HW1

$s(t) = 5\sin(10t + 2)$。

$= 5\sin(2\pi \times f \times t + 2)$

$2\pi f = 10$

$\Rightarrow f = \frac{10}{2\pi}$ (Hz)

HW1 Problem 2

Bandwidth is determined by $\cos(200\pi t)^2$

frequency of $\cos(200\pi t)$ is 100 Hz

$(= \cos(2\pi \times 100 t))$

highest freq. $100 + 100 = 200$ Hz

lowest freq $= 100 - 100 = 0$ Hz

$\Rightarrow BW = 200 - 0$

$= 200$ Hz

HW1 Problem 5

receive $I(t)\cos(2\pi f t) + Q(t)\sin(2\pi f t)$

$1^\circ \times \sin(2\pi f t) \Rightarrow I(t)\cos(2\pi f t)\sin(2\pi f t) + Q(t)\sin(2\pi f t)\sin(2\pi f t)$

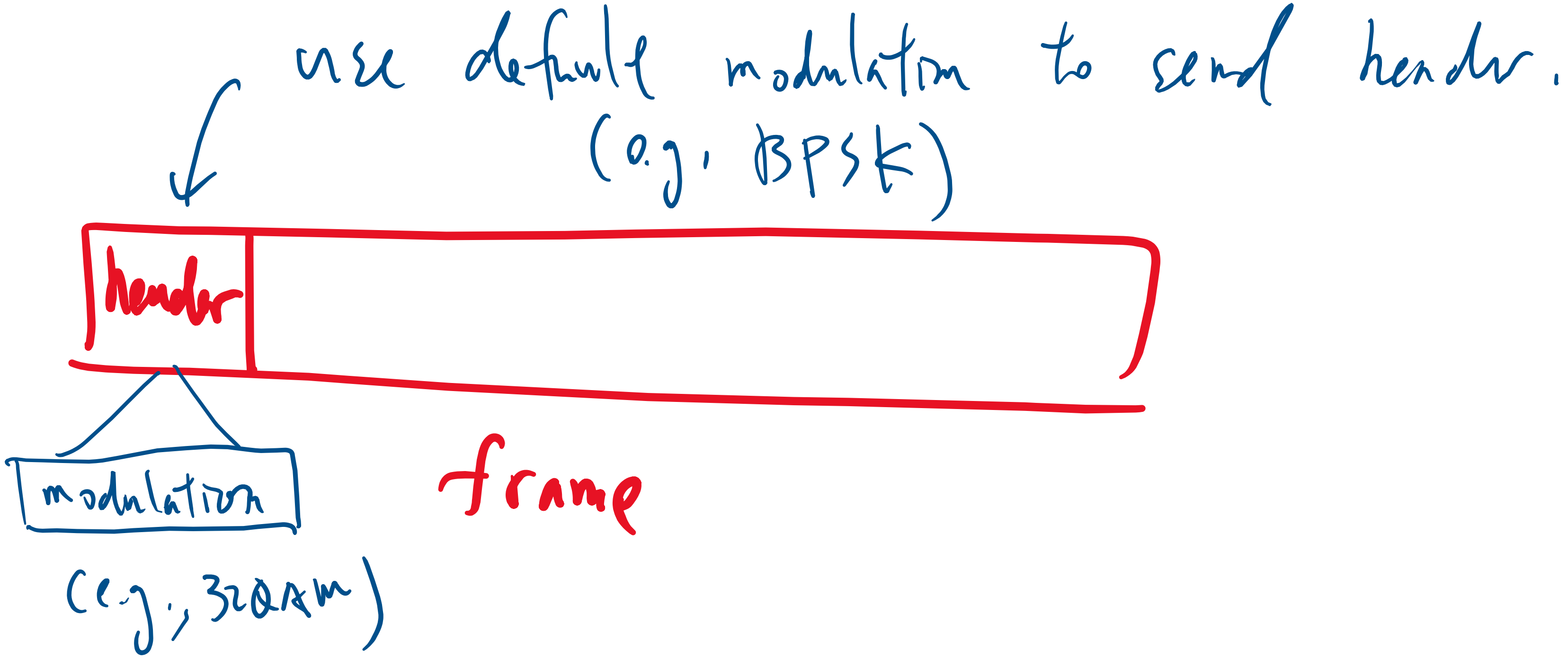
$$\Rightarrow \frac{I(t)}{2}\left(\underbrace{\cancel{\sin(2\pi(2f)t)}}_{\text{filter}} - \underbrace{\cancel{\sin(2\pi \times 0 \times t)}}_{=0}\right)$$

$$+ \frac{Q(t)}{2}\left(\underbrace{\cos(2\pi \times 0 \times t)}_{=1} - \underbrace{\cancel{\cos(2\pi \times (2f) \times t)}}_{\text{filter}}\right)$$

$\xrightarrow{\text{filter}} \frac{Q(t)}{2}$

$Q(t) \xleftarrow{\text{amplifi}} \frac{Q(t)}{2}$

HW1 Problem 7



HW1 Problem 8

a. 530 MHz

b. BW (before modulation) = 50 MHz

$$\frac{10}{T} = 50 \text{ MHz}$$

$$\Rightarrow T = \frac{10}{50M} (\text{sec}) = 0.2 \times 10^{-6} (\text{sec})$$

HW1 Problem 8

c. $\frac{1}{T} \Rightarrow$ symbols/sec

$\frac{1}{0.2\times10^{-6}}$ symbols per second.

5×10^{6} symbols per sec.

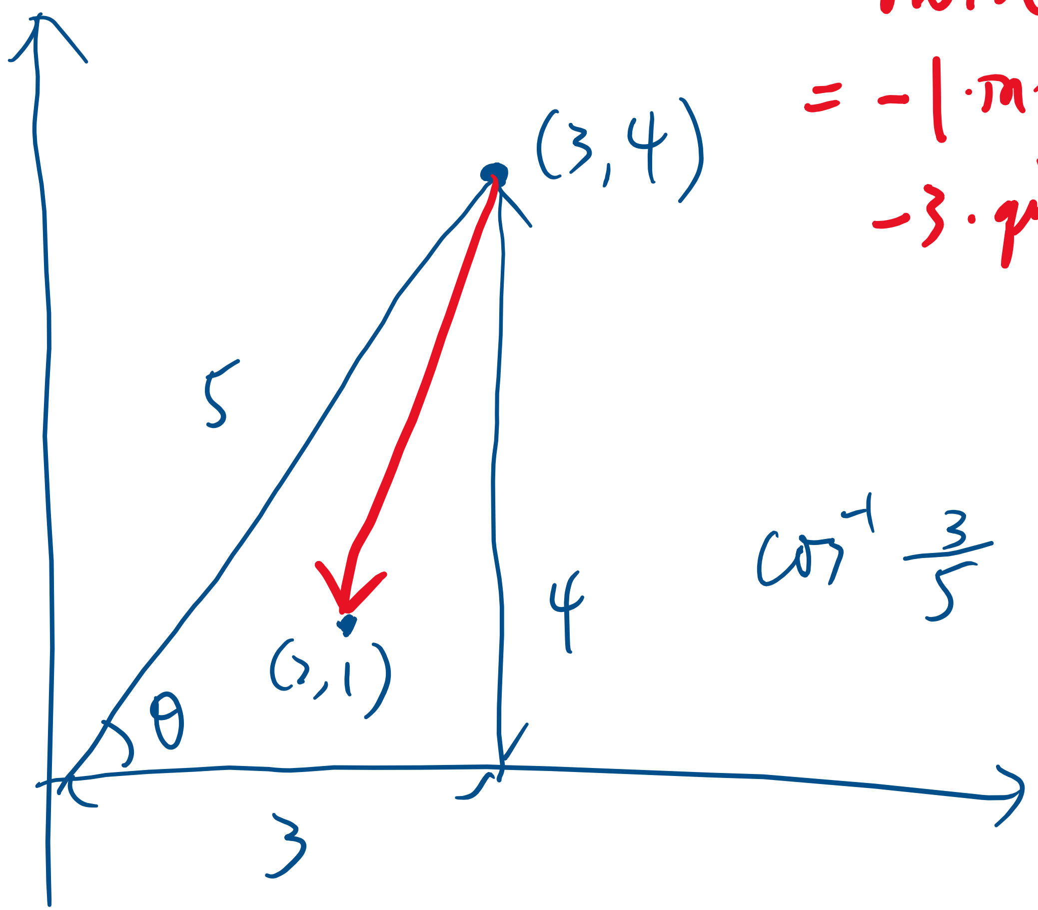
BPSK

$\Rightarrow 5\times10^{6}$ bits per sec.

1024 QAM

$\Rightarrow 50\times10^{6}$ bits per se.

HW1 Problem 8

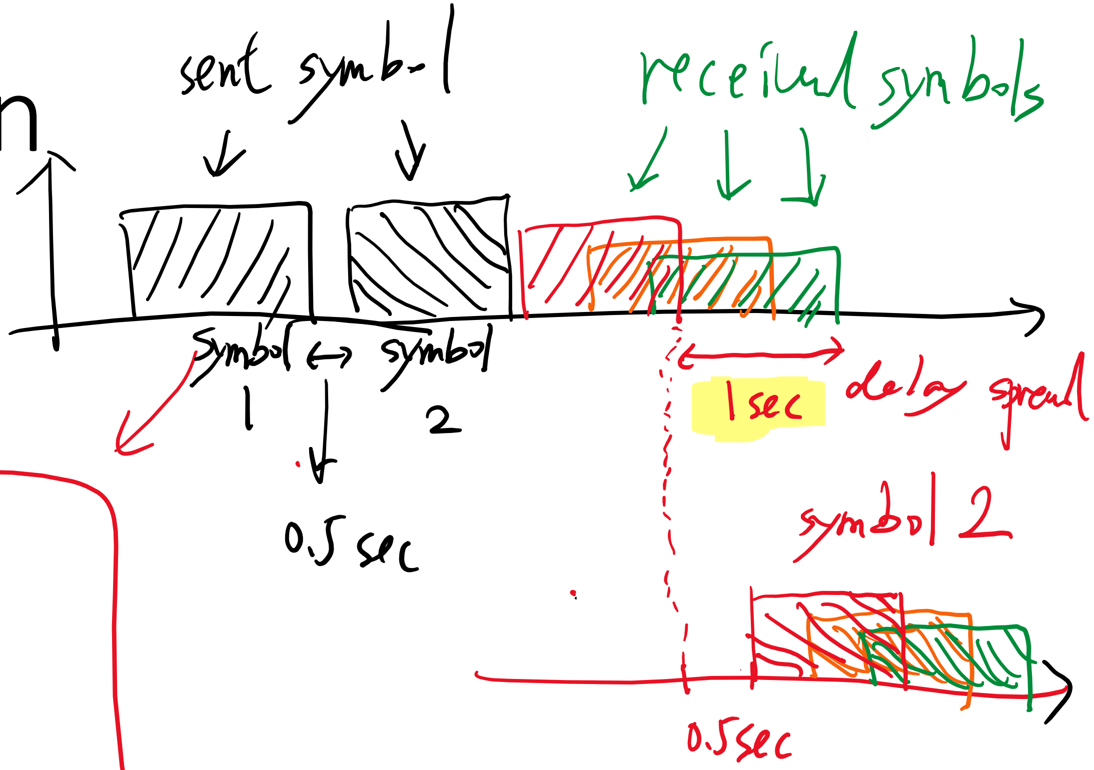


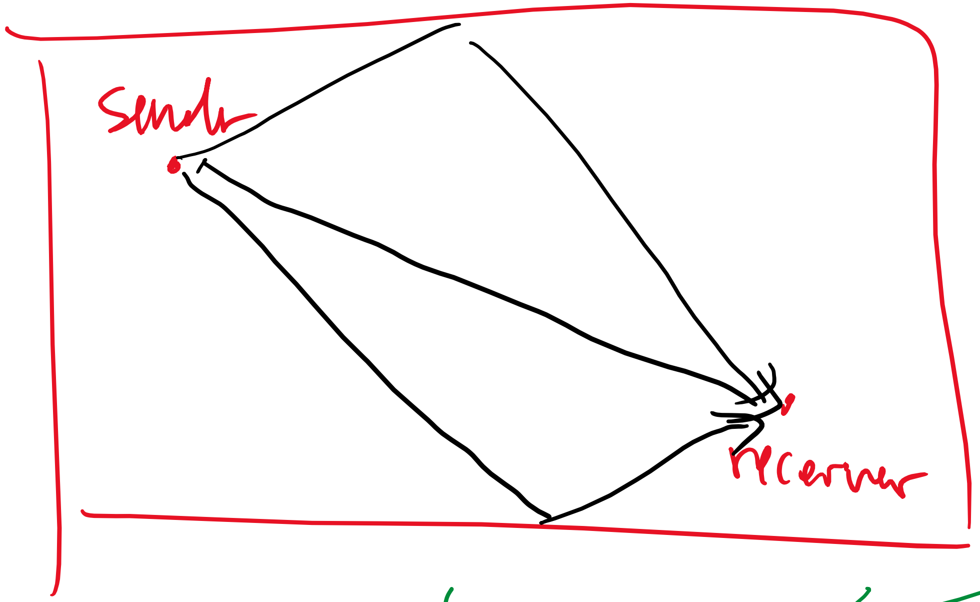
noise $= -1 \cdot$ inphase carrier $-3 \cdot$ quadrature carrier.

OFDM

Broadband Communication

challenge: multipath





2 types of interference:

1° intra-symbol interference

copies of symbol 1 interfere with each other.

2° inter-symbol interference

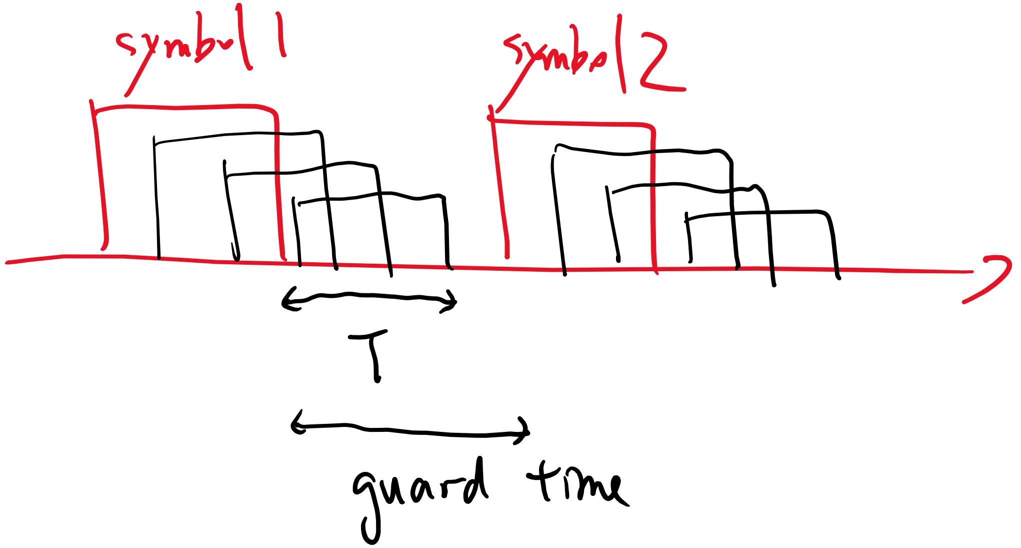
symbols 1 & 2 interfere.

OFDM can solve it →

Guard Time (to avoid inter symbol interference)

suppose max delay spread is T

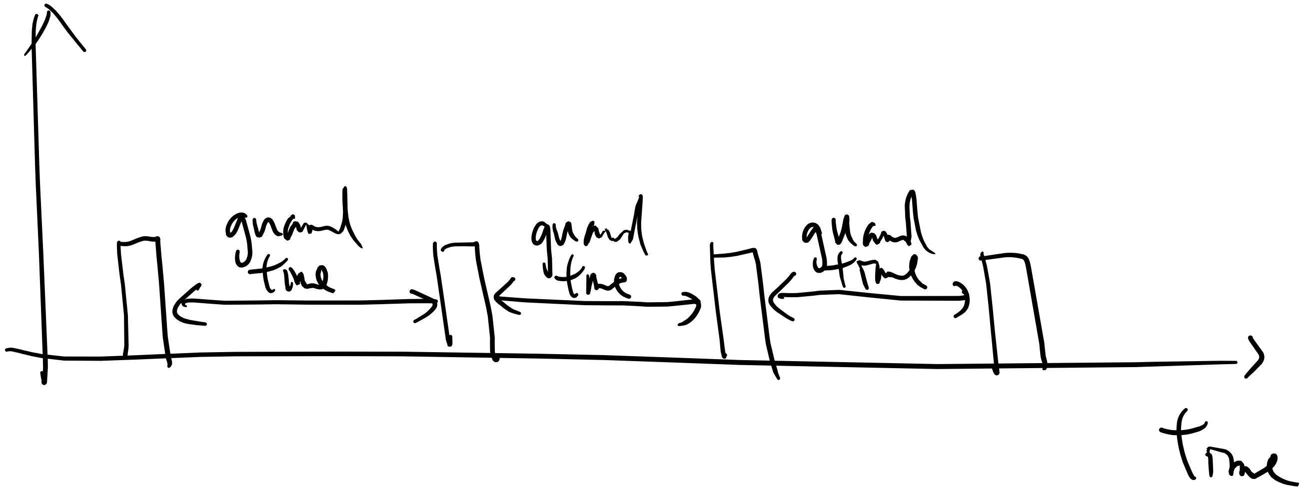
guard time $\geq T$



Multipath on Broadband Communication

Delay spread T is fixed (regardless of the bandwidth)

Bandwidth ↑ ⇒ symbol duration ↓



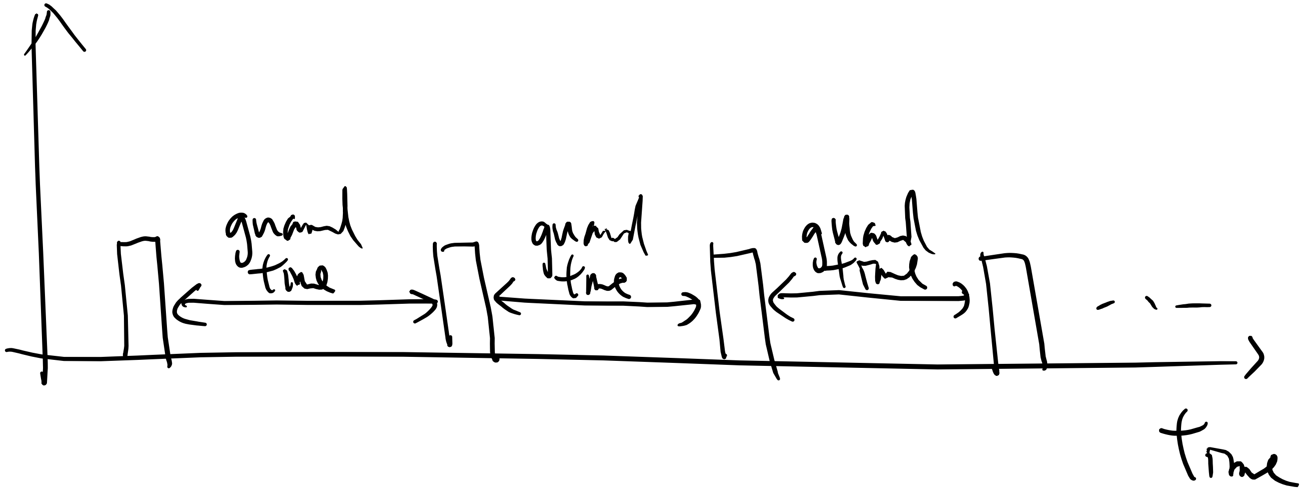
Broadband

narrowband ...

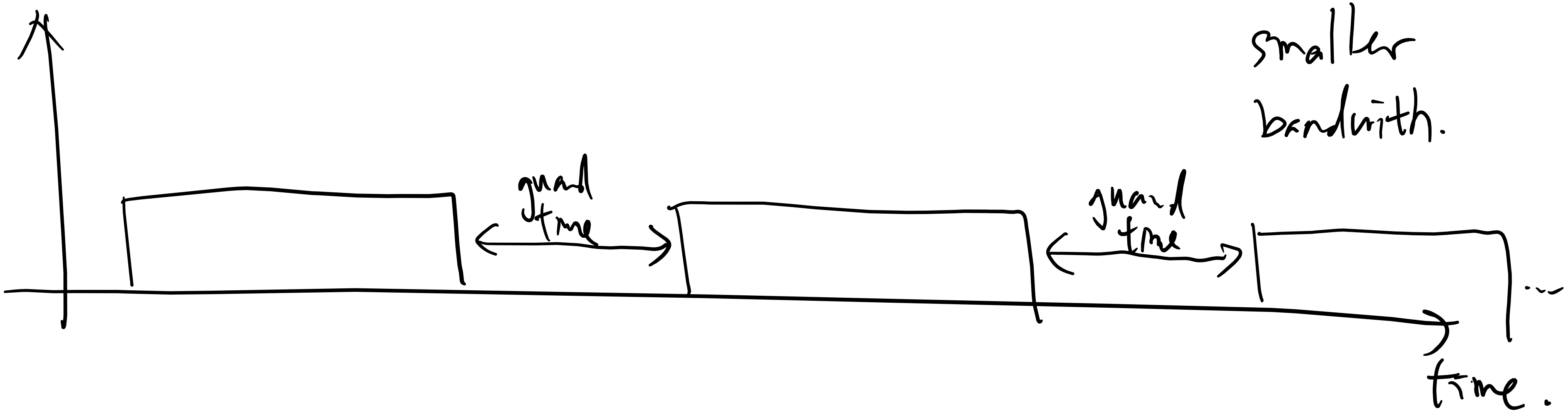
(most time is idle)

utilization is low

larger bandwidth

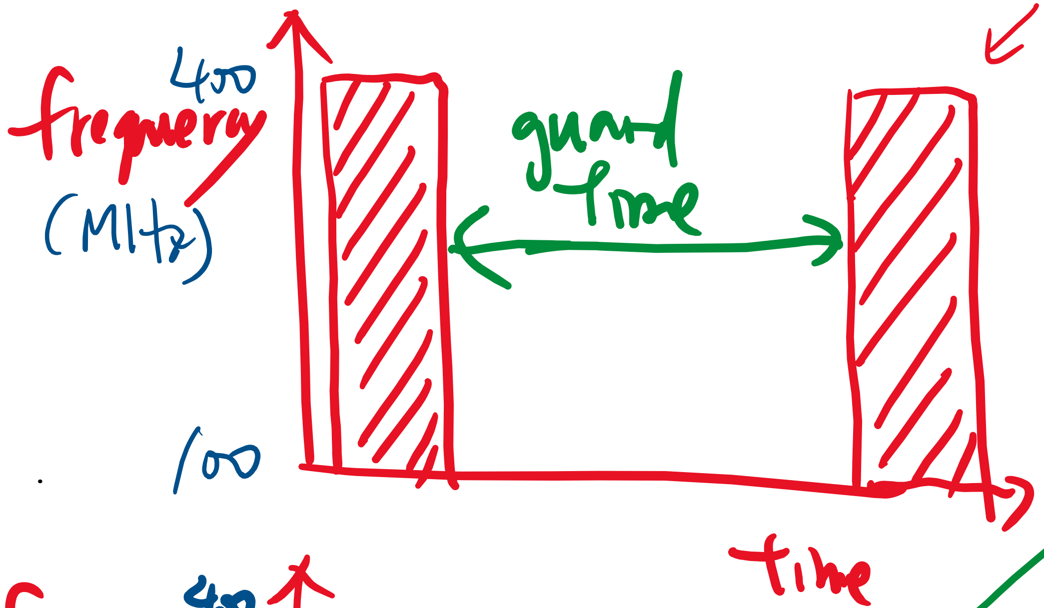


smaller bandwidth.



Solution: divide a large channel into smaller (sub)channels

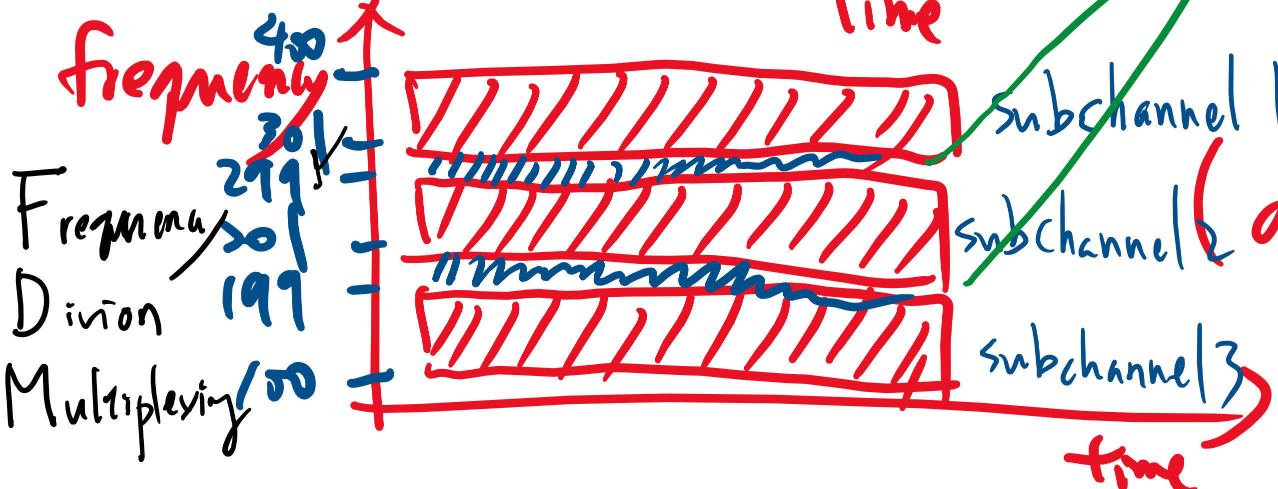
single channel



(Broadband)

guard band (to avoid interference between subchannels)

smaller BW ⇒ longer symbol duration



(divide the channel into 3 subchannels)

Comparison

1° single channel (broadband) : pros: better spectrum utilization

cons: poor time utilation. (guard time)

2° multi channel (subchannels)

pros : better time utilization

cons: 1° poor spectrum utilization (guard band)

2° higher hardware cost.

Example

$$\text{eff. BW} = \frac{1}{T}$$

channel : 100 MHz ~ 130 MHz

$BW = 30\ MHz$

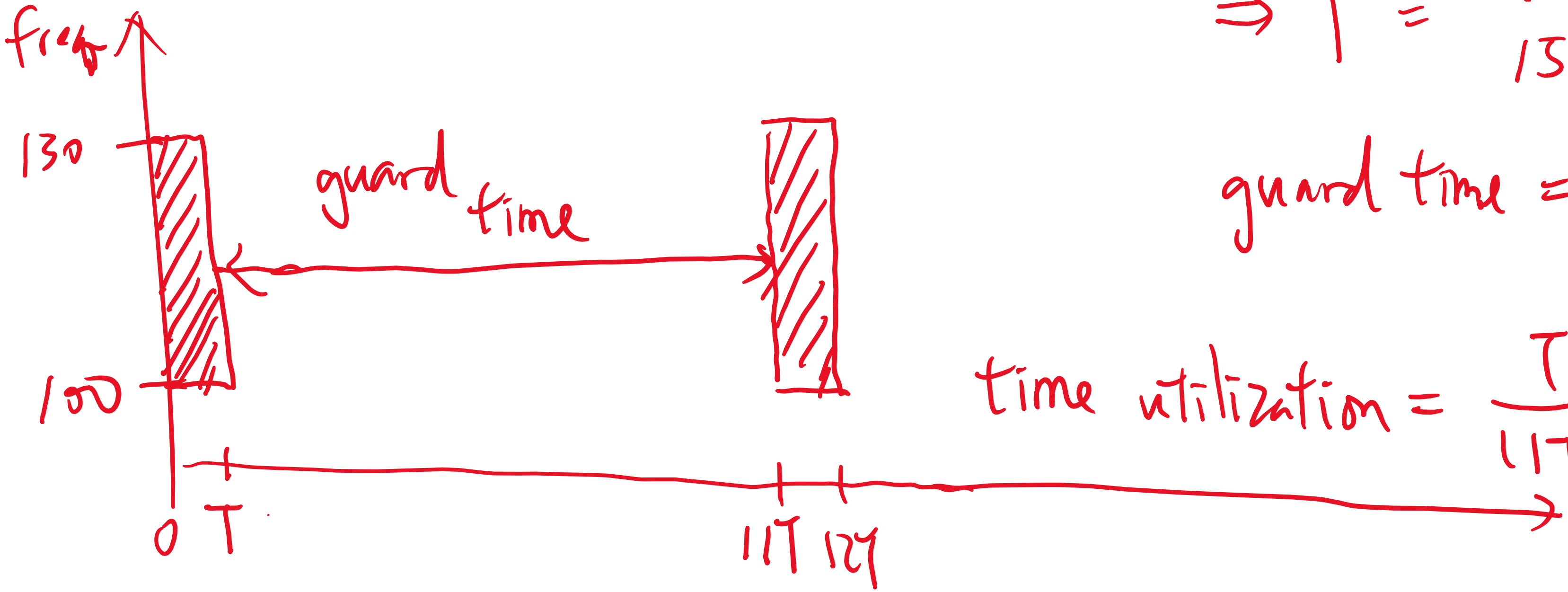
$\Rightarrow$ BW before modulation = 15 MHz

$$\Rightarrow T = \frac{1}{15\times10^6}\ sec = \frac{1}{1.5\times10^7}$$

$$= 0.66\times10^{-7}$$

guard time = 0.66×10^{-6} sec

$$\text{time utilization} = \frac{T}{11T} = \frac{1}{11}$$



Example

3 subchannels : $100 \sim 110$

$110 \sim 120$ $\Rightarrow$ BW of each subchanel = 10MHz

$120 \sim 130$ $\Rightarrow$ BW of each subchanel = 5MHz before modulation

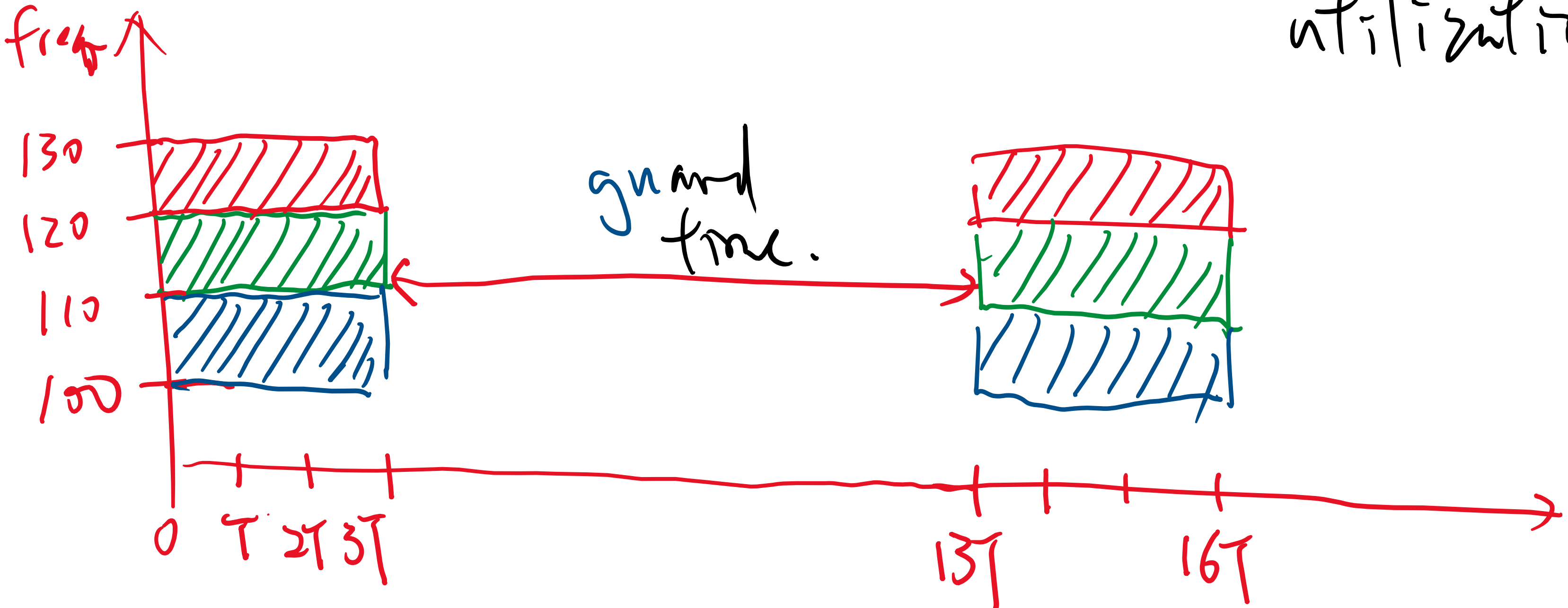
$$T' = 3T$$

$$\frac{1}{T'} = 5 \times 10^{6}\ Hz$$

$$T' = \frac{1}{5 \times 10^{6}} = 0.2 \times 10^{-6}\ (sec)$$

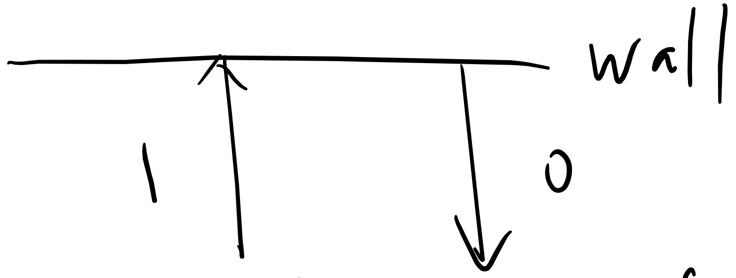
$$= 2 \times 10^{-7}\ (sec)$$

Example



$$\text{utilization} = \frac{3}{13}$$

Phase and reflection



$\cos(2\pi f t)$ $\cos(2\pi f t + 180^\circ)$

BPSK	bits		0	1	1	0
	phase		0°	180°	180°	0°
modification		0	0	1	1	0
	sent	0°	0°	180°	180°	0°

Phase and Reflection

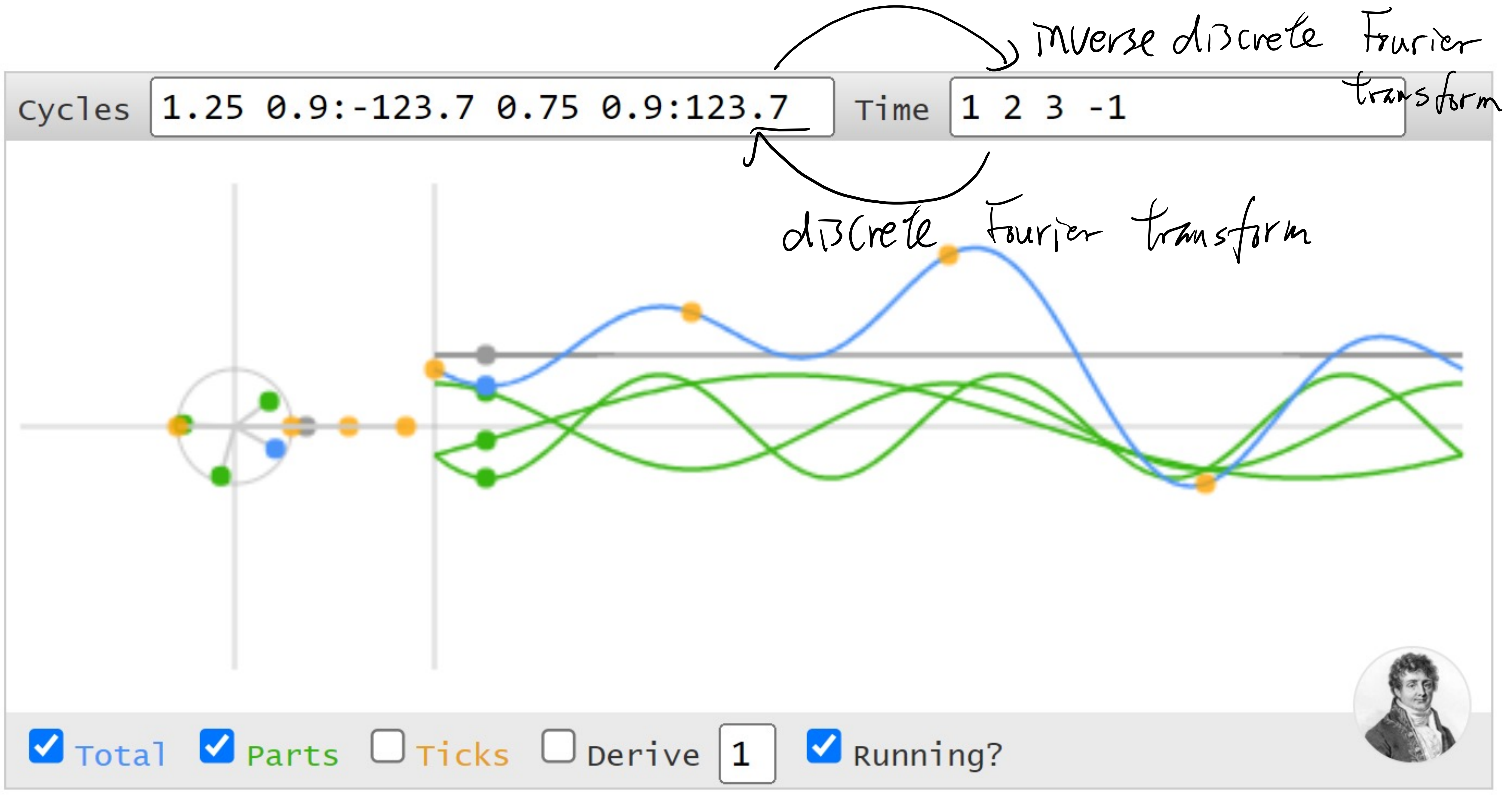
BPSK	bits		0	1	1	0
	phase		0°	180°	180°	0°
		0	0	1	1	0
modification						
sent		0°	0°	180°	180°	0°
received		180°	180°	0°	0°	180°
		0	0	1	1	0

Differential BPSK

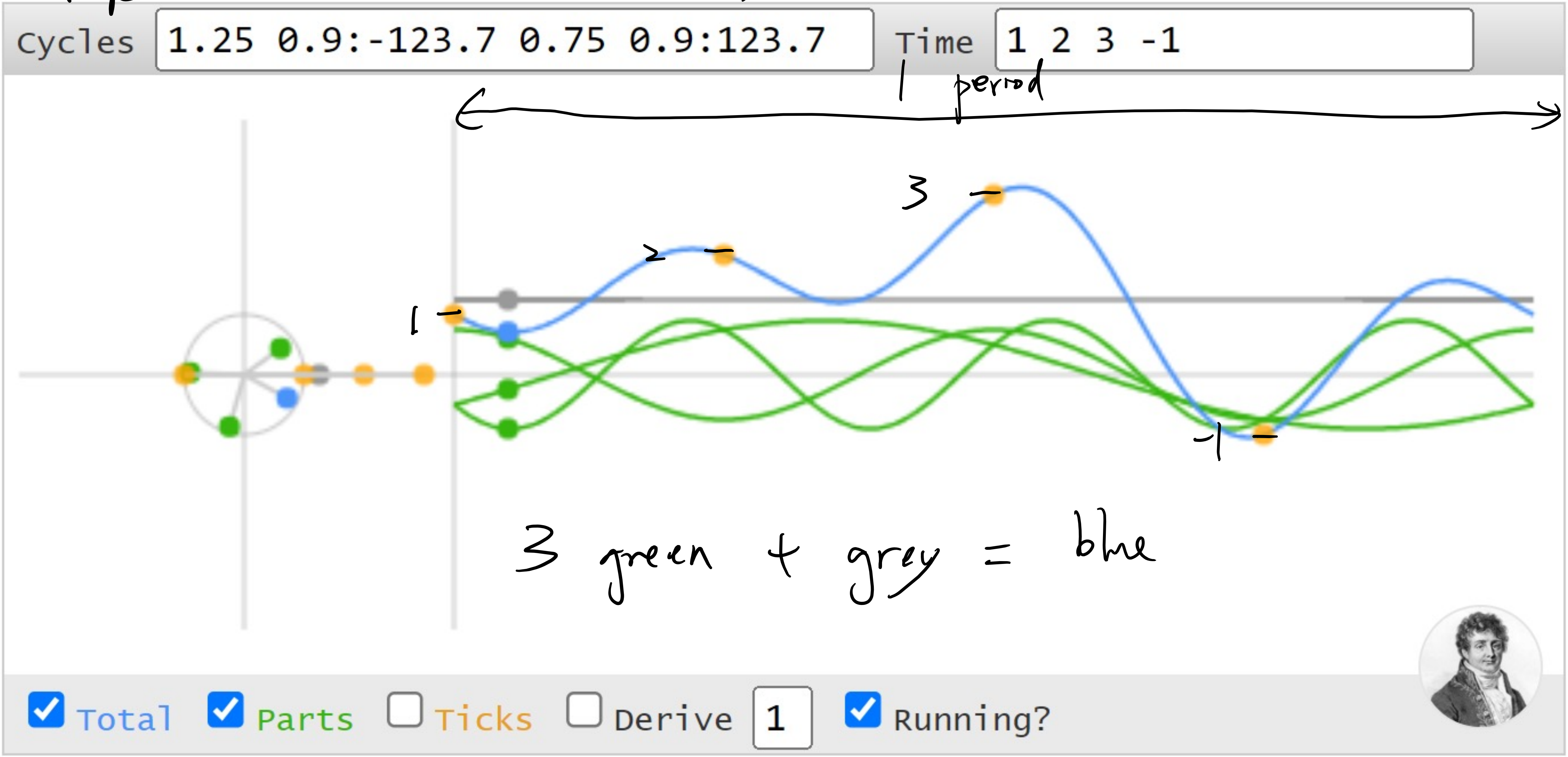
BPSK	bits		0	1	1	0
	phase		0°	180°	180°	0°
DBPSK			0	1	1	0
	sent	0°	0°	180°	0°	0°
	recv: (reflection) +60°	60°	60°	240°	60°	60°
			0	1	1	0

0 → phase = previous symbol's phase

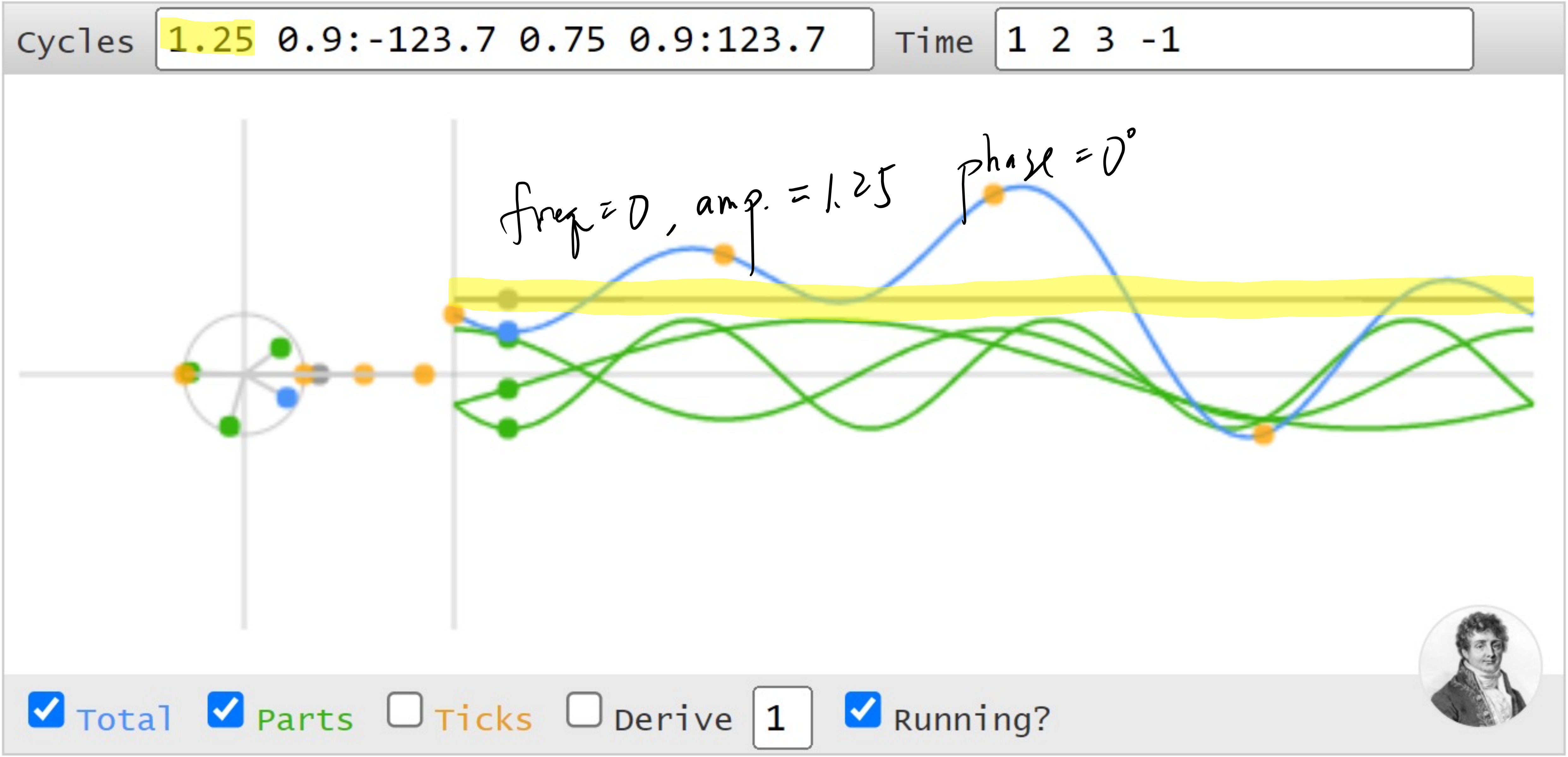
1 → phase = previous symbol's phase + 180°

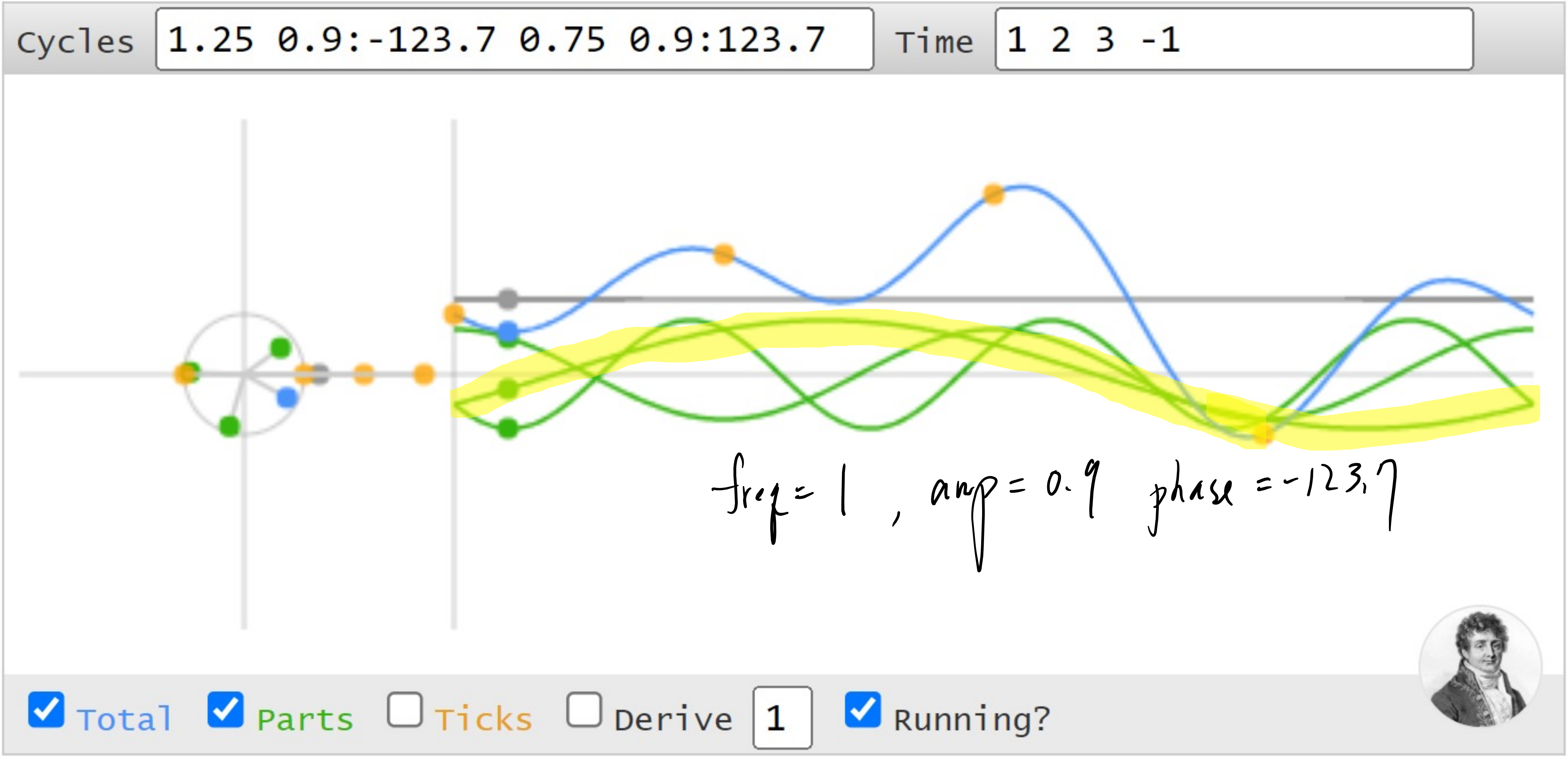

inverse discrete Fourier transform
Cycles
1.25 0.9:-123.7 0.75 0.9:123.7
Time
1 2 3 -1
discrete Fourier transform
Total
Parts
Ticks
Derive
1
Running?

freq 0 1 2 3

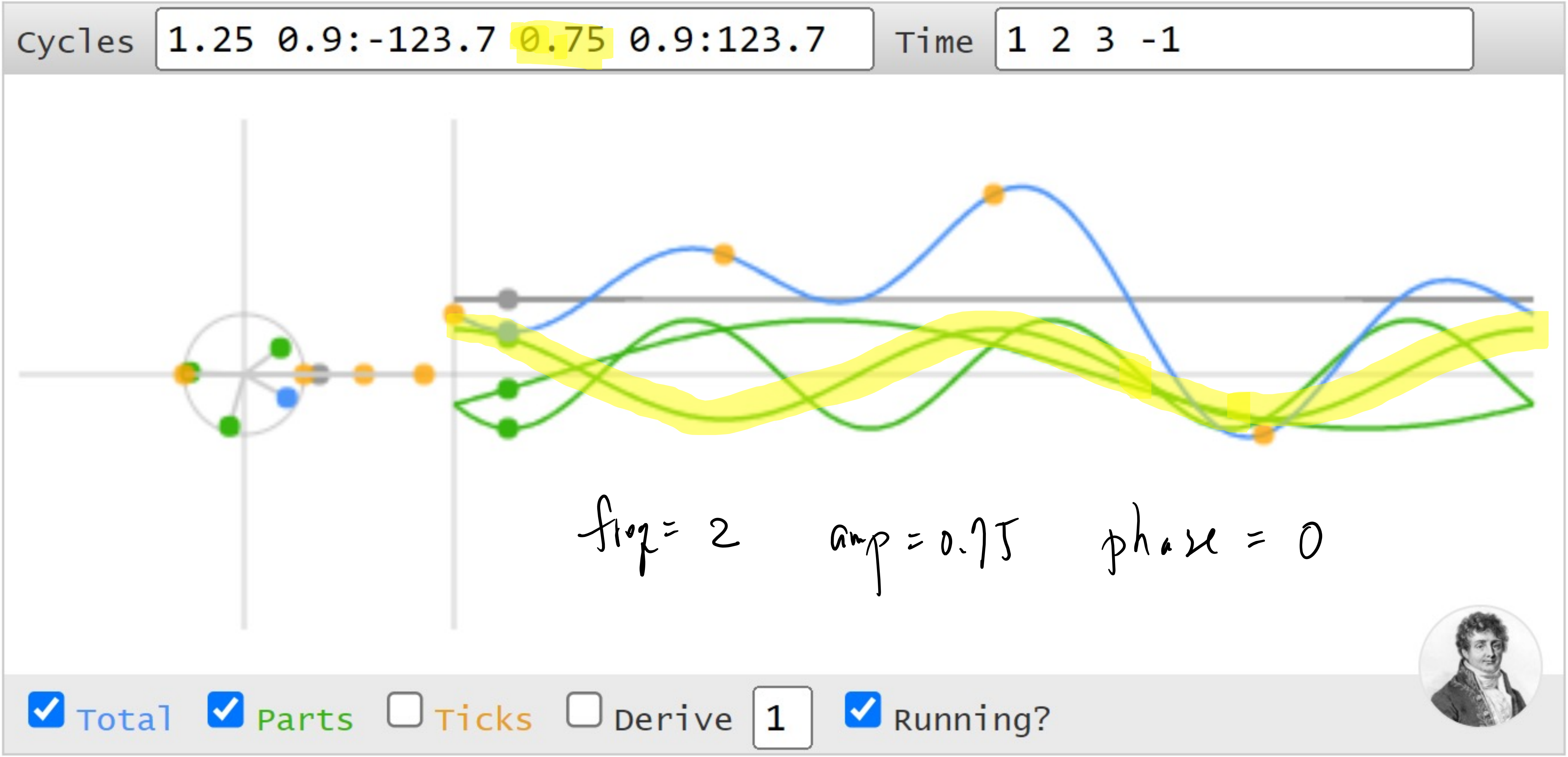


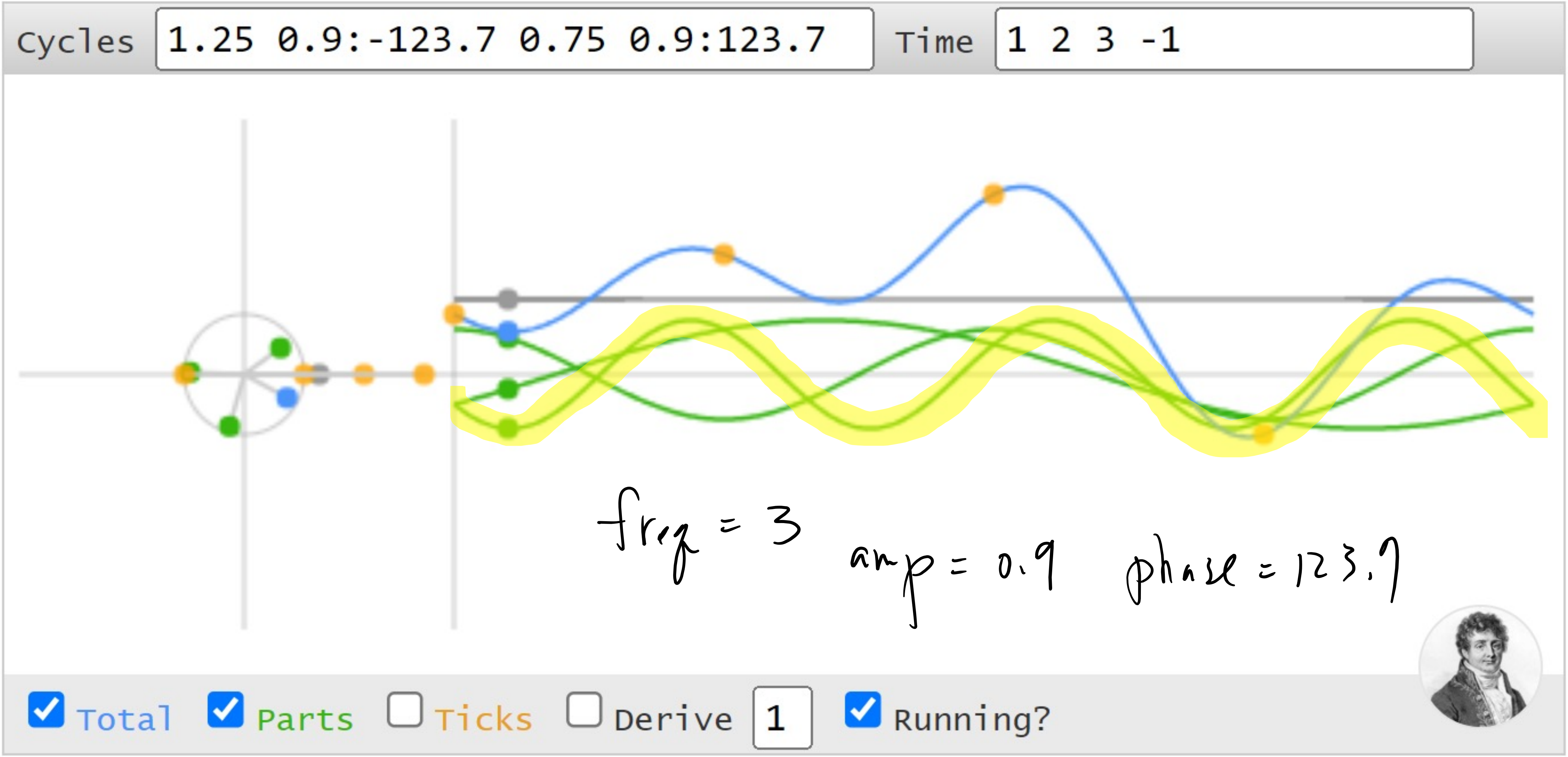
https://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/

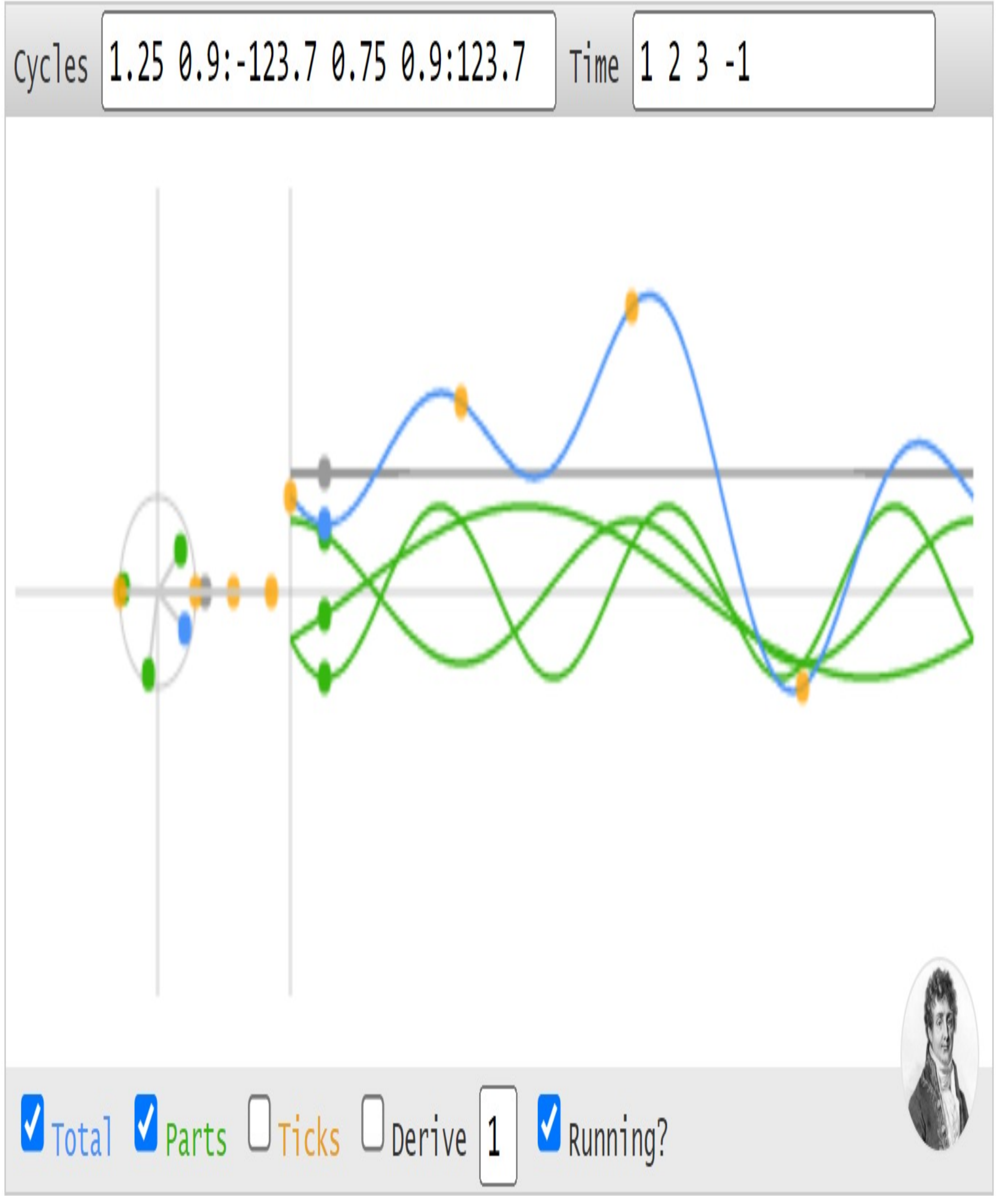

Cycles
1.25 0.9:-123.7 0.75 0.9:123.7
Time
1 2 3 -1
freq = 0, amp. = 1.25
phase = 0°
Total
Parts
Ticks
Derive
1
Running?

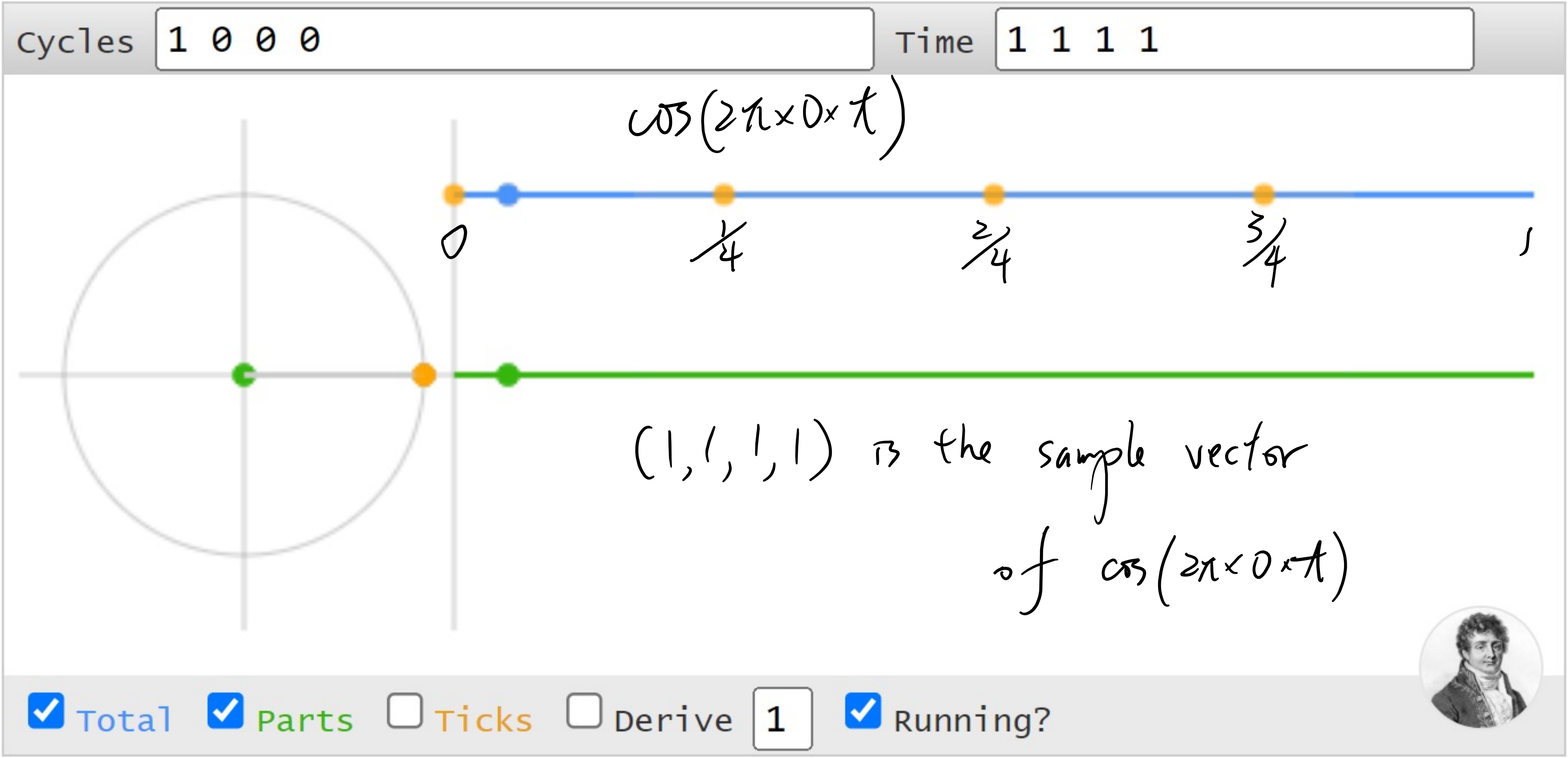


Cycles
1.25 0.9:-123.7 0.75 0.9:123.7
Time
1 2 3 -1
freq = 1 , amp = 0.9 phase = -123.7
Total
Parts
Ticks
Derive
1
Running?


Cycles
1.25 0.9:-123.7 0.75 0.9:123.7
Time
1 2 3 -1
freq = 2   amp = 0.75   phase = 0
Total
Parts
Ticks
Derive
1
Running?

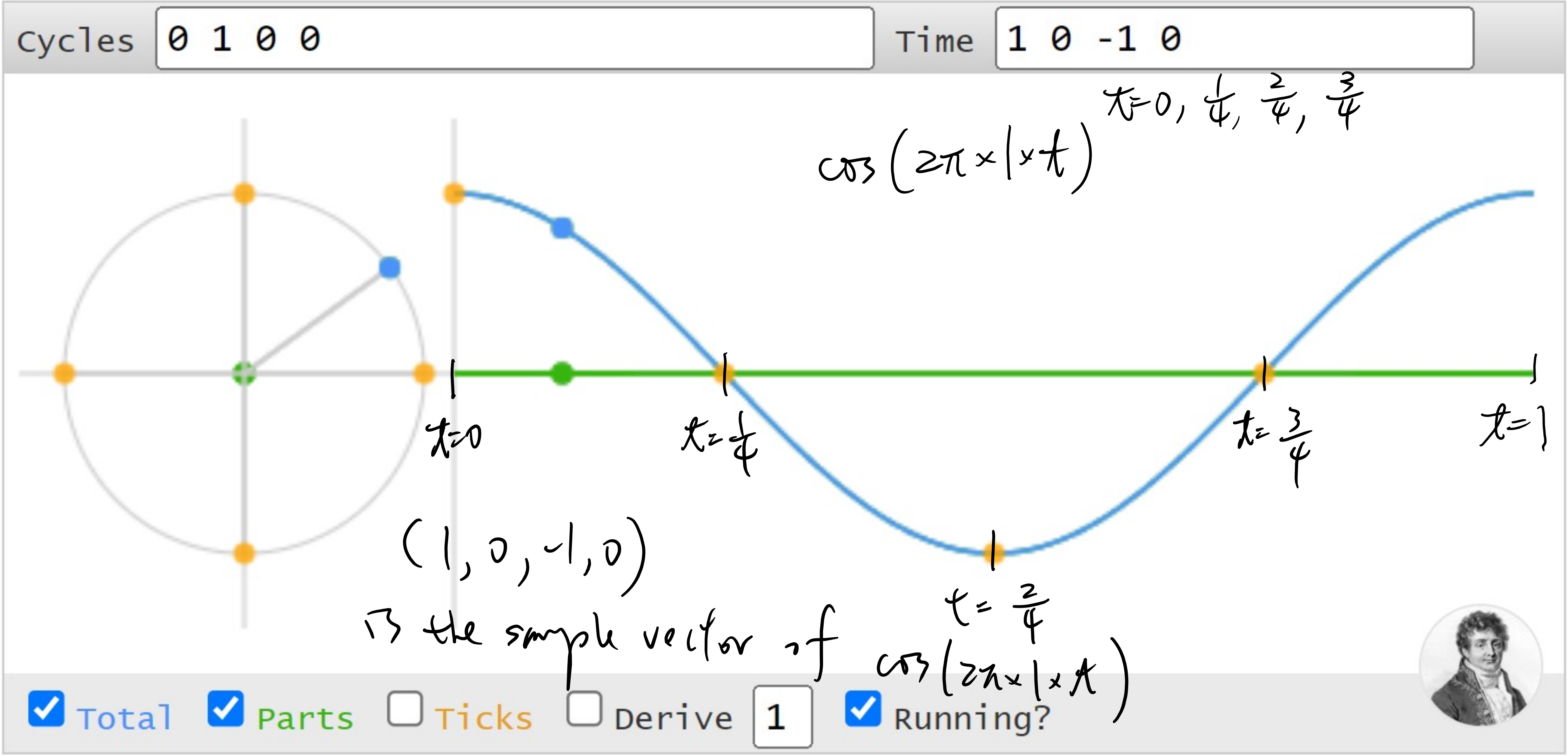

Cycles
1.25 0.9:-123.7 0.75 0.9:123.7
Time
1 2 3 -1
freq = 3
amp = 0.9
phase = 123.7
Total
Parts
Ticks
Derive
1
Running?


Cycles
1.25 0.9:-123.7 0.75 0.9:123.7
Time
1 2 3 -1
Total
Parts
Ticks
Derive
1
Running?



$\cos(2\pi \times 0 \times t)$

$(1,1,1,1)$ is the sample vector of $\cos(2\pi \times 0 \times t)$

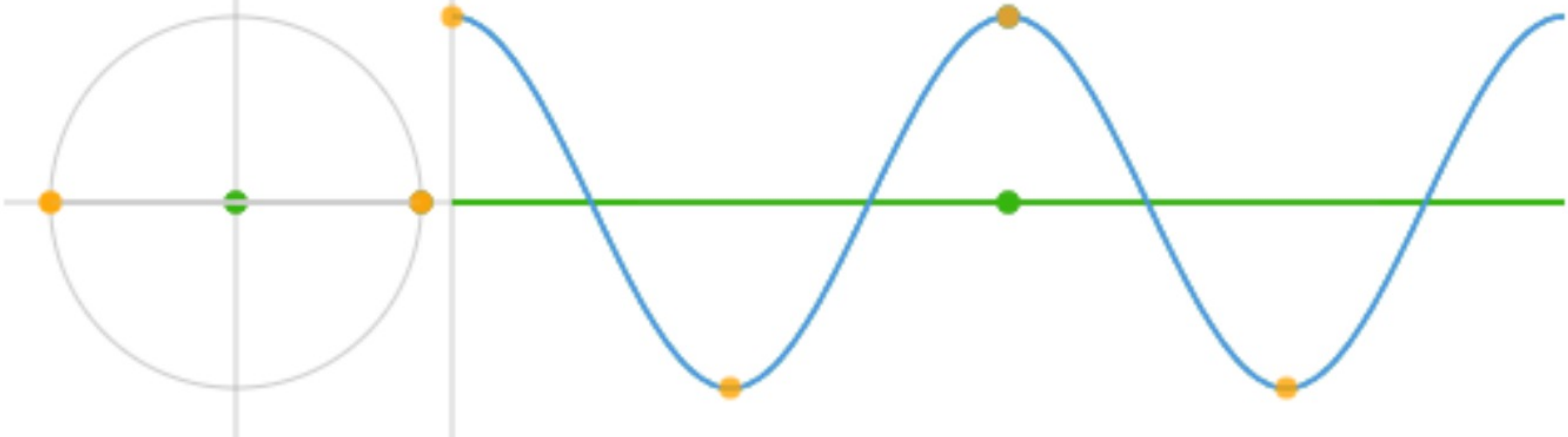


https://www.desmos.com/calculator/cuuntonqrv

Cycles 0 0 1 0

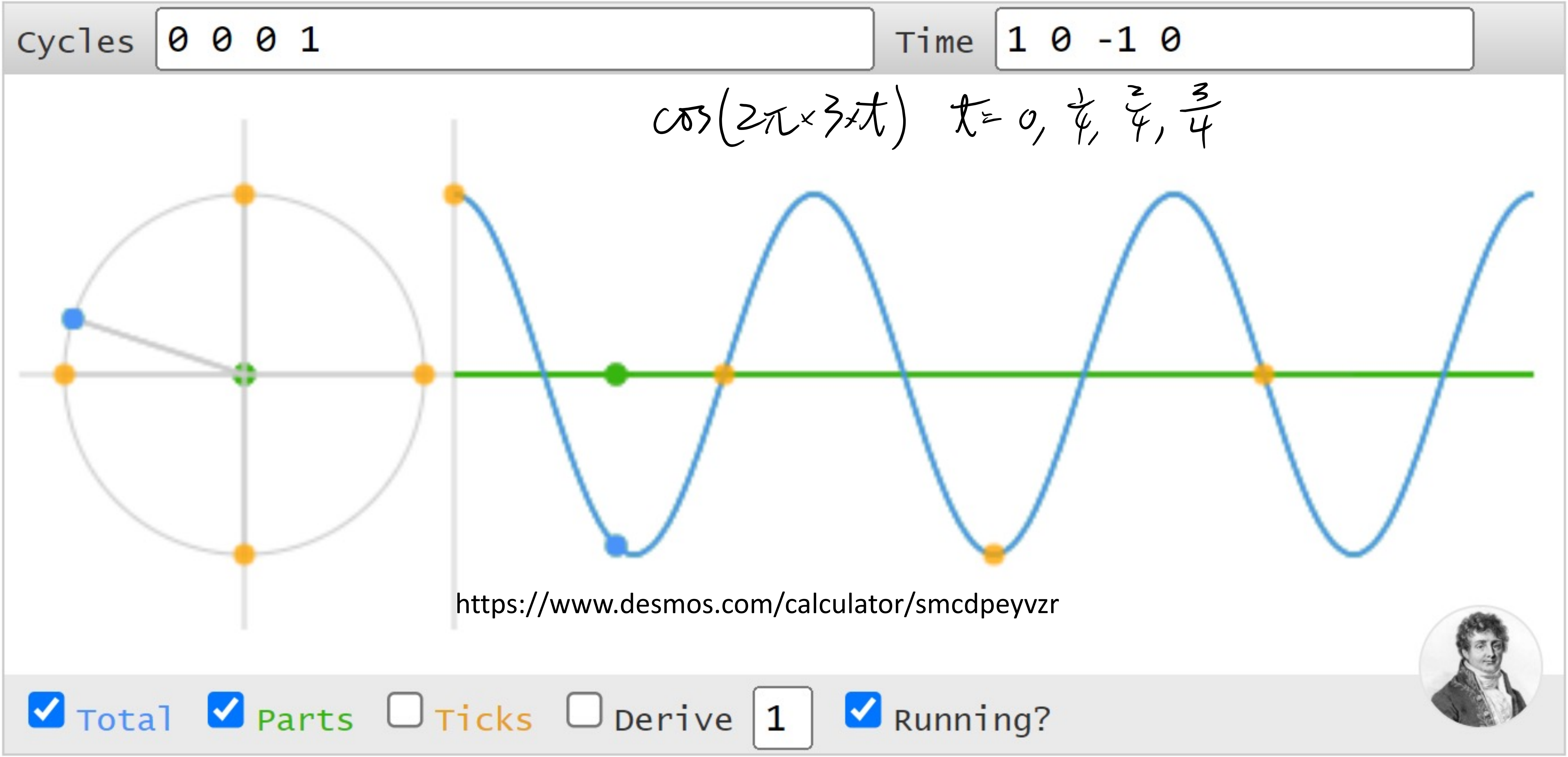
Time 1 -1 1 -1

$\cos(2\pi \times 2 \times t)$ $\quad t = 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}$



https://www.desmos.com/calculator/8vhkoyvrke

☑ Total ☑ Parts ☐ Ticks ☐ Derive 1 ☑ Running?



$\cos(2\pi \times 3 \times t)$ $t = 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}$

https://www.desmos.com/calculator/smcdpeyvzr

n-point Discrete Fourier Transform

C_f: sample vector of $\cos(2\pi f t)$ at time $0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \cdots \frac{n-1}{n}$

For example: $n=4$

vector of cos(2π·0·t)'s samples

$C_0 = (1, 1, 1, 1)$

$C_1 = (1, 0, -1, 0)$

vector of cos(2π·2·t)'s samples

$C_2 = (1, -1, 1, -1)$

$C_3 = (1, 0, -1, 0)$

Input: n time-domain samples:

$s_0, s_1, \cdots, s_{n-1}$

$S = (s_0, s_1, \cdots, s_{n-1})$

output: n amplitudes $a_0, a_1, \cdots, a_{n-1}$

such that

$S = a_0 C_0 + a_1 C_1 + \cdots + a_{n-1} C_{n-1}$

Cycles 1 2 3 -1 Time 5 -2 3 -2

$$\begin{cases} 5 = 1x + 1y + 1z + 1w \\ -2 = 1x + 0y - 1z + 0w \\ \vdots \end{cases}$$

$n = 4$

input $S = (5, -2, 3, -2)$

output $a_0 = 1, a_1 = 2, a_2 = 3, a_3 = -1$

samples of $\boxed{a_0 \cos(2\pi \times 0 \times t)}$ at time $\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}$

$$\begin{pmatrix} 5 \\ -2 \\ 3 \\ -2 \end{pmatrix} = \underset{a_0}{x} \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{C_0} + \underset{a_1}{y} \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}}_{C_1} + \underset{a_2}{z} \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}}_{C_2} + \underset{a_3}{w} \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}}_{C_3}$$

Basis of Discrete Fourier Transform

consider $\cos(2\pi f t) + i\sin(2\pi f t)$

$$\boxed{i = \sqrt{-1}}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$a\cos(2\pi f t + \theta) + i \cdot a\sin(2\pi f t + \theta)$$

$$e^{i 2\pi f t}$$

To change amplitude & phase $\Rightarrow a e^{i\theta} \times e^{i 2\pi f t}$

a $\qquad$ θ

$$= a e^{i(2\pi f t + \theta)}$$

$\omega = e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$\omega = e^{i\frac{2\pi}{n}}$

Basis of Discrete Fourier Transform

Basis: samples of $e^{i2\pi f t}$ at time: $\frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \dots \frac{n-1}{n}$

Example: $n=4$

$b_0 = \left(e^{i2\pi\times0\times\frac{0}{4}}, e^{i2\pi\times0\times\frac{1}{4}}, e^{i2\pi\times0\times\frac{2}{4}}, e^{i2\pi\times0\times\frac{3}{4}}\right) = (1, 1, 1, 1)$

$b_1 = \left(e^{i2\pi\times1\times\frac{0}{4}}, \underbrace{e^{i2\pi\times1\times\frac{1}{4}}}_{\omega}, e^{i2\pi\times1\times\frac{2}{4}}, e^{i2\pi\times1\times\frac{3}{4}}\right) = (1, i, -1, -i)$

$b_2 = \left(e^{i2\pi\times2\times\frac{0}{4}}, e^{i2\pi\times2\times\frac{1}{4}}, e^{i2\pi\times2\times\frac{2}{4}}, e^{i2\pi\times2\times\frac{3}{4}}\right) = (1, -1, 1, -1)$

$b_3 = \left(e^{i2\pi\times3\times\frac{0}{4}}, e^{i2\pi\times3\times\frac{1}{4}}, e^{i2\pi\times3\times\frac{2}{4}}, e^{i2\pi\times3\times\frac{3}{4}}\right) = (1, -i, -1, i)$

Sdvcm ,

Basis of Discrete Fourier Transform

$$\omega = e^{i\frac{2\pi}{n}}$$

$$b_0 = (\underbrace{1, 1, 1, 1, \cdots, 1}_{n})$$

when $n=4$

$$\omega = e^{i\frac{2\pi}{4}} = i$$

$$b_1 = (1, \omega^1, \omega^2, \omega^3, \cdots, \omega^{n-1})$$

$$b_2 = (1, \omega^2, \omega^4, \omega^6, \cdots, \omega^{2(n-1)})$$

$\vdots$

$$b_k = (1, \omega^k, \omega^{2k}, \omega^{3k}, \cdots, \omega^{k(n-1)})$$

$\vdots$

$$b_{n-1} = (1, \omega^{n-1}, \omega^{2(n-1)} \cdots, \omega^{(n-1)(n-1)})$$

Inner Product

$\overline{a+bi} = a-bi$

$b_1 \cdot b_1 = (1, i, -1, -i) \cdot (1, \bar{i}, -1, \overline{-i})$

$= 1 \cdot 1 + i \cdot (-i) + (-1)(-1) + (-i)(i)$

$= 1 + 1 + 1 + 1 = 4$

Inner Product

$$b_1 = (1, i, -1, -i)$$

$$b_2 = (1, -1, 1, -1)$$

$$b_1 \cdot b_2 = 1 \cdot 1 + i(-1) - 1(1) + (-i)(-1)$$

$$= 1 - i - 1 + i = 0$$

$$b_2 \cdot b_1 = 1 \cdot 1 + (-1)(-i) + 1(-1) + (-1)(i)$$

$$= 1 + i - 1 - i = 0$$

Inner Product

$$b_3 = (1, -i, -1, i)$$

$$b_1 = (1, i, -1, -i)$$

$$b_3 \cdot b_1 = 1 \times 1 + (-i)(-i) + (-1)(-1) + (i)(i)$$

$$= 1 - 1 + 1 - 1 = 0$$

Discrete Fourier Transform

$n=4$

input: s_0, s_1, s_2, s_3 , output: a_0, a_1, a_2, a_3

$$(s_0, s_1, s_2, s_3) = a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$(s_0, s_1, s_2, s_3) \cdot b_0 = (a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3) \cdot b_0$$

$$= a_0 b_0 \cdot b_0 + a_1 b_1 \cdot b_0 + a_2 b_2 \cdot b_0 + a_3 b_3 \cdot b_0$$

$$= a_0 b_0 \cdot b_0 = 4 a_0 \Rightarrow a_0 = \frac{(s_0, s_1, s_2, s_3) \cdot b_0}{4}$$

Discrete Fourier Transform

$O(n)$

$$a_j = \frac{\overbrace{(s_0, s_1, \cdots, s_{n-1})}^{\text{input}} \cdot b_j}{n} \leftarrow b_j \cdot b_j = n$$

Discrete Fourier Transform

$\Downarrow$

inner product.

(alternative def: ① $a_j = (s_0, s_1, \cdots, s_{n-1}) \cdot b_j$)

② $a_j = \dfrac{(s_0, s_1, \cdots s_{n-1}) \cdot b_j}{\sqrt{n}}$

Discrete Fourier Transform

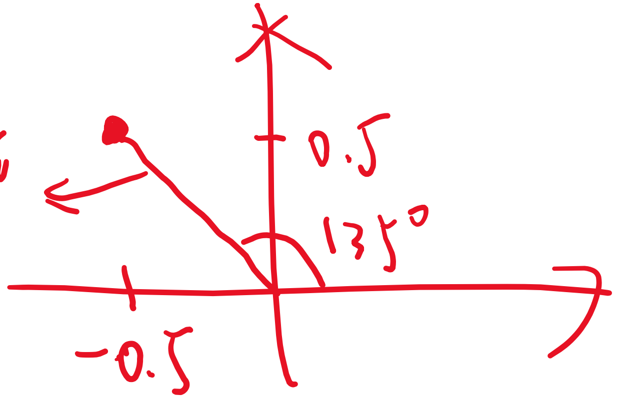
Example: $s=(1, 2, 3, 4)$

$$a_0 = \frac{(1,2,3,4)\cdot(1,1,1,1)}{4} = \frac{10}{4} = 2.5$$

$$a_1 = \frac{(1,2,3,4)\cdot(1,i,-1,-i)}{4} = \frac{1-2i-3+4i}{4} = \frac{-2+2i}{4}$$

$$= -0.5+0.5i$$

$\sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{1}{2}}$

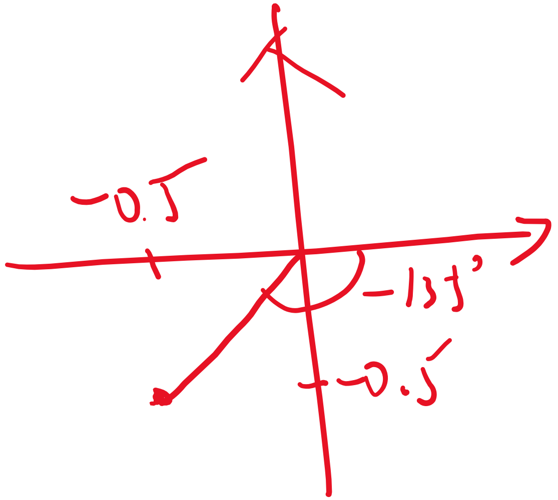


Discrete Fourier Transform

$$a_2 = \frac{(1,2,3,4)\cdot(1,-1,1,-1)}{4} = \frac{1-2+3-4}{4} = -0.5$$

$$a_3 = \frac{(1,2,3,4)(1,-i,-1,i)}{4} = \frac{1+2i-3-4i}{4} = \frac{-2-2i}{4}$$

$$= -0.5-0.5i$$



Conjugate

$$\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$$

$$= \cos(-\theta) + i\sin(-\theta)$$

$$= e^{-i\theta}$$

$$\overline{\omega^k} = \overline{e^{\frac{i2\pi k}{n}}} = e^{-\frac{i2\pi k}{n}} = \omega^{-k}$$

$$\hookrightarrow e^{\frac{i2\pi(-k)}{n}}$$

Orthogonality

$$\omega = e^{\frac{i2\pi}{n}}$$

$$\omega^{n(k-j)} = e^{\frac{i2\pi(k-j)\times n}{n}}$$

$$= e^{i2\pi(k-j)}$$

$$= \cos(2\pi(k-j)) + i\sin(2\pi(k-j))$$

$$= 1 + 0i = 1$$

$k \neq j, \in \{0, 1, \ldots, n-1\}$

$$b_k = (1, \omega^k, \omega^{2k}, \cdots, \omega^{k(n-1)})$$

$$b_j = (1, \omega^j, \omega^{2j}, \cdots, \omega^{j(n-1)})$$

$$b_k \cdot b_j = 1 \cdot 1 + \omega^{k-j} + \omega^{2(k-j)} + \cdots + \omega^{(n-1)(k-j)}$$

G.P.

initial term = 1

common ratio $= \omega^{k-j}$

$$= \frac{1(1-\omega^{(k-j)n})}{1-\omega^{k-j}} = \frac{1-\omega^{n(k-j)}}{1-\omega^{k-j}} = \frac{1-1}{1-\omega^{k-j}} = 0$$

goal : $\omega^{k-j} \neq 1$

Orthogonality

goal: $\omega^{k-j} \neq 1$

$\iff \frac{k-j}{n} \neq 0, 1, 2, \cdots,$

$\omega^{k-j} = e^{\frac{i2\pi(k-j)}{n}}$

$= \cos\left(2\pi \times \frac{k-j}{n}\right) + i\sin\left(2\pi \times \frac{(k-j)}{n}\right)$

$\neq 1$

Properties of Fourier Basis

(1) $b_k \cdot b_j = 0 \quad \text{if } k \neq j$

(2) $b_k \cdot b_k = n$

Inverse Discrete Fourier Transform

Input: $a_0, a_1, \cdots, a_{n-1}$

output: $s_0, \cdots, s_{n-1}$

$O(n^2)$ time

$$S = a_0 b_0 + a_1 b_1 + a_2 b_2 + \cdots + a_{n-1} b_{n-1}$$

alter. def: ① $S = \frac{1}{n}(a_0 b_0 + a_1 b_1 + \cdots + a_{n-1} b_{n-1})$

② $S = \frac{1}{\sqrt{n}}(a_0 b_0 + a_1 b_1 + \cdots + a_{n-1} b_{n-1})$

Inverse Discrete Fourier Transform

Example: input: $a_0 = 1, \quad a_1 = 2, \quad a_2 = 1, \quad a_3 = 3$

$$S = 1 \cdot (1, 1, 1, 1) + 2(1, i, -1, -i) + 1(1, -1, 1, -1)$$

$$+ 3(1, -i, -1, i) = (1+2+1+3, \quad 1+2i-1-3i, \quad 1-2+1-3, \quad 1-2i-1+3i)$$

$$= (7, -i, -3, i)$$

Sazu

Discrete Fourier Transform (Matrix Representation)

Inverse Discrete Fourier Transform

Take $O(n^2)$ time

$$\begin{array}{c} \begin{pmatrix} 1, & 1, & 1, & \cdots & , 1 \\ 1, & w, & w^2, & w^3, & \cdots & w^{n-1} \\ \vdots & & & & & \\ 1, & w^{n-1} & w^{2(n-1)} & \cdots & & w^{(n-1)(n-1)} \end{pmatrix} \\ \begin{array}{cccccc} b_0 & b_1 & b_2 & & & b_{n-1} \end{array} \end{array} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix}$$

Discrete Fourier Transform (Matrix Representation)

Discrete Fourier Transform

Take $O(n^2)$ time

conjugate

$$\frac{1}{n}\begin{pmatrix} 1, & 1, & 1, & \cdots, & 1 \\ 1, & w^{-1}, & w^{-2}, & w^{-3}, \cdots & w^{-(n-1)} \\ \vdots & & & & \\ 1, & w^{-(n-1)} & w^{-2(n-1)} & \cdots & w^{-(n-1)(n-1)} \end{pmatrix}\begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$$

input size = n

4-point FFT → 2×(2-point FFT) ⇒ Divide & Conquer

Fast Inverse Fourier Transform (IFFT)

$\omega_4 = e^{i\frac{2\pi}{4}} = i$

$\omega_4^4 = e^{i\frac{2\pi}{4}\times 4} = e^{i2\pi} = 1$

$n=4$ (4-point FFT)

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} X_0 + y_0 \\ X_1 + w y_1 \\ X_0 + \underbrace{w^2}_{-1} y_0 \\ X_1 + \underbrace{w^3}_{-w} y_1 \end{pmatrix}$$

$w^9 = w^5$

$a_0 + a_1 + a_2 + a_3$

$a_0 + wa_1 - a_2 - wa_3$

$a_0 - a_1 + a_2 - a_3$

$a_0 - wa_1 - a_2 + wa_3$

2-point FFT $\begin{pmatrix} 1 & 1 \\ 1 & w_4^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_2 \end{pmatrix} = \begin{pmatrix} X_0 \\ X_1 \end{pmatrix}$

$w_4^2 = (-1)$ — 4-point FFT

even terms

$X_0 = a_0 + a_2$

$X_1 = a_0 - a_2$

2-point FFT $\begin{pmatrix} 1 & 1 \\ 1 & w_4^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_3 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$

odd terms

$y_0 = a_1 + a_3$

$y_1 = a_1 - a_3$

2-point IFFT

$$\left(\omega_n = e^{\frac{i2\pi}{n}}\right)$$

$n=2$

$$\omega_2 = e^{\frac{i2\pi}{2}} = e^{\frac{i2\pi\times 2}{4}}$$
$$= \omega_4^2$$

$$\begin{pmatrix} 1 & 1 \\ 1 & \omega_2 \end{pmatrix}$$

2-point FFT

$$\parallel$$

$$\begin{pmatrix} 1 & 1 \\ 1 & \omega_4^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$e^{\frac{i2\pi}{2}} = e^{i\pi} = \cos\pi + i\sin\pi$$
$$= -1$$

8-Point IFFT

input: $a_0, a_1, a_2, \cdots, a_7$

output: $s_0, s_1, \cdots, s_7$ such that

$$\omega = e^{\frac{i2\pi}{8}}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \cdots & & & & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & & & \omega^{2\times7} \\ 1 & \vdots & & \ddots & & & & \\ 1 & \vdots & & & \ddots & & & \\ 1 & \vdots & & & & \ddots & & \\ 1 & & & & & & & \\ 1 & \omega^7 & \omega^{2\times7} & \omega^{3\times7} & \cdots & & & \omega^{7\times7} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} = \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_7 \end{pmatrix}$$

8-Point IFFT

compute $\omega_4 = e^{\frac{i 2\pi}{4}}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} a_0 \\ a_2 \\ a_4 \\ a_6 \end{pmatrix} = \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

even terms

8-Point IFFT

compute $\omega_4 = e^{\frac{i2\pi}{4}}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} a_1 \\ a_3 \\ a_5 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

odd terms

8-Point IFFT

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & \cdots & & & & w^7 \\ 1 & w^2 & w^4 & w^6 & \cdots & & & w^{2\times 7} \\ 1 & \vdots & & \ddots & & & & \\ 1 & & & & & & & \\ 1 & \vdots & & & \ddots & & & \\ 1 & & & & & & & \\ 1 & w^7 & w^{2\times 7} & w^{3\times 7} & \cdots & & & w^{7\times 7} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ \\ \\ \\ a_7 \end{bmatrix} = \begin{pmatrix} X_0 + 1\, y_0 \\ X_1 + w\, y_1 \\ X_2 + w^2\, y_2 \\ X_3 + w^3\, y_3 \\ X_0 + w^4\, y_0 \\ X_1 + w^5\, y_1 \\ X_2 + w^6\, y_2 \\ X_3 + w^7\, y_3 \end{pmatrix}$$

IFFT

see next slide

$\mathrm{IFFT}(a_0, a_1, \dots, a_{n-1})$ {

$x \leftarrow \mathrm{IFFT}(a_0, a_2, a_4, a_6, \dots, a_{n-2})$ even terms

$y \leftarrow \mathrm{IFFT}(a_1, a_3, a_5, a_7, \dots, a_{n-1})$ odd terms

recursion

return
$$\left(\begin{array}{c}
x_0 + w^0 y_0 \\
x_1 + w\, y_1 \\
\vdots \\
x_{\frac{n}{2}-1} + w^{\frac{n}{2}-1} y_{\frac{n}{2}-1} \\
\hline
x_0 + w^{\frac{n}{2}} y_0 \\
x_1 + w^{\frac{n}{2}+1} y_1 \\
\vdots \\
x_{\frac{n}{2}-1} + w^{n-1} y_{\frac{n}{2}-1}
\end{array}\right)$$

}

$$w = e^{\frac{i2\pi}{n}}$$

IFFT

$$\text{if } (n == 2) \{$$

$$\text{return: } \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

base case

$$\}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

$$T(n) = O(n \log n)$$

FFT

see next slide

$$\mathrm{FFT}(s_0, s_1, \ldots, s_{n-1}) \; \{$$

$$x \leftarrow \mathrm{FFT}(s_0, s_2, s_4, s_6, \ldots, s_{n-2})$$

$$y \leftarrow \mathrm{FFT}(s_1, s_3, s_5, s_7, \ldots, s_{n-1})$$

$$\text{return} \begin{pmatrix} x_0 + w^{-0} y_0 \\ x_1 + w^{-1} y_1 \\ \vdots \\ x_{\frac{n}{2}-1} + w^{-(\frac{n}{2}-1)} y_{\frac{n}{2}-1} \\ x_0 + w^{-(\frac{n}{2})} y_0 \\ x_1 + w^{-(\frac{n}{2}+1)} y_1 \\ \vdots \\ x_{\frac{n}{2}-1} + w^{-(n-1)} y_{\frac{n}{2}-1} \end{pmatrix} \qquad w = e^{\frac{i2\pi}{n}}$$

$$\}$$

FFT

if (n == 2) {

return : $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \end{pmatrix}$

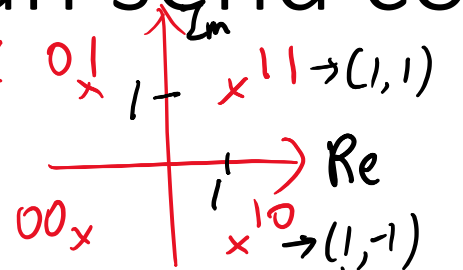
}

$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$

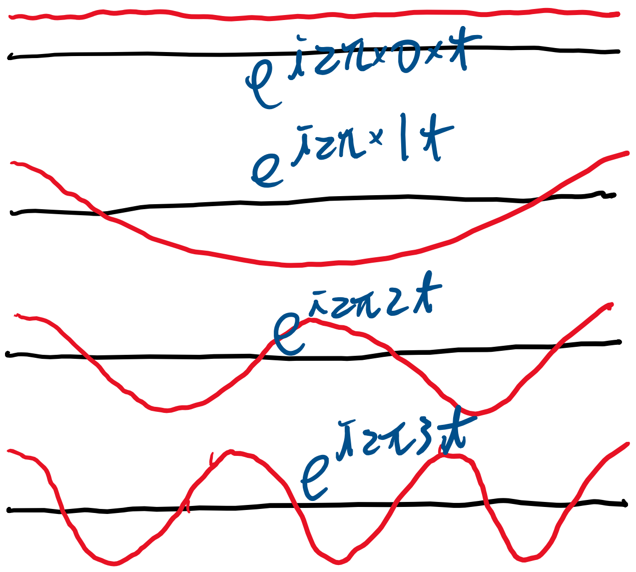
$T(n) = O(n \log n)$

OFDM (assuming that we can send complex signal)

QPSK



Divide the channel into n subchannels. $n=4$



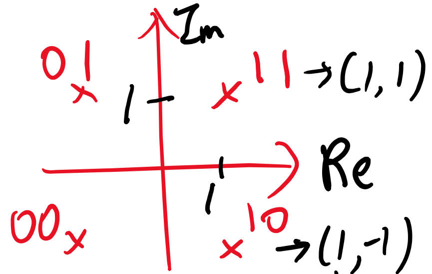
subchannel 0	$f=0$	carrier	$e^{i2\pi\times 0\times t}$	$\times\, a_0 = -1-i$
subchannel 1	$f=1$		$e^{i2\pi\times 1 t}$	$\times\, a_1 = -1+i$
subchannel 2	$f=2$		$e^{i2\pi 2t}$	$\times\, a_2 = 1-i$
subchannel 3	$f=3$		$e^{i2\pi 3t}$	$\times\, a_3 = 1+i$

send to receiver

assume sender sends $\boxed{00}\,\boxed{01}\,\boxed{10}\,\boxed{11}$ 11 10 11 01 ....

OFDM sender

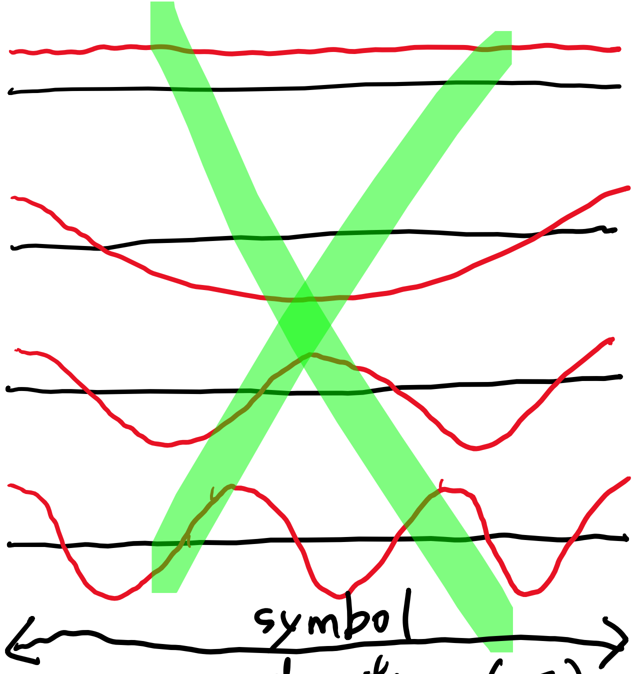
QPSK



Divide the channel into n subchannels. $n=4$

simulate n concurrent transmission (∵ to reduce cost)

subchannel 0	$f=0$	0 Hz	carrier
subchannel 1	$f=1$	$\frac{1}{T}$ Hz	
subchannel 2	$f=2$	$\frac{2}{T}$ Hz	
subchannel 3	$f=3$	$\frac{3}{T}$ Hz	



$a_0 = -1-i$

$a_1 = -1+i$

$a_2 = 1-i$

$a_3 = 1+i$

IFFT $\Rightarrow S_0 \sim S_3$

complex numbers

assume sender sends 00 01 10 11 11 10 11 01 ....

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} -1-i \\ -1+i \\ 1-i \\ 1+i \end{pmatrix} = \begin{pmatrix} 0 \\ -2-2i \\ -4i \\ -2+2i \end{pmatrix}$$

$$(-1-i)-i(-1+i)-(1-i)+i(1+i)$$

$$-1-i+i+1-1+i+i-1$$

OFDM sender (cont.)

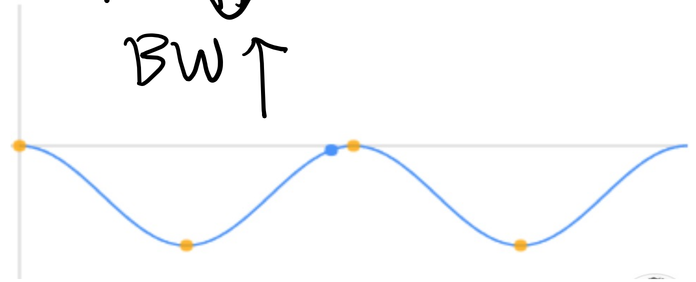
Digital to Analog Converter

IFFT ⇒ S_0, S_1, S_2, S_3

$S_0 = 0$, $S_1 = -2-2i$, $S_2 = -4i$, $S_3 = -2+2i$

$\xrightarrow{Re}$ $Re(S_0), Re(S_1), Re(S_2), Re(S_3)$ $\xrightarrow{DAC}$

0, −2, 0, −2



sample interval ↓

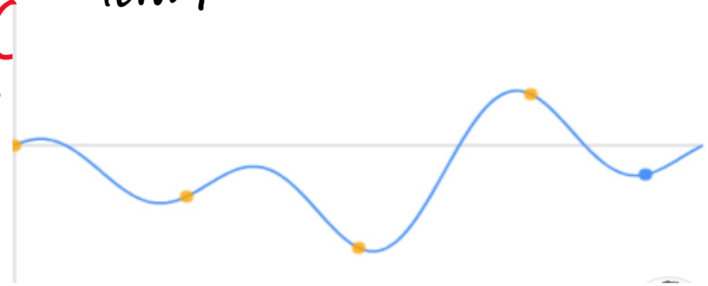
⇓

BW ↑

⟵ sample interval ⟶

$\xrightarrow{Im}$ $Im(S_0), Im(S_1), Im(S_2), Im(S_3)$ $\xrightarrow{DAC}$

0, −2, −4, 2



⟵ symbol duration (T) ⟶

symbol duration ↓ ⇒ BW ↑

‖

n × sample intervals.

OFDM sender (cont.)

f_c is the center frequency of the channel.

$I(t)$

in phase-carrier

$\times \cos(2\pi f_c t)$

$Q(t)$

$\times \sin(2\pi f_c t)$

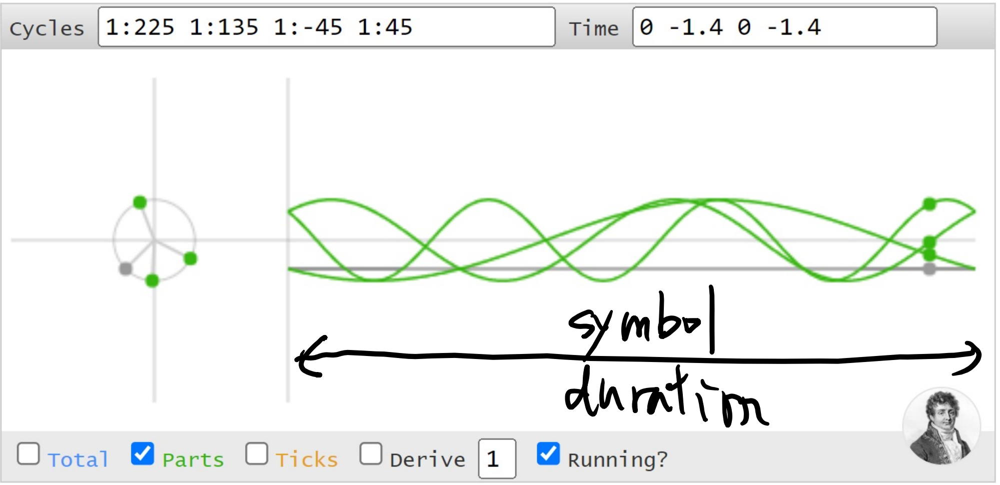
quadrature carrier

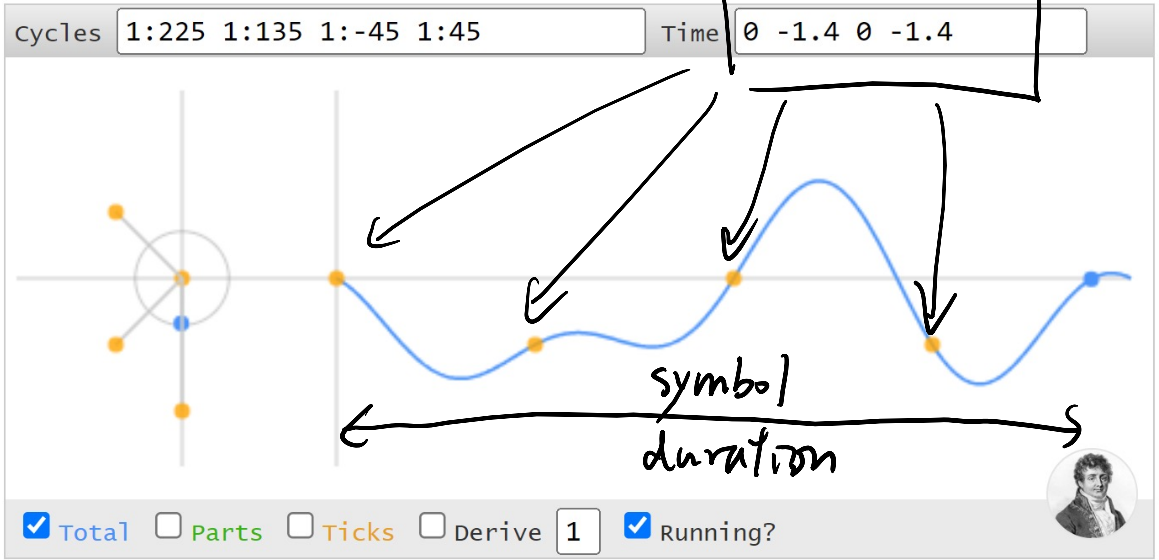
$\Longrightarrow$ send to receiver

OFDM vs FDM

during the transmission of a symbol,
we do not change
the amp. of subchannels

IFFT





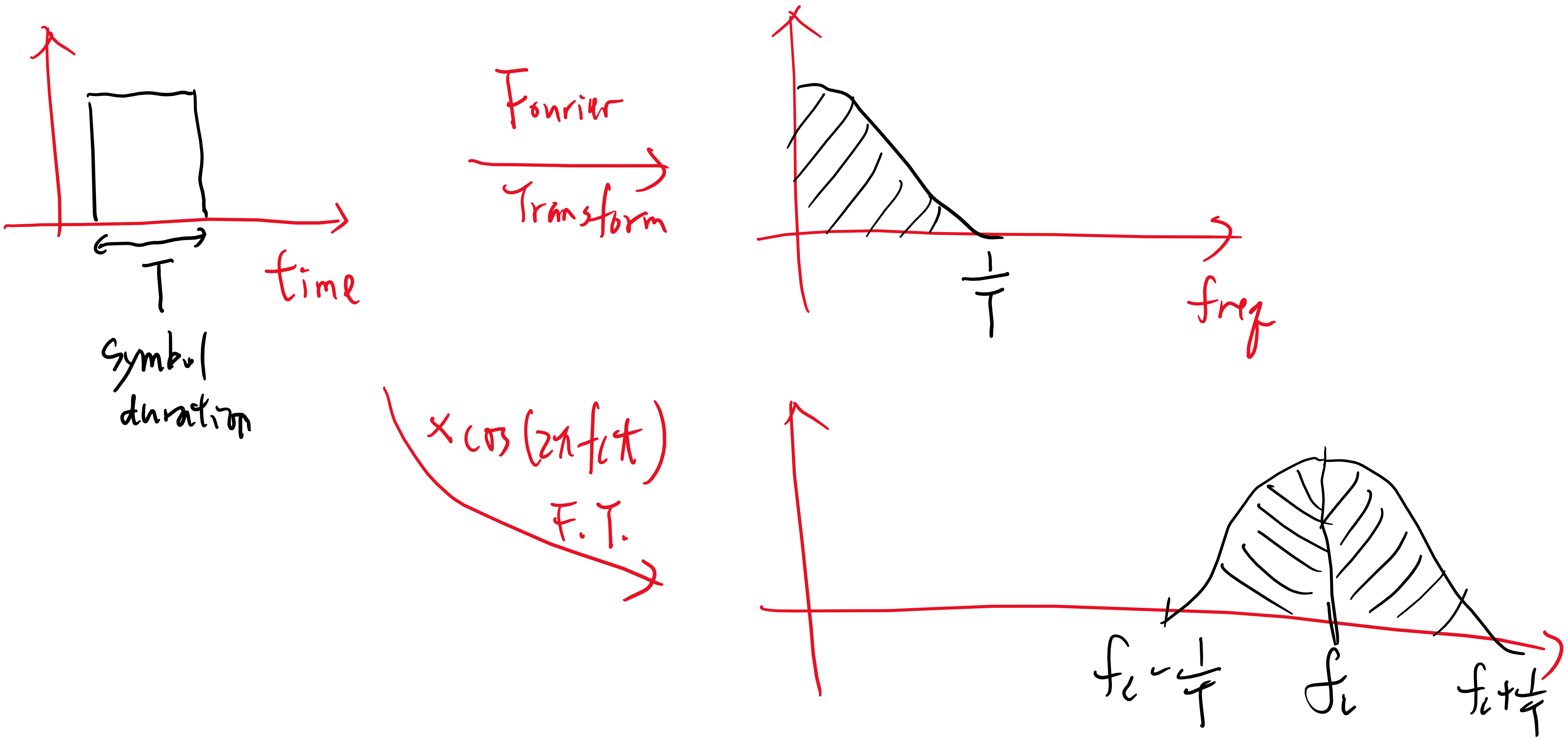
FDM

4 concurrent
subchannels

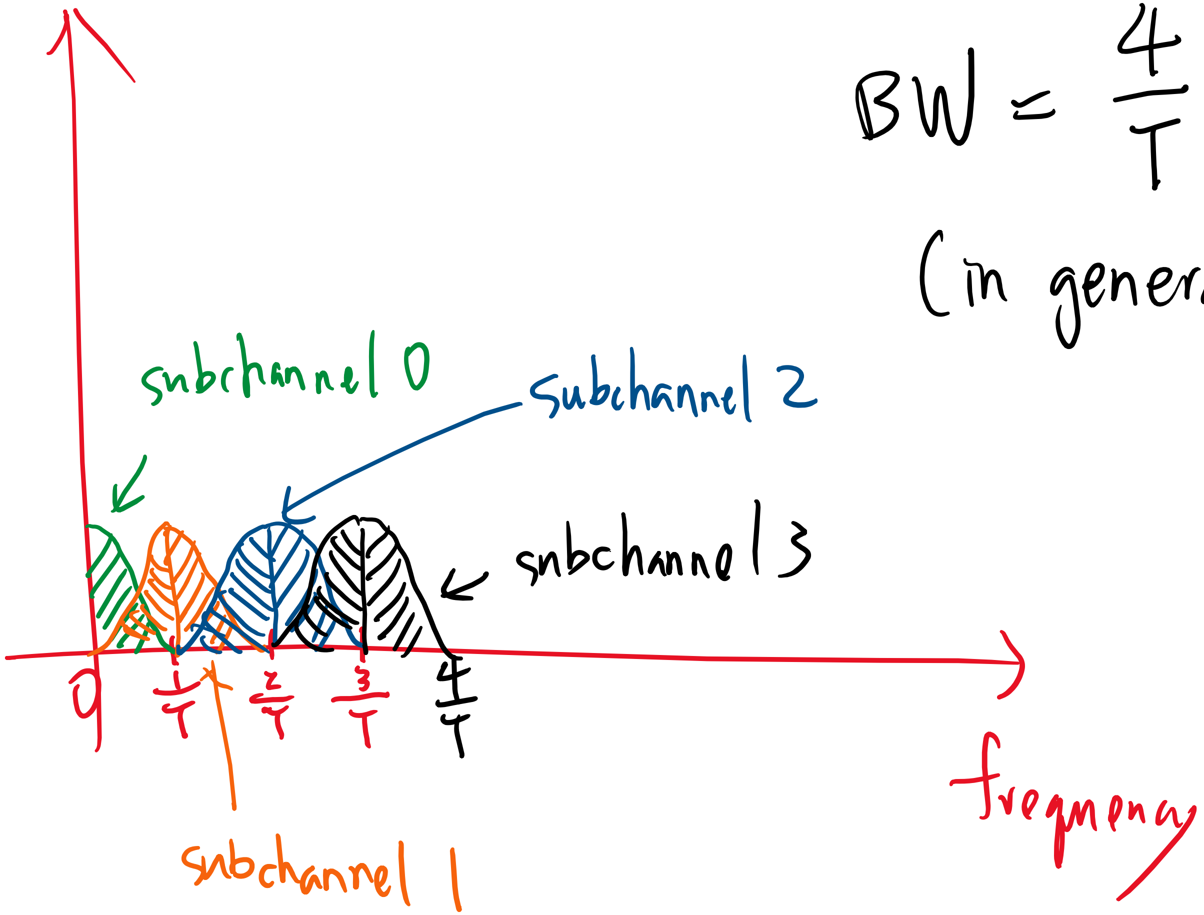
OFDM
has
lower hardware
cost

OFDM
use IFFT to
simulate 4 concurrent
subchannels

OFDM vs FDM


T
symbol duration
time
Fourier
Transform
1/T
freq
× cos(2πfct)
F.T.
fc − 1/T
fc
fc + 1/T

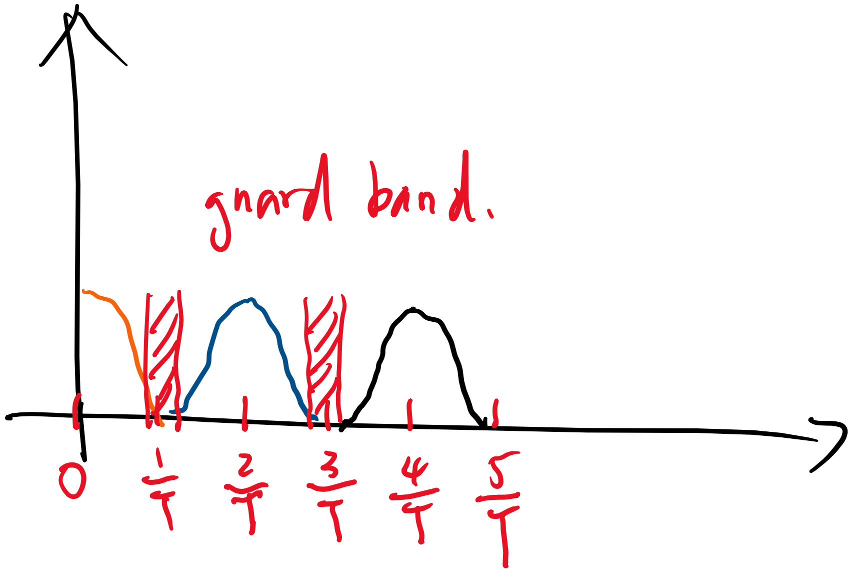
OFDM Spectrum (before the final modulation)

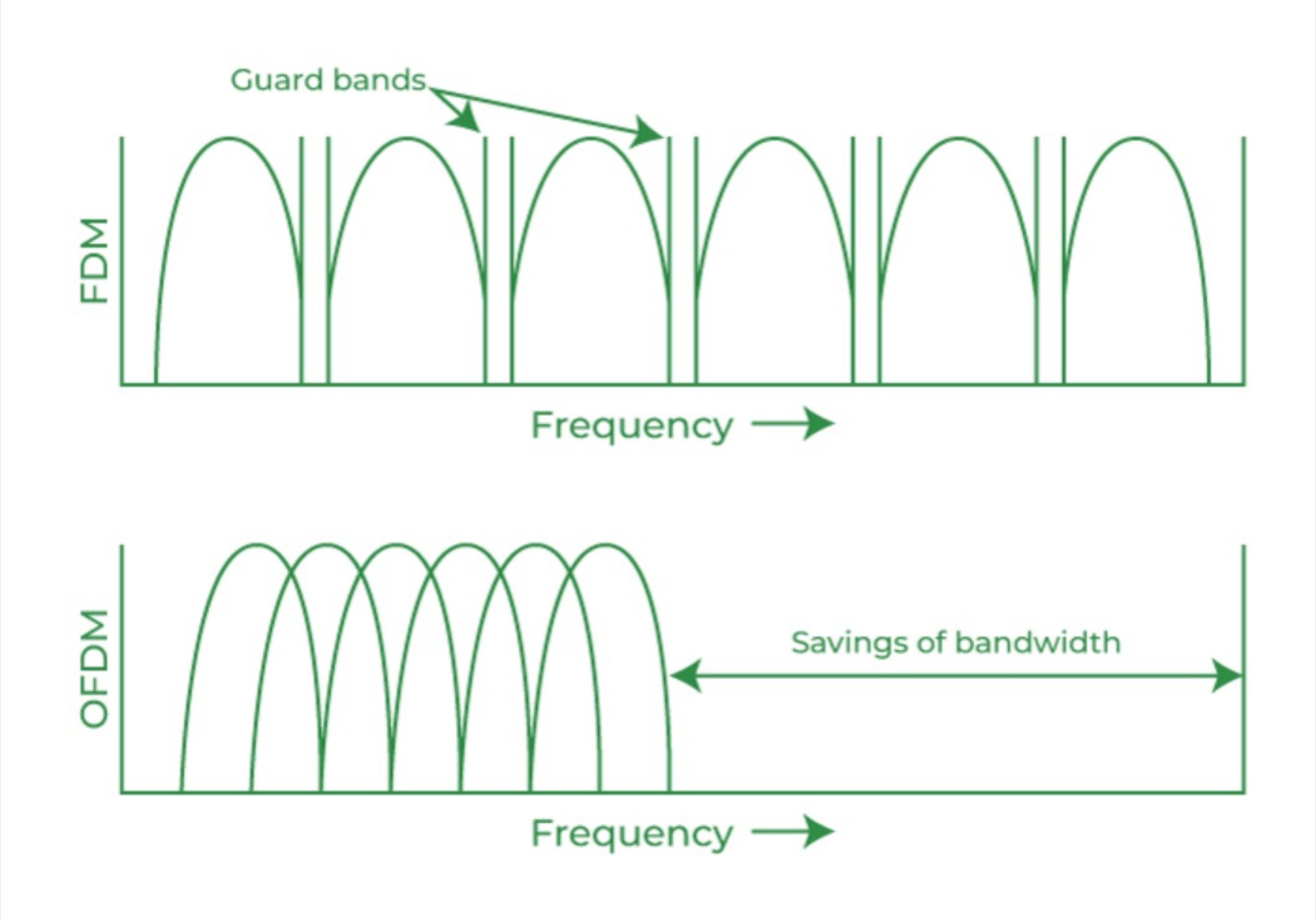


$$BW = \frac{4}{T} \quad \left(\text{or } \frac{4}{T}\right)$$

$$\left(\text{in general: } \frac{n}{T}\right)$$

FDM Spectrum




Guard bands
FDM
Frequency
OFDM
Savings of bandwidth
Frequency

OFDM vs Single channel

assume channel $BW = B$

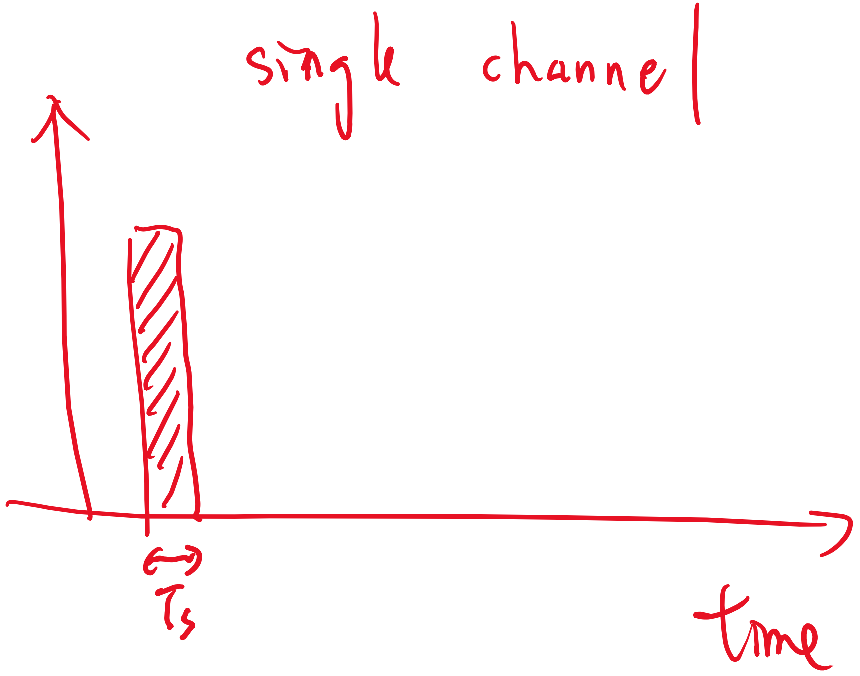
single channel : $T_s = \frac{1}{B}$

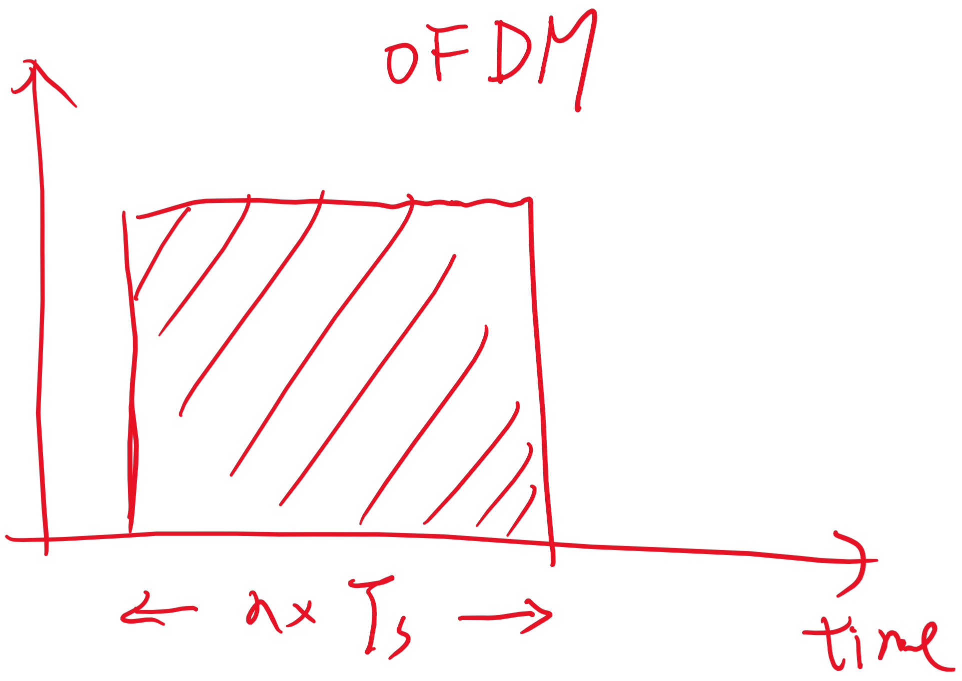
$$\left(\frac{1}{T_s} = B\right)$$

OFDM : $\frac{n}{T_{OFDM}} = B \Rightarrow T_{OFDM} = \frac{1}{B} \times n$

$$= n \times T_s$$

OFDM vs Single channel





OFDM sender

① given channel bandwidth B, # subchannels n

② $T \Rightarrow \frac{n}{B}$

③ divide bits into n chunks (according to modulation (e.g., QPSK, 32QAM))

④ from constellation diagram $\Rightarrow a_0 \sim a_{n-1}$

⑤ $S_0, S_1, \cdots, S_{n-1} \leftarrow \text{IFFT}(a_0, \sim a_{n-1})$

OFDM sender

⑥ $\mathrm{Re}(S_0, S_1, \dots S_{n-1}) \xrightarrow{DAC} I(t) \times \cos(2\pi f_i t)$

$\Rightarrow$ send to receiver.

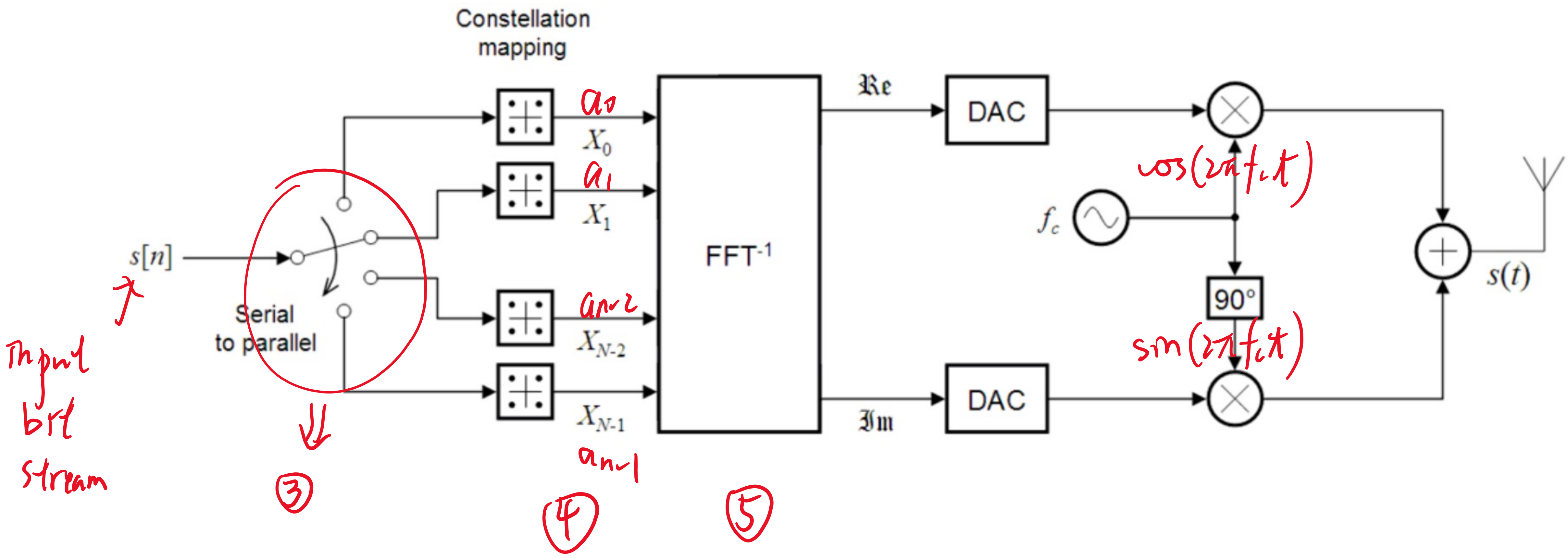
⑦ $\mathrm{Im}(S_0, S_1, \dots, S_{n-1}) \xrightarrow{DAC} Q(t) \times \sin(2\pi f_i t)$

$\Downarrow$

sample interval

$= \frac{T}{n}$

Sender diagram



OFDM receiver

1° receiver receives $I(t) \times$ inphase carrier $+ Q(t) \times$ quadrature carrier $\Rightarrow r(t)$

2° $I(t) \leftarrow r(t) \times$ inphase carrier + filter

$Q(t) \leftarrow r(t) \times$ quadrature carrier + filter

goal: get the bits sent by the sender

OFDM receiver

Analog to digital Converter
(ADC)

FFT ← $\underset{0}{\overset{}{s_0}}, \overset{-2-2i}{s_1}, \underset{-4i}{s_2}, \overset{-2+2i}{s_3}$

Re

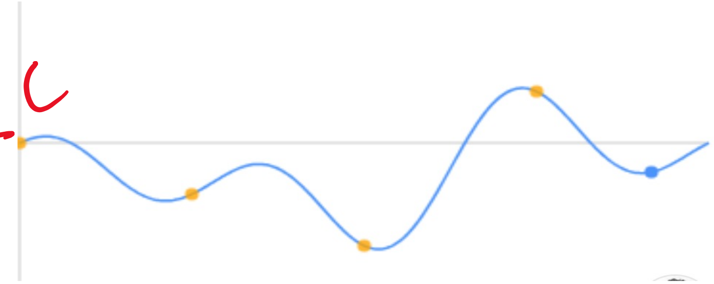
Im

$\overset{0}{Re(s_0)}, \overset{-2}{Re(s_1)}, \overset{0}{Re(s_2)}, \overset{-2}{Re(s_3)}$ ← ADC

$\underset{0}{Im(s_0)}, \underset{-2}{Im(s_1)}, \underset{-4}{Im(s_2)}, \underset{2}{Im(s_3)}$ ← ADC

$S * h$

I(t)

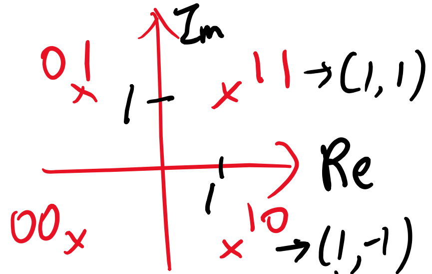


Q(t)

require time synchronization
(by correlation)

OFDM receiver

QPSK



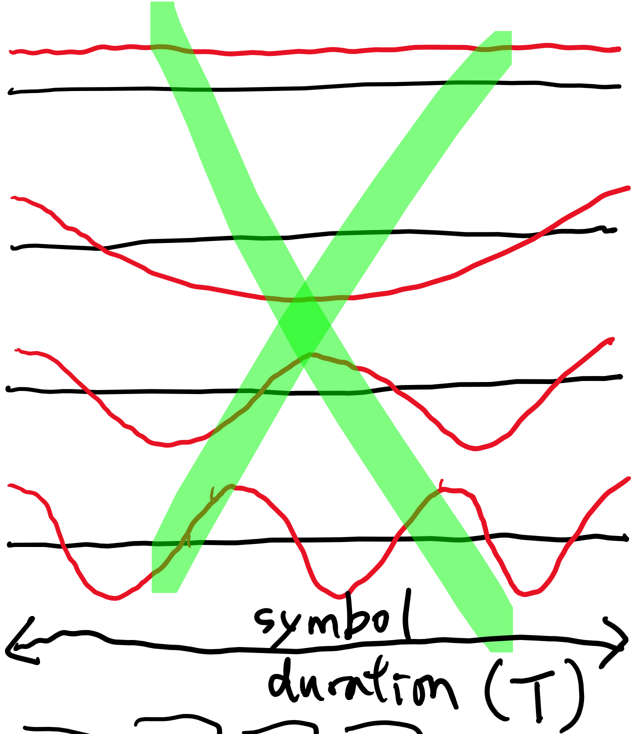
Divide the channel into n subchannels. n=4

subchannel 0 f=0 0 Hz carrier

subchannel 1 f=1 $\frac{1}{T}$ Hz

subchannel 2 f=2 $\frac{2}{T}$ Hz

subchannel 3 f=3 $\frac{3}{T}$ Hz



$a_0 = -1-i$

$a_1 = -1+i$

$a_2 = 1-i$

$a_3 = 1+i$

FFT ← $S_0 \ldots S_3$

complex numbers

assume sender sends 00 01 10 11 11 10 11 01 ...

Linear Time-Invariant System

① Linear:

a. if $x \to \boxed{\text{LTI system}} \to y$

then $a \cdot x \to \boxed{\text{LTI system}} \to a \cdot y$

b. if $x_1 \to \boxed{\text{LTI system}} \to y_1$

& $x_2 \to \boxed{\text{LTI system}} \to y_2$, then $x_1 + x_2 \to \boxed{\text{LTI system}} \to y_1 + y_2$

② Time invariant:

the behavior of the system does not change over time.

Channel Impulse Response

$[1] \longrightarrow \boxed{\text{LTI system}} \longrightarrow h = [h_0, h_1, \ldots, h_d]$

delay spread $= d$

$[1] \longrightarrow \boxed{\text{LTI system}} \longrightarrow [1, \frac{1}{2}, \frac{1}{4}]$

delay spread $= 2$

h can be obtained by learning (e.g., sender sends $[1]$ receivers observes the corresponding received samples)

Example

convolution
$\downarrow$
$s * h$

$s = [1, 2, 4, 8]$

$h = [1, \frac{1}{2}, \frac{1}{4}]$

$$s \longrightarrow \boxed{\begin{matrix} \text{LTI} \\ h \end{matrix}} \longrightarrow [1, 2.5, 5.25, 10.5, 5, 2]$$

$$\begin{array}{rcccccc}
1 \rightarrow & 1 & \frac{1}{2} & \frac{1}{4} & & & \\
2 \rightarrow & & 2 & 1 & \frac{1}{2} & & \\
4 \rightarrow & & & 4 & 2 & 1 & \\
+)\ 8 \rightarrow & & & & 8 & 4 & 2 \\
\hline
& (1 & 2.5 & 5.25 & 10.5) & 5 & 2
\end{array}$$

Convolution

$s_j : s[j]$

$s = [s_0, s_1, s_2, \ldots, s_{n-1}]$

$h = [h_0, h_1, h_2, \ldots, h_d]$

$$(s * h)[j] = s[j]h[0] + s[j-1]h[1] + s[j-2]h[2] + \cdots + s[0]h[j]$$

$$h[j] = 0 \text{ if } j > d$$

OFDM receiver design v2

receiver receives $s * h$

if $FFT(s * h)[j] = FFT(s)[j] \times FFT(h)[j] \times N$

then $FFT(s)[j] = \dfrac{FFT(s * h)[j]}{N \cdot FFT(h)[j]}$

goal: get $FFT(s)$

Circular Convolution

$s = [1, 2, 4, 8]$

$h = [1, \frac{1}{2}, \frac{1}{4}, 0]$

s and h must have the same length.

	1	$\rightarrow$	1	$\frac{1}{2}$	$\frac{1}{4}$	0
	2	$\rightarrow$	0	2	1	$\frac{1}{2}$
	4	$\rightarrow$	1	0	4	2
+)	8	$\rightarrow$	4	2	0	8
			6	4.5	5.25	10.5

Circular Convolution

$s = [1, 2, 4, 8]$

$h = [1, \frac{1}{2}, \frac{1}{4}, 0]$

s and h must have the same length.

	2	4	8	1	2	4	8	]		
	s[-3]	s[-2]	s[-1]	s[0]	s[1]	s[2]	s[3]			
s[0] →				1	$\frac{1}{2}$	$\frac{1}{4}$	0			
s[1] →					2	1	$\frac{1}{2}$	0		
s[2] →						4	2	1	0	
s[3] →							8	4	2	0
s[-1] →			8	4	2	0				
s[-2] →		4	2	1	0					
s[-3] →	2	1	$\frac{1}{2}$	0						

How to receive circular convolution?

sender:

originally send: $s_0, s_1, s_2, \cdots, s_{N-1}$

now: $s_{-d}, \cdots, s_{-2}, s_{-1}, s_0, s_1, s_2, \cdots, s_{N-1}$

d: delay spread

$h = [1, 0.5, 0.1]$

Example: $d = 2$

$[1, -1, \boxed{1, -2}]$

$\boxed{1, -2}, 1, -1, 1, -2$

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Sender:

originally send: $s_0, s_1, s_2, \cdots, s_{N-1}$

now: $\boxed{s_{-d}, \cdots, s_{-2}, s_{-1}}, s_0, s_1, s_2, \cdots, s_{N-1}$

d: delay spread

cyclic prefix (CP)

$h = [1, 0.5]$

Example: $d = 1$

$[1, -1, 1, \boxed{-2}]$

$\boxed{-2}, 1, -1, 1, -2$

Example

Assume sender wants to send 3 OFDM symbols x, y, z

x: x_0, x_1, x_2, x_3

y: y_0, y_1, y_2, y_3 (x, y, z are obtained by IFFT)

z: z_0, z_1, z_2, z_3

delay spread = 2

Example

sender sends

$\underbrace{x_2 \quad x_3}_{\text{CP}} \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad \underbrace{y_2 \quad y_3}_{\text{CP}} \quad y_0 \quad y_1 \quad y_2 \quad y_3 \quad \underbrace{z_2 \quad z_3}_{\text{CP}} \quad z_0 \quad z_1 \quad z_2 \cdots$

guard time   guard time   guard time

Example

$h = [1, \frac{1}{2}, \frac{1}{4}]$

receiver receives

$\underbrace{x_2 \quad x_3} \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad \underbrace{y_2 \quad y_3} \;|\; y_0 \quad y_1 \quad y_2 \quad y_3 \;|\; z_2 \quad z_3 \;|\; z_0 \quad z_1 \quad z_2$

$x_2 \cdot 1$	$x_3 \cdot 1$	x_0	x_1	x_2	x_3	y_2	y_3	y_0	y_1	y_2	y_3	
	$x_2 \cdot \frac{1}{2}$	$\frac{x_3}{2}$	$\frac{x_0}{2}$	$\frac{x_1}{2}$	$\frac{x_2}{2}$	$\frac{x_3}{2}$	$\frac{y_2}{2}$	$\frac{y_3}{2}$	$\frac{y_0}{2}$	$\frac{y_1}{2}$	$\frac{y_2}{2}$	
		$\frac{x_2}{4}$	$\frac{x_3}{4}$	$\frac{x_0}{4}$	$\frac{x_1}{4}$	$\frac{x_2}{4}$	$\frac{x_3}{4}$	$\frac{y_2}{4}$	$\frac{y_3}{4}$	$\frac{y_0}{4}$	$\frac{y_1}{4}$	
		$x \circledast h$				guard time		$y \circledast h$				guard time

Example

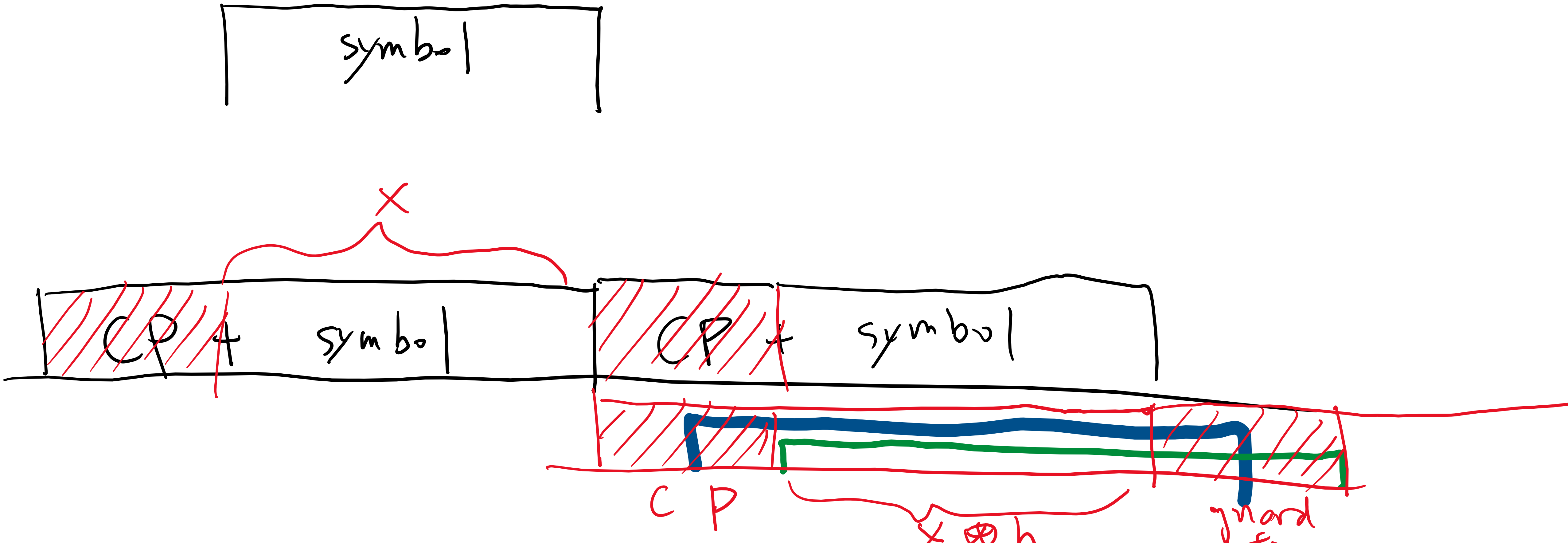
$$X = [X_0, X_1, X_2, X_3]$$

$$h = [1, \tfrac{1}{2}, \tfrac{1}{4}, 0]$$

$$\begin{matrix} X_0 & X_1 & X_2 & X_3 \\ \frac{X_3}{2} & \frac{X_0}{2} & \frac{X_1}{2} & \frac{X_2}{2} \\ \frac{X_2}{4} & \frac{X_3}{4} & \frac{X_0}{4} & \frac{X_1}{4} \end{matrix}$$

$$\begin{matrix} X_0 \to & X_0 & \frac{X_0}{2} & \frac{X_0}{4} & 0 \\ X_1 \to & 0 & X_1 & \frac{X_1}{2} & \frac{X_1}{4} \\ X_2 \to & \frac{X_2}{4} & 0 & X_2 & \frac{X_2}{2} \\ X_3 \to & \frac{X_3}{2} & \frac{X_3}{4} & 0 & X_3 \end{matrix}$$

$$\begin{matrix} X_0 & X_1 & X_2 & X_3 \\ \frac{X_3}{2} & \frac{X_0}{2} & \frac{X_1}{2} & \frac{X_2}{2} \\ \frac{X_2}{4} & \frac{X_3}{4} & \frac{X_0}{4} & \frac{X_1}{4} \end{matrix}$$


symbol
x
CP + symbol
CP + symbol
CP
x ⊛ h
guard