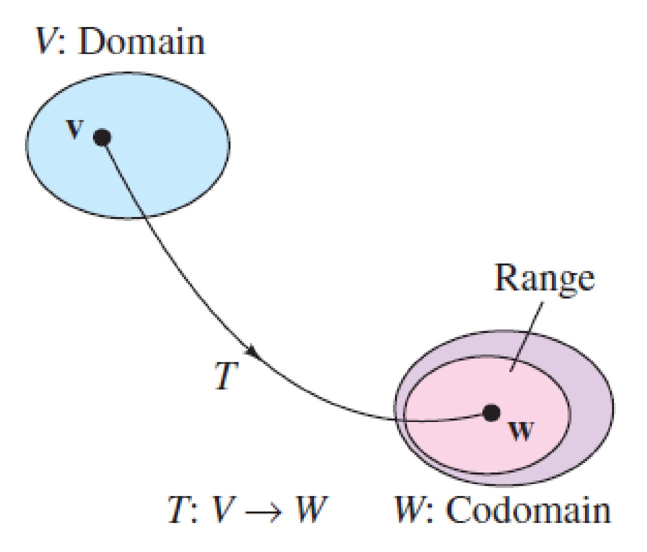


- Onto:

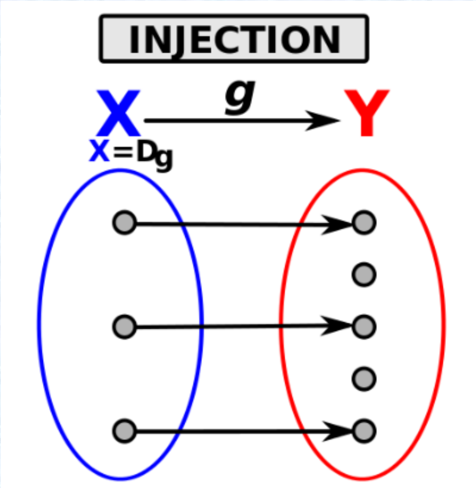
A function $T: V \rightarrow W$ is said to be onto if every element in $\mathbf{w}$ has a preimage in V

(T is onto W when W is equal to the range of T.)





- Thm 6.6: (One-to-one linear transformation)



$$\text{Let } T: V \rightarrow W \text{ be a L.T.}$$

$$\text{Then T is 1-1 iff } \ker(T) = \{0\}$$

Pf:

$$\text{Suppose } T \text{ is 1-1}$$

$$\text{Then } T(v) = 0 \text{ can have only one solution} : v = 0$$

$$\text{i.e. } \ker(T) = \{0\}$$

$$\text{Suppose } \ker(T) = \{0\} \text{ and } T(u) = T(v)$$

$$T(u - v) = T(u) - T(v) = 0$$

T is a L.T.

$$\because u - v \in \ker(T) \Rightarrow u - v = 0$$

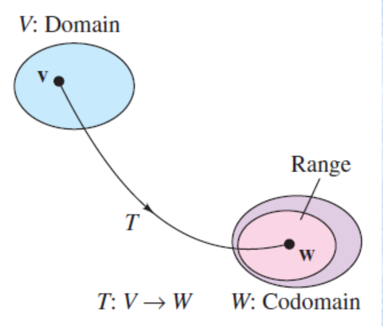
$$\Rightarrow T \text{ is 1-1}$$

- Ex 10: (One-to-one or not one-to-one linear transformation)

(a) The L.T. $T: M_{m\times n} \rightarrow M_{n\times m}$ given by $T(A) = A^T$

is

(b) The zero transformation $T: R^3 \rightarrow R^3$ is



- **Thm 6.7: (Onto linear transformation)**

Let $T: V \rightarrow W$ be a L.T., where W is finite dimensional.

Then T is onto iff the rank of T is equal to the dimension of W.

- **Thm 6.8: (One-to-one and onto linear transformation)**

Let $T: V \rightarrow W$ be a L.T. with vector space V and W both of dimension n. Then T is one-to-one if and only if it is onto.

Pf:

If T is one-to-one, then $\ker(T) = \{0\}$ and $\dim(\ker(T)) = 0$

$$\dim(\text{range}(T)) = n - \dim(\ker(T)) = n = \dim(W)$$

Consequently, T is onto.

If T is onto, then $\dim(\text{range of } T) = \dim(W) = n$

$$\dim(\ker(T)) = n - \dim(\text{range of } T) = n - n = 0$$

Therefore, T is one-to-one.

- Ex 11:

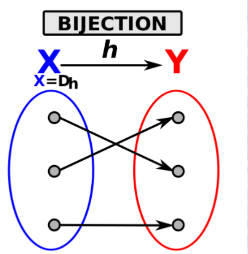
The L. T. $T : R^n \to R^m$ is given by $T(\mathbf{x}) = A\mathbf{x}$, Find the nullity and rank of T and determine whether T is one − to − one, onto, or neither.

$$(a) A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad (b) A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(c) A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \qquad (d) A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol:

$T: R^n \to R^m$	dim(domain of T)	rank(T)	nullity(T)	1-1	onto
$(a) T: R^3 \to R^3$					
$(b) T: R^2 \to R^3$					
$(c) T: R^3 \to R^2$					
$(d) T: R^3 \to R^3$					



- Isomorphism:

  A linear transformation $T : V \to W$ that is one to one and onto is called an isomorphism. Moreover, if V and W are vector spaces such that there exists an isomorphism from V to W, then V and W are said to be isomorphic to each other.

- Thm 6.9: (Isomorphic spaces and dimension)

  Two finite-dimensional vector space V and W are isomorphic if and only if they are of the same dimension.

Pf:

Assume that V is isomorphic to W, where V has dimension n.

$\Rightarrow$ There exists a L.T. $T : V \to W$ that is one to one and onto.

$\because T$ is one - to - one

$\Rightarrow \dim(Ker(T)) = 0$

$\Rightarrow \dim(\text{range of } T) = \dim(\text{domain of } T) - \dim(Ker(T)) = n - 0 = n$

$\because T$ is onto.

$\Rightarrow \dim(\text{range of } T) = \dim(W) = n$

Thus $\dim(V) = \dim(W) = n$

- Assume that V and W both have dimension n.

  Let $\{v_1, v_2, \cdots, v_n\}$ be a basis of V, and

  let $\{w_1, w_2, \cdots, w_n\}$ be a basis of W.

  Then an arbitrary vector in V can be represented as

  $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$

  and you can define a L.T. $T: V \rightarrow W$ as follows.

  $T(\mathbf{v}) = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_n$

  It can be shown that this L.T. is both 1-1 and onto.

  Thus V and W are isomorphic.

- Ex 12: (Isomorphic vector spaces)

The following vector spaces are isomorphic to each other.

$$(a)\ R^4 = 4\text{-space}$$

$$(b)\ M_{4\times 1} = \text{space of all } 4\times 1 \text{ matrices}$$

$$(c)\ M_{2\times 2} = \text{space of all } 2\times 2 \text{ matrices}$$

$$(d)\ P_3(x) = \text{space of all polynomials of degree 3 or less}$$

$$(e)\ V = \{(x_1, x_2, x_3, x_4, 0), x_i \text{ is a real number}\}\ (\text{subspace of } R^5)$$

6.3 Matrices for Linear Transformations

- Two representations of the linear transformation $T: R^3 \rightarrow R^3$:

$$(1) T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

$$(2) T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Five reasons for matrix representation of a linear transformation:
  - Simpler to write.
  - Simpler to read.
  - More easily adapted for computer use.
  - Easy to represent using a basis representation of a matrix
  - Easy to represent the inverse of a linear transformation

- Thm 6.10: (Standard matrix for a linear transformation)

$$\mathbf{T(v)=Av}$$

$$\mathbf{A=?}$$

Pf:

$$\mathbf{v}=\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}=v_1e_1+v_2e_2+\cdots+v_ne_n$$

$$\begin{aligned} T \text{ is a L.T.} \Rightarrow T(\boldsymbol{v}) &= T(v_1e_1+v_2e_2+\cdots+v_ne_n) \\ &= T(v_1e_1)+T(v_2e_2)+\cdots+T(v_ne_n) \\ &= v_1T(e_1)+v_2T(e_2)+\cdots+v_nT(e_n) \end{aligned}$$

$$A\mathbf{v}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}=\begin{bmatrix} a_{11}v_1+a_{12}v_2+\cdots+a_{1n}v_n \\ a_{21}v_1+a_{22}v_2+\cdots+a_{2n}v_n \\ \vdots \\ a_{m1}v_1+a_{m2}v_2+\cdots+a_{mn}v_n \end{bmatrix}$$

$$= v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= v_1 T(e_1) + v_2 T(e_2) + \cdots + v_n T(e_n)$$

Therefore, $T(\mathbf{v}) = A\mathbf{v}$ for each $\mathbf{v}$ in R^n

- Ex 1: (Finding the standard matrix of a linear transformation)

Find the standard matrix for the L.T. $T: R^3 \rightarrow R^2$ define by

$$T(x, y, z) = (x - 2y, 2x + y)$$

Sol:

Vector Notation | Matrix Notation

$$T(e_1) = T(1, 0, 0) = (1, 2) \qquad T(e_1) = T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T(e_2) = T(0, 1, 0) = (-2, 1) \qquad T(e_2) = T(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$T(e_3) = T(0, 0, 1) = (0, 0) \qquad T(e_3) = T(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = \left[T(e_1) \mid T(e_2) \mid T(e_3) \right] = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

- Check:

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-2y \\ 2x+y \end{bmatrix}$$

$$\text{i.e. } T(x, y, z) = (x-2y, 2x+y)$$

- Ex 2: (Finding the standard matrix of a linear transformation)

  The linear transformation $T: R^2 \to R^2$ is given by projecting each point in R^2 onto the x-axis. Find the standard matrix for T.

  Sol:

$$T(x, y) = (x, 0)$$

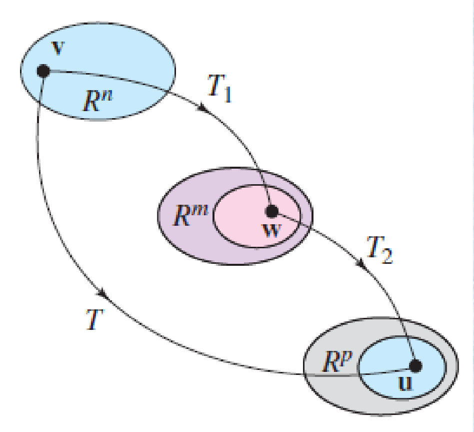
$$A = \left[T(e_1) \mid T(e_2)\right] = \left[T(1, 0) \mid T(0, 1)\right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- Notes:

  (1) The standard matrix for the zero transformation from R^n into R^m is the $m \times n$ zero matrix.

  (2) The standard matrix for the identity transformation from R^n into R^n is the $n \times n$ identity matrix I_n.

- Composition of $T_1 : R^n \to R^m$ with $T_2 : R^m \to R^p$:



$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})), \quad \mathbf{v} \in R^n$$

$$T = T_2 \circ T_1, \quad \text{domain of } T = \text{domain of } T_1$$

- Thm 6.11: (Composition of linear transformations)

Let $T_1 : R^n \to R^m$ and $T_2 : R^m \to R^p$ be L.T.

with standard matrices A_1 and A_2 , then

(1)The composition $T : R^n \to R^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is a L.T.

(2) The standard matrix A for T is given by the matrix product $A = \boxed{?}$

Pf:

(1)(T is a L.T.)

Let **u** and **v** be vectors in R^n and let c be any scalar then

$$T(\mathbf{u}+\mathbf{v}) = T_2(T_1(\mathbf{u}+\mathbf{v})) = T_2(T_1(\mathbf{u})+T_1(\mathbf{v}))$$
$$= T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) = T(\mathbf{u}) + T(\mathbf{v})$$

$$T(c\mathbf{v}) = T_2(T_1(c\mathbf{v})) = T_2(cT_1(\mathbf{v})) = cT_2(T_1(\mathbf{v})) = cT(\mathbf{v})$$

(2)(A_2A_1 is the standard matrix for T)

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})) = T_2(A_1\mathbf{v}) = A_2A_1\mathbf{v} = (A_2A_1)\mathbf{v}$$

- Note: (1) $T_1 \circ T_2 \neq T_2 \circ T_1$

  (2) $\mathrm{T(v)} = \mathrm{T_n}(T_{\mathrm{n}-1} \cdots (T_2(T_1(\mathrm{v}))) \cdots)$

  $\mathrm{A} = \mathrm{A_n A_{n-1}} \cdots \mathrm{A_2 A_1}$

- Ex 3: (The standard matrix of a composition)

Let T_1 and T_2 be L.T. from R^3 into R^3 s.t.

$$T_1(x, y, z) = (2x + y, 0, x + z)$$

$$T_2(x, y, z) = (x - y, z, y)$$

Find the standard matrices for the compositions

$T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$,

Sol:

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{(standard matrix for } T_1\text{)}$$

$$A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{(standard matrix for } T_2\text{)}$$

The standard matrix for $T = T_2 \circ T_1$

$$A = A_2 A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The standard matrix for $T' = T_1 \circ T_2$

$$A' = A_1 A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- Inverse linear transformation:

If $T_1 : R^n \to R^n$ and $T_2 : R^n \to R^n$ are L.T. s.t. for every $\mathbf{v}$ in R^n

$$T_2(T_1(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T_1(T_2(\mathbf{v})) = \mathbf{v}$$

Then T_2 is called the inverse of T_1 and T_1 is said to be invertible

- Note:

If the transformation T is invertible, then the inverse is unique and denoted by T^{-1} .

- Thm 6.12: (Existence of an inverse transformation)

Let $T : R^n \to R^n$ be a L.T. with standard matrix A,

Then the following condition are equivalent.

(1) T is invertible.

(2) T is an isomorphism.

(3) A is invertible.

- Note:

If T is invertible with standard matrix A, then the standard matrix for T^{-1} is A^{-1}.

- Ex 4: (Finding the inverse of a linear transformation)

The L.T. $T: R^3 \to R^3$ is defined by

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3, 3x_1 + 3x_2 + x_3, 2x_1 + 4x_2 + x_3)$$

Show that T is invertible, and find its inverse.

Sol:

The standard matrix for T

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \quad \begin{matrix} \leftarrow 2x_1 + 3x_2 + x_3 \\ \leftarrow 3x_1 + 3x_2 + x_3 \\ \leftarrow 2x_1 + 4x_2 + x_3 \end{matrix}$$

$$\left[A \mid I_3\right] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{G.J.E}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & 1 & 0\\ 0 & 1 & 0 & -1 & 0 & 1\\ 0 & 0 & 1 & 6 & -2 & -3\end{array}\right]=\left[I \mid A^{-1}\right]$$

Therefore T is invertible and the standard matrix for T^{-1} is A^{-1}

$$A^{-1}=\begin{bmatrix}-1 & 1 & 0\\ -1 & 0 & 1\\ 6 & -2 & -3\end{bmatrix}$$

$$T^{-1}(\mathbf{v})=A^{-1}\mathbf{v}=\begin{bmatrix}-1 & 1 & 0\\ -1 & 0 & 1\\ 6 & -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-x_1+x_2\\ -x_1+x_3\\ 6x_1-2x_2-3x_3\end{bmatrix}$$

In other words,

$$T^{-1}(x_1,x_2,x_3)=(-x_1+x_2,-x_1+x_3,6x_1-2x_2-3x_3)$$

- the matrix of T relative to the bases B *and* B':

$$T: V \rightarrow W \qquad \text{(a L.T.)}$$

$$B = \{v_1, v_2, \cdots, v_n\} \qquad \text{(a basis for } V\text{)}$$

$$B' = \{w_1, w_2, \cdots, w_m\} \qquad \text{(a basis for } W\text{)}$$

Thus, the matrix of T relative to the bases B *and* B' is

$$A = \left[[T(v_1)]_{B'}, [T(v_2)]_{B'}, \cdots, [T(v_n)]_{B'} \right] \in M_{m\times n}$$

- Transformation matrix for nonstandard bases:

Let V and W be finite - dimensional vector spaces with basis B and B', respectively, where $B=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$

If $T: V \to W$ is a L.T. s.t.

$$\big[T(\mathbf{v}_1)\big]_{B'} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \big[T(\mathbf{v}_2)\big]_{B'} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, \quad \big[T(\mathbf{v}_n)\big]_{B'} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

then the $m \times n$ matrix whose n columns correspond to $\big[T(\mathbf{v}_i)\big]_{B'}$

$$A = \left[T(v_1)_{B'} \middle| \; T(v_2)_{B'} \middle| \; \cdots \; \middle| \; T(v_n)_{B'} \right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $[T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B$ for every $\mathbf{v}$ in V.

- Ex 5: (Finding a matrix relative to nonstandard bases)

Let $T: R^2 \to R^2$ be a L.T. defined by

$$T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$$

Find the matrix of T relative to the basis

$B = \{(1, 2), (-1, 1)\}$ and $B' = \{(1, 0), (0, 1)\}$

Sol:

$$T(1, 2) = (3, 0) = 3(1, 0) + 0(0, 1)$$

$$T(-1, 1) = (0, -3) = 0(1, 0) - 3(0, 1)$$

$$\left[T(1, 2)\right]_{B'} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \left[T(-1, 1)\right]_{B'} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

the matrix for T relative to B and B'

$$A = \left[\left[T(1, 2)\right]_{B'} \quad \left[T(1, 2)\right]_{B'}\right] = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

- Ex 6:

For the L.T. $T: R^2 \to R^2$ given in Example 5, use the matrix A to find $T(\mathbf{v})$, where $\mathbf{v} = (2, 1)$

Sol:

$$\mathbf{v} = (2,1) = 1(1,2) - 1(-1,1) \qquad B = \{(1,2), (-1,1)\}$$

$$\Rightarrow [\mathbf{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow [T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow T(\mathbf{v}) = 3(1,0) + 3(0,1) = (3,3) \qquad B' = \{(1,0), (0,1)\}$$

- Check:

$$T(2,1) = (2+1, 2(2)-1) = (3,3)$$

- Notes:

(1) In the special case where $V = W$ and $B = B'$,

the matrix A is called the matrix of T relative to the basis B

(2) $T : V \rightarrow V$: the identity transformation

$B = \{v_1, v_2, \cdots, v_n\}$: a basis for V

$\Rightarrow$ the matrix of T relative to the basis B

$$A = \left[[T(v_1)]_B, [T(v_2)]_B, \cdots, [T(v_n)]_B\right] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n$$