14 □ PARTIAL DERIVATIVES

14.1 Functions of Several Variables

1. (a) $f(x,y) = \dfrac{x^2y}{2x-y^2} \quad \Rightarrow \quad f(1,3) = \dfrac{1^2(3)}{2(1)-3^2} = -\dfrac{3}{7}$ (b) $f(-2,-1) = \dfrac{(-2)^2(-1)}{2(-2)-(-1)^2} = \dfrac{4}{5}$

(c) $f(x+h,y) = \dfrac{(x+h)^2y}{2(x+h)-y^2}$ (d) $f(x,x) = \dfrac{x^2x}{2x-x^2} = \dfrac{x^3}{x(2-x)} = \dfrac{x^2}{2-x}$

2. (a) $g(x,y) = x\sin y + y\sin x \quad \Rightarrow \quad g(\pi,0) = \pi\sin 0 + 0\sin\pi = \pi\cdot 0 + 0\cdot 0 = 0$

(b) $g\left(\dfrac{\pi}{2},\dfrac{\pi}{4}\right) = \dfrac{\pi}{2}\sin\dfrac{\pi}{4} + \dfrac{\pi}{4}\sin\dfrac{\pi}{2} = \dfrac{\pi}{2}\left(\dfrac{\sqrt{2}}{2}\right) + \dfrac{\pi}{4}(1) = \dfrac{\pi\left(\sqrt{2}+1\right)}{4}$

(c) $g(0,y) = 0\sin y + y\sin 0 = 0 + y\cdot 0 = 0$

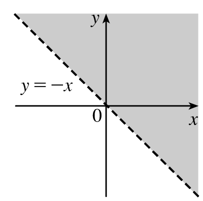
(d) $g(x,y+h) = x\sin(y+h) + (y+h)\sin x$

3. (a) $g(x,y) = x^2\ln(x+y) \quad \Rightarrow \quad g(3,1) = 3^2\ln(3+1) = 9\ln 4$

(b) $\ln(x+y)$ is defined only when $x+y>0 \quad \Rightarrow \quad y>-x$.

Thus, the domain of g is $\{(x,y) \mid y>-x\}$.

(c) The range of $\ln(x+y)$ is $\mathbb{R}$, so the range of g is $\mathbb{R}$.

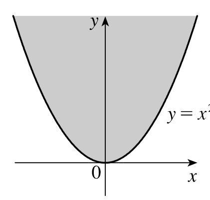


4. (a) $h(x,y) = e^{\sqrt{y-x^2}} \quad \Rightarrow \quad h(-2,5) = e^{\sqrt{5-(-2)^2}} = e^{\sqrt{1}} = e$

(b) $\sqrt{y-x^2}$ is defined only when $y-x^2\geq 0 \quad \Rightarrow \quad y\geq x^2$.

Thus, the domain of h is $\{(x,y) \mid y\geq x^2\}$.

(c) We know $\sqrt{y-x^2}\geq 0 \quad \Rightarrow \quad e^{\sqrt{y-x^2}}\geq 1$. Thus, the range of h is $[1,\infty]$.



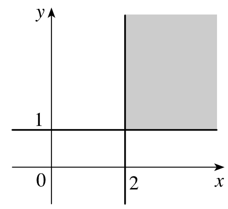
5. (a) $F(x,y,z) = \sqrt{y} - \sqrt{x-2z} \quad \Rightarrow \quad F(3,4,1) = \sqrt{4} - \sqrt{3-2(1)} = 2-1 = 1$

(b) $\sqrt{y}$ is defined only when $y\geq 0$. $\sqrt{x-2z}$ is defined only when $x-2z\geq 0 \quad \Rightarrow \quad z\leq\frac{1}{2}x$. Thus, the domain is $\{(x,y,z) \mid x\geq 2z, y\geq 0\}$, which is the set of points on or below the plane $z=\frac{1}{2}x$ and on or to the right of the xz-plane.

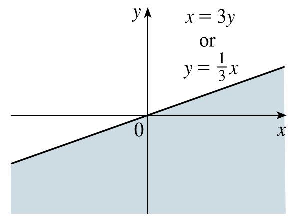
6. (a) $f(x,y,z) = \ln\left(z-\sqrt{x^2+y^2}\right) \quad \Rightarrow \quad f(4,-3,6) = \ln\left(6-\sqrt{4^2+(-3)^2}\right) = \ln\left(6-\sqrt{25}\right) = \ln 1 = 0$

(b) $\ln\left(z-\sqrt{x^2+y^2}\right)$ is defined only when $z-\sqrt{x^2+y^2}>0 \quad \Leftrightarrow \quad z>\sqrt{x^2+y^2} \quad \Rightarrow \quad z^2>x^2+y^2$. Thus, the domain is $\{(x,y,z) \mid z>\sqrt{x^2+y^2}\}$, which is the set of points inside the top half of the cone $z^2=x^2+y^2$.

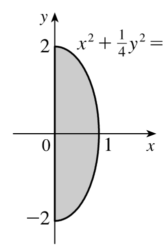
7. $f(x,y)=\sqrt{x-2}+\sqrt{y-1}$. $\sqrt{x-2}$ is defined only when $x-2\geq 0$, or $x\geq 2$, and $\sqrt{y-1}$ is defined only when $y-1\geq 0$, or $y\geq 1$. So the domain of f is $\{(x,y) \mid x\geq 2,\ y\geq 1\}$.



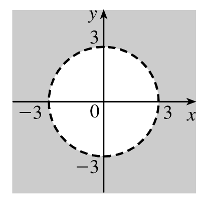
8. $f(x,y)=\sqrt[4]{x-3y}$. $\sqrt[4]{x-3y}$ is defined only when $x-3y\geq 0$, or $x\geq 3y$. So the domain of f is $\{(x,y) \mid x\geq 3y\}$ or equivalently $\left\{(x,y) \mid y\leq \frac{1}{3}x\right\}$.



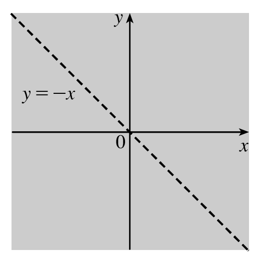
9. $q(x,y)=\sqrt{x}+\sqrt{4-4x^2-y^2}$. $\sqrt{x}$ is defined only when $x\geq 0$. $\sqrt{4-4x^2-y^2}$ is defined only when $4-4x^2-y^2\geq 0 \quad\Leftrightarrow\quad 1\geq x^2+\frac{1}{4}y^2$. So the domain of q is $\left\{(x,y) \mid x^2+\frac{1}{4}y^2\leq 1, x\geq 0\right\}$.



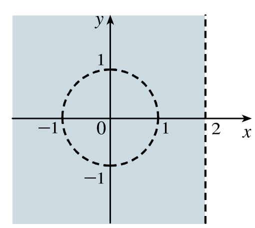
10. $g(x,y)=\ln(x^2+y^2-9)$. $\ln(x^2+y^2-9)$ is defined only when $x^2+y^2-9>0 \quad\Leftrightarrow\quad x^2+y^2>9$. So the domain of g is $\{(x,y) \mid x^2+y^2>9\}$.



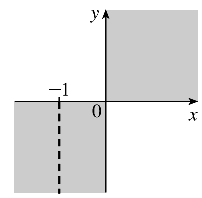
11. $g(x,y)=\dfrac{x-y}{x+y}$. g is not defined if $x+y=0 \quad\Leftrightarrow\quad y=-x$ (and is defined otherwise). Thus, the domain of g is $\{(x,y) \mid y\neq -x\}$, the set of all points in $\mathbb{R}^2$ that are not on the line $y=-x$.



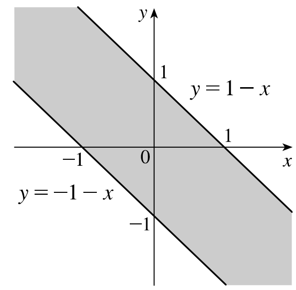
12. $g(x,y)=\dfrac{\ln(2-x)}{1-x^2-y^2}$. $\ln(2-x)$ is defined only when $2-x>0 \quad\Leftrightarrow\quad x<2$. In addition, g is not defined if $1-x^2-y^2=0 \quad\Leftrightarrow\quad x^2+y^2=1$. Thus, the domain of g is $\left\{(x,y) \mid x<2,\, x^2+y^2\neq 1\right\}$, the set of all points to the left of the line $x=2$ and not on the unit circle.



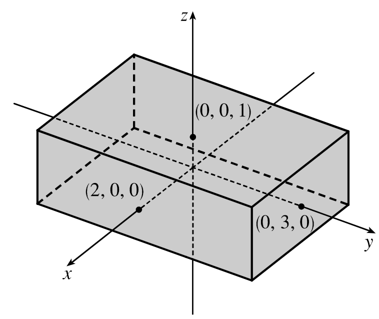
13. $p(x,y) = \dfrac{\sqrt{xy}}{x+1}$. $\sqrt{xy}$ is defined only when $xy \geq 0$. Further, p is defined only when $x + 1 \neq 0 \quad \Leftrightarrow \quad x \neq -1$. So the domain of p is $\{(x,y) \mid xy \geq 0, x \neq -1\}$.



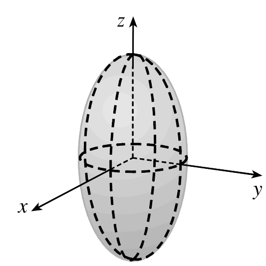
14. $f(x,y) = \sin^{-1}(x+y)$. $\sin^{-1}(x+y)$ is defined only when $-1 \leq x + y \leq 1 \quad \Leftrightarrow \quad -1 - x \leq y \leq 1 - x$. Thus, the domain of f is $\{(x,y) \mid -1 - x \leq y \leq 1 - x\}$, which consists of those points on or between the parallel lines $y = -1 - x$ and $y = 1 - x$.



15. $f(x,y,z) = \sqrt{4 - x^2} + \sqrt{9 - y^2} + \sqrt{1 - z^2}$. f is defined only when $4 - x^2 \geq 0 \quad \Leftrightarrow \quad -2 \leq x \leq 2$, and $9 - y^2 \geq 0 \quad \Leftrightarrow \quad -3 \leq y \leq 3$, and $1 - z^2 \geq 0 \quad \Leftrightarrow \quad -1 \leq z \leq 1$. Thus, the domain of f is $\{(x,y,z) \mid -2 \leq x \leq 2,\ -3 \leq y \leq 3,\ -1 \leq z \leq 1\}$, which is a solid rectangular box with vertices $(\pm 2, \pm 3, \pm 1)$ (all 8 combinations).



16. $f(x,y,z) = \ln(16 - 4x^2 - 4y^2 - z^2)$. f is defined only when $16 - 4x^2 - 4y^2 - z^2 > 0 \quad \Rightarrow \quad \dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{16} < 1$. Thus, $D = \left\{ (x,y,z) \,\middle|\, \dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{16} < 1 \right\}$, that is, the points inside the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{16} = 1$.



17. (a) $f(73, 178) = 0.0072(73)^{0.425}(178)^{0.725} \approx 1.9$, which means that the surface area of a person 178 centimeters tall who weighs 73 kilograms is approximately 1.91 square meters.

(b) Answers will vary depending on the height and weight of the reader.

18. $P(120, 20) = 1.47(120)^{0.65}(20)^{0.35} \approx 94.2$, so when the manufacturer invests \$20 million in capital and 120,000 hours of labor are completed yearly, the monetary value of the production is about \$94.2 million.

19. (a) From Table 1, $f(-15, 40) = -27$, which means that if the temperature is -15°C and the wind speed is 40 km/h, then the air would feel equivalent to approximately -27°C without wind.

(b) The question is asking: when the temperature is -20°C, what wind speed gives a wind-chill index of -30°C? From Table 1, the speed is 20 km/h.

(c) The question is asking: when the wind speed is 20 km/h, what temperature gives a wind-chill index of $-49°C$? From Table 1, the temperature is $-35°C$.

(d) The function $W = f(-5, v)$ means that we fix T at -5 and allow v to vary, resulting in a function of one variable. In other words, the function gives wind-chill index values for different wind speeds when the temperature is $-5°C$. From Table 1 (look at the row corresponding to $T = -5$), the function decreases and appears to approach a constant value as v increases.

(e) The function $W = f(T, 50)$ means that we fix v at 50 and allow T to vary, again giving a function of one variable. In other words, the function gives wind-chill index values for different temperatures when the wind speed is 50 km/h . From Table 1 (look at the column corresponding to $v = 50$), the function increases almost linearly as T increases.

20. (a) From Table 3, $f(35, 70) = 51$, which means that when the actual temperature is $35°C$ and the relative humidity is 70%, the perceived air temperature is approximately $51°C$.

(b) Looking at the row corresponding to $T = 30$, we see that $f(30, h) = 38$ when $h = 60$.

(c) Looking at the column corresponding to $h = 50$, we see that $f(T, 50) = 28$ when $T = 25$.

(d) $I = f(20, h)$ means that T is fixed at 20 and h is allowed to vary, resulting in a function of h that gives the humidex values for different relative humidities when the actual temperature is $20°C$. Similarly, $I = f(40, h)$ is a function of one variable that gives the humidex values for different relative humidities when the actual temperature is $40°C$. Looking at the rows of the table corresponding to $T = 20$ and $T = 40$, we see that $f(20, h)$ increases at a relatively constant rate of approximately $1°C$ per 10% relative humidity, while $f(40, h)$ increases more quickly (at first with an average rate of change of $4°C$ per 10% relative humidity).

21. (a) According to Table 4, $f(80, 15) = 7.7$, which means that if a 80 km/h wind has been blowing in the open sea for 15 hours, it will create waves with estimated heights of 7.7 meters.

(b) $h = f(60, t)$ means we fix v at 60 and allow t to vary, resulting in a function of one variable. Thus here, $h = f(60, t)$ gives the wave heights produced by 60 km/h winds blowing for t hours. From the table (look at the row corresponding to $v = 60$), the function increases but at a declining rate as t increases. In fact, the function values appear to be approaching a limiting value of approximately 5.9, which suggests that 60 km/h winds cannot produce waves higher than about 5.9 meters.

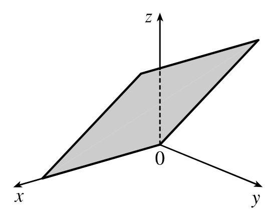
(c) $h = f(v, 30)$ means we fix t at 30, again giving a function of one variable. So, $h = f(v, 30)$ gives the wave heights produced by winds of speed v blowing for 30 hours. From the table (look at the column corresponding to $t = 30$), the function appears to increase at an increasing rate, with no apparent limiting value. This suggests that faster winds (lasting 30 hours) always create higher waves.

22. (a) The cost of making x small boxes, y medium boxes, and z large boxes is $C = f(x, y, z) = 8000 + 2.5x + 4y + 4.5z$ dollars.

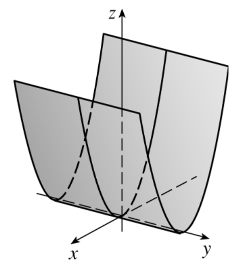
(b) $f(3000, 5000, 4000) = 8000 + 2.5(3000) + 4(5000) + 4.5(4000) = 53{,}500$ which means that it costs \$53,500 to make 3000 small boxes, 5000 medium boxes, and 4000 large boxes.

(c) Because no partial boxes will be produced, each of x, y, and z must be a positive integer or zero.

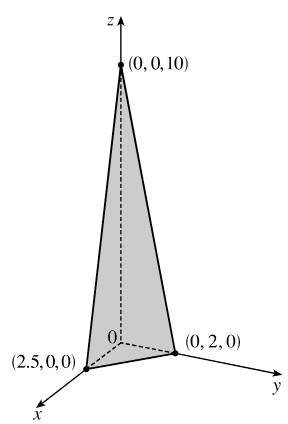
23. The graph of f has equation $z = y$, a plane which intersects the yz-plane in the line $z = y$, $x = 0$. The portion of this plane in the first octant is shown.



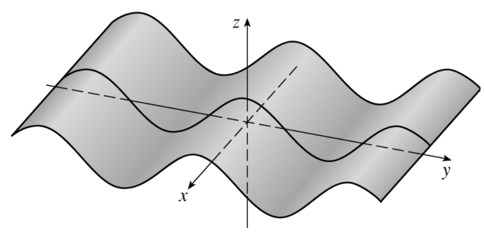
24. The graph of f has equation $z = x^2$, a parabolic cylinder.



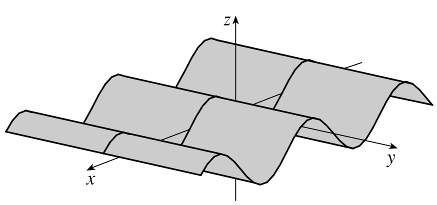
25. $z = 10 - 4x - 5y$ or $4x + 5y + z = 10$, a plane with intercepts 2.5, 2, and 10.



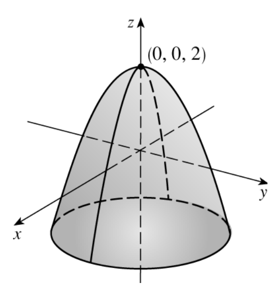
26. $z = \cos y$, a cylinder.



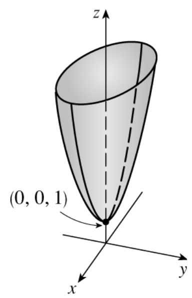
27. $z = \sin x$, a cylinder.



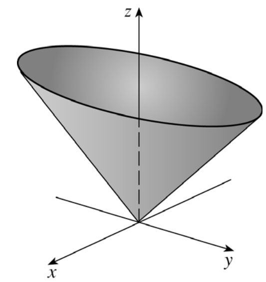
28. $z = 2 - x^2 - y^2$, a circular paraboloid opening downward with vertex at $(0, 0, 2)$.



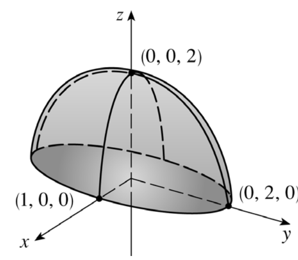
29. $z = x^2 + 4y^2 + 1$, an elliptic paraboloid opening upward with vertex at $(0, 0, 1)$.



30. $z = \sqrt{4x^2 + y^2}$ so $4x^2 + y^2 = z^2$ and $z \geq 0$, the top half of an elliptic cone.



31. $z = \sqrt{4 - 4x^2 - y^2}$ so $4x^2 + y^2 + z^2 = 4$ or $x^2 + \dfrac{y^2}{4} + \dfrac{z^2}{4} = 1$ and $z \geq 0$, the top half of an ellipsoid.



32. (a) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$. The trace in $x = 0$ is $z = \dfrac{1}{1 + y^2}$, and the trace in $y = 0$ is $z = \dfrac{1}{1 + x^2}$. The only possibility is graph III. Notice also that the level curves of f are $\dfrac{1}{1 + x^2 + y^2} = k \quad \Leftrightarrow \quad x^2 + y^2 = \dfrac{1}{k} - 1$, a family of circles for $k < 1$.

(b) $f(x, y) = \dfrac{1}{1 + x^2 y^2}$. The trace in $x = 0$ is the horizontal line $z = 1$, and the trace in $y = 0$ is also $z = 1$. Both graphs I and II have these traces; however, notice that here $z > 0$, so the graph is I.

(c) $f(x, y) = \ln(x^2 + y^2)$. The trace in $x = 0$ is $z = \ln y^2$, and the trace in $y = 0$ is $z = \ln x^2$. The level curves of f are $\ln(x^2 + y^2) = k \quad \Leftrightarrow \quad x^2 + y^2 = e^k$, a family of circles. In addition, f is large negative when $x^2 + y^2$ is small, so this is graph IV.

(d) $f(x, y) = \cos\sqrt{x^2 + y^2}$. The trace in $x = 0$ is $z = \cos\sqrt{y^2} = \cos|y| = \cos y$, and the trace in $y = 0$ is $z = \cos\sqrt{x^2} = \cos|x| = \cos x$. Notice also that the level curve $f(x, y) = 0$ is $\cos\sqrt{x^2 + y^2} = 0 \quad \Leftrightarrow$ $x^2 + y^2 = \left(\frac{\pi}{2} + n\pi\right)^2$, a family of circles, so this is graph V.

(e) $f(x, y) = |xy|$. The trace in $x = 0$ is $z = 0$, and the trace in $y = 0$ is $z = 0$, so it must be graph VI.

(f) $f(x, y) = \cos(xy)$. The trace in $x = 0$ is $z = \cos 0 = 1$, and the trace in $y = 0$ is $z = 1$. As mentioned in part (b), these traces match both graphs I and II. Here z can be negative, so the graph is II. (Also notice that the trace in $x = 1$ is $z = \cos y$, and the trace in $y = 1$ is $z = \cos x$.)

33. The point $(-3, 3)$ lies between the level curves with z-values 50 and 60. Since the point is a little closer to the level curve with $z = 60$, we estimate that $f(-3, 3) \approx 56$. The point $(3, -2)$ appears to be just about halfway between the level curves with z-values 30 and 40, so we estimate $f(3, -2) \approx 35$. The graph rises as we approach the origin, gradually from above, steeply from below.

34. (a) C (Chicago) lies between level curves with pressures 1012 and 1016 mb, and since C appears to be located about one-fourth the distance from the 1012 mb isobar to the 1016 mb isobar, we estimate the pressure at Chicago to be about 1013 mb. N lies very close to a level curve with pressure 1012 mb so we estimate the pressure at Nashville to be approximately 1012 mb. S appears to be just about halfway between level curves with pressures 1008 and 1012 mb, so we estimate the pressure at San Francisco to be about 1010 mb. V lies close to a level curve with pressure 1016 mb but we can't see a level curve to its left so it is more difficult to make an accurate estimate. There are lower pressures to the right of V and V is a short distance to the left of the level curve with pressure 1016 mb, so we might estimate that the pressure at Vancouver is about 1017 mb.

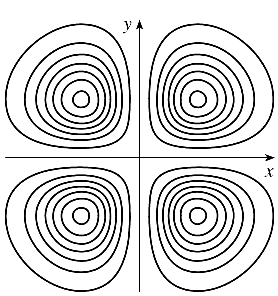
(b) Winds are stronger where the isobars are closer together (see Figure 13), and the level curves are closer near S than at the other locations, so the winds were strongest at San Francisco.

35. The point $(160, 10)$, corresponding to day 160 and a depth of 10 m, lies between the isothermals with temperature values of 8 and 12°C. Since the point appears to be located about three-fourths the distance from the 8°C isothermal to the 12°C isothermal, we estimate the temperature at that point to be approximately 11°C. The point $(180, 5)$ lies between the 16 and 20°C isothermals, very close to the 20°C level curve, so we estimate the temperature there to be about 19.5°C.

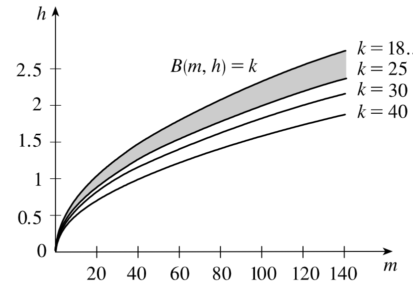
36. If we start at the origin and move along the x-axis, for example, the z-values of a cone centered at the origin increase at a constant rate, so we would expect its level curves to be equally spaced. A paraboloid with vertex the origin, on the other hand, has z-values which change slowly near the origin and more quickly as we move farther away. Thus, we would expect its level curves near the origin to be spaced more widely apart than those farther from the origin. Therefore contour map I must correspond to the paraboloid, and contour map II the cone.

37. Near A, the level curves are very close together, indicating that the terrain is quite steep. At B, the level curves are much farther apart, so we would expect the terrain to be much less steep than near A, perhaps almost flat.

38.



39. The level curves of $B(m,h) = \dfrac{m}{h^2}$ are $\dfrac{m}{h^2} = k \;\Leftrightarrow\; m = kh^2$ or equivalently $h = \sqrt{m/k} = \dfrac{1}{\sqrt{k}}\sqrt{m}$ since $m > 0$, $h > 0$. We draw the level curves for $k = 18.5$, 25, 30, and 40.

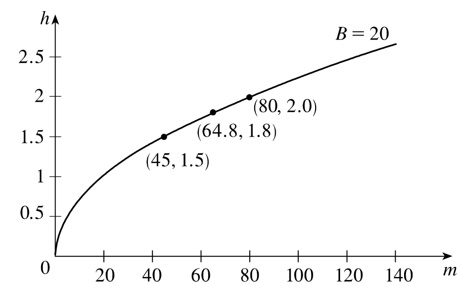


The shaded region corresponds to BMI values between 18.5 and 25, those considered optimal. For a mass of 62 kg and a height of 152 cm (1.52 m), the BMI is $B(62, 1.52) = \dfrac{62}{1.52^2} \approx 26.8$, which is outside the optimal range.

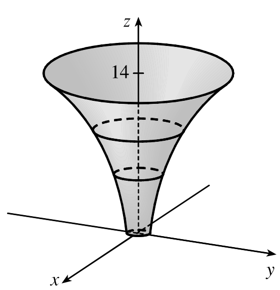
40. From Exercise 39, the body mass index function is $B(m,h) = m/h^2$. The BMI for a person 200 cm (2.0 m) tall and with mass 80 kg is $B(80, 2.0) = 80/(2.0)^2 = 20$.

The level curve $B(m,h) = 20 \;\Leftrightarrow\; m = 20h^2$ is shown in the graph.

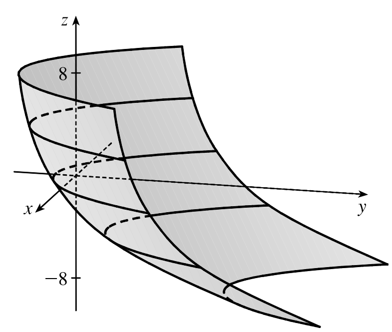
A person 1.5 m tall has a BMI on the same level curve if their mass is $m = 20(1.5)^2 = 45$ kg, and a person 1.8 m tall would have mass $m = 20(1.8)^2 = 64.8$ kg. (See the graph.)



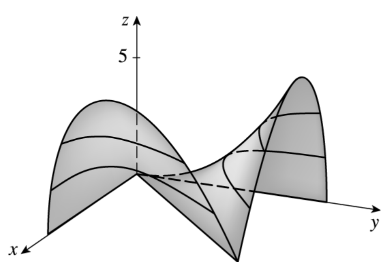
41.



42.



43.



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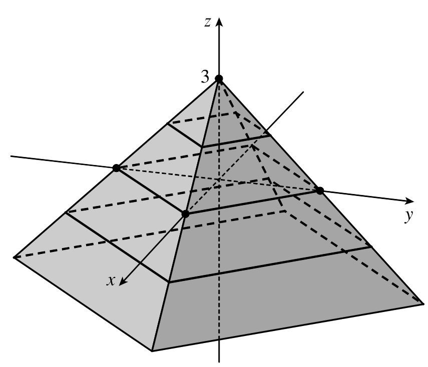
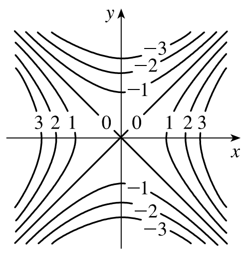
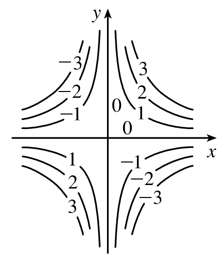


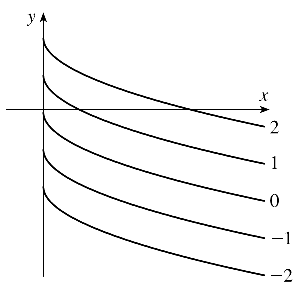
45. The level curves are $x^2 - y^2 = k$. When $k = 0$ the level curve is the pair of lines $y = \pm x$, and when $k \neq 0$ the level curves are a family of hyperbolas (oriented differently for $k > 0$ than for $k < 0$).



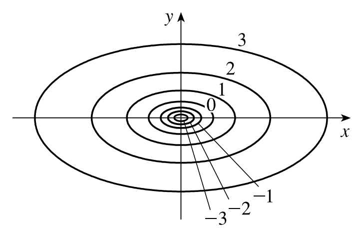
46. The level curves are $xy = k$ or $y = k/x$. When $k \neq 0$ the level curves are a family of hyperbolas. When $k = 0$ the level curve is the pair of lines $x = 0$, $y = 0$.



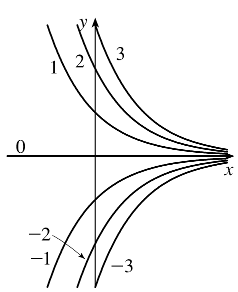
47. The level curves are $\sqrt{x} + y = k$ or $y = -\sqrt{x} + k$, a family of vertical translations of the graph of the root function $y = -\sqrt{x}$.



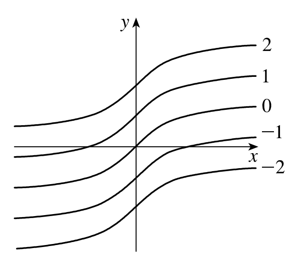
48. The level curves are $\ln(x^2 + 4y^2) = k$ or $x^2 + 4y^2 = e^k$, a family of ellipses.



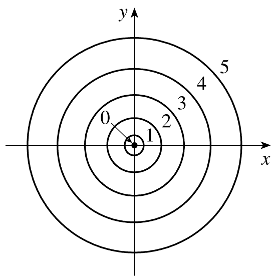
49. The level curves are $ye^x = k$ or $y = ke^{-x}$, a family of exponential curves.



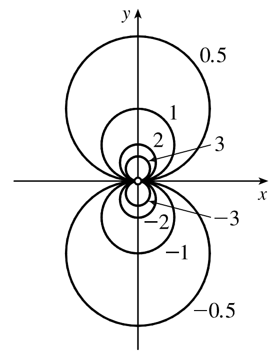
50. The level curves are $y - \arctan x = k$ or $y = (\arctan x) + k$, a family of vertical translations of the graph of the inverse tangent function $y = \arctan x$.



51. The level curves are $\sqrt[3]{x^2+y^2}=k$ or $x^2+y^2=k^3$ $(k \geq 0)$, a family of circles centered at the origin with radius $k^{3/2}$.

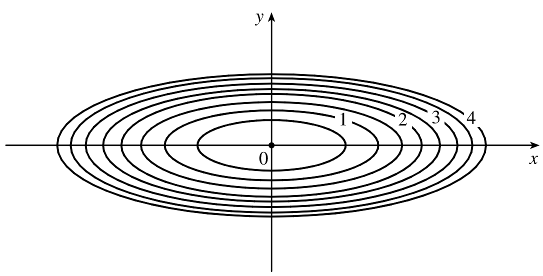


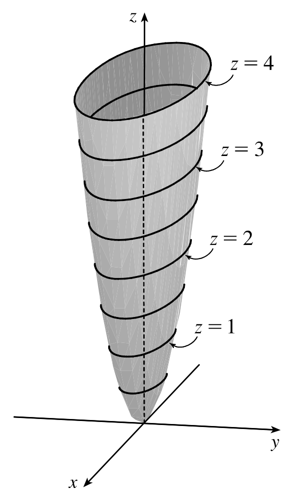
52. For $k \neq 0$ and $(x,y) \neq (0,0)$, $k = \dfrac{y}{x^2+y^2}$ $\Leftrightarrow$ $x^2+y^2-\dfrac{y}{k}=0$ $\Leftrightarrow$ $x^2+\left(y-\frac{1}{2k}\right)^2=\dfrac{1}{4k^2}$, a family of circles with center $\left(0,\frac{1}{2k}\right)$ and radius $\frac{1}{2k}$ (without the origin). If $k=0$, the level curve is the x-axis.



53. The contour map consists of the level curves $k=x^2+9y^2$, a family of ellipses with major axis the x-axis. (Or, if $k=0$, the origin.)

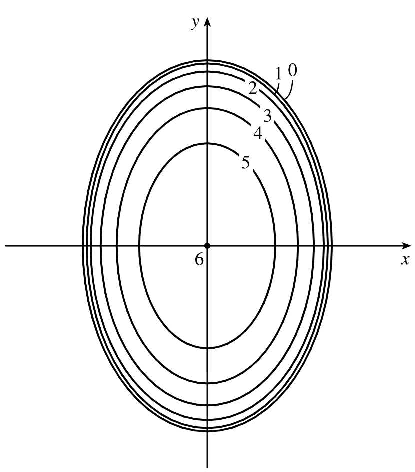
The graph of $f(x,y)$ is the surface $z=x^2+9y^2$, an elliptic paraboloid.



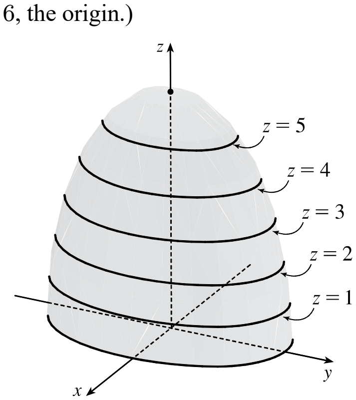


If we visualize lifting each ellipse $k=x^2+9y^2$ of the contour map to the plane $z=k$, we have horizontal traces that indicate the shape of the graph of f.

54.



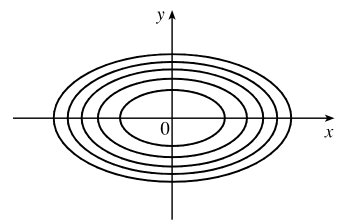
The contour map consists of the level curves $k=\sqrt{36-9x^2-4y^2}$ $\Rightarrow$ $9x^2+4y^2=36-k^2$, $k \geq 0$, a family of ellipses with major axis the y-axis. (Or, if $k=6$, the origin.)



[continued]

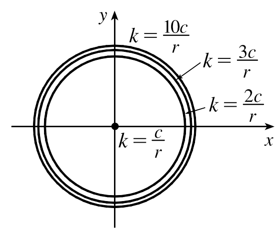
The graph of $f(x, y)$ is the surface $z = \sqrt{36 - 9x^2 - 4y^2}$, or equivalently the upper half of the ellipsoid $9x^2 + 4y^2 + z^2 = 36$. If we visualize lifting each ellipse $k = \sqrt{36 - 9x^2 - 4y^2}$ of the contour map to the plane $z = k$, we have horizontal traces that indicate the shape of the graph of f.

55. The isothermals are given by $k = 100/(1 + x^2 + 2y^2)$ or $x^2 + 2y^2 = (100 - k)/k$ $[0 < k \leq 100]$, a family of ellipses.

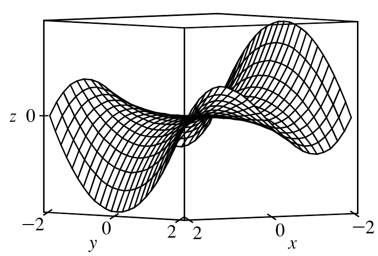


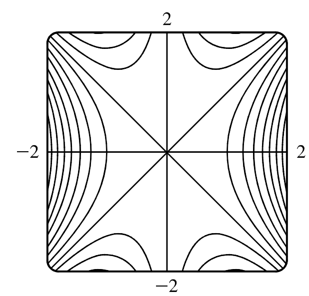
56. The equipotential curves are $k = \dfrac{c}{\sqrt{r^2 - x^2 - y^2}}$ or $x^2 + y^2 = r^2 - \left(\dfrac{c}{k}\right)^2$, a family of circles ($k \geq c/r$).

Note: As $k \to \infty$, the radius of the circle approaches r.



57. $f(x, y) = xy^2 - x^3$

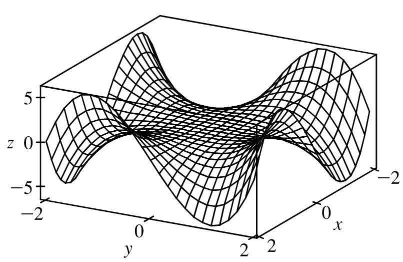


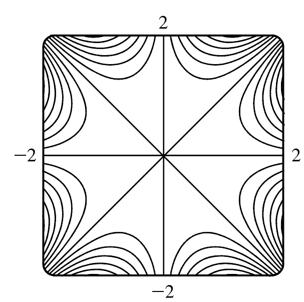


The traces parallel to the yz-plane (such as the left-front trace in the graph above) are parabolas; those parallel to the xz-plane (such as the right-front trace) are cubic curves. The surface is called a monkey saddle because a monkey sitting on the surface near the origin has places for both legs and tail to rest.

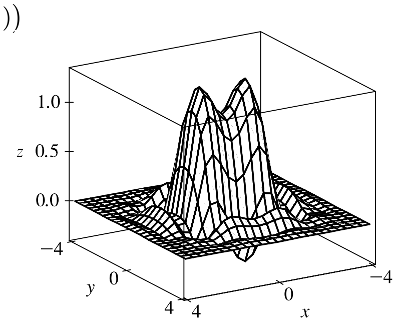
58. $f(x, y) = xy^3 - yx^3$

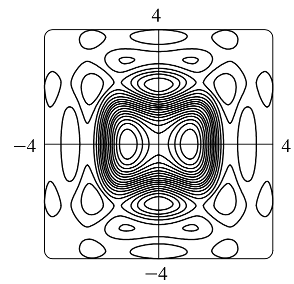
The traces parallel to either the yz-plane or the xz-plane are cubic curves.





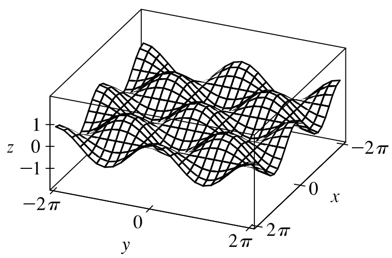
59. $f(x,y) = e^{-(x^2+y^2)/3}\left(\sin(x^2) + \cos(y^2)\right)$

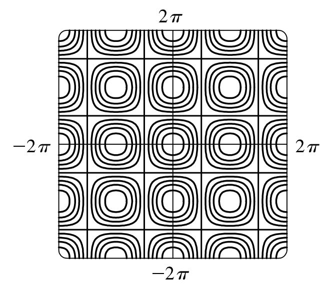




60. $f(x,y) = \cos x \,\cos y$

The traces parallel to either the yz- or xz-plane are cosine curves with amplitudes that vary from 0 to 1.





61. $z = \sin(xy)$ (a) C (b) II

Reasons: This function is periodic in both x and y, and the function is the same when x is interchanged with y, so its graph is symmetric about the plane $y = x$. In addition, the function is 0 along the x- and y-axes. These conditions are satisfied only by C and II.

62. $z = e^x \cos y$ (a) A (b) IV

Reasons: This function is periodic in y but not x, a condition satisfied only by A and IV. Also, note that traces in $x = k$ are cosine curves with amplitude that increases as x increases.

63. $z = \sin(x - y)$ (a) F (b) I

Reasons: This function is periodic in both x and y but is constant along the lines $y = x + k$, a condition satisfied only by F and I.

64. $z = \sin x - \sin y$ (a) E (b) III

Reasons: This function is periodic in both x and y, but unlike the function in Exercise 63, it is not constant along lines such as $y = x + \pi$, so the contour map is III. Also notice that traces in $y = k$ are vertically shifted copies of the sine wave $z = \sin x$, so the graph must be E.

65. $z = (1 - x^2)(1 - y^2)$ (a) B (b) VI

Reasons: This function is 0 along the lines $x = \pm 1$ and $y = \pm 1$. The only contour map in which this could occur is VI. Also note that the trace in the xz-plane is the parabola $z = 1 - x^2$ and the trace in the yz-plane is the parabola $z = 1 - y^2$, so the graph is B.

66. $z = \dfrac{x - y}{1 + x^2 + y^2}$ (a) D (b) V

Reasons: This function is not periodic, ruling out the graphs in A, C, E, and F. Also, the values of z approach 0 as we use points farther from the origin. The only graph that shows this behavior is D, which corresponds to V.

67. $k = 2y - z + 1$ is a family of parallel planes with normal vector $\langle 0, 2, -1 \rangle$.

68. $k = x + y^2 - z^2$ is a family of hyperbolic paraboloids with saddle point $(k, 0, 0)$.

69. Equations for the level surfaces are $k = x^2 + y^2 - z^2$. For $k = 0$, the equation becomes $z^2 = x^2 + y^2$ and the surface is a right circular cone with center the origin and axis the z-axis. For $k > 0$, we have a family of hyperboloids of one sheet with axis the z-axis. For $k < 0$ we have a family of hyperboloids of two sheets with axis the z-axis.

70. $k = x^2 + 2y^2 + 3z^2$ is a family of ellipsoids with major axis the x-axis for $k > 0$ and the origin for $k = 0$.

71. (a) The graph of g is the graph of f shifted upward 2 units.

(b) The graph of g is the graph of f stretched vertically by a factor of 2.

(c) The graph of g is the graph of f reflected about the xy-plane.

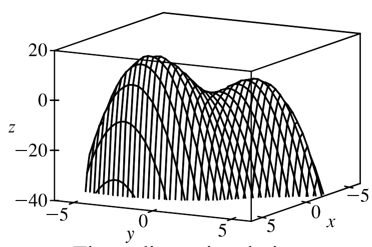
(d) The graph of $g(x, y) = -f(x, y) + 2$ is the graph of f reflected about the xy-plane and then shifted upward 2 units.

72. (a) The graph of g is the graph of f shifted 2 units in the positive x-direction.

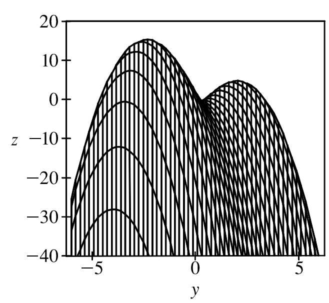
(b) The graph of g is the graph of f shifted 2 units in the negative y-direction.

(c) The graph of g is the graph of f shifted 3 units in the negative x-direction and 4 units in the positive y-direction.

73. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$



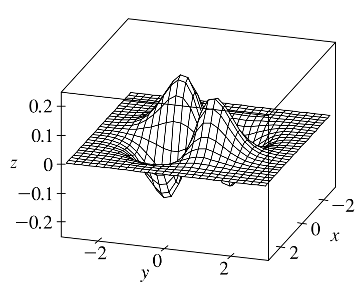
Three-dimensional view



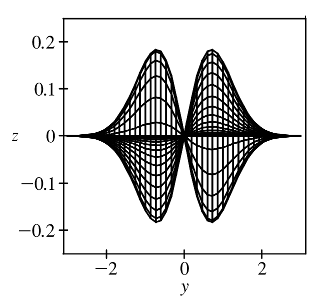
Front view

It does appear that the function has a maximum value, at the higher of the two "hilltops." From the front view graph, the maximum value appears to be approximately 15. Both hilltops could be considered local maximum points, as the values of f there are larger than at the neighboring points. There does not appear to be any local minimum point; although the valley shape between the two peaks looks like a minimum of some kind, some neighboring points have lower function values.

74. $f(x, y) = xye^{-x^2-y^2}$

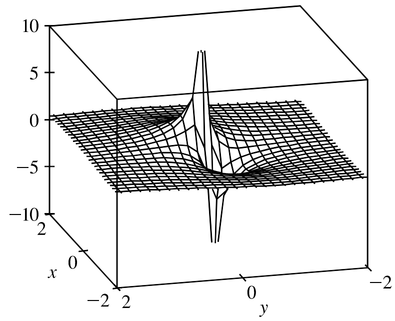


Three-dimensional view

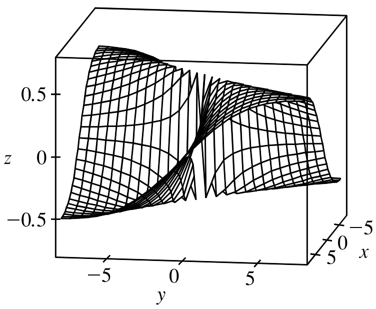


Front view

The function does have a maximum value, which it appears to achieve at two different points (the two "hilltops"). From the front view graph, we can estimate the maximum value to be approximately 0.18. These same two points can also be considered local maximum points. The two "valley bottoms" visible in the graph can be considered local minimum points, as all the neighboring points give greater values of f.

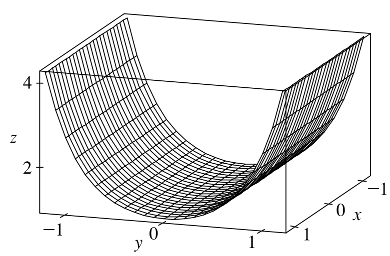
75. 

$f(x, y) = \dfrac{x + y}{x^2 + y^2}$. As both x and y become large, the function values appear to approach 0, regardless of which direction is considered. As (x, y) approaches the origin, the graph exhibits asymptotic behavior. From some directions, $f(x, y) \to \infty$, while in others $f(x, y) \to -\infty$. (These are the vertical spikes visible in the graph.) If the graph is examined carefully, however, one can see that $f(x, y)$ approaches 0 along the line $y = -x$.

76. 

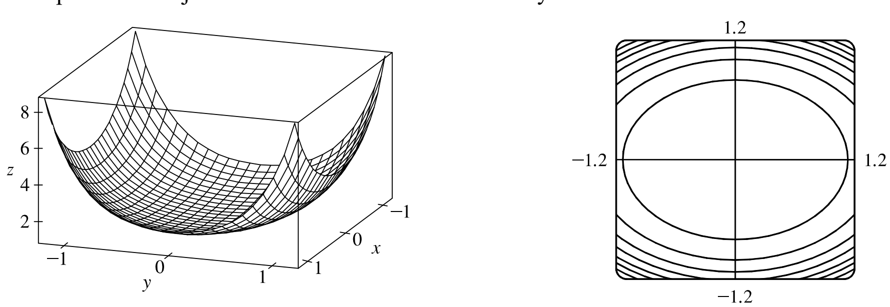
$f(x, y) = \dfrac{xy}{x^2 + y^2}$. The graph exhibits different limiting values as x and y become large or as (x, y) approaches the origin, depending on the direction being examined. For example, although f is undefined at the origin, the function values appear to be $\frac{1}{2}$ along the line $y = x$, regardless of the distance from the origin. Along the line $y = -x$, the value is always $-\frac{1}{2}$. Along the axes, $f(x, y) = 0$ for all values of (x, y) except the origin. Other directions, heading toward the origin or away from the origin, give various limiting values between $-\frac{1}{2}$ and $\frac{1}{2}$.

77. $f(x, y) = e^{cx^2 + y^2}$. First, if $c = 0$, the graph is the cylindrical surface $z = e^{y^2}$ (whose level curves are parallel lines). When $c > 0$, the vertical trace above the y-axis remains fixed while the sides of the surface in the x-direction "curl" upward, giving the graph a shape resembling an elliptic paraboloid. The level curves of the surface are ellipses centered at the origin.



$c = 0$

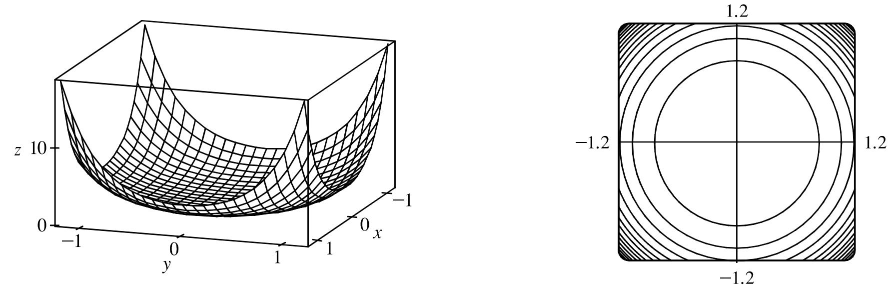
For $0 < c < 1$, the ellipses have major axis the x-axis and the eccentricity increases as $c \to 0$.



$c = 0.5$ (level curves in increments of 1)

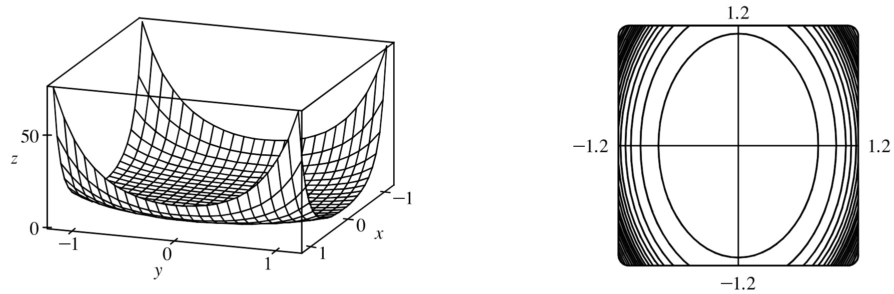
[continued]

For $c = 1$ the level curves are circles centered at the origin.



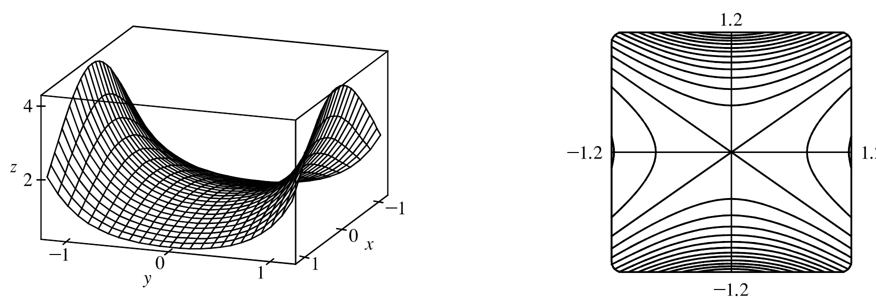
$c = 1$ (level curves in increments of 1)

When $c > 1$, the level curves are ellipses with major axis the y-axis, and the eccentricity increases as c increases.

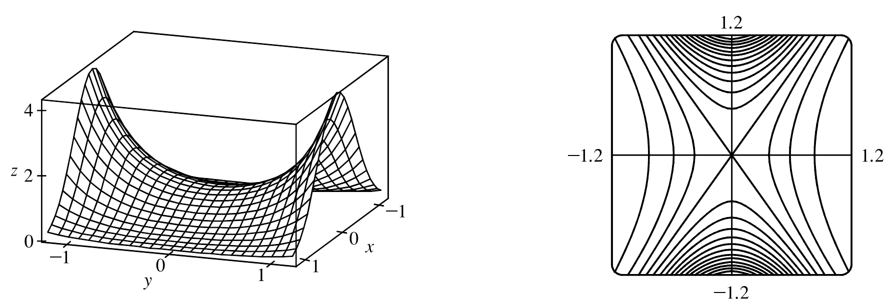


$c = 2$ (level curves in increments of 4)

For values of $c < 0$, the sides of the surface in the x-direction curl downward and approach the xy-plane (while the vertical trace $x = 0$ remains fixed), giving a saddle-shaped appearance to the graph near the point $(0, 0, 1)$. The level curves consist of a family of hyperbolas. As c decreases, the surface becomes flatter in the x-direction and the surface's approach to the curve in the trace $x = 0$ becomes steeper, as the graphs demonstrate.

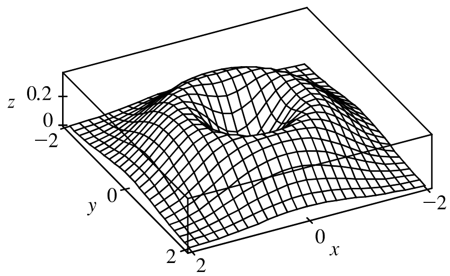


$c = -0.5$ (level curves in increments of 0.25)

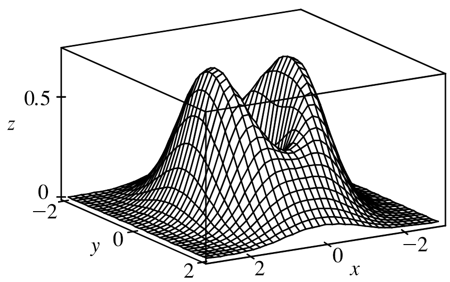


$c = -2$ (level curves in increments of 0.25)

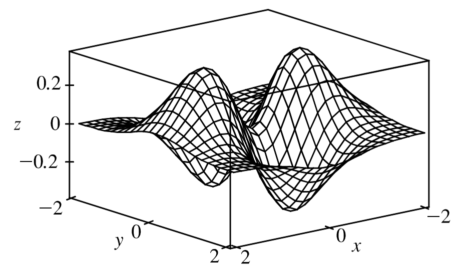
78. $z = (ax^2 + by^2)e^{-x^2-y^2}$. There are only three basic shapes which can be obtained (the fourth and fifth graphs are the reflections of the first and second ones in the xy-plane). Interchanging a and b rotates the graph by 90° about the z-axis.



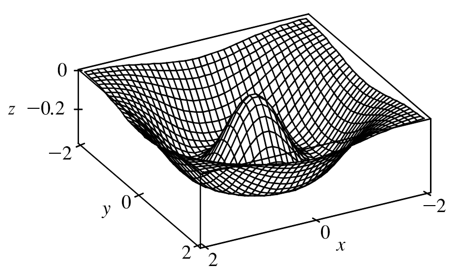
$a = 1, b = 1$



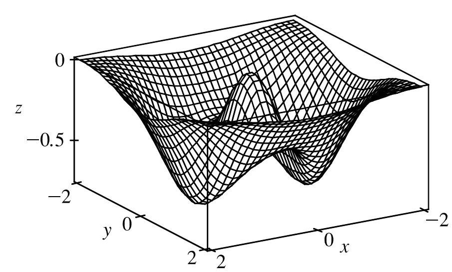
$a = 2, b = 1$



$a = 1, b = -1$



$a = -1, b = -1$



$a = -2, b = -1$

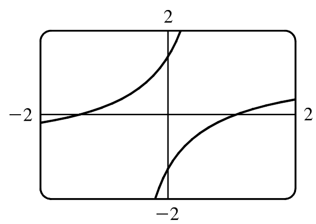
If a and b are both positive ($a \neq b$), we see that the graph has two maximum points whose height increases as a and b increase. If a and b have opposite signs, the graph has two maximum points and two minimum points, and if a and b are both negative, the graph has one maximum point and two minimum points.

79. $z = x^2 + y^2 + cxy$. When $c < -2$, the surface intersects the plane $z = k \neq 0$ in a hyperbola. (See the following graph.) It intersects the plane $x = y$ in the parabola $z = (2 + c)x^2$, and the plane $x = -y$ in the parabola $z = (2 - c)x^2$. These parabolas open in opposite directions, so the surface is a hyperbolic paraboloid.

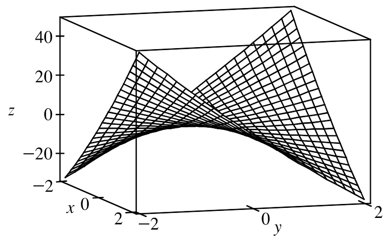
When $c = -2$ the surface is $z = x^2 + y^2 - 2xy = (x - y)^2$. So the surface is constant along each line $x - y = k$. That is, the surface is a cylinder with axis $x - y = 0$, $z = 0$. The shape of the cylinder is determined by its intersection with the

[continued]

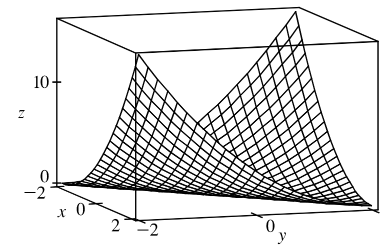
plane $x + y = 0$, where $z = 4x^2$, and hence the cylinder is parabolic with minimums of 0 on the line $y = x$.



$c = -5, z = 2$



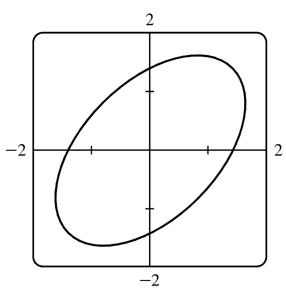
$c = -10$



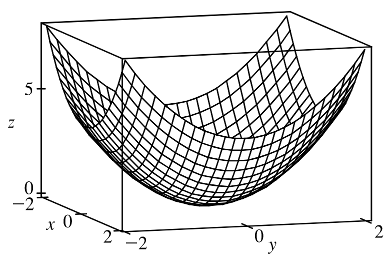
$c = -2$

When $-2 < c \leq 0$, $z \geq 0$ for all x and y. If x and y have the same sign, then $x^2 + y^2 + cxy \geq x^2 + y^2 - 2xy = (x - y)^2 \geq 0$. If they have opposite signs, then $cxy \geq 0$. The intersection with the surface and the plane $z = k > 0$ is an ellipse (see graph below). The intersection with the surface and the planes $x = 0$ and $y = 0$ are parabolas $z = y^2$ and $z = x^2$ respectively, so the surface is an elliptic paraboloid.

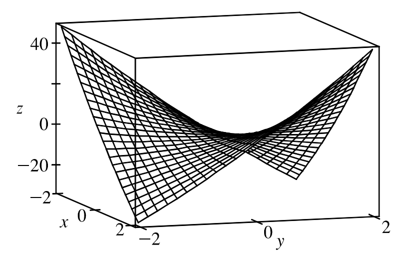
When $c > 0$ the graphs have the same shape, but are reflected in the plane $x = 0$, because $x^2 + y^2 + cxy = (-x)^2 + y^2 + (-c)(-x)y$. That is, the value of z is the same for c at (x, y) as it is for $-c$ at $(-x, y)$.



$c = -1, z = 2$



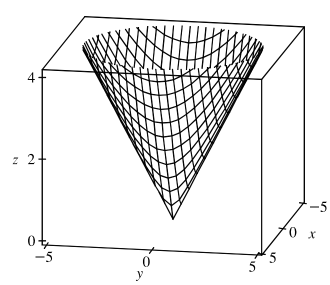
$c = 0$



$c = 10$

So the surface is an elliptic paraboloid for $0 < c < 2$, a parabolic cylinder for $c = 2$, and a hyperbolic paraboloid for $c > 2$.

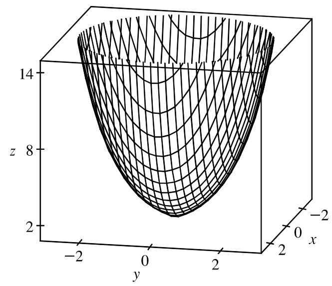
80. First, we graph $f(x, y) = \sqrt{x^2 + y^2}$.



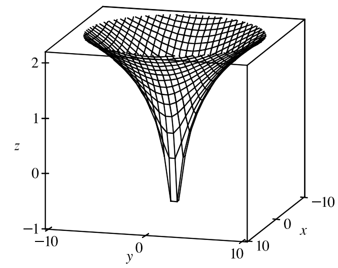
$f(x, y) = \sqrt{x^2 + y^2}$

[continued]

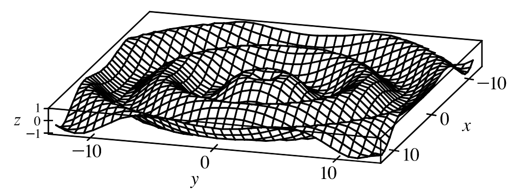
Graphs of the other four functions follow.



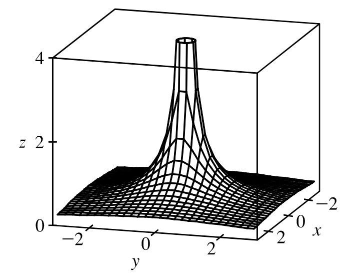
$f(x, y) = e^{\sqrt{x^2+y^2}}$



$f(x, y) = \ln\sqrt{x^2+y^2}$



$f(x, y) = \sin\left(\sqrt{x^2+y^2}\right)$



$f(x, y) = \dfrac{1}{\sqrt{x^2+y^2}}$

Notice that each graph $f(x, y) = g\left(\sqrt{x^2+y^2}\right)$ exhibits radial symmetry about the z-axis and the trace in the xz-plane for $x \geq 0$ is the graph of $z = g(x)$, $x \geq 0$. This suggests that the graph of $f(x, y) = g\left(\sqrt{x^2+y^2}\right)$ is obtained from the graph of g by graphing $z = g(x)$ in the xz-plane and rotating the curve about the z-axis.

81. (a) $P = bL^{\alpha}K^{1-\alpha} \quad \Rightarrow \quad \dfrac{P}{K} = bL^{\alpha}K^{-\alpha} \quad \Rightarrow \quad \dfrac{P}{K} = b\left(\dfrac{L}{K}\right)^{\alpha} \quad \Rightarrow \quad \ln\dfrac{P}{K} = \ln\left(b\left(\dfrac{L}{K}\right)^{\alpha}\right) \quad \Rightarrow$

$$\ln\frac{P}{K} = \ln b + \alpha \ln\left(\frac{L}{K}\right)$$

(b) We list the values for $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. (Historically, these values were rounded to 2 decimal places.)

Year	$x = \ln(L/K)$	$y = \ln(P/K)$
1899	0	0
1900	−0.02	−0.06
1901	−0.04	−0.02
1902	−0.04	0
1903	−0.07	−0.05
1904	−0.13	−0.12
1905	−0.18	−0.04
1906	−0.20	−0.07
1907	−0.23	−0.15
1908	−0.41	−0.38
1909	−0.33	−0.24
1910	−0.35	−0.27

Year	$x = \ln(L/K)$	$y = \ln(P/K)$
1911	−0.38	−0.34
1912	−0.38	−0.24
1913	−0.41	−0.25
1914	−0.47	−0.37
1915	−0.53	−0.34
1916	−0.49	−0.28
1917	−0.53	−0.39
1918	−0.60	−0.50
1919	−0.68	−0.57
1920	−0.74	−0.57
1921	−1.05	−0.85
1922	−0.98	−0.59

[continued]

After entering the (x, y) pairs into a calculator or CAS, the resulting least squares regression line through the points is approximately $y = 0.75136x + 0.01053$, which we round to $y = 0.75x + 0.01$.

(c) Comparing the regression line from part (b) to the equation $y = \ln b + \alpha x$ with $x = \ln(L/K)$ and $y = \ln(P/K)$, we have $\alpha = 0.75$ and $\ln b = 0.01 \quad \Rightarrow \quad b = e^{0.01} \approx 1.01$. Thus, the Cobb-Douglas production function is $P = bL^{\alpha}K^{1-\alpha} = 1.01L^{0.75}K^{0.25}$.

14.2 Limits and Continuity

1. In general, we can't say anything about $f(3, 1)$! $\lim\limits_{(x,y)\to(3,1)} f(x, y) = 6$ means that the values of $f(x, y)$ approach 6 as (x, y) approaches, but is not equal to, $(3, 1)$. If f is continuous, we know that $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$, so $\lim\limits_{(x,y)\to(3,1)} f(x, y) = f(3, 1) = 6$.

2. (a) The outdoor temperature as a function of longitude, latitude, and time is continuous. Small changes in longitude, latitude, or time can produce only small changes in temperature, as the temperature doesn't jump abruptly from one value to another.

(b) Elevation is not necessarily continuous. If we think of a cliff with a sudden drop-off, a very small change in longitude or latitude can produce a comparatively large change in elevation, without all the intermediate values being attained. Elevation *can* jump from one value to another.

(c) The cost of a taxi ride is usually discontinuous. The cost normally increases in jumps, so small changes in distance traveled or time can produce a jump in cost. A graph of the function would show breaks in the surface.

3. We make a table of values of $f(x, y) = \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy}$ for a set of (x, y) points near the origin.

x \ y	-0.2	-0.1	-0.05	0	0.05	0.1	0.2
-0.2	-2.551	-2.525	-2.513	-2.500	-2.488	-2.475	-2.451
-0.1	-2.525	-2.513	-2.506	-2.500	-2.494	-2.488	-2.475
-0.05	-2.513	-2.506	-2.503	-2.500	-2.497	-2.494	-2.488
0	-2.500	-2.500	-2.500		-2.500	-2.500	-2.500
0.05	-2.488	-2.494	-2.497	-2.500	-2.503	-2.506	-2.513
0.1	-2.475	-2.488	-2.494	-2.500	-2.506	-2.513	-2.525
0.2	-2.451	-2.475	-2.488	-2.500	-2.513	-2.525	-2.551

As the table shows, the values of $f(x, y)$ seem to approach -2.5 as (x, y) approaches the origin from a variety of different directions. This suggests that $\lim\limits_{(x,y)\to(0,0)} f(x, y) = -2.5$. Since f is a rational function, it is continuous on its domain. f is defined at $(0, 0)$, so we can use direct substitution to establish that $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \dfrac{0^2 0^3 + 0^3 0^2 - 5}{2 - 0 \cdot 0} = -\dfrac{5}{2}$, verifying our guess.

4. We make a table of values of $f(x,y) = \dfrac{2xy}{x^2+2y^2}$ for a set of (x,y) points near the origin.

$x \backslash y$	-0.3	-0.2	-0.1	0	0.1	0.2	0.3
-0.3	0.667	0.706	0.545	0.000	-0.545	-0.706	-0.667
-0.2	0.545	0.667	0.667	0.000	-0.667	-0.667	-0.545
-0.1	0.316	0.444	0.667	0.000	-0.667	-0.444	-0.316
0	0.000	0.000	0.000		0.000	0.000	0.000
0.1	-0.316	-0.444	-0.667	0.000	0.667	0.444	0.316
0.2	-0.545	-0.667	-0.667	0.000	0.667	0.667	0.545
0.3	-0.667	-0.706	-0.545	0.000	0.545	0.706	0.667

It appears from the table that the values of $f(x,y)$ are not approaching a single value as (x,y) approaches the origin. For verification, if we first approach $(0,0)$ along the x-axis, we have $f(x,0)=0$, so $f(x,y)\to 0$. But if we approach $(0,0)$ along the line $y=x$, $f(x,x)=\dfrac{2x^2}{x^2+2x^2}=\dfrac{2}{3}$ $(x\neq 0)$, so $f(x,y)\to \frac{2}{3}$. Since f approaches different values along different paths to the origin, this limit does not exist.

5. $f(x,y)=x^2y^3-4y^2$ is a polynomial, and hence continuous, so we can find the limit by direct substitution:
$\lim\limits_{(x,y)\to(3,2)} f(x,y)=f(3,2)=(3)^2(2)^3-4(2)^2=56.$

6. $f(x,y)=x^2y+3xy^2+4$ is a polynomial, and hence continuous, so we can find the limit by direct substitution:
$\lim\limits_{(x,y)\to(5,-2)} f(x,y)=f(5,-2)=5^2(-2)+3(5)(-2)^2+4=14.$

7. $f(x,y)=\dfrac{x^2y-xy^3}{x-y+2}$ is a rational function, and hence, continuous on its domain. $(-3,1)$ is in the domain of f, so we can find the limit by direct substitution: $\lim\limits_{(x,y)\to(-3,1)} f(x,y)=f(-3,1)=\dfrac{(-3)^2(1)-(-3)(1)^3}{-3-1+2}=\dfrac{12}{-2}=-6.$

8. $f(x,y)=\dfrac{x^2y+xy^2}{x^2-y^2}$ is a rational function, and hence, continuous on its domain. $(2,-1)$ is in the domain of f, so we can find the limit by direct substitution: $\lim\limits_{(x,y)\to(2,-1)} f(x,y)=f(2,-1)=\dfrac{(2)^2(-1)+(2)(-1)^2}{(2)^2-(-1)^2}=-\dfrac{2}{3}.$

9. $x-y$ is a polynomial and therefore continuous. Since $\sin t$ is a continuous function, the composition $\sin(x-y)$ is also continuous. The function y is a polynomial, and hence continuous, and the product of continuous functions is continuous, so $f(x,y)=y\sin(x-y)$ is a continuous function. Then $\lim\limits_{(x,y)\to(\pi,\pi/2)} f(x,y)=f\left(\pi,\frac{\pi}{2}\right)=\frac{\pi}{2}\sin\left(\pi-\frac{\pi}{2}\right)=\frac{\pi}{2}\sin\frac{\pi}{2}=\frac{\pi}{2}.$

10. $2x-y$ is a polynomial and therefore continuous. Since $\sqrt{t}$ is continuous for $t\geq 0$, the composition $\sqrt{2x-y}$ is continuous where $2x-y\geq 0$. The function e^u is continuous everywhere, so the composition $f(x,y)=e^{\sqrt{2x-y}}$ is a continuous function for $2x-y\geq 0$. If $x=3$ and $y=2$ then $2x-y\geq 0$, so $\lim\limits_{(x,y)\to(3,2)} f(x,y)=f(3,2)=e^{\sqrt{2(3)-2}}=e^2.$

11. $f(x,y) = \dfrac{x^2y^3 - x^3y^2}{x^2 - y^2} = \dfrac{x^2y^2(y-x)}{(x-y)(x+y)} = -\dfrac{x^2y^2}{x+y}$ for $x - y \neq 0$; in particular, $(x,y) \neq (1,1)$. Thus,

$$\lim_{(x,y)\to(1,1)} f(x,y) = \lim_{(x,y)\to(1,1)} \left(-\frac{x^2y^2}{x+y}\right) = -\frac{(1)^2(1)^2}{1+1} = -\frac{1}{2}.$$

12. $f(x,y) = \dfrac{\cos y - \sin 2y}{\cos x \cos y} = \dfrac{\cos y - 2\sin y \cos y}{\cos x \cos y} = \dfrac{1 - 2\sin y}{\cos x}$ for $\cos y \neq 0$; in particular, $(x,y) \neq (\pi, \pi/2)$. Thus,

$$\lim_{(x,y)\to(\pi,\pi/2)} f(x,y) = \lim_{(x,y)\to(\pi,\pi/2)} \frac{1 - 2\sin y}{\cos x} = \frac{1 - 2\sin(\pi/2)}{\cos \pi} = \frac{-1}{-1} = 1.$$

13. $f(x,y) = \dfrac{y^2}{x^2 + y^2}$. First approach $(0,0)$ along the x-axis. Then $f(x,0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x,y) \to 0$. Now approach $(0,0)$ along the y-axis. Then $f(0,y) = y^2/y^2 = 1$ for $y \neq 0$, so $f(x,y) \to 1$. Since f has two different limits along two different lines, the limit does not exist.

14. $f(x,y) = \dfrac{2xy}{x^2 + 3y^2}$. First approach $(0,0)$ along the x-axis. Then $f(x,0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x,y) \to 0$. Now approach $(0,0)$ along the line $y = x$. Then $f(x,x) = 2x^2/4x^2 = 1/2$ for $x \neq 0$. Since f has two different limits along two different lines, the limit does not exist.

15. $f(x,y) = \dfrac{(x+y)^2}{x^2 + y^2}$. First approach $(0,0)$ along the x-axis. Then $f(x,0) = x^2/x^2 = 1$ for $x \neq 0$, so $f(x,y) \to 1$. Now approach $(0,0)$ along the line $y = x$. Then $f(x,x) = 4x^2/(2x^2) = 2$ for $x \neq 0$, so $f(x,y) \to 2$. Since f has two different limits along two different lines, the limit does not exist.

16. $f(x,y) = \dfrac{x^2 + xy^2}{x^4 + y^2}$. First approach $(0,0)$ along the y-axis. Then $f(0,y) = 0/y^2 = 0$ for $y \neq 0$, so $f(x,y) \to 0$. Now approach $(0,0)$ along the line $y = x$. Then $f(x,x) = \dfrac{x^2 + x^3}{x^4 + x^2} = \dfrac{x^2(1+x)}{x^2(x^2+1)} = \dfrac{1+x}{1+x^2}$ for $x \neq 0$, so $f(x,y) \to 1$. Since f has two different limits along two different lines, the limit does not exist.

17. $f(x,y) = \dfrac{y^2 \sin^2 x}{x^4 + y^4}$. First approach $(0,0)$ along the y-axis. Then $f(0,y) = 0/y^4 = 0$ for $y \neq 0$, so $f(x,y) \to 0$. Now approach $(0,0)$ along the line $y = x$. Then $f(x,x) = \dfrac{x^2 \sin^2 x}{2x^4} = \dfrac{1}{2}\left(\dfrac{\sin x}{x}\right)^2$ for $x \neq 0$, so by Equation 3.3.5, $f(x,y) \to \frac{1}{2}(1)^2 = \frac{1}{2}$. Since f has two different limits along two different lines, the limit does not exist.

18. $f(x,y) = \dfrac{y - x}{1 - y + \ln x}$. First approach $(1,1)$ along the line $x = 1$. Then $f(1,y) = \dfrac{y-1}{1-y+0} = -1$ for $y \neq 1$, so $f(x,y) \to -1$. Now approach $(1,1)$ along the line $y = x$. Then $f(x,x) = \dfrac{x - x}{1 - x + \ln x} = 0$ for $1 - x + \ln x \neq 0$. So $f(x,y) \to 0$. Since f has two different limits along two different lines, the limit does not exist.

19. $x^2y - xy^2 + 3$ is a polynomial and therefore continuous. t^3 is also a polynomial and therefore continuous. so the composition $(x^2y - xy^2 + 3)^3$ is continuous. Thus, $\displaystyle\lim_{(x,y)\to(-1,-2)} (x^2y - xy^2 + 3)^3 = [(-1)^2(-2) - (-1)(-2)^2 + 3]^3 = 5^3 = 125.$

20. e^{xy} is a composition of continuous functions and therefore continuous. $\sin xy$ is also a composition of continuous functions and therefore continuous. Thus, the product $e^{xy}\sin xy$ is continuous, and $\displaystyle\lim_{(x,y)\to(\pi,1/2)} e^{xy}\sin xy = e^{\pi/2}\sin\tfrac{\pi}{2} = e^{\pi/2}.$

21. $f(x,y) = \dfrac{3x-2y}{4x^2-y^2}$ is a rational function and continuous on its domain. $(2,3)$ is in the domain of f, so

$$\lim_{(x,y)\to(2,3)} f(x,y) = \frac{3(2)-2(3)}{4(2)^2-3^2} = \frac{0}{7} = 0.$$

22. $f(x,y) = \dfrac{2x-y}{4x^2-y^2} = \dfrac{2x-y}{(2x-y)(2x+y)} = \dfrac{1}{2x+y}$ for $2x - y \neq 0$. Thus, $\displaystyle\lim_{(x,y)\to(1,2)} f(x,y) = \frac{1}{2(1)+2} = \frac{1}{4}.$

23. Let $f(x,y) = \dfrac{xy^2\cos y}{x^2+y^4}$. Then $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. Approaching $(0,0)$ along the y-axis or the line $y = x$ also gives a limit of 0. But $f\left(y^2,y\right) = \dfrac{y^2y^2\cos y}{(y^2)^2+y^4} = \dfrac{y^4\cos y}{2y^4} = \dfrac{\cos y}{2}$ for $y \neq 0$, so $f(x,y) \to \frac{1}{2}\cos 0 = \frac{1}{2}$ as $(x,y) \to (0,0)$ along the parabola $x = y^2$. Thus the limit doesn't exist.

24. $f(x,y) = \dfrac{x^3-y^3}{x^2+xy+y^2} = \dfrac{(x-y)(x^2+xy+y^2)}{x^2+xy+y^2} = x - y$ for $(x,y) \neq (0,0)$. [Note that $x^2 + xy + y^2 = 0$ only when $(x,y) = (0,0)$.] Thus $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} (x-y) = 0 - 0 = 0.$

25.
$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} &= \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}\cdot\frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1} \\ &= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\left(\sqrt{x^2+y^2+1}+1\right)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \left(\sqrt{x^2+y^2+1}+1\right) = 2\end{aligned}$$

26. $f(x,y) = \dfrac{xy^4}{x^2+y^8}$. On the x-axis, $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. Approaching $(0,0)$ along the curve $x = y^4$ gives $f(y^4,y) = y^8/2y^8 = \frac{1}{2}$ for $y \neq 0$, so along this path $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$. Thus, the limit does not exist.

27. $x + z$ is a polynomial and therefore, continuous. $\sqrt{t}$ is continuous on its domain. Thus, the composition is continuous at $(6,1,-2)$. $\cos\pi y$ is continuous, so the product of $\sqrt{x+z}$ and $\cos\pi y$ is also continuous. Then

$$\lim_{(x,y,z)\to(6,1,-2)} \sqrt{x+z}\cos\pi y = \sqrt{6+(-2)}\cos\pi = -2.$$

28. $f(x,y,z)=\dfrac{xy+yz}{x^2+y^2+z^2}$. Then $f(x,0,0)=0/x^2=0$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the x-axis, $f(x,y,z)\to 0$. But $f(x,x,0)=x^2/(2x^2)=\frac{1}{2}$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the line $y=x$, $z=0$, $f(x,y,z)\to\frac{1}{2}$. Thus, the limit doesn't exist.

29. $f(x,y,z)=\dfrac{xy+yz^2+xz^2}{x^2+y^2+z^4}$. Then $f(x,0,0)=0/x^2=0$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the x-axis, $f(x,y,z)\to 0$. But $f(x,x,0)=x^2/(2x^2)=\frac{1}{2}$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the line $y=x$, $z=0$, $f(x,y,z)\to\frac{1}{2}$. Thus, the limit doesn't exist.

30. $f(x,y,z)=\dfrac{x^4+y^2+z^3}{x^4+2y^2+z}$. Then $f(x,0,0)=x^4/x^4=1$ for $x\neq 0$, so $f(x,y,z)\to 1$ as $(x,y,z)\to(0,0,0)$ along the x-axis. But $f(0,y,0)=y^2/(2y^2)=\frac{1}{2}$ for $y\neq 0$, so $f(x,y,z)\to\frac{1}{2}$ as $(x,y,z)\to(0,0,0)$ along the y-axis. Since f has two different limits along two different lines, the limit does not exist.

31. $-1\leq\sin\left(\dfrac{1}{x^2+y^2}\right)\leq 1\quad\Rightarrow\quad -xy\leq xy\sin\left(\dfrac{1}{x^2+y^2}\right)\leq xy$ for $xy>0$. If $xy<0$, we have $-xy\geq xy\sin\left(\dfrac{1}{x^2+y^2}\right)\geq xy$. In either case, $\displaystyle\lim_{(x,y)\to(0,0)} xy=0$ and $\displaystyle\lim_{(x,y)\to(0,0)}(-xy)=0$. Thus, $\displaystyle\lim_{(x,y)\to(0,0)} xy\sin\left(\frac{1}{x^2+y^2}\right)=0$ by the Squeeze Theorem.

32. $f(x,y)=\dfrac{xy}{\sqrt{x^2+y^2}}$. We can see that the limit along any line through $(0,0)$ is 0, as well as along other paths through $(0,0)$ such as $x=y^2$ and $y=x^2$. So we suspect that the limit exists and equals 0; we use the Squeeze Theorem to prove our assertion. Since $|y|\leq\sqrt{x^2+y^2}$, we have $\dfrac{|y|}{\sqrt{x^2+y^2}}\leq 1$ and so $0\leq\left|\dfrac{xy}{\sqrt{x^2+y^2}}\right|\leq|x|$. Now $|x|\to 0$ as $(x,y)\to(0,0)$, so $\left|\dfrac{xy}{\sqrt{x^2+y^2}}\right|\to 0$ and hence $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)=0$.

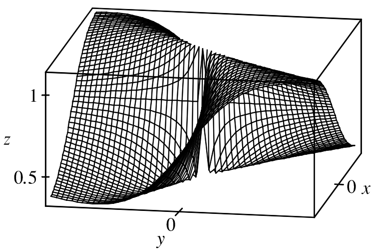
33. We use the Squeeze Theorem to show that $\displaystyle\lim_{(x,y)\to(0,0)}\frac{xy^4}{x^4+y^4}=0$:

$0\leq\dfrac{|x|\,y^4}{x^4+y^4}\leq|x|$ since $0\leq\dfrac{y^4}{x^4+y^4}\leq 1$, and $|x|\to 0$ as $(x,y)\to(0,0)$, so $\dfrac{|x|\,y^4}{x^4+y^4}\to 0\quad\Rightarrow\quad\dfrac{xy^4}{x^4+y^4}\to 0$ as $(x,y)\to(0,0)$.

34. We use the Squeeze Theorem to show that $\displaystyle\lim_{(x,y,z)\to(0,0,0)}\frac{x^2y^2z^2}{x^2+y^2+z^2}=0$:

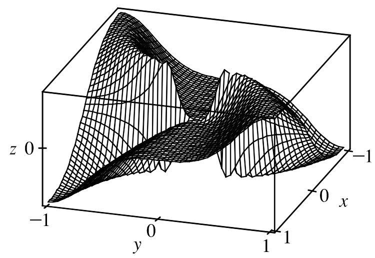
$0\leq\dfrac{x^2y^2z^2}{x^2+y^2+z^2}\leq x^2y^2$ since $0\leq\dfrac{z^2}{x^2+y^2+z^2}\leq 1$, and $x^2y^2\to 0$ as $(x,y,z)\to(0,0,0)$, so $\dfrac{x^2y^2z^2}{x^2+y^2+z^2}\to 0$ as $(x,y,z)\to(0,0,0)$.

35.



From the ridges on the graph, we see that as $(x, y) \to (0, 0)$ along the lines under the two ridges, $f(x, y) = \dfrac{2x^2 + 3xy + 4y^2}{3x^2 + 5y^2}$ approaches different values. Since the function approaches different values depending on the path of approach, the limit does not exist.

36.

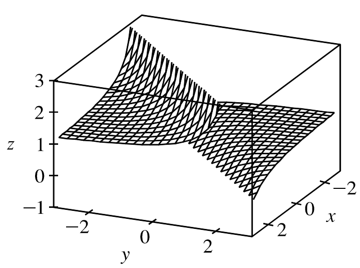


From the graph, it appears that as we approach the origin along the lines $x = 0$ or $y = 0$, the function $f(x, y) = \dfrac{xy^3}{x^2 + y^6}$ is everywhere 0, whereas if we approach the origin along a certain curve it has a constant value of about $\frac{1}{2}$. [In fact, $f(y^3, y) = y^6/(2y^6) = \frac{1}{2}$ for $y \neq 0$, so $f(x, y) \to \frac{1}{2}$ as $(x, y) \to (0, 0)$ along the curve $x = y^3$.] Since the function approaches different values depending on the path of approach, the limit does not exist.

37. $h(x, y) = g(f(x, y)) = (2x + 3y - 6)^2 + \sqrt{2x + 3y - 6}$. Since f is a polynomial, it is continuous on $\mathbb{R}^2$ and g is continuous on its domain $\{t \mid t \geq 0\}$. Thus, h is continuous on its domain $\{(x, y) \mid 2x + 3y - 6 \geq 0\} = \left\{(x, y) \mid y \geq -\frac{2}{3}x + 2\right\}$, which consists of all points on or above the line $y = -\frac{2}{3}x + 2$.

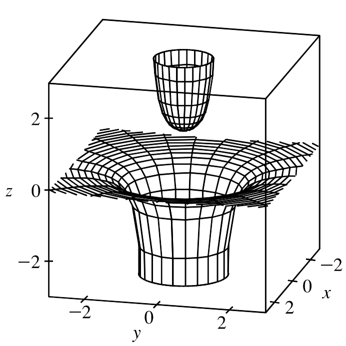
38. $h(x, y) = g(f(x, y)) = \dfrac{1 - xy}{1 + x^2y^2} + \ln\left(\dfrac{1 - xy}{1 + x^2y^2}\right)$. f is a rational function, so it is continuous on its domain. Because $1 + x^2y^2 > 0$, the domain of f is $\mathbb{R}^2$, so f is continuous everywhere. g is continuous on its domain $\{t \mid t > 0\}$. Thus, h is continuous on its domain $\left\{(x, y) \;\middle|\; \dfrac{1 - xy}{1 + x^2y^2} > 0\right\} = \{(x, y) \mid xy < 1\}$, which consists of all points between (but not on) the two branches of the hyperbola $y = 1/x$.

39.



From the graph, it appears that f is discontinuous along the line $y = x$. If we consider $f(x, y) = e^{1/(x-y)}$ as a composition of functions, $g(x, y) = 1/(x - y)$ is a rational function and therefore continuous except where $x - y = 0 \;\Leftrightarrow\; y = x$. Since the function $h(t) = e^t$ is continuous everywhere, the composition $h(g(x, y)) = e^{1/(x-y)} = f(x, y)$ is continuous except along the line $y = x$, as we suspected.

40.



We can see a circular break in the graph, corresponding approximately to the unit circle, where f is discontinuous. Since $f(x, y) = \dfrac{1}{1 - x^2 - y^2}$ is a rational function, it is continuous except where $1 - x^2 - y^2 = 0 \;\Leftrightarrow\; x^2 + y^2 = 1$, confirming our observation that f is discontinuous on the circle $x^2 + y^2 = 1$.

41. The functions xy and $1+e^{x-y}$ are continuous everywhere, and $1+e^{x-y}$ is never zero, so $F(x,y)=\dfrac{xy}{1+e^{x-y}}$ is continuous on its domain $\mathbb{R}^2$.

42. $F(x,y)=\cos\sqrt{1+x-y}=g(f(x,y))$ where $f(x,y)=\sqrt{1+x-y}$, continuous on its domain $\{(x,y)\mid 1+x-y\geq 0\}=\{(x,y)\mid y\leq x+1\}$, and $g(t)=\cos t$ is continuous everywhere. Thus F is continuous on its domain $\{(x,y)\mid y\leq x+1\}$.

43. $F(x,y)=\dfrac{1+x^2+y^2}{1-x^2-y^2}$ is a rational function and thus is continuous on its domain $\{(x,y)\mid 1-x^2-y^2\neq 0\}=\{(x,y)\mid x^2+y^2\neq 1\}$.

44. The functions e^x+e^y and $e^{xy}-1$ are continuous everywhere, so $H(x,y)=\dfrac{e^x+e^y}{e^{xy}-1}$ is continuous except where $e^{xy}-1=0 \quad\Rightarrow\quad xy=0 \quad\Rightarrow\quad x=0$ or $y=0$. Thus H is continuous on its domain $\{(x,y)\mid x\neq 0,\ y\neq 0\}$.

45. $\sqrt{x}$ is continuous on its domain $\{(x,y)\mid x\geq 0\}$ and $\sqrt{1-x^2-y^2}$ is continuous on its domain $\{(x,y)\mid 1-x^2-y^2\geq 0\}=\{(x,y)\mid x^2+y^2\leq 1\}$, so the sum $G(x,y)=\sqrt{x}+\sqrt{1-x^2-y^2}$ is continuous for $x\geq 0$ and $x^2+y^2\leq 1$, that is, $\{(x,y)\mid x^2+y^2\leq 1,\ x\geq 0\}$. This is the right half of the unit disk.

46. $G(x,y)=\ln(1+x-y)=g(f(x,y))$ where $f(x,y)=1+x-y$, a polynomial and hence continuous on $\mathbb{R}^2$, and $g(t)=\ln t$, continuous on its domain $\{t\mid t>0\}$. Thus G is continuous on its domain $\{(x,y)\mid 1+x-y>0\}=\{(x,y)\mid y<x+1\}$, the region in $\mathbb{R}^2$ below the line $y=x+1$.

47. $f(x,y,z)=h(g(x,y,z))$ where $g(x,y,z)=x^2+y^2+z^2$, a polynomial that is continuous everywhere, and $h(t)=\arcsin t$, continuous on $[-1,1]$. Thus, f is continuous on its domain $\{(x,y,z)\mid -1\leq x^2+y^2+z^2\leq 1\}=\{(x,y,z)\mid x^2+y^2+z^2\leq 1\}$, so f is continuous on the unit ball.

48. $\sqrt{y-x^2}$ is continuous on its domain $\{(x,y)\mid y-x^2\geq 0\}=\{(x,y)\mid y\geq x^2\}$ and $\ln z$ is continuous on its domain $\{z\mid z>0\}$, so the product $f(x,y,z)=\sqrt{y-x^2}\ln z$ is continuous for $y\geq x^2$ and $z>0$, that is, $\{(x,y,z)\mid y\geq x^2,\ z>0\}$.

49. $f(x,y)=\begin{cases}\dfrac{x^2y^3}{2x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 1 & \text{if } (x,y)=(0,0)\end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on $\mathbb{R}^2$ except possibly at the origin. Since $x^2\leq 2x^2+y^2$, we have $\left|x^2y^3/(2x^2+y^2)\right|\leq\left|y^3\right|$. We know that $\left|y^3\right|\to 0$ as $(x,y)\to(0,0)$. So, by the Squeeze Theorem, $\displaystyle\lim_{(x,y)\to(0,0)}f(x,y)=\lim_{(x,y)\to(0,0)}\frac{x^2y^3}{2x^2+y^2}=0$. But $f(0,0)=1$, so f is discontinuous at $(0,0)$. Therefore, f is continuous on the set $\{(x,y)\mid (x,y)\neq(0,0)\}$.

50. $f(x,y)=\begin{cases}\dfrac{xy}{x^2+xy+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on $\mathbb{R}^2$ except possibly at the origin. $f(x,0)=0/x^2=0$ for $x\neq 0$, so $f(x,y)\to 0$ as

$(x,y) \to (0,0)$ along the x-axis. But $f(x,x) = x^2/(3x^2) = \frac{1}{3}$ for $x \neq 0$, so $f(x,y) \to \frac{1}{3}$ as $(x,y) \to (0,0)$ along the line $y = x$. Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is not continuous at $(0,0)$ and the largest set on which f is continuous is $\{(x,y) \mid (x,y) \neq (0,0)\}$.

51. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0^+} \frac{(r\cos\theta)^3 + (r\sin\theta)^3}{r^2} = \lim_{r\to 0^+} (r\cos^3\theta + r\sin^3\theta) = 0$

52. $\displaystyle \lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim_{r\to 0^+} r^2 \ln r^2 = \lim_{r\to 0^+} \frac{\ln r^2}{1/r^2} = \lim_{r\to 0^+} \frac{(1/r^2)(2r)}{-2/r^3}$ [using l'Hospital's Rule]

$\displaystyle = \lim_{r\to 0^+} (-r^2) = 0$

53. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2} = \lim_{r\to 0^+} \frac{e^{-r^2}-1}{r^2} = \lim_{r\to 0^+} \frac{e^{-r^2}(-2r)}{2r}$ [using l'Hospital's Rule]

$\displaystyle = \lim_{r\to 0^+} -e^{-r^2} = -e^0 = -1$

54. **1.** $\displaystyle \lim_{(x,y)\to(a,b)} x = a$: Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$, then $|x-a| < \varepsilon$.

But $|x-a| \leq \sqrt{(x-a)^2+(y-b)^2} < \delta$, so choose $\varepsilon = \delta$. Then $|x-a| \leq \sqrt{(x-a)^2+(y-b)^2} < \delta = \varepsilon$.

Thus, $\displaystyle \lim_{(x,y)\to(a,b)} x = a$.

2. $\displaystyle \lim_{(x,y)\to(a,b)} y = b$: Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$, then $|y-b| < \varepsilon$.

But $|y-b| \leq \sqrt{(x-a)^2+(y-b)^2} < \delta$, so choose $\varepsilon = \delta$. Then $|y-b| \leq \sqrt{(x-a)^2+(y-b)^2} < \delta = \varepsilon$.

Thus, $\displaystyle \lim_{(x,y)\to(a,b)} y = b$.

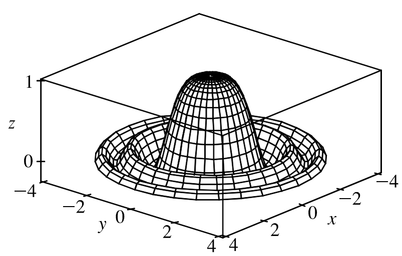
3. $\displaystyle \lim_{(x,y)\to(a,b)} c = c$: Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$, then $|c-c| < \varepsilon$.

But $|c-c| = 0$, so this will be true no matter what δ we choose.

55. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0^+} \frac{\sin(r^2)}{r^2}$, which is an indeterminate form of type $0/0$. Using l'Hospital's Rule, we get

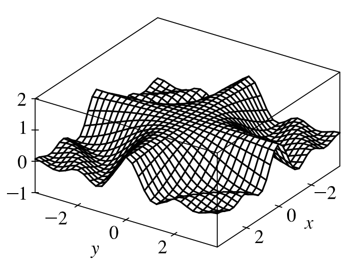
$\displaystyle \lim_{r\to 0^+} \frac{\sin(r^2)}{r^2} \overset{\text{H}}{=} \lim_{r\to 0^+} \frac{2r\cos(r^2)}{2r} = \lim_{r\to 0^+} \cos(r^2) = 1.$

Or: Use the fact that $\displaystyle \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.



56. $f(x,y) = \begin{cases} \dfrac{\sin xy}{xy} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$

From the graph, it appears that f is continuous everywhere. We know xy is continuous on $\mathbb{R}^2$ and $\sin t$ is continuous everywhere, so $\sin xy$ is continuous on $\mathbb{R}^2$ and $\dfrac{\sin xy}{xy}$ is continuous on $\mathbb{R}^2$ except



possibly where $xy = 0$. To show that f is continuous at those points, consider any point (a, b) in $\mathbb{R}^2$ where $ab = 0$. Because xy is continuous, $xy \to ab = 0$ as $(x, y) \to (a, b)$. If we let $t = xy$, then $t \to 0$ as $(x, y) \to (a, b)$ and $\displaystyle\lim_{(x,y)\to(a,b)} \frac{\sin xy}{xy} = \lim_{t\to 0} \frac{\sin t}{t} = 1$ by Equation 3.3.5. Thus $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ and f is continuous on $\mathbb{R}^2$.

57. (a) $f(x,y) = \begin{cases} 0 & \text{if } y \le 0 \quad \text{or} \quad y \ge x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$ Consider the path $y = mx^a$, $0 < a < 4$. [The path does not pass through $(0, 0)$ if $a \le 0$ except for the trivial case where $m = 0$.] If $mx^a \le 0$, then $f(x, mx^a) = 0$. If $mx^a > 0$, then $mx^a = |mx^a| = |m|\ |x^a|$ and $mx^a \ge x^4 \quad \Leftrightarrow \quad |m|\ |x^a| \ge x^4 \quad \Leftrightarrow \quad \dfrac{x^4}{|x^a|} \le |m| \quad \Leftrightarrow \quad |x|^{4-a} \le |m|$ whenever x^a is defined. Then $mx^a \ge x^4 \quad \Leftrightarrow \quad |x| \le |m|^{1/(4-a)}$ so $f(x, mx^a) = 0$ for $|x| \le |m|^{1/(4-a)}$ and $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along this path.

(b) If we approach $(0, 0)$ along the path $y = x^5$, $x > 0$, then we have $f(x, x^5) = 1$ for $0 < x < 1$ because $0 < x^5 < x^4$ there. Thus, $f(x, y) \to 1$ as $(x, y) \to (0, 0)$ along this path, but in part (a) we found a limit of 0 along other paths, so $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist and f is discontinuous at $(0, 0)$.

(c) First we show that f is discontinuous at any point $(a, 0)$ on the x-axis. If we approach $(a, 0)$ along the path $x = a$, $y > 0$, then $f(a, y) = 1$ for $0 < y < a^4$, so $f(x, y) \to 1$ as $(x, y) \to (a, 0)$ along this path. If we approach $(a, 0)$ along the path $x = a$, $y < 0$, then $f(a, y) = 0$ since $y < 0$ and $f(x, y) \to 0$ as $(x, y) \to (a, 0)$. Thus, the limit does not exist and f is discontinuous on the line $y = 0$. f is also discontinuous on the curve $y = x^4$: For any point (a, a^4) on this curve, approaching the point along the path $x = a$, $y > a^4$ gives $f(a, y) = 0$ since $y > a^4$, so $f(x, y) \to 0$ as $(x, y) \to (a, a^4)$. But approaching the point along the path $x = a$, $y < a^4$ gives $f(a, y) = 1$ for $y > 0$, so $f(x, y) \to 1$ as $(x, y) \to (a, a^4)$ and the limit does not exist there.

58. Since $|\mathbf{x} - \mathbf{a}|^2 = |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2\,|\mathbf{x}|\,|\mathbf{a}|\cos\theta \ge |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2\,|\mathbf{x}|\,|\mathbf{a}| = (|\mathbf{x}| - |\mathbf{a}|)^2$, we have $\big||\mathbf{x}| - |\mathbf{a}|\big| \le |\mathbf{x} - \mathbf{a}|$. Let $\varepsilon > 0$ be given and set $\delta = \varepsilon$. Then if $0 < |\mathbf{x} - \mathbf{a}| < \delta$, $\big||\mathbf{x}| - |\mathbf{a}|\big| \le |\mathbf{x} - \mathbf{a}| < \delta = \varepsilon$. Hence $\lim_{\mathbf{x}\to\mathbf{a}} |\mathbf{x}| = |\mathbf{a}|$ and $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$.

59. $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that if $0 < |\mathbf{x} - \mathbf{a}| < \delta$, then $|f(\mathbf{x}) - f(\mathbf{a})| = |\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| < \varepsilon$. But $|\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| = |\mathbf{c} \cdot (\mathbf{x} - \mathbf{a})|$ and $|\mathbf{c} \cdot (\mathbf{x} - \mathbf{a})| \le |\mathbf{c}|\,|\mathbf{x} - \mathbf{a}|$ by Exercise 12.3.61 (the Cauchy-Schwartz Inequality). Set $\delta = \varepsilon / |\mathbf{c}|$. Then if $0 < |\mathbf{x} - \mathbf{a}| < \delta$,
$|f(\mathbf{x}) - f(\mathbf{a})| = |\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| \le |\mathbf{c}|\,|\mathbf{x} - \mathbf{a}| < |\mathbf{c}|\,\delta = |\mathbf{c}|\,(\varepsilon / |\mathbf{c}|) = \varepsilon$. So f is continuous on $\mathbb{R}^n$.

14.3 Partial Derivatives

1. By Definition 4, $f_T(92, 60) = \lim\limits_{h \to 0} \dfrac{f(92+h, 60) - f(92, 60)}{h}$, which we can approximate by considering $h = 2$ and $h = -2$ and using the values given in Table 1: $f_T(92, 60) \approx \dfrac{f(94, 60) - f(92, 60)}{2} = \dfrac{111 - 105}{2} = 3$, $f_T(92, 60) \approx \dfrac{f(90, 60) - f(92, 60)}{-2} = \dfrac{100 - 105}{-2} = 2.5$. Averaging these values, we estimate $f_T\,(92, 60)$ to be approximately 2.75. Thus, when the actual temperature is $92°\text{F}$ and the relative humidity is 60%, the apparent temperature rises by about $2.75°\text{F}$ for every degree that the actual temperature rises.

Similarly, $f_H(92, 60) = \lim\limits_{h \to 0} \dfrac{f(92, 60+h) - f(92, 60)}{h}$ which we can approximate by considering $h = 5$ and $h = -5$: $f_H(92, 60) \approx \dfrac{f(92, 65) - f(92, 60)}{5} = \dfrac{108 - 105}{5} = 0.6$, $f_H(92, 60) \approx \dfrac{f(92, 55) - f(92, 60)}{-5} = \dfrac{103 - 105}{-5} = 0.4$. Averaging these values, we estimate $f_H(92, 60)$ to be approximately 0.5. Thus, when the actual temperature is $92°\text{F}$ and the relative humidity is 60%, the apparent temperature rises by about $0.5°\text{F}$ for every percent that the relative humidity increases.

2. (a) $\partial h/\partial v$ represents the rate of change of h when we fix t and consider h as a function of v, which describes how quickly the wave heights change when the wind speed changes for fixed time duration. $\partial h/\partial v$ represents the rate of change of h when we fix v and consider h as a function of t, which describes how quickly the wave heights change when the duration of time changes, but the wind speed is constant.

(b) By Definition 4, $f_v(40, 15) = lim_{h \to 0} \frac{f(40+h,15) - f(40,15)}{h}$ which we can approximate by considering $h = 20$ and $h = -20$ and using the values given in the table: $f_v(40, 15) \approx \frac{f(60,15) - f(40,15)}{20} = \frac{4.9 - 2.4}{20} = 0.125$, $f_v(40, 15) \approx \frac{f(20,15) - f(40,15)}{-20} = \frac{0.6 - 2.4}{-20} = 0.09$. Averaging these values, we have $f_v(40, 15) \approx 0.1075$. Thus, when a 40 km/h wind has been blowing for 15 hours, the wave heights should increase by about 0.1075 km/h for every km/h that the wind speed increases (with the same time duration). Similarly, $f_t(40, 15) = lim_{h \to 0} \frac{f(40,15+h) - f(40,15)}{h}$, which we can approximate by considering $h = 5$ and $h = -5$: $f_t(40, 15) \approx \frac{f(40,20) - f(40,15)}{5} = \frac{2.5 - 2.4}{5} = 0.02$, $f_t(40, 15) \approx \frac{f(40,10) - f(40,15)}{-5} = \frac{2.2 - 2.4}{-5} = 0.04$. Averaging these values, we have $f_t(40, 15) \approx 0.03$. Thus, when a 40 km/h wind has been blowing for 15 hours, the wave heights increase by about 0.03 meters for every additional hour that the wind blows.

(c) For fixed values of v, the function values $f(v, t)$ appear to increase in smaller and smaller increments, becoming nearly constant as t increases. Thus, the corresponding rate of change is nearly 0 as t increases, suggesting that $lim_{t \to \infty}(\partial h/\partial t) = 0$.

3. (a) $\partial T/\partial x$ represents the rate of change of T when we fix y and t and consider T as a function of the single variable x, which describes how quickly the temperature changes when longitude changes but latitude and time are constant. $\partial T/\partial y$ represents the rate of change of T when we fix x and t and consider T as a function of y, which describes how quickly the temperature changes when latitude changes but longitude and time are constant. $\partial T/\partial t$ represents the rate of change of T when we fix x and y and consider T as a function of t, which describes how quickly the temperature changes over time for a constant longitude and latitude.

(b) $f_x(158, 21, 9)$ represents the rate of change of temperature at longitude $158°$W, latitude $21°$N at 9:00 AM when only longitude varies. Since the air is warmer to the west than to the east, increasing longitude results in an increased air temperature, so we would expect $f_x(158, 21, 9)$ to be positive. $f_y(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only latitude varies. Since the air is warmer to the south and cooler to the north, increasing latitude results in a decreased air temperature, so we would expect $f_y(158, 21, 9)$ to be negative. $f_t(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only time varies. Since typically air temperature increases from the morning to the afternoon as the sun warms it, we would expect $f_t(158, 21, 9)$ to be positive.

4. (a) If we start at $(1, 2)$ and move in the positive x-direction, the graph of f increases. Thus $f_x(1, 2)$ is positive.

(b) If we start at $(1, 2)$ and move in the positive y-direction, the graph of f decreases. Thus $f_y(1, 2)$ is negative.

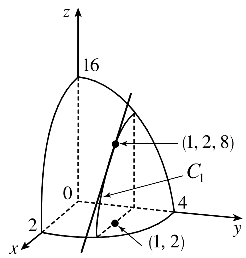
5. (a) The graph of f decreases if we start at $(-1, 2)$ and move in the positive x-direction, so $f_x(-1, 2)$ is negative.

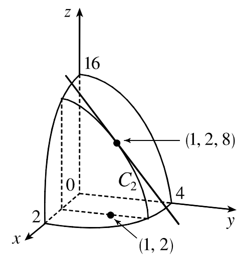
(b) The graph of f decreases if we start at $(-1, 2)$ and move in the positive y-direction, so $f_y(-1, 2)$ is negative.

6. $f_x(2, 1)$ is the rate of change of f at $(2, 1)$ in the x-direction. If we start at $(2, 1)$, where $f(2, 1) = 10$, and move in the positive x-direction, we reach the next contour line [where $f(x, y) = 12$] after approximately 0.6 units. This represents an average rate of change of about $\frac{2}{0.6}$. If we approach the point $(2, 1)$ from the left (moving in the positive x-direction) the output values increase from 8 to 10 with an increase in x of approximately 0.9 units, corresponding to an average rate of change of $\frac{2}{0.9}$. A good estimate for $f_x(2, 1)$ would be the average of these two, so $f_x(2, 1) \approx 2.8$. Similarly, $f_y(2, 1)$ is the rate of change of f at $(2, 1)$ in the y-direction. If we approach $(2, 1)$ from below, the output values decrease from 12 to 10 with a change in y of approximately 1 unit, corresponding to an average rate of change of -2. If we start at $(2, 1)$ and move in the positive y-direction, the output values decrease from 10 to 8 after approximately 0.9 units, a rate of change of $\frac{-2}{0.9}$. Averaging these two results, we estimate $f_y(2, 1) \approx -2.1$.

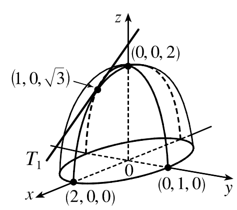
7. $f(x, y) = 16 - 4x^2 - y^2 \quad \Rightarrow \quad f_x(x, y) = -8x$ and $f_y(x, y) = -2y \quad \Rightarrow \quad f_x(1, 2) = -8$ and $f_y(1, 2) = -4$. The graph of f is the paraboloid $z = 16 - 4x^2 - y^2$ and the vertical plane $y = 2$ intersects it in the parabola $z = 12 - 4x^2$, $y = 2$

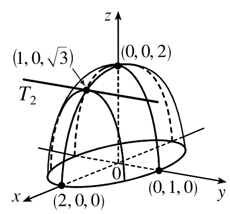
(the curve C_1 in the first figure). The slope of the tangent line to this parabola at $(1, 2, 8)$ is $f_x(1, 2) = -8$. Similarly the plane $x = 1$ intersects the paraboloid in the parabola $z = 12 - y^2$, $x = 1$ (the curve C_2 in the second figure) and the slope of the tangent line at $(1, 2, 8)$ is $f_y(1, 2) = -4$.





8. $f(x, y) = (4 - x^2 - 4y^2)^{1/2} \;\Rightarrow\; f_x(x, y) = -x(4 - x^2 - 4y^2)^{-1/2}$ and $f_y(x, y) = -4y(4 - x^2 - 4y^2)^{-1/2} \;\Rightarrow$ $f_x(1, 0) = -\frac{1}{\sqrt{3}}$, $f_y(1, 0) = 0$. The graph of f is the upper half of the ellipsoid $z^2 + x^2 + 4y^2 = 4$ and the plane $y = 0$ intersects the graph in the semicircle $x^2 + z^2 = 4$, $z \geq 0$ and the slope of the tangent line T_1 to this semicircle at $\left(1, 0, \sqrt{3}\right)$ is $f_x(1, 0) = -\frac{1}{\sqrt{3}}$. Similarly the plane $x = 1$ intersects the graph in the semi-ellipse $z^2 + 4y^2 = 3$, $z \geq 0$ and the slope of the tangent line T_2 to this semi-ellipse at $\left(1, 0, \sqrt{3}\right)$ is $f_y(1, 0) = 0$.





9. $f(x, y) = x^4 + 5xy^3 \;\Rightarrow\; f_x(x, y) = 4x^3 + 5y^3,\; f_y(x, y) = 0 + 5x \cdot 3y^2 = 15xy^2$

10. $f(x, y) = x^2y - 3y^4 \;\Rightarrow\; f_x(x, y) = 2x \cdot y - 0 = 2xy,\; f_y(x, y) = x^2 \cdot 1 - 3 \cdot 4y^3 = x^2 - 12y^3$

11. $g(x, y) = x^3 \sin y \;\Rightarrow\; g_x(x, y) = 3x^2 \sin y,\; g_y(x, y) = x^3 \cos y$

12. $g(x, t) = e^{xt} \;\Rightarrow\; g_x(x, t) = e^{xt} \cdot t = te^{xt},\; g_t(x, t) = e^{xt} \cdot x = xe^{xt}$

13. $z = \ln(x + t^2) \;\Rightarrow\; \dfrac{\partial z}{\partial x} = \dfrac{1}{x + t^2} \cdot 1 = \dfrac{1}{x + t^2},\; \dfrac{\partial z}{\partial t} = \dfrac{1}{x + t^2} \cdot 2t = \dfrac{2t}{x + t^2}$

14. $w = \dfrac{u}{v^2} \;\Rightarrow\; w_u = \dfrac{1}{v^2} \cdot 1 = \dfrac{1}{v^2},\; w_v = u \cdot (-2v^{-3}) = -\dfrac{2u}{v^3}$

15. $f(x, y) = ye^{xy} \;\Rightarrow\; f_x(x, y) = y \cdot e^{xy} \cdot y = y^2e^{xy},\; f_y(x, y) = y \cdot e^{xy} \cdot x + e^{xy} \cdot 1 = e^{xy} + xye^{xy}$

16. $g(x, y) = (x^2 + xy)^3 \;\Rightarrow\; g_x(x, y) = 3(x^2 + xy)^2(2x + y),\; g_y(x, y) = 3(x^2 + xy)^2(0 + x) = 3x(x^2 + xy)^2$

17. $g(x, y) = y(x + x^2y)^5 \;\Rightarrow\; g_x(x, y) = 5y(x + x^2y)^4(1 + 2xy)$,

$g_y(x, y) = y \cdot 5(x + x^2y)^4 \cdot x^2 + (x + x^2y)^5 \cdot 1 = 5x^2y(x + x^2y)^4 + (x + x^2y)^5$

18. $f(x, y) = \dfrac{x}{(x + y)^2} \;\Rightarrow\; f_x(x, y) = \dfrac{(x + y)^2(1) - (x)(2)(x + y)}{[(x + y)^2]^2} = \dfrac{x + y - 2x}{(x + y)^3} = \dfrac{y - x}{(x + y)^3}$,

$f_y(x, y) = \dfrac{(x + y)^2(0) - (x)(2)(x + y)}{[(x + y)^2]^2} = -\dfrac{2x}{(x + y)^3}$

19. $f(x,y) = \dfrac{ax+by}{cx+dy} \quad \Rightarrow \quad f_x(x,y) = \dfrac{(cx+dy)(a)-(ax+by)(c)}{(cx+dy)^2} = \dfrac{(ad-bc)y}{(cx+dy)^2},$

$f_y(x,y) = \dfrac{(cx+dy)(b)-(ax+by)(d)}{(cx+dy)^2} = \dfrac{(bc-ad)x}{(cx+dy)^2}$

20. $w = \dfrac{e^v}{u+v^2} \quad \Rightarrow \quad \dfrac{\partial w}{\partial u} = \dfrac{0(u+v^2)-e^v(1)}{(u+v^2)^2} = -\dfrac{e^v}{(u+v^2)^2}, \quad \dfrac{\partial w}{\partial v} = \dfrac{e^v(u+v^2)-e^v(2v)}{(u+v^2)^2} = \dfrac{e^v(u+v^2-2v)}{(u+v^2)^2}$

21. $g(u,v) = (u^2v-v^3)^5 \quad \Rightarrow \quad g_u(u,v) = 5(u^2v-v^3)^4 \cdot 2uv = 10uv(u^2v-v^3)^4,$

$g_v(u,v) = 5(u^2v-v^3)^4(u^2-3v^2) = 5(u^2-3v^2)(u^2v-v^3)^4$

22. $u(r,\theta) = \sin(r\cos\theta) \quad \Rightarrow \quad u_r(r,\theta) = \cos(r\cos\theta)\cdot\cos\theta = \cos\theta\cos(r\cos\theta),$

$u_\theta(r,\theta) = \cos(r\cos\theta)(-r\sin\theta) = -r\sin\theta\cos(r\cos\theta)$

23. $R(p,q) = \tan^{-1}(pq^2) \quad \Rightarrow \quad R_p(p,q) = \dfrac{1}{1+(pq^2)^2}\cdot q^2 = \dfrac{q^2}{1+p^2q^4}, \quad R_q(p,q) = \dfrac{1}{1+(pq^2)^2}\cdot 2pq = \dfrac{2pq}{1+p^2q^4}$

24. $f(x,y) = x^y \quad \Rightarrow \quad f_x(x,y) = yx^{y-1}, \; f_y(x,y) = x^y\ln x$

25. $F(x,y) = \displaystyle\int_y^x \cos(e^t)\,dt \quad \Rightarrow \quad F_x(x,y) = \dfrac{\partial}{\partial x}\int_y^x \cos(e^t)\,dt = \cos(e^x)$ by the Fundamental Theorem of Calculus, Part 1;

$F_y(x,y) = \dfrac{\partial}{\partial y}\displaystyle\int_y^x \cos(e^t)\,dt = \dfrac{\partial}{\partial y}\left[-\int_x^y \cos(e^t)\,dt\right] = -\dfrac{\partial}{\partial y}\int_x^y \cos(e^t)\,dt = -\cos(e^y).$

26. $F(\alpha,\beta) = \displaystyle\int_\alpha^\beta \sqrt{t^3+1}\,dt \quad \Rightarrow$

$F_\alpha(\alpha,\beta) = \dfrac{\partial}{\partial\alpha}\displaystyle\int_\alpha^\beta \sqrt{t^3+1}\,dt = \dfrac{\partial}{\partial\alpha}\left[-\int_\beta^\alpha \sqrt{t^3+1}\,dt\right] = -\dfrac{\partial}{\partial\alpha}\int_\beta^\alpha \sqrt{t^3+1}\,dt = -\sqrt{\alpha^3+1}$ by the Fundamental Theorem of Calculus, Part 1; $\quad F_\beta(\alpha,\beta) = \dfrac{\partial}{\partial\beta}\displaystyle\int_\alpha^\beta \sqrt{t^3+1}\,dt = \sqrt{\beta^3+1}.$

27. $f(x,y,z) = x^3yz^2+2yz \quad \Rightarrow \quad f_x(x,y,z) = 3x^2yz^2, \; f_y(x,y,z) = x^3z^2+2z, \; f_z(x,y,z) = 2x^3yz+2y$

28. $f(x,y,z) = xy^2e^{-xz} \quad \Rightarrow \quad f_x(x,y,z) = y^2\left[x\cdot e^{-xz}(-z)+e^{-xz}\cdot 1\right] = (1-xz)y^2e^{-xz}, \quad f_y(x,y,z) = 2xye^{-xz},$

$f_z(x,y,z) = xy^2e^{-xz}(-x) = -x^2y^2e^{-xz}$

29. $w = \ln(x+2y+3z) \quad \Rightarrow \quad \dfrac{\partial w}{\partial x} = \dfrac{1}{x+2y+3z}, \quad \dfrac{\partial w}{\partial y} = \dfrac{2}{x+2y+3z}, \quad \dfrac{\partial w}{\partial z} = \dfrac{3}{x+2y+3z}$

30. $w = y\tan(x+2z) \quad \Rightarrow \quad \dfrac{\partial w}{\partial x} = y\left[\sec^2(x+2z)\right](1) = y\sec^2(x+2z), \quad \dfrac{\partial w}{\partial y} = \tan(x+2z),$

$\dfrac{\partial w}{\partial z} = y\left[\sec^2(x+2z)\right](2) = 2y\sec^2(x+2z)$

31. $p = \sqrt{t^4+u^2\cos v} \quad \Rightarrow \quad \dfrac{\partial p}{\partial t} = \frac{1}{2}(t^4+u^2\cos v)^{-1/2}(4t^3) = \dfrac{2t^3}{\sqrt{t^4+u^2\cos v}},$

$\dfrac{\partial p}{\partial u} = \frac{1}{2}(t^4+u^2\cos v)^{-1/2}(2u\cos v) = \dfrac{u\cos v}{\sqrt{t^4+u^2\cos v}}, \quad \dfrac{\partial p}{\partial v} = \frac{1}{2}(t^4+u^2\cos v)^{-1/2}[u^2(-\sin v)] = -\dfrac{u^2\sin v}{2\sqrt{t^4+u^2\cos v}}$

32. $u = x^{y/z} \quad \Rightarrow \quad u_x = \dfrac{y}{z}\,x^{(y/z)-1},\; u_y = x^{y/z}\ln x \cdot \dfrac{1}{z} = \dfrac{x^{y/z}}{z}\ln x,\;\; u_z = x^{y/z}\ln x \cdot \dfrac{-y}{z^2} = -\dfrac{yx^{y/z}}{z^2}\ln x$

33. $h(x,y,z,t) = x^2 y\cos(z/t) \quad \Rightarrow \quad h_x(x,y,z,t) = 2xy\cos(z/t),\; h_y(x,y,z,t) = x^2\cos(z/t),$

$h_z(x,y,z,t) = -x^2 y\sin(z/t)(1/t) = (-x^2 y/t)\sin(z/t),\; h_t(x,y,z,t) = -x^2 y\sin(z/t)(-zt^{-2}) = (x^2 yz/t^2)\sin(z/t)$

34. $\phi(x,y,z,t) = \dfrac{\alpha x + \beta y^2}{\gamma z + \delta t^2} \quad \Rightarrow \quad \phi_x(x,y,z,t) = \dfrac{1}{\gamma z + \delta t^2}(\alpha) = \dfrac{\alpha}{\gamma z + \delta t^2},$

$\phi_y(x,y,z,t) = \dfrac{1}{\gamma z + \delta t^2}(2\beta y) = \dfrac{2\beta y}{\gamma z + \delta t^2},\; \phi_z(x,y,z,t) = \dfrac{(\gamma z + \delta t^2)(0) - (\alpha x + \beta y^2)(\gamma)}{(\gamma z + \delta t^2)^2} = \dfrac{-\gamma(\alpha x + \beta y^2)}{(\gamma z + \delta t^2)^2},$

$\phi_t(x,y,z,t) = \dfrac{(\gamma z + \delta t^2)(0) - (\alpha x + \beta y^2)(2\delta t)}{(\gamma z + \delta t^2)^2} = -\dfrac{2\delta t(\alpha x + \beta y^2)}{(\gamma z + \delta t^2)^2}$

35. $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. For each $i = 1, \ldots, n$, $u_{x_i} = \frac{1}{2}\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{-1/2}(2x_i) = \dfrac{x_i}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}$.

36. $u = \sin(x_1 + 2x_2 + \cdots + nx_n)$. For each $i = 1, \ldots, n$, $u_{x_i} = i\cos(x_1 + 2x_2 + \cdots + nx_n)$.

37. $R(s,t) = te^{s/t} \quad \Rightarrow \quad R_t(s,t) = t \cdot e^{s/t}(-s/t^2) + e^{s/t} \cdot 1 = \left(1 - \dfrac{s}{t}\right)e^{s/t}$, so $R_t(0,1) = \left(1 - \dfrac{0}{1}\right)e^{0/1} = 1$.

38. $f(x,y) = y\sin^{-1}(xy) \quad \Rightarrow \quad f_y(x,y) = y \cdot \dfrac{1}{\sqrt{1-(xy)^2}}(x) + \sin^{-1}(xy) \cdot 1 = \dfrac{xy}{\sqrt{1 - x^2y^2}} + \sin^{-1}(xy),$

so $f_y\left(1, \frac{1}{2}\right) = \dfrac{1 \cdot \frac{1}{2}}{\sqrt{1 - 1^2\left(\frac{1}{2}\right)^2}} + \sin^{-1}\left(1 \cdot \frac{1}{2}\right) = \dfrac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} + \sin^{-1}\frac{1}{2} = \frac{1}{\sqrt{3}} + \frac{\pi}{6}$.

39. $f(x,y,z) = \ln\dfrac{1 - \sqrt{x^2+y^2+z^2}}{1 + \sqrt{x^2+y^2+z^2}} \quad \Rightarrow$

$$f_y(x,y,z) = \frac{1}{\dfrac{1 - \sqrt{x^2+y^2+z^2}}{1 + \sqrt{x^2+y^2+z^2}}} \cdot \frac{\left(1 + \sqrt{x^2+y^2+z^2}\right)\left(-\frac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2y\right) - \left(1 - \sqrt{x^2+y^2+z^2}\right)\left(\frac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2y\right)}{\left(1 + \sqrt{x^2+y^2+z^2}\right)^2}$$

$$= \frac{1 + \sqrt{x^2+y^2+z^2}}{1 - \sqrt{x^2+y^2+z^2}} \cdot \frac{-y(x^2+y^2+z^2)^{-1/2}\left(1 + \sqrt{x^2+y^2+z^2} + 1 - \sqrt{x^2+y^2+z^2}\right)}{\left(1 + \sqrt{x^2+y^2+z^2}\right)^2}$$

$$= \frac{-y(x^2+y^2+z^2)^{-1/2}\,(2)}{\left(1 - \sqrt{x^2+y^2+z^2}\right)\left(1 + \sqrt{x^2+y^2+z^2}\right)} = \frac{-2y}{\sqrt{x^2+y^2+z^2}\,[1 - (x^2+y^2+z^2)]}$$

so $f_y(1,2,2) = \dfrac{-2(2)}{\sqrt{1^2+2^2+2^2}\,[1 - (1^2+2^2+2^2)]} = \dfrac{-4}{\sqrt{9}\,(1-9)} = \dfrac{1}{6}$.

40. $f(x,y,z) = x^{yz} \;\Rightarrow\; f_z(x,y,z) = (x^{yz}\ln x)(y) = yx^{yz}\ln x$, so $f_z(e,1,0) = 1e^{(1)(0)}\ln e = 1$.

41. $x^2 + 2y^2 + 3z^2 = 1 \;\Rightarrow\; \dfrac{\partial}{\partial x}\left(x^2 + 2y^2 + 3z^2\right) = \dfrac{\partial}{\partial x}(1) \;\Rightarrow\; 2x + 0 + 6z\,\dfrac{\partial z}{\partial x} = 0 \;\Rightarrow\; 6z\,\dfrac{\partial z}{\partial x} = -2x \;\Rightarrow$

$\dfrac{\partial z}{\partial x} = \dfrac{-2x}{6z} = -\dfrac{x}{3z}$, and $\dfrac{\partial}{\partial y}\left(x^2 + 2y^2 + 3z^2\right) = \dfrac{\partial}{\partial y}(1) \;\Rightarrow\; 0 + 4y + 6z\,\dfrac{\partial z}{\partial y} = 0 \;\Rightarrow\; 6z\,\dfrac{\partial z}{\partial y} = -4y \;\Rightarrow$

$\dfrac{\partial z}{\partial y} = \dfrac{-4y}{6z} = -\dfrac{2y}{3z}$.

42. $x^2 - y^2 + z^2 - 2z = 4 \;\Rightarrow\; \dfrac{\partial}{\partial x}\left(x^2 - y^2 + z^2 - 2z\right) = \dfrac{\partial}{\partial x}(4) \;\Rightarrow\; 2x - 0 + 2z\,\dfrac{\partial z}{\partial x} - 2\,\dfrac{\partial z}{\partial x} = 0 \;\Rightarrow$

$(2z-2)\,\dfrac{\partial z}{\partial x} = -2x \;\Rightarrow\; \dfrac{\partial z}{\partial x} = \dfrac{-2x}{2z-2} = \dfrac{x}{1-z}$, and $\dfrac{\partial}{\partial y}\left(x^2 - y^2 + z^2 - 2z\right) = \dfrac{\partial}{\partial y}(4) \;\Rightarrow$

$0 - 2y + 2z\,\dfrac{\partial z}{\partial y} - 2\,\dfrac{\partial z}{\partial y} = 0 \;\Rightarrow\; (2z-2)\,\dfrac{\partial z}{\partial y} = 2y \;\Rightarrow\; \dfrac{\partial z}{\partial y} = \dfrac{2y}{2z-2} = \dfrac{y}{z-1}$.

43. $e^z = xyz \;\Rightarrow\; \dfrac{\partial}{\partial x}(e^z) = \dfrac{\partial}{\partial x}(xyz) \;\Rightarrow\; e^z\,\dfrac{\partial z}{\partial x} = y\left(x\,\dfrac{\partial z}{\partial x} + z\cdot 1\right) \;\Rightarrow\; e^z\,\dfrac{\partial z}{\partial x} - xy\,\dfrac{\partial z}{\partial x} = yz \;\Rightarrow$

$(e^z - xy)\,\dfrac{\partial z}{\partial x} = yz$, so $\dfrac{\partial z}{\partial x} = \dfrac{yz}{e^z - xy}$.

$\dfrac{\partial}{\partial y}(e^z) = \dfrac{\partial}{\partial y}(xyz) \;\Rightarrow\; e^z\,\dfrac{\partial z}{\partial y} = x\left(y\,\dfrac{\partial z}{\partial x} + z\cdot 1\right) \;\Rightarrow\; e^z\,\dfrac{\partial z}{\partial y} - xy\,\dfrac{\partial z}{\partial y} = xz \;\Rightarrow\; (e^z - xy)\,\dfrac{\partial z}{\partial y} = xz$, so

$\dfrac{\partial z}{\partial y} = \dfrac{xz}{e^z - xy}$.

44. $yz + x\ln y = z^2 \;\Rightarrow\; \dfrac{\partial}{\partial x}(yz + x\ln y) = \dfrac{\partial}{\partial x}(z^2) \;\Rightarrow\; y\,\dfrac{\partial z}{\partial x} + \ln y = 2z\,\dfrac{\partial z}{\partial x} \;\Rightarrow\; \ln y = 2z\,\dfrac{\partial z}{\partial x} - y\,\dfrac{\partial z}{\partial x} \;\Rightarrow$

$\ln y = (2z - y)\,\dfrac{\partial z}{\partial x}$, so $\dfrac{\partial z}{\partial x} = \dfrac{\ln y}{2z - y}$.

$\dfrac{\partial}{\partial y}(yz + x\ln y) = \dfrac{\partial}{\partial y}(z^2) \;\Rightarrow\; y\,\dfrac{\partial z}{\partial y} + z\cdot 1 + x\cdot\dfrac{1}{y} = 2z\,\dfrac{\partial z}{\partial y} \;\Rightarrow\; z + \dfrac{x}{y} = 2z\,\dfrac{\partial z}{\partial y} - y\,\dfrac{\partial z}{\partial y} \;\Rightarrow$

$z + \dfrac{x}{y} = (2z - y)\,\dfrac{\partial z}{\partial y}$, so $\dfrac{\partial z}{\partial y} = \dfrac{z + (x/y)}{2z - y} = \dfrac{x + yz}{y(2z - y)}$.

45. (a) $z = f(x) + g(y) \;\Rightarrow\; \dfrac{\partial z}{\partial x} = f'(x), \quad \dfrac{\partial z}{\partial y} = g'(y)$

(b) $z = f(x+y)$. Let $u = x + y$. Then $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = \dfrac{df}{du}(1) = f'(u) = f'(x+y)$,

$\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = \dfrac{df}{du}(1) = f'(u) = f'(x+y)$.

46. (a) $z = f(x)g(y) \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = f'(x)g(y), \quad \dfrac{\partial z}{\partial y} = f(x)g'(y)$

(b) $z = f(xy)$. Let $u = xy$. Then $\dfrac{\partial u}{\partial x} = y$ and $\dfrac{\partial u}{\partial y} = x$. Hence $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = \dfrac{df}{du} \cdot y = yf'(u) = yf'(xy)$

and $\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = \dfrac{df}{du} \cdot x = xf'(u) = xf'(xy)$.

(c) $z = f\left(\dfrac{x}{y}\right)$. Let $u = \dfrac{x}{y}$. Then $\dfrac{\partial u}{\partial x} = \dfrac{1}{y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{x}{y^2}$. Hence $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = f'(u)\dfrac{1}{y} = \dfrac{f'(x/y)}{y}$

and $\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = f'(u)\left(-\dfrac{x}{y^2}\right) = -\dfrac{xf'(x/y)}{y^2}$.

47. $f(x,y) = x^4y - 2x^3y^2 \quad \Rightarrow \quad f_x(x,y) = 4x^3y - 6x^2y^2,\;\; f_y(x,y) = x^4 - 4x^3y$. Then $f_{xx}(x,y) = 12x^2y - 12xy^2$,

$f_{xy}(x,y) = 4x^3 - 12x^2y,\;\; f_{yx}(x,y) = 4x^3 - 12x^2y$, and $f_{yy}(x,y) = -4x^3$.

48. $f(x,y) = \ln(ax+by) \quad \Rightarrow \quad f_x(x,y) = \dfrac{a}{ax+by} = a(ax+by)^{-1},\;\; f_y(x,y) = \dfrac{b}{ax+by} = b(ax+by)^{-1}$. Then

$f_{xx}(x,y) = -a(ax+by)^{-2}(a) = -\dfrac{a^2}{(ax+by)^2},\;\; f_{xy}(x,y) = -a(ax+by)^{-2}(b) = -\dfrac{ab}{(ax+by)^2}$,

$f_{yx}(x,y) = -b(ax+by)^{-2}(a) = -\dfrac{ab}{(ax+by)^2}$, and $f_{yy}(x,y) = -b(ax+by)^{-2}(b) = -\dfrac{b^2}{(ax+by)^2}$.

49. $z = \dfrac{y}{2x+3y} = y(2x+3y)^{-1} \quad \Rightarrow \quad z_x = y(-1)(2x+3y)^{-2}(2) = -\dfrac{2y}{(2x+3y)^2}$,

$z_y = \dfrac{(2x+3y)\cdot 1 - y\cdot 3}{(2x+3y)^2} = \dfrac{2x}{(2x+3y)^2}$. Then $z_{xx} = -2y(-2)(2x+3y)^{-3}(2) = \dfrac{8y}{(2x+3y)^3}$,

$z_{xy} = -\dfrac{(2x+3y)^2\cdot 2 - 2y\cdot 2(2x+3y)(3)}{[(2x+3y)^2]^2} = -\dfrac{(2x+3y)\,(4x+6y-12y)}{(2x+3y)^4} = \dfrac{6y-4x}{(2x+3y)^3}$,

$z_{yx} = \dfrac{(2x+3y)^2\cdot 2 - 2x\cdot 2(2x+3y)(2)}{[(2x+3y)^2]^2} = \dfrac{6y-4x}{(2x+3y)^3},\;\; z_{yy} = 2x(-2)(2x+3y)^{-3}(3) = -\dfrac{12x}{(2x+3y)^3}$.

50. $T = e^{-2r}\cos\theta \quad \Rightarrow \quad T_r = -2e^{-2r}\cos\theta,\;\; T_\theta = -e^{-2r}\sin\theta$. Then $T_{rr} = -2e^{-2r}(-2)\cos\theta = 4e^{-2r}\cos\theta$,

$T_{r\theta} = 2e^{-2r}\sin\theta,\;\; T_{\theta r} = -e^{-2r}(-2)\sin\theta = 2e^{-2r}\sin\theta,\;\; T_{\theta\theta} = -e^{-2r}\cos\theta$.

51. $v = \sin(s^2 - t^2) \quad \Rightarrow \quad v_s = \cos(s^2-t^2)\cdot 2s = 2s\cos(s^2-t^2),\;\; v_t = \cos(s^2-t^2)\cdot(-2t) = -2t\cos(s^2-t^2)$. Then

$v_{ss} = 2s\big[-\sin(s^2-t^2)\cdot 2s\big] + \cos(s^2-t^2)\cdot 2 = 2\cos(s^2-t^2) - 4s^2\sin(s^2-t^2)$,

$v_{st} = 2s\big[-\sin(s^2-t^2)\cdot(-2t)\big] + \cos(s^2-t^2)\cdot 0 = 4st\sin(s^2-t^2)$,

$v_{ts} = -2t\big[-\sin(s^2-t^2)\cdot 2s\big] + \cos(s^2-t^2)\cdot 0 = 4st\sin(s^2-t^2)$,

$v_{tt} = -2t\big[-\sin(s^2-t^2)\cdot(-2t)\big] + \cos(s^2-t^2)\cdot(-2) = -2\cos(s^2-t^2) - 4t^2\sin(s^2-t^2)$.

52. $z = \arctan \dfrac{x+y}{1-xy} \quad \Rightarrow$

$$z_x = \frac{1}{1+\left(\dfrac{x+y}{1-xy}\right)^2} \cdot \frac{(1-xy)(1)-(x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2+(x+y)^2} = \frac{1+y^2}{1+x^2+y^2+x^2y^2}$$

$$= \frac{1+y^2}{(1+x^2)(1+y^2)} = \frac{1}{1+x^2},$$

$$z_y = \frac{1}{1+\left(\dfrac{x+y}{1-xy}\right)^2} \cdot \frac{(1-xy)(1)-(x+y)(-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2+(x+y)^2} = \frac{1+x^2}{(1+x^2)(1+y^2)} = \frac{1}{1+y^2}.$$

Then $z_{xx} = -(1+x^2)^{-2} \cdot 2x = -\dfrac{2x}{(1+x^2)^2}$, $z_{xy} = 0$, $z_{yx} = 0$, $z_{yy} = -(1+y^2)^{-2} \cdot 2y = -\dfrac{2y}{(1+y^2)^2}$.

53. $u = x^4y^3 - y^4 \quad \Rightarrow \quad u_x = 4x^3y^3$, $u_{xy} = 12x^3y^2$ and $u_y = 3x^4y^2 - 4y^3$, $u_{yx} = 12x^3y^2$.

Thus, $u_{xy} = u_{yx}$.

54. $u = e^{xy}\sin y \quad \Rightarrow \quad u_x = ye^{xy}\sin y$, $u_{xy} = ye^{xy}\cos y + (\sin y)(y \cdot xe^{xy} + e^{xy} \cdot 1) = e^{xy}(y\cos y + xy\sin y + \sin y)$,

$u_y = e^{xy}\cos y + (\sin y)(xe^{xy}) = e^{xy}(\cos y + x\sin y)$,

$u_{yx} = e^{xy} \cdot \sin y + (\cos y + x\sin y) \cdot ye^{xy} = e^{xy}(\sin y + y\cos y + xy\sin y)$. Thus, $u_{xy} = u_{yx}$.

55. $u = \cos(x^2y) \quad \Rightarrow \quad u_x = -\sin(x^2y) \cdot 2xy = -2xy\sin(x^2y)$,

$u_{xy} = -2xy \cdot \cos(x^2y) \cdot x^2 + \sin(x^2y) \cdot (-2x) = -2x^3y\cos(x^2y) - 2x\sin(x^2y)$ and

$u_y = -\sin(x^2y) \cdot x^2 = -x^2\sin(x^2y)$, $u_{yx} = -x^2 \cdot \cos(x^2y) \cdot 2xy + \sin(x^2y) \cdot (-2x) = -2x^3y\cos(x^2y) - 2x\sin(x^2y)$.

Thus, $u_{xy} = u_{yx}$.

56. $u = \ln(x+2y) \quad \Rightarrow \quad u_x = \dfrac{1}{x+2y} = (x+2y)^{-1}$, $u_{xy} = (-1)(x+2y)^{-2}(2) = -\dfrac{2}{(x+2y)^2}$ and

$u_y = \dfrac{1}{x+2y} \cdot 2 = 2(x+2y)^{-1}$, $u_{yx} = (-2)(x+2y)^{-2} = -\dfrac{2}{(x+2y)^2}$. Thus, $u_{xy} = u_{yx}$.

57. $f(x,y) = x^4y^2 - x^3y \quad \Rightarrow \quad f_x = 4x^3y^2 - 3x^2y$, $f_{xx} = 12x^2y^2 - 6xy$, $f_{xxx} = 24xy^2 - 6y$ and

$f_{xy} = 8x^3y - 3x^2$, $f_{xyx} = 24x^2y - 6x$.

58. $f(x,y) = \sin(2x+5y) \quad \Rightarrow \quad f_y = \cos(2x+5y) \cdot 5 = 5\cos(2x+5y)$, $f_{yx} = -5\sin(2x+5y) \cdot 2 = -10\sin(2x+5y)$,

$f_{yxy} = -10\cos(2x+5y) \cdot 5 = -50\cos(2x+5y)$

59. $f(x,y,z) = e^{xyz^2} \quad \Rightarrow \quad f_x = e^{xyz^2} \cdot yz^2 = yz^2e^{xyz^2}$, $f_{xy} = yz^2 \cdot e^{xyz^2}(xz^2) + e^{xyz^2} \cdot z^2 = (xyz^4 + z^2)e^{xyz^2}$,

$f_{xyz} = (xyz^4 + z^2) \cdot e^{xyz^2}(2xyz) + e^{xyz^2} \cdot (4xyz^3 + 2z) = (2x^2y^2z^5 + 6xyz^3 + 2z)e^{xyz^2}$.

60. $g(r,s,t) = e^r\sin(st) \quad \Rightarrow \quad g_r = e^r\sin(st)$, $g_{rs} = e^r\cos(st) \cdot t = te^r\cos(st)$,

$g_{rst} = te^r(-\sin(st) \cdot s) + \cos(st) \cdot e^r = e^r[\cos(st) - st\sin(st)]$.

61. $W = \sqrt{u + v^2} \quad \Rightarrow \quad \dfrac{\partial W}{\partial v} = \frac{1}{2}(u + v^2)^{-1/2}(2v) = v(u + v^2)^{-1/2}$,

$\dfrac{\partial^2 W}{\partial u\,\partial v} = v\left(-\frac{1}{2}\right)(u + v^2)^{-3/2}(1) = -\frac{1}{2}v(u + v^2)^{-3/2}, \quad \dfrac{\partial^3 W}{\partial u^2\,\partial v} = -\frac{1}{2}v\left(-\frac{3}{2}\right)(u + v^2)^{-5/2}(1) = \frac{3}{4}v(u + v^2)^{-5/2}.$

62. $V = \ln(r + s^2 + t^3) \quad \Rightarrow \quad \dfrac{\partial V}{\partial t} = \dfrac{3t^2}{r + s^2 + t^3} = 3t^2(r + s^2 + t^3)^{-1}$,

$\dfrac{\partial^2 V}{\partial s\,\partial t} = 3t^2(-1)(r + s^2 + t^3)^{-2}(2s) = -6st^2(r + s^2 + t^3)^{-2}$,

$\dfrac{\partial^3 V}{\partial r\,\partial s\,\partial t} = -6st^2(-2)(r + s^2 + t^3)^{-3}(1) = 12st^2(r + s^2 + t^3)^{-3} = \dfrac{12st^2}{(r + s^2 + t^3)^3}.$

63. $w = \dfrac{x}{y + 2z} = x(y + 2z)^{-1} \quad \Rightarrow \quad \dfrac{\partial w}{\partial x} = (y + 2z)^{-1}, \quad \dfrac{\partial^2 w}{\partial y\,\partial x} = -(y + 2z)^{-2}(1) = -(y + 2z)^{-2}$,

$\dfrac{\partial^3 w}{\partial z\,\partial y\,\partial x} = -(-2)(y + 2z)^{-3}(2) = 4(y + 2z)^{-3} = \dfrac{4}{(y + 2z)^3}$ and $\dfrac{\partial w}{\partial y} = x(-1)(y + 2z)^{-2}(1) = -x(y + 2z)^{-2}$,

$\dfrac{\partial^2 w}{\partial x\,\partial y} = -(y + 2z)^{-2}, \quad \dfrac{\partial^3 w}{\partial x^2\,\partial y} = 0.$

64. $u = x^a y^b z^c$. If $a = 0$, or if $b = 0$ or 1, or if $c = 0$, 1, or 2, then $\dfrac{\partial^6 u}{\partial x\,\partial y^2\,\partial z^3} = 0$. Otherwise $\dfrac{\partial u}{\partial z} = cx^a y^b z^{c-1}$,

$\dfrac{\partial^2 u}{\partial z^2} = c(c-1)x^a y^b z^{c-2}, \quad \dfrac{\partial^3 u}{\partial z^3} = c(c-1)(c-2)x^a y^b z^{c-3}, \quad \dfrac{\partial^4 u}{\partial y\,\partial z^3} = bc(c-1)(c-2)x^a y^{b-1} z^{c-3}$,

$\dfrac{\partial^5 u}{\partial y^2\,\partial z^3} = b(b-1)c(c-1)(c-2)x^a y^{b-2} z^{c-3}$, and $\dfrac{\partial^6 u}{\partial x\,\partial y^2\,\partial z^3} = ab(b-1)c(c-1)(c-2)x^{a-1} y^{b-2} z^{c-3}.$

65. $f(x, y) = xy^2 - x^3 y \quad \Rightarrow$

$$\begin{aligned} f_x(x, y) &= \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h} = \lim_{h\to 0} \frac{(x + h)y^2 - (x + h)^3 y - (xy^2 - x^3 y)}{h} \\ &= \lim_{h\to 0} \frac{h(y^2 - 3x^2 y - 3xyh - yh^2)}{h} = \lim_{h\to 0}(y^2 - 3x^2 y - 3xyh - yh^2) = y^2 - 3x^2 y \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h} = \lim_{h\to 0} \frac{x(y + h)^2 - x^3(y + h) - (xy^2 - x^3 y)}{h} = \lim_{h\to 0} \frac{h(2xy + xh - x^3)}{h} \\ &= \lim_{h\to 0}(2xy + xh - x^3) = 2xy - x^3 \end{aligned}$$

66. $f(x, y) = \dfrac{x}{x + y^2} \quad \Rightarrow$

$$\begin{aligned} f_x(x, y) &= \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h} = \lim_{h\to 0} \frac{\frac{x+h}{x+h+y^2} - \frac{x}{x+y^2}}{h} \cdot \frac{(x + h + y^2)(x + y^2)}{(x + h + y^2)(x + y^2)} \\ &= \lim_{h\to 0} \frac{(x + h)(x + y^2) - x(x + h + y^2)}{h(x + h + y^2)(x + y^2)} = \lim_{h\to 0} \frac{y^2 h}{h(x + h + y^2)(x + y^2)} \\ &= \lim_{h\to 0} \frac{y^2}{(x + h + y^2)(x + y^2)} = \frac{y^2}{(x + y^2)^2} \end{aligned}$$

[continued]

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} = \lim_{h\to 0} \frac{\frac{x}{x+(y+h)^2} - \frac{x}{x+y^2}}{h} \cdot \frac{\left[x+(y+h)^2\right](x+y^2)}{\left[x+(y+h)^2\right](x+y^2)}$$

$$= \lim_{h\to 0} \frac{x(x+y^2) - x\left[x+(y+h)^2\right]}{h[x+(y+h)^2](x+y^2)} = \lim_{h\to 0} \frac{h(-2xy-xh)}{h[x+(y+h)^2](x+y^2)}$$

$$= \lim_{h\to 0} \frac{-2xy-xh}{\left[x+(y+h)^2\right](x+y^2)} = \frac{-2xy}{(x+y^2)^2}$$

67. Assuming that the third partial derivatives of f are continuous (easily verified), we can write $f_{xzy} = f_{yxz}$. Then

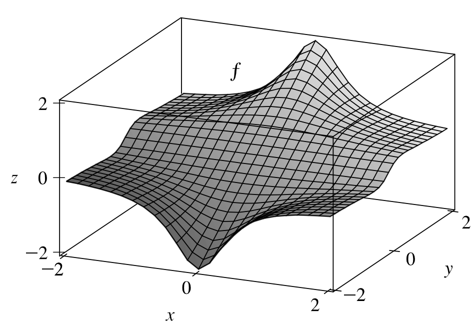
$f(x,y,z) = xy^2z^3 + \arcsin\left(x\sqrt{z}\right) \quad\Rightarrow\quad f_y = 2xyz^3 + 0$, $f_{yx} = 2yz^3$, and $f_{yxz} = 6yz^2 = f_{xzy}$.

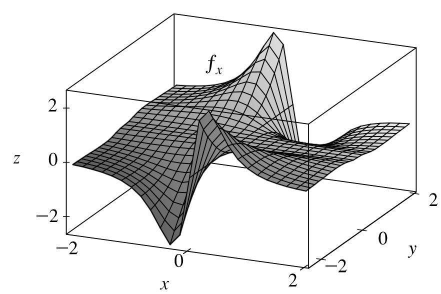
68. Let $f(x,y,z) = \sqrt{1+xz}$ and $h(x,y,z) = \sqrt{1-xy}$ so that $g = f + h$. Then $f_y = 0 = f_{yx} = f_{yxz}$ and $h_z = 0 = h_{zx} = h_{zxy}$. But (since the partial derivatives are continous on their domains) $f_{xyz} = f_{yxz}$ and $h_{xyz} = h_{zxy}$, so $g_{xyz} = f_{xyz} + h_{xyz} = 0 + 0 = 0$.

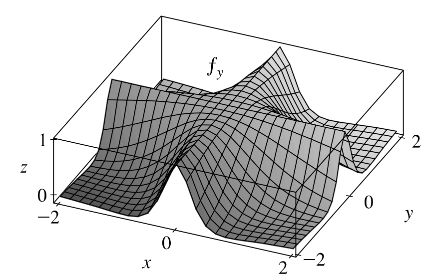
69. First of all, if we start at the point $(3,-3)$ and move in the positive y-direction, we see that both b and c decrease, while a increases. Both b and c have a low point at about $(3,-1.5)$, while a is 0 at this point. So a is definitely the graph of f_y, and one of b and c is the graph of f. To see which is which, we start at the point $(-3,-1.5)$ and move in the positive x-direction. b traces out a line with negative slope, while c traces out a parabola opening downward. This tells us that b is the x-derivative of c. So c is the graph of f, b is the graph of f_x, and a is the graph of f_y.

70. $f(x,y) = \dfrac{y}{1+x^2y^2} \quad\Rightarrow\quad f_x = \dfrac{(1+x^2y^2)(0) - y(2xy^2)}{(1+x^2y^2)^2} = -\dfrac{2xy^3}{(1+x^2y^2)^2}$,

$f_y = \dfrac{(1+x^2y^2)(1) - y(2x^2y)}{(1+x^2y^2)^2} = \dfrac{1-x^2y^2}{(1+x^2y^2)^2}$

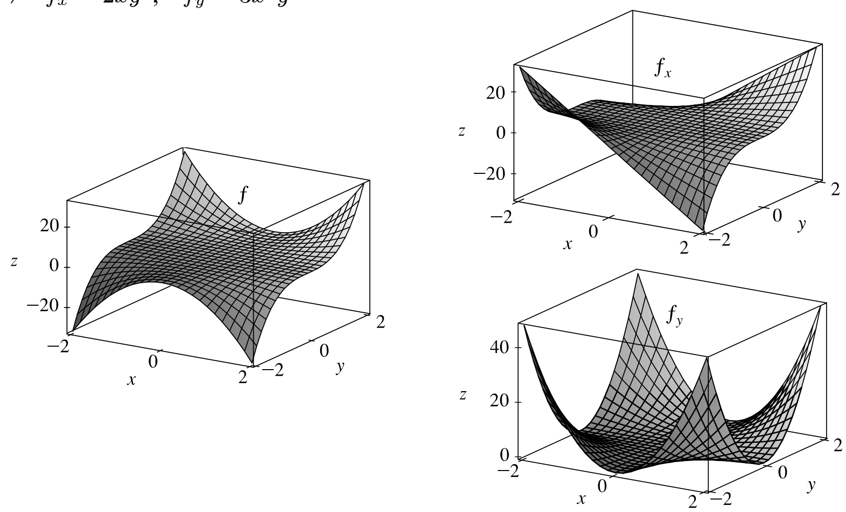






Note that traces of f in planes parallel to the xz-plane have only one extreme value (a minimum for $y < 0$, a maximum for $y > 0$), and the traces of f_x in these planes have only one zero (going from negative to positive if $y < 0$ and from positive to negative if $y > 0$). The traces of f in planes parallel to the yz-plane have two extreme values, and the traces of f_y in these planes have two zeros.

71. $f(x,y) = x^2y^3 \;\Rightarrow\; f_x = 2xy^3,\;\; f_y = 3x^2y^2$



Note that traces of f in planes parallel to the xz-plane are parabolas which open downward for $y < 0$ and upward for $y > 0$, and the traces of f_x in these planes are straight lines, which have negative slopes for $y < 0$ and positive slopes for $y > 0$. The traces of f in planes parallel to the yz-plane are cubic curves, and the traces of f_y in these planes are parabolas.

72. (a) $f_{xx} = \frac{\partial}{\partial x}(f_x)$, so f_{xx} is the rate of change of f_x in the x-direction. f_x is negative at $(-1,2)$ and if we move in the positive x-direction, the surface becomes less steep. Thus, the values of f_x are increasing and $f_{xx}(-1,2)$ is positive.

(b) f_{yy} is the rate of change of f_y in the y-direction. f_y is negative at $(-1,2)$ and if we move in the positive y-direction, the surface becomes steeper. Thus, the values of f_y are decreasing, and $f_{yy}(-1,2)$ is negative.

(c) $f_{xy} = \frac{\partial}{\partial y}(f_x)$, so f_{xy} is the rate of change of f_x in the y-direction. f_x is positive at $(1,2)$ and if we move in the positive y-direction, the surface becomes steeper, looking in the positive x-direction. Thus, the values of f_x are increasing and $f_{xy}(1,2)$ is positive.

(d) f_x is negative at $(-1,2)$ and if we move in the positive y-direction, the surface gets steeper (with negative slope), looking in the positive x-direction. This means that the values of f_x are decreasing as y increases, so $f_{xy}(-1,2)$ is negative.

73. By Definition 4, $f_x(3,2) = \lim\limits_{h\to 0} \dfrac{f(3+h,2) - f(3,2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$:

$f_x(3,2) \approx \dfrac{f(3.5,2) - f(3,2)}{0.5} = \dfrac{22.4 - 17.5}{0.5} = 9.8$, $f_x(3,2) \approx \dfrac{f(2.5,2) - f(3,2)}{-0.5} = \dfrac{10.2 - 17.5}{-0.5} = 14.6$. Averaging these values, we estimate $f_x(3,2)$ to be approximately 12.2.

Similarly, $f_x(3,2.2) = \lim\limits_{h\to 0} \dfrac{f(3+h,2.2) - f(3,2.2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$:

$f_x(3,2.2) \approx \dfrac{f(3.5,2.2) - f(3,2.2)}{0.5} = \dfrac{26.1 - 15.9}{0.5} = 20.4$, $f_x(3,2.2) \approx \dfrac{f(2.5,2.2) - f(3,2.2)}{-0.5} = \dfrac{9.3 - 15.9}{-0.5} = 13.2$.

Averaging these values, we have $f_x(3,2.2) \approx 16.8$.

[continued]

To estimate $f_{xy}(3,2)$, we first need an estimate for $f_x(3,1.8)$:

$$f_x(3,1.8) \approx \frac{f(3.5,1.8)-f(3,1.8)}{0.5} = \frac{20.0-18.1}{0.5} = 3.8,\ f_x(3,1.8) \approx \frac{f(2.5,1.8)-f(3,1.8)}{-0.5} = \frac{12.5-18.1}{-0.5} = 11.2.$$

Averaging these values, we get $f_x(3,1.8) \approx 7.5$. Now $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\,[f_x(x,y)]$ and $f_x(x,y)$ is itself a function of two variables, so Definition 4 says that $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\,[f_x(x,y)] = \displaystyle\lim_{h\to 0} \frac{f_x(x,y+h)-f_x(x,y)}{h} \quad\Rightarrow$

$f_{xy}(3,2) = \displaystyle\lim_{h\to 0} \frac{f_x(3,2+h)-f_x(3,2)}{h}$. We can estimate this value using our previous work with $h=0.2$ and $h=-0.2$:

$$f_{xy}(3,2) \approx \frac{f_x(3,2.2)-f_x(3,2)}{0.2} = \frac{16.8-12.2}{0.2} = 23,\ f_{xy}(3,2) \approx \frac{f_x(3,1.8)-f_x(3,2)}{-0.2} = \frac{7.5-12.2}{-0.2} = 23.5.$$

Averaging these values, we estimate $f_{xy}(3,2)$ to be approximately 23.25.

74. (a) If we fix y and allow x to vary, the level curves indicate that the value of f decreases as we move through P in the positive x-direction, so f_x is negative at P.

(b) If we fix x and allow y to vary, the level curves indicate that the value of f increases as we move through P in the positive y-direction, so f_y is positive at P.

(c) $f_{xx} = \dfrac{\partial}{\partial x}\,(f_x)$, so if we fix y and allow x to vary, f_{xx} is the rate of change of f_x as x increases. Note that at points to the right of P the level curves are spaced farther apart (in the x-direction) than at points to the left of P, demonstrating that f decreases less quickly with respect to x to the right of P. So as we move through P in the positive x-direction the (negative) value of f_x increases, hence $\dfrac{\partial}{\partial x}\,(f_x) = f_{xx}$ is positive at P.

(d) $f_{xy} = \dfrac{\partial}{\partial y}\,(f_x)$, so if we fix x and allow y to vary, f_{xy} is the rate of change of f_x as y increases. The level curves are closer together (in the x-direction) at points above P than at those below P, demonstrating that f decreases more quickly with respect to x for y-values above P. So as we move through P in the positive y-direction, the (negative) value of f_x decreases, hence f_{xy} is negative.

(e) $f_{yy} = \dfrac{\partial}{\partial y}\,(f_y)$, so if we fix x and allow y to vary, f_{yy} is the rate of change of f_y as y increases. The level curves are closer together (in the y-direction) at points above P than at those below P, demonstrating that f increases more quickly with respect to y above P. So as we move through P in the positive y-direction the (positive) value of f_y increases, hence $\dfrac{\partial}{\partial y}\,(f_y) = f_{yy}$ is positive at P.

75. (a) $f(x,y) = 4 - x^2 - 2y^2$. In the plane $y=1$, $f(x,1) = 4 - x^2 - 2(1^2) = 2 - x^2$, so a vector equation for C_1 is given by $\mathbf{r}(t) = \langle t, 1, 2-t^2\rangle$ where the point $(1,1,1)$ corresponds to $t=1$. Then $\mathbf{r}'(t) = \langle 1, 0, -2t\rangle \quad\Rightarrow\quad \mathbf{r}'(1) = \langle 1,0,-2\rangle$ and parametric equations of the tangent line are $x = t+1$, $y=1$, $z = -2t+1$. Thus, $x = t+1 \quad\Rightarrow\quad x-1=t \quad\Rightarrow$

$z = -2(x-1)+1 = -2x+3$. So the equation of the tangent line can be given by $z = -2x+3$, $y = 1$ which has a slope $m = -2$. Therefore, $f_x(1,1) = -2$.

(b) In the plane $x = 1$, $f(1,y) = 4 - 1^2 - 2y^2 = 3 - 2y^2$, so a vector equation for C_2 is given by $\mathbf{r}(t) = \left\langle 1, t, 3 - 2t^2 \right\rangle$ where the point $(1,1,1)$ corresponds to $t = 1$. Then $\mathbf{r}'(t) = \langle 0, 1, -4t \rangle \;\Rightarrow\; \mathbf{r}'(1) = \langle 0, 1, -4 \rangle$ and parametric equations of the tangent line are $x = 1, y = t+1, z = -4t+1$. Thus, $y = t+1 \;\Rightarrow\; y - 1 = t \;\Rightarrow$ $z = -4(y-1)+1 = -4y+5$. So the equation of the tangent line can be given by $z = -4y+5$, $x = 1$ which has a slope $m = -4$. Therefore, $f_y(1,1) = -4$.

76. For each i, $i = 1, \ldots, n$, $\partial u/\partial x_i = a_i e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$ and $\partial^2 u/\partial x_i^2 = a_i^2 e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$.

Then $\dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} + \cdots + \dfrac{\partial^2 u}{\partial x_n^2} = \left(a_1^2 + a_2^2 + \cdots + a_n^2\right) e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n} = e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n} = u$

since $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$.

77. (a) $u = \sin(kx)\,\sin(akt) \;\Rightarrow\; u_t = ak\sin(kx)\,\cos(akt)$, $u_{tt} = -a^2k^2\sin(kx)\,\sin(akt)$, $u_x = k\cos(kx)\,\sin(akt)$, $u_{xx} = -k^2\sin(kx)\,\sin(akt)$. Thus $u_{tt} = a^2u_{xx}$.

(b) $u = \dfrac{t}{a^2t^2 - x^2} \;\Rightarrow\; u_t = \dfrac{(a^2t^2 - x^2) - t(2a^2t)}{(a^2t^2 - x^2)^2} = -\dfrac{a^2t^2 + x^2}{(a^2t^2 - x^2)^2}$,

$$u_{tt} = \frac{-2a^2t(a^2t^2 - x^2)^2 + (a^2t^2 + x^2)(2)(a^2t^2 - x^2)(2a^2t)}{(a^2t^2 - x^2)^4} = \frac{2a^4t^3 + 6a^2tx^2}{(a^2t^2 - x^2)^3},$$

$$u_x = t(-1)(a^2t^2 - x^2)^{-2}(-2x) = \frac{2tx}{(a^2t^2 - x^2)^2},$$

$$u_{xx} = \frac{2t(a^2t^2 - x^2)^2 - 2tx\,(2)(a^2t^2 - x^2)(-2x)}{(a^2t^2 - x^2)^4} = \frac{2a^2t^3 - 2tx^2 + 8tx^2}{(a^2t^2 - x^2)^3} = \frac{2a^2t^3 + 6tx^2}{(a^2t^2 - x^2)^3}.$$

Thus, $u_{tt} = a^2u_{xx}$.

(c) $u = (x - at)^6 + (x + at)^6 \;\Rightarrow\; u_t = -6a(x - at)^5 + 6a(x + at)^5$, $u_{tt} = 30a^2(x - at)^4 + 30a^2(x + at)^4$, $u_x = 6(x - at)^5 + 6(x + at)^5$, $u_{xx} = 30(x - at)^4 + 30(x + at)^4$. Thus, $u_{tt} = a^2u_{xx}$.

(d) $u = \sin(x - at) + \ln(x + at) \;\Rightarrow\; u_t = -a\cos(x - at) + \dfrac{a}{x + at}$, $u_{tt} = -a^2\sin(x - at) - \dfrac{a^2}{(x + at)^2}$, $u_x = \cos(x - at) + \dfrac{1}{x + at}$, $u_{xx} = -\sin(x - at) - \dfrac{1}{(x + at)^2}$. Thus, $u_{tt} = a^2u_{xx}$.

78. (a) $u = x^2 + y^2 \;\Rightarrow\; u_x = 2x$, $u_{xx} = 2$; $u_y = 2y$, $u_{yy} = 2$. Thus, $u_{xx} + u_{yy} \neq 0$ and $u = x^2 + y^2$ does not satisfy Laplace's Equation.

(b) $u = x^2 - y^2$ is a solution: $u_{xx} = 2$, $u_{yy} = -2$ so $u_{xx} + u_{yy} = 0$.

(c) $u = x^3 + 3xy^2$ is not a solution: $u_x = 3x^2 + 3y^2$, $u_{xx} = 6x$; $u_y = 6xy$, $u_{yy} = 6x$.

(d) $u = \ln\sqrt{x^2+y^2}$ is a solution: $u_x = \dfrac{1}{\sqrt{x^2+y^2}}\left(\dfrac{1}{2}\right)(x^2+y^2)^{-1/2}(2x) = \dfrac{x}{x^2+y^2}$,

$u_{xx} = \dfrac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$. By symmetry, $u_{yy} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$, so $u_{xx}+u_{yy}=0$.

(e) $u = \sin x\,\cosh y + \cos x\,\sinh y$ is a solution: $u_x = \cos x\,\cosh y - \sin x\,\sinh y$, $u_{xx} = -\sin x\,\cosh y - \cos x\,\sinh y$, and $u_y = \sin x\,\sinh y + \cos x\,\cosh y$, $u_{yy} = \sin x\,\cosh y + \cos x\,\sinh y$.

(f) $u = e^{-x}\cos y - e^{-y}\cos x$ is a solution: $u_x = -e^{-x}\cos y + e^{-y}\sin x$, $\quad u_{xx} = e^{-x}\cos y + e^{-y}\cos x$, and $u_y = -e^{-x}\sin y + e^{-y}\cos x$, $\quad u_{yy} = -e^{-x}\cos y - e^{-y}\cos x$.

79. $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}} \quad\Rightarrow\quad u_x = \left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2}$ and

$$u_{xx} = -(x^2+y^2+z^2)^{-3/2} - x\left(-\tfrac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}.$$

By symmetry, $u_{yy} = \dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ and $u_{zz} = \dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$.

Thus, $u_{xx}+u_{yy}+u_{zz} = \dfrac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}} = 0$.

80. $u = e^{-\alpha^2k^2t}\sin kx \quad\Rightarrow\quad u_x = ke^{-\alpha^2k^2t}\cos kx$, $\quad u_{xx} = -k^2e^{-\alpha^2k^2t}\sin kx$, $\quad$ and $\quad u_t = -\alpha^2k^2e^{-\alpha^2k^2t}\sin kx$.

Thus, $\alpha^2u_{xx} = u_t$.

81. $c(x,t) = \dfrac{1}{\sqrt{4\pi Dt}}\,e^{-x^2/(4Dt)} \quad\Rightarrow$

$$\begin{aligned}\frac{\partial c}{\partial t} &= \frac{1}{\sqrt{4\pi Dt}}\cdot e^{-x^2/(4Dt)}\left[-x^2(-1)(4Dt)^{-2}(4D)\right] + e^{-x^2/(4Dt)}\cdot\left(-\tfrac{1}{2}\right)(4\pi Dt)^{-3/2}\,(4\pi D)\\ &= (4\pi Dt)^{-3/2}\left(4\pi Dt\cdot\frac{x^2}{4Dt^2} - 2\pi D\right)e^{-x^2/(4Dt)} = \frac{2\pi D}{(4\pi Dt)^{3/2}}\left(\frac{x^2}{2Dt}-1\right)e^{-x^2/(4Dt)},\end{aligned}$$

$$\frac{\partial c}{\partial x} = \frac{1}{\sqrt{4\pi Dt}}\,e^{-x^2/(4Dt)}\cdot\frac{-2x}{4Dt} = \frac{-2\pi x}{(4\pi Dt)^{3/2}}\,e^{-x^2/(4Dt)}, \text{ and}$$

$$\begin{aligned}\frac{\partial^2 c}{\partial x^2} &= \frac{-2\pi}{(4\pi Dt)^{3/2}}\left[x\cdot e^{-x^2/(4Dt)}\cdot\frac{-2x}{4Dt} + e^{-x^2/(4Dt)}\cdot 1\right]\\ &= \frac{-2\pi}{(4\pi Dt)^{3/2}}\left(-\frac{x^2}{2Dt}+1\right)e^{-x^2/(4Dt)} = \frac{2\pi}{(4\pi Dt)^{3/2}}\left(\frac{x^2}{2Dt}-1\right)e^{-x^2/(4Dt)}.\end{aligned}$$

Thus, $\dfrac{\partial c}{\partial t} = \dfrac{2\pi D}{(4\pi Dt)^{3/2}}\left(\dfrac{x^2}{2Dt}-1\right)e^{-x^2/(4Dt)} = D\left[\dfrac{2\pi}{(4\pi Dt)^{3/2}}\left(\dfrac{x^2}{2Dt}-1\right)e^{-x^2/(4Dt)}\right] = D\,\dfrac{\partial^2 c}{\partial x^2}$.

82. (a) $T(x,y) = \dfrac{60}{1+x^2+y^2} \quad\Rightarrow\quad \dfrac{\partial T}{\partial x} = -\dfrac{60(2x)}{(1+x^2+y^2)^2}$, so at $(2,1)$, $T_x = -\dfrac{240}{(1+4+1)^2} = -\dfrac{20}{3}$.

(b) $\dfrac{\partial T}{\partial y} = -\dfrac{60(2y)}{(1+x^2+y^2)^2}$, so at $(2,1)$, $T_y = -\dfrac{120}{36} = -\dfrac{10}{3}$. Thus, from the point $(2,1)$ the temperature is decreasing at a rate of $\frac{20}{3}\,^\circ\mathrm{C/m}$ in the x-direction and is decreasing at a rate of $\frac{10}{3}\,^\circ\mathrm{C/m}$ in the y-direction.

83. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$. By the Chain Rule, taking the partial derivative of both sides with respect to R_1 gives

$\dfrac{\partial R^{-1}}{\partial R}\dfrac{\partial R}{\partial R_1} = \dfrac{\partial\,[(1/R_1)+(1/R_2)+(1/R_3)]}{\partial R_1}$ or $-R^{-2}\dfrac{\partial R}{\partial R_1} = -R_1^{-2}$. Thus, $\dfrac{\partial R}{\partial R_1} = \dfrac{R^2}{R_1^2}$.

84. (a) $P = \dfrac{mRT}{V}$ so $\dfrac{\partial P}{\partial V} = \dfrac{-mRT}{V^2}$; $V = \dfrac{mRT}{P}$, so $\dfrac{\partial V}{\partial T} = \dfrac{mR}{P}$; $T = \dfrac{PV}{mR}$, so $\dfrac{\partial T}{\partial P} = \dfrac{V}{mR}$.

Thus, $\dfrac{\partial P}{\partial V}\dfrac{\partial V}{\partial T}\dfrac{\partial T}{\partial P} = \dfrac{-mRT}{V^2}\dfrac{mR}{P}\dfrac{V}{mR} = \dfrac{-mRT}{PV} = -1$, since $PV = mRT$.

(b) By part (a), $PV = mRT \;\Rightarrow\; P = \dfrac{mRT}{V}$, so $\dfrac{\partial P}{\partial T} = \dfrac{mR}{V}$. Also, $PV = mRT \;\Rightarrow\; V = \dfrac{mRT}{P}$ and $\dfrac{\partial V}{\partial T} = \dfrac{mR}{P}$.

Since $T = \dfrac{PV}{mR}$, we have $T\,\dfrac{\partial P}{\partial T}\,\dfrac{\partial V}{\partial T} = \dfrac{PV}{mR}\cdot\dfrac{mR}{V}\cdot\dfrac{mR}{P} = mR$.

85. $\left(P + \dfrac{n^2a}{V^2}\right)(V - nb) = nRT \;\Rightarrow\; T = \dfrac{1}{nR}\left(P + \dfrac{n^2a}{V^2}\right)(V - nb)$, so $\dfrac{\partial T}{\partial P} = \dfrac{1}{nR}(1)(V - nb) = \dfrac{V - nb}{nR}$.

We can also write $P + \dfrac{n^2a}{V^2} = \dfrac{nRT}{V - nb} \;\Rightarrow\; P = \dfrac{nRT}{V - nb} - \dfrac{n^2a}{V^2} = nRT(V - nb)^{-1} - n^2aV^{-2}$, so

$\dfrac{\partial P}{\partial V} = -nRT(V - nb)^{-2}(1) + 2n^2aV^{-3} = \dfrac{2n^2a}{V^3} - \dfrac{nRT}{(V - nb)^2}$.

86. $W = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$. $\dfrac{\partial W}{\partial T} = 0.6215 + 0.3965v^{0.16}$. When $T = -15^\circ\mathrm{C}$ and $v = 30$ km/h, $\dfrac{\partial W}{\partial T} = 0.6215 + 0.3965(30)^{0.16} \approx 1.3048$, so we would expect the apparent temperature to drop by approximately $1.3^\circ\mathrm{C}$ if the actual temperature decreases by $1^\circ\mathrm{C}$. $\dfrac{\partial W}{\partial v} = -11.37(0.16)v^{-0.84} + 0.3965T(0.16)v^{-0.84}$ and when $T = -15^\circ\mathrm{C}$ and $v = 30$ km/h, $\dfrac{\partial W}{\partial v} = -11.37(0.16)(30)^{-0.84} + 0.3965(-15)(0.16)(30)^{-0.84} \approx -0.1592$, so we would expect the apparent temperature to drop by approximately $0.16^\circ\mathrm{C}$ if the wind speed increases by 1 km/h.

87. (a) $S = f(w,h) = 0.0072w^{0.425}h^{0.725} \;\Rightarrow\; \dfrac{\partial S}{\partial w} = 0.0072(0.425)w^{0.425-1}h^{0.725} = 0.00306w^{-0.575}h^{0.725}$, so

$\dfrac{\partial S}{\partial w}(73, 178) = 0.00306(73)^{-0.575}(178)^{0.725} \approx 0.0111$. This means that for a person 178 centimeters tall who weighs 73 kilograms, an increase in weight (while height remains constant) causes the surface area to increase at a rate of about 0.0111 square meters (about 111 square centimeters) per kilogram.

(b) $\dfrac{\partial S}{\partial h} = 0.0072(0.725)w^{0.425}h^{0.725-1} = 0.00522w^{0.425}h^{-0.275}$, so

$\dfrac{\partial S}{\partial h}(73, 178) = 0.00522(73)^{0.425}(178)^{-0.275} \approx 0.0078$. This means that for a person 178 centimeters tall who weighs

73 kilograms, an increase in height (while weight remains unchanged at 73 kilograms) causes the surface area to increase at a rate of about 0.0078 square meters (about 78 square centimeters) per centimeter of height.

88. $R = C\dfrac{L}{r^4} \quad\Rightarrow\quad \dfrac{\partial R}{\partial L} = \dfrac{C}{r^4}$ and $\dfrac{\partial R}{\partial r} = CL\left(-4r^{-5}\right) = -4C\dfrac{L}{r^5}$.

$\partial R/\partial L$ is the rate at which the resistance of the flowing blood increases with respect to the length of the artery when the radius stays constant. $\partial R/\partial r$ is the rate of change of the resistance with respect to the radius of the artery when the length remains unchanged. Because $\partial R/\partial r$ is negative, the resistance decreases if the radius increases.

89. $P(v, x, m) = Av^3 + \dfrac{B(mg/x)^2}{v} = Av^3 + Bm^2g^2x^{-2}v^{-1}$.

$\partial P/\partial v = 3Av^2 - \dfrac{B(mg/x)^2}{v^2}$ is the rate of change of the power needed during flapping mode with respect to the bird's velocity when the mass and fraction of flapping time remain constant. $\partial P/\partial x = -2Bm^2g^2x^{-3}v^{-1} = -\dfrac{2Bm^2g^2}{x^3v}$ is the rate at which the power changes with respect to the fraction of time spent in flapping mode when the mass and velocity are held constant. $\partial P/\partial m = 2Bmg^2x^{-2}v^{-1} = \dfrac{2Bmg^2}{x^2v}$ is the rate of change of the power with respect to mass when the velocity and fraction of flapping time remain constant.

90. $T(x, t) = T_0 + T_1e^{-\lambda x}\sin(\omega t - \lambda x)$

(a) $\partial T/\partial x = T_1e^{-\lambda x}\left[\cos(\omega t - \lambda x)(-\lambda)\right] + T_1(-\lambda e^{-\lambda x})\sin(\omega t - \lambda x) = -\lambda T_1e^{-\lambda x}\left[\sin(\omega t - \lambda x) + \cos(\omega t - \lambda x)\right]$.

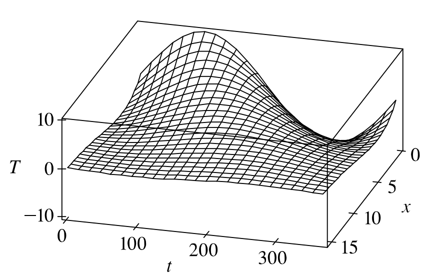
This quantity represents the rate of change of temperature with respect to depth below the surface, at a given time t.

(b) $\partial T/\partial t = T_1e^{-\lambda x}\left[\cos(\omega t - \lambda x)(\omega)\right] = \omega T_1e^{-\lambda x}\cos(\omega t - \lambda x)$. This quantity represents the rate of change of temperature with respect to time at a fixed depth x.

(c)
$$\begin{aligned} T_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) \\ &= -\lambda T_1\left(e^{-\lambda x}\left[\cos(\omega t - \lambda x)(-\lambda) - \sin(\omega t - \lambda x)(-\lambda)\right] + e^{-\lambda x}(-\lambda)\left[\sin(\omega t - \lambda x) + \cos(\omega t - \lambda x)\right]\right) \\ &= 2\lambda^2T_1e^{-\lambda x}\cos(\omega t - \lambda x) \end{aligned}$$

But from part (b), $T_t = \omega T_1e^{-\lambda x}\cos(\omega t - \lambda x) = \dfrac{\omega}{2\lambda^2}T_{xx}$. So with $k = \dfrac{\omega}{2\lambda^2}$, the function T satisfies the heat equation.

(d)



Note that near the surface (that is, for small x) the temperature varies greatly as t changes, but deeper (for large x) the temperature is more stable.

(e) The term $-\lambda x$ is a phase shift: it represents the fact that since heat diffuses slowly through soil, it takes time for changes in the surface temperature to affect the temperature at deeper points. As x increases, the phase shift also increases. For example, when $\lambda = 0.2$, the highest temperature at the surface is reached when $t \approx 91$, whereas at a depth of 1.5 meters the peak temperature is attained at $t \approx 149$, and at a depth of 3 meters, at $t \approx 207$.

91. $\dfrac{\partial K}{\partial m} = \frac{1}{2}v^2$, $\dfrac{\partial K}{\partial v} = mv$, $\dfrac{\partial^2 K}{\partial v^2} = m$. Thus $\dfrac{\partial K}{\partial m} \cdot \dfrac{\partial^2 K}{\partial v^2} = \frac{1}{2}v^2 m = K$.

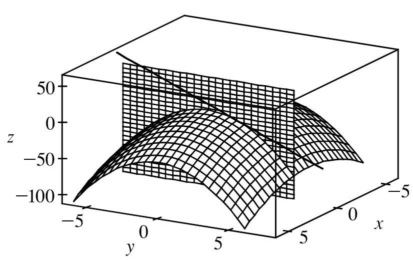
92. $E(m,v) = 2.65m^{0.66} + \dfrac{3.5m^{0.75}}{v} \quad \Rightarrow$

$E_m(m,v) = 2.65(0.66)m^{0.66-1} + \dfrac{3.5(0.75)m^{0.75-1}}{v} = 1.749m^{-0.34} + \dfrac{2.625m^{-0.25}}{v}$,

$E_v(m,v) = 3.5m^{0.75}\left(-v^{-2}\right) = -\dfrac{3.5m^{0.75}}{v^2}$. Then $E_m(400,8) = 1.749(400)^{-0.34} + \dfrac{2.625(400)^{-0.25}}{8} \approx 0.301$ which means that the average energy needed for a lizard to walk or run 1 km increases at a rate of about 0.301 kcal per gram of body mass increase from 400 g if the speed is 8 km/h. $E_v(400,8) = -\dfrac{3.5(400)^{0.75}}{8^2} \approx -4.89$, which means that the average energy needed by a lizard with body mass 400 g decreases at a rate of about 4.89 kcal per km/h when the speed increases from 8 km/h.

93. By the geometry of partial derivatives, the slope of the tangent line is $f_x(1,2)$. By implicit differentiation of $4x^2 + 2y^2 + z^2 = 16$, we get $8x + 2z\,(\partial z/\partial x) = 0 \quad \Rightarrow \quad \partial z/\partial x = -4x/z$, so when $x = 1$ and $z = 2$ we have $\partial z/\partial x = -2$. So the slope is $f_x(1,2) = -2$. Thus the tangent line is given by $z - 2 = -2(x-1)$, $y = 2$. Taking the parameter to be $t = x - 1$, we can write parametric equations for this line: $x = 1 + t$, $y = 2$, $z = 2 - 2t$.

94. $z = 6 - x - x^2 - 2y^2$. Setting $x = 1$, the equation of the parabola of intersection is $z = 6 - 1 - 1 - 2y^2 = 4 - 2y^2$. The slope of the tangent is $\partial z/\partial y = -4y$, so at $(1,2,-4)$ the slope is -8. Parametric equations for the line are therefore $x = 1$, $y = 2 + t$, $z = -4 - 8t$.



95. $f_x(x,y) = x + 4y \quad \Rightarrow \quad f_{xy}(x,y) = 4$ and $f_y(x,y) = 3x - y \quad \Rightarrow \quad f_{yx}(x,y) = 3$. Since f_{xy} and f_{yx} are continuous everywhere but $f_{xy}(x,y) \neq f_{yx}(x,y)$, Clairaut's Theorem implies that such a function $f(x,y)$ does not exist.

96. The Law of Cosines says that $a^2 = b^2 + c^2 - 2bc\cos A$. Thus $\dfrac{\partial(a^2)}{\partial a} = \dfrac{\partial(b^2 + c^2 - 2ab\cos A)}{\partial a}$ or

$2a = -2bc\,(-\sin A)\,\dfrac{\partial A}{\partial a}$, implying that $\dfrac{\partial A}{\partial a} = \dfrac{a}{bc\sin A}$. Taking the partial derivative of both sides with respect to b gives

$0 = 2b - 2c(\cos A) - 2bc\,(-\sin A)\,\dfrac{\partial A}{\partial b}$. Thus, $\dfrac{\partial A}{\partial b} = \dfrac{c\cos A - b}{bc\sin A}$. By symmetry, $\dfrac{\partial A}{\partial c} = \dfrac{b\cos A - c}{bc\sin A}$.

97. By Clairaut's Theorem, $f_{xyy} = (f_{xy})_y = (f_{yx})_y = f_{yxy} = (f_y)_{xy} = (f_y)_{yx} = f_{yyx}$.

98. (a) Since we are differentiating n times, with two choices of variable at each differentiation, there are 2^n nth-order partial derivatives.

(b) If these partial derivatives are all continuous, then the order in which the partials are taken doesn't affect the value of the result, that is, all nth-order partial derivatives with p partials with respect to x and $n - p$ partials with respect to y are equal. Since the number of partials taken with respect to x for an nth-order partial derivative can range from 0 to n, a function of two variables has $n + 1$ distinct partial derivatives of order n if these partial derivatives are all continuous.

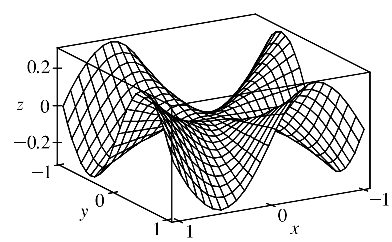
(c) Since n differentiations are to be performed with three choices of variable at each differentiation, there are 3^n nth-order partial derivatives of a function of three variables.

99. $f(x, y) = x(x^2 + y^2)^{-3/2} e^{\sin(x^2 y)}$. Let $g(x) = f(x, 0) = x(x^2)^{-3/2} e^0 = x\,|x|^{-3}$. To find $f_x(1, 0)$, we are using the point $(1, 0)$, so near $(1, 0)$, $g(x) = x^{-2}$. Then $g'(x) = -2x^{-3}$ and $g'(1) = -2$, so using Equation 1, we have $f_x(1, 0) = g'(1) = -2$.

100. $f(x, y) = \sqrt[3]{x^3 + y^3}$. $f_x(0, 0) = \lim\limits_{h \to 0} \dfrac{f(0 + h, 0) - f(0, 0)}{h} = \lim\limits_{h \to 0} \dfrac{(h^3 + 0)^{1/3} - 0}{h} = \lim\limits_{h \to 0} \dfrac{h}{h} = 1$.

Or: Let $g(x) = f(x, 0) = \sqrt[3]{x^3 + 0} = x$. Then $g'(x) = 1$ and $g'(0) = 1$ so, by (1), $f_x(0, 0) = g'(0) = 1$.

101. (a) $f(x, y) = \begin{cases} \dfrac{x^3 y - x y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$



(b) For $(x, y) \neq (0, 0)$,

$$f_x(x, y) = \frac{(3x^2 y - y^3)(x^2 + y^2) - (x^3 y - x y^3)(2x)}{(x^2 + y^2)^2} = \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2}$$

and, by symmetry, $f_y(x, y) = \dfrac{x^5 - 4x^3 y^2 - x y^4}{(x^2 + y^2)^2}$.

(c) $f_x(0, 0) = \lim\limits_{h \to 0} \dfrac{f(h, 0) - f(0, 0)}{h} = \lim\limits_{h \to 0} \dfrac{(0/h^2) - 0}{h} = 0$ and $f_y(0, 0) = \lim\limits_{h \to 0} \dfrac{f(0, h) - f(0, 0)}{h} = 0$.

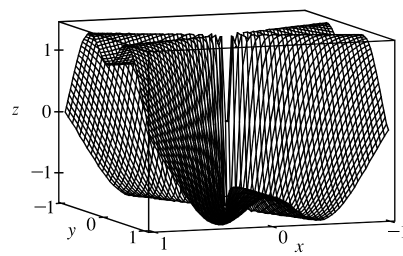
(d) By (3), $f_{xy}(0, 0) = \dfrac{\partial f_x}{\partial y} = \lim\limits_{h \to 0} \dfrac{f_x(0, h) - f_x(0, 0)}{h} = \lim\limits_{h \to 0} \dfrac{(-h^5 - 0)/h^4}{h} = -1$ while by (2),

$$f_{yx}(0, 0) = \frac{\partial f_y}{\partial x} = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = \lim_{h \to 0} \frac{h^5/h^4}{h} = 1.$$

(e) For $(x, y) \neq (0, 0)$, we use a CAS to compute

$$f_{xy}(x, y) = \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{(x^2 + y^2)^3}$$

Now as $(x, y) \to (0, 0)$ along the x-axis, $f_{xy}(x, y) \to 1$ while as $(x, y) \to (0, 0)$ along the y-axis, $f_{xy}(x, y) \to -1$. Thus f_{xy} isn't



continuous at $(0, 0)$ and Clairaut's Theorem doesn't apply, so there is no contradiction. The graphs of f_{xy} and f_{yx} are identical except at the origin, where we observe the discontinuity.

DISCOVERY PROJECT Deriving the Cobb-Douglas Production Function

1. For $K = K_0$, we have $\dfrac{dP}{dL} = \alpha\,\dfrac{P}{L} \;\Rightarrow\; \dfrac{dP}{P} = \alpha\,\dfrac{dL}{L} \;\Rightarrow\; \displaystyle\int \frac{dP}{P} = \int \alpha\,\frac{dL}{L} \;\Rightarrow\; \ln|P| = \alpha \ln|L| + A_1(K_0)$.

Then $e^{\ln|P|} = e^{\ln|L|^{\alpha} + A_1(K_0)} \;\Rightarrow\; P = e^{A_1(K_0)} L^{\alpha} \;\Rightarrow\; P(L, K_0) = C_1(K_0) L^{\alpha}$, where $e^{A_1(K_0)} = C_1(K_0)$.

2. For $L = L_0$, we have $\dfrac{dP}{dK} = \beta\,\dfrac{P}{K} \;\Rightarrow\; \dfrac{dP}{P} = \beta\,\dfrac{dK}{K} \;\Rightarrow\; \displaystyle\int \frac{dP}{P} = \int \beta\,\frac{dK}{K} \;\Rightarrow\; \ln|P| = \beta \ln|K| + A_2(L_0)$.

Then $e^{\ln|P|} = e^{\ln|K|^{\beta} + A_2(L_0)} \;\Rightarrow\; P = e^{A_2(L_0)} K^{\beta} \;\Rightarrow\; P(L_0, K) = C_2(L_0) K^{\beta}$, where $e^{A_2(L_0)} = C_2(L_0)$.

3. Suppose both labor and capital are increased by a factor of m. Then

$$\begin{aligned} P(mL, mk) &= b(mL)^{\alpha}(mK)^{1-\alpha} = bm^{\alpha}L^{\alpha}m^{1-\alpha}K^{1-\alpha} = m^{\alpha}m^{1-\alpha}bL^{\alpha}K^{1-\alpha} \\ &= m(bL^{\alpha}K^{1-\alpha}) = mP(L, K) \end{aligned}$$

Therefore, production is also increased by a factor of m.

4. For $P(L, K) = bL^{\alpha}K^{1-\alpha}$,

$$\begin{aligned} L\,\frac{\partial P}{\partial L} + K\,\frac{\partial P}{\partial K} &= L(b\alpha L^{\alpha-1}K^{1-\alpha}) + K(bL^{\alpha}(1-\alpha)K^{1-\alpha-1}) \\ &= b\alpha L^{\alpha}K^{1-\alpha} + b(1-\alpha)L^{\alpha}K^{1-\alpha} = bL^{\alpha}K^{1-\alpha}[\alpha + (1-\alpha)] \\ &= bL^{\alpha}K^{1-\alpha} = P(L, K) \end{aligned}$$

5. $P(L, K) = 1.01L^{0.75}K^{0.25}$. Marginal productivity of labor is given by $\partial P/\partial L = 1.01(0.75)L^{-0.25}K^{0.25}$. With $L = 194$ and $K = 407$, we have $\partial P/\partial L \approx 0.911656$. When capital is held constant at $K = 407$, as labor increases from $L = 194$, production will increase 0.911656 per unit of labor.

Marginal productivity of capital is given by $\partial P/\partial K = 1.01(0.25)L^{0.75}K^{-0.75}$. With $L = 194$ and $K = 407$, we have $\partial P/\partial K \approx 0.144849$. When labor is held constant at $L = 194$, as capital increases from $K = 407$, production will increase 0.144849 per unit of capital.

As the marginal productivity of labor is higher, it would be more beneficial to increase labor.

14.4 Tangent Planes and Linear Approximations

1. $z = f(x, y) = 16 - x^2 - y^2 \;\Rightarrow\; f_x(x, y) = -2x, f_y(x, y) = -2y$, so $f_x(2, 2) = -4$ and $f_y(2, 2) = -4$. By Equation 2, an equation of the tangent plane is $z - 8 = f_x(2, 2)(x - 2) + f_y(2, 2)(y - 2) \;\Rightarrow\; z - 8 = -4(x - 2) - 4(y - 2)$, or $z = -4x - 4y + 24$.

2. $z = f(x,y) = y^2 \sin x \quad \Rightarrow \quad f_x(x,y) = y^2 \cos x$, $f_y(x,y) = 2y \sin x$, so $f_x(\pi/2, -2) = 0$ and $f_y(\pi/2, -2) = -4$. By Equation 2, an equation of the tangent plane is $z - 4 = f_x(\pi/2,-2)(x - \pi/2) + f_y(\pi/2,-2)(y - (-2)) \quad \Rightarrow$ $z - 4 = -4(y+2)$, or $z = -4y - 4$.

3. $z = f(x,y) = 2x^2 + y^2 - 5y \quad \Rightarrow \quad f_x(x,y) = 4x,\ f_y(x,y) = 2y - 5$, so $f_x(1,2) = 4$ and $f_y(1,2) = -1$. By Equation 2, an equation of the tangent plane is $z - (-4) = f_x(1,2)(x-1) + f_y(1,2)(y-2) \quad \Rightarrow$ $z + 4 = 4(x-1) + (-1)(y-2)$, or $z = 4x - y - 6$.

4. $z = f(x,y) = (x+2)^2 - 2(y-1)^2 - 5 \quad \Rightarrow \quad f_x(x,y) = 2(x+2),\ f_y(x,y) = -4(y-1)$, so $f_x(2,3) = 8$ and $f_y(2,3) = -8$. By Equation 2, an equation of the tangent plane is $z - 3 = f_x(2,3)(x-2) + f_y(2,3)(y-3) \quad \Rightarrow$ $z - 3 = 8(x-2) + (-8)(y-3)$, or $z = 8x - 8y + 11$.

5. $z = f(x,y) = e^{x-y} \quad \Rightarrow \quad f_x(x,y) = e^{x-y}(1) = e^{x-y},\ f_y(x,y) = e^{x-y}(-1) = -e^{x-y}$, so $f_x(2,2) = 1$ and $f_y(2,2) = -1$. Thus, an equation of the tangent plane is $z - 1 = f_x(2,2)(x-2) + f_y(2,2)(y-2) \quad \Rightarrow$ $z - 1 = 1(x-2) + (-1)(y-2)$, or $z = x - y + 1$.

6. $z = f(x,y) = y^2 e^x \quad \Rightarrow \quad f_x(x,y) = y^2 e^x$, $f_y(x,y) = 2ye^x$, so $f_x(0,3) = 9$ and $f_y(0,3) = 6$. Thus, an equation of the tangent plane is $z - 9 = f_x(0,3)(x-0) + f_y(0,3)(y-3) \quad \Rightarrow \quad z - 9 = 9x + 6(y-3)$, or $z = 9x + 6y - 9$.

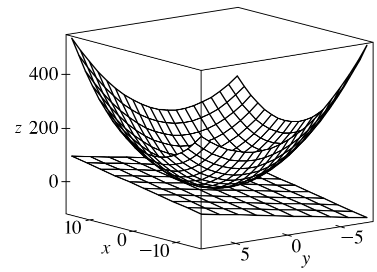
7. $z = f(x,y) = \dfrac{2\sqrt{y}}{x} \quad \Rightarrow \quad f_x(x,y) = -\dfrac{2\sqrt{y}}{x^2}$, $f_y(x,y) = \dfrac{1}{x\sqrt{y}}$, so $f_x(-1,1) = -2$ and $f_y(-1,1) = -1$. Thus, an equation of the tangent plane is $z - (-2) = f_x(-1,1)(x - (-1)) + f_y(-1,1)(y-1) \quad \Rightarrow$ $z + 2 = -2(x+1) - 1(y-1)$, or $z = -2x - y - 3$.

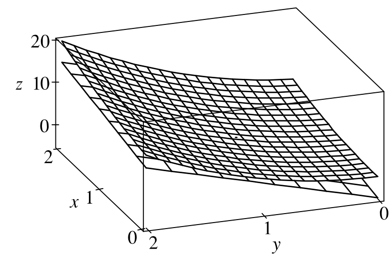
8. $z = f(x,y) = x/y^2 = xy^{-2} \quad \Rightarrow \quad f_x(x,y) = 1/y^2,\ f_y(x,y) = -2xy^{-3} = -2x/y^3$, so $f_x(-4,2) = \frac{1}{4}$ and $f_y(-4,2) = 1$. Thus, an equation of the tangent plane is $z - (-1) = f_x(-4,2)\,[x - (-4)] + f_y(-4,2)(y-2) \quad \Rightarrow$ $z + 1 = \frac{1}{4}(x+4) + 1(y-2)$, or $z = \frac{1}{4}x + y - 2$.

9. $z = f(x,y) = x\sin(x+y) \quad \Rightarrow \quad f_x(x,y) = x \cdot \cos(x+y) \cdot 1 + \sin(x+y) \cdot 1 = x\cos(x+y) + \sin(x+y)$ and $f_y(x,y) = x\cos(x+y) \cdot 1$, so $f_x(-1,1) = (-1)\cos 0 + \sin 0 = -1,\ f_y(-1,1) = (-1)\cos 0 = -1$. Thus, an equation of the tangent plane is $z - 0 = f_x(-1,1)(x - (-1)) + f_y(-1,1)(y-1) \quad \Rightarrow \quad z = (-1)(x+1) + (-1)(y-1)$, or $x + y + z = 0$.

10. $z = f(x,y) = \ln(x - 2y) \quad \Rightarrow \quad f_x(x,y) = 1/(x-2y),\ f_y(x,y) = -2/(x-2y)$, so $f_x(3,1) = 1$ and $f_y(3,1) = -2$. Thus, an equation of the tangent plane is $z - 0 = f_x(3,1)(x-3) + f_y(3,1)(y-1) \quad \Rightarrow \quad z = 1(x-3) + (-2)(y-1)$, or $z = x - 2y - 1$.

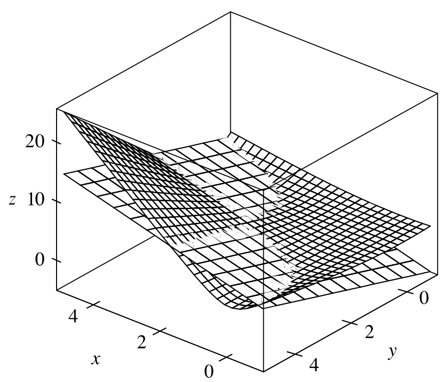
11. $z = f(x,y) = x^2 + xy + 3y^2$, so $f_x(x,y) = 2x + y \quad \Rightarrow \quad f_x(1,1) = 3,\ f_y(x,y) = x + 6y \quad \Rightarrow \quad f_y(1,1) = 7$ and an equation of the tangent plane is $z - 5 = 3(x-1) + 7(y-1)$, or $z = 3x + 7y - 5$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is below the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.

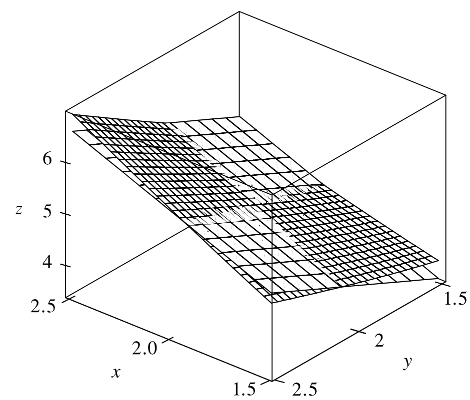




12. $z = f(x,y) = \sqrt{9 + x^2y^2} \;\;\Rightarrow\;\; f_x(x,y) = \frac{1}{2}\left(9 + x^2y^2\right)^{-1/2}\left(2xy^2\right) = xy^2/\sqrt{9 + x^2y^2}$,

$f_y(x,y) = \frac{1}{2}\left(9 + x^2y^2\right)^{-1/2}\left(2x^2y\right) = x^2y/\sqrt{9 + x^2y^2}$, so $f_x(2,2) = \frac{8}{5}$ and $f_y(2,2) = \frac{8}{5}$. Thus, an equation of the tangent plane is $z - 5 = f_x(2,2)(x-2) + f_y(2,2)(y-2) \;\;\Rightarrow\;\; z - 5 = \frac{8}{5}(x-2) + \frac{8}{5}(y-2)$ or $z = \frac{8}{5}x + \frac{8}{5}y - \frac{7}{5}$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here the tangent plane is shown with fewer traces than the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.





13. $f(x,y) = \dfrac{1 + \cos^2(x-y)}{1 + \cos^2(x+y)}$. A CAS gives

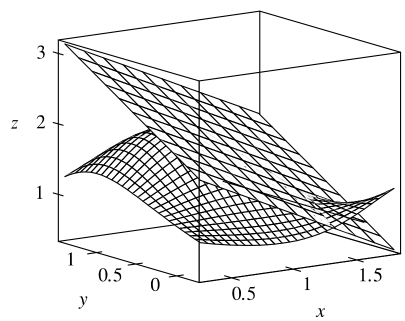
$$f_x(x,y) = -\frac{2\cos(x-y)\sin(x-y)}{1 + \cos^2(x+y)} + \frac{2\left[1 + \cos^2(x-y)\right]\cos(x+y)\sin(x+y)}{\left[1 + \cos^2(x+y)\right]^2} \quad \text{and}$$

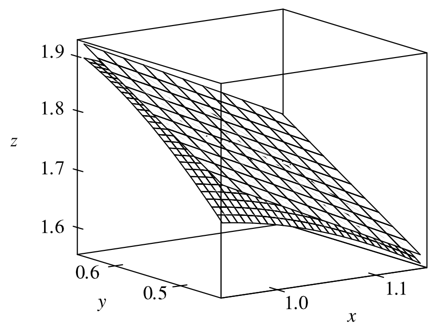
$$f_y(x,y) = \frac{2\cos(x-y)\sin(x-y)}{1 + \cos^2(x+y)} + \frac{2\left[1 + \cos^2(x-y)\right]\cos(x+y)\sin(x+y)}{\left[1 + \cos^2(x+y)\right]^2}.$$

We use the CAS to evaluate these at $(\pi/3, \pi/6)$, giving $f_x(\pi/3, \pi/6) = -\sqrt{3}/2$ and $f_y(\pi/3, \pi/6) = \sqrt{3}/2$. Substituting into Equation 2, an equation of the tangent plane is $z = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}\left(y - \frac{\pi}{6}\right) + \frac{7}{4}$. The surface and tangent plane are shown in the first graph.

[continued]

After zooming in, the surface and the tangent plane become almost indistinguishable, as shown in the second graph. (Here, the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.



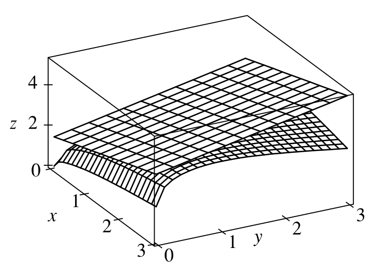


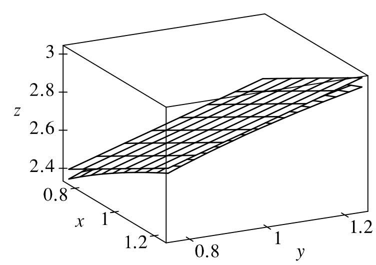
14. $f(x,y) = e^{-xy/10}\left(\sqrt{x} + \sqrt{y} + \sqrt{xy}\right)$. A CAS gives

$f_x(x,y) = -\frac{1}{10}ye^{-xy/10}\left(\sqrt{x} + \sqrt{y} + \sqrt{xy}\right) + e^{-xy/10}\left(\frac{1}{2\sqrt{x}} + \frac{y}{2\sqrt{xy}}\right)$ and

$f_y(x,y) = -\frac{1}{10}xe^{-xy/10}\left(\sqrt{x} + \sqrt{y} + \sqrt{xy}\right) + e^{-xy/10}\left(\frac{1}{2\sqrt{y}} + \frac{x}{2\sqrt{xy}}\right)$. We use the CAS to evaluate these at $(1,1)$, and then substitute the results into Equation 2 to get an equation of the tangent plane:

$z - 3e^{-0.1} = 0.7e^{-0.1}(x-1) + 0.7e^{-0.1}(y-1) \quad\Rightarrow\quad z = 0.7e^{-0.1}x + 0.7e^{-0.1}y + 1.6e^{-0.1}$. The surface and tangent plane are shown in the first graph below. After zooming in, the surface and the tangent plane become almost indistinguishable, as shown in the second graph. (Here, the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.





15. $f(x,y) = x^3y^2$. The partial derivatives are $f_x(x,y) = 3x^2y^2$ and $f_y(x,y) = 2x^3y$, so $f_x(-2,1) = 12$ and $f_y(-2,1) = -16$. Both f_x and f_y are continuous functions, so by Theorem 8, f is differentiable at $(-2,1)$. By Equation 3, the linearization of f at $(-2,1)$ is given by

$L(x,y) = f(-2,1) + f_x(-2,1)(x-(-2)) + f_y(-2,1)(y-1) = -8 + 12(x+2) - 16(y-1) = 12x - 16y + 32$.

16. $f(x,y) = y\tan x$. The partial derivatives are $f_x(x,y) = y\sec^2 x$ and $f_y = \tan x$, so $f_x\left(\frac{\pi}{4},2\right) = 4$ and $f_y\left(\frac{\pi}{4},2\right) = 1$. Both f_x and f_y are continuous for $x \neq \frac{\pi}{2} + n\pi$, so by Theorem 8, f is differentiable at $\left(\frac{\pi}{4},2\right)$. By Equation 3, the linearization of f at $\left(\frac{\pi}{4},2\right)$ is given by $L(x,y) = f\left(\frac{\pi}{4},2\right) + f_x\left(\frac{\pi}{4},2\right)\left(x - \frac{\pi}{4}\right) + f_y\left(\frac{\pi}{4},2\right)(y-2) = 2 + 4\left(x - \frac{\pi}{4}\right) + 1(y-2) = 4x + y - \pi$.

17. $f(x,y)=1+x\ln(xy-5)$. The partial derivatives are $f_x(x,y)=x\cdot\dfrac{1}{xy-5}(y)+\ln(xy-5)\cdot 1=\dfrac{xy}{xy-5}+\ln(xy-5)$ and $f_y(x,y)=x\cdot\dfrac{1}{xy-5}(x)=\dfrac{x^2}{xy-5}$, so $f_x(2,3)=6$ and $f_y(2,3)=4$. Both f_x and f_y are continuous functions for $xy>5$, so by Theorem 8, f is differentiable at $(2,3)$. By Equation 3, the linearization of f at $(2,3)$ is given by

$L(x,y)=f(2,3)+f_x(2,3)(x-2)+f_y(2,3)(y-3)=1+6(x-2)+4(y-3)=6x+4y-23.$

18. $f(x,y)=\sqrt{xy}=(xy)^{1/2}$. The partial derivatives are $f_x(x,y)=\frac{1}{2}(xy)^{-1/2}(y)=y/\left(2\sqrt{xy}\right)$ and $f_y(x,y)=\frac{1}{2}(xy)^{-1/2}(x)=x/\left(2\sqrt{xy}\right)$, so $f_x(1,4)=4/\left(2\sqrt{4}\right)=1$ and $f_y(1,4)=1/\left(2\sqrt{4}\right)=\frac{1}{4}$. Both f_x and f_y are continuous functions for $xy>0$, so f is differentiable at $(1,4)$ by Theorem 8. By Equation 3, the linearization of f at $(1,4)$ is given by $L(x,y)=f(1,4)+f_x(1,4)(x-1)+f_y(1,4)(y-4)=2+1(x-1)+\frac{1}{4}(y-4)=x+\frac{1}{4}y$.

19. $f(x,y)=x^2e^y$. The partial derivatives are $f_x(x,y)=2xe^y$ and $f_y(x,y)=x^2e^y$, so $f_x(1,0)=2$ and $f_y(1,0)=1$. Both f_x and f_y are continuous functions, so by Theorem 8, f is differentiable at $(1,0)$. By Equation 3, the linearization of f at $(1,0)$ is given by $L(x,y)=f(1,0)+f_x(1,0)(x-1)+f_y(1,0)(y-0)=1+2(x-1)+1(y-0)=2x+y-1$.

20. $f(x,y)=\dfrac{1+y}{1+x}=(1+y)(1+x)^{-1}$. The partial derivatives are $f_x(x,y)=(1+y)(-1)(1+x)^{-2}=-\dfrac{1+y}{(1+x)^2}$ and $f_y(x,y)=(1)(1+x)^{-1}=\dfrac{1}{1+x}$, so $f_x(1,3)=-1$ and $f_y(1,3)=\frac{1}{2}$. Both f_x and f_y are continuous functions for $x\neq -1$, so by Theorem 8, f is differentiable at $(1,3)$. By Equation 3, the linearization of f at $(1,3)$ is given by

$L(x,y)=f(1,3)+f_x(1,3)(x-1)+f_y(1,3)(y-3)=2+(-1)(x-1)+\frac{1}{2}(y-3)=-x+\frac{1}{2}y+\frac{3}{2}.$

21. $f(x,y)=4\arctan(xy)$. The partial derivatives are $f_x(x,y)=4\cdot\dfrac{1}{1+(xy)^2}(y)=\dfrac{4y}{1+x^2y^2}$, and $f_y(x,y)=\dfrac{4x}{1+x^2y^2}$, so $f_x(1,1)=2$ and $f_y(1,1)=2$. Both f_x and f_y are continuous functions, so by Theorem 8, f is differentiable at $(1,1)$. By Equation 3, the linearization of f at $(1,1)$ is given by

$L(x,y)=f(1,1)+f_x(1,1)(x-1)+f_y(1,1)(y-1)=4(\pi/4)+2(x-1)+2(y-1)=2x+2y+\pi-4.$

22. $f(x,y)=y+\sin(x/y)$. The partial derivatives are $f_x(x,y)=(1/y)\cos(x/y)$ and $f_y(x,y)=1+(-x/y^2)\cos(x/y)$, so $f_x(0,3)=\frac{1}{3}$ and $f_y(0,3)=1$. Both f_x and f_y are continuous functions for $y\neq 0$, so by Theorem 8, f is differentiable at $(0,3)$. By Equation 3, the linearization of f at $(0,3)$ is given by

$L(x,y)=f(0,3)+f_x(0,3)(x-0)+f_y(0,3)(y-3)=3+\frac{1}{3}(x-0)+1(y-3)=\frac{1}{3}x+y.$

23. Let $f(x,y)=e^x\cos(xy)$. Then $f_x(x,y)=e^x[-\sin(xy)](y)+e^x\cos(xy)=e^x[\cos(xy)-y\sin(xy)]$ and $f_y(x,y)=e^x[-\sin(xy)](x)=-xe^x\sin(xy)$. Both f_x and f_y are continuous functions, so by Theorem 8, f is differentiable at $(0,0)$. We have $f_x(0,0)=e^0(\cos 0-0)=1$, $f_y(0,0)=0$ and the linear approximation of f at $(0,0)$ is

$f(x,y)\approx f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0)=1+1x+0y=x+1.$

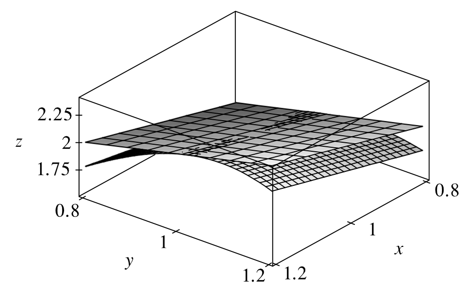
24. Let $f(x,y)=\dfrac{y-1}{x+1}$. Then $f_x(x,y)=(y-1)(-1)(x+1)^{-2}=\dfrac{1-y}{(x+1)^2}$ and $f_y(x,y)=\dfrac{1}{x+1}$. Both f_x and f_y are continuous functions for $x\neq -1$, so by Theorem 8, f is differentiable at $(0,0)$. We have $f_x(0,0)=1$, $f_y(0,0)=1$ and the linear approximation of f at $(0,0)$ is $f(x,y)\approx f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0)=-1+1x+1y=x+y-1$.

25. We can estimate $f(2.2,4.9)$ using a linear approximation of f at $(2,5)$, given by $f(x,y)\approx f(2,5)+f_x(2,5)(x-2)+f_y(2,5)(y-5)=6+1(x-2)+(-1)(y-5)=x-y+9$. Thus, $f(2.2,4.9)\approx 2.2-4.9+9=6.3$.

26. $f(x,y)=1-xy\cos\pi y \quad\Rightarrow\quad f_x(x,y)=-y\cos\pi y$ and $f_y(x,y)=-x[y(-\pi\sin\pi y)+(\cos\pi y)(1)]=\pi xy\sin\pi y-x\cos\pi y$, so $f_x(1,1)=1$, $f_y(1,1)=1$. Then the linear approximation of f at $(1,1)$ is given by

$$\begin{aligned}f(x,y)&\approx f(1,1)+f_x(1,1)(x-1)+f_y(1,1)(y-1)\\&=2+(1)(x-1)+(1)(y-1)=x+y\end{aligned}$$

Thus $f(1.02,0.97)\approx 1.02+0.97=1.99$. We graph f and its tangent plane near the point $(1,1,2)$ below. Notice near $y=1$ the surfaces are almost identical.



27. $f(x,y,z)=\sqrt{x^2+y^2+z^2} \quad\Rightarrow\quad f_x(x,y,z)=\dfrac{x}{\sqrt{x^2+y^2+z^2}}$, $f_y(x,y,z)=\dfrac{y}{\sqrt{x^2+y^2+z^2}}$, and $f_z(x,y,z)=\dfrac{z}{\sqrt{x^2+y^2+z^2}}$, so $f_x(3,2,6)=\frac{3}{7}$, $f_y(3,2,6)=\frac{2}{7}$, $f_z(3,2,6)=\frac{6}{7}$. Then the linear approximation of f at $(3,2,6)$ is given by

$$\begin{aligned}f(x,y,z)&\approx f(3,2,6)+f_x(3,2,6)(x-3)+f_y(3,2,6)(y-2)+f_z(3,2,6)(z-6)\\&=7+\tfrac{3}{7}(x-3)+\tfrac{2}{7}(y-2)+\tfrac{6}{7}(z-6)=\tfrac{3}{7}x+\tfrac{2}{7}y+\tfrac{6}{7}z\end{aligned}$$

Thus, $\sqrt{(3.02)^2+(1.97)^2+(5.99)^2}=f(3.02,1.97,5.99)\approx\frac{3}{7}(3.02)+\frac{2}{7}(1.97)+\frac{6}{7}(5.99)\approx 6.9914$.

28. From the table, $f(80,20)=8.6$. To estimate $f_v(80,20)$ and $f_t(80,20)$, we follow the procedure used in Exercise 14.3.4. Since $f_v(80,20)=lim_{h\to 0}\frac{f(80+h,\,20)-f(80,\,20)}{h}$, we approximate this quantity with $h=\pm 20$ and use the values given in the table:

$$\begin{aligned}f_v(80,20)&\approx\frac{f(100,20)-f(80,20)}{20}=\frac{12.2-8.6}{20}=0.18,\\f_v(80,20)&\approx\frac{f(60,20)-f(80,20)}{-20}=\frac{5.2-8.6}{-20}=0.17\end{aligned}$$

[continued]

Averaging these values gives $f_v(40, 20) \approx 0.175$. Similarly, $f_t(80, 20) = lim_{h \to 0} \frac{f(80,\,20+h) - f(80,\,20)}{h}$, so we use $h = 10$ and $h = -5$:

$$\begin{aligned} f_t(80,20) &\approx \frac{f(80,30) - f(80,20)}{10} = \frac{9.5 - 8.6}{10} = 0.09 \\ f_t(80,20) &\approx \frac{f(80,15) - f(80,20)}{-5} = \frac{7.7 - 8.6}{-5} = 0.18 \end{aligned}$$

Averaging these values gives $f_t(80, 20) \approx 0.135$. The linear approximation, then, is

$$f(v,t) \approx f(80,20) + f_v(80,20)(v - 80) + f_t(80,20)(t - 20) \approx 8.6 + 0.175(v - 80) + 0.135(t - 20)$$

When $v = 83$ and $t = 24$, we estimate $f(83, 24) \approx 8.6 + 0.175(83 - 80) + 0.135(24 - 20) = 9.665$, so we would expect the wave heights to be approximately 9.665 meters.

29. From the table, $f(30, 65) = 40$. To estimate $f_T(30, 65)$ and $f_H(30, 65)$, we follow the procedure used in Section 14.3. Since $f_T(30, 65) = lim_{h \to 0} \frac{f(30+h, 65) - f(30, 65)}{h}$, we approximate this quantity with $h = \pm 2$ and use the values given in the table:

$$\begin{aligned} f_T(30,65) &\approx \frac{f(32,65) - f(30,65)}{2} = \frac{43 - 40}{2} = 1.5 \\ f_T(30,65) &\approx \frac{f(28,65) - f(30,65)}{-2} = \frac{36 - 40}{-2} = 2 \end{aligned}$$

Averaging these values gives $f_T(30, 65) \approx 1.75$. Similarly, $f_H(30, 65) = lim_{h \to 0} \frac{f(30,\,65+h) - f(30,\,65)}{h}$, so we use $h = \pm 5$:

$$\begin{aligned} f_H(30,65) &\approx \frac{f(30,70 - f(30,65)}{5} = \frac{41 - 40}{5} = 0.2 \\ f_H(30,65) &\approx \frac{f(30,60) - f(30,65)}{-5} = \frac{38 - 40}{-5} = 0.4 \end{aligned}$$

Averaging these values gives $f_H(30, 65) \approx 0.3$. The linear approximation, then, is

$$\begin{aligned} f(T,H) &\approx f(30,65) + f_T(30,65)(T - 30) + f_H(30,65)(H - 65) \\ &\approx 40 + 1.75(T - 30) + 0.3(H - 65)\ [or\ 1.75T + 0.3H - 32] \end{aligned}$$

Thus when $T = 33$ and $H = 63$, $f(33, 63) \approx 40 + 1.75(33 - 30) + 0.3(63 - 65) = 44.65$, so we estimate the heat index to be approximately $44.65°$C.

30. From the table, $f(-15, 50) = -29$. To estimate $f_T(-15, 50)$ and $f_v(-15, 50)$ we follow the procedure used in Section 14.3. Since $f_T(-15, 50) = \lim\limits_{h \to 0} \dfrac{f(-15 + h, 50) - f(-15, 50)}{h}$, we approximate this quantity with $h = \pm 5$ and use the values given in the table:

$$f_T(-15,50) \approx \frac{f(-10,50) - f(-15,50)}{5} = \frac{-22 - (-29)}{5} = 1.4$$

$$f_T(-15,50) \approx \frac{f(-20,50) - f(-15,50)}{-5} = \frac{-35 - (-29)}{-5} = 1.2$$

Averaging these values gives $f_T(-15, 50) \approx 1.3$. Similarly $f_v(-15, 50) = \lim\limits_{h \to 0} \dfrac{f(-15, 50 + h) - f(-15, 50)}{h}$,

[continued]

so we use $h = \pm 10$:

$$f_v(-15, 50) \approx \frac{f(-15, 60) - f(-15, 50)}{10} = \frac{-30 - (-29)}{10} = -0.1$$

$$f_v(-15, 50) \approx \frac{f(-15, 40) - f(-15, 50)}{-10} = \frac{-27 - (-29)}{-10} = -0.2$$

Averaging these values gives $f_v(-15, 50) \approx -0.15$. The linear approximation to the wind-chill index function, then, is $f(T, v) \approx f(-15, 50) + f_T(-15, 50)(T - (-15)) + f_v(-15, 50)(v - 50) \approx -29 + (1.3)(T + 15) - (0.15)(v - 50)$. Thus when $T = -17^\circ\text{C}$ and $v = 55$ km/h, $f(-17, 55) \approx -29 + (1.3)(-17 + 15) - (0.15)(55 - 50) = -32.35$, so we estimate the wind-chill index to be approximately -32.35°C.

31. $m = p^5 q^3 \quad \Rightarrow \quad dm = \dfrac{\partial m}{\partial p}\, dp + \dfrac{\partial m}{\partial q}\, dq = 5p^4 q^3\, dp + 3p^5 q^2\, dq$

32. $z = x \ln(y^2 + 1) \quad \Rightarrow \quad dz = \dfrac{\partial z}{\partial x}\, dx + \dfrac{\partial z}{\partial y}\, dy = \ln(y^2 + 1)\, dx + \dfrac{x}{y^2 + 1}(2y)\, dy = \ln(y^2 + 1)\, dx + \dfrac{2xy}{y^2 + 1}\, dy$

33. $z = e^{-2x} \cos 2\pi t \quad \Rightarrow$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial t} dt = e^{-2x}(-2)\cos 2\pi t\, dx + e^{-2x}(-\sin 2\pi t)(2\pi)\, dt = -2e^{-2x}\cos 2\pi t\, dx - 2\pi e^{-2x}\sin 2\pi t\, dt$$

34. $u = \sqrt{x^2 + 3y^2} = (x^2 + 3y^2)^{1/2} \quad \Rightarrow$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \tfrac{1}{2}(x^2 + 3y^2)^{-1/2}(2x)\, dx + \tfrac{1}{2}(x^2 + 3y^2)^{-1/2}(6y)\, dy = \frac{x}{\sqrt{x^2 + 3y^2}}\, dx + \frac{3y}{\sqrt{x^2 + 3y^2}}\, dy$$

35. $H = x^2 y^4 + y^3 z^5 \quad \Rightarrow \quad dH = \dfrac{\partial H}{\partial x}\, dx + \dfrac{\partial H}{\partial y}\, dy + \dfrac{\partial H}{\partial z}\, dz = 2xy^4\, dx + (4x^2 y^3 + 3y^2 z^5)\, dy + 5y^3 z^4\, dz$

36. $w = xze^{-y^2 - z^2} \quad \Rightarrow$

$$\begin{aligned} dw &= \frac{\partial w}{\partial x}\, dx + \frac{\partial w}{\partial y}\, dy + \frac{\partial w}{\partial z}\, dz = ze^{-y^2 - z^2}\, dx + xze^{-y^2 - z^2}(-2y)\, dy + x[z \cdot e^{-y^2 - z^2}(-2z) + e^{-y^2 - z^2} \cdot 1]\, dz \\ &= ze^{-y^2 - z^2}\, dx - 2xyze^{-y^2 - z^2}\, dy + x(1 - 2z^2)e^{-y^2 - z^2}\, dz \end{aligned}$$

37. $R = \alpha\beta^2 \cos\gamma \quad \Rightarrow \quad dR = \dfrac{\partial R}{\partial \alpha}\, d\alpha + \dfrac{\partial R}{\partial \beta}\, d\beta + \dfrac{\partial R}{\partial \gamma}\, d\gamma = \beta^2 \cos\gamma\, d\alpha + 2\alpha\beta\, \cos\gamma\, d\beta - \alpha\beta^2 \sin\gamma\, d\gamma$

38. $T = \dfrac{v}{1 + uvw} \quad \Rightarrow$

$$\begin{aligned} dT &= \frac{\partial T}{\partial u}\, du + \frac{\partial T}{\partial v}\, dv + \frac{\partial T}{\partial w}\, dw \\ &= v(-1)(1 + uvw)^{-2}(vw)\, du + \frac{1(1 + uvw) - v(uw)}{(1 + uvw)^2}\, dv + v(-1)(1 + uvw)^{-2}(uv)\, dw \\ &= -\frac{v^2 w}{(1 + uvw)^2}\, du + \frac{1}{(1 + uvw)^2}\, dv - \frac{uv^2}{(1 + uvw)^2}\, dw \end{aligned}$$

39. $dx = \Delta x = 0.05$, $dy = \Delta y = 0.1$, $z = 5x^2 + y^2$, $z_x = 10x$, $z_y = 2y$. Thus when $x = 1$ and $y = 2$,

$dz = z_x(1,2)\,dx + z_y(1,2)\,dy = (10)(0.05) + (4)(0.1) = 0.9$ while

$\Delta z = f(1.05, 2.1) - f(1,2) = 5(1.05)^2 + (2.1)^2 - 5 - 4 = 0.9225.$

40. $dx = \Delta x = -0.04$, $dy = \Delta y = 0.05$, $z = x^2 - xy + 3y^2$, $z_x = 2x - y$, $z_y = 6y - x$. Thus when $x = 3$ and $y = -1$,

$dz = (7)(-0.04) + (-9)(0.05) = -0.73$ while $\Delta z = (2.96)^2 - (2.96)(-0.95) + 3(-0.95)^2 - (9 + 3 + 3) = -0.7189.$

41. $dA = \dfrac{\partial A}{\partial x}\,dx + \dfrac{\partial A}{\partial y}\,dy = y\,dx + x\,dy$ and $|\Delta x| \le 0.1$, $|\Delta y| \le 0.1$. We use $dx = 0.1$, $dy = 0.1$ with $x = 30$, $y = 24$; then the maximum error in the area is about $dA = 24(0.1) + 30(0.1) = 5.4\ \text{cm}^2$.

42. Let V be the volume. Then $V = \pi r^2 h$ and $\Delta V \approx dV = 2\pi r h\,dr + \pi r^2\,dh$ is an estimate of the amount of metal. With $dr = 0.05$ and $dh = 0.2$ (0.1 on top, 0.1 on bottom), we get $dV = 2\pi(2)(10)(0.05) + \pi(2)^2(0.2) = 2.80\pi \approx 8.8\ \text{cm}^3$.

43. The volume of a can is $V = \pi r^2 h$ and $\Delta V \approx dV$ is an estimate of the amount of tin. Here $dV = 2\pi r h\,dr + \pi r^2\,dh$, so put $dr = 0.04$, $dh = 0.08$ (0.04 on top, 0.04 on bottom) and then $\Delta V \approx dV = 2\pi(48)(0.04) + \pi(16)(0.08) \approx 16.08\ \text{cm}^3$. Thus the amount of tin is about $16\ \text{cm}^3$.

44. (a) Let A be the area, then $A = \frac{1}{2}bh$ and $\Delta A \approx dA = \frac{\partial A}{\partial b}db + \frac{\partial A}{\partial h}dh = \frac{1}{2}h\,db + \frac{1}{2}b\,dh$. We have $|\Delta h| = |\Delta b| \le \epsilon$. So we take $dh = db = \epsilon$ with $b = 70$ centimeters, $h = 40$ centimeters. Then the maximum error in the area is $dA = \frac{1}{2}(40)\epsilon + \frac{1}{2}(70)\epsilon = 55\epsilon$.

(b) With $\epsilon = \frac{64}{100} = \frac{16}{25}$, we have $55(\frac{16}{25}) = \frac{176}{5} = 35.2$ cm.

45. (a) Let V be the volume. Then $V = \pi r^2 h$ and $\Delta V \approx dV = \dfrac{\partial V}{\partial r}\,dr + \dfrac{\partial V}{\partial h}\,dh = 2\pi r h\,dr + \pi r^2\,dh$. We have $|\Delta r| = |\Delta h| = \varepsilon$. So we take $dr = dh = \varepsilon$ with $r = 1$ meter and $h = 4$ meters. Then the maximum error in the volume is $dV = 2\pi(1)(4)\varepsilon + \pi(1)^2\varepsilon = 9\pi\varepsilon\ \text{m}^3$.

(b) We need $9\pi\varepsilon \le 1 \quad \Rightarrow \quad \varepsilon \le 0.03536$ meters or 3.537 centimeters.

46. $W = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$, so the differential of W is

$$\begin{aligned} dW &= \frac{\partial W}{\partial T}\,dT + \frac{\partial W}{\partial v}\,dv = (0.6215 + 0.3965v^{0.16})\,dT + \left[-11.37(0.16)v^{-0.84} + 0.3965T(0.16)v^{-0.84}\right]dv \\ &= (0.6215 + 0.3965v^{0.16})\,dT + (-1.8192 + 0.06344T)v^{-0.84}\,dv \end{aligned}$$

Here we have $|\Delta T| \le 1$, $|\Delta v| \le 2$, so we take $dT = 1$, $dv = 2$ with $T = -11$, $v = 26$. The maximum error in the calculated value of W is about $dW = (0.6215 + 0.3965(26)^{0.16})(1) + (-1.8192 + 0.06344(-11))(26)^{-0.84}(2) \approx 0.96$.

47. $T = \dfrac{mgR}{2r^2 + R^2}$, so the differential of T is

$$dT = \frac{\partial T}{\partial R}\,dR + \frac{\partial T}{\partial r}\,dr = \frac{(2r^2+R^2)(mg) - mgR(2R)}{(2r^2+R^2)^2}\,dR + \frac{(2r^2+R^2)(0) - mgR(4r)}{(2r^2+R^2)^2}\,dr$$
$$= \frac{mg(2r^2 - R^2)}{(2r^2+R^2)^2}\,dR - \frac{4mgRr}{(2r^2+R^2)^2}\,dr$$

Here we have $\Delta R = 0.1$ and $\Delta r = 0.1$, so we take $dR = 0.1$, $dr = 0.1$ with $R = 3$, $r = 0.7$. Then the change in the tension T is approximately

$$dT = \frac{mg[2(0.7)^2 - (3)^2]}{[2(0.7)^2 + (3)^2]^2}\,(0.1) - \frac{4mg(3)(0.7)}{[2(0.7)^2 + (3)^2]^2}\,(0.1)$$
$$= -\frac{0.802mg}{(9.98)^2} - \frac{0.84mg}{(9.98)^2} = -\frac{1.642}{99.6004}mg \approx -0.0165mg$$

Because the change is negative, tension decreases.

48. Here $dV = \Delta V = 0.3$, $dT = \Delta T = -5$, $P = 8.31\,\dfrac{T}{V}$, so

$dP = \left(\dfrac{8.31}{V}\right) dT - \dfrac{8.31 \cdot T}{V^2}\,dV = 8.31\left[-\dfrac{5}{12} - \dfrac{310}{144}\cdot\dfrac{3}{10}\right] \approx -8.83$. Thus, the pressure will drop by about 8.83 kPa.

49. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$. First we find $\dfrac{\partial R}{\partial R_1}$ implicitly by taking partial derivatives of both sides with respect to R_1:

$\dfrac{\partial}{\partial R_1}\left(\dfrac{1}{R}\right) = \dfrac{\partial\left[(1/R_1) + (1/R_2) + (1/R_3)\right]}{\partial R_1} \quad\Rightarrow\quad -R^{-2}\dfrac{\partial R}{\partial R_1} = -R_1^{-2} \quad\Rightarrow\quad \dfrac{\partial R}{\partial R_1} = \dfrac{R^2}{R_1^2}$. Then by symmetry,

$\dfrac{\partial R}{\partial R_2} = \dfrac{R^2}{R_2^2}$, $\dfrac{\partial R}{\partial R_3} = \dfrac{R^2}{R_3^2}$. When $R_1 = 25$, $R_2 = 40$ and $R_3 = 50$, $\dfrac{1}{R} = \dfrac{17}{200} \quad\Leftrightarrow\quad R = \frac{200}{17}\ \Omega$. Since the possible error for each R_i is 0.5%, the maximum error of R is attained by setting $\Delta R_i = 0.005R_i$. So

$$\Delta R \approx dR = \frac{\partial R}{\partial R_1}\Delta R_1 + \frac{\partial R}{\partial R_2}\Delta R_2 + \frac{\partial R}{\partial R_3}\Delta R_3 = (0.005)R^2\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) = (0.005)R = \tfrac{1}{17} \approx 0.059\ \Omega.$$

50. The errors in measurement are at most 2%, so $\left|\dfrac{\Delta w}{w}\right| \le 0.02$ and $\left|\dfrac{\Delta h}{h}\right| \le 0.02$. The relative error in the calculated surface area is

$$\frac{\Delta S}{S} \approx \frac{dS}{S} = \frac{0.0072(0.425w^{0.425-1})h^{0.725}\,dw + 0.0072w^{0.425}(0.725h^{0.725-1})\,dh}{0.0072w^{0.425}h^{0.725}} = 0.425\frac{dw}{w} + 0.725\frac{dh}{h}$$

To estimate the maximum relative error, we use $\dfrac{dw}{w} = \left|\dfrac{\Delta w}{w}\right| = 0.02$ and $\dfrac{dh}{h} = \left|\dfrac{\Delta h}{h}\right| = 0.02 \quad\Rightarrow$

$\dfrac{dS}{S} = 0.425\,(0.02) + 0.725\,(0.02) = 0.023$. Thus, the maximum percentage error is approximately 2.3%.

51. (a) $B(m,h) = m/h^2 \quad\Rightarrow\quad B_m(m,h) = 1/h^2$ and $B_h(m,h) = -2m/h^3$. Since $h > 0$, both B_m and B_h are continuous functions, so B is differentiable at $(23, 1.10)$. We have $B(23, 1.10) = 23/(1.10)^2 \approx 19.01$, $B_m(23, 1.10) = 1/(1.10)^2 \approx 0.8264$, and $B_h(23, 1.10) = -2(23)/(1.10)^3 \approx -34.56$, so the linear approximation of B at $(23, 1.10)$ is

$B\,(m,h) \approx B(23,1.10)+B_m(23,1.10)(m-23)+B_h(23,1.10)(h-1.10) \approx 19.01+0.8264(m-23)-34.56(h-1.10)$

or $\;B(m,h) \approx 0.8264m - 34.56h + 38.02$.

(b) From part (a), for values near $m = 23$ and $h = 1.10$, $B(m,h) \approx 0.8264m - 34.56h + 38.02$. If m increases by 1 kg to 24 kg and h increases by 0.03 m to 1.13 m, we estimate the BMI to be $B(24, 1.13) \approx 0.8264(24) - 34.56(1.13) + 38.02 \approx 18.801$. This is very close to the actual computed BMI: $B(24, 1.13) = 24/(1.13)^2 \approx 18.796$.

52. $\mathbf{r}_1(t) = \left\langle 2+3t, 1-t^2, 3-4t+t^2 \right\rangle \;\Rightarrow\; \mathbf{r}_1'(t) = \langle 3, -2t, -4+2t \rangle$, $\;\mathbf{r}_2(u) = \left\langle 1+u^2, 2u^3-1, 2u+1 \right\rangle \;\Rightarrow$ $\mathbf{r}_2'(u) = \left\langle 2u, 6u^2, 2 \right\rangle$. Both curves pass through $P(2,1,3)$ since $\mathbf{r}_1(0) = \mathbf{r}_2(1) = \langle 2,1,3 \rangle$, so the tangent vectors $\mathbf{r}_1'(0) = \langle 3, 0, -4 \rangle$ and $\mathbf{r}_2'(1) = \langle 2, 6, 2 \rangle$ are both parallel to the tangent plane to S at P. A normal vector for the tangent plane is $\mathbf{r}_1'(0) \times \mathbf{r}_2'(1) = \langle 3, 0, -4 \rangle \times \langle 2, 6, 2 \rangle = \langle 24, -14, 18 \rangle$, so an equation of the tangent plane is $24(x-2) - 14(y-1) + 18(z-3) = 0$ or $12x - 7y + 9z = 44$.

53. To show that f is continuous at (a,b) we need to show that $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ or equivalently $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} f(a+\Delta x, b+\Delta y) = f(a,b)$. Since f is differentiable at (a,b), $f(a+\Delta x, b+\Delta y) - f(a,b) = \Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$, where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. Thus, $f(a+\Delta x, b+\Delta y) = f(a,b) + f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$. Taking the limit of both sides as $(\Delta x, \Delta y) \to (0,0)$ gives $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} f(a+\Delta x, b+\Delta y) = f(a,b)$. Thus, f is continuous at (a,b).

54. (a) $\displaystyle\lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$ and $\displaystyle\lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$. Thus, $f_x(0,0) = f_y(0,0) = 0$. To show that f isn't differentiable at $(0,0)$ we need only show that f is not continuous at $(0,0)$ and apply the contrapositive of Exercise 53. As $(x,y) \to (0,0)$ along the x-axis $f(x,y) = 0/x^2 = 0$ for $x \neq 0$ so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. But as $(x,y) \to (0,0)$ along the line $y = x$, $f(x,x) = x^2/\left(2x^2\right) = \frac{1}{2}$ for $x \neq 0$ so $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along this line. Thus, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is discontinuous at $(0,0)$ and thus not differentiable there.

(b) For $(x,y) \neq (0,0)$, $f_x(x,y) = \dfrac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2} = \dfrac{y(y^2-x^2)}{(x^2+y^2)^2}$. If we approach $(0,0)$ along the y-axis, then $f_x(x,y) = f_x(0,y) = \dfrac{y^3}{y^4} = \dfrac{1}{y}$, so $f_x(x,y) \to \pm\infty$ as $(x,y) \to (0,0)$. Thus, $\displaystyle\lim_{(x,y)\to(0,0)} f_x(x,y)$ does not exist and $f_x(x,y)$ is not continuous at $(0,0)$. Similarly, $f_y(x,y) = \dfrac{(x^2+y^2)x - xy(2y)}{(x^2+y^2)^2} = \dfrac{x(x^2-y^2)}{(x^2+y^2)^2}$ for $(x,y) \neq (0,0)$, and if we approach $(0,0)$ along the x-axis, then $f_y(x,y) = f_x(x,0) = \dfrac{x^3}{x^4} = \dfrac{1}{x}$. Thus, $\displaystyle\lim_{(x,y)\to(0,0)} f_y(x,y)$ does not exist and $f_y(x,y)$ is not continuous at $(0,0)$.

APPLIED PROJECT The Speedo LZR Racer

1. $v(P,C) = \left(\dfrac{2P}{kC}\right)^{1/3} \quad \Rightarrow$

$$f(x,y) = \frac{v(P+xP, C+yC) - v(P,C)}{v(P,C)} = \frac{v(P+xP, C+yC)}{v(P,C)} - \frac{v(P,C)}{v(P,C)} = \frac{\left(\dfrac{2(P+xP)}{k(C+yC)}\right)^{1/3}}{\left(\dfrac{2P}{kC}\right)^{1/3}} - 1$$

$$= \left(\frac{2P(1+x)}{kC(1+y)} \cdot \frac{kC}{2P}\right)^{1/3} - 1 = \left(\frac{1+x}{1+y}\right)^{1/3} - 1$$

Both power and drag cannot be reduced by more than 100%, but both could be increased by any percentage, so $x \geq -1$ and $y \geq -1$. But f is undefined when $y = -1$, so the domain is $\{(x,y) \mid x \geq -1,\ y > -1\}$.

2. If x and y are small, then we can say they are near zero and we can use a linear approximation to f at $(0,0)$.

We have $f(x,y) = (1+x)^{1/3}(1+y)^{-1/3} - 1$ so the partial derivatives are

$f_x(x,y) = \frac{1}{3}(1+x)^{-2/3}(1+y)^{-1/3} = \dfrac{1}{3(1+x)^{2/3}(1+y)^{1/3}}$ and

$f_y(x,y) = -\frac{1}{3}(1+x)^{1/3}(1+y)^{-4/3} = -\dfrac{(1+x)^{1/3}}{3(1+y)^{4/3}}$. Note that f_x and f_y are continuous functions for $x > -1, y > -1$

so f is differentiable at $(0,0)$. Then $f_x(0,0) = \frac{1}{3}$ and $f_y(0,0) = -\frac{1}{3}$, and the linear approximation is

$f(x,y) \approx f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 0 + \frac{1}{3}(x-0) - \frac{1}{3}(y-0) = \frac{1}{3}x - \frac{1}{3}y$. According to the linear approximation, a small fractional increase in power results in 1/3 that fractional increase in speed, and a small decrease in drag has the same effect.

3. $f_{xx}(x,y) = \dfrac{1}{3(1+y)^{1/3}} \cdot \left(-\frac{2}{3}\right)(1+x)^{-5/3} = -\dfrac{2}{9(1+x)^{5/3}(1+y)^{1/3}}$,

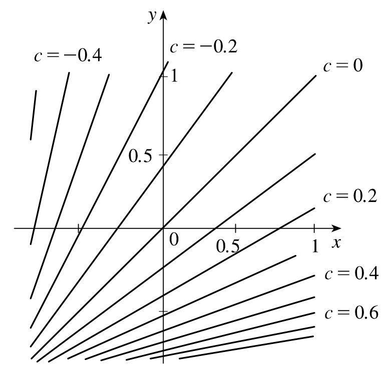
$f_{yy}(x,y) = -\frac{1}{3}(1+x)^{1/3} \cdot \left(-\frac{4}{3}\right)(1+y)^{-7/3} = \dfrac{4(1+x)^{1/3}}{9(1+y)^{7/3}}$. Because $f_x(x,y)$ is positive in the domain of f, an increase in power results in an increase in speed, but $f_{xx}(x,y)$ is negative, so as the fractional power increases, the fractional speed increases at a declining rate. (We can say that in the positive x-direction, f is increasing and concave downward.) Thus the linear approximation gives an overestimate for an increase in power. Since $f_y(x,y)$ is negative, a *decrease* in drag *increases* speed. But $f_{yy}(x,y)$ is positive, so f_y increases as y increases and f_y decreases (f_y becomes larger and larger negative) as y decreases. (In the positive y-direction, f is decreasing and concave upward.) Thus as the fractional drag decreases, the fractional speed increases at an accelerating pace and the linear approximation gives an underestimate of the increase in power. This explains why a decrease in drag is more effective than an increase in power: Reducing drag improves speed at an increasing rate while adding power improves speed at a declining rate.

4. The level curves of $f(x,y) = \left(\dfrac{1+x}{1+y}\right)^{1/3} - 1$ are

$$\left(\frac{1+x}{1+y}\right)^{1/3} - 1 = c \quad \Rightarrow \quad \frac{1+x}{1+y} = (1+c)^3 \quad \Rightarrow$$

$$y = \frac{1+x}{(1+c)^3} - 1.$$

From the level curves, we see that increasing x (from 0) by a small amount has a similar effect on the value of f as decreasing y by a small amount. However, for larger changes, a decrease in y gives greater values of f than a similar increase in x.



14.5 The Chain Rule

1. Find dz/dt using the Chain Rule: $z = x^2y + xy^2, \quad x = 3t, \quad y = t^2 \quad \Rightarrow$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} = (2xy+y^2)(3) + (x^2+2xy)(2t)\\ &= 3\left[2(3t)(t^2) + (t^2)^2\right] + 2t\left[(3t)^2 + 2(3t)(t^2)\right] \qquad [\text{with } x = 3t \text{ and } y = t^2]\\ &= 18t^3 + 3t^4 + 18t^3 + 12t^4 = 36t^3 + 15t^4\end{aligned}$$

Find dz/dt by substituting first: $z\,(x(t), y(t)) = (3t)^2(t^2) + (3t)(t^2)^2 = 9t^4 + 3t^5 \quad \Rightarrow \quad \dfrac{dz}{dt} = 36t^3 + 15t^4$

Yes, the two answers agree.

2. Find dz/dt using the Chain Rule: $z = xye^y, \quad x = t^2, \quad y = 5t \quad \Rightarrow$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} = ye^y(2t) + x(ye^y + e^y)(5)\\ &= (5t)(e^{5t})(2t) + t^2(5te^{5t} + e^{5t})(5) \qquad [\text{with } x = t^2 \text{ and } y = 5t]\\ &= 10t^2e^{5t} + 25t^3e^{5t} + 5t^2e^{5t} = 15t^2e^{5t} + 25t^3e^{5t} = 5t^2e^{5t}(3+5t)\end{aligned}$$

Find dz/dt by substituting first: $z\,(x(t), y(t)) = (t^2)(5t)e^{5t} = 5t^3e^{5t} \quad \Rightarrow \quad \dfrac{dz}{dt} = 5t^3e^{5t}(5) + 15t^2e^{5t} = 5t^2e^{5t}(3+5t)$

Yes, the two answers agree.

3. $z = xy^3 - x^2y, \quad x = t^2+1, \quad y = t^2 - 1 \quad \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (y^3 - 2xy)(2t) + (3xy^2 - x^2)(2t) = 2t(y^3 - 2xy + 3xy^2 - x^2)$$

4. $z = \dfrac{x-y}{x+2y}, \quad x = e^{\pi t}, \quad y = e^{-\pi t} \quad \Rightarrow$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{(x+2y)(1) - (x-y)(1)}{(x+2y)^2}(\pi e^{\pi t}) + \frac{(x+2y)(-1) - (x-y)(2)}{(x+2y)^2}(-\pi e^{-\pi t})\\ &= \frac{3y}{(x+2y)^2}(\pi e^{\pi t}) + \frac{-3x}{(x+2y)^2}(-\pi e^{-\pi t}) = \frac{3\pi}{(x+2y)^2}\left(ye^{\pi t} + xe^{-\pi t}\right)\end{aligned}$$

5. $z = \sin x \cos y, \quad x = \sqrt{t}, \quad y = 1/t \quad \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (\cos x \cos y)\left(\tfrac{1}{2}t^{-1/2}\right) + (-\sin x \sin y)\left(-t^{-2}\right) = \frac{1}{2\sqrt{t}}\cos x \cos y + \frac{1}{t^2}\sin x \sin y$$

6. $z = \sqrt{1 + xy}, \quad x = \tan t, \quad y = \arctan t \quad \Rightarrow$

$$\begin{aligned} \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} &= \tfrac{1}{2}(1+xy)^{-1/2}(y)\cdot \sec^2 t + \tfrac{1}{2}(1+xy)^{-1/2}(x)\cdot \frac{1}{1+t^2} \\ &= \frac{1}{2\sqrt{1+xy}}\left(y\sec^2 t + \frac{x}{1+t^2}\right) \end{aligned}$$

7. $w = xe^{y/z}, \quad x = t^2, \quad y = 1 - t, \quad z = 1 + 2t \quad \Rightarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = e^{y/z}\cdot 2t + xe^{y/z}\left(\frac{1}{z}\right)\cdot(-1) + xe^{y/z}\left(-\frac{y}{z^2}\right)\cdot 2 = e^{y/z}\left(2t - \frac{x}{z} - \frac{2xy}{z^2}\right)$$

8. $w = \ln\sqrt{x^2+y^2+z^2} = \frac{1}{2}\ln(x^2+y^2+z^2), \quad x = \sin t, \quad y = \cos t, \quad z = \tan t \quad \Rightarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = \frac{1}{2}\cdot\frac{2x}{x^2+y^2+z^2}\cdot\cos t + \frac{1}{2}\cdot\frac{2y}{x^2+y^2+z^2}\cdot(-\sin t) + \frac{1}{2}\cdot\frac{2z}{x^2+y^2+z^2}\cdot\sec^2 t$$

$$= \frac{x\cos t - y\sin t + z\sec^2 t}{x^2+y^2+z^2}$$

9. First we find $\partial z/\partial s$ in two ways. $z = x^2 + y^2, \quad x = 2s + 3t, \quad y = s + t \quad \Rightarrow$

$$\begin{aligned} \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} &= 2x(2) + 2y(1) \\ &= 2(2s+3t)(2) + 2(s+t)(1) \qquad [\text{with } x = 2s+3t \text{ and } y = s+t] \\ &= 8s + 12t + 2s + 2t = 10s + 14t \end{aligned}$$

$$z(x(s,t), y(s,t)) = (2s+3t)^2 + (s+t)^2 \quad \Rightarrow \quad \frac{\partial z}{\partial s} = 2(2s+3t)(2) + 2(s+t)(1) = 8s + 12t + 2s + 2t = 10s + 14t$$

Yes, the two answers agree. Now we find $\partial z/\partial t$ in two ways.

$$\begin{aligned} \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} &= 2x(3) + 2y(1) \\ &= 2(2s+3t)(3) + 2(s+t)(1) \qquad [\text{with } x = 2s+3t \text{ and } y = s+t] \\ &= 12s + 18t + 2s + 2t = 14s + 20t \end{aligned}$$

$$z(x(s,t), y(s,t)) = (2s+3t)^2 + (s+t)^2 \quad \Rightarrow \quad \frac{\partial z}{\partial t} = 2(2s+3t)(3) + 2(s+t)(1) = 12s + 18t + 2s + 2t = 14s + 20t$$

Yes, the two answers agree.

10. First we find $\partial z/\partial s$ in two ways. $z = x^2 \sin y, \quad x = s^2 t, \quad y = st \quad \Rightarrow$

$$\begin{aligned} \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} &= 2x\sin y(2st) + x^2\cos y(t) \\ &= 2(s^2t)\sin(st)(2st) + (s^2t)^2\cos(st)(t) \qquad [\text{with } x = s^2t \text{ and } y = st] \\ &= 4s^3t^2\sin(st) + s^4t^3\cos(st) \end{aligned}$$

[continued]

$z(x(s,t), y(s,t)) = (s^2t)^2 \sin(st) = s^4t^2 \sin(st) \quad \Rightarrow$

$\dfrac{\partial z}{\partial s} = s^4t^2 \cos(st)(t) + 4s^3t^2 \sin(st) = 4s^3t^2 \sin(st) + s^4t^3 \cos(st)$

Yes, the two answers agree. Now we find $\partial z/\partial t$ in two ways.

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} = 2x \sin y(s^2) + x^2 \cos y(s) \\
&= 2(s^2t)\sin(st)(s^2) + (s^2t)^2 \cos(st)(s) \qquad \text{[with } x = s^2t \text{ and } y = st\text{]} \\
&= 2s^4t \sin(st) + s^5t^2 \cos(st)
\end{aligned}$$

$z(x(s,t), y(s,t)) = (s^2t)^2 \sin(st) = s^4t^2 \sin(st) \quad \Rightarrow$

$\dfrac{\partial z}{\partial t} = 2s^4t \sin(st) + s^4t^2 \cos(st)(s) = 2s^4t \sin(st) + s^5t^2 \cos(st)$

Yes, the two answers agree.

11. $z = (x-y)^5, \quad x = s^2t, \quad y = st^2 \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 5(x-y)^4(1) \cdot 2st + 5(x-y)^4(-1) \cdot t^2 = 5(x-y)^4\left(2st - t^2\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = 5(x-y)^4(1) \cdot s^2 + 5(x-y)^4(-1) \cdot 2st = 5(x-y)^4\left(s^2 - 2st\right)$$

12. $z = \tan^{-1}(x^2+y^2), \quad x = s\ln t, \quad y = te^s \quad \Rightarrow$

$$\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{2x}{1+(x^2+y^2)^2} \cdot \ln t + \frac{2y}{1+(x^2+y^2)^2} \cdot te^s \\
&= \frac{2}{1+(x^2+y^2)^2}\left(x \ln t + yte^s\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{2x}{1+(x^2+y^2)^2} \cdot \frac{s}{t} + \frac{2y}{1+(x^2+y^2)^2} \cdot e^s \\
&= \frac{2}{1+(x^2+y^2)^2}\left(\frac{xs}{t} + ye^s\right)
\end{aligned}$$

13. $z = \ln(3x+2y), \quad x = s\sin t, \quad y = t\cos s \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{3}{3x+2y}(\sin t) + \frac{2}{3x+2y}(-t\sin s) = \frac{3\sin t - 2t\sin s}{3x+2y}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{3}{3x+2y}(s\cos t) + \frac{2}{3x+2y}(\cos s) = \frac{3s\cos t + 2\cos s}{3x+2y}$$

14. $z = \sqrt{x}\, e^{xy}, \quad x = 1+st, \quad y = s^2 - t^2 \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \left(\sqrt{x} \cdot e^{xy}(y) + e^{xy} \cdot \tfrac{1}{2}x^{-1/2}\right)(t) + \sqrt{x}\, e^{xy}(x)\,(2s) = \left(yt\sqrt{x} + \frac{t}{2\sqrt{x}} + 2x^{3/2}s\right)e^{xy}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left(\sqrt{x} \cdot e^{xy}(y) + e^{xy} \cdot \tfrac{1}{2}x^{-1/2}\right)(s) + \sqrt{x}\, e^{xy}(x)\,(-2t) = \left(ys\sqrt{x} + \frac{s}{2\sqrt{x}} - 2x^{3/2}t\right)e^{xy}$$

15. $z = (\sin\theta)/r, \quad r = st, \quad \theta = s^2 + t^2 \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial s} = -\frac{\sin\theta}{r^2}(t) + \frac{\cos\theta}{r}(2s) = -\frac{t\sin\theta}{r^2} + \frac{2s\cos\theta}{r}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial t} = -\frac{\sin\theta}{r^2}(s) + \frac{\cos\theta}{r}(2t) = -\frac{s\sin\theta}{r^2} + \frac{2t\cos\theta}{r}$$

16. $z = \tan(u/v), \quad u = 2s + 3t, \quad v = 3s - 2t \quad \Rightarrow$

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial s} = \sec^2(u/v)(1/v)\cdot 2 + \sec^2(u/v)(-uv^{-2})\cdot 3\\ &= \frac{2}{v}\sec^2\left(\frac{u}{v}\right) - \frac{3u}{v^2}\sec^2\left(\frac{u}{v}\right) = \frac{2v-3u}{v^2}\sec^2\left(\frac{u}{v}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} = \sec^2(u/v)(1/v)\cdot 3 + \sec^2(u/v)(-uv^{-2})\cdot(-2)\\ &= \frac{3}{v}\sec^2\left(\frac{u}{v}\right) + \frac{2u}{v^2}\sec^2\left(\frac{u}{v}\right) = \frac{2u+3v}{v^2}\sec^2\left(\frac{u}{v}\right)\end{aligned}$$

17. Let $x = g(t)$ and $y = h(t)$. Then $p(t) = f(x,y)$ and the Chain Rule (2) gives $\dfrac{dp}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$. When $t = 2$,

$x = g(2) = 4$ and $y = h(2) = 5$, so $p'(2) = f_x(4,5)\,g'(2) + f_y(4,5)\,h'(2) = (2)(-3) + (8)(6) = 42$.

18. $R(s,t) = G\left(u(s,t), v(s,t)\right) \quad \Rightarrow \quad \dfrac{\partial R}{\partial s} = \dfrac{\partial R}{\partial u}\dfrac{\partial u}{\partial s} + \dfrac{\partial R}{\partial v}\dfrac{\partial v}{\partial s}$ and $\dfrac{\partial R}{\partial t} = \dfrac{\partial R}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial R}{\partial v}\dfrac{\partial v}{\partial t}$ by the

Chain Rule (3). When $s = 1$ and $t = 2$, $u(1,2) = 5$ and $v(1,2) = 7$.

Thus $R_s(1,2) = G_u(5,7)\,u_s(1,2) + G_v(5,7)\,v_s(1,2) = (9)(4) + (-2)(2) = 32$ and

$R_t(1,2) = G_u(5,7)\,u_t(1,2) + G_v(5,7)\,v_t(1,2) = (9)(-3) + (-2)(6) = -39$.

19. $g(u,v) = f(x(u,v), y(u,v))$ where $x = e^u + \sin v$, $y = e^u + \cos v \quad \Rightarrow$

$\dfrac{\partial x}{\partial u} = e^u$, $\dfrac{\partial x}{\partial v} = \cos v$, $\dfrac{\partial y}{\partial u} = e^u$, $\dfrac{\partial y}{\partial v} = -\sin v$. By the Chain Rule (3), $\dfrac{\partial g}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u}$. Then

$g_u(0,0) = f_x(x(0,0), y(0,0))\,x_u(0,0) + f_y(x(0,0), y(0,0))\,y_u(0,0) = f_x(1,2)(e^0) + f_y(1,2)(e^0) = 2(1) + 5(1) = 7$.

Similarly, $\dfrac{\partial g}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v}$. Then

$$\begin{aligned}g_v(0,0) &= f_x(x(0,0), y(0,0))\,x_v(0,0) + f_y(x(0,0), y(0,0))\,y_v(0,0) = f_x(1,2)(\cos 0) + f_y(1,2)(-\sin 0)\\ &= 2(1) + 5(0) = 2\end{aligned}$$

20. $g(r,s) = f(x(r,s), y(r,s))$ where $x = 2r - s$, $y = s^2 - 4r \quad \Rightarrow \quad \dfrac{\partial x}{\partial r} = 2$, $\dfrac{\partial x}{\partial s} = -1$, $\dfrac{\partial y}{\partial r} = -4$, $\dfrac{\partial y}{\partial s} = 2s$.

By the Chain Rule (3), $\dfrac{\partial g}{\partial r} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial r}$. Then

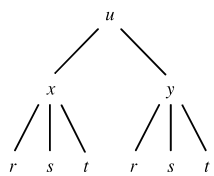
$$\begin{aligned}g_r(1,2) &= f_x(x(1,2), y(1,2))\,x_r(1,2) + f_y(x(1,2), y(1,2))\,y_r(1,2) = f_x(0,0)(2) + f_y(0,0)(-4)\\ &= 4(2) + 8(-4) = -24\end{aligned}$$

[continued]

Similarly, $\dfrac{\partial g}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}$. Then

$$\begin{aligned} g_s(1,2) &= f_x(x(1,2), y(1,2))\, x_s(1,2) + f_y(x(1,2), y(1,2))\, y_s(1,2) = f_x(0,0)(-1) + f_y(0,0)(4) \\ &= 4(-1) + 8(4) = 28 \end{aligned}$$

21.

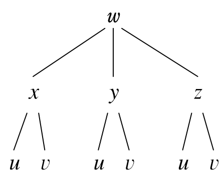


$u = f(x,y),\ x = x(r,s,t),\ y = y(r,s,t) \quad \Rightarrow$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

22.

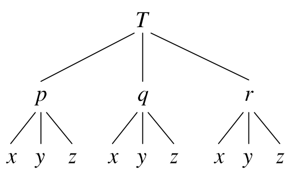


$w = f(x,y,z),\ \ x = x(u,v),\ \ y = y(u,v),\ \ z = z(u,v) \quad \Rightarrow$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u},$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

23.



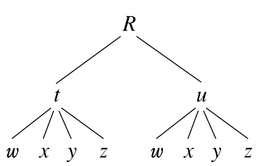
$T = F(p,q,r),\ p = p(x,y,z),\ \ q = q(x,y,z),\ r = r(x,y,z) \quad \Rightarrow$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial T}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial T}{\partial r}\frac{\partial r}{\partial x},$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial T}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial T}{\partial r}\frac{\partial r}{\partial y},$$

$$\frac{\partial T}{\partial z} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial T}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial T}{\partial r}\frac{\partial r}{\partial z}$$

24.



$R = F(t,u),\ \ t = t(w,x,y,z),\ \ u = u(w,x,y,z) \quad \Rightarrow$

$$\frac{\partial R}{\partial w} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial w} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial w}, \quad \frac{\partial R}{\partial x} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial x},$$

$$\frac{\partial R}{\partial y} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial y} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial y}, \quad \frac{\partial R}{\partial z} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial z} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial z}$$

25. $z = x^4 + x^2 y,\ x = s + 2t - u,\ y = stu^2 \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (4x^3 + 2xy)(1) + (x^2)(tu^2),$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (4x^3 + 2xy)(2) + (x^2)(su^2),$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (4x^3 + 2xy)(-1) + (x^2)(2stu).$$

When $s = 4$, $t = 2$, and $u = 1$ we have $x = 7$ and $y = 8$,

so $\dfrac{\partial z}{\partial s} = (1484)(1) + (49)(2) = 1582,\ \ \dfrac{\partial z}{\partial t} = (1484)\,(2) + (49)(4) = 3164,\ \ \dfrac{\partial z}{\partial u} = (1484)(-1) + (49)(16) = -700.$

26. $T = v/(2u+v) = v(2u+v)^{-1}$, $u = pq\sqrt{r}$, $v = p\sqrt{q}\,r$ $\Rightarrow$

$$\frac{\partial T}{\partial p} = \frac{\partial T}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial T}{\partial v}\frac{\partial v}{\partial p} = [-v(2u+v)^{-2}(2)](q\sqrt{r}) + \frac{(2u+v)(1) - v(1)}{(2u+v)^2}\,(\sqrt{q}\,r)$$

$$= \frac{-2v}{(2u+v)^2}\,(q\sqrt{r}) + \frac{2u}{(2u+v)^2}\,(\sqrt{q}\,r),$$

$$\frac{\partial T}{\partial q} = \frac{\partial T}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial T}{\partial v}\frac{\partial v}{\partial q} = \frac{-2v}{(2u+v)^2}\,(p\sqrt{r}) + \frac{2u}{(2u+v)^2}\,\frac{pr}{2\sqrt{q}},$$

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial T}{\partial v}\frac{\partial v}{\partial r} = \frac{-2v}{(2u+v)^2}\,\frac{pq}{2\sqrt{r}} + \frac{2u}{(2u+v)^2}\,(p\sqrt{q}).$$

When $p = 2$, $q = 1$, and $r = 4$ we have $u = 4$ and $v = 8$,

so $\dfrac{\partial T}{\partial p} = \left(-\frac{1}{16}\right)(2) + \left(\frac{1}{32}\right)(4) = 0$, $\dfrac{\partial T}{\partial q} = \left(-\frac{1}{16}\right)(4) + \left(\frac{1}{32}\right)(4) = -\frac{1}{8}$, $\dfrac{\partial T}{\partial r} = \left(-\frac{1}{16}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{32}\right)(2) = \frac{1}{32}$.

27. $w = xy + yz + zx$, $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$ $\Rightarrow$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = (y+z)(\cos\theta) + (x+z)(\sin\theta) + (y+x)(\theta),$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta} = (y+z)(-r\sin\theta) + (x+z)(r\cos\theta) + (y+x)(r).$$

When $r = 2$ and $\theta = \pi/2$ we have $x = 0$, $y = 2$, and $z = \pi$, so $\dfrac{\partial w}{\partial r} = (2+\pi)(0) + (0+\pi)(1) + (2+0)(\pi/2) = 2\pi$

and $\dfrac{\partial w}{\partial \theta} = (2+\pi)(-2) + (0+\pi)(0) + (2+0)(2) = -2\pi$.

28. $P = \sqrt{u^2+v^2+w^2} = (u^2+v^2+w^2)^{1/2}$, $u = xe^y$, $v = ye^x$, $w = e^{xy}$ $\Rightarrow$

$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial P}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial P}{\partial w}\frac{\partial w}{\partial x}$$

$$= \tfrac{1}{2}(u^2+v^2+w^2)^{-1/2}(2u)(e^y) + \tfrac{1}{2}(u^2+v^2+w^2)^{-1/2}(2v)(ye^x) + \tfrac{1}{2}(u^2+v^2+w^2)^{-1/2}(2w)(ye^{xy})$$

$$= \frac{ue^y + vye^x + wye^{xy}}{\sqrt{u^2+v^2+w^2}},$$

$$\frac{\partial P}{\partial y} = \frac{\partial P}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial P}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial P}{\partial w}\frac{\partial w}{\partial y}$$

$$= \frac{u}{\sqrt{u^2+v^2+w^2}}\,(xe^y) + \frac{v}{\sqrt{u^2+v^2+w^2}}\,(e^x) + \frac{w}{\sqrt{u^2+v^2+w^2}}\,(xe^{xy})$$

$$= \frac{uxe^y + ve^x + wxe^{xy}}{\sqrt{u^2+v^2+w^2}}.$$

When $x = 0$ and $y = 2$ we have $u = 0$, $v = 2$, and $w = 1$, so $\dfrac{\partial P}{\partial x} = \dfrac{0+4+2}{\sqrt{5}} = \dfrac{6}{\sqrt{5}}$ and $\dfrac{\partial P}{\partial y} = \dfrac{0+2+0}{\sqrt{5}} = \dfrac{2}{\sqrt{5}}$.

29. $N = \dfrac{p+q}{p+r},\ p = u + vw,\ q = v + uw,\ r = w + uv \quad \Rightarrow$

$$\begin{aligned}\frac{\partial N}{\partial u} &= \frac{\partial N}{\partial p}\frac{\partial p}{\partial u} + \frac{\partial N}{\partial q}\frac{\partial q}{\partial u} + \frac{\partial N}{\partial r}\frac{\partial r}{\partial u} \\ &= \frac{(p+r)(1)-(p+q)(1)}{(p+r)^2}\,(1) + \frac{(p+r)(1)-(p+q)(0)}{(p+r)^2}\,(w) + \frac{(p+r)(0)-(p+q)(1)}{(p+r)^2}\,(v) \\ &= \frac{(r-q)+(p+r)w-(p+q)v}{(p+r)^2},\end{aligned}$$

$$\frac{\partial N}{\partial v} = \frac{\partial N}{\partial p}\frac{\partial p}{\partial v} + \frac{\partial N}{\partial q}\frac{\partial q}{\partial v} + \frac{\partial N}{\partial r}\frac{\partial r}{\partial v} = \frac{r-q}{(p+r)^2}\,(w) + \frac{p+r}{(p+r)^2}\,(1) + \frac{-(p+q)}{(p+r)^2}\,(u) = \frac{(r-q)w+(p+r)-(p+q)u}{(p+r)^2},$$

$$\frac{\partial N}{\partial w} = \frac{\partial N}{\partial p}\frac{\partial p}{\partial w} + \frac{\partial N}{\partial q}\frac{\partial q}{\partial w} + \frac{\partial N}{\partial r}\frac{\partial r}{\partial w} = \frac{r-q}{(p+r)^2}\,(v) + \frac{p+r}{(p+r)^2}\,(u) + \frac{-(p+q)}{(p+r)^2}\,(1) = \frac{(r-q)v+(p+r)u-(p+q)}{(p+r)^2}.$$

When $u = 2$, $v = 3$, and $w = 4$ we have $p = 14$, $q = 11$, and $r = 10$, so $\dfrac{\partial N}{\partial u} = \dfrac{-1+(24)(4)-(25)(3)}{(24)^2} = \dfrac{20}{576} = \dfrac{5}{144}$,

$\dfrac{\partial N}{\partial v} = \dfrac{(-1)(4)+24-(25)(2)}{(24)^2} = \dfrac{-30}{576} = -\dfrac{5}{96}$, and $\dfrac{\partial N}{\partial w} = \dfrac{(-1)(3)+(24)(2)-25}{(24)^2} = \dfrac{20}{576} = \dfrac{5}{144}$.

30. $u = xe^{ty},\ x = \alpha^2\beta,\ y = \beta^2\gamma,\ t = \gamma^2\alpha \quad \Rightarrow$

$$\frac{\partial u}{\partial \alpha} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \alpha} = e^{ty}(2\alpha\beta) + xte^{ty}(0) + xye^{ty}(\gamma^2) = e^{ty}(2\alpha\beta + xy\gamma^2),$$

$$\frac{\partial u}{\partial \beta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \beta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \beta} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \beta} = e^{ty}(\alpha^2) + xte^{ty}(2\beta\gamma) + xye^{ty}(0) = e^{ty}(\alpha^2 + 2xt\beta\gamma),$$

$$\frac{\partial u}{\partial \gamma} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \gamma} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \gamma} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \gamma} = e^{ty}(0) + xte^{ty}(\beta^2) + xye^{ty}(2\gamma\alpha) = e^{ty}(xt\beta^2 + 2xy\alpha\gamma).$$

When $\alpha = -1$, $\beta = 2$, and $\gamma = 1$ we have $x = 2$, $y = 4$, and $t = -1$, so $\dfrac{\partial u}{\partial \alpha} = e^{-4}(-4+8) = 4e^{-4}$,

$\dfrac{\partial u}{\partial \beta} = e^{-4}(1-8) = -7e^{-4}$, and $\dfrac{\partial u}{\partial \gamma} = e^{-4}(-8-16) = -24e^{-4}$.

31. $y\cos x = x^2 + y^2$, so let $F(x,y) = y\cos x - x^2 - y^2 = 0$. Then by Equation 5,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-y\sin x - 2x}{\cos x - 2y} = \frac{2x + y\sin x}{\cos x - 2y}.$$

32. $\cos(xy) = 1 + \sin y$, so let $F(x,y) = \cos(xy) - 1 - \sin y = 0$. Then by Equation 5,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-\sin(xy)(y)}{-\sin(xy)(x) - \cos y} = -\frac{y\sin(xy)}{\cos y + x\sin(xy)}.$$

33. $\tan^{-1}(x^2y) = x + xy^2$, so let $F(x,y) = \tan^{-1}(x^2y) - x - xy^2 = 0$. Then

$$F_x(x,y) = \frac{1}{1+(x^2y)^2}\,(2xy) - 1 - y^2 = \frac{2xy}{1+x^4y^2} - 1 - y^2 = \frac{2xy - (1+y^2)(1+x^4y^2)}{1+x^4y^2},$$

$$F_y(x,y) = \frac{1}{1+(x^2y)^2}\,(x^2) - 2xy = \frac{x^2}{1+x^4y^2} - 2xy = \frac{x^2 - 2xy(1+x^4y^2)}{1+x^4y^2}$$

[continued]

and $$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{[2xy - (1+y^2)(1+x^4y^2)]/(1+x^4y^2)}{[x^2 - 2xy(1+x^4y^2)]/(1+x^4y^2)} = \frac{(1+y^2)(1+x^4y^2) - 2xy}{x^2 - 2xy(1+x^4y^2)}$$
$$= \frac{1 + x^4y^2 + y^2 + x^4y^4 - 2xy}{x^2 - 2xy - 2x^5y^3}$$

34. $e^y \sin x = x + xy$, so let $F(x,y) = e^y \sin x - x - xy = 0$. Then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{e^y \cos x - 1 - y}{e^y \sin x - x} = \dfrac{1 + y - e^y \cos x}{e^y \sin x - x}$.

35. $x^2 + 2y^2 + 3z^2 = 1$, so let $F(x,y,z) = x^2 + 2y^2 + 3z^2 - 1 = 0$. Then by Equations 6,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{6z} = -\frac{x}{3z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{4y}{6z} = -\frac{2y}{3z}.$$

36. $x^2 - y^2 + z^2 - 2z = 4$, so let $F(x,y,z) = x^2 - y^2 + z^2 - 2z - 4 = 0$. Then by Equations 6,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{2z-2} = \frac{x}{1-z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-2y}{2z-2} = \frac{y}{z-1}.$$

37. $e^z = xyz$, so let $F(x,y,z) = e^z - xyz = 0$. Then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{-yz}{e^z - xy} = \dfrac{yz}{e^z - xy}$ and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-xz}{e^z - xy} = \frac{xz}{e^z - xy}.$$

38. $yz + x \ln y = z^2$, so let $F(x,y,z) = yz + x \ln y - z^2 = 0$. Then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{\ln y}{y - 2z} = \dfrac{\ln y}{2z - y}$ and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z + (x/y)}{y - 2z} = \frac{x + yz}{2yz - y^2}.$$

39. Since x and y are each functions of t, $T(x,y)$ is a function of t, so by the Chain Rule, $\dfrac{dT}{dt} = \dfrac{\partial T}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial T}{\partial y}\dfrac{dy}{dt}$. After 3 seconds, $x = \sqrt{1+t} = \sqrt{1+3} = 2$, $y = 2 + \frac{1}{3}t = 2 + \frac{1}{3}(3) = 3$, $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{1+t}} = \dfrac{1}{2\sqrt{1+3}} = \dfrac{1}{4}$, and $\dfrac{dy}{dt} = \dfrac{1}{3}$.

Then $\dfrac{dT}{dt} = T_x(2,3)\,\dfrac{dx}{dt} + T_y(2,3)\,\dfrac{dy}{dt} = 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{3}\right) = 2$. Thus, the temperature is rising at a rate of 2°C/s.

40. (a) Since $\partial W/\partial T$ is negative, a rise in average temperature (while annual rainfall remains constant) causes a decrease in wheat production at the current production levels. Since $\partial W/\partial R$ is positive, an increase in annual rainfall (while the average temperature remains constant) causes an increase in wheat production.

(b) Since the average temperature is rising at a rate of 0.15°C/year, we know that $dT/dt = 0.15$. Since rainfall is decreasing at a rate of $0.1\ \text{cm/year}$, we know $dR/dt = -0.1$. Then, by the Chain Rule,

$\dfrac{dW}{dt} = \dfrac{\partial W}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial W}{\partial R}\dfrac{dR}{dt} = (-2)(0.15) + (8)(-0.1) = -1.1$. Thus, we estimate that wheat production will decrease at a rate of $1.1\ \text{units/year}$.

41. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$, so $\dfrac{\partial C}{\partial T} = 4.6 - 0.11T + 0.00087T^2$ and $\dfrac{\partial C}{\partial D} = 0.016$.

According to the graph, the diver is experiencing a temperature of approximately 12.5°C at $t = 20$ minutes, so

$\dfrac{\partial C}{\partial T} = 4.6 - 0.11(12.5) + 0.00087(12.5)^2 \approx 3.36$. By sketching tangent lines at $t = 20$ to the graphs given, we estimate

$\dfrac{dD}{dt} \approx \dfrac{1}{2}$ and $\dfrac{dT}{dt} \approx -\dfrac{1}{10}$. Then, by the Chain Rule, $\dfrac{dC}{dt} = \dfrac{\partial C}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial C}{\partial D}\dfrac{dD}{dt} \approx (3.36)\left(-\frac{1}{10}\right) + (0.016)\left(\frac{1}{2}\right) \approx -0.33$.

Thus, the speed of sound experienced by the diver is decreasing at a rate of approximately 0.33 m/s per minute.

42. $V = \pi r^2 h/3$, so $\dfrac{dV}{dt} = \dfrac{\partial V}{\partial r}\dfrac{dr}{dt} + \dfrac{\partial V}{\partial h}\dfrac{dh}{dt} = \dfrac{2\pi rh}{3}\, 4.6 + \dfrac{\pi r^2}{3}(-6.5) = 127{,}000\pi\ \text{cm}^3/\text{s}$.

43. (a) $V = \ell wh$, so by the Chain Rule,

$$\frac{dV}{dt} = \frac{\partial V}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = wh\,\frac{d\ell}{dt} + \ell h\,\frac{dw}{dt} + \ell w\,\frac{dh}{dt} = 2\cdot 2\cdot 2 + 1\cdot 2\cdot 2 + 1\cdot 2\cdot(-3) = 6\ \text{m}^3/\text{s}.$$

(b) $S = 2(\ell w + \ell h + wh)$, so by the Chain Rule,

$$\begin{aligned}\frac{dS}{dt} &= \frac{\partial S}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial S}{\partial w}\frac{dw}{dt} + \frac{\partial S}{\partial h}\frac{dh}{dt} = 2(w+h)\,\frac{d\ell}{dt} + 2(\ell+h)\,\frac{dw}{dt} + 2(\ell+w)\,\frac{dh}{dt}\\ &= 2(2+2)2 + 2(1+2)2 + 2(1+2)(-3) = 10\ \text{m}^2/\text{s}\end{aligned}$$

(c) $L^2 = \ell^2 + w^2 + h^2 \quad\Rightarrow\quad 2L\,\dfrac{dL}{dt} = 2\ell\,\dfrac{d\ell}{dt} + 2w\,\dfrac{dw}{dt} + 2h\,\dfrac{dh}{dt} = 2(1)(2) + 2(2)(2) + 2(2)(-3) = 0 \quad\Rightarrow$

$dL/dt = 0$ m/s.

44. $I = \dfrac{V}{R} \quad\Rightarrow$

$$\frac{dI}{dt} = \frac{\partial I}{\partial V}\frac{dV}{dt} + \frac{\partial I}{\partial R}\frac{dR}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{V}{R^2}\frac{dR}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{I}{R}\frac{dR}{dt} = \frac{1}{400}(-0.01) - \frac{0.08}{400}(0.03) = -0.000031\ \text{A/s}$$

45. $\dfrac{dP}{dt} = 0.05$, $\dfrac{dT}{dt} = 0.15$, $V = 8.31\dfrac{T}{P}$ and $\dfrac{dV}{dt} = \dfrac{8.31}{P}\dfrac{dT}{dt} - 8.31\dfrac{T}{P^2}\dfrac{dP}{dt}$. Thus when $P = 20$ and $T = 320$,

$\dfrac{dV}{dt} = 8.31\left[\dfrac{0.15}{20} - \dfrac{(0.05)(320)}{400}\right] \approx -0.27$ L/s.

46. $P = 1.47L^{0.65}K^{0.35}$ and considering P, L, and K as functions of time t we have

$\dfrac{dP}{dt} = \dfrac{\partial P}{\partial L}\dfrac{dL}{dt} + \dfrac{\partial P}{\partial K}\dfrac{dK}{dt} = 1.47(0.65)L^{-0.35}K^{0.35}\dfrac{dL}{dt} + 1.47(0.35)L^{0.65}K^{-0.65}\dfrac{dK}{dt}$. We are given

that $\dfrac{dL}{dt} = -2$ and $\dfrac{dK}{dt} = 0.5$, so when $L = 30$ and $K = 8$, the rate of change of production $\dfrac{dP}{dt}$ is

$1.47(0.65)(30)^{-0.35}(8)^{0.35}(-2) + 1.47(0.35)(30)^{0.65}(8)^{-0.65}(0.5) \approx -0.596$. Thus production at that time

is decreasing at a rate of about \$596,000 per year.

47. Let x be the length of the first side of the triangle and y the length of the second side. The area A of the triangle is given by $A = \frac{1}{2}xy\sin\theta$ [Formula 6 in Appendix D], where θ is the angle between the two sides. Thus A is a function of x, y, and θ, and x, y, and θ are each in turn functions of time t. We are given that $\dfrac{dx}{dt} = 3$, $\dfrac{dy}{dt} = -2$, and because A is constant, $\dfrac{dA}{dt} = 0$.

By the Chain Rule, $\dfrac{dA}{dt} = \dfrac{\partial A}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial A}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial A}{\partial \theta}\dfrac{d\theta}{dt} \;\Rightarrow\; \dfrac{dA}{dt} = \frac{1}{2}y\sin\theta \cdot \dfrac{dx}{dt} + \frac{1}{2}x\sin\theta \cdot \dfrac{dy}{dt} + \frac{1}{2}xy\cos\theta \cdot \dfrac{d\theta}{dt}$.

When $x = 20$, $y = 30$, and $\theta = \pi/6$, we have

$$0 = \tfrac{1}{2}(30)\left(\sin\tfrac{\pi}{6}\right)(3) + \tfrac{1}{2}(20)\left(\sin\tfrac{\pi}{6}\right)(-2) + \tfrac{1}{2}(20)(30)\left(\cos\tfrac{\pi}{6}\right)\frac{d\theta}{dt}$$
$$= 45\cdot\tfrac{1}{2} - 20\cdot\tfrac{1}{2} + 300\cdot\frac{\sqrt{3}}{2}\cdot\frac{d\theta}{dt} = \tfrac{25}{2} + 150\sqrt{3}\,\frac{d\theta}{dt}$$

Solving for $\dfrac{d\theta}{dt}$ gives $\dfrac{d\theta}{dt} = \dfrac{-25/2}{150\sqrt{3}} = -\dfrac{1}{12\sqrt{3}}$, so the angle between the sides is decreasing at a rate of $1/\left(12\sqrt{3}\right) \approx 0.048\ \text{rad/s}$.

48. $f_o = \left(\dfrac{c+v_o}{c-v_s}\right)f_s = \left(\frac{332+34}{332-40}\right)460 \approx 576.6$ Hz. v_o and v_s are functions of time t, so

$$\frac{df_o}{dt} = \frac{\partial f_o}{\partial v_o}\frac{dv_o}{dt} + \frac{\partial f_o}{\partial v_s}\frac{dv_s}{dt} = \left(\frac{1}{c-v_s}\right)f_s\cdot\frac{dv_o}{dt} + \frac{c+v_o}{(c-v_s)^2}f_s\cdot\frac{dv_s}{dt}$$
$$= \left(\tfrac{1}{332-40}\right)(460)\,(1.2) + \tfrac{332+34}{(332-40)^2}\,(460)\,(1.4) \approx 4.65\ \text{Hz/s}$$

49. (a) By the Chain Rule, $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$, $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}(-r\sin\theta) + \dfrac{\partial z}{\partial y}r\cos\theta$.

(b) $\left(\dfrac{\partial z}{\partial r}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\,\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}\cos\theta\,\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2\sin^2\theta$,

$\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2\,\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}r^2\cos\theta\,\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$. Thus,

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right](\cos^2\theta + \sin^2\theta) = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

50. By the Chain Rule, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}e^s\cos t + \dfrac{\partial u}{\partial y}e^s\sin t$, $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}(-e^s\sin t) + \dfrac{\partial u}{\partial y}e^s\cos t$. Then

$\left(\dfrac{\partial u}{\partial s}\right)^2 = \left(\dfrac{\partial u}{\partial x}\right)^2 e^{2s}\cos^2 t + 2\,\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}e^{2s}\cos t\,\sin t + \left(\dfrac{\partial u}{\partial y}\right)^2 e^{2s}\sin^2 t$ and

$\left(\dfrac{\partial u}{\partial t}\right)^2 = \left(\dfrac{\partial u}{\partial x}\right)^2 e^{2s}\sin^2 t - 2\,\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}e^{2s}\cos t\,\sin t + \left(\dfrac{\partial u}{\partial y}\right)^2 e^{2s}\sin^2 t$. Thus,

$$\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]e^{-2s} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

51. Let $u = x + at,\ v = x - at$. Then $z = f(u) + g(v)$, so $\partial z/\partial u = f'(u)$ and $\partial z/\partial v = g'(v)$.

Thus, $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial t} = af'(u) - ag'(v)$ and

$$\frac{\partial^2 z}{\partial t^2} = a\,\frac{\partial}{\partial t}\,[f'(u) - g'(v)] = a\left(\frac{df'(u)}{du}\frac{\partial u}{\partial t} - \frac{dg'(v)}{dv}\frac{\partial v}{\partial t}\right) = a^2 f''(u) + a^2 g''(v).$$

Similarly, $\dfrac{\partial z}{\partial x} = f'(u) + g'(v)$ and $\dfrac{\partial^2 z}{\partial x^2} = f''(u) + g''(v)$. Thus, $\dfrac{\partial^2 z}{\partial t^2} = a^2\,\dfrac{\partial^2 z}{\partial x^2}$.

52. By the Chain Rule, $\dfrac{\partial u}{\partial s} = e^s\cos t\,\dfrac{\partial u}{\partial x} + e^s\sin t\,\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial u}{\partial t} = -e^s\sin t\,\dfrac{\partial u}{\partial x} + e^s\cos t\,\dfrac{\partial u}{\partial y}$.

Then $\dfrac{\partial^2 u}{\partial s^2} = e^s\cos t\,\dfrac{\partial u}{\partial x} + e^s\cos t\,\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial x}\right) + e^s\sin t\,\dfrac{\partial u}{\partial y} + e^s\sin t\,\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial y}\right)$. But

$\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial x^2}\dfrac{\partial x}{\partial s} + \dfrac{\partial^2 u}{\partial y\,\partial x}\dfrac{\partial y}{\partial s} = e^s\cos t\,\dfrac{\partial^2 u}{\partial x^2} + e^s\sin t\,\dfrac{\partial^2 u}{\partial y\,\partial x}$ and

$\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial^2 u}{\partial y^2}\dfrac{\partial y}{\partial s} + \dfrac{\partial^2 u}{\partial x\,\partial y}\dfrac{\partial x}{\partial s} = e^s\sin t\,\dfrac{\partial^2 u}{\partial y^2} + e^s\cos t\,\dfrac{\partial^2 u}{\partial x\,\partial y}$.

Also, by continuity of the partials, $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$. Thus,

$$\begin{aligned}\frac{\partial^2 u}{\partial s^2} &= e^s\cos t\,\frac{\partial u}{\partial x} + e^s\cos t\left(e^s\cos t\,\frac{\partial^2 u}{\partial x^2} + e^s\sin t\,\frac{\partial^2 u}{\partial x\,\partial y}\right) + e^s\sin t\,\frac{\partial u}{\partial y} + e^s\sin t\left(e^s\sin t\,\frac{\partial^2 u}{\partial y^2} + e^s\cos t\,\frac{\partial^2 u}{\partial x\,\partial y}\right)\\ &= e^s\cos t\frac{\partial u}{\partial x} + e^s\sin t\,\frac{\partial u}{\partial y} + e^{2s}\cos^2 t\,\frac{\partial^2 u}{\partial x^2} + 2e^{2s}\cos t\sin t\,\frac{\partial^2 u}{\partial x\,\partial y} + e^{2s}\sin^2 t\,\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= -e^s\cos t\,\frac{\partial u}{\partial x} - e^s\sin t\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) - e^s\sin t\,\frac{\partial u}{\partial y} + e^s\cos t\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)\\ &= -e^s\cos t\,\frac{\partial u}{\partial x} - e^s\sin t\left(-e^s\sin t\,\frac{\partial^2 u}{\partial x^2} + e^s\cos t\,\frac{\partial^2 u}{\partial x\,\partial y}\right)\\ &\qquad -e^s\sin t\,\frac{\partial u}{\partial y} + e^s\cos t\left(e^s\cos t\,\frac{\partial^2 u}{\partial y^2} - e^s\sin t\,\frac{\partial^2 u}{\partial x\,\partial y}\right)\\ &= -e^s\cos t\,\frac{\partial u}{\partial x} - e^s\sin t\,\frac{\partial u}{\partial y} + e^{2s}\sin^2 t\,\frac{\partial^2 u}{\partial x^2} - 2e^{2s}\cos t\sin t\,\frac{\partial^2 u}{\partial x\,\partial y} + e^{2s}\cos^2 t\,\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

Thus, $e^{-2s}\left(\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2}\right) = (\cos^2 t + \sin^2 t)\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$, as desired.

53. $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\,2s + \dfrac{\partial z}{\partial y}\,2r$. Then

$$\begin{aligned}\frac{\partial^2 z}{\partial r\,\partial s} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}2s\right) + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}2r\right)\\ &= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r}\,2s + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r}\,2s + \frac{\partial z}{\partial x}\frac{\partial}{\partial r}\,2s + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\,2r + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r}\,2r + \frac{\partial z}{\partial y}\,2\\ &= 4rs\,\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y\,\partial x}\,4s^2 + 0 + 4rs\,\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x\,\partial y}\,4r^2 + 2\,\frac{\partial z}{\partial y}\end{aligned}$$

By the continuity of the partials, $\dfrac{\partial^2 z}{\partial r\partial s} = 4rs\,\dfrac{\partial^2 z}{\partial x^2} + 4rs\,\dfrac{\partial^2 z}{\partial y^2} + (4r^2 + 4s^2)\,\dfrac{\partial^2 z}{\partial x\,\partial y} + 2\dfrac{\partial z}{\partial y}$.

54. By the Chain Rule,

(a) $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$ (b) $\dfrac{\partial z}{\partial \theta} = -\dfrac{\partial z}{\partial x}\,r\sin\theta + \dfrac{\partial z}{\partial y}\,r\cos\theta$

(c) $$\begin{aligned}\frac{\partial^2 z}{\partial r\,\partial\theta} &= \frac{\partial^2 z}{\partial\theta\,\partial r} = \frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right) = -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\,\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\right) + \cos\theta\,\frac{\partial z}{\partial y} + \sin\theta\,\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\right)\\
&= -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial\theta} + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial y}{\partial\theta}\right) + \cos\theta\,\frac{\partial z}{\partial y} + \sin\theta\,\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial\theta} + \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial\theta}\\
&= -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\left(-r\sin\theta\,\frac{\partial^2 z}{\partial x^2} + r\cos\theta\,\frac{\partial^2 z}{\partial y\,\partial x}\right) + \cos\theta\,\frac{\partial z}{\partial y} + \sin\theta\left(r\cos\theta\,\frac{\partial^2 z}{\partial y^2} - r\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y}\right)\\
&= -\sin\theta\,\frac{\partial z}{\partial x} - r\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x^2} + r\cos^2\theta\,\frac{\partial^2 z}{\partial y\,\partial x} + \cos\theta\,\frac{\partial z}{\partial y} + r\cos\,\theta\sin\theta\,\frac{\partial^2 z}{\partial y^2} - r\sin^2\theta\,\frac{\partial^2 z}{\partial y\,\partial x}\\
&= \cos\theta\,\frac{\partial z}{\partial y} - \sin\theta\,\frac{\partial z}{\partial x} + r\cos\,\theta\sin\theta\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2}\right) + r(\cos^2\theta - \sin^2\theta)\,\frac{\partial^2 z}{\partial y\,\partial x}\end{aligned}$$

55. $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$ and $\dfrac{\partial z}{\partial\theta} = -\dfrac{\partial z}{\partial x}\,r\sin\theta + \dfrac{\partial z}{\partial y}\,r\cos\theta$. Then

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y\,\partial x}\sin\theta\right) + \sin\theta\left(\frac{\partial^2 z}{\partial y^2}\sin\theta + \frac{\partial^2 z}{\partial x\,\partial y}\cos\theta\right)\\
&= \cos^2\theta\,\frac{\partial^2 z}{\partial x^2} + 2\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + \sin^2\theta\,\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 z}{\partial\theta^2} &= -r\cos\theta\,\frac{\partial z}{\partial x} + (-r\sin\theta)\left(\frac{\partial^2 z}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 z}{\partial y\,\partial x}\,r\cos\theta\right)\\
&\qquad -r\sin\theta\,\frac{\partial z}{\partial y} + r\cos\theta\left(\frac{\partial^2 z}{\partial y^2}\,r\cos\theta + \frac{\partial^2 z}{\partial x\,\partial y}(-r\sin\theta)\right)\\
&= -r\cos\theta\,\frac{\partial z}{\partial x} - r\sin\theta\,\frac{\partial z}{\partial y} + r^2\sin^2\theta\,\frac{\partial^2 z}{\partial x^2} - 2r^2\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + r^2\cos^2\theta\,\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} &= (\cos^2\theta + \sin^2\theta)\,\frac{\partial^2 z}{\partial x^2} + \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 z}{\partial y^2}\\
&\qquad -\frac{1}{r}\cos\theta\,\frac{\partial z}{\partial x} - \frac{1}{r}\sin\theta\,\frac{\partial z}{\partial y} + \frac{1}{r}\left(\cos\theta\,\frac{\partial z}{\partial x} + \sin\theta\,\frac{\partial z}{\partial y}\right)\\
&= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\ \text{as desired.}\end{aligned}$$

56. Since f is a polynomial, it has continuous second-order partial derivatives, and

$f(tx, ty) = (tx)^2(ty) + 2(tx)(ty)^2 + 5(ty)^3 = t^3x^2y + 2t^3xy^2 + 5t^3y^3 = t^3(x^2y + 2xy^2 + 5y^3) = t^3 f(x, y)$.

Thus, f is homogeneous of degree 3.

57. (a) Differentiating both sides of $f(tx, ty) = t^n f(x, y)$ with respect to t using the Chain Rule, we get

$$\frac{\partial}{\partial t} f(tx, ty) = \frac{\partial}{\partial t}\left[t^n f(x, y)\right] \quad \Leftrightarrow$$

$$\frac{\partial}{\partial(tx)}f(tx,ty)\cdot\frac{\partial(tx)}{\partial t}+\frac{\partial}{\partial(ty)}f(tx,ty)\cdot\frac{\partial(ty)}{\partial t}=x\frac{\partial}{\partial(tx)}f(tx,ty)+y\frac{\partial}{\partial(ty)}f(tx,ty)=nt^{n-1}f(x,y).$$

Setting $t=1$: $x\dfrac{\partial}{\partial x}f(x,y)+y\dfrac{\partial}{\partial y}f(x,y)=nf(x,y)$.

(b) Differentiating both sides of $f(tx,ty)=t^nf(x,y)$ with respect to t using the Chain Rule, we get

$\dfrac{\partial}{\partial(tx)}f(tx,ty)\cdot\dfrac{\partial(tx)}{\partial t}+\dfrac{\partial}{\partial(ty)}f(tx,ty)\cdot\dfrac{\partial(ty)}{\partial t}=x\dfrac{\partial}{\partial(tx)}f(tx,ty)+y\dfrac{\partial}{\partial(ty)}f(tx,ty)=nt^{n-1}f(x,y)$ and

differentiating again with respect to t gives

$$x\left[\frac{\partial^2}{\partial(tx)^2}f(tx,ty)\cdot\frac{\partial(tx)}{\partial t}+\frac{\partial^2}{\partial(ty)\,\partial(tx)}f(tx,ty)\cdot\frac{\partial(ty)}{\partial t}\right]$$
$$+y\left[\frac{\partial^2}{\partial(tx)\,\partial(ty)}f(tx,ty)\cdot\frac{\partial(tx)}{\partial t}+\frac{\partial^2}{\partial(ty)^2}f(tx,ty)\cdot\frac{\partial(ty)}{\partial t}\right]=n(n-1)t^{n-1}f(x,y).$$

Setting $t=1$ and using the fact that $f_{yx}=f_{xy}$, we have $x^2f_{xx}+2xyf_{xy}+y^2f_{yy}=n(n-1)f(x,y)$.

58. Differentiating both sides of $f(tx,ty)=t^nf(x,y)$ with respect to x using the Chain Rule, we get

$$\frac{\partial}{\partial x}f(tx,ty)=\frac{\partial}{\partial x}[t^nf(x,y)]\quad\Leftrightarrow$$
$$\frac{\partial}{\partial(tx)}f(tx,ty)\cdot\frac{\partial(tx)}{\partial x}+\frac{\partial}{\partial(ty)}f(tx,ty)\cdot\frac{\partial(ty)}{\partial x}=t^n\frac{\partial}{\partial x}f(x,y)\quad\Leftrightarrow\quad tf_x(tx,ty)=t^nf_x(x,y).$$

Thus, $f_x(tx,ty)=t^{n-1}f_x(x,y)$.

59. $F(x,y,z)=0$ is assumed to define z as a function of x and y, that is, $z=f(x,y)$. So by (6), $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}$ since $F_z\neq 0$.

Similarly, it is assumed that $F(x,y,z)=0$ defines x as a function of y and z, that is $x=h(x,z)$. Then $F(h(y,z),y,z)=0$

and by the Chain Rule, $F_x\dfrac{\partial x}{\partial y}+F_y\dfrac{\partial y}{\partial y}+F_z\dfrac{\partial z}{\partial y}=0$. But $\dfrac{\partial z}{\partial y}=0$ and $\dfrac{\partial y}{\partial y}=1$, so $F_x\dfrac{\partial x}{\partial y}+F_y=0\quad\Rightarrow\quad\dfrac{\partial x}{\partial y}=-\dfrac{F_y}{F_x}$.

A similar calculation shows that $\dfrac{\partial y}{\partial z}=-\dfrac{F_z}{F_y}$. Thus, $\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial y}\dfrac{\partial y}{\partial z}=\left(-\dfrac{F_x}{F_z}\right)\left(-\dfrac{F_y}{F_x}\right)\left(-\dfrac{F_z}{F_y}\right)=-1$.

60. Given a function defined implicitly by $F(x,y)=0$, where F is differentiable and $F_y\neq 0$, we know that $\dfrac{dy}{dx}=-\dfrac{F_x}{F_y}$. Let

$G(x,y)=-\dfrac{F_x}{F_y}$ so $\dfrac{dy}{dx}=G(x,y)$. Differentiating both sides with respect to x and using the Chain Rule gives

$\dfrac{d^2y}{dx^2}=\dfrac{\partial G}{\partial x}\dfrac{dx}{dx}+\dfrac{\partial G}{\partial y}\dfrac{dy}{dx}$ where $\dfrac{\partial G}{\partial x}=\dfrac{\partial}{\partial x}\left(-\dfrac{F_x}{F_y}\right)=-\dfrac{F_yF_{xx}-F_xF_{yx}}{F_y^2}$, $\dfrac{\partial G}{\partial y}=\dfrac{\partial}{\partial y}\left(-\dfrac{F_x}{F_y}\right)=-\dfrac{F_yF_{xy}-F_xF_{yy}}{F_y^2}$.

Thus,
$$\begin{aligned}\frac{d^2y}{dx^2}&=\left(-\frac{F_yF_{xx}-F_xF_{yx}}{F_y^2}\right)(1)+\left(-\frac{F_yF_{xy}-F_xF_{yy}}{F_y^2}\right)\left(-\frac{F_x}{F_y}\right)\\&=-\frac{F_{xx}F_y^2-F_{yx}F_xF_y-F_{xy}F_yF_x+F_{yy}F_x^2}{F_y^3}\end{aligned}$$

But F has continuous second derivatives, so by Clauraut's Theorem, $F_{yx}=F_{xy}$ and we have

$\dfrac{d^2y}{dx^2}=-\dfrac{F_{xx}F_y^2-2F_{xy}F_xF_y+F_{yy}F_x^2}{F_y^3}$ as desired.

14.6 Directional Derivatives and the Gradient Vector

1. We can approximate the directional derivative of the pressure function at K in the direction of S by the average rate of change of pressure between the points where the red line intersects the contour lines closest to K (extend the red line slightly at the left). In the direction of S, the pressure changes from 1000 millibars to 996 millibars and we estimate the distance between these two points to be approximately 50 km (using the fact that the distance from K to S is 300 km). Then the rate of change of pressure in the direction given is approximately $\frac{996-1000}{50} = -0.08$ millibar/km.

2. First we draw a line passing through Dubbo and Sydney. We approximate the directional derivative at Dubbo in the direction of Sydney by the average rate of change of temperature between the points where the line intersects the contour lines closest to Dubbo. In the direction of Sydney, the temperature changes from 30°C to 27°C. We estimate the distance between these two points to be approximately 120 km, so the rate of change of maximum temperature in the direction given is approximately $\frac{27-30}{120} = -0.025^\circ\text{C/km}$.

3. $D_{\mathbf{u}}\, f(-20,30) = \nabla f\,(-20,30)\cdot\mathbf{u} = f_T(-20,30)\left(\frac{1}{\sqrt{2}}\right) + f_v(-20,30)\left(\frac{1}{\sqrt{2}}\right)$.

$f_T(-20,30) = \lim\limits_{h\to 0}\dfrac{f(-20+h,30)-f(-20,30)}{h}$, so we can approximate $f_T(-20,30)$ by considering $h=\pm 5$ and using the values given in the table: $f_T(-20,30) \approx \dfrac{f(-15,30)-f(-20,30)}{5} = \dfrac{-26-(-33)}{5} = 1.4$,

$f_T(-20,30) \approx \dfrac{f(-25,30)-f(-20,30)}{-5} = \dfrac{-39-(-33)}{-5} = 1.2$. Averaging these values gives $f_T(-20,30)\approx 1.3$.

Similarly, $f_v(-20,30) = \lim\limits_{h\to 0}\dfrac{f(-20,30+h)-f(-20,30)}{h}$, so we can approximate $f_v(-20,30)$ with $h=\pm 10$:

$f_v(-20,30) \approx \dfrac{f(-20,40)-f(-20,30)}{10} = \dfrac{-34-(-33)}{10} = -0.1$,

$f_v(-20,30) \approx \dfrac{f(-20,20)-f(-20,30)}{-10} = \dfrac{-30-(-33)}{-10} = -0.3$. Averaging these values gives $f_v(-20,30)\approx -0.2$.

Then $D_{\mathbf{u}}f(-20,30) \approx 1.3\left(\frac{1}{\sqrt{2}}\right) + (-0.2)\left(\frac{1}{\sqrt{2}}\right) \approx 0.778$.

4. $f(x,y) = xy^3 - x^2 \quad\Rightarrow\quad f_x(x,y) = y^3 - 2x$ and $f_y(x,y) = 3xy^2$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \pi/3$, then from Equation 6, $D_{\mathbf{u}}f(1,2) = f_x(1,2)\cos\left(\frac{\pi}{3}\right) + f_y(1,2)\sin\left(\frac{\pi}{3}\right) = 6\cdot\frac{1}{2} + 12\cdot\frac{\sqrt{3}}{2} = 3 + 6\sqrt{3}$.

5. $f(x,y) = y\cos(xy) \quad\Rightarrow\quad f_x(x,y) = y[-\sin(xy)](y) = -y^2\sin(xy)$ and

$f_y(x,y) = y[-\sin(xy)](x) + [\cos(xy)](1) = \cos(xy) - xy\sin(xy)$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \pi/4$, then from Equation 6, $D_{\mathbf{u}}f(0,1) = f_x(0,1)\cos\left(\frac{\pi}{4}\right) + f_y(0,1)\sin\left(\frac{\pi}{4}\right) = 0\cdot\frac{\sqrt{2}}{2} + 1\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$.

6. $f(x,y) = \sqrt{2x+3y} \quad\Rightarrow\quad f_x(x,y) = \frac{1}{2}(2x+3y)^{-1/2}(2) = 1/\sqrt{2x+3y}$ and

$f_y(x,y) = \frac{1}{2}(2x+3y)^{-1/2}(3) = 3/\left(2\sqrt{2x+3y}\right)$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = -\pi/6$, then from Equation 6,

$D_{\mathbf{u}}f(3,1) = f_x(3,1)\cos\left(-\frac{\pi}{6}\right) + f_y(3,1)\sin\left(-\frac{\pi}{6}\right) = \frac{1}{3}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\left(-\frac{1}{2}\right) = \frac{\sqrt{3}}{6} - \frac{1}{4}$.

7. $f(x,y) = \arctan(xy) \quad \Rightarrow \quad f_x(x,y) = \dfrac{y}{1+(xy)^2}$ and $f_y(x,y) = \dfrac{x}{1+(xy)^2}$. If $\mathbf{u}$ is a unit vector in the direction $\theta = 3\pi/4$, then from Equation 6,

$$D_{\mathbf{u}} f(2,-3) = f_x(2,-3)\cos\left(\frac{3\pi}{4}\right) + f_y(2,-3)\sin\left(\frac{3\pi}{4}\right) = -\frac{3}{37}\left(-\frac{\sqrt{2}}{2}\right) + \frac{2}{37}\left(\frac{\sqrt{2}}{2}\right) = \frac{5\sqrt{2}}{74}.$$

8. $f(x,y) = x^2 e^y$

(a) $\nabla f(x,y) = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} = 2xe^y\,\mathbf{i} + x^2 e^y\,\mathbf{j}$

(b) $\nabla f(3,0) = 2(3)e^0\,\mathbf{i} + 3^2 e^0\,\mathbf{j} = 6\,\mathbf{i} + 9\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}} f(3,0) = \nabla f(3,0)\cdot\mathbf{u} = (6\mathbf{i} + 9\,\mathbf{j})\cdot\left(\frac{3}{5}\,\mathbf{i} - \frac{4}{5}\,\mathbf{j}\right) = \frac{18}{5} - \frac{36}{5} = -\frac{18}{5}$.

9. $f(x,y) = x/y = xy^{-1}$

(a) $\nabla f(x,y) = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} = y^{-1}\,\mathbf{i} + (-xy^{-2})\,\mathbf{j} = \dfrac{1}{y}\,\mathbf{i} - \dfrac{x}{y^2}\,\mathbf{j}$

(b) $\nabla f(2,1) = \dfrac{1}{1}\,\mathbf{i} - \dfrac{2}{1^2}\,\mathbf{j} = \mathbf{i} - 2\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\, f(2,1) = \nabla f(2,1)\cdot\mathbf{u} = (\mathbf{i} - 2\,\mathbf{j})\;\cdot\left(\frac{3}{5}\,\mathbf{i} + \frac{4}{5}\,\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1$.

10. $f(x,y) = x^2 \ln y$

(a) $\nabla f(x,y) = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} = 2x\ln y\,\mathbf{i} + (x^2/y)\,\mathbf{j}$

(b) $\nabla f(3,1) = 0\,\mathbf{i} + (9/1)\,\mathbf{j} = 9\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\, f(3,1) = \nabla f(3,1)\cdot\mathbf{u} = 9\,\mathbf{j}\;\cdot\left(-\frac{5}{13}\,\mathbf{i} + \frac{12}{13}\,\mathbf{j}\right) = 0 + \frac{108}{13} = \frac{108}{13}$.

11. $f(x,y,z) = x^2yz - xyz^3$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \langle 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2\rangle$

(b) $\nabla f(2,-1,1) = \langle -4+1, 4-2, -4+6\rangle = \langle -3,2,2\rangle$

(c) By Equation 14, $D_{\mathbf{u}} f(2,-1,1) = \nabla f(2,-1,1)\cdot\mathbf{u} = \langle -3,2,2\rangle\cdot\left\langle 0, \frac{4}{5}, -\frac{3}{5}\right\rangle = 0 + \frac{8}{5} - \frac{6}{5} = \frac{2}{5}$.

12. $f(x,y,z) = y^2 e^{xyz}$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \left\langle y^2e^{xyz}(yz), y^2\cdot e^{xyz}(xz) + e^{xyz}\cdot 2y, y^2e^{xyz}(xy)\right\rangle$

$\qquad = \left\langle y^3ze^{xyz}, (xy^2z + 2y)e^{xyz}, xy^3e^{xyz}\right\rangle$

(b) $\nabla f(0,1,-1) = \langle -1,2,0\rangle$

(c) $D_{\mathbf{u}} f(0,1,-1) = \nabla f(0,1,-1)\cdot\mathbf{u} = \langle -1,2,0\rangle\cdot\left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right\rangle = -\frac{3}{13} + \frac{8}{13} + 0 = \frac{5}{13}$

13. $f(x,y) = e^x \sin y \quad \Rightarrow \quad \nabla f(x,y) = \langle e^x\sin y, e^x\cos y\rangle$, $\nabla f(0,\pi/3) = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{(-6)^2+8^2}}\langle -6,8\rangle = \frac{1}{10}\langle -6,8\rangle = \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle$, so

$D_{\mathbf{u}}\, f(0,\pi/3) = \nabla f(0,\pi/3)\cdot\mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle\cdot\left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle = -\frac{3\sqrt{3}}{10} + \frac{4}{10} = \frac{4-3\sqrt{3}}{10}$.

14. $f(x,y) = \dfrac{x}{x^2+y^2} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle \dfrac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}, \dfrac{0 - x(2y)}{(x^2+y^2)^2} \right\rangle = \left\langle \dfrac{y^2-x^2}{(x^2+y^2)^2}, -\dfrac{2xy}{(x^2+y^2)^2} \right\rangle,$

$\nabla f(1,2) = \left\langle \frac{3}{25}, -\frac{4}{25} \right\rangle$, and a unit vector in the direction of $\mathbf{v} = \langle 3,5 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{9+25}} \langle 3,5 \rangle = \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$, so

$D_{\mathbf{u}} f(1,2) = \nabla f(1,2) \cdot \mathbf{u} = \left\langle \frac{3}{25}, -\frac{4}{25} \right\rangle \cdot \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle = \frac{9}{25\sqrt{34}} - \frac{20}{25\sqrt{34}} = -\frac{11}{25\sqrt{34}}.$

15. $g(s,t) = s\sqrt{t} \quad \Rightarrow \quad \nabla g(s,t) = \left(\sqrt{t}\,\right)\mathbf{i} + \left(s/(2\sqrt{t}\,)\right)\mathbf{j}$, $\nabla g(2,4) = 2\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{2^2+(-1)^2}}(2\,\mathbf{i} - \mathbf{j}) = \frac{1}{\sqrt{5}}(2\,\mathbf{i} - \mathbf{j})$, so $D_{\mathbf{u}}\, g(2,4) = \nabla g(2,4) \cdot \mathbf{u} = (2\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}) \cdot \frac{1}{\sqrt{5}}(2\,\mathbf{i} - \mathbf{j}) = \frac{1}{\sqrt{5}}\left(4 - \frac{1}{2}\right) = \frac{7}{2\sqrt{5}}$ or $\frac{7\sqrt{5}}{10}$.

16. $g(u,v) = u^2 e^{-v} \quad \Rightarrow \quad \nabla g(u,v) = \left(2ue^{-v}\right)\mathbf{i} + \left(-u^2 e^{-v}\right)\mathbf{j}$, $\nabla g(3,0) = 6\,\mathbf{i} - 9\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{3^2+4^2}}(3\,\mathbf{i} + 4\,\mathbf{j}) = \frac{1}{5}(3\,\mathbf{i} + 4\,\mathbf{j})$, so $D_{\mathbf{u}}\, g(3,0) = \nabla g(3,0) \cdot \mathbf{u} = (6\,\mathbf{i} - 9\,\mathbf{j}) \cdot \frac{1}{5}(3\,\mathbf{i} + 4\,\mathbf{j}) = \frac{1}{5}\,(18 - 36) = -\frac{18}{5}.$

17. $f(x,y,z) = x^2 y + y^2 z \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle 2xy, x^2 + 2yz, y^2 \right\rangle$, $\nabla f(1,2,3) = \langle 4,13,4 \rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{4+1+4}}\langle 2,-1,2 \rangle = \frac{1}{3}\langle 2,-1,2 \rangle$, so

$D_{\mathbf{u}}\, f(1,2,3) = \nabla f(1,2,3) \cdot \mathbf{u} = \langle 4,13,4 \rangle \cdot \frac{1}{3}\langle 2,-1,2 \rangle = \frac{1}{3}\,(8 - 13 + 8) = \frac{3}{3} = 1.$

18. $f(x,y,z) = xy^2 \tan^{-1} z \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle y^2 \tan^{-1} z, 2xy \tan^{-1} z, \dfrac{xy^2}{1+z^2} \right\rangle,$

$\nabla f(2,1,1) = \left\langle 1 \cdot \frac{\pi}{4}, 4 \cdot \frac{\pi}{4}, \frac{2}{1+1} \right\rangle = \left\langle \frac{\pi}{4}, \pi, 1 \right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{1+1+1}}\langle 1,1,1 \rangle = \frac{1}{\sqrt{3}}\,\langle 1,1,1 \rangle$, so $D_{\mathbf{u}}\, f(2,1,1) = \nabla f(2,1,1) \cdot \mathbf{u} = \left\langle \frac{\pi}{4}, \pi, 1 \right\rangle \cdot \frac{1}{\sqrt{3}}\,\langle 1,1,1 \rangle = \frac{1}{\sqrt{3}}\left(\frac{\pi}{4} + \pi + 1\right) = \frac{1}{\sqrt{3}}\left(\frac{5\pi}{4} + 1\right).$

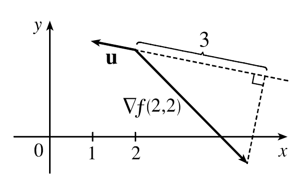
19. $h(r,s,t) = \ln(3r + 6s + 9t) \quad \Rightarrow \quad \nabla h(r,s,t) = \langle 3/(3r+6s+9t), 6/(3r+6s+9t), 9/(3r+6s+9t) \rangle,$

$\nabla h(1,1,1) = \left\langle \frac{1}{6}, \frac{1}{3}, \frac{1}{2} \right\rangle$, and a unit vector in the direction of $\mathbf{v} = 4\,\mathbf{i} + 12\,\mathbf{j} + 6\,\mathbf{k}$ is

$\mathbf{u} = \frac{1}{\sqrt{16+144+36}}\,(4\,\mathbf{i} + 12\,\mathbf{j} + 6\,\mathbf{k}) = \frac{2}{7}\,\mathbf{i} + \frac{6}{7}\,\mathbf{j} + \frac{3}{7}\,\mathbf{k}$, so

$D_{\mathbf{u}}\, h(1,1,1) = \nabla h(1,1,1) \cdot \mathbf{u} = \left\langle \frac{1}{6}, \frac{1}{3}, \frac{1}{2} \right\rangle \cdot \left\langle \frac{2}{7}, \frac{6}{7}, \frac{3}{7} \right\rangle = \frac{1}{21} + \frac{2}{7} + \frac{3}{14} = \frac{23}{42}.$

20. $D_{\mathbf{u}} f(2,2) = \nabla f(2,2) \cdot \mathbf{u}$, the scalar projection of $\nabla f(2,2)$ onto $\mathbf{u}$, so we draw a perpendicular from the tip of $\nabla f(2,2)$ to the line containing $\mathbf{u}$. We can use the point $(2,2)$ to determine the scale of the axes, and we estimate the length of the projection to be approximately 3.0 units. Since the angle between $\nabla f(2,2)$ and $\mathbf{u}$ is greater than 90°, the scalar projection is negative. Thus $D_{\mathbf{u}}\, f(2,2) \approx -3$.



21. $f(x,y) = x^2 y^2 - y^3 \quad \Rightarrow \quad \nabla f(x,y) = \left\langle 2xy^2, 2x^2 y - 3y^2 \right\rangle$, so $\nabla f(1,2) = \langle 8,-8 \rangle$. The unit vector in the direction of $\overrightarrow{PQ} = \langle -3-1, 5-2 \rangle = \langle -4,3 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{(-4)^2 + 3^2}}\,\langle -4,3 \rangle = \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle$, so

$D_{\mathbf{u}} f(1,2) = \nabla f(1,2) \cdot \mathbf{u} = \langle 8,-8 \rangle \cdot \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle = -\frac{56}{5}.$

22. $f(x,y) = \dfrac{x}{y^2} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle \dfrac{1}{y^2}, -\dfrac{2x}{y^3} \right\rangle$, so $\nabla f(3,-1) = \langle 1, 6 \rangle$. The unit vector in the direction of

$\overrightarrow{PQ} = \langle -2-3, 11-(-1) \rangle = \langle -5, 12 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{(-5)^2 + 12^2}} \langle -5, 12 \rangle = \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle$, so

$D_{\mathbf{u}} f(3,-1) = \nabla f(3,-1) \cdot \mathbf{u} = \langle 1, 6 \rangle \cdot \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle = \frac{67}{13}$.

23. $f(x,y) = \sqrt{xy} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle \frac{1}{2}(xy)^{-1/2}(y), \frac{1}{2}(xy)^{-1/2}(x) \right\rangle = \left\langle \dfrac{y}{2\sqrt{xy}}, \dfrac{x}{2\sqrt{xy}} \right\rangle$, so $\nabla f(2,8) = \left\langle 1, \frac{1}{4} \right\rangle$.

The unit vector in the direction of $\overrightarrow{PQ} = \langle 5-2, 4-8 \rangle = \langle 3, -4 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{3^2 + (-4)^2}} \langle 3, -4 \rangle = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$,

so $D_{\mathbf{u}}\, f(2,8) = \nabla f(2,8) \cdot \mathbf{u} = \left\langle 1, \frac{1}{4} \right\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle = \frac{2}{5}$.

24. $f(x,y,z) = xy^2z^3 \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle y^2z^3, 2xyz^3, 3xy^2z^2 \right\rangle$, so $\nabla f(2,1,1) = \langle 1, 4, 6 \rangle$. The unit vector in the direction

of $\overrightarrow{PQ} = \langle 0-2, -3-1, 5-1 \rangle = \langle -2, -4, 4 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{(-2)^2 + (-4)^2 + 4^2}} \langle -2, -4, 4 \rangle = \frac{1}{6} \langle -2, -4, 4 \rangle$, so

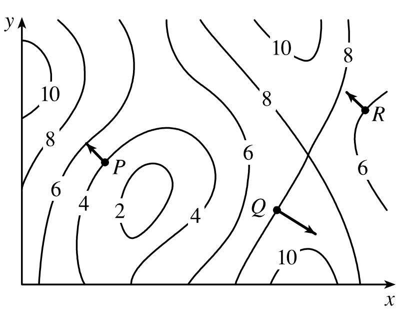
$D_{\mathbf{u}}\, f(2,1,1) = \nabla f(2,1,1) \cdot \mathbf{u} = \langle 1, 4, 6 \rangle \cdot \frac{1}{6} \langle -2, -4, 4 \rangle = \frac{1}{6}(-2 - 16 + 24) = 1$.

25. $f(x,y,z) = xy - xy^2z^2 \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle y - y^2z^2, x - 2xyz^2, -2xy^2z \right\rangle$, so $\nabla f(2,-1,1) = \langle -2, 6, -4 \rangle$.

The unit vector in the direction of $\overrightarrow{PQ} = \langle 5-2, 1-(-1), 7-1 \rangle = \langle 3, 2, 6 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{3^2 + 2^2 + 6^2}} \langle 3, 2, 6 \rangle = \left\langle \frac{3}{7}, \frac{2}{7}, \frac{6}{7} \right\rangle$, so

$D_{\mathbf{u}} f(2,-1,1) = \nabla f(2,-1,1) \cdot \mathbf{u} = \langle -2, 6, -4 \rangle \cdot \left\langle \frac{3}{7}, \frac{2}{7}, \frac{6}{7} \right\rangle = -\frac{18}{7}$.

26.



Note that the vectors drawn at P, Q, and R are perpendicular to the curves and represent the direction of maximum increase.

27. $f(x,y) = 5xy^2 \quad \Rightarrow \quad \nabla f(x,y) = \left\langle 5y^2, 10xy \right\rangle$. Then $\nabla f(3,-2) = \langle 20, -60 \rangle$, or equivalently, $\langle 1, -3 \rangle$ is the direction of maximum rate of change, and the maximum rate of change is $|\nabla f(3,-2)| = \sqrt{20^2 + (-60)^2} = \sqrt{4000} = 20\sqrt{10}$.

28. $f(s,t) = \dfrac{s}{s^2 + t^2} = s(s^2 + t^2)^{-1} \quad \Rightarrow$

$$\nabla f(s,t) = \left\langle s(-1)(s^2+t^2)^{-2}(2s) + (1)(s^2+t^2)^{-1},\ s(-1)(s^2+t^2)^{-2}(2t) \right\rangle = \left\langle \frac{t^2 - s^2}{(s^2+t^2)^2}, -\frac{2st}{(s^2+t^2)^2} \right\rangle.$$

Then $\nabla f(-1,1) = \left\langle 0, \frac{1}{2} \right\rangle$ is the direction of maximum rate of change, and the maximum rate of change is $|\nabla f(-1,1)| = \frac{1}{2}$.

29. $f(x,y) = \sin(xy) \quad \Rightarrow \quad \nabla f(x,y) = \langle y\cos(xy), x\cos(xy) \rangle$, $\nabla f(1,0) = \langle 0, 1 \rangle$. Thus, the maximum rate of change is $|\nabla f(1,0)| = 1$ in the direction $\langle 0, 1 \rangle$.

30. $f(x,y,z) = x\ln(yz) \quad\Rightarrow\quad \nabla f(x,y,z) = \left\langle \ln(yz), x\cdot\dfrac{z}{yz}, x\cdot\dfrac{y}{yz}\right\rangle = \left\langle \ln(yz), \dfrac{x}{y}, \dfrac{x}{z}\right\rangle,\ \ \nabla f(1,2,\frac{1}{2}) = \langle 0, \frac{1}{2}, 2\rangle.$

Thus, the maximum rate of change is $\left|\nabla f(1,2,\frac{1}{2})\right| = \sqrt{0+\frac{1}{4}+4} = \sqrt{\frac{17}{4}} = \frac{\sqrt{17}}{2}$ in the direction $\langle 0, \frac{1}{2}, 2\rangle$, or equivalently, $\langle 0,1,4\rangle$.

31. $f(x,y,z) = x/(y+z) = x(y+z)^{-1} \quad\Rightarrow$

$\nabla f(x,y,z) = \langle 1/(y+z), -x(y+z)^{-2}(1), -x(y+z)^{-2}(1)\rangle = \left\langle \dfrac{1}{y+z}, -\dfrac{x}{(y+z)^2}, -\dfrac{x}{(y+z)^2}\right\rangle,$

$\nabla f(8,1,3) = \langle \frac{1}{4}, -\frac{8}{4^2}, -\frac{8}{4^2}\rangle = \langle \frac{1}{4}, -\frac{1}{2}, -\frac{1}{2}\rangle$. Thus, the maximum rate of change is

$|\nabla f(8,1,3)| = \sqrt{\frac{1}{16}+\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$ in the direction $\langle \frac{1}{4}, -\frac{1}{2}, -\frac{1}{2}\rangle$, or equivalently, $\langle 1,-2,-2\rangle$.

32. $f(p,q,r) = \arctan(pqr) \quad\Rightarrow\quad \nabla f(p,q,r) = \left\langle \dfrac{qr}{1+(pqr)^2}, \dfrac{pr}{1+(pqr)^2}, \dfrac{pq}{1+(pqr)^2}\right\rangle, \nabla f(1,2,1) = \langle \frac{2}{5}, \frac{1}{5}, \frac{2}{5}\rangle.$

Thus, the maximum rate of change is $|\nabla f(1,2,1)| = \sqrt{\frac{4}{25}+\frac{1}{25}+\frac{4}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$ in the direction $\langle \frac{2}{5}, \frac{1}{5}, \frac{2}{5}\rangle$, or equivalently, $\langle 2,1,2\rangle$.

33. (a) As in the proof of Theorem 15, $D_{\mathbf{u}}f = |\nabla f|\cos\theta$. Since the minimum value of $\cos\theta$ is -1, occurring when $\theta = \pi$, the maximum rate of decrease of $D_{\mathbf{u}}f$ is $-|\nabla f|$ occurring when $\theta = \pi$; that is, when $\mathbf{u}$ is in the opposite direction of ∇f (assuming $\nabla f \neq \mathbf{0}$).

(b) $f(x,y) = x^4y - x^2y^3 \quad\Rightarrow\quad \nabla f(x,y) = \langle 4x^3y - 2xy^3, x^4 - 3x^2y^2\rangle$, so f decreases fastest at the point $(2,-3)$ in the direction $-\nabla f(2,-3) = -\langle 12,-92\rangle = \langle -12, 92\rangle$. The maximum rate of decrease is

$-|\nabla f(2,-3)| = -|\langle 12,-92\rangle| = -\sqrt{12^2+(-92)^2} = -\sqrt{8608} = -4\sqrt{538}$.

34. $f(x,y) = x^2 + xy^3 \quad\Rightarrow\quad \nabla f(x,y) = \langle 2x+y^3, 3xy^2\rangle$ so $\nabla f(2,1) = \langle 5,6\rangle$. If $\mathbf{u} = \langle a,b\rangle$ is a unit vector in the desired direction then $D_{\mathbf{u}}f(2,1) = 2 \quad\Leftrightarrow\quad \langle 5,6\rangle\cdot\langle a,b\rangle = 2 \quad\Leftrightarrow\quad 5a+6b = 2 \quad\Leftrightarrow\quad b = \frac{1}{3} - \frac{5}{6}a$. But $a^2+b^2 = 1 \quad\Leftrightarrow$ $a^2 + \left(\frac{1}{3} - \frac{5}{6}a\right)^2 = 1 \quad\Leftrightarrow\quad \frac{61}{36}a^2 - \frac{5}{9}a + \frac{1}{9} = 1 \quad\Leftrightarrow\quad 61a^2 - 20a - 32 = 0$. By the quadratic formula, the solutions are

$a = \dfrac{-(-20)\pm\sqrt{(-20)^2 - 4(61)(-32)}}{2(61)} = \dfrac{20\pm\sqrt{8208}}{122} = \dfrac{10\pm 6\sqrt{57}}{61}$. If $a = \dfrac{10+6\sqrt{57}}{61} \approx 0.9065$, then

$b = \dfrac{1}{3} - \dfrac{5}{6}\left(\dfrac{10+6\sqrt{57}}{61}\right) \approx -0.4221$, and if $a = \dfrac{10-6\sqrt{57}}{61} \approx -0.5787$ then $b = \dfrac{1}{3} - \dfrac{5}{6}\left(\dfrac{10-6\sqrt{57}}{61}\right) \approx 0.8156$.

Thus, the two directions giving a directional derivative of 2 are approximately $\langle 0.9065, -0.4221\rangle$ and $\langle -0.5787, 0.8156\rangle$.

35. For $f(x,y) = x^2 + y^2 - 2x - 4y$, the direction of greatest rate of change is $\nabla f(x,y) = (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j}$, so we need to find all points (x,y) where $\nabla f(x,y)$ is parallel to $\mathbf{i}+\mathbf{j} \quad\Leftrightarrow\quad (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j} = c\,(\mathbf{i}+\mathbf{j}) \quad\Leftrightarrow\quad c = 2x-2$ and $c = 2y-4$. Then $2x - 2 = 2y - 4 \quad\Rightarrow\quad y = x+1$, so at all points on the line $y = x+1$, the direction of greatest rate of change of f is $\mathbf{i}+\mathbf{j}$.

36. The fisherman is traveling in the direction $\langle -80, -60\rangle$. A unit vector in this direction is $\mathbf{u} = \frac{1}{100}\langle -80, -60\rangle = \left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle$, and if the depth of the lake is given by $f(x, y) = 200 + 0.02x^2 - 0.001y^3$, then $\nabla f(x, y) = \left\langle 0.04x, -0.003y^2\right\rangle$. $D_{\mathbf{u}}\, f(80, 60) = \nabla f(80, 60) \cdot \mathbf{u} = \langle 3.2, -10.8\rangle \cdot \left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle = 3.92$. Since $D_{\mathbf{u}}\, f(80, 60)$ is positive, the depth of the lake is increasing near $(80, 60)$ in the direction toward the buoy.

37. $T = \dfrac{k}{\sqrt{x^2 + y^2 + z^2}}$ and $120 = T(1, 2, 2) = \dfrac{k}{3}$ so $k = 360$.

(a) $\mathbf{u} = \dfrac{\langle 1, -1, 1\rangle}{\sqrt{3}}$,

$$D_{\mathbf{u}}T(1, 2, 2) = \nabla T(1, 2, 2) \cdot \mathbf{u} = \left[-360\left(x^2 + y^2 + z^2\right)^{-3/2}\langle x, y, z\rangle\right]_{(1,2,2)} \cdot \mathbf{u} = -\tfrac{40}{3}\langle 1, 2, 2\rangle \cdot \tfrac{1}{\sqrt{3}}\langle 1, -1, 1\rangle = -\tfrac{40}{3\sqrt{3}}$$

(b) From (a), $\nabla T = -360\left(x^2 + y^2 + z^2\right)^{-3/2}\langle x, y, z\rangle$, and since $\langle x, y, z\rangle$ is the position vector of the point (x, y, z), the vector $-\langle x, y, z\rangle$, and thus ∇T, always points toward the origin.

38. $\nabla T = -400e^{-x^2-3y^2-9z^2}\langle x, 3y, 9z\rangle$

(a) $\mathbf{u} = \frac{1}{\sqrt{6}}\langle 1, -2, 1\rangle$, $\nabla T(2, -1, 2) = -400e^{-43}\langle 2, -3, 18\rangle$ and

$$D_{\mathbf{u}}\, T(2, -1, 2) = \left(-\frac{400e^{-43}}{\sqrt{6}}\right)(26) = -\frac{5200\sqrt{6}}{3e^{43}}\,{}^\circ\mathrm{C/m}.$$

(b) $\nabla T(2, -1, 2) = 400e^{-43}\langle -2, 3, -18\rangle$ or equivalently $\langle -2, 3, -18\rangle$.

(c) $|\nabla T| = 400e^{-x^2-3y^2-9z^2}\sqrt{x^2 + 9y^2 + 81z^2}\,{}^\circ\mathrm{C/m}$ is the maximum rate of increase. At $(2, -1, 2)$ the maximum rate of increase is $400e^{-43}\sqrt{337}\,{}^\circ\mathrm{C/m}$.

39. $\nabla V(x, y, z) = \langle 10x - 3y + yz, xz - 3x, xy\rangle$, $\nabla V(3, 4, 5) = \langle 38, 6, 12\rangle$

(a) $D_{\mathbf{u}}\, V(3, 4, 5) = \langle 38, 6, 12\rangle \cdot \frac{1}{\sqrt{3}}\langle 1, 1, -1\rangle = \frac{32}{\sqrt{3}}$

(b) $\nabla V(3, 4, 5) = \langle 38, 6, 12\rangle$, or equivalently, $\langle 19, 3, 6\rangle$.

(c) $|\nabla V(3, 4, 5)| = \sqrt{38^2 + 6^2 + 12^2} = \sqrt{1624} = 2\sqrt{406}$

40. $z = f(x, y) = 1000 - 0.005x^2 - 0.01y^2 \;\Rightarrow\; \nabla f(x, y) = \langle -0.01x, -0.02y\rangle$ and $\nabla f(60, 40) = \langle -0.6, -0.8\rangle$.

(a) Due south is in the direction of the unit vector $\mathbf{u} = -\mathbf{j}$ and $D_{\mathbf{u}}f(60, 40) = \nabla f(60, 40) \cdot \langle 0, -1\rangle = \langle -0.6, -0.8\rangle \cdot \langle 0, -1\rangle = 0.8$. Thus, if you walk due south from $(60, 40, 966)$ you will ascend at a rate of 0.8 vertical meters per horizontal meter.

(b) Northwest is in the direction of the unit vector $\mathbf{u} = \frac{1}{\sqrt{2}}\langle -1, 1\rangle$ and $D_{\mathbf{u}}f(60, 40) = \nabla f(60, 40) \cdot \frac{1}{\sqrt{2}}\langle -1, 1\rangle = \langle -0.6, -0.8\rangle \cdot \frac{1}{\sqrt{2}}\langle -1, 1\rangle = -\frac{0.2}{\sqrt{2}} \approx -0.14$. Thus, if you walk northwest from $(60, 40, 966)$ you will descend at a rate of approximately 0.14 vertical meters per horizontal meter.

(c) $\nabla f(60,40) = \langle -0.6, -0.8 \rangle$ is the direction of largest slope with a rate of ascent given by

$|\nabla f(60,40)| = \sqrt{(-0.6)^2 + (-0.8)^2} = 1$. The angle above the horizontal in which the path begins is given by

$\tan\theta = 1 \quad \Rightarrow \quad \theta = 45°$.

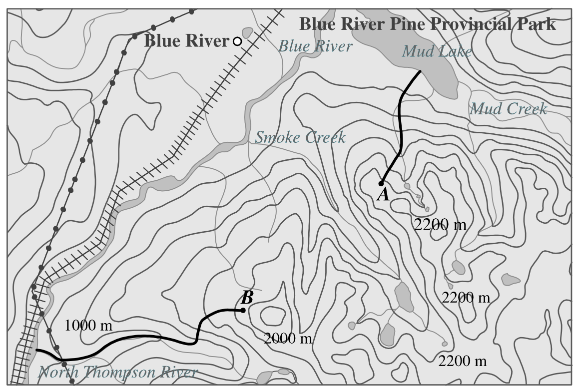
41. A unit vector in the direction of $\overrightarrow{AB}$ is $\mathbf{i}$ and a unit vector in the direction of $\overrightarrow{AC}$ is $\mathbf{j}$. Thus $D_{\overrightarrow{AB}}\, f(1,3) = f_x(1,3) = 3$ and

$D_{\overrightarrow{AC}}\, f(1,3) = f_y(1,3) = 26$. Therefore $\nabla f(1,3) = \langle f_x(1,3), f_y(1,3)\rangle = \langle 3, 26\rangle$, and by definition,

$D_{\overrightarrow{AD}}\, f(1,3) = \nabla f \cdot \mathbf{u}$ where $\mathbf{u}$ is a unit vector in the direction of $\overrightarrow{AD}$, which is $\left\langle \frac{5}{13}, \frac{12}{13}\right\rangle$. Therefore,

$D_{\overrightarrow{AD}}\, f\,(1,3) = \langle 3, 26\rangle \cdot \left\langle \frac{5}{13}, \frac{12}{13}\right\rangle = 3 \cdot \frac{5}{13} + 26 \cdot \frac{12}{13} = \frac{327}{13}$.

42. The curves of steepest ascent or descent are perpendicular to all of the contour lines (see Figure 13) so we sketch curves beginning at A and B that head toward lower elevations, crossing each contour line at a right angle.



43. (a) $\nabla(au+bv) = \left\langle \dfrac{\partial(au+bv)}{\partial x}, \dfrac{\partial(au+bv)}{\partial y}\right\rangle = \left\langle a\,\dfrac{\partial u}{\partial x} + b\,\dfrac{\partial v}{\partial x}, a\,\dfrac{\partial u}{\partial y} + b\,\dfrac{\partial v}{\partial y}\right\rangle = a\left\langle \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right\rangle + b\left\langle \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}\right\rangle$

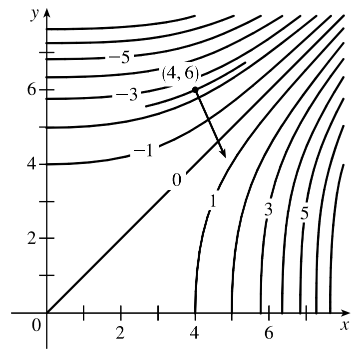
$= a\,\nabla u + b\,\nabla v$

(b) $\nabla(uv) = \left\langle v\,\dfrac{\partial u}{\partial x} + u\,\dfrac{\partial v}{\partial x}, v\,\dfrac{\partial u}{\partial y} + u\,\dfrac{\partial v}{\partial y}\right\rangle = v\left\langle \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right\rangle + u\left\langle \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}\right\rangle = v\,\nabla u + u\,\nabla v$

(c) $\nabla\left(\dfrac{u}{v}\right) = \left\langle \dfrac{v\,\dfrac{\partial u}{\partial x} - u\,\dfrac{\partial v}{\partial x}}{v^2}, \dfrac{v\,\dfrac{\partial u}{\partial y} - u\,\dfrac{\partial v}{\partial y}}{v^2}\right\rangle = \dfrac{v\left\langle \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right\rangle - u\left\langle \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}\right\rangle}{v^2} = \dfrac{v\,\nabla u - u\,\nabla v}{v^2}$

(d) $\nabla u^n = \left\langle \dfrac{\partial(u^n)}{\partial x}, \dfrac{\partial(u^n)}{\partial y}\right\rangle = \left\langle nu^{n-1}\,\dfrac{\partial u}{\partial x}, nu^{n-1}\,\dfrac{\partial u}{\partial y}\right\rangle = nu^{n-1}\,\nabla u$

44. If we place the initial point of the gradient vector $\nabla f(4,6)$ at $(4,6)$, the vector is perpendicular to the level curve of f that includes $(4,6)$, so we sketch a portion of the level curve through $(4,6)$ (using the nearby level curves as a guideline) and draw a line perpendicular to the curve at $(4,6)$. The gradient vector is parallel to this line, pointing in the direction of increasing function values, and with length equal to the maximum value of the directional derivative of f at $(4, 6)$. We can estimate this length by finding the average rate of change in the direction of the gradient. The line intersects the contour lines corresponding to -2 and -3 with an estimated distance of 0.5 units. Thus the rate of change is approximately $\dfrac{-2-(-3)}{0.5} = 2$, and we sketch the gradient vector with length 2.



45. $f(x, y) = x^3 + 5x^2y + y^3 \quad \Rightarrow$

$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = \left\langle 3x^2 + 10xy, 5x^2 + 3y^2 \right\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \frac{9}{5}x^2 + 6xy + 4x^2 + \frac{12}{5}y^2 = \frac{29}{5}x^2 + 6xy + \frac{12}{5}y^2.$

Then

$$\begin{aligned} D^2_{\mathbf{u}}f(x, y) &= D_{\mathbf{u}}\left[D_{\mathbf{u}}f(x, y)\right] = \nabla\left[D_{\mathbf{u}}f(x, y)\right] \cdot \mathbf{u} = \left\langle \tfrac{58}{5}x + 6y, 6x + \tfrac{24}{5}y \right\rangle \cdot \left\langle \tfrac{3}{5}, \tfrac{4}{5} \right\rangle \\ &= \tfrac{174}{25}x + \tfrac{18}{5}y + \tfrac{24}{5}x + \tfrac{96}{25}y = \tfrac{294}{25}x + \tfrac{186}{25}y \end{aligned}$$

and $D^2_{\mathbf{u}}f(2, 1) = \frac{294}{25}(2) + \frac{186}{25}(1) = \frac{774}{25}$.

46. (a) From Equation 9 we have $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \langle f_x, f_y \rangle \cdot \langle a, b \rangle = f_x a + f_y b$ and from Exercise 39 we have

$$\begin{aligned} D^2_{\mathbf{u}}f &= D_{\mathbf{u}}\left[D_{\mathbf{u}}f\right] = \nabla\left[D_{\mathbf{u}}f\right] \cdot \mathbf{u} \\ &= \langle f_{xx}a + f_{yx}b, f_{xy}a + f_{yy}b \rangle \cdot \langle a, b \rangle \\ &= f_{xx}a^2 + f_{yx}ab + f_{xy}ab + f_{yy}b^2 \end{aligned}$$

But $f_{yx} = f_{xy}$ by Clairaut's Theorem, so $D^2_{\mathbf{u}}f = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2$.

(b) $f(x, y) = xe^{2y} \quad \Rightarrow \quad f_x = e^{2y}, f_y = 2xe^{2y}, f_{xx} = 0, f_{xy} = 2e^{2y}, f_{yy} = 4xe^{2y}$ and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{4^2+6^2}}\langle 4, 6 \rangle = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle = \langle a, b \rangle$. Then

$$\begin{aligned} D^2_{\mathbf{u}}f &= f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2 \\ &= 0 \cdot \left(\tfrac{2}{\sqrt{13}}\right)^2 + 2 \cdot 2e^{2y}\left(\tfrac{2}{\sqrt{13}}\right)\left(\tfrac{3}{\sqrt{13}}\right) + 4xe^{2y}\left(\tfrac{3}{\sqrt{13}}\right)^2 = \tfrac{24}{13}e^{2y} + \tfrac{36}{13}xe^{2y} \end{aligned}$$

47. Let $F(x, y, z) = 2(x-2)^2 + (y-1)^2 + (z-3)^2$. Then $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10$ is a level surface of F.

$F_x(x, y, z) = 4(x-2) \quad \Rightarrow \quad F_x(3, 3, 5) = 4$, $F_y(x, y, z) = 2(y-1) \quad \Rightarrow \quad F_y(3, 3, 5) = 4$, and

$F_z(x, y, z) = 2(z-3) \quad \Rightarrow \quad F_z(3, 3, 5) = 4$.

(a) Equation 19 gives an equation of the tangent plane at $(3, 3, 5)$ as $4(x-3) + 4(y-3) + 4(z-5) = 0 \quad \Leftrightarrow$

$4x + 4y + 4z = 44$ or equivalently $x + y + z = 11$.

(b) By Equation 20, the normal line has symmetric equations $\dfrac{x-3}{4} = \dfrac{y-3}{4} = \dfrac{z-5}{4}$ or equivalently

$x - 3 = y - 3 = z - 5$. Corresponding parametric equations are $x = 3 + t$, $y = 3 + t$, $z = 5 + t$.

48. Let $F(x,y,z) = y^2 + z^2 - x$. Then $x = y^2 + z^2 + 1 \quad \Leftrightarrow \quad y^2 + z^2 - x = -1$ is a level surface of F.

$F_x(x,y,z) = -1 \quad \Rightarrow \quad F_x(3,1,-1) = -1$, $F_y(x,y,z) = 2y \quad \Rightarrow \quad F_y(3,1,-1) = 2$, and $F_z(x,y,z) = 2z \quad \Rightarrow$ $F_z(3,1,-1) = -2$.

(a) By Equation 19, an equation of the tangent plane at $(3,1,-1)$ is $(-1)(x-3) + 2(y-1) + (-2)[z-(-1)] = 0$ or $-x + 2y - 2z = 1$ or $x - 2y + 2z = -1$.

(b) By Equation 20, the normal line has symmetric equations $\dfrac{x-3}{-1} = \dfrac{y-1}{2} = \dfrac{z-(-1)}{-2}$ or equivalently $x - 3 = \dfrac{y-1}{-2} = \dfrac{z+1}{2}$ and parametric equations $x = 3 - t$, $y = 1 + 2t$, $z = -1 - 2t$.

49. Let $F(x,y,z) = xy^2z^3$. Then $xy^2z^3 = 8$ is a level surface of F and $\nabla F(x,y,z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$.

(a) $\nabla F(2,2,1) = \langle 4, 8, 24 \rangle$ is a normal vector for the tangent plane at $(2,2,1)$, so an equation of the tangent plane is $4(x-2) + 8(y-2) + 24(z-1) = 0$ or $4x + 8y + 24z = 48$ or equivalently $x + 2y + 6z = 12$.

(b) The normal line has direction $\nabla F(2,2,1) = \langle 4, 8, 24 \rangle$ or equivalently $\langle 1, 2, 6 \rangle$, so parametric equations are $x = 2 + t$, $y = 2 + 2t$, $z = 1 + 6t$, and symmetric equations are $x - 2 = \dfrac{y-2}{2} = \dfrac{z-1}{6}$.

50. Let $F(x,y,z) = xy + yz + zx$. Then $xy + yz + zx = 5$ is a level surface of F and $\nabla F(x,y,z) = \langle y+z, x+z, x+y \rangle$.

(a) $\nabla F(1,2,1) = \langle 3, 2, 3 \rangle$ is a normal vector for the tangent plane at $(1,2,1)$, so an equation of the tangent plane is $3(x-1) + 2(y-2) + 3(z-1) = 0$ or $3x + 2y + 3z = 10$.

(b) The normal line has direction $\langle 3, 2, 3 \rangle$, so parametric equations are $x = 1 + 3t$, $y = 2 + 2t$, $z = 1 + 3t$, and symmetric equations are $\dfrac{x-1}{2} = \dfrac{y-2}{1} = \dfrac{z-1}{3}$.

51. Let $F(x,y,z) = x + y + z - e^{xyz}$. Then $x + y + z = e^{xyz}$ is the level surface $F(x,y,z) = 0$, and $\nabla F(x,y,z) = \langle 1 - yze^{xyz}, 1 - xze^{xyz}, 1 - xye^{xyz} \rangle$.

(a) $\nabla F(0,0,1) = \langle 1, 1, 1 \rangle$ is a normal vector for the tangent plane at $(0,0,1)$, so an equation of the tangent plane is $1(x-0) + 1(y-0) + 1(z-1) = 0$ or $x + y + z = 1$.

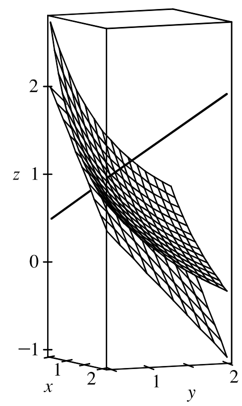
(b) The normal line has direction $\langle 1, 1, 1 \rangle$, so parametric equations are $x = t$, $y = t$, $z = 1 + t$, and symmetric equations are $x = y = z - 1$.

52. Let $F(x,y,z) = x^4 + y^4 + z^4 - 3x^2y^2z^2$. Then $x^4 + y^4 + z^4 = 3x^2y^2z^2$ is the level surface $F(x,y,z) = 0$, and $\nabla F(x,y,z) = \langle 4x^3 - 6xy^2z^2, 4y^3 - 6x^2yz^2, 4z^3 - 6x^2y^2z \rangle$.

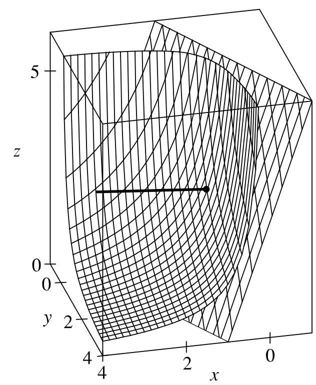
(a) $\nabla F(1,1,1) = \langle -2, -2, -2 \rangle$ or equivalently $\langle 1, 1, 1 \rangle$ is a normal vector for the tangent plane at $(1,1,1)$, so an equation of the tangent plane is $1(x-1) + 1(y-1) + 1(z-1) = 0$ or $x + y + z = 3$.

(b) The normal line has direction $\langle 1, 1, 1 \rangle$, so parametric equations are $x = 1 + t$, $y = 1 + t$, $z = 1 + t$, and symmetric equations are $x - 1 = y - 1 = z - 1$ or equivalently $x = y = z$.

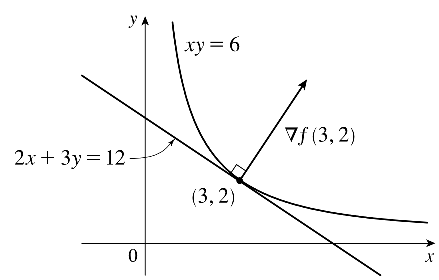
53. $F(x,y,z) = xy + yz + zx \quad \Rightarrow \quad xy + yz + zx = 3$ is the level surface $F(x,y,z) = 3$. $\nabla F(x,y,z) = \langle y+z, x+z, y+x \rangle \quad \Rightarrow$ $\nabla F(1,1,1) = \langle 2,2,2 \rangle$, and an equation of the tangent plane is $2x + 2y + 2z = 6$ or $x + y + z = 3$. The normal line is given by $x - 1 = y - 1 = z - 1$ or $x = y = z$. To graph the surface we solve for z: $z = \dfrac{3 - xy}{x + y}$.



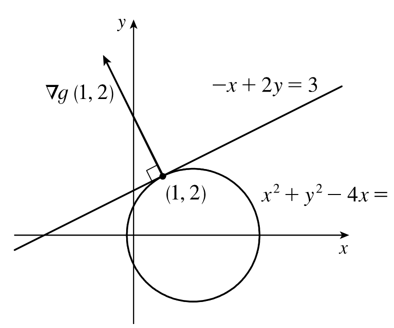
54. $F(x,y,z) = xyz \quad \Rightarrow \quad xyz = 6$ is the level surface $F(x,y,z) = 6$. $\nabla F(x,y,z) = \langle yz, xz, yx \rangle \quad \Rightarrow \quad \nabla F(1,2,3) = \langle 6,3,2 \rangle$, and an equation of the tangent plane is $6x + 3y + 2z = 18$. The normal line is given by $\dfrac{x-1}{6} = \dfrac{y-2}{3} = \dfrac{z-3}{2}$ or $x = 1 + 6t,\ y = 2 + 3t,$ $z = 3 + 2t$. To graph the surface we solve for z: $z = \dfrac{6}{xy}$.



55. $f(x,y) = xy \quad \Rightarrow \quad \nabla f(x,y) = \langle y, x \rangle$ and $\nabla f(3,2) = \langle 2,3 \rangle$. Since $\nabla f(3,2)$ is perpendicular to the tangent line, the tangent line has equation $\nabla f(3,2) \cdot \langle x - 3, y - 2 \rangle = 0 \quad \Rightarrow \quad \langle 2,3 \rangle \cdot \langle x - 3, x - 2 \rangle = 0 \quad \Rightarrow$ $2(x-3) + 3(y-2) = 0 \quad$ or $\quad 2x + 3y = 12$.



56. $g(x,y) = x^2 + y^2 - 4x \quad \Rightarrow \quad \nabla g(x,y) = \langle 2x - 4, 2y \rangle$ and $\nabla g(1,2) = \langle -2, 4 \rangle$. Since $\nabla g(1,2)$ is perpendicular to the tangent line, the tangent line has equation $\nabla g(1,2) \cdot \langle x - 1, y - 2 \rangle = 0 \quad \Rightarrow$ $\langle -2, 4 \rangle \cdot \langle x - 1, y - 2 \rangle = 0 \quad \Rightarrow \quad -2(x-1) + 4(y-2) = 0 \quad \Leftrightarrow$ $-2x + 4y = 6$ or equivalently $-x + 2y = 3$.



57. $F(x,y,z) = x^2/a^2 + y^2/b^2 + z^2/c^2$. Then $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is the level surface $F(x,y,z) = 1$ and

$\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{2z_0}{c^2} \right\rangle$. Thus, an equation of the tangent plane at (x_0, y_0, z_0) is

$\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y + \dfrac{2z_0}{c^2}\,z = 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} + \dfrac{z_0^2}{c^2}\right) = 2(1) = 2$ since (x_0, y_0, z_0) is a point on the ellipsoid. Hence

$\dfrac{x_0}{a^2}\,x + \dfrac{y_0}{b^2}\,y + \dfrac{z_0}{c^2}\,z = 1$ is an equation of the tangent plane.

58. $F(x,y,z) = x^2/a^2 + y^2/b^2 - z^2/c^2$. Then $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ is the level surface $F(x,y,z) = 1$ and

$\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{-2z_0}{c^2} \right\rangle$, so an equation of the tangent plane at (x_0, y_0, z_0) is

$\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y - \dfrac{2z_0}{c^2}\,z = 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} - \dfrac{z_0^2}{c^2}\right) = 2$ or $\dfrac{x_0}{a^2}\,x + \dfrac{y_0}{b^2}\,y - \dfrac{z_0}{c^2}\,z = 1$.

59. $F(x,y,z) = x^2/a^2 + y^2/b^2 - z/c$. Then $z/c = x^2/a^2 + y^2/b^2$ is the level surface $F(x,y,z) = 0$ and

$\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{-1}{c} \right\rangle$, so an equation of the tangent plane is $\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y - \dfrac{1}{c}\,z = \dfrac{2x_0^2}{a^2} + \dfrac{2y_0^2}{b^2} - \dfrac{z_0}{c}$

or $\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y = \dfrac{z}{c} + 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}\right) - \dfrac{z_0}{c}$. But $\dfrac{z_0}{c} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}$, so the equation can be written as

$\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y = \dfrac{z + z_0}{c}$.

60. Let $F(x,y,z) = x^2 + y^2 + 2z^2$; then the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is a level surface of F. $\nabla F(x,y,z) = \langle 2x, 2y, 4z\rangle$ is a normal vector to the surface at (x,y,z) and so it is a normal vector for the tangent plane there. The tangent plane is parallel to the plane $x + 2y + z = 1$ when the normal vectors of the planes are parallel, so we need a point (x_0, y_0, z_0) on the ellipsoid where $\langle 2x_0, 2y_0, 4z_0\rangle = k\,\langle 1, 2, 1\rangle$ for some $k \neq 0$. Comparing components we have $2x_0 = k \;\Rightarrow\; x_0 = k/2$, $2y_0 = 2k \;\Rightarrow\; y_0 = k$, $4z_0 = k \;\Rightarrow\; z_0 = k/4$. $(x_0, y_0, z_0) = (k/2, k, k/4)$ lies on the ellipsoid, so $(k/2)^2 + k^2 + 2(k/4)^2 = 1 \;\Rightarrow\; \frac{11}{8}k^2 = 1 \;\Rightarrow\; k^2 = \frac{8}{11} \;\Rightarrow\; k = \pm 2\sqrt{\frac{2}{11}}$. Thus the tangent planes at the points $\left(\sqrt{\frac{2}{11}}, 2\sqrt{\frac{2}{11}}, \frac{1}{2}\sqrt{\frac{2}{11}}\right)$ and $\left(-\sqrt{\frac{2}{11}}, -2\sqrt{\frac{2}{11}}, -\frac{1}{2}\sqrt{\frac{2}{11}}\right)$ are parallel to the given plane.

61. The hyperboloid $x^2 - y^2 - z^2 = 1$ is a level surface of $F(x,y,z) = x^2 - y^2 - z^2$ and $\nabla F\,(x,y,z) = \langle 2x, -2y, -2z\rangle$ is a normal vector to the surface and hence a normal vector for the tangent plane at (x,y,z). The tangent plane is parallel to the plane $z = x + y$ or $x + y - z = 0$ if and only if the corresponding normal vectors are parallel, so we need a point (x_0, y_0, z_0) on the hyperboloid where $\langle 2x_0, -2y_0, -2z_0\rangle = c\,\langle 1, 1, -1\rangle$ or equivalently $\langle x_0, -y_0, -z_0\rangle = k\,\langle 1, 1, -1\rangle$ for some $k \neq 0$. Then we must have $x_0 = k$, $y_0 = -k$, $z_0 = k$ and substituting into the equation of the hyperboloid gives $k^2 - (-k)^2 - k^2 = 1 \;\Leftrightarrow\; -k^2 = 1$, an impossibility. Thus there is no such point on the hyperboloid.

62. First note that the point $(1, 1, 2)$ is on both surfaces. The ellipsoid $3x^2 + 2y^2 + z^2 = 9$ is a level surface of $F(x,y,z) = 3x^2 + 2y^2 + z^2$ and $\nabla F(x,y,z) = \langle 6x, 4y, 2z\rangle$. A normal vector to the surface at $(1,1,2)$ is $\nabla F(1,1,2) = \langle 6, 4, 4\rangle$ and an equation of the tangent plane there is $6(x-1) + 4(y-1) + 4(z-2) = 0$ or

$6x + 4y + 4z = 18$ or $3x + 2y + 2z = 9$. The sphere is a level surface of $G(x,y,z) = x^2 + y^2 + z^2 - 8x - 6y - 8z + 24$ and $\nabla G(x,y,z) = \langle 2x - 8, 2y - 6, 2z - 8 \rangle$. A normal vector to the sphere at $(1,1,2)$ is $\nabla G(1,1,2) = \langle -6, -4, -4 \rangle$ and the tangent plane there is $-6(x-1) - 4(y-1) - 4(z-2) = 0$ or $3x + 2y + 2z = 9$. Since these tangent planes are identical, the surfaces are tangent to each other at the point $(1,1,2)$.

63. Let (x_0, y_0, z_0) be a point on the cone [other than $(0,0,0)$]. The cone $x^2 + y^2 = z^2$ is a level surface of $F(x,y,z) = x^2 + y^2 - z^2$ and $\nabla F(x,y,z) = \langle 2x, 2y, -2z \rangle$, so $\nabla F(x_0,y_0,z_0) = \langle 2x_0, 2y_0, -2z_0 \rangle$ is a normal vector to the cone at this point and an equation of the tangent plane there is $2x_0\,(x - x_0) + 2y_0\,(y - y_0) - 2z_0\,(z - z_0) = 0$ or $x_0x + y_0y - z_0z = x_0^2 + y_0^2 - z_0^2$. But $x_0^2 + y_0^2 = z_0^2$ so the tangent plane is given by $x_0x + y_0y - z_0z = 0$, a plane which always contains the origin.

64. Let (x_0, y_0, z_0) be a point on the sphere and $F(x,y,z) = x^2 + y^2 + z^2$. Then $\nabla F(x,y,z) = \langle 2x, 2y, 2z \rangle$ and $\nabla F(x_0,y_0,z_0) = \langle 2x_0, 2y_0, 2z_0 \rangle$, so the normal line is given by $\dfrac{x - x_0}{2x_0} = \dfrac{y - y_0}{2y_0} = \dfrac{z - z_0}{2z_0}$. For the center $(0,0,0)$ to be on the line, we need $-\dfrac{x_0}{2x_0} = -\dfrac{y_0}{2y_0} = -\dfrac{z_0}{2z_0}$ or equivalently $1 = 1 = 1$, which is true.

65. Let $F(x,y,z) = x^2 + y^2 - z$. Then the paraboloid is the level surface $F(x,y,z) = 0$ and $\nabla F(x,y,z) = \langle 2x, 2y, -1 \rangle$, so $\nabla F(1,1,2) = \langle 2, 2, -1 \rangle$ is a normal vector to the surface. Thus the normal line at $(1,1,2)$ is given by $x = 1 + 2t$, $y = 1 + 2t$, $z = 2 - t$. Substitution into the equation of the paraboloid $z = x^2 + y^2$ gives $2 - t = (1 + 2t)^2 + (1 + 2t)^2 \quad \Leftrightarrow$ $2 - t = 2 + 8t + 8t^2 \quad \Leftrightarrow \quad 8t^2 + 9t = 0 \quad \Leftrightarrow \quad t(8t + 9) = 0$. Thus the line intersects the paraboloid when $t = 0$, corresponding to the given point $(1,1,2)$, or when $t = -\frac{9}{8}$, corresponding to the point $\left(-\frac{5}{4}, -\frac{5}{4}, \frac{25}{8}\right)$.

66. The ellipsoid $4x^2 + y^2 + 4z^2 = 12$ is a level surface of $F(x,y,z) = 4x^2 + y^2 + 4z^2$ and $\nabla F(x,y,z) = \langle 8x, 2y, 8z \rangle$, so $\nabla F(1,2,1) = \langle 8, 4, 8 \rangle$ or equivalently $\langle 2, 1, 2 \rangle$ is a normal vector to the surface. Thus, the normal line to the ellipsoid at $(1,2,1)$ is given by $x = 1 + 2t$, $y = 2 + t$, $z = 1 + 2t$. Substitution into the equation of the sphere gives $(1 + 2t)^2 + (2 + t)^2 + (1 + 2t)^2 = 102 \quad \Leftrightarrow \quad 6 + 12t + 9t^2 = 102 \quad \Leftrightarrow \quad 9t^2 + 12t - 96 = 0 \quad \Leftrightarrow$ $3(t + 4)(3t - 8) = 0$. Thus, the line intersects the sphere when $t = -4$, corresponding to the point $(-7, -2, -7)$, and when $t = \frac{8}{3}$, corresponding to the point $\left(\frac{19}{3}, \frac{14}{3}, \frac{19}{3}\right)$.

67. Let (x_0, y_0, z_0) be a point on the surface and $F(x,y,z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$. Then $\nabla F(x,y,z) = \left\langle \dfrac{1}{2\sqrt{x}}, \dfrac{1}{2\sqrt{y}}, \dfrac{1}{2\sqrt{z}} \right\rangle$ and

$\nabla F(x_0,y_0,z_0) = \left\langle \dfrac{1}{2\sqrt{x_0}}, \dfrac{1}{2\sqrt{y_0}}, \dfrac{1}{2\sqrt{z_0}} \right\rangle$, so an equation of the tangent plane at the point is

$$\frac{x - x_0}{2\sqrt{x_0}} + \frac{y - y_0}{2\sqrt{y_0}} + \frac{z - z_0}{2\sqrt{z_0}} = 0 \quad \Leftrightarrow \quad \frac{x}{2\sqrt{x_0}} + \frac{y}{2\sqrt{y_0}} + \frac{z}{2\sqrt{z_0}} = \frac{x_0}{2\sqrt{x_0}} + \frac{y_0}{2\sqrt{y_0}} + \frac{z_0}{2\sqrt{z_0}} \quad \Leftrightarrow$$

$\dfrac{x}{2\sqrt{x_0}} + \dfrac{y}{2\sqrt{y_0}} + \dfrac{z}{2\sqrt{z_0}} = \dfrac{\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}}{2}$. But $\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = \sqrt{c}$, so the equation is

$\dfrac{x}{\sqrt{x_0}} + \dfrac{y}{\sqrt{y_0}} + \dfrac{z}{\sqrt{z_0}} = \sqrt{c}$. The x-, y-, and z-intercepts are $\sqrt{cx_0}$, $\sqrt{cy_0}$, and $\sqrt{cz_0}$, respectively. (The x-intercept is found by setting $y = z = 0$ and solving the resulting equation for x, and the y- and z-intercepts are found similarly.) So the sum of the intercepts is $\sqrt{c}\left(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}\right) = c$, a constant.

68. The surface $xyz = 1$ is a level surface of $F(x,y,z) = xyz$ and $\nabla F(x,y,z) = \langle yz, xz, xy \rangle$ is normal to the surface, so a normal vector for the tangent plane to the surface at (x_0, y_0, z_0) is $\langle y_0z_0, x_0z_0, x_0y_0 \rangle$. An equation for the tangent plane there is $y_0z_0(x - x_0) + x_0z_0(y - y_0) + x_0y_0(z - z_0) = 0 \quad \Rightarrow \quad y_0z_0x + x_0z_0y + x_0y_0z = 3x_0y_0z_0$ or $\dfrac{x}{x_0} + \dfrac{y}{y_0} + \dfrac{z}{z_0} = 3$.

If (x_0, y_0, z_0) is in the first octant, then the tangent plane cuts off a pyramid in the first octant with vertices $(0,0,0)$, $(3x_0, 0, 0)$, $(0, 3y_0, 0)$, $(0, 0, 3z_0)$. The base in the xy-plane is a triangle with area $\frac{1}{2}(3x_0)(3y_0)$ and the height (along the z-axis) of the pyramid is $3z_0$. The volume of the pyramid for any point (x_0, y_0, z_0) on the surface $xyz = 1$ in the first octant is $\frac{1}{3}\,(\text{base})\,(\text{height}) = \frac{1}{3} \cdot \frac{1}{2}(3x_0)(3y_0) \cdot 3z_0 = \frac{9}{2}x_0y_0z_0 = \frac{9}{2}$ since $x_0y_0z_0 = 1$.

69. If $f(x,y,z) = z - x^2 - y^2$ and $g(x,y,z) = 4x^2 + y^2 + z^2$, then the tangent line is perpendicular to both ∇f and ∇g at $(-1, 1, 2)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ will therefore be parallel to the tangent line.

We have $\nabla f(x,y,z) = \langle -2x, -2y, 1 \rangle \quad \Rightarrow \quad \nabla f(-1,1,2) = \langle 2, -2, 1 \rangle$, and $\nabla g(x,y,z) = \langle 8x, 2y, 2z \rangle \quad \Rightarrow$

$\nabla g(-1,1,2) = \langle -8, 2, 4 \rangle$. Hence $\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = -10\,\mathbf{i} - 16\,\mathbf{j} - 12\,\mathbf{k}$.

Parametric equations are: $x = -1 - 10t,\ \ y = 1 - 16t,\ \ z = 2 - 12t$.

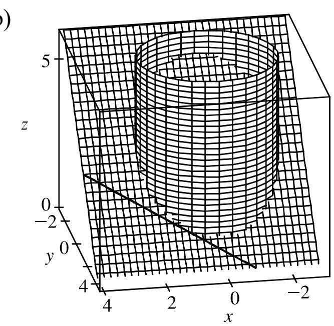
70. (a) Let $f(x,y,z) = y + z$ and $g(x,y,z) = x^2 + y^2$. The plane is a level surface of f and the cylinder is a level surface of g. Then the required tangent line is perpendicular to both ∇f and ∇g at $(1, 2, 1)$ and the vector $\mathbf{v} = \nabla f \times \nabla g$ is parallel to the tangent line. We have

$\nabla f(x,y,z) = \langle 0, 1, 1 \rangle \quad \Rightarrow \quad \nabla f(1,2,1) = \langle 0, 1, 1 \rangle$, and

$\nabla g(x,y,z) = \langle 2x, 2y, 0 \rangle \quad \Rightarrow \quad \nabla g(1,2,1) = \langle 2, 4, 0 \rangle$. Hence

$\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 2 & 4 & 0 \end{vmatrix} = -4\,\mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}$. So parametric equations of the desired tangent line are $x = 1 - 4t,\ \ y = 2 + 2t,\ \ z = 1 - 2t$.

(b)



71. Parametric equations for the helix are $x = \cos \pi t,\ \ y = \sin \pi t,\ \ z = t$, and substituting into the equation of the paraboloid gives $t = \cos^2 \pi t + \sin^2 \pi t \quad \Rightarrow \quad t = 1$. Thus the helix intersects the surface at the point $(\cos \pi, \sin \pi, 1) = (-1, 0, 1)$. Here $\mathbf{r}'(t) = \langle -\pi \sin \pi t, \pi \cos \pi t, 1 \rangle$, so the tangent vector to the helix at that point is $\mathbf{r}'(1) = \langle -\pi \sin \pi, \pi \cos \pi, 1 \rangle = \langle 0, -\pi, 1 \rangle$.

[continued]

The paraboloid $z = x^2 + y^2 \quad \Leftrightarrow \quad x^2 + y^2 - z = 0$ is a level surface of $F(x, y, z) = x^2 + y^2 - z$ and $\nabla F(x, y, z) = \langle 2x, 2y, -1 \rangle$, so a normal vector to the tangent plane at $(-1, 0, 1)$ is $\nabla F(-1, 0, 1) = \langle -2, 0, -1 \rangle$. The angle θ between $\mathbf{r}'(1)$ and $\nabla F(-1, 0, 1)$ is given by

$$\cos\theta = \frac{\langle 0, -\pi, 1 \rangle \cdot \langle -2, 0, -1 \rangle}{|\langle 0, -\pi, 1 \rangle|\,|\langle -2, 0, -1 \rangle|} = \frac{0 + 0 - 1}{\sqrt{0 + \pi^2 + 1}\,\sqrt{4 + 0 + 1}} = \frac{-1}{\sqrt{5(\pi^2 + 1)}} \quad \Rightarrow$$

$\theta = \cos^{-1} \dfrac{-1}{\sqrt{5(\pi^2 + 1)}} \approx 97.8^\circ$. Because $\nabla F(-1, 0, 1)$ is perpendicular to the tangent plane, the angle of intersection between the helix and the paraboloid is approximately $97.8^\circ - 90^\circ = 7.8^\circ$.

72. Parametric equations for the helix are $x = \cos(\pi t/2)$, $\;y = \sin(\pi t/2)$, $\;z = t$, and substituting into the equation of the sphere gives $\cos^2(\pi t/2) + \sin^2(\pi t/2) + t^2 = 2 \quad \Rightarrow \quad 1 + t^2 = 2 \quad \Rightarrow \quad t = \pm 1$. Thus, the helix intersects the sphere at two points: $(\cos(\pi/2), \sin(\pi/2), 1) = (0, 1, 1)$, when $t = 1$, and $(\cos(-\pi/2), \sin(-\pi/2), -1) = (0, -1, -1)$, when $t = -1$. Here $\mathbf{r}'(t) = \left\langle -\frac{\pi}{2}\sin(\pi t/2), \frac{\pi}{2}\cos(\pi t/2), 1 \right\rangle$, so the tangent vector to the helix at $(0, 1, 1)$ is $\mathbf{r}'(1) = \langle -\pi/2, 0, 1 \rangle$. The sphere $x^2 + y^2 + z^2 = 2$ is a level surface of $F(x, y, z) = x^2 + y^2 + z^2$ and $\nabla F(x, y, z) = \langle 2x, 2y, 2z \rangle$, so a normal vector to the tangent plane at $(0, 1, 1)$ is $\nabla F(0, 1, 1) = \langle 0, 2, 2 \rangle$. As in Exercise 71, the angle of intersection between the helix and the sphere is the angle between the tangent vector to the helix and the tangent plane to the sphere. The angle θ between $\mathbf{r}'(1)$ and $\nabla F(0, 1, 1)$ is given by

$$\cos\theta = \frac{\langle -\pi/2, 0, 1 \rangle \cdot \langle 0, 2, 2 \rangle}{|\langle -\pi/2, 0, 1 \rangle|\,|\langle 0, 2, 2 \rangle|} = \frac{2}{\sqrt{(\pi^2/4) + 1}\,\sqrt{8}} = \frac{2}{\sqrt{2\pi^2 + 8}} \quad \Rightarrow \quad \theta = \cos^{-1} \frac{2}{\sqrt{2\pi^2 + 8}} \approx 67.7^\circ$$

Because $\nabla F(0, 1, 1)$ is perpendicular to the tangent plane, the angle between $\mathbf{r}'(1)$ and the tangent plane is approximately $90^\circ - 67.7^\circ = 22.3^\circ$.

At $(0, -1, -1)$, $\;\mathbf{r}'(-1) = \langle \pi/2, 0, 1 \rangle$ and $\nabla F(0, -1, -1) = \langle 0, -2, -2 \rangle$, and the angle ϕ between these vectors is given by $\cos\phi = \dfrac{\langle \pi/2, 0, 1 \rangle \cdot \langle 0, -2, -2 \rangle}{|\langle \pi/2, 0, 1 \rangle|\,|\langle 0, -2, -2 \rangle|} = \dfrac{-2}{\sqrt{2\pi^2 + 8}} \quad \Rightarrow \quad \phi = \cos^{-1} \dfrac{-2}{\sqrt{2\pi^2 + 8}} \approx 112.3^\circ$. Thus the angle between the helix and the sphere at $(0, -1, -1)$ is approximately $112.3^\circ - 90^\circ = 22.3^\circ$. (By symmetry, we would expect the angles to be identical.)

73. The direction of the normal line of F is given by ∇F, and that of G by ∇G. Assuming that $\nabla F \neq 0 \neq \nabla G$, the two normal lines are perpendicular at P if $\nabla F \cdot \nabla G = 0$ at $P \quad \Leftrightarrow \quad \langle \partial F/\partial x, \partial F/\partial y, \partial F/\partial z \rangle \cdot \langle \partial G/\partial x, \partial G/\partial y, \partial G/\partial z \rangle = 0$ at $P \quad \Leftrightarrow \quad F_x G_x + F_y G_y + F_z G_z = 0$ at P.

74. Here $F(x, y, z) = x^2 + y^2 - z^2$ and $G(x, y, z) = x^2 + y^2 + z^2 - r^2$, so $z^2 = x^2 + y^2$ is a level surface of F and $x^2 + y^2 + z^2 = r^2$ is a level surface of $G \quad \Rightarrow \quad \nabla F \cdot \nabla G = \langle 2x, 2y, -2z \rangle \cdot \langle 2x, 2y, 2z \rangle = 4x^2 + 4y^2 - 4z^2 = 4F = 0$, since the point (x, y, z) lies on the graph of $F = 0$. To see that this is true without using calculus, note that $G = 0$ is the equation of a sphere centered at the origin and $F = 0$ is the equation of a right circular cone with vertex at the origin (which is generated by lines through the origin). At any point of intersection, the sphere's normal line (which passes through the origin) lies on the cone, and thus is perpendicular to the cone's normal line. So the surfaces with equations $F = 0$ and $G = 0$ are everywhere orthogonal.

75. Let $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$. Then we know that at the given point, $D_{\mathbf{u}}\, f = \nabla f \cdot \mathbf{u} = af_x + bf_y$ and $D_{\mathbf{v}}\, f = \nabla f \cdot \mathbf{v} = cf_x + df_y$. But these are just two linear equations in the two unknowns f_x and f_y, and since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, we can solve the equations to find $\nabla f = \langle f_x, f_y \rangle$ at the given point. In fact,

$$\nabla f = \left\langle \frac{d\, D_{\mathbf{u}}\, f - b\, D_{\mathbf{v}}\, f}{ad - bc}, \frac{a\, D_{\mathbf{v}}\, f - c\, D_{\mathbf{u}}\, f}{ad - bc} \right\rangle.$$

76. (a) The function $f(x, y) = (xy)^{1/3}$ is continuous on $\mathbb{R}^2$ since it is a composition of a polynomial and the cube root function, both of which are continuous. (See the text just after Example 14.2.9.)

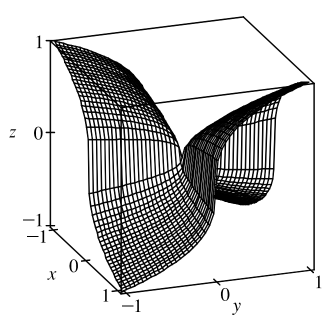
$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h} = \lim_{h \to 0} \frac{(h \cdot 0)^{1/3} - 0}{h} = 0,$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0, 0+h) - f(0,0)}{h} = \lim_{h \to 0} \frac{(0 \cdot h)^{1/3} - 0}{h} = 0.$$

Therefore, $f_x(0,0)$ and $f_y(0,0)$ do exist and are equal to 0. Now let $\mathbf{u}$ be any unit vector other than $\mathbf{i}$ and $\mathbf{j}$ (these correspond to f_x and f_y respectively.) Then $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j}$ where $a \neq 0$ and $b \neq 0$. Thus,

$D_{\mathbf{u}}\, f(0,0) = \lim\limits_{h \to 0} \dfrac{f(0 + ha, 0 + hb) - f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt[3]{(ha)(hb)}}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt[3]{ab}}{h^{1/3}}$ and this limit does not exist, so $D_{\mathbf{u}}\, f(0,0)$ does not exist.

(b)



Notice that if we start at the origin and proceed in the direction of the x- or y-axis, then the graph is flat. But if we proceed in any other direction, then the graph is extremely steep.

77. Since $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = (x_0, y_0)$, by Definition 14.4.7 we have $\Delta z = f_x(x_0, y_0)\, \Delta x + f_y(x_0, y_0)\, \Delta y + \varepsilon_1\, \Delta x + \varepsilon_2\, \Delta y$ where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Now $\Delta z = f(\mathbf{x}) - f(\mathbf{x}_0)$, $\langle \Delta x, \Delta y \rangle = \mathbf{x} - \mathbf{x}_0$ so $(\Delta x, \Delta y) \to (0, 0)$ is equivalent to $\mathbf{x} \to \mathbf{x}_0$ and $\langle f_x(x_0, y_0)\,, f_y(x_0, y_0) \rangle = \nabla f(\mathbf{x}_0)$. Substituting into 14.4.7 gives $f(\mathbf{x}) - f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \langle \varepsilon_1, \varepsilon_2 \rangle \cdot \langle \Delta x, \Delta y \rangle$ or $\langle \varepsilon_1, \varepsilon_2 \rangle \cdot (\mathbf{x} - \mathbf{x}_0) = f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$,

and so $\dfrac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = \dfrac{\langle \varepsilon_1, \varepsilon_2 \rangle \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|}$. But $\dfrac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}$ is a unit vector so

$\lim\limits_{\mathbf{x} \to \mathbf{x}_0} \dfrac{\langle \varepsilon_1, \varepsilon_2 \rangle \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0$ since $\varepsilon_1, \varepsilon_2 \to 0$ as $\mathbf{x} \to \mathbf{x}_0$. Hence $\lim\limits_{\mathbf{x} \to \mathbf{x}_0} \dfrac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0.$

14.7 Maximum and Minimum Values

1. (a) First we compute $D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (1)^2 = 7$. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$ by the Second Derivatives Test.

(b) $D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (3)^2 = -1$. Since $D(1,1) < 0$, f has a saddle point at $(1,1)$ by the Second Derivatives Test.

2. (a) $D(0,2) = g_{xx}(0,2)\, g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (-1)(1) - (6)^2 = -37$. Since $D(0,2) < 0$, g has a saddle point at $(0,2)$ by the Second Derivatives Test.

(b) $D(0,2) = g_{xx}(0,2)\, g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (-1)(-8) - (2)^2 = 4$. Since $D(0,2) > 0$ and $g_{xx}(0,2) < 0$, g has a local maximum at $(0,2)$ by the Second Derivatives Test.

(c) $D(0,2) = g_{xx}(0,2)\, g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (4)(9) - (6)^2 = 0$. In this case the Second Derivatives Test gives no information about g at the point $(0,2)$.

3. In the figure, a point at approximately $(1,1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of f are increasing. Hence we would expect a local minimum at or near $(1,1)$. The level curves near $(0,0)$ resemble hyperbolas, and as we move away from the origin, the values of f increase in some directions and decrease in others, so we would expect to find a saddle point there.

To verify our predictions, we have $f(x,y) = 4 + x^3 + y^3 - 3xy \;\Rightarrow\; f_x(x,y) = 3x^2 - 3y$, $f_y(x,y) = 3y^2 - 3x$. We have critical points where these partial derivatives are equal to 0: $3x^2 - 3y = 0$, $3y^2 - 3x = 0$. Substituting $y = x^2$ from the first equation into the second equation gives $3(x^2)^2 - 3x = 0 \;\Rightarrow\; 3x(x^3 - 1) = 0 \;\Rightarrow\; x = 0$ or $x = 1$. Then we have two critical points, $(0,0)$ and $(1,1)$. The second partial derivatives are $f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = -3$, and $f_{yy}(x,y) = 6y$, so $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. Then $D(0,0) = 36(0)(0) - 9 = -9$, and $D(1,1) = 36(1)(1) - 9 = 27$. Since $D(0,0) < 0$, f has a saddle point at $(0,0)$ by the Second Derivatives Test. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$.

4. In the figure, points at approximately $(-1,1)$ and $(-1,-1)$ are enclosed by oval-shaped level curves which indicate that as we move away from either point in any direction, the values of f are increasing. Hence we would expect local minimums at or near $(-1,\pm 1)$. Similarly, the point $(1,0)$ appears to be enclosed by oval-shaped level curves which indicate that as we move away from the point in any direction the values of f are decreasing, so we should have a local maximum there. We also show hyperbola-shaped level curves near the points $(-1,0)$, $(1,1)$, and $(1,-1)$. The values of f increase along some paths leaving these points and decrease in others, so we should have a saddle point at each of these points.

To confirm our predictions, we have $f(x,y) = 3x - x^3 - 2y^2 + y^4 \;\Rightarrow\; f_x(x,y) = 3 - 3x^2$, $\;f_y(x,y) = -4y + 4y^3$. Setting these partial derivatives equal to 0, we have $3 - 3x^2 = 0 \;\Rightarrow\; x = \pm 1$ and $-4y + 4y^3 = 0 \;\Rightarrow$
$y\left(y^2 - 1\right) = 0 \;\Rightarrow\; y = 0, \pm 1$. So our critical points are $(\pm 1, 0)$, $(\pm 1, 1)$, $(\pm 1, -1)$.
The second partial derivatives are $f_{xx}(x,y) = -6x$, $f_{xy}(x,y) = 0$, and $f_{yy}(x,y) = 12y^2 - 4$, so

$D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (-6x)(12y^2 - 4) - (0)^2 = -72xy^2 + 24x.$

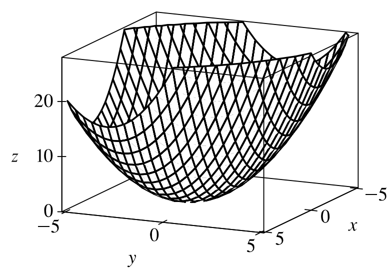
We use the Second Derivatives Test to classify the 6 critical points:

Critical Point	D	f_{xx}	Conclusion
$(1,0)$	24	-6	$D > 0,\ f_{xx} < 0 \quad \Rightarrow \quad f$ has a local maximum at $(1,0)$
$(1,1)$	-48		$D < 0 \quad \Rightarrow \quad f$ has a saddle point at $(1,1)$
$(1,-1)$	-48		$D < 0 \quad \Rightarrow \quad f$ has a saddle point at $(1,-1)$
$(-1,0)$	-24		$D < 0 \quad \Rightarrow \quad f$ has a saddle point at $(-1,0)$
$(-1,1)$	48	6	$D > 0,\ f_{xx} > 0 \quad \Rightarrow \quad f$ has a local minimum at $(-1,1)$
$(-1,-1)$	48	6	$D > 0,\ f_{xx} > 0 \quad \Rightarrow \quad f$ has a local minimum at $(-1,-1)$

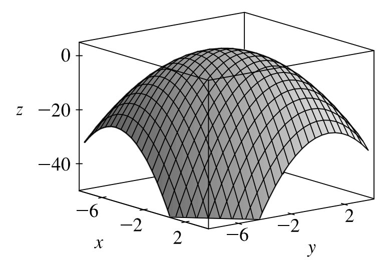
5. $f(x,y) = x^2 + xy + y^2 + y \quad \Rightarrow \quad f_x = 2x + y,\ f_y = x + 2y + 1,\ f_{xx} = 2,\ f_{xy} = 1,\ f_{yy} = 2$. Then $f_x = 0$ implies $y = -2x$, and substitution into $f_y = x + 2y + 1 = 0$ gives $x + 2(-2x) + 1 = 0 \quad \Rightarrow \quad -3x = -1 \quad \Rightarrow \quad x = \frac{1}{3}$.

Then $y = -\frac{2}{3}$ and the only critical point is $\left(\frac{1}{3}, -\frac{2}{3}\right)$.

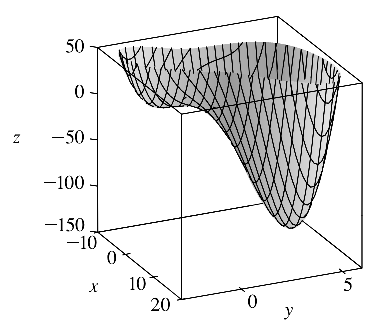
$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = (2)(2) - (1)^2 = 3$, and since $D\left(\frac{1}{3}, -\frac{2}{3}\right) = 3 > 0$ and $f_{xx}\left(\frac{1}{3}, -\frac{2}{3}\right) = 2 > 0$, $f\left(\frac{1}{3}, -\frac{2}{3}\right) = -\frac{1}{3}$ is a local minimum by the Second Derivatives Test.



6. $f(x,y) = xy - 2x - 2y - x^2 - y^2 \quad \Rightarrow \quad f_x = y - 2 - 2x$, $f_y = x - 2 - 2y,\ f_{xx} = -2,\ f_{xy} = 1,\ f_{yy} = -2$. Then $f_x = 0$ implies $y = 2x + 2$, and substitution into $f_y = 0$ gives $x - 2 - 2(2x + 2) = 0 \quad \Rightarrow$ $-3x = 6 \quad \Rightarrow \quad x = -2$. Then $y = -2$ and the only critical point is $(-2,-2)$. $D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = (-2)(-2) - 1^2 = 3$, and since $D(-2,-2) = 3 > 0$ and $f_{xx}(-2,-2) = -2 < 0$, $f(-2,-2) = 4$ is a local maximum by the Second Derivatives Test.



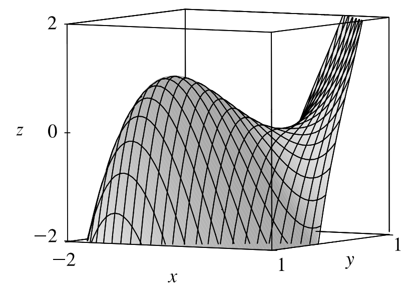
7. $f(x,y) = 2x^2 - 8xy + y^4 - 4y^3 \quad \Rightarrow \quad f_x = 4x - 8y$, $f_y = -8x + 4y^3 - 12y^2$, $f_{xx} = 4$, $f_{xy} = -8$, $f_{yy} = 12y^2 - 24y$. Then $f_x = 0$ implies $x = 2y$, and sustitution into $f_y = 0$ gives $-16y + 4y^3 - 12y^2 = 0 \quad \Rightarrow \quad 4y(y^2 - 3y - 4) = 0 \quad \Rightarrow \quad y = 0, -1, 4$. Then $x = 0, -2, 8$ and the critical points are $(0,0)$, $(-2,-1)$, and $(8,4)$. $D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$. $D(0,0) = 4(0) - 64 = -64 < 0$, so $(0,0)$ is a saddle point. $D(-2,-1) = 4(36) - 64 = 80 > 0$ and



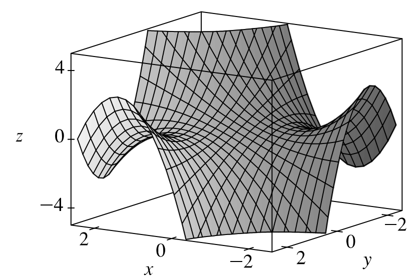
$f_{xx}(-2,-1) = 4 > 0$, so $f(-2,-1) = -3$ is a local minimum by the Second Derivatives Test.

$D(8,4) = 4(96) - 64 = 320 > 0$ and $f_{xx}(8,4) = 4 > 0$, so $f(8,4) = -128$ is a local minimum.

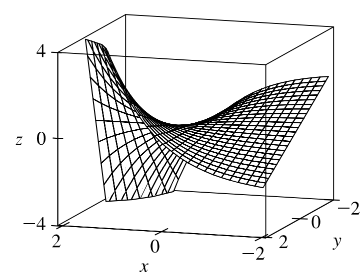
8. $f(x,y) = x^3 + y^3 + 3xy \quad \Rightarrow \quad f_x = 3x^2 + 3y$, $f_y = 3y^2 + 3x$, $f_{xx} = 6x$, $f_{xy} = 3$, $f_{yy} = 6y$. Then $f_x = 0$ implies $y = -x^2$ and substitution into $f_y = 0$ gives $3(-x^2)^2 + 3x = 3x^4 + 3x = 0 \quad \Rightarrow$ $3x(x^3 + 1) = 0 \quad \Rightarrow \quad x = 0$ or $x = -1$. Thus, the critical points are $(0,0)$ and $(-1,-1)$. $D(x,y) = (6x)(6y) - 3^2 = 36xy - 9$. $D(0,0) = -9 < 0$, so $(0,0)$ is a saddle point. $D(-1,-1) = 27 > 0$ and $f_{xx}(-1,-1) = -6 < 0$, so $f(-1,-1) = 1$ is a local maximum by the Second Derivatives Test.



9. $f(x,y) = (x-y)(1-xy) = x - y - x^2y + xy^2 \quad \Rightarrow \quad f_x = 1 - 2xy + y^2,\ f_y = -1 - x^2 + 2xy,\ f_{xx} = -2y,$ $f_{xy} = -2x + 2y,\ f_{yy} = 2x$. Then $f_x = 0$ implies $1 - 2xy + y^2 = 0$ and $f_y = 0$ implies $-1 - x^2 + 2xy = 0$. Adding the two equations gives $1 + y^2 - 1 - x^2 = 0 \quad \Rightarrow \quad y^2 = x^2 \quad \Rightarrow \quad y = \pm x$, but if $y = -x$ then $f_x = 0$ implies $1 + 2x^2 + x^2 = 0 \quad \Rightarrow \quad 3x^2 = -1$ which has no real solution. If $y = x$, then substitution into $f_x = 0$ gives $1 - 2x^2 + x^2 = 0 \quad \Rightarrow$ $x^2 = 1 \quad \Rightarrow \quad x = \pm 1$, so the critical points are $(1,1)$ and $(-1,-1)$. Now $D(1,1) = (-2)(2) - 0^2 = -4 < 0$ and $D(-1,-1) = (2)(-2) - 0^2 = -4 < 0$, so $(1,1)$ and $(-1,-1)$ are saddle points.



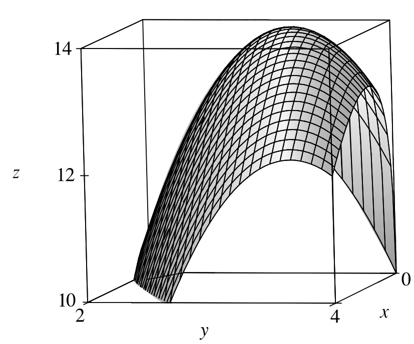
10. $f(x,y) = y(e^x - 1) \quad \Rightarrow \quad f_x = ye^x,\ f_y = e^x - 1,\ f_{xx} = ye^x,$ $f_{xy} = e^x,\ f_{yy} = 0$. Because e^x is never zero, $f_x = 0$ only when $y = 0$, and $f_y = 0$ when $e^x = 1 \quad \Rightarrow \quad x = 0$, so the only critical point is $(0,0)$. $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (ye^x)(0) - (e^x)^2 = -e^{2x}$, and since $D(0,0) = -1 < 0$, $(0,0)$ is a saddle point.



11. $f(x,y) = y\sqrt{x} - y^2 - 2x + 7y \quad \Rightarrow \quad f_x = \frac{1}{2}yx^{-1/2} - 2,$ $f_y = \sqrt{x} - 2y + 7$, $f_{xx} = -\frac{1}{4}yx^{-3/2}$, $f_{xy} = \frac{1}{2}x^{-1/2}$, $f_{yy} = -2$. Then $f_x = 0 \quad \Rightarrow \quad y = 4\sqrt{x}$ and substitution into $f_y = 0$ gives $\sqrt{x} - 2\left(4\sqrt{x}\right) + 7 = -7\sqrt{x} + 7 = 0 \quad \Rightarrow \quad x = 1$, so the only critical point is $(1,4)$.

$$D(x,y) = -\frac{1}{4}yx^{-3/2}(-2) - \left(\frac{1}{2}x^{-1/2}\right)^2 = \frac{1}{2}yx^{-3/2} - \frac{1}{4x}.$$

$D(1,4) = \frac{7}{4} > 0$, and $f_{xx}(1,4) = -1 < 0$, so $f(1,4) = 14$ is a local maximum by the Second Derivatives Test.



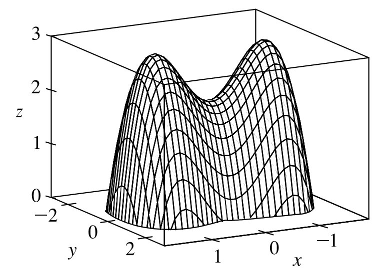
12. $f(x,y) = 2 - x^4 + 2x^2 - y^2 \quad \Rightarrow \quad f_x = -4x^3 + 4x,\ f_y = -2y,\ f_{xx} = -12x^2 + 4,\ f_{xy} = 0,\ f_{yy} = -2$. Then $f_x = 0$ implies $-4x(x^2-1) = 0$, so $x = 0$ or $x = \pm 1$, and $f_y = 0$ implies $y = 0$. Thus, the critical points are $(0,0)$, $(\pm 1, 0)$.

$D(0,0) = (4)(-2) - 0^2 = -8 < 0$, so $(0,0)$ is a saddle point.

$D(1,0) = D(-1,0) = (-8)(-2) - (0)^2 = 16 > 0$, and

$f_{xx}(1,0) = f_{xx}(-1,0) = -8 < 0$, so $f(1,0) = 3$ and $f(-1,0) = 3$

are local maximums.

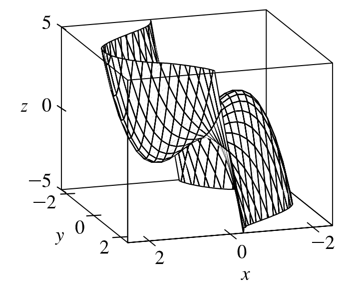


13. $f(x,y) = x^3 - 3x + 3xy^2 \quad \Rightarrow \quad f_x = 3x^2 - 3 + 3y^2,\ f_y = 6xy,\ f_{xx} = 6x,\ f_{xy} = 6y,\ f_{yy} = 6x$. Then $f_y = 0$ implies $x = 0$ or $y = 0$. If $x = 0$, substitution into $f_x = 0$ gives $3y^2 = 3 \quad \Rightarrow \quad y = \pm 1$, and if $y = 0$, substitution into $f_x = 0$ gives $x = \pm 1$. Thus the critical points are $(0, \pm 1)$ and $(\pm 1, 0)$.

$D(0, \pm 1) = 0 - 36 < 0$, so $(0, \pm 1)$ are saddle points.

$D(\pm 1, 0) = 36 - 0 > 0$, $f_{xx}(1,0) = 6 > 0$, and $f_{xx}(-1,0) = -6 < 0$,

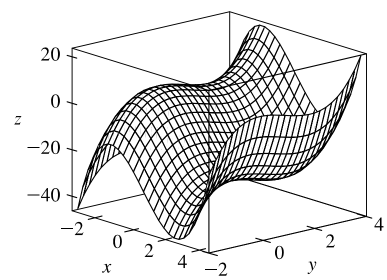
so $f(1,0) = -2$ is a local minimum and $f(-1,0) = 2$ is a local maximum.



14. $f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x \quad \Rightarrow \quad f_x = 3x^2 - 6x - 9,\ f_y = 3y^2 - 6y,\ f_{xx} = 6x - 6,\ f_{xy} = 0,\ f_{yy} = 6y - 6$. Then $f_x = 0$ implies $3(x+1)(x-3) = 0 \quad \Rightarrow \quad x = -1$ or $x = 3$, and $f_y = 0$ implies $3y(y-2) = 0 \quad \Rightarrow \quad y = 0$ or $y = 2$. Thus, the critical points are $(-1,0)$, $(-1,2)$, $(3,0)$, and $(3,2)$. $D(-1,2) = (-12)(6) - (0)^2 = -72 < 0$ and

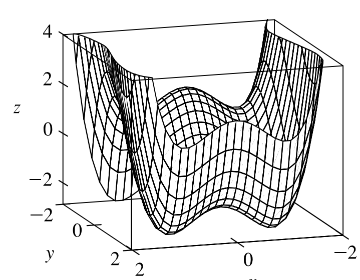
$D(3,0) = (12)(-6) - (0)^2 = -72 < 0$, so $(-1,2)$ and $(3,0)$ are saddle points. $D(-1,0) = (-12)(-6) - (0)^2 = 72 > 0$ and $f_{xx}(-1,0) = -12 < 0$, so $f(-1,0) = 5$ is a local maximum.

$D(3,2) = (12)(6) - (0)^2 = 72 > 0$ and $f_{xx}(3,2) = 12 > 0$, so $f(3,2) = -31$ is a local minimum.



15. $f(x,y) = x^4 - 2x^2 + y^3 - 3y \quad \Rightarrow \quad f_x = 4x^3 - 4x,\ f_y = 3y^2 - 3,\ f_{xx} = 12x^2 - 4,\ f_{xy} = 0,\ f_{yy} = 6y$. Then $f_x = 0$ implies $4x(x^2-1) = 0 \quad \Rightarrow \quad x = 0$ or $x = \pm 1$, and $f_y = 0$ implies $3(y^2-1) = 0 \quad \Rightarrow \quad y = \pm 1$. Thus, there are six critical points: $(0, \pm 1)$, $(\pm 1, 1)$, and $(\pm 1, -1)$.

$D(0,1) = (-4)(6) - (0)^2 = -24 < 0$ and $D(\pm 1, -1) = (8)(-6) = -48 < 0$, so $(0,1)$ and $(\pm 1, -1)$ are saddle points. $D(0,-1) = (-4)(-6) = 24 > 0$ and $f_{xx}(0,-1) = -4 < 0$, so $f(0,-1) = 2$ is a local maximum. $D(\pm 1, 1) = (8)(6) = 48 > 0$ and $f_{xx}(\pm 1, 1) = 8 > 0$, so $f(\pm 1, 1) = -3$ are local minimums.



16. $f(x,y) = x^2 + y^4 + 2xy \quad \Rightarrow \quad f_x = 2x + 2y,\ f_y = 4y^3 + 2x,\ f_{xx} = 2,\ f_{xy} = 2,\ f_{yy} = 12y^2$. Then $f_x = 0$ implies $y = -x$, and substitution into $f_y = 4y^3 + 2x = 0$ gives $-4x^3 + 2x = 0 \quad \Rightarrow \quad 2x\left(1 - 2x^2\right) = 0 \quad \Rightarrow \quad x = 0$ or $x = \pm\frac{1}{\sqrt{2}}$. Thus, the critical points are $(0,0)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Now

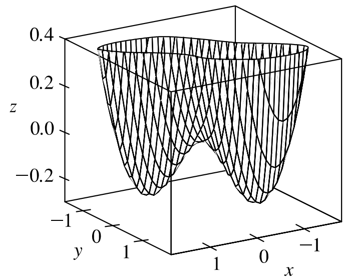
$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (2)(12y^2) - (2)^2 = 24y^2 - 4$,

so $D(0,0) = -4 < 0$ and $(0,0)$ is a saddle point.

$D\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = D\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 24\left(\frac{1}{2}\right) - 4 = 8 > 0$ and

$f_{xx}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = f_{xx}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 > 0$, so $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{4}$

and $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{4}$ are local minimums.



17. $f(x,y) = xy - x^2y - xy^2 \quad \Rightarrow \quad f_x = y - 2xy - y^2$,

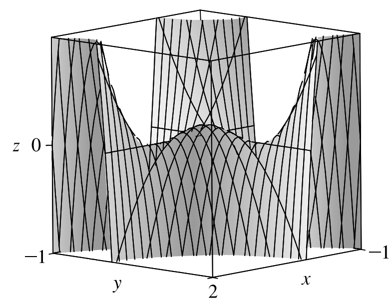
$f_y = x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{xy} = 1 - 2x - 2y$, $f_{yy} = -2x$. Then

$f_x = y - 2xy - y^2 = 0 \quad \Rightarrow \quad y(1 - 2x - y) = 0 \quad \Rightarrow \quad y = 0$ or

$y = 1 - 2x$. Substituting $y = 0$ into $f_y = 0$ gives $x - x^2 = 0 \quad \Rightarrow$

$x = 0$ or $x = 1$. Next, substituting $y = 1 - 2x$ into $f_y = 0$ gives

$x - x^2 - 2x(1 - 2x) = 0 \quad \Rightarrow \quad 3x^2 - x = 0 \quad \Rightarrow \quad x = 0$ or $x = \frac{1}{3}$.



Thus, the critical points are $(0,0)$, $(1,0)$, $(0,1)$ and $\left(\frac{1}{3}, \frac{1}{3}\right)$.

$D(x,y) = (-2y)(-2x) - (1 - 2x - 2y)^2 = 4xy - (1 - 2x - 2y)^2$. $D(0,0) = D(1,0) = D(0,1) = -1 < 0$, so $(0,0)$, $(1,0)$, and $(0,1)$ are saddle points. $D\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3} > 0$ and $f_{xx}\left(\frac{1}{3}, \frac{1}{3}\right) = -\frac{2}{3} < 0$, so $f\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27}$ is a local maximum by the Second Derivatives Test.

18. $f(x,y) = (6x - x^2)(4y - y^2) \quad \Rightarrow \quad f_x = (6 - 2x)(4y - y^2)$,

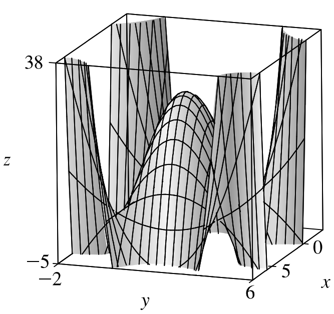
$f_y = (6x - x^2)(4 - 2y)$, $f_{xx} = -2(4y - y^2)$, $f_{xy} = (6 - 2x)(4 - 2y)$,

$f_{yy} = -2(6x - x^2)$. Then $f_x = 0 \quad \Rightarrow \quad 6 - 2x = 0 \quad \Rightarrow \quad x = 3$ or

$4y - y^2 = 0 \quad \Rightarrow \quad y = 0$ or $y = 4$. Substituting $x = 3$ into $f_y = 0 \quad \Rightarrow$

$y = 2$. Substituting $y = 0$ into $f_y = 0 \quad \Rightarrow \quad x = 0$ or $x = 6$. $y = 4 \quad \Rightarrow$

$x = 0$ or $x = 6$ also. This gives critical points $(3,2)$, $(0,0)$, $(6,0)$, $(0,4)$,



and $(6,4)$. $D(x,y) = 4(4y - y^2)(6x - x^2) - [(6 - 2x)(4 - 2y)]^2$.

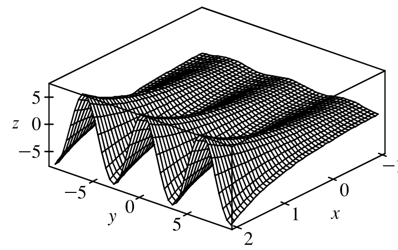
$D(0,0) = D(6,0) = D(0,4) = D(6,4) = -576 < 0$, so $(0,0)$, $(6,0)$, $(0,4)$, and $(6,4)$ are saddle points.

$D(3,2) = 144 > 0$ and $f_{xx}(3,2) = -8 < 0$, so $f(3,2) = 36$ is a local maximum by the Second Derivatives Test.

19. $f(x,y)=e^x\cos y \quad\Rightarrow\quad f_x=e^x\cos y,\ f_y=-e^x\sin y.$

Now $f_x=0$ implies $\cos y=0$ or $y=\frac{\pi}{2}+n\pi$ for n an integer.

But $\sin\left(\frac{\pi}{2}+n\pi\right)\neq 0$, so there are no critical points.



20. $f(x,y)=(x^2+y^2)e^{-x} \quad\Rightarrow\quad f_x=(x^2+y^2)(-e^{-x})+e^{-x}(2x)=(2x-x^2-y^2)e^{-x},\quad f_y=2ye^{-x},$

$f_{xx}=(2x-x^2-y^2)(-e^{-x})+e^{-x}(2-2x)=(x^2+y^2-4x+2)e^{-x},\quad f_{xy}=-2ye^{-x},\quad f_{yy}=2e^{-x}$. Then $f_y=0$

implies $y=0$ and substituting into $f_x=0$ gives $(2x-x^2)e^{-x}=0 \quad\Rightarrow$

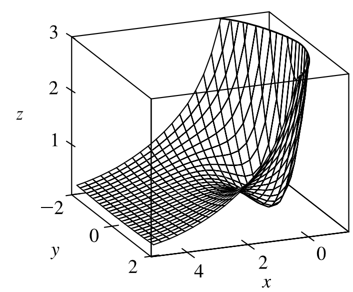
$x(2-x)=0 \quad\Rightarrow\quad x=0$ or $x=2$, so the critical points are $(0,0)$ and

$(2,0)$. $\quad D(0,0)=(2)(2)-(0)^2=4>0$ and $f_{xx}(0,0)=2>0$, so

$f(0,0)=0$ is a local minimum.

$D(2,0)=(-2e^{-2})(2e^{-2})-(0)^2=-4e^{-4}<0$ so $(2,0)$ is a saddle

point.



21. $f(x,y)=y^2-2y\cos x \quad\Rightarrow\quad f_x=2y\sin x,\ f_y=2y-2\cos x,$

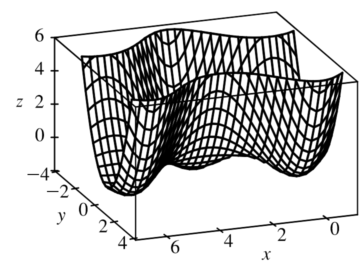
$f_{xx}=2y\cos x,\ f_{xy}=2\sin x,\ f_{yy}=2$. Then $f_x=0$ implies $y=0$ or

$\sin x=0 \quad\Rightarrow\quad x=0,\ \pi,$ or 2π for $-1\le x\le 7$. Substituting $y=0$ into

$f_y=0$ gives $\cos x=0 \quad\Rightarrow\quad x=\frac{\pi}{2}$ or $\frac{3\pi}{2}$, substituting $x=0$ or $x=2\pi$

into $f_y=0$ gives $y=1$, and substituting $x=\pi$ into $f_y=0$ gives $y=-1$.

Thus the critical points are $(0,1)$, $\left(\frac{\pi}{2},0\right)$, $(\pi,-1)$, $\left(\frac{3\pi}{2},0\right)$, and $(2\pi,1)$.



$D\left(\frac{\pi}{2},0\right)=D\left(\frac{3\pi}{2},0\right)=-4<0$ so $\left(\frac{\pi}{2},0\right)$ and $\left(\frac{3\pi}{2},0\right)$ are saddle points. $D(0,1)=D(\pi,-1)=D(2\pi,1)=4>0$ and

$f_{xx}(0,1)=f_{xx}(\pi,-1)=f_{xx}(2\pi,1)=2>0$, so $f(0,1)=f(\pi,-1)=f(2\pi,1)=-1$ are local minimums.

22. $f(x,y)=\sin x\,\sin y \quad\Rightarrow\quad f_x=\cos x\,\sin y,\ f_y=\sin x\,\cos y,\ f_{xx}=-\sin x\,\sin y,\ f_{xy}=\cos x\,\cos y,$

$f_{yy}=-\sin x\,\sin y$. Here we have $-\pi<x<\pi$ and $-\pi<y<\pi$, so $f_x=0$ implies $\cos x=0$ or $\sin y=0$. If $\cos x=0$

then $x=-\frac{\pi}{2}$ or $\frac{\pi}{2}$, and if $\sin y=0$ then $y=0$. Substituting $x=\pm\frac{\pi}{2}$ into $f_y=0$ gives $\cos y=0 \quad\Rightarrow\quad y=-\frac{\pi}{2}$ or $\frac{\pi}{2}$, and

substituting $y=0$ into $f_y=0$ gives $\sin x=0 \quad\Rightarrow\quad x=0$. Thus the critical points are $\left(-\frac{\pi}{2},\pm\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2},\pm\frac{\pi}{2}\right)$, and $(0,0)$.

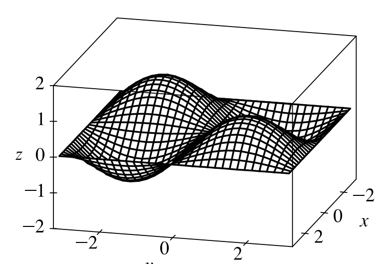
$D(0,0)=-1<0$ so $(0,0)$ is a saddle point.

$D\left(-\frac{\pi}{2},\pm\frac{\pi}{2}\right)=D\left(\frac{\pi}{2},\pm\frac{\pi}{2}\right)=1>0$ and

$f_{xx}\left(-\frac{\pi}{2},-\frac{\pi}{2}\right)=f_{xx}\left(\frac{\pi}{2},\frac{\pi}{2}\right)=-1<0$ while

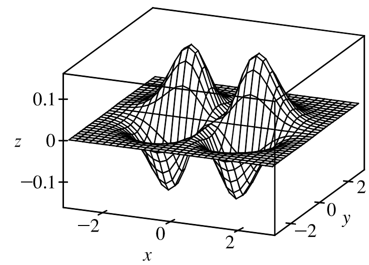
$f_{xx}\left(-\frac{\pi}{2},\frac{\pi}{2}\right)=f_{xx}\left(\frac{\pi}{2},-\frac{\pi}{2}\right)=1>0$, so $f\left(-\frac{\pi}{2},-\frac{\pi}{2}\right)=f\left(\frac{\pi}{2},\frac{\pi}{2}\right)=1$

are local maximums and $f\left(-\frac{\pi}{2},\frac{\pi}{2}\right)=f\left(\frac{\pi}{2},-\frac{\pi}{2}\right)=1$ are local minimums.



23. $f(x,y)=x^2+4y^2-4xy+2 \quad\Rightarrow\quad f_x=2x-4y$, $f_y=8y-4x$, $f_{xx}=2$, $f_{xy}=-4$, $f_{yy}=8$. Then $f_x=0$ and $f_y=0$ each implies $y=\frac{1}{2}x$, so all points of the form $\left(x_0,\frac{1}{2}x_0\right)$ are critical points and for each of these we have $D\left(x_0,\frac{1}{2}x_0\right)=(2)(8)-(-4)^2=0$. The Second Derivatives Test gives no information, but $f(x,y)=x^2+4y^2-4xy+2=(x-2y)^2+2\geq 2$ with equality if and only if $y=\frac{1}{2}x$. Thus $f\left(x_0,\frac{1}{2}x_0\right)=2$ are all local (and absolute) minimums.

24. $f(x,y)=x^2ye^{-x^2-y^2} \quad\Rightarrow$



$f_x=x^2ye^{-x^2-y^2}(-2x)+2xye^{-x^2-y^2}=2xy(1-x^2)e^{-x^2-y^2}$,

$f_y=x^2ye^{-x^2-y^2}(-2y)+x^2e^{-x^2-y^2}=x^2(1-2y^2)e^{-x^2-y^2}$,

$f_{xx}=2y(2x^4-5x^2+1)e^{-x^2-y^2}$,

$f_{xy}=2x(1-x^2)(1-2y^2)e^{-x^2-y^2}$, $f_{yy}=2x^2y(2y^2-3)e^{-x^2-y^2}$.

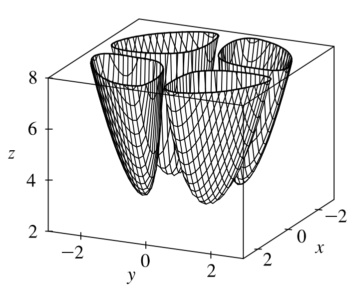
$f_x=0$ implies $x=0$, $y=0$, or $x=\pm1$. If $x=0$ then $f_y=0$ for any y-value, so all points of the form $(0,y)$ are critical points. If $y=0$ then $f_y=0 \quad\Rightarrow\quad x^2e^{-x^2}=0 \quad\Rightarrow\quad x=0$, so $(0,0)$ (already included above) is a critical point. If $x=\pm1$ then $(1-2y^2)e^{-1-y^2}=0 \quad\Rightarrow\quad y=\pm\frac{1}{\sqrt{2}}$, so $\left(\pm1,\frac{1}{\sqrt{2}}\right)$ and $\left(\pm1,-\frac{1}{\sqrt{2}}\right)$ are critical points. Now

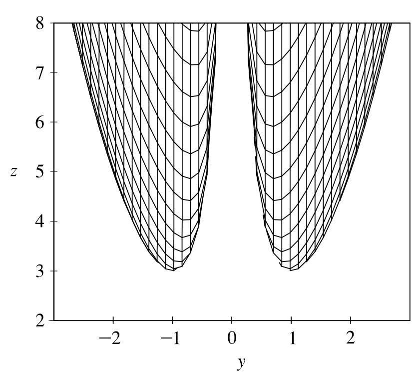
$D\left(\pm1,\frac{1}{\sqrt{2}}\right)=8e^{-3}>0$, $f_{xx}\left(\pm1,\frac{1}{\sqrt{2}}\right)=-2\sqrt{2}\,e^{-3/2}<0$ and $D\left(\pm1,-\frac{1}{\sqrt{2}}\right)=8e^{-3}>0$,

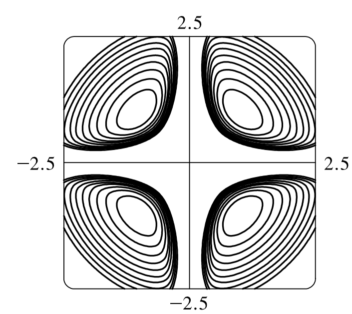
$f_{xx}\left(\pm1,-\frac{1}{\sqrt{2}}\right)=2\sqrt{2}\,e^{-3/2}>0$, so $f\left(\pm1,\frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}e^{-3/2}$ are local maximum points while

$f\left(\pm1,-\frac{1}{\sqrt{2}}\right)=-\frac{1}{\sqrt{2}}e^{-3/2}$ are local minimum points. At all critical points $(0,y)$ we have $D(0,y)=0$, so the Second Derivatives Test gives no information. However, if $y>0$ then $x^2ye^{-x^2-y^2}\geq 0$ with equality only when $x=0$, so we have local minimum values $f(0,y)=0$, $y>0$. Similarly, if $y<0$ then $x^2ye^{-x^2-y^2}\leq 0$ with equality when $x=0$ so $f(0,y)=0$, $y<0$ are local maximum values, and $(0,0)$ is a saddle point.

25. $f(x,y)=x^2+y^2+x^{-2}y^{-2}$



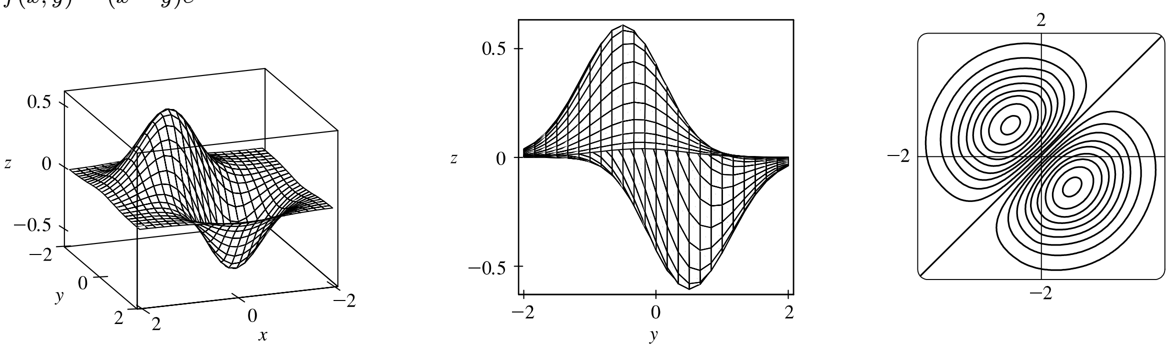




From the graphs, there appear to be local minimums of about $f(1,\pm1)=f(-1,\pm1)\approx 3$ (and no local maximums or saddle points). $f_x=2x-2x^{-3}y^{-2}$, $f_y=2y-2x^{-2}y^{-3}$, $f_{xx}=2+6x^{-4}y^{-2}$, $f_{xy}=4x^{-3}y^{-3}$, $f_{yy}=2+6x^{-2}y^{-4}$. Then

$f_x = 0$ implies $2x^4y^2 - 2 = 0$ or $x^4y^2 = 1$ or $y^2 = x^{-4}$. Note that neither x nor y can be zero. Now $f_y = 0$ implies $2x^2y^4 - 2 = 0$, and with $y^2 = x^{-4}$ this implies $2x^{-6} - 2 = 0$ or $x^6 = 1$. Thus, $x = \pm 1$ and if $x = 1$, $y = \pm 1$; if $x = -1$, $y = \pm 1$. So the critical points are $(1, 1)$, $(1, -1)$,$(-1, 1)$ and $(-1, -1)$. Now $D(1, \pm 1) = D(-1, \pm 1) = 64 - 16 > 0$ and $f_{xx} > 0$ always, so $f(1, \pm 1) = f(-1, \pm 1) = 3$ are local minimums.

26. $f(x, y) = (x - y)e^{-x^2-y^2}$



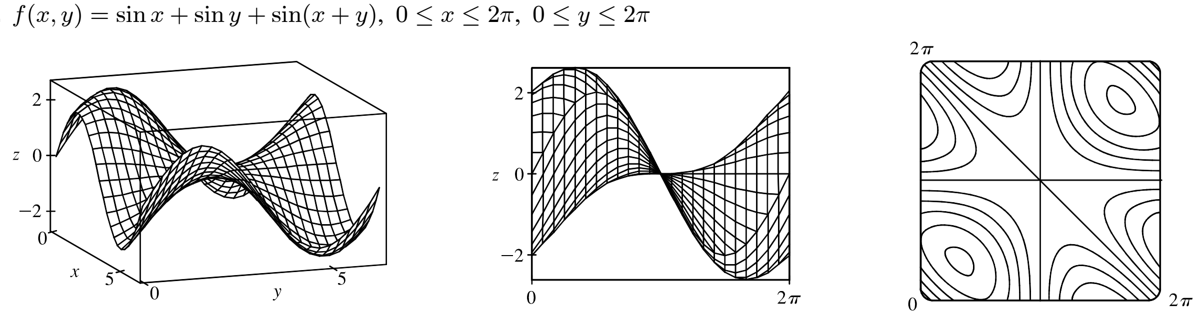
From the graphs, there appears to be a local maximum of about $f(0.5, -0.5) \approx 0.6$ and a local minimum of about $f(-0.5, 0.5) \approx -0.6$.

$f_x = (x - y)e^{-x^2-y^2}(-2x) + e^{-x^2-y^2}(1) = e^{-x^2-y^2}(1 - 2x^2 + 2xy)$,

$f_y = (x - y)e^{-x^2-y^2}(-2y) + e^{-x^2-y^2}(-1) = -e^{-x^2-y^2}(1 - 2y^2 + 2xy)$, $f_{xx} = 2e^{-x^2-y^2}(2x^3 - 3x + y - 2x^2y)$,

$f_{xy} = 2e^{-x^2-y^2}(x - y + 2x^2y - 2xy^2)$, $f_{yy} = -2e^{-x^2-y^2}(2y^3 - 3y + x - 2xy^2)$. Then $f_x = 0$ implies $1 - 2x^2 + 2xy = 0$ and $f_y = 0$ implies $1 - 2y^2 + 2xy = 0$. Subtracting these two equations gives $-2x^2 + 2y^2 = 0 \quad \Rightarrow \quad y = \pm x$. If $y = x$, then substituting into $f_x = 0$ gives $1 - 2x^2 + 2x^2 = 0$, an impossibility. Substituting $y = -x$ gives $1 - 2x^2 - 2x^2 = 0 \quad \Rightarrow \quad x^2 = \frac{1}{4} \quad \Rightarrow \quad x = \pm\frac{1}{2}$. Thus,the critical points are $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Now $D\left(\frac{1}{2}, -\frac{1}{2}\right) = (-3e^{-1/2})(-3e^{-1/2}) - (e^{-1/2})^2 = 8e^{-1} > 0$ with $f_{xx}\left(\frac{1}{2}, -\frac{1}{2}\right) = -3e^{-1/2} < 0$, so $f\left(\frac{1}{2}, -\frac{1}{2}\right) = e^{-1/2} \approx 0.607$ is a local maximum, and $D\left(-\frac{1}{2}, \frac{1}{2}\right) = (3e^{-1/2})(3e^{-1/2}) - (-e^{-1/2})^2 = 8e^{-1} > 0$ with $f_{xx}\left(-\frac{1}{2}, \frac{1}{2}\right) = 3e^{-1/2} > 0$, so $f\left(-\frac{1}{2}, \frac{1}{2}\right) = -e^{-1/2} \approx -0.607$ is a local minimum.

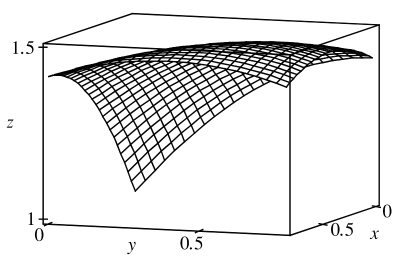
27. $f(x, y) = \sin x + \sin y + \sin(x + y)$, $0 \le x \le 2\pi$, $0 \le y \le 2\pi$

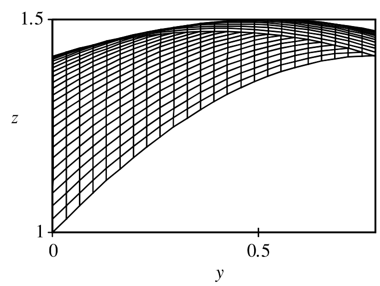


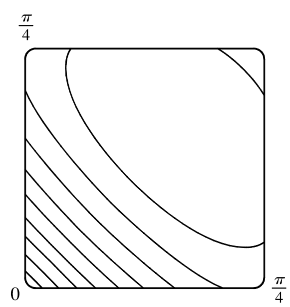
From the graphs it appears that f has a local maximum at about $(1, 1)$ with value approximately 2.6, a local minimum at about $(5, 5)$ with value approximately -2.6, and a saddle point at about $(3, 3)$.

$f_x = \cos x + \cos(x + y)$, $f_y = \cos y + \cos(x + y)$, $f_{xx} = -\sin x - \sin(x + y)$, $f_{yy} = -\sin y - \sin(x + y)$, $f_{xy} = -\sin(x + y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x - \cos y = 0$ or $\cos x = \cos y$. Thus, $x = y$ or $x = 2\pi - y$. If $x = y$, $f_x = 0$ becomes $\cos x + \cos 2x = 0$ or $2\cos^2 x + \cos x - 1 = 0$, a quadratic in $\cos x$. Thus, $\cos x = -1$ or $\frac{1}{2}$ and $x = \pi$, $\frac{\pi}{3}$, or $\frac{5\pi}{3}$, giving the critical points (π, π), $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right)$. Similarly, if $x = 2\pi - y$, $f_x = 0$ becomes $(\cos x) + 1 = 0$ and the resulting critical point is (π, π). Now $D(x, y) = \sin x\, \sin y + \sin x\, \sin(x + y) + \sin y\, \sin(x + y)$. So $D(\pi, \pi) = 0$ and the Second Derivatives Test doesn't apply. However, along the line $y = x$ we have $f(x, x) = 2\sin x + \sin 2x = 2\sin x + 2\sin x\, \cos x = 2\sin x(1 + \cos x)$, and $f(x, x) > 0$ for $0 < x < \pi$ while $f(x, x) < 0$ for $\pi < x < 2\pi$. Thus, every disk with center (π, π) contains points where f is positive as well as points where f is negative, so the graph crosses its tangent plane ($z = 0$) there and (π, π) is a saddle point. $D\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{9}{4} > 0$ and $f_{xx}\left(\frac{\pi}{3}, \frac{\pi}{3}\right) < 0$ so $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$ is a local maximum while $D\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) = \frac{9}{4} > 0$ and $f_{xx}\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) > 0$, so $f\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) = -\frac{3\sqrt{3}}{2}$ is a local minimum.

28. $f(x, y) = \sin x + \sin y + \cos(x + y)$, $0 \le x \le \frac{\pi}{4}$, $0 \le y \le \frac{\pi}{4}$



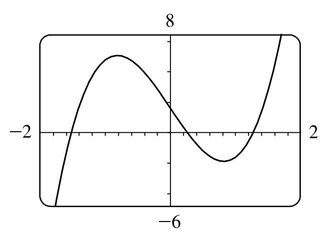


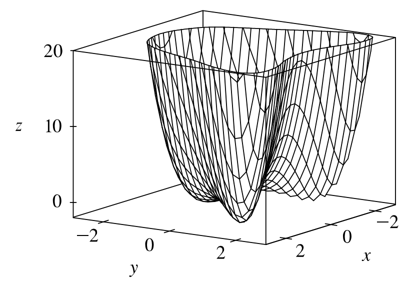


From the graphs, it seems that f has a local maximum at about $(0.5, 0.5)$.

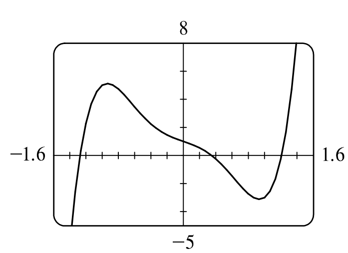
$f_x = \cos x - \sin(x + y)$, $f_y = \cos y - \sin(x + y)$, $f_{xx} = -\sin x - \cos(x + y)$, $f_{yy} = -\sin y - \cos(x + y)$, $f_{xy} = -\cos(x + y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x = \cos y$. Thus, $x = y$. Substituting $x = y$ into $f_x = 0$ gives $\cos x - \sin 2x = 0$ or $\cos x(1 - 2\sin x) = 0$. But $\cos x \neq 0$ for $0 \le x \le \frac{\pi}{4}$ and $1 - 2\sin x = 0$ implies $x = \frac{\pi}{6}$, so the only critical point is $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$. Here $f_{xx}\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = -1 < 0$ and $D\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = (-1)^2 - \frac{1}{4} > 0$. Thus, $f\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{3}{2}$ is a local maximum.

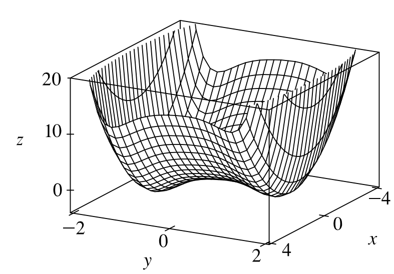
29. $f(x,y) = x^4 + y^4 - 4x^2y + 2y \Rightarrow f_x(x,y) = 4x^3 - 8xy$ and $f_y(x,y) = 4y^3 - 4x^2 + 2$. $f_x = 0 \Rightarrow$ $4x(x^2 - 2y) = 0$, so $x = 0$ or $x^2 = 2y$. If $x = 0$ then substitution into $f_y = 0$ gives $4y^3 = -2 \Rightarrow y = -\frac{1}{\sqrt[3]{2}}$, so $\left(0, -\frac{1}{\sqrt[3]{2}}\right)$ is a critical point. Substituting $x^2 = 2y$ into $f_y = 0$ gives $4y^3 - 8y + 2 = 0$. Using a graph, solutions are approximately $y = -1.526$, 0.259, and 1.267. (Alternatively, we could have used a calculator or a CAS to find these roots.) We have $x^2 = 2y \Rightarrow x = \pm\sqrt{2y}$, so $y = -1.526$ gives no real-valued solution for x, but $y = 0.259 \Rightarrow x \approx \pm 0.720$ and $y = 1.267 \Rightarrow x \approx \pm 1.592$. Thus to three decimal places, the critical points are $\left(0, -\frac{1}{\sqrt[3]{2}}\right) \approx (0, -0.794)$, $(\pm 0.720, 0.259)$, and $(\pm 1.592, 1.267)$. Now since $f_{xx} = 12x^2 - 8y$, $f_{xy} = -8x$, $f_{yy} = 12y^2$, and $D = (12x^2 - 8y)(12y^2) - 64x^2$, we have $D(0, -0.794) > 0$, $f_{xx}(0, -0.794) > 0$, $D(\pm 0.720, 0.259) < 0$, $D(\pm 1.592, 1.267) > 0$, and $f_{xx}(\pm 1.592, 1.267) > 0$. Therefore, $f(0, -0.794) \approx -1.191$ and $f(\pm 1.592, 1.267) \approx -1.310$ are local minimums, and $(\pm 0.720, 0.259)$ are saddle points. There is no highest point on the graph, but the lowest points are approximately $(\pm 1.592, 1.267, -1.310)$.



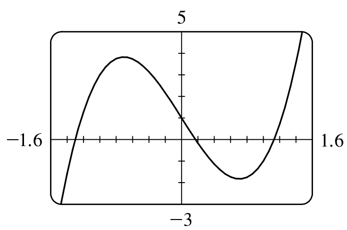


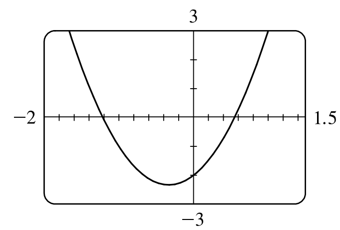
30. $f(x,y) = y^6 - 2y^4 + x^2 - y^2 + y \Rightarrow f_x(x,y) = 2x$ and $f_y(x,y) = 6y^5 - 8y^3 - 2y + 1$. $f_x = 0$ implies $x = 0$, and the graph of f_y shows that the roots of $f_y = 0$ are approximately $y = -1.273$, 0.347, and 1.211. (Alternatively, we could have found the roots of $f_y = 0$ directly, using a calculator or CAS.) So to three decimal places, the critical points are $(0, -1.273)$, $(0, 0.347)$, and $(0, 1.211)$. Now since $f_{xx} = 2$, $f_{xy} = 0$, $f_{yy} = 30y^4 - 24y^2 - 2$, and $D = 60y^4 - 48y^2 - 4$, we have $D(0, -1.273) > 0$, $f_{xx}(0, -1.273) > 0$, $D(0, 0.347) < 0$, $D(0, 1.211) > 0$, and $f_{xx}(0, 1.211) > 0$, so $f(0, -1.273) \approx -3.890$ and $f(0, 1.211) \approx -1.403$ are local minimums, and $(0, 0.347)$ is a saddle point. The lowest point on the graph is approximately $(0, -1.273, -3.890)$.

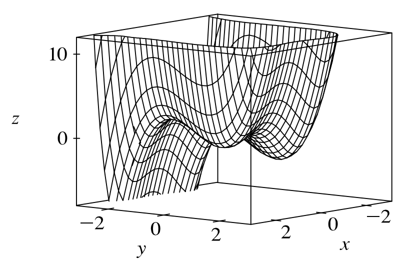




31. $f(x,y) = x^4 + y^3 - 3x^2 + y^2 + x - 2y + 1 \quad \Rightarrow \quad f_x(x,y) = 4x^3 - 6x + 1$ and $f_y(x,y) = 3y^2 + 2y - 2$. From the graphs, we see that to three decimal places, $f_x = 0$ when $x \approx -1.301$, 0.170, or 1.131, and $f_y = 0$ when $y \approx -1.215$ or 0.549. (Alternatively, we could have used a calculator or a CAS to find these roots. We could also use the quadratic formula to find the solutions of $f_y = 0$.) So, to three decimal places, f has critical points at $(-1.301, -1.215)$, $(-1.301, 0.549)$, $(0.170, -1.215)$, $(0.170, 0.549)$, $(1.131, -1.215)$, and $(1.131, 0.549)$. Now since $f_{xx} = 12x^2 - 6$, $f_{xy} = 0$, $f_{yy} = 6y + 2$, and $D = (12x^2 - 6)(6y + 2)$, we have $D(-1.301, -1.215) < 0$, $D(-1.301, 0.549) > 0$, $f_{xx}(-1.301, 0.549) > 0$, $D(0.170, -1.215) > 0$, $f_{xx}(0.170, -1.215) < 0$, $D(0.170, 0.549) < 0$, $D(1.131, -1.215) < 0$, $D(1.131, 0.549) > 0$, and $f_{xx}(1.131, 0.549) > 0$. Therefore, to three decimal places, $f(-1.301, 0.549) \approx -3.145$ and $f(1.131, 0.549) \approx -0.701$ are local minimums, $f(0.170, -1.215) \approx 3.197$ is a local maximum, and $(-1.301, -1.215)$, $(0.170, 0.549)$, and $(1.131, -1.215)$ are saddle points. There is no highest or lowest point on the graph.







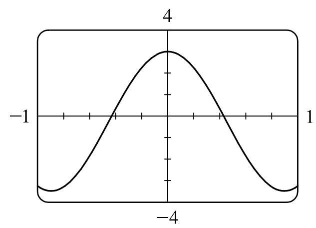
32. $f(x,y) = 20e^{-x^2-y^2} \sin 3x \cos 3y \quad \Rightarrow$

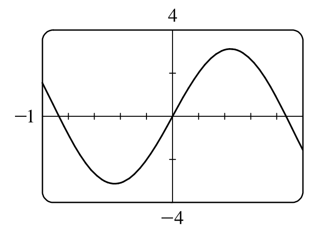
$$\begin{aligned} f_x(x,y) &= 20\cos 3y \left[e^{-x^2-y^2}(3\cos 3x) + (\sin 3x)e^{-x^2-y^2}(-2x)\right] \\ &= 20e^{-x^2-y^2} \cos 3y\,(3\cos 3x - 2x\sin 3x) \\ f_y(x,y) &= 20\sin 3x \left[e^{-x^2-y^2}(-3\sin 3y) + (\cos 3y)e^{-x^2-y^2}(-2y)\right] \\ &= -20e^{-x^2-y^2} \sin 3x\,(3\sin 3y + 2y\cos 3y) \end{aligned}$$

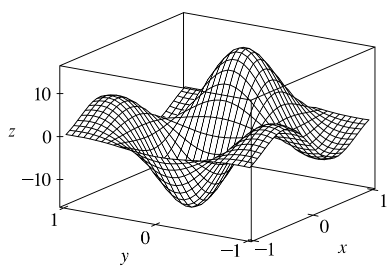
Now $f_x = 0$ implies $\cos 3y = 0$ or $3\cos 3x - 2x\sin 3x = 0$. For $|y| \le 1$, the solutions to $\cos 3y = 0$ are $y = \pm\frac{\pi}{6} \approx \pm 0.524$. Using a graph (or a calculator or CAS), we estimate the roots of $3\cos 3x - 2x\sin 3x$ for $|x| \le 1$ to be $x \approx \pm 0.430$. $f_y = 0$ implies $\sin 3x = 0$, so $x = 0$, or $3\sin 3y + 2y\cos 3y = 0$. From a graph (or calculator or CAS), the roots of $3\sin 3y + 2y\cos 3y$ between -1 and 1 are approximately 0 and ± 0.872. So to three decimal places, f has critical points at $(\pm 0.430, 0)$, $(0.430, \pm 0.872)$, $(-0.430, \pm 0.872)$, and $(0, \pm 0.524)$. Now

$$\begin{aligned} f_{xx} &= 20e^{-x^2-y^2} \cos 3y[(4x^2 - 11)\sin 3x - 12x\cos 3x] \\ f_{xy} &= -20e^{-x^2-y^2}(3\cos 3x - 2x\sin 3x)(3\sin 3y + 2y\cos 3y) \\ f_{yy} &= 20e^{-x^2-y^2} \sin 3x[(4y^2 - 11)\cos 3y - 12y\sin 3y] \end{aligned}$$

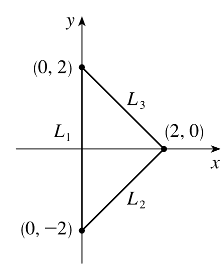
and $D = f_{xx} f_{yy} - f_{xy}^2$. Then $D(\pm 0.430, 0) > 0$, $f_{xx}(0.430, 0) < 0$, $f_{xx}(-0.430, 0) > 0$, $D(0.430, \pm 0.872) > 0$, $f_{xx}(0.430, \pm 0.872) > 0$, $D(-0.430, \pm 0.872) > 0$, $f_{xx}(-0.430, \pm 0.872) < 0$, and $D(0, \pm 0.524) < 0$, so $f(0.430, 0) \approx 15.973$ and $f(-0.430, \pm 0.872) \approx 6.459$ are local maximums, $f(-0.430, 0) \approx -15.973$ and $f(0.430, \pm 0.872) \approx -6.459$ are local minimums, and $(0, \pm 0.524)$ are saddle points. The highest point on the graph is approximately $(0.430, 0, 15.973)$ and the lowest point is approximately $(-0.430, 0, -15.973)$.







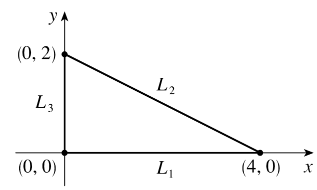
33. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. Here $f_x = 2x - 2$, $f_y = 2y$, and setting $f_x = f_y = 0$ gives $(1, 0)$ as the only critical point (which is inside D), where $f(1, 0) = -1$. Along L_1: $x = 0$ and $f(0, y) = y^2$ for $-2 \le y \le 2$, a quadratic function which attains its minimum at $y = 0$, where $f(0, 0) = 0$, and its maximum at $y = \pm 2$, where $f(0, \pm 2) = 4$. Along L_2: $y = x - 2$ for $0 \le x \le 2$, and $f(x, x - 2) = 2x^2 - 6x + 4 = 2\left(x - \frac{3}{2}\right)^2 - \frac{1}{2}$, a quadratic which attains its minimum at $x = \frac{3}{2}$, where $f\left(\frac{3}{2}, -\frac{1}{2}\right) = -\frac{1}{2}$, and its maximum at $x = 0$, where $f(0, -2) = 4$. Along L_3: $y = 2 - x$ for $0 \le x \le 2$, and $f(x, 2 - x) = 2x^2 - 6x + 4 = 2\left(x - \frac{3}{2}\right)^2 - \frac{1}{2}$, a quadratic which attains its minimum at $x = \frac{3}{2}$, where $f\left(\frac{3}{2}, \frac{1}{2}\right) = -\frac{1}{2}$, and its maximum at $x = 0$, where $f(0, 2) = 4$. Thus, the absolute maximum of f on D is $f(0, \pm 2) = 4$ and the absolute minimum is $f(1, 0) = -1$.



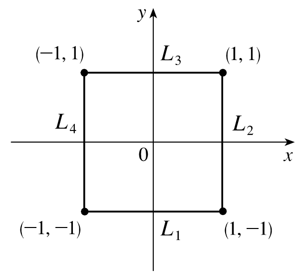
34. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. $f_x = 1 - y$, $f_y = 1 - x$, and setting $f_x = f_y = 0$ gives $(1, 1)$ as the only critical point (which is inside D), where $f(1, 1) = 1$. Along L_1: $y = 0$ and $f(x, 0) = x$ for $0 \le x \le 4$, an increasing function in x, so the maximum value is $f(4, 0) = 4$ and the minimum value is $f(0, 0) = 0$. Along L_2: $y = 2 - \frac{1}{2}x$ and $f\left(x, 2 - \frac{1}{2}x\right) = \frac{1}{2}x^2 - \frac{3}{2}x + 2 = \frac{1}{2}\left(x - \frac{3}{2}\right)^2 + \frac{7}{8}$ for $0 \le x \le 4$, a quadratic function which has a minimum at $x = \frac{3}{2}$, where $f\left(\frac{3}{2}, \frac{5}{4}\right) = \frac{7}{8}$, and a maximum at $x = 4$, where $f(4, 0) = 4$.

[continued]

Along L_3: $x = 0$ and $f(0, y) = y$ for $0 \le y \le 2$, an increasing function in y, so the maximum value is $f(0, 2) = 2$ and the minimum value is $f(0, 0) = 0$. Thus the absolute maximum of f on D is $f(4, 0) = 4$ and the absolute minimum is $f(0, 0) = 0$.



35. $f(x, y) = x^2 + y^2 + x^2y + 4 \quad \Rightarrow \quad f_x(x, y) = 2x + 2xy$, $f_y(x, y) = 2y + x^2$, and setting $f_x = f_y = 0$ gives $(0, 0)$ as the only critical point in D, with $f(0, 0) = 4$.



On L_1: $y = -1$, $f(x, -1) = 5$, a constant.

On L_2: $x = 1$, $f(1, y) = y^2 + y + 5$, a quadratic in y which attains its maximum at $(1, 1)$, $f(1, 1) = 7$ and its minimum at $\left(1, -\frac{1}{2}\right)$, $f\left(1, -\frac{1}{2}\right) = \frac{19}{4}$.

On L_3: $f(x, 1) = 2x^2 + 5$ which attains its maximum at $(-1, 1)$ and $(1, 1)$ with $f(\pm 1, 1) = 7$ and its minimum at $(0, 1)$, $f(0, 1) = 5$.

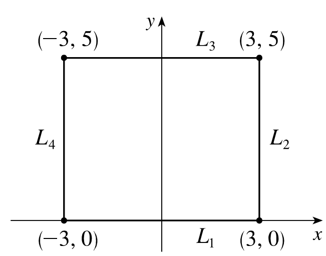
On L_4: $f(-1, y) = y^2 + y + 5$ with maximum at $(-1, 1)$, $f(-1, 1) = 7$ and minimum at $\left(-1, -\frac{1}{2}\right)$, $f\left(-1, -\frac{1}{2}\right) = \frac{19}{4}$.

Thus, the absolute maximum is attained at both $(\pm 1, 1)$ with $f(\pm 1, 1) = 7$ and the absolute minimum on D is attained at $(0, 0)$ with $f(0, 0) = 4$.

36. $f(x, y) = x^2 + xy + y^2 - 6y \quad \Rightarrow \quad f_x = 2x + y, \;\; f_y = x + 2y - 6$. Then $f_x = 0$ implies $y = -2x$, and substituting into $f_y = 0$ gives $x - 4x - 6 = 0 \quad \Rightarrow \quad x = -2$, so the only critical point is $(-2, 4)$ (which is in D) where $f(-2, 4) = -12$.

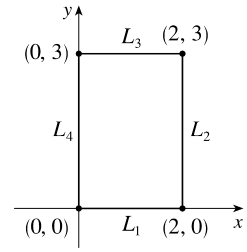
Along L_1: $y = 0$, so $f(x, 0) = x^2$, $-3 \le x \le 3$, which has a maximum value at $x = \pm 3$ where $f(\pm 3, 0) = 9$ and a minimum value at $x = 0$, where $f(0, 0) = 0$. Along L_2: $x = 3$, so $f(3, y) = 9 - 3y + y^2 = \left(y - \frac{3}{2}\right)^2 + \frac{27}{4}$, $0 \le y \le 5$, which has a maximum value at $y = 5$ where $f(3, 5) = 19$ and a minimum value at $y = \frac{3}{2}$ where $f(3, \frac{3}{2}) = \frac{27}{4}$.

Along L_3: $y = 5$, so $f(x, 5) = x^2 + 5x - 5 = \left(x + \frac{5}{2}\right)^2 - \frac{45}{4}$, $-3 \le x \le 3$, which has a maximum value at $x = 3$ where $f(3, 5) = 19$ and a minimum value at $x = -\frac{5}{2}$, where $f\left(-\frac{5}{2}, 5\right) = -\frac{45}{4}$. Along L_4: $x = -3$, so $f(-3, y) = 9 - 9y + y^2 = \left(y - \frac{9}{2}\right)^2 - \frac{45}{4}$, $0 \le y \le 5$, which has a maximum value at $y = 0$ where $f(-3, 0) = 9$ and a minimum value at $y = \frac{9}{2}$ where $f\left(-3, \frac{9}{2}\right) = -\frac{45}{4}$. Thus, the absolute maximum of f on D is $f(3, 5) = 19$ and the absolute minimum is $f(-2, 4) = -12$.

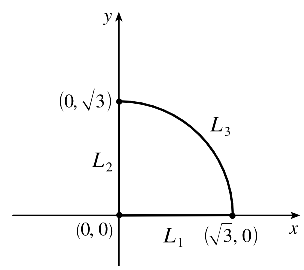


37. $f(x, y) = x^2 + 2y^2 - 2x - 4y + 1 \quad \Rightarrow \quad f_x = 2x - 2, \;\; f_y = 4y - 4$. Setting $f_x = 0$ and $f_y = 0$ gives $(1, 1)$ as the only critical point (which is inside D), where $f(1, 1) = -2$. Along L_1: $y = 0$, so $f(x, 0) = x^2 - 2x + 1 = (x - 1)^2$, $0 \le x \le 2$, which has a maximum value both at $x = 0$ and $x = 2$ where $f(0, 0) = f(2, 0) = 1$ and a minimum value at $x = 1$, where $f(1, 0) = 0$. Along L_2: $x = 2$, so $f(2, y) = 2y^2 - 4y + 1 = 2(y - 1)^2 - 1$, $0 \le y \le 3$, which has a maximum value at

$y = 3$ where $f(2,3) = 7$ and a minimum value at $y = 1$ where $f(2,1) = -1$. Along L_3: $y = 3$, so $f(x,3) = x^2 - 2x + 7 = (x-1)^2 + 6$, $0 \le x \le 2$, which has a maximum value both at $x = 0$ and $x = 2$ where $f(0,3) = f(2,3) = 7$ and a minimum value at $x = 1$, where $f(1,3) = 6$. Along L_4: $x = 0$, so $f(0,y) = 2y^2 - 4y + 1 = 2(y-1)^2 - 1$, $0 \le y \le 3$, which has a maximum value at $y = 3$ where $f(0,3) = 7$ and a minimum value at $y = 1$ where $f(0,1) = -1$. Thus, the absolute maximum is attained at both $(0,3)$ and $(2,3)$, where $f(0,3) = f(2,3) = 7$, and the absolute minimum is $f(1,1) = -2$.



38. $f(x,y) = xy^2 \quad \Rightarrow \quad f_x = y^2$ and $f_y = 2xy$, and since $f_x = 0 \quad \Leftrightarrow$ $y = 0$, there are no critical points in the interior of D. Along L_1: $y = 0$ and $f(x,0) = 0$. Along L_2: $x = 0$ and $f(0,y) = 0$. Along L_3: $y = \sqrt{3 - x^2}$, so let $g(x) = f\left(x, \sqrt{3 - x^2}\right) = 3x - x^3$ for $0 \le x \le \sqrt{3}$. Then $g'(x) = 3 - 3x^2 = 0 \quad \Leftrightarrow \quad x = 1$. The maximum value is $f\left(1, \sqrt{2}\right) = 2$ and the minimum occurs both at $x = 0$ and $x = \sqrt{3}$ where $f\left(0, \sqrt{3}\right) = f\left(\sqrt{3}, 0\right) = 0$. Thus, the absolute maximum of f on D is $f\left(1, \sqrt{2}\right) = 2$, and the absolute minimum is 0 which occurs at all points along L_1 and L_2.

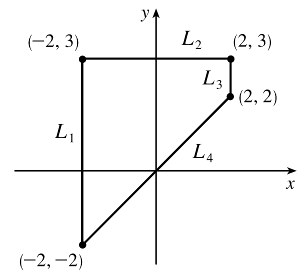


39. $f(x,y) = 2x^3 + y^4 \quad \Rightarrow \quad f_x(x,y) = 6x^2$ and $f_y(x,y) = 4y^3$. And so $f_x = 0$ and $f_y = 0$ only occur when $x = y = 0$. Hence, the only critical point inside the disk is at $x = y = 0$ where $f(0,0) = 0$. Now on the circle $x^2 + y^2 = 1$, $y^2 = 1 - x^2$ so let $g(x) = f(x,y) = 2x^3 + (1 - x^2)^2 = x^4 + 2x^3 - 2x^2 + 1$, $-1 \le x \le 1$. Then $g'(x) = 4x^3 + 6x^2 - 4x = 0 \quad \Rightarrow$ $x = 0, -2$, or $\frac{1}{2}$. $f(0, \pm 1) = g(0) = 1$, $f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = g\left(\frac{1}{2}\right) = \frac{13}{16}$, and $(-2, -3)$ is not in D. Checking the endpoints, we get $f(-1,0) = g(-1) = -2$ and $f(1,0) = g(1) = 2$. Thus, the absolute maximum and minimum of f on D are $f(1,0) = 2$ and $f(-1,0) = -2$.

Another method: On the boundary $x^2 + y^2 = 1$ we can write $x = \cos\theta$, $y = \sin\theta$, so $f(\cos\theta, \sin\theta) = 2\cos^3\theta + \sin^4\theta$, $0 \le \theta \le 2\pi$.

40. $f(x,y) = x^3 - 3x - y^3 + 12y \quad \Rightarrow \quad f_x(x,y) = 3x^2 - 3$ and $f_y(x,y) = -3y^2 + 12$ and the critical points are $(1,2)$, $(1,-2)$, $(-1,2)$, and $(-1,-2)$. But only $(1,2)$ and $(-1,2)$ are in D and $f(1,2) = 14$, $f(-1,2) = 18$. Along L_1: $x = -2$ and $f(-2,y) = -2 - y^3 + 12y$, $-2 \le y \le 3$, which has a maximum at $y = 2$ where $f(-2,2) = 14$ and a minimum at $y = -2$ where $f(-2,-2) = -18$. Along L_2: $x = 2$ and $f(2,y) = 2 - y^3 + 12y$, $2 \le y \le 3$, which has a maximum at $y = 2$ where $f(2,2) = 18$ and a minimum at $y = 3$ where $f(2,3) = 11$. Along L_3: $y = 3$ and $f(x,3) = x^3 - 3x + 9$, $-2 \le x \le 2$, which has a maximum at $x = -1$ and $x = 2$ where $f(-1,3) = f(2,3) = 11$ and a minimum at $x = 1$

and $x = -2$ where $f(1,3) = f(-2,3) = 7$. Along L_4: $y = x$ and $f(x,x) = 9x$, $-2 \le x \le 2$, which has a maximum at $x = 2$ where $f(2,2) = 18$ and a minimum at $x = -2$ where $f(-2,-2) = -18$. So the absolute maximum value of f on D is $f(2,2) = 18$ and the minimum is $f(-2,-2) = -18$.



41. $f(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2 \quad \Rightarrow \quad f_x(x,y) = -2(x^2-1)(2x) - 2(x^2y-x-1)(2xy-1)$ and $f_y(x,y) = -2(x^2y-x-1)x^2$. Setting $f_y(x,y) = 0$ gives either $x = 0$ or $x^2y - x - 1 = 0$.

There are no critical points for $x = 0$, since $f_x(0,y) = -2$, so we set $x^2y - x - 1 = 0 \quad \Leftrightarrow \quad y = \dfrac{x+1}{x^2} \quad [x \ne 0]$,

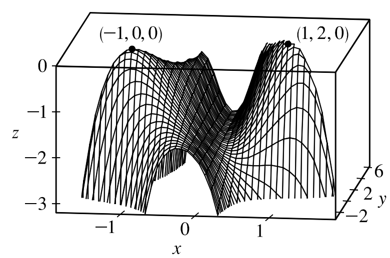
so $f_x\left(x, \dfrac{x+1}{x^2}\right) = -2(x^2-1)(2x) - 2\left(x^2\,\dfrac{x+1}{x^2} - x - 1\right)\left(2x\,\dfrac{x+1}{x^2} - 1\right) = -4x(x^2-1)$. Therefore,

$f_x(x,y) = f_y(x,y) = 0$ at the points $(1,2)$ and $(-1,0)$. To classify these critical points, we calculate

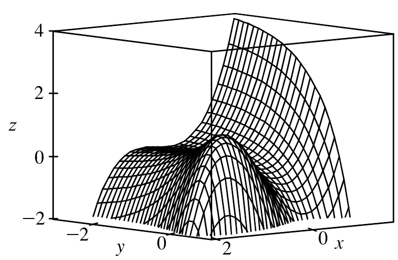
$f_{xx}(x,y) = -12x^2 - 12x^2y^2 + 12xy + 4y + 2$, $f_{yy}(x,y) = -2x^4$, and $f_{xy}(x,y) = -8x^3y + 6x^2 + 4x$. In order to use the Second Derivatives Test we calculate

$D(-1,0) = f_{xx}(-1,0)\,f_{yy}(-1,0) - [f_{xy}(-1,0)]^2 = 16 > 0$,

$f_{xx}(-1,0) = -10 < 0$, $D(1,2) = 16 > 0$, and $f_{xx}(1,2) = -26 < 0$, so both $(-1,0)$ and $(1,2)$ give local maximums.



42. $f(x,y) = 3xe^y - x^3 - e^{3y}$ is differentiable everywhere, so the requirement for critical points is that $f_x = 3e^y - 3x^2 = 0$ **(1)** and $f_y = 3xe^y - 3e^{3y} = 0$ **(2)**. From **(1)** we obtain $e^y = x^2$, and then **(2)** gives $3x^3 - 3x^6 = 0 \quad \Rightarrow \quad x = 1$ or 0, but only $x = 1$ is valid, since $x = 0$ makes **(1)** impossible. So substituting $x = 1$ into **(1)** gives $y = 0$, and the only critical point is $(1,0)$.



The Second Derivatives Test shows that this gives a local maximum, since $D(1,0) = \left[-6x(3xe^y - 9e^{3y}) - (3e^y)^2\right]_{(1,0)} = 27 > 0$ and $f_{xx}(1,0) = [-6x]_{(1,0)} = -6 < 0$. But $f(1,0) = 1$ is not an absolute maximum because, for instance, $f(-3,0) = 17$. This can also be seen from the graph.

43. Let d be the distance from $(2,0,-3)$ to any point (x,y,z) on the plane $x+y+z=1$, so $d = \sqrt{(x-2)^2 + y^2 + (z+3)^2}$ where $z = 1 - x - y$, and we minimize $d^2 = f(x,y) = (x-2)^2 + y^2 + (4-x-y)^2$. Then $f_x(x,y) = 2(x-2) + 2(4-x-y)(-1) = 4x + 2y - 12$, $f_y(x,y) = 2y + 2(4-x-y)(-1) = 2x + 4y - 8$. Solving $4x + 2y - 12 = 0$ and $2x + 4y - 8 = 0$ simultaneously gives $x = \frac{8}{3}$, $y = \frac{2}{3}$, so the only critical point is $\left(\frac{8}{3}, \frac{2}{3}\right)$. An absolute minimum exists (since there is a minimum distance from the point to the plane) and it must occur at a critical point, so the shortest distance occurs for $x = \frac{8}{3}$, $y = \frac{2}{3}$ for which $d = \sqrt{\left(\frac{8}{3} - 2\right)^2 + \left(\frac{2}{3}\right)^2 + \left(4 - \frac{8}{3} - \frac{2}{3}\right)^2} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$.

44. Here the distance d from a point on the plane to the point $(0,1,1)$ is $d = \sqrt{x^2 + (y-1)^2 + (z-1)^2}$, where $z = 2 - \frac{1}{3}x + \frac{2}{3}y$. We can minimize $d^2 = f(x,y) = x^2 + (y-1)^2 + \left(1 - \frac{1}{3}x + \frac{2}{3}y\right)^2$, so

$f_x(x,y) = 2x + 2\left(1 - \frac{1}{3}x + \frac{2}{3}y\right)\left(-\frac{1}{3}\right) = \frac{20}{9}x - \frac{4}{9}y - \frac{2}{3}$ and

$f_y(x,y) = 2(y-1) + 2\left(1 - \frac{1}{3}x + \frac{2}{3}y\right)\left(\frac{2}{3}\right) = -\frac{4}{9}x + \frac{26}{9}y - \frac{2}{3}$. Solving $\frac{20}{9}x - \frac{4}{9}y - \frac{2}{3} = 0$ and $-\frac{4}{9}x + \frac{26}{9}y - \frac{2}{3} = 0$ simultaneously gives $x = \frac{5}{14}$ and $y = \frac{2}{7}$, so the only critical point is $\left(\frac{5}{14}, \frac{2}{7}\right)$.

This point must correspond to the minimum distance, so the point on the plane closest to $(0,1,1)$ is $\left(\frac{5}{14}, \frac{2}{7}, \frac{29}{14}\right)$.

45. Let d be the distance from the point $(4,2,0)$ to any point (x,y,z) on the cone, so $d = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$ where $z^2 = x^2 + y^2$, and we minimize $d^2 = (x-4)^2 + (y-2)^2 + x^2 + y^2 = f(x,y)$. Then $f_x(x,y) = 2(x-4) + 2x = 4x - 8$, $f_y(x,y) = 2(y-2) + 2y = 4y - 4$, and the critical points occur when $f_x = 0 \Rightarrow x = 2$, $f_y = 0 \Rightarrow y = 1$. Thus, the only critical point is $(2,1)$. An absolute minimum exists (since there is a minimum distance from the cone to the point) which must occur at a critical point, so the points on the cone closest to $(4,2,0)$ are $\left(2, 1, \pm\sqrt{5}\right)$.

46. The distance from the origin to a point (x,y,z) on the surface is $d = \sqrt{x^2 + y^2 + z^2}$ where $y^2 = 9 + xz$, so we minimize $d^2 = x^2 + 9 + xz + z^2 = f(x,z)$. Then $f_x = 2x + z$, $f_z = x + 2z$, and $f_x = 0$, $f_z = 0 \Rightarrow x = 0$, $z = 0$, so the only critical point is $(0,0)$. $D(0,0) = (2)(2) - 1 = 3 > 0$ with $f_{xx}(0,0) = 2 > 0$, so this is a minimum. Thus, $y^2 = 9 + 0 \Rightarrow y = \pm 3$ and the points on the surface closest to the origin are $(0, \pm 3, 0)$.

47. Let x, y, z be the positive numbers. Then $x + y + z = 100 \Rightarrow z = 100 - x - y$, and we want to maximize $xyz = xy(100 - x - y) = 100xy - x^2y - xy^2 = f(x,y)$ for $0 < x, y, z < 100$. $f_x = 100y - 2xy - y^2$, $f_y = 100x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 100 - 2x - 2y$. Then $f_x = 0$ implies $y(100 - 2x - y) = 0 \Rightarrow y = 100 - 2x$ (since $y > 0$). Substituting into $f_y = 0$ gives $x[100 - x - 2(100 - 2x)] = 0 \Rightarrow 3x - 100 = 0$ (since $x > 0$) $\Rightarrow x = \frac{100}{3}$. Then $y = 100 - 2\left(\frac{100}{3}\right) = \frac{100}{3}$, and the only critical point is $\left(\frac{100}{3}, \frac{100}{3}\right)$.

$D\left(\frac{100}{3}, \frac{100}{3}\right) = \left(-\frac{200}{3}\right)\left(-\frac{200}{3}\right) - \left(-\frac{100}{3}\right)^2 = \frac{10{,}000}{3} > 0$ and $f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) = -\frac{200}{3} < 0$. Thus, $f\left(\frac{100}{3}, \frac{100}{3}\right)$ is a local maximum. It is also the absolute maximum (compare to the values of f as x, y, or $z \to 0$ or 100), so the numbers are $x = y = z = \frac{100}{3}$.

48. Let x, y, z, be the positive numbers. Then $x + y + z = 12$ and we want to minimize $x^2 + y^2 + z^2 = x^2 + y^2 + (12 - x - y)^2 = f(x,y)$ for $0 < x, y < 12$. $f_x = 2x + 2(12 - x - y)(-1) = 4x + 2y - 24$, $f_y = 2y + 2(12 - x - y)(-1) = 2x + 4y - 24$, $f_{xx} = 4$, $f_{xy} = 2$, $f_{yy} = 4$. Then $f_x = 0$ implies $4x + 2y = 24$ or $y = 12 - 2x$ and substituting into $f_y = 0$ gives $2x + 4(12 - 2x) = 24 \Rightarrow 6x = 24 \Rightarrow x = 4$ and then $y = 4$, so the only critical point is $(4, 4)$. $D(4, 4) = 16 - 4 > 0$ and $f_{xx}(4, 4) = 4 > 0$, so $f(4, 4)$ is a local minimum. $f(4, 4)$ is also the absolute minimum [compare to the values of f as $x, y \to 0$ or 12] so the numbers are $x = y = z = 4$.

49. Center the sphere at the origin so that its equation is $x^2 + y^2 + z^2 = r^2$, and orient the inscribed rectangular box so that its edges are parallel to the coordinate axes. Any vertex of the box satisfies $x^2 + y^2 + z^2 = r^2$, so take (x, y, z) to be the vertex in the first octant. Then the box has length $2x$, width $2y$, and height $2z = 2\sqrt{r^2 - x^2 - y^2}$ with volume given by

$V(x, y) = (2x)(2y)\left(2\sqrt{r^2 - x^2 - y^2}\right) = 8xy\sqrt{r^2 - x^2 - y^2}$ for $0 < x < r$, $0 < y < r$. Then

$$V_x = (8xy) \cdot \tfrac{1}{2}(r^2 - x^2 - y^2)^{-1/2}(-2x) + \sqrt{r^2 - x^2 - y^2} \cdot 8y = \frac{8y(r^2 - 2x^2 - y^2)}{\sqrt{r^2 - x^2 - y^2}} \text{ and } V_y = \frac{8x(r^2 - x^2 - 2y^2)}{\sqrt{r^2 - x^2 - y^2}}.$$

Setting $V_x = 0$ gives $y = 0$ or $2x^2 + y^2 = r^2$, but $y > 0$ so only the latter solution applies. Similarly, $V_y = 0$ with $x > 0$ implies $x^2 + 2y^2 = r^2$. Substituting, we have $2x^2 + y^2 = x^2 + 2y^2 \;\Rightarrow\; x^2 = y^2 \;\Rightarrow\; y = x$. Then $x^2 + 2y^2 = r^2 \;\Rightarrow\; 3x^2 = r^2 \;\Rightarrow\; x = \sqrt{r^2/3} = r/\sqrt{3} = y$. Thus, the only critical point is $\left(r/\sqrt{3}, r/\sqrt{3}\right)$. There must be a maximum volume and here it must occur at a critical point, so the maximum volume occurs when $x = y = r/\sqrt{3}$ and the maximum volume is $V\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right) = 8\left(\frac{r}{\sqrt{3}}\right)\left(\frac{r}{\sqrt{3}}\right)\sqrt{r^2 - \left(\frac{r}{\sqrt{3}}\right)^2 - \left(\frac{r}{\sqrt{3}}\right)^2} = \dfrac{8}{3\sqrt{3}}\,r^3$.

50. Let x, y, and z be the dimensions of the box. We wish to minimize surface area $= 2xy + 2xz + 2yz$, but we have volume $= xyz = 1000 \;\Rightarrow\; z = \dfrac{1000}{xy}$ so we minimize

$f(x, y) = 2xy + 2x\left(\dfrac{1000}{xy}\right) + 2y\left(\dfrac{1000}{xy}\right) = 2xy + \dfrac{2000}{y} + \dfrac{2000}{x}$. Then $f_x = 2y - \dfrac{2000}{x^2}$ and $f_y = 2x - \dfrac{2000}{y^2}$. Setting $f_x = 0$ implies $y = \dfrac{1000}{x^2}$ and substituting into $f_y = 0$ gives $x - \dfrac{x^4}{1000} = 0 \;\Rightarrow\; x^3 = 1000$ [since $x \neq 0$] $\Rightarrow\; x = 10$.

The surface area has a minimum but no maximum and it must occur at a critical point, so the minimal surface area occurs for a box with dimensions $x = 10$ cm, $y = 1000/10^2 = 10$ cm, $z = 1000/10^2 = 10$ cm.

51. The volume of the box is $V = xyz$. Since one vertex is in the plane $x + 2y + 3z = 6 \;\Leftrightarrow\; z = \frac{1}{3}(6 - x - 2y)$, the volume is given by $V(x, y) = \frac{1}{3}xy(6 - x - 2y) = \frac{1}{3}(6xy - x^2y - 2xy^2)$. Now maximize V.

$V_x = \frac{1}{3}(6y - 2xy - 2y^2) = \frac{1}{3}y(6 - 2x - 2y)$ and $V_y = \frac{1}{3}x\,(6 - x - 4y)$. Setting $f_x = 0$ and $f_y = 0$ gives $y = 3 - x$ and $x = 6 - 4y \;\Rightarrow\; y = 1$ and $x = 2$, so the critical point is $(2, 1)$, which geometrically must give a maximum. Thus, the volume of the largest such box is $V = (2)(1)\left(\frac{2}{3}\right) = \frac{4}{3}$.

52. Surface area $= 2(xy + xz + yz) = 64\text{ cm}^2$, so $xy + xz + yz = 32$ or $z = \dfrac{32 - xy}{x + y}$. Maximize the volume

$f(x, y) = xy\dfrac{32 - xy}{x + y}$. Then $f_x = \dfrac{32y^2 - 2xy^3 - x^2y^2}{(x + y)^2} = y^2\dfrac{32 - 2xy - x^2}{(x + y)^2}$ and $f_y = x^2\dfrac{32 - 2xy - y^2}{(x + y)^2}$. Setting $f_x = 0$ implies $y = \dfrac{32 - x^2}{2x}$ and substituting into $f_y = 0$ gives $32(4x^2) - (32 - x^2)(4x^2) - (32 - x^2)^2 = 0$ or

$3x^4 + 64x^2 - (32)^2 = 0$. Thus, $x^2 = \frac{64}{6}$ or $x = \frac{8}{\sqrt{6}}$, $y = \frac{64/3}{16/\sqrt{6}} = \frac{8}{\sqrt{6}}$ and $z = \frac{8}{\sqrt{6}}$. Thus, the box is a cube with edge length $\frac{8}{\sqrt{6}}$ cm.

53. Let the dimensions be x, y, and z; then $4x + 4y + 4z = c$ and the volume is

$V = xyz = xy\left(\frac{1}{4}c - x - y\right) = \frac{1}{4}cxy - x^2y - xy^2$, $x > 0$, $y > 0$. Then $V_x = \frac{1}{4}cy - 2xy - y^2$ and $V_y = \frac{1}{4}cx - x^2 - 2xy$, so $V_x = 0 = V_y$ when $2x + y = \frac{1}{4}c$ and $x + 2y = \frac{1}{4}c$. Solving, we get $x = \frac{1}{12}c$, $y = \frac{1}{12}c$ and $z = \frac{1}{4}c - x - y = \frac{1}{12}c$. From the geometrical nature of the problem, this critical point must give an absolute maximum. Thus, the box is a cube with edge length $\frac{1}{12}c$.

54. The cost equals $5xy + 2(xz + yz)$ and $xyz = V$, so $C(x, y) = 5xy + 2V(x + y)/(xy) = 5xy + 2V(x^{-1} + y^{-1})$. Then $C_x = 5y - 2Vx^{-2}$, $C_y = 5x - 2\,Vy^{-2}$, $C_x = 0$ implies $y = 2\,V/(5x^2)$, $C_y = 0$ implies $x = \sqrt[3]{\frac{2}{5}V} = y$. Thus, the dimensions of the aquarium which minimize the cost are $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$.

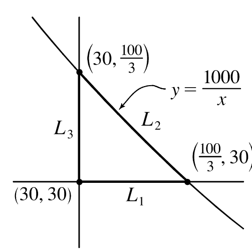
55. Let the dimensions be x, y and z, then minimize $xy + 2(xz + yz)$ if $xyz = 32{,}000 \text{ cm}^3$. Then

$f(x, y) = xy + [64{,}000(x + y)/xy] = xy + 64{,}000(x^{-1} + y^{-1})$, $f_x = y - 64{,}000x^{-2}$, $f_y = x - 64{,}000y^{-2}$.

And $f_x = 0$ implies $y = 64{,}000/x^2$; substituting into $f_y = 0$ implies $x^3 = 64{,}000$ or $x = 40$ and then $y = 40$. Now $D(x, y) = [(2)(64{,}000)]^2x^{-3}y^{-3} - 1 > 0$ for $(40, 40)$ and $f_{xx}(40, 40) > 0$ so this is indeed a minimum. Thus, the dimensions of the box are $x = y = 40$ cm, $z = 20$ cm.

56. Let x be the length of the north and south walls, y the length of the east and west walls, and z the height of the building. The heat loss is given by $h = 10(2yz) + 8(2xz) + 1(xy) + 5(xy) = 6xy + 16xz + 20yz$. The volume is 4000 m^3, so $xyz = 4000$, and we substitute $z = \frac{4000}{xy}$ to obtain the heat loss function $h(x, y) = 6xy + 80{,}000/x + 64{,}000/y$.

(a) Since $z = \frac{4000}{xy} \geq 4$, $xy \leq 1000 \quad\Rightarrow\quad y \leq 1000/x$. Also $x \geq 30$ and $y \geq 30$, so the domain of h is $D = \{(x, y) \mid x \geq 30, 30 \leq y \leq 1000/x\}$.



(b) $h(x, y) = 6xy + 80{,}000x^{-1} + 64{,}000y^{-1} \quad\Rightarrow$

$h_x = 6y - 80{,}000x^{-2}$, $h_y = 6x - 64{,}000y^{-2}$.

$h_x = 0$ implies $6x^2y = 80{,}000 \quad\Rightarrow\quad y = \dfrac{80{,}000}{6x^2}$ and substituting into

$h_y = 0$ gives $6x = 64{,}000\left(\dfrac{6x^2}{80{,}000}\right)^2 \quad\Rightarrow\quad x^3 = \dfrac{80{,}000^2}{6 \cdot 64{,}000} = \dfrac{50{,}000}{3}$, so

$x = \sqrt[3]{\dfrac{50{,}000}{3}} = 10\sqrt[3]{\dfrac{50}{3}} \quad\Rightarrow\quad y = \dfrac{80}{\sqrt[3]{60}}$, and the only critical point of h is $\left(10\sqrt[3]{\dfrac{50}{3}}, \dfrac{80}{\sqrt[3]{60}}\right) \approx (25.54, 20.43)$

which is not in D. Next we check the boundary of D.

[continued]

On L_1: $y = 30$, $h(x, 30) = 180x + 80{,}000/x + 6400/3$, $30 \le x \le \frac{100}{3}$. Since $h'(x, 30) = 180 - 80{,}000/x^2 > 0$ for $30 \le x \le \frac{100}{3}$, $h(x, 30)$ is an increasing function with minimum $h(30, 30) = 10{,}200$ and maximum $h\left(\frac{100}{3}, 30\right) \approx 10{,}533$.

On L_2: $y = 1000/x$, $h(x, 1000/x) = 6000 + 64x + 80{,}000/x$, $30 \le x \le \frac{100}{3}$.

Since $h'(x, 1000/x) = 64 - 80{,}000/x^2 < 0$ for $30 \le x \le \frac{100}{3}$, $h(x, 1000/x)$ is a decreasing function with minimum $h\left(\frac{100}{3}, 30\right) \approx 10{,}533$ and maximum $h\left(30, \frac{100}{3}\right) \approx 10{,}587$.

On L_3: $x = 30$, $h(30, y) = 180y + 64{,}000/y + 8000/3$, $30 \le y \le \frac{100}{3}$. $h'(30, y) = 180 - 64{,}000/y^2 > 0$ for $30 \le y \le \frac{100}{3}$, so $h(30, y)$ is an increasing function of y with minimum $h(30, 30) = 10{,}200$ and maximum $h\left(30, \frac{100}{3}\right) \approx 10{,}587$.

Thus, the absolute minimum of h is $h(30, 30) = 10{,}200$, and the dimensions of the building that minimize heat loss are walls 30 m in length and height $\frac{4000}{30^2} = \frac{40}{9} \approx 4.44$ m.

(c) From part (b), the only critical point of h, which gives a local (and absolute) minimum, is approximately $h(25.54, 20.43) \approx 9396$. So a building of volume 4000 m^2 with dimensions $x \approx 25.54$ m, $y \approx 20.43$ m, $z \approx \frac{4000}{(25.54)(20.43)} \approx 7.67$ m has the least amount of heat loss.

57. Let x, y, z be the dimensions of the rectangular box. Then the volume of the box is xyz and

$L = \sqrt{x^2 + y^2 + z^2} \quad \Rightarrow \quad L^2 = x^2 + y^2 + z^2 \quad \Rightarrow \quad z = \sqrt{L^2 - x^2 - y^2}$.

Substituting, we have volume $V(x, y) = xy\sqrt{L^2 - x^2 - y^2}$ $(x, y > 0)$.

$$V_x = xy \cdot \tfrac{1}{2}(L^2 - x^2 - y^2)^{-1/2}(-2x) + y\sqrt{L^2 - x^2 - y^2} = y\sqrt{L^2 - x^2 - y^2} - \frac{x^2 y}{\sqrt{L^2 - x^2 - y^2}},$$

$V_y = x\sqrt{L^2 - x^2 - y^2} - \dfrac{xy^2}{\sqrt{L^2 - x^2 - y^2}}$. $V_x = 0$ implies $y(L^2 - x^2 - y^2) = x^2 y \quad \Rightarrow \quad y(L^2 - 2x^2 - y^2) = 0 \quad \Rightarrow$ $2x^2 + y^2 = L^2$ (since $y > 0$), and $V_y = 0$ implies $x(L^2 - x^2 - y^2) = xy^2 \quad \Rightarrow \quad x(L^2 - x^2 - 2y^2) = 0 \quad \Rightarrow$ $x^2 + 2y^2 = L^2$ (since $x > 0$). Substituting $y^2 = L^2 - 2x^2$ into $x^2 + 2y^2 = L^2$ gives $x^2 + 2L^2 - 4x^2 = L^2 \quad \Rightarrow$ $3x^2 = L^2 \quad \Rightarrow \quad x = L/\sqrt{3}$ (since $x > 0$) and then $y = \sqrt{L^2 - 2\left(L/\sqrt{3}\right)^2} = L/\sqrt{3}$.

So the only critical point is $\left(L/\sqrt{3}, L/\sqrt{3}\right)$ which, from the geometrical nature of the problem, must give an absolute maximum. Thus, the maximum volume is $V\left(L/\sqrt{3}, L/\sqrt{3}\right) = \left(L/\sqrt{3}\right)^2\sqrt{L^2 - \left(L/\sqrt{3}\right)^2 - \left(L/\sqrt{3}\right)^2} = L^3/\left(3\sqrt{3}\right)$ cubic units.

58. $Y(N, P) = kNPe^{-N-P} \quad \Rightarrow \quad Y_N = kP\left[N(-e^{-N-P}) + e^{-N-P}(1)\right] = kP(1 - N)e^{-N-P}$,

$Y_P = kN\left[P(-e^{-N-P}) + e^{-N-P}(1)\right] = kN(1 - P)e^{-N-P}$. Here $N \ge 0$ and $P \ge 0$, but if either $N = 0$ or $P = 0$ then the yield is zero. Assuming that $N > 0$ and $P > 0$, $Y_N = 0$ implies $N = 1$ and $Y_P = 0$ implies $P = 1$, so the only critical point in $\{(N, P) \mid N > 0, P > 0\}$ is $(1, 1)$ where $Y(1, 1) = ke^{-2}$.

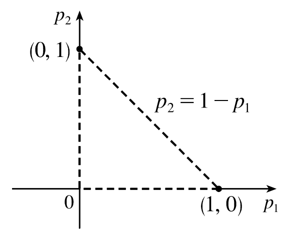
$D(N, P) = Y_{NN}Y_{PP} - (Y_{NP})^2 = \left[kP(N-2)e^{-N-P}\right]\left[kN(P-2)e^{-N-P}\right] - \left[k(1-N)(1-P)e^{-N-P}\right]^2 \quad \Rightarrow$

$D(1, 1) = \left(-ke^{-2}\right)\left(-ke^{-2}\right) - (0)^2 = k^2e^{-4} > 0$ and $Y_{NN}(1, 1) = -ke^{-2} < 0$, so $Y(1, 1) = ke^{-2}$ is a local maximum. $Y(1, 1)$ is also the absolute maximum (we have only one critical point, and $Y \to 0$ as $N \to 0$ or $P \to 0$ and $Y \to 0$ as N or P grow large) so the best yield is achieved when both the nitrogen and phosphorus levels are 1 (measured in appropriate units).

59. (a) We are given that $p_1 + p_2 + p_3 = 1 \quad \Rightarrow \quad p_3 = 1 - p_1 - p_2$, so

$H = -p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3 = -p_1 \ln p_1 - p_2 \ln p_2 - (1 - p_1 - p_2) \ln (1 - p_1 - p_2)$.

(b) Because p_i is a proportion we have $0 \leq p_i \leq 1$, but H is undefined unless $p_1 > 0$, $p_2 > 0$, and $1 - p_1 - p_2 > 0 \quad \Leftrightarrow \quad p_1 + p_2 < 1$. This last restriction forces $p_1 < 1$ and $p_2 < 1$, so the domain of H is $\{(p_1, p_2) \mid 0 < p_1 < 1,\ p_2 < 1 - p_1\}$. It is the interior of the triangle drawn in the figure.



(c)
$$\begin{aligned} H_{p_1} &= -\left[p_1 \cdot (1/p_1) + (\ln p_1) \cdot 1\right] - \left[(1 - p_1 - p_2) \cdot (-1)/(1 - p_1 - p_2) + \ln(1 - p_1 - p_2) \cdot (-1)\right] \\ &= -1 - \ln p_1 + 1 + \ln(1 - p_1 - p_2) = \ln(1 - p_1 - p_2) - \ln p_1 \end{aligned}$$

Similarly, $H_{p_2} = \ln(1 - p_1 - p_2) - \ln p_2$. Then $H_{p_1} = 0$ implies

$\ln(1 - p_1 - p_2) = \ln p_1 \quad \Rightarrow \quad 1 - p_1 - p_2 = p_1 \quad \Rightarrow \quad p_2 = 1 - 2p_1$, and $H_{p_2} = 0$ implies

$\ln(1 - p_1 - p_2) = \ln p_2 \quad \Rightarrow \quad p_1 = 1 - 2p_2$. Substituting, we have $p_1 = 1 - 2(1 - 2p_1) \quad \Rightarrow$

$3p_1 = 1 \quad \Rightarrow \quad p_1 = \frac{1}{3}$, and then $p_2 = 1 - 2\left(\frac{1}{3}\right) = \frac{1}{3}$. Thus, the only critical point is $\left(\frac{1}{3}, \frac{1}{3}\right)$.

$D(p_1, p_2) = H_{p_1p_1}H_{p_2p_2} - (H_{p_1p_2})^2 = \left(\dfrac{-1}{1 - p_1 - p_2} - \dfrac{1}{p_1}\right)\left(\dfrac{-1}{1 - p_1 - p_2} - \dfrac{1}{p_2}\right) - \left(\dfrac{-1}{1 - p_1 - p_2}\right)^2$, so

$D\left(\frac{1}{3}, \frac{1}{3}\right) = (-6)(-6) - (-3)^2 = 27 > 0$ and $H_{p_1p_1}\left(\frac{1}{3}, \frac{1}{3}\right) = -6 < 0$. Thus,

$H\left(\frac{1}{3}, \frac{1}{3}\right) = -\frac{1}{3}\ln\frac{1}{3} - \frac{1}{3}\ln\frac{1}{3} - \frac{1}{3}\ln\frac{1}{3} = -\ln\frac{1}{3} = \ln 3$ is a local maximum. Here it is also the absolute maximum, so the maximum value of H is $\ln 3$, which occurs for $p_1 = p_2 = p_3 = \frac{1}{3}$ (all three species have equal proportion in the ecosystem).

60. Since $p + q + r = 1$, we can substitute $p = 1 - r - q$ into P giving

$P = P(q, r) = 2(1 - r - q)q + 2(1 - r - q)r + 2rq = 2q - 2q^2 + 2r - 2r^2 - 2rq$. Since p, q and r represent proportions and $p + q + r = 1$, we know $q \geq 0$, $r \geq 0$, and $q + r \leq 1$. Thus, we want to find the absolute maximum of the continuous function $P(q, r)$ on the closed set D enclosed by the lines $q = 0$, $r = 0$, and $q + r = 1$. To find any critical points, we set the partial derivatives equal to zero: $P_q(q, r) = 2 - 4q - 2r = 0$ and $P_r(q, r) = 2 - 4r - 2q = 0$. The first equation gives

$r = 1 - 2q$, and substituting into the second equation we have $2 - 4(1 - 2q) - 2q = 0 \Rightarrow q = \frac{1}{3}$. Then we have one critical point, $\left(\frac{1}{3}, \frac{1}{3}\right)$, where $P\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{2}{3}$. Next we find the maximum values of P on the boundary of D which consists of three line segments. For the segment given by $r = 0$, $0 \leq q \leq 1$, $P(q, r) = P(q, 0) = 2q - 2q^2$, $0 \leq q \leq 1$. This represents a parabola with maximum value $P\left(\frac{1}{2}, 0\right) = \frac{1}{2}$. On the segment $q = 0$, $0 \leq r \leq 1$ we have $P(0, r) = 2r - 2r^2$, $0 \leq r \leq 1$. This represents a parabola with maximum value $P\left(0, \frac{1}{2}\right) = \frac{1}{2}$. Finally, on the segment $q + r = 1$, $0 \leq q \leq 1$, $P(q, r) = P(q, 1 - q) = 2q - 2q^2$, $0 \leq q \leq 1$ which has a maximum value of $P\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$. Comparing these values with the value of P at the critical point, we see that the absolute maximum value of $P(q, r)$ on D is $\frac{2}{3}$.

61. Note that here the variables are m and b, and $f(m, b) = \sum\limits_{i=1}^{n} [y_i - (mx_i + b)]^2$. Then $f_m = \sum\limits_{i=1}^{n} -2x_i[y_i - (mx_i + b)] = 0$ implies $\sum\limits_{i=1}^{n} \left(x_i y_i - mx_i^2 - bx_i\right) = 0$ or $\sum\limits_{i=1}^{n} x_i y_i = m \sum\limits_{i=1}^{n} x_i^2 + b \sum\limits_{i=1}^{n} x_i$ and $f_b = \sum\limits_{i=1}^{n} -2[y_i - (mx_i + b)] = 0$ implies $\sum\limits_{i=1}^{n} y_i = m \sum\limits_{i=1}^{n} x_i + \sum\limits_{i=1}^{n} b = m\left(\sum\limits_{i=1}^{n} x_i\right) + nb$. Thus, we have the two desired equations.

Now $f_{mm} = \sum\limits_{i=1}^{n} 2x_i^2$, $f_{bb} = \sum\limits_{i=1}^{n} 2 = 2n$ and $f_{mb} = \sum\limits_{i=1}^{n} 2x_i$. And $f_{mm}(m, b) > 0$ always and

$D(m, b) = 4n\left(\sum\limits_{i=1}^{n} x_i^2\right) - 4\left(\sum\limits_{i=1}^{n} x_i\right)^2 = 4\left[n\left(\sum\limits_{i=1}^{n} x_i^2\right) - \left(\sum\limits_{i=1}^{n} x_i\right)^2\right] > 0$ always so the solutions of these two equations do indeed minimize $\sum\limits_{i=1}^{n} d_i^2$.

62. Any such plane must cut out a tetrahedron in the first octant. We need to minimize the volume of the tetrahedron that passes through the point $(1, 2, 3)$. Writing the equation of the plane as $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$, the volume of the tetrahedron is given by $V = \dfrac{abc}{6}$. But $(1, 2, 3)$ must lie on the plane, so we need $\dfrac{1}{a} + \dfrac{2}{b} + \dfrac{3}{c} = 1$ $(\star)$ and thus can think of c as a function of a and b. Then $V_a = \dfrac{b}{6}\left(c + a\,\dfrac{\partial c}{\partial a}\right)$ and $V_b = \dfrac{a}{6}\left(c + b\,\dfrac{\partial c}{\partial b}\right)$. Differentiating $(\star)$ with respect to a we get $-a^{-2} - 3c^{-2}\,\dfrac{\partial c}{\partial a} = 0 \Rightarrow$ $\dfrac{\partial c}{\partial a} = \dfrac{-c^2}{3a^2}$, and differentiating $(\star)$ with respect to b gives $-2b^{-2} - 3c^{-2}\,\dfrac{\partial c}{\partial b} = 0 \Rightarrow \dfrac{\partial c}{\partial b} = \dfrac{-2c^2}{3b^2}$. Then $V_a = \dfrac{b}{6}\left(c + a\,\dfrac{-c^2}{3a^2}\right) = 0 \Rightarrow c = 3a$, and $V_b = \dfrac{a}{6}\left(c + b\,\dfrac{-2c^2}{3b^2}\right) = 0 \Rightarrow c = \frac{3}{2}b$. Thus, $3a = \frac{3}{2}b$ or $b = 2a$. Putting these into $(\star)$ gives $\frac{3}{a} = 1$ or $a = 3$ and then $b = 6$, $c = 9$. Thus ,the equation of the required plane is $\dfrac{x}{3} + \dfrac{y}{6} + \dfrac{z}{9} = 1$ or $6x + 3y + 2z = 18$.

DISCOVERY PROJECT Quadratic Approximations and Critical Points

1. $Q(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2} f_{xx}(a,b)(x-a)^2$
$$+ f_{xy}(a,b)(x-a)(y-b) + \tfrac{1}{2} f_{yy}(a,b)(y-b)^2,$$

so

$$Q_x(x,y) = f_x(a,b) + \tfrac{1}{2} f_{xx}(a,b)(2)(x-a) + f_{xy}(a,b)(y-b) = f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b)$$

At (a,b) we have $Q_x(a,b) = f_x(a,b) + f_{xx}(a,b)\,(a-a) + f_{xy}(a,b)(b-b) = f_x(a,b)$.

Similarly, $Q_y(x,y) = f_y(a,b) + f_{xy}(a,b)(x-a) + f_{yy}(a,b)(y-b) \quad \Rightarrow$

$Q_y(a,b) = f_y(a,b) + f_{xy}(a,b)(a-a) + f_{yy}(a,b)(b-b) = f_y(a,b)$.

For the second-order partial derivatives we have

$$Q_{xx}(x,y) = \frac{\partial}{\partial x}\left[f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b)\right] = f_{xx}(a,b) \quad \Rightarrow \quad Q_{xx}(a,b) = f_{xx}(a,b)$$

$$Q_{xy}(x,y) = \frac{\partial}{\partial y}\left[f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b)\right] = f_{xy}(a,b) \quad \Rightarrow \quad Q_{xy}(a,b) = f_{xy}(a,b)$$

$$Q_{yy}(x,y) = \frac{\partial}{\partial y}\left[f_y(a,b) + f_{xy}(a,b)(x-a) + f_{yy}(a,b)(y-b)\right] = f_{yy}(a,b) \quad \Rightarrow \quad Q_{yy}(a,b) = f_{yy}(a,b)$$

2. (a) First, we find the partial derivatives and values that will be needed:

$$\begin{array}{ll} f(x,y) = e^{-x^2-y^2} & f(0,0) = 1 \\ f_x(x,y) = -2xe^{-x^2-y^2} & f_x(0,0) = 0 \\ f_y(x,y) = -2ye^{-x^2-y^2} & f_y(0,0) = 0 \\ f_{xx}(x,y) = (4x^2-2)e^{-x^2-y^2} & f_{xx}(0,0) = -2 \\ f_{xy}(x,y) = 4xye^{-x^2-y^2} & f_{xy}(0,0) = 0 \\ f_{yy}(x,y) = (4y^2-2)e^{-x^2-y^2} & f_{yy}(0,0) = -2 \end{array}$$

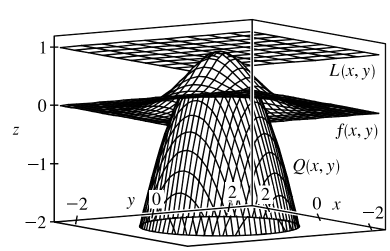
Then the first-degree Taylor polynomial of f at $(0,0)$ is

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 1 + (0)(x-0) + (0)(y-0) = 1$$

The second-degree Taylor polynomial is given by

$$Q(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + \tfrac{1}{2} f_{xx}(0,0)(x-0)^2$$
$$+ f_{xy}(0,0)(x-0)(y-0) + \tfrac{1}{2} f_{yy}(0,0)(y-0)^2$$
$$= 1 - x^2 - y^2$$

(b)



As we see from the graph, L approximates f well only for points (x,y) extremely close to the origin. Q is a much better approximation; the shape of its graph looks similar to that of the graph of f near the origin, and the values of Q appear to be good estimates for the values of f within a significant radius of the origin.

3. (a) First we find the partial derivatives and values that will be needed:

$$\begin{array}{llll} f(x,y)=xe^y & f(1,0)=1 & f_{xx}(x,y)=0 & f_{xx}(1,0)=0 \\ f_x(x,y)=e^y & f_x(1,0)=1 & f_{xy}(x,y)=e^y & f_{xy}(1,0)=1 \\ f_y(x,y)=xe^y & f_y(1,0)=1 & f_{yy}(x,y)=xe^y & f_{yy}(1,0)=1 \end{array}$$

Then the first-degree Taylor polynomial of f at $(1,0)$ is

$$L(x,y)=f(1,0)+f_x(1,0)(x-1)+f_y(1,0)(y-0)=1+(1)(x-1)+(1)(y-0)=x+y$$

The second-degree Taylor polynomial is given by

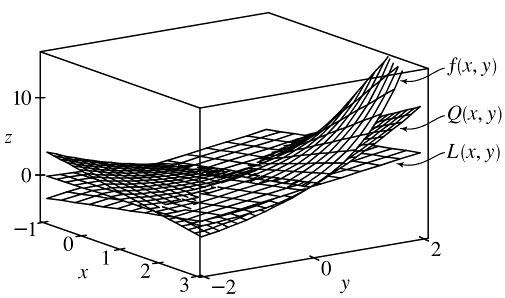
$$\begin{aligned} Q(x,y) &= f(1,0)+f_x(1,0)(x-1)+f_y(1,0)(y-0)+\tfrac{1}{2}f_{xx}(1,0)(x-1)^2 \\ &\qquad + f_{xy}(1,0)(x-1)(y-0)+\tfrac{1}{2}f_{yy}(1,0)(y-0)^2 \\ &= \tfrac{1}{2}y^2+x+xy \end{aligned}$$

(b) $L(0.9,0.1)=0.9+0.1=1.0$

$Q(0.9,0.1)=\frac{1}{2}(0.1)^2+0.9+(0.9)(0.1)=0.995$

$f(0.9,0.1)=0.9e^{0.1}\approx 0.9947$

(c)



As we see from the graph, L and Q both approximate f reasonably well near the point $(1,0)$. As we venture farther from the point, the graph of Q follows the shape of the graph of f more closely than L.

4. (a) $f(x,y)=ax^2+bxy+cy^2=a\left[x^2+\dfrac{b}{a}\,xy+\dfrac{c}{a}\,y^2\right]=a\left[x^2+\dfrac{b}{a}\,xy+\left(\dfrac{b}{2a}\,y\right)^2-\left(\dfrac{b}{2a}\,y\right)^2+\dfrac{c}{a}\,y^2\right]$

$$=a\left[\left(x+\frac{b}{2a}\,y\right)^2-\frac{b^2}{4a^2}\,y^2+\frac{c}{a}\,y^2\right]=a\left[\left(x+\frac{b}{2a}\,y\right)^2+\left(\frac{4ac-b^2}{4a^2}\right)y^2\right]$$

(b) For $D=4ac-b^2$, from part (a) we have $f(x,y)=a\left[\left(x+\dfrac{b}{2a}\,y\right)^2+\left(\dfrac{D}{4a^2}\right)y^2\right]$. If $D>0$,

$\left(\dfrac{D}{4a^2}\right)y^2\geq 0$ and $\left(x+\dfrac{b}{2a}\,y\right)^2\geq 0$, so $\left[\left(x+\dfrac{b}{2a}\,y\right)^2+\left(\dfrac{D}{4a^2}\right)y^2\right]\geq 0$. Here $a>0$, thus

$f(x,y)=a\left[\left(x+\dfrac{b}{2a}\,y\right)^2+\left(\dfrac{D}{4a^2}\right)y^2\right]\geq 0$. We know $f(0,0)=0$, so $f(0,0)\leq f(x,y)$ for all (x,y), and by definition f has a local minimum at $(0,0)$.

(c) As in part (b), $\left[\left(x + \dfrac{b}{2a}\,y\right)^2 + \left(\dfrac{D}{4a^2}\right)y^2\right] \geq 0$, and since $a < 0$, we have

$f(x, y) = a\left[\left(x + \dfrac{b}{2a}\,y\right)^2 + \left(\dfrac{D}{4a^2}\right)y^2\right] \leq 0$. Since $f(0, 0) = 0$, we must have $f(0, 0) \geq f(x, y)$ for all (x, y), so by definition f has a local maximum at $(0, 0)$.

(d) $f(x, y) = ax^2 + bxy + cy^2$, so $f_x(x, y) = 2ax + by \;\Rightarrow\; f_x(0, 0) = 0$ and $f_y(x, y) = bx + 2cy \;\Rightarrow\; f_y(0, 0) = 0$. Since $f(0, 0) = 0$ and f and its partial derivatives are continuous, we know from Equation 14.4.2 that the tangent plane to the graph of f at $(0, 0)$ is the plane $z = 0$. Then f has a saddle point at $(0, 0)$ if the graph of f crosses the tangent plane at $(0, 0)$, or equivalently, if some paths to the origin have positive function values while other paths have negative function values. Suppose we approach the origin along the x-axis; then we have $y = 0 \;\Rightarrow\; f(x, 0) = ax^2$ which has the same sign as a. We must now find at least one path to the origin where $f(x, y)$ gives values with sign opposite that of a. Since

$f(x, y) = a\left[\left(x + \dfrac{b}{2a}\,y\right)^2 + \left(\dfrac{D}{4a^2}\right)y^2\right]$, if we approach the origin along the line $x = -\dfrac{b}{2a}\,y$, we have

$f\left(-\dfrac{b}{2a}\,y, y\right) = a\left[\left(-\dfrac{b}{2a}\,y + \dfrac{b}{2a}\,y\right)^2 + \left(\dfrac{D}{4a^2}\right)y^2\right] = \dfrac{D}{4a}\,y^2$. Since $D < 0$, these values have signs opposite that of a. Thus, f has a saddle point at $(0, 0)$.

5. (a) Since the partial derivatives of f exist at $(0, 0)$ and $(0, 0)$ is a critical point, we know $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$. Then the second-degree Taylor polynomial of f at $(0, 0)$ can be expressed as

$$\begin{aligned} Q(x, y) &= f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) + \tfrac{1}{2} f_{xx}(0, 0)(x - 0)^2 \\ &\qquad + f_{xy}(0, 0)(x - 0)(y - 0) + \tfrac{1}{2} f_{yy}(0, 0)(y - 0)^2 \\ &= \tfrac{1}{2} f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \tfrac{1}{2} f_{yy}(0, 0)y^2 \end{aligned}$$

(b) $Q(x, y) = \frac{1}{2}\, f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}\, f_{yy}(0, 0)y^2$ fits the form of the polynomial function in Problem 4 with $a = \frac{1}{2}\, f_{xx}(0, 0)$, $b = f_{xy}(0, 0)$, and $c = \frac{1}{2}\, f_{yy}(0, 0)$. Then we know Q is a paraboloid, and that Q has a local maximum, local minimum, or saddle point at $(0, 0)$. Here,

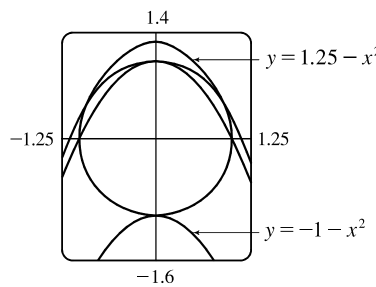
$D = 4ac - b^2 = 4\left(\frac{1}{2}\right) f_{xx}(0, 0) \left(\frac{1}{2}\right) f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = f_{xx}(0, 0) f_{yy}(0, 0) - [f_{xy}(0, 0)]^2$, and if $D > 0$ with $a = \frac{1}{2}\, f_{xx}(0, 0) > 0 \;\Rightarrow\; f_{xx}(0, 0) > 0$, we know from Problem 4 that Q has a local minimum at $(0, 0)$. Similarly, if $D > 0$ and $a < 0 \;\Rightarrow\; f_{xx}(0, 0) < 0$, Q has a local maximum at $(0, 0)$, and if $D < 0$, Q has a saddle point at $(0, 0)$.

(c) Since $f(x, y) \approx Q(x, y)$ near $(0, 0)$, part (b) suggests that for $D = f_{xx}(0, 0) f_{yy}(0, 0) - [f_{xy}(0, 0)]^2$, if $D > 0$ and $f_{xx}(0, 0) > 0$, f has a local minimum at $(0, 0)$. If $D > 0$ and $f_{xx}(0, 0) < 0$, f has a local maximum at $(0, 0)$, and if $D < 0$, f has a saddle point at $(0, 0)$. Together with the conditions given in part (a), this is precisely the Second Derivatives Test from Section 14.7.

14.8 Lagrange Multipliers

1. At the extreme values of f, the level curves of f just touch the curve $g(x,y) = 8$ with a common tangent line. (See Figure 1 and the accompanying discussion.) We can observe several such occurrences on the contour map, but the level curve $f(x,y) = c$ with the largest value of c which still intersects the curve $g(x,y) = 8$ is approximately $c = 59$, and the smallest value of c corresponding to a level curve which intersects $g(x,y) = 8$ appears to be $c = 30$. Thus, we estimate the maximum value of f subject to the constraint $g(x,y) = 8$ to be about 59 and the minimum to be 30.

2. (a) The values $c = \pm 1$ and $c = 1.25$ seem to give curves which are tangent to the circle. These values represent possible extreme values of the function $x^2 + y$ subject to the constraint $x^2 + y^2 = 1$.



(b) $\nabla f = \langle 2x, 1 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. So $2x = 2\lambda x \quad \Rightarrow \quad$ either $\lambda = 1$ or $x = 0$. If $\lambda = 1$, then $y = \frac{1}{2}$ and so $x = \pm\frac{\sqrt{3}}{2}$ (from the constraint). If $x = 0$, then $y = \pm 1$. Therefore f has possible extreme values at the points $(0, \pm 1)$ and $\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. We calculate $f\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{5}{4}$ (the maximum value), $f(0,1) = 1$, and $f(0,-1) = -1$ (the minimum value). These are our answers from part (a).

3. We want to find the extreme values of $f(x,y) = x^2 - y^2$ subject to the constraint $g(x,y) = x^2 + y^2 = 1$. Then $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 2x, -2y \rangle = \lambda \langle 2x, 2y \rangle$, so we solve the equations $2x = 2\lambda x$, $-2y = 2\lambda y$, and $x^2 + y^2 = 1$. From the first equation we have $2x(\lambda - 1) = 0 \quad \Rightarrow \quad x = 0$ or $\lambda = 1$. If $x = 0$ then substitution into the constraint gives $y^2 = 1 \quad \Rightarrow \quad y = \pm 1$. If $\lambda = 1$ then substitution into the second equation gives $-2y = 2y \quad \Rightarrow \quad y = 0$, and from the constraint we must have $x = \pm 1$. Thus, the possible points for the extreme values of f are $(0, \pm 1)$ and $(\pm 1, 0)$. Evaluating f at these points, we see that the maximum value of f is $f(\pm 1, 0) = 1$ and the minimum is $f(0, \pm 1) = -1$.

4. $f(x,y) = x^2 y$, $g(x,y) = x^2 + y^4 = 5$, and $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 2xy, x^2 \rangle = \langle 2\lambda x, 4\lambda y^3 \rangle$, so we get the three equations $2xy = 2\lambda x$, $x^2 = 4\lambda y^3$, and $x^2 + y^4 = 5$. $2xy = 2\lambda x \quad \Rightarrow \quad x = 0$ or $y = \lambda$. If $x = 0$, the second equation implies $y = 0$ or $\lambda = 0$. The point $(0,0)$ does not satisfy the constraint, so $x = \lambda = 0$ and the constraint gives a possible extreme value at the point $\left(0, \sqrt[4]{5}\right)$. Next, suppose $y = \lambda$. Substituting into the second equation $\Rightarrow \quad x^2 = 4\lambda^4$ and substituting into the third equation gives $4\lambda^4 + \lambda^4 = 5 \quad \Rightarrow \quad \lambda = \pm 1$. From the second equation with $y = \lambda = 1$, we get $x = \pm 2$. From the second equation with $y = \lambda = -1$, we get $x = \pm 2$. So f also has possible extreme values at $(\pm 2, 1)$ and $(\pm 2, -1)$. Evaluating f at these 5 points, we see $f(\pm 2, 1) = 4$ is the maximum value and $f(\pm 2, -1) = -4$ is the minimum value.

5. $f(x,y) = xy$, $g(x,y) = 4x^2 + y^2 = 8$, and $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle y, x \rangle = \langle 8\lambda x, 2\lambda y \rangle$, so $y = 8\lambda x$, $x = 2\lambda y$, and $4x^2 + y^2 = 8$. First note that if $x = 0$, then $y = 0$ by the first equation, and if $y = 0$, then $x = 0$ by the second equation. But this contradicts the third equation, so $x \neq 0$ and $y \neq 0$. Then from the first two equations we have $\dfrac{y}{8x} = \lambda = \dfrac{x}{2y} \quad \Rightarrow$

$2y^2 = 8x^2 \quad\Rightarrow\quad y^2 = 4x^2$, and substitution into the third equation gives $4x^2 + 4x^2 = 8 \quad\Rightarrow\quad x = \pm 1$. If $x = \pm 1$ then $y^2 = 4 \quad\Rightarrow\quad y = \pm 2$, so f has possible extreme values at $(1, \pm 2)$ and $(-1, \pm 2)$. Evaluating f at these points, we see that the maximum value is $f(1, 2) = f(-1, -2) = 2$ and the minimum is $f(1, -2) = f(-1, 2) = -2$.

6. $f(x, y) = xe^y$, $g(x, y) = x^2 + y^2 = 2$, and $\nabla f = \lambda \nabla g \quad\Rightarrow\quad \langle e^y, xe^y \rangle = \langle 2\lambda x, 2\lambda y \rangle$, so $e^y = 2\lambda x$, $xe^y = 2\lambda y$, and $x^2 + y^2 = 2$. First note that from the first equation $x \neq 0$. If $y = 0$, the second equation implies $x = 0$, so $y \neq 0$. Then from the first two equations we have $\dfrac{e^y}{2x} = \lambda = \dfrac{xe^y}{2y} \quad\Rightarrow\quad 2ye^y = 2x^2 e^y \quad\Rightarrow\quad y = x^2$, and substituting into the third equation gives $x^2 + (x^2)^2 = 2 \quad\Rightarrow\quad x^4 + x^2 - 2 = 0 \quad\Rightarrow\quad (x^2 + 2)(x^2 - 1) = 0 \quad\Rightarrow\quad x = \pm 1$. From $y = x^2$ we have $y = 1$, so f has possible extreme values at $(\pm 1, 1)$. Evaluating f at these points, we see that the maximum value is $f(1, 1) = e$ and the minimum is $f(-1, 1) = -e$.

7. $f(x, y) = 2x^2 + 6y^2$, $g(x, y) = x^4 + 3y^4 = 1$, and $\nabla f = \lambda \nabla g \quad\Rightarrow\quad \langle 4x, 12y \rangle = \left\langle 4\lambda x^3, 12\lambda y^3 \right\rangle$, so we get the three equations $4x = 4\lambda x^3$, $12y = 12\lambda y^3$, and $x^4 + 3y^4 = 1$. The first equation implies that $x = 0$ or $x^2 = \dfrac{1}{\lambda}$. The second equation implies that $y = 0$ or $y^2 = \dfrac{1}{\lambda}$. Note that x and y cannot both be zero as this contradicts the third equation. If $x = 0$, the third equation implies $y = \pm\dfrac{1}{\sqrt[4]{3}}$. If $y = 0$, the third equation implies that $x = \pm 1$. Thus, f has possible extreme values at $\left(0, \pm\dfrac{1}{\sqrt[4]{3}}\right)$ and $(\pm 1, 0)$. Next, suppose $x^2 = y^2 = \dfrac{1}{\lambda}$. Then the third equation gives $\left(\dfrac{1}{\lambda}\right)^2 + 3\left(\dfrac{1}{\lambda}\right)^2 = 1 \quad\Rightarrow\quad \lambda = \pm 2$. $\lambda = -2$ results in a nonreal solution, so consider $\lambda = 2 \quad\Rightarrow\quad x = y = \pm\dfrac{1}{\sqrt{2}}$. Therefore, f also has possible extreme values at $\left(\pm\dfrac{1}{\sqrt{2}}, \pm\dfrac{1}{\sqrt{2}}\right)$ (all 4 combinations). Substituting all 8 points into f, we find the maximum value is $f\left(\pm\dfrac{1}{\sqrt{2}}, \pm\dfrac{1}{\sqrt{2}}\right) = 4$ and the minimum value is $f(\pm 1, 0) = 2$.

8. $f(x, y) = xye^{-x^2-y^2}$, $g(x, y) = 2x - y = 0$, and $\nabla f = \lambda \nabla g \quad\Rightarrow$
$\left\langle ye^{-x^2-y^2} - 2x^2ye^{-x^2-y^2}, xe^{-x^2-y^2} - 2xy^2e^{-x^2-y^2} \right\rangle = \langle 2\lambda, -\lambda \rangle$, so we get the three equations $ye^{-x^2-y^2} - 2x^2ye^{-x^2-y^2} = 2\lambda$, $xe^{-x^2-y^2} - 2xy^2e^{-x^2-y^2} = -\lambda$, and $2x - y = 0$. Multiplying the second equation by 2 and adding it to the first gives $2xe^{-x^2-y^2} - 4xy^2e^{-x^2-y^2} + ye^{-x^2-y^2} - 2x^2ye^{-x^2-y^2} = -2\lambda + 2\lambda = 0 \quad\Rightarrow$
$2x - 4xy^2 + y - 2x^2y = 0$ (as $e^{-x^2-y^2} \neq 0$). From the third equation, $2x = y$, and substituting into the new equation, we have $2x - 4x(2x)^2 + 2x - 2x^2(2x) = 0 \quad\Rightarrow\quad 4x - 20x^3 = 0 \quad\Rightarrow\quad 4x(1 - 5x^2) = 0 \quad\Rightarrow\quad x = 0$ or $x = \pm\dfrac{1}{\sqrt{5}}$, so f has possible extreme values at $(0, 0)$, $\left(\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right)$, and $\left(-\dfrac{1}{\sqrt{5}}, -\dfrac{2}{\sqrt{5}}\right)$. Substituting these into f, we see that the minimum value is $f(0, 0) = 0$ and the maximum value is $f\left(\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right) = f\left(-\dfrac{1}{\sqrt{5}}, -\dfrac{2}{\sqrt{5}}\right) = \dfrac{2}{5e}$.

9. $f(x, y, z) = 2x + 2y + z$, $g(x, y, z) = x^2 + y^2 + z^2 = 9$, and $\nabla f = \lambda \nabla g \Rightarrow \langle 2, 2, 1 \rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$, so $2\lambda x = 2$, $2\lambda y = 2$, $2\lambda z = 1$, and $x^2 + y^2 + z^2 = 9$. The first three equations imply $x = \dfrac{1}{\lambda}$, $y = \dfrac{1}{\lambda}$, and $z = \dfrac{1}{2\lambda}$. But substitution into the fourth equation gives $\left(\dfrac{1}{\lambda}\right)^2 + \left(\dfrac{1}{\lambda}\right)^2 + \left(\dfrac{1}{2\lambda}\right)^2 = 9 \Rightarrow \dfrac{9}{4\lambda^2} = 9 \Rightarrow \lambda = \pm\frac{1}{2}$, so f has possible extreme values at the points $(2, 2, 1)$ and $(-2, -2, -1)$. The maximum value of f on $x^2 + y^2 + z^2 = 9$ is $f(2, 2, 1) = 9$, and the minimum is $f(-2, -2, -1) = -9$.

10. $f(x, y, z) = e^{xyz}$, $g(x, y, z) = 2x^2 + y^2 + z^2 = 24$, and $\nabla f = \lambda \nabla g \Rightarrow \langle yze^{xyz}, xze^{xyz}, xye^{xyz} \rangle = \langle 4\lambda x, 2\lambda y, 2\lambda z \rangle$. Then $yze^{xyz} = 4\lambda x$, $xze^{xyz} = 2\lambda y$, $xye^{xyz} = 2\lambda z$, and $2x^2 + y^2 + z^2 = 24$. If any of x, y, z, or λ is zero, then the first three equations imply that two of the variables x, y, z must be zero. If $x = y = z = 0$ it contradicts the fourth equation, so exactly two are zero, and from the fourth equation the possibilities are $\left(\pm 2\sqrt{3}, 0, 0\right)$, $\left(0, \pm 2\sqrt{6}, 0\right)$, $\left(0, 0, \pm 2\sqrt{6}\right)$, all with an f-value of $e^0 = 1$. If none of x, y, z, λ is zero, then from the first three equations we have $\dfrac{4\lambda x}{yz} = e^{xyz} = \dfrac{2\lambda y}{xz} = \dfrac{2\lambda z}{xy} \Rightarrow \dfrac{2x}{yz} = \dfrac{y}{xz} = \dfrac{z}{xy}$. This gives $2x^2 z = y^2 z \Rightarrow 2x^2 = y^2$ and $xy^2 = xz^2 \Rightarrow y^2 = z^2$. Substituting into the fourth equation, we have $y^2 + y^2 + y^2 = 24 \Rightarrow y^2 = 8 \Rightarrow y = \pm 2\sqrt{2}$, so $x^2 = 4 \Rightarrow x = \pm 2$ and $z^2 = y^2 \Rightarrow z = \pm 2\sqrt{2}$, giving possible points $\left(\pm 2, \pm 2\sqrt{2}, \pm 2\sqrt{2}\right)$ (all combinations). The value of f is e^{16} when all coordinates are positive or exactly two are negative, and the value is e^{-16} when all are negative or exactly one of the coordinates is negative. Thus, the maximum of f subject to the constraint is e^{16} and the minimum is e^{-16}.

11. $f(x, y, z) = xy^2 z$, $g(x, y, z) = x^2 + y^2 + z^2 = 4$, and $\nabla f = \lambda \nabla g \Rightarrow \left\langle y^2 z, 2xyz, xy^2 \right\rangle = \lambda \langle 2x, 2y, 2z \rangle$. Then $y^2 z = 2\lambda x$, $2xyz = 2\lambda y$, $xy^2 = 2\lambda z$, and $x^2 + y^2 + z^2 = 4$.

Case 1: If $\lambda = 0$, then the first equation implies that $y = 0$ or $z = 0$. If $y = 0$, then any values of x and z satisfy the first three equations, so from the fourth equation all points $(x, 0, z)$ such that $x^2 + z^2 = 4$ are possible points. If $z = 0$, then from the third equation $x = 0$ or $y = 0$, and from the fourth equation, the possible points are $(0, \pm 2, 0)$, $(\pm 2, 0, 0)$. The f-value in all these cases is 0.

Case 2: If $\lambda \neq 0$ but any one of x, y, z is zero, the first three equations imply that all three coordinates must be zero, contradicting the fourth equation. Thus if $\lambda \neq 0$, none of x, y, z is zero and from the first three equations we have $\dfrac{y^2 z}{2x} = \lambda = xz = \dfrac{xy^2}{2z}$. This gives $y^2 z = 2x^2 z \Rightarrow y^2 = 2x^2$ and $2y^2 z^2 = 2x^2 y^2 \Rightarrow z^2 = x^2$. Substituting into the fourth equation, we have $x^2 + 2x^2 + x^2 = 4 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$, so $y = \pm\sqrt{2}$ and $z = \pm 1$, giving possible points $\left(\pm 1, \pm\sqrt{2}, \pm 1\right)$ (all combinations). The value of f is 2 when x and z are the same sign and -2 when they are opposite. Thus, the maximum of f subject to the constraint is $f(1, \pm\sqrt{2}, 1) = f(-1, \pm\sqrt{2}, -1) = 2$ and the minimum is $f(1, \pm\sqrt{2}, -1) = f(-1, \pm\sqrt{2}, 1) = -2$.

12. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = x^2 + y^2 + z^2 + xy = 12$, and $\nabla f = \lambda \nabla g \quad \Rightarrow$

$\langle 2x, 2y, 2z \rangle = \langle \lambda(2x + y), \lambda(2y + x), \lambda 2z \rangle$, so $(\mathbf{1})$ $2x = \lambda(2x + y)$, $(\mathbf{2})$ $2y = \lambda(2y + x)$, $(\mathbf{3})$ $2z = \lambda 2z$, and $(\mathbf{4})$ $x^2 + y^2 + z^2 + xy = 12$. First note that $\lambda = 0 \quad \Rightarrow \quad x = y = z = 0$, which contradicts $(\mathbf{4})$, so assume $\lambda \neq 0$. Then $(\mathbf{3})$ implies $\lambda = 1$ or $z = 0$. If $\lambda = 1$, then $(\mathbf{1})$ and $(\mathbf{2})$ imply $x = y = 0 \quad \Rightarrow \quad 0^2 + 0^2 + z^2 + 0 = 12 \quad \Rightarrow \quad z = \pm\sqrt{12}$. If $z = 0$, $y(\mathbf{1}) - x(\mathbf{2}) \quad \Rightarrow \quad 0 = \lambda y^2 - \lambda x^2 \quad \Rightarrow \quad x = y$ or $x = -y$. Substituting $x = y$ into $(\mathbf{4}) \quad \Rightarrow \quad y = x = \pm 2$. Substituting $x = -y$ into $(\mathbf{4}) \quad \Rightarrow \quad y = \pm\sqrt{12}$. Thus, f has possible extreme values at $\left(0, 0, \pm\sqrt{12}\right)$, $(2, 2, 0)$, $(-2, -2, 0)$, $\left(\sqrt{12}, -\sqrt{12}, 0\right)$, and $\left(-\sqrt{12}, \sqrt{12}, 0\right)$. Evaluating f at these points, we see that the maximum value is $f\left(\sqrt{12}, -\sqrt{12}, 0\right) = f\left(-\sqrt{12}, \sqrt{12}, 0\right) = 24$ and the minimum is $f(2, 2, 0) = f(-2, -2, 0) = 8$.

13. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = x^4 + y^4 + z^4 = 1 \quad \Rightarrow \quad \nabla f = \langle 2x, 2y, 2z \rangle$, $\lambda \nabla g = \left\langle 4\lambda x^3, 4\lambda y^3, 4\lambda z^3 \right\rangle$.

Case 1: If $x \neq 0$, $y \neq 0$, and $z \neq 0$, then $\nabla f = \lambda \nabla g$ implies $\lambda = 1/(2x^2) = 1/(2y^2) = 1/(2z^2)$ or $x^2 = y^2 = z^2$ and $3x^4 = 1$ or $x = \pm\frac{1}{\sqrt[4]{3}}$ giving the points $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$ all with an f-value of $\sqrt{3}$.

Case 2: If one of the variables equals zero and the other two are not zero, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{\sqrt{2}}$ and corresponding f-value of $\sqrt{2}$.

Case 3: If exactly two of the variables are zero, then the third variable has value ± 1 with the corresponding f-value of 1.

Thus on $x^4 + y^4 + z^4 = 1$, the maximum value of f is $\sqrt{3}$ and the minimum value is 1.

14. $f(x,y,z) = x^4 + y^4 + z^4$, $g(x,y,z) = x^2 + y^2 + z^2 = 1 \quad \Rightarrow \quad \nabla f = \left\langle 4x^3, 4y^3, 4z^3 \right\rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$.

Case 1: If $x \neq 0$, $y \neq 0$, and $z \neq 0$, then $\nabla f = \lambda \nabla g$ implies $\lambda = 2x^2 = 2y^2 = 2z^2$ or $x^2 = y^2 = z^2 = \frac{1}{3}$ giving 8 points each with an f-value of $\frac{1}{3}$.

Case 2: If one of the variables is 0 and the other two are not, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{2}$ and the corresponding f-value is $\frac{1}{2}$.

Case 3: If exactly two of the variables are 0, then the third variable has value ± 1 with corresponding f-value of 1.

Thus on $x^2 + y^2 + z^2 = 1$, the maximum value of f is 1 and the minimum value is $\frac{1}{3}$.

15. $f(x,y,z,t) = x + y + z + t$, $g(x,y,z,t) = x^2 + y^2 + z^2 + t^2 = 1 \quad \Rightarrow \quad \langle 1, 1, 1, 1 \rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z, 2\lambda t \rangle$, so $\lambda = 1/(2x) = 1/(2y) = 1/(2z) = 1/(2t)$ and $x = y = z = t$. But $x^2 + y^2 + z^2 + t^2 = 1$, so the possible points are $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$. Thus, the maximum value of f is $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 2$ and the minimum value is $f\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = -2$.

16. $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$, $g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \quad \Rightarrow$

$\langle 1, 1, \ldots, 1 \rangle = \langle 2\lambda x_1, 2\lambda x_2, \ldots, 2\lambda x_n \rangle$, so $\lambda = 1/(2x_1) = 1/(2x_2) = \cdots = 1/(2x_n)$ and $x_1 = x_2 = \cdots = x_n$.

[continued]

But $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, so $x_i = \pm 1/\sqrt{n}$ for $i = 1, \ldots, n$. Thus, the maximum value of f is $f(1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n}) = \sqrt{n}$ and the minimum value is $f(-1/\sqrt{n}, -1/\sqrt{n}, \ldots, -1/\sqrt{n}) = -\sqrt{n}$.

17. If the two numbers are x and y, we want to minimize $f(x,y) = x + y$, $x > 0$, $y > 0$ subject to $g(x,y) = xy = 100$. Then $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 1, 1 \rangle = \langle \lambda y, \lambda x \rangle$, so $1 = \lambda y$, $1 = \lambda x$, and $xy = 100$. The first two equations imply $y = \dfrac{1}{\lambda} = x$ and substitution into the third equation gives $\dfrac{1}{\lambda^2} = 100 \quad \Rightarrow \quad \lambda = \pm\dfrac{1}{10}$. Since $x > 0$ and $y > 0$, we have $\lambda = \dfrac{1}{10}$ and hence, $x = y = \dfrac{1}{\lambda} = 10$. Thus, the minimum value of f is $f(10, 10) = 20$. By comparing nearby values we can confirm that this gives a minimum and not a maximum. Therefore, the two numbers are 10 and 10.

18. If x and y are the dimensions of the rectangle in meters, we want to minimize $f(x,y) = 2x + 2y$, $x > 0$, $y > 0$ subject to $g(x,y) = xy = 1000$. Then $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 2, 2 \rangle = \langle \lambda y, \lambda x \rangle$, so $2 = \lambda y$, $2 = \lambda x$, and $xy = 1000$. The first two equations imply $y = \dfrac{2}{\lambda} = x$ and substitution into the third equation gives $\dfrac{4}{\lambda^2} = 1000 \quad \Rightarrow \quad \lambda = \pm\dfrac{1}{5\sqrt{10}}$. Since $x > 0$ and $y > 0$, we have $\lambda = \dfrac{1}{5\sqrt{10}}$ and hence, $x = y = \dfrac{2}{\lambda} = 10\sqrt{10}$. Thus, the minimum value of f is $f\left(10\sqrt{10}, 10\sqrt{10}\right) = 40\sqrt{10}$. By comparing nearby values we can confirm that this gives a minimum and not a maximum. Therefore, the dimensions of the rectangle that minimize the perimeter are $10\sqrt{10}$ m by $10\sqrt{10}$ m.

19. Let x and y be the dimensions of the rectangle in meters. Then the perimeter constraint is given by $2x + 2y = 100 \quad \Rightarrow$ $x + y = 50$. We want to maximize $f(x,y) = xy$, $x > 0$, $y > 0$ subject to $g(x,y) = x + y = 50$. Then $\nabla f = \lambda \nabla g \quad \Rightarrow$ $\langle y, x \rangle = \langle \lambda, \lambda \rangle$, so $y = \lambda$, $x = \lambda$, and $x + y = 50$. Substituting the first two equations into the third gives $2\lambda = 50 \quad \Rightarrow$ $\lambda = 25$. Thus, the area is maximized when $x = y = 25$ meters and $f(25, 25) = 625\text{ m}^2$.

20. Let x, y, and z be the dimensions of the box. The box has a square bottom, so $x = y$. We want to minimize $f(x,z) = x^2 + 4xz$ subject to $g(x,z) = x^2 z = 32{,}000$. Then $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 2x + 4z, 4x \rangle = \left\langle 2\lambda xz, \lambda x^2 \right\rangle$, so $2x + 4z = 2\lambda xz$, $4x = \lambda x^2$, and $x^2 z = 32{,}000$. The second equation implies $x = 0$ or $\lambda x = 4$. Note that $x = 0$ results in a zero volume. So let $\lambda = 4/x$. Substitution into the first equation gives $2x + 4z = 8z \quad \Rightarrow \quad x = 2z$, and substituting $2z$ for x into the third equation gives $4z^3 = 32{,}000 \quad \Rightarrow \quad z = 20 \quad \Rightarrow \quad x = 40$. Therefore, the dimensions of the box that will minimize the surface area are 40 cm by 40 cm by 20 cm.

21. The distance d from any point (x,y) in the xy-plane to the origin is given by $d = \sqrt{x^2 + y^2}$. We will minimize $d^2 = f(x,y) = x^2 + y^2$ subject to $g(x,y) = -2x + y = 3$. Then $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 2x, 2y \rangle = \langle -2\lambda, \lambda \rangle$, so $2x = -2\lambda$, $2y = \lambda$, and $-2x + y = 3$. The first equation implies $x = -\lambda$, and the second, $y = \dfrac{\lambda}{2}$. Then substitution into the third equation gives $2\lambda + \dfrac{\lambda}{2} = 3 \quad\Rightarrow\quad \lambda = \frac{6}{5} \quad\Rightarrow\quad x = -\frac{6}{5}$ and $y = \frac{3}{5}$. Thus, the point on the line $y = 2x + 3$ that is closest to the origin is $\left(-\frac{6}{5}, \frac{3}{5}\right)$.

22. Let r and h be the radius and height of a right circular cylinder. We want to maximize $V(r,h) = \pi r^2 h$ subject to $g(r,h) = h + 2\pi r = 108$. Then $\nabla V = \lambda \nabla g \quad\Rightarrow\quad \left\langle 2\pi rh, \pi r^2\right\rangle = \langle 2\pi\lambda, \lambda\rangle$, so $2\pi rh = 2\pi\lambda$, $\pi r^2 = \lambda$, and $h + 2\pi r = 108$. The first equation implies $rh = \lambda$ and substitution into the second gives $\pi r^2 = rh \quad\Rightarrow\quad r = 0$ or $h = \pi r$. $r \neq 0$ (else, $V = 0$), so substitute $h = \pi r$ into the third equation. Then $\pi r + 2\pi r = 108 \quad\Rightarrow\quad \pi r = 36 \quad\Rightarrow\quad h = 36$. Therefore, the dimensions that maximize the volume are $r = \dfrac{36}{\pi}$ inches and $h = 36$ inches for a volume of $V = \dfrac{46{,}656}{\pi}$ in^3.

23. $f(x,y) = x^2 + y^2,\ g(x,y) = xy = 1$, and $\nabla f = \lambda \nabla g \quad\Rightarrow\quad \langle 2x, 2y\rangle = \langle \lambda y, \lambda x\rangle$, so $2x = \lambda y$, $2y = \lambda x$, and $xy = 1$. From the last equation, $x \neq 0$ and $y \neq 0$, so $2x = \lambda y \quad\Rightarrow\quad \lambda = 2x/y$. Substituting, we have $2y = (2x/y)\,x \quad\Rightarrow$ $y^2 = x^2 \quad\Rightarrow\quad y = \pm x$. But $xy = 1$, so $x = y = \pm 1$ and the possible points for the extreme values of f are $(1,1)$ and $(-1,-1)$. Here there is no maximum value, since the constraint $xy = 1 \quad\Leftrightarrow\quad y = 1/x$ allows x or y to become arbitrarily large, and hence $f(x,y) = x^2 + y^2$ can be made arbitrarily large. The minimum value is $f(1,1) = f(-1,-1) = 2$.

24. $f(x,y,z) = x^2 + 2y^2 + 3z^2,\ g(x,y) = x + 2y + 3z = 10$, and $\nabla f = \lambda \nabla g \quad\Rightarrow\quad \langle 2x, 4y, 6z\rangle = \langle \lambda, 2\lambda, 3\lambda\rangle$, so $2x = \lambda$, $4y = 2\lambda$, $6z = 3\lambda$, and $x + 2y + 3z = 10$. From the first three equations we have $2x = \lambda = 2y = 2z \quad\Rightarrow\quad x = y = z$, and substituting into the fourth equation gives $x + 2x + 3x = 10 \quad\Rightarrow\quad x = \frac{5}{3} = y = z$. Thus, the only possible point for an extreme value of f is $\left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right)$. Notice here that the constraint $x + 2y + 3z = 10$ allows any of $|x|$, $|y|$, or $|z|$ to be arbitrarily large, and hence $f(x,y,z) = x^2 + 2y^2 + 3z^2$ can be made arbitrarily large. So f has no maximum value subject to the constraint. The minimum value is $f\left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right) = 6\left(\frac{5}{3}\right)^2 = \frac{50}{3}$.

25. $f(x,y) = e^{xy}, g(x,y) = x^3 + y^3 = 16$. Then $\nabla f = \lambda \nabla g \quad\Rightarrow\quad \langle ye^{xy}, xe^{xy}\rangle = \left\langle 3\lambda x^2, 3\lambda y^2\right\rangle$, so $ye^{xy} = 3\lambda x^2$, $xe^{xy} = 3\lambda y^2$, and $x^3 + y^3 = 16$. Multiplying the first equation by x and the second by y gives $xye^{xy} = 3\lambda x^3$ and $xye^{xy} = 3\lambda y^3 \quad\Rightarrow\quad 3\lambda x^3 = 3\lambda y^3 \quad\Rightarrow\quad x = y$ and substituting x for y into the third equation, we have $x^3 + x^3 = 16 \quad\Rightarrow$ $x = y = 2$. Therefore, f has an extreme value at the point $(2,2)$ and evaluating f at that point we see $f(2,2) = e^4$. Notice from the constraint that $y^3 = 16 - x^3$ and y^3 becomes increasingly negative as x^3 becomes arbitrarily large (similarly, x^3 can be increasingly negative while y^3 is arbitrarily large). Thus, for any small value $\varepsilon > 0$, we can find x and y that satisfy the constraint such that $0 < e^{xy} < \varepsilon$. Thus, f has no minimum value subject to the constraint and $f(2,2) = e^4$ is a maximum.

26. $f(x,y,z) = 4x + 2y + z$, $g(x,y,z) = x^2 + y + z^2 = 1$. Then $\nabla f = \lambda \nabla g \quad\Rightarrow\quad \langle 4, 2, 1\rangle = \langle 2\lambda x, \lambda, 2\lambda z\rangle$, so $4 = 2\lambda x$, $2 = \lambda$, $1 = 2\lambda z$, and $x^2 + y + z^2 = 1$. As $\lambda = 2$ by equation two, we have $x = 1$ and $z = \frac{1}{4}$ by equations one and three, respectively. Substituting these values into equation four gives $1^2 + y + \left(\frac{1}{4}\right)^2 = 1 \quad\Rightarrow\quad y = -\frac{1}{16}$. Thus, f has an extreme value at $\left(1, -\frac{1}{16}, \frac{1}{4}\right)$. Notice that $y = 1 - (x^2 + z^2) < 0$ for $x^2 + z^2 > 1$, and substituting into f, we see that

$f(x,y,z) = 4x + 2 + z - 2(x^2 + z^2)$ can decrease without bound for values of (x,y,z) that satisfy the constraint. Thus, f has no minimum value subject to the constraint and $f\left(1, -\frac{1}{16}, \frac{1}{4}\right) = \frac{33}{8}$ is a maximum.

27. $f(x,y) = x^2 + y^2 + 4x - 4y$. For the interior of the region, we find the critical points: $f_x = 2x + 4$, $f_y = 2y - 4$, so the only critical point is $(-2, 2)$ (which is inside the region) and $f(-2,2) = -8$. For the boundary, we use Lagrange multipliers. $g(x,y) = x^2 + y^2 = 9$, so $\nabla f = \lambda \nabla g \;\Rightarrow\; \langle 2x + 4, 2y - 4 \rangle = \langle 2\lambda x, 2\lambda y \rangle$. Thus, $2x + 4 = 2\lambda x$ and $2y - 4 = 2\lambda y$. Adding the two equations gives $2x + 2y = 2\lambda x + 2\lambda y \;\Rightarrow\; x + y = \lambda(x + y) \;\Rightarrow\; (x + y)(\lambda - 1) = 0$, so $x + y = 0 \;\Rightarrow\; y = -x$ or $\lambda - 1 = 0 \;\Rightarrow\; \lambda = 1$. But $\lambda = 1$ leads to a contradition in $2x + 4 = 2\lambda x$, so $y = -x$ and $x^2 + y^2 = 9$ implies $2y^2 = 9 \;\Rightarrow\; y = \pm\frac{3}{\sqrt{2}}$. We have $f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = 9 + 12\sqrt{2} \approx 25.97$ and $f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = 9 - 12\sqrt{2} \approx -7.97$, so the maximum value of f on the disk $x^2 + y^2 \leq 9$ is $f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = 9 + 12\sqrt{2}$ and the minimum is $f(-2,2) = -8$.

28. $f(x,y) = 2x^2 + 3y^2 - 4x - 5 \;\Rightarrow\; \nabla f = \langle 4x - 4, 6y \rangle = \langle 0, 0 \rangle \;\Rightarrow\; x = 1, y = 0$. Thus, $(1,0)$ is the only critical point of f, and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x,y) = x^2 + y^2 = 16 \;\Rightarrow\; \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$, so $6y = 2\lambda y \;\Rightarrow\;$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \;\Rightarrow\; x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1,0) = -7$, $f(4,0) = 11$, $f(-4,0) = 43$, and $f\left(-2, \pm 2\sqrt{3}\right) = 47$. Thus, the maximum value of $f(x,y)$ on the disk $x^2 + y^2 \leq 16$ is $f\left(-2, \pm 2\sqrt{3}\right) = 47$, and the minimum value is $f(1,0) = -7$.

29. $f(x,y) = e^{-xy}$. For the interior of the region, we find the critical points: $f_x = -ye^{-xy}$, $f_y = -xe^{-xy}$, so the only critical point is $(0,0)$, and $f(0,0) = 1$. For the boundary, we use Lagrange multipliers. $g(x,y) = x^2 + 4y^2 = 1 \;\Rightarrow\; \lambda \nabla g = \langle 2\lambda x, 8\lambda y \rangle$, so setting $\nabla f = \lambda \nabla g$ we get $-ye^{-xy} = 2\lambda x$ and $-xe^{-xy} = 8\lambda y$. The first of these gives $e^{-xy} = -2\lambda x/y$, and then the second gives $-x(-2\lambda x/y) = 8\lambda y \;\Rightarrow\; x^2 = 4y^2$. Solving this last equation with the constraint $x^2 + 4y^2 = 1$ gives $x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\frac{1}{2\sqrt{2}}$. Now $f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right) = e^{1/4} \approx 1.284$ and $f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right) = e^{-1/4} \approx 0.779$. The former are the maximums on the region and the latter are the minimums.

30. $f(x,y,z) = z$, $g(x,y,z) = x^2 + y^2 - z^2 = 0$, $h(x,y,z) = x + y + z = 24$, and $\nabla f = \lambda \nabla g + \mu \nabla h \;\Rightarrow\; \langle 0, 0, 1 \rangle = \langle 2\lambda x, 2\lambda y, -2\lambda z \rangle + \langle \mu, \mu, \mu \rangle$. Then $0 = 2\lambda x + \mu$, $0 = 2\lambda y + \mu$, $1 = -2\lambda z + \mu$, $x^2 + y^2 - z^2 = 0$, and $x + y + z = 24$. From the first two equations we have $-2\lambda x = \mu = -2\lambda y \;\Rightarrow\; \lambda = 0$ or $x = y$. But $\lambda = 0 \;\Rightarrow\; \mu = 0$ which contradicts the third equation, so $x = y$ and substitution into the last equation gives $z = 24 - 2x$. From the fourth equation we have $x^2 + x^2 - (24 - 2x)^2 = 0 \;\Rightarrow\; -2x^2 + 96x - 576 = 0 \;\Rightarrow\; x^2 - 48x + 288 = 0 \;\Rightarrow$

$$x = \frac{48 \pm \sqrt{1152}}{2} = 24 \pm 12\sqrt{2} = y.$$

Now $z = 24 - 2x$, so the possible points are $\left(24 + 12\sqrt{2}, 24 + 12\sqrt{2}, -24 - 24\sqrt{2}\right)$ and $\left(24 - 12\sqrt{2}, 24 - 12\sqrt{2}, -24 + 24\sqrt{2}\right)$. The maximum of f subject to the constraints is

$f\left(24-12\sqrt{2}, 24-12\sqrt{2}, -24+24\sqrt{2}\right) = -24+24\sqrt{2} \approx 9.94$ and the minimum is

$f\left(24+12\sqrt{2}, 24+12\sqrt{2}, -24-24\sqrt{2}\right) = -24-24\sqrt{2} \approx -57.94$.

31. $f(x,y,z) = x+y+z,\ g(x,y,z) = x^2+z^2 = 2,\ h(x,y,z) = x+y = 1$, and $\nabla f = \lambda\nabla g + \mu\nabla h \Rightarrow$ $\langle 1,1,1\rangle = \langle 2\lambda x, 0, 2\lambda z\rangle + \langle \mu,\mu,0\rangle$. Then $1 = 2\lambda x + \mu,\ \ 1 = \mu,\ \ 1 = 2\lambda z,\ \ x^2+z^2 = 2$, and $x+y=1$. Substituting $\mu = 1$ into the first equation gives $\lambda = 0$ or $x = 0$. But $\lambda = 0$ contradicts $1 = 2\lambda z$, so $x = 0$. Then $x+y=1 \Rightarrow y = 1$ and $x^2+z^2 = 2 \Rightarrow z = \pm\sqrt{2}$, so the possible points are $\left(0,1,\pm\sqrt{2}\,\right)$. The maximum value of f subject to the constraints is $f(0,1,\sqrt{2}\,) = 1+\sqrt{2} \approx 2.41$ and the minimum is $f(0,1,-\sqrt{2}\,) = 1-\sqrt{2} \approx -0.41$.

Note: Since $x+y=1$ is one of the constraints, we could have solved the problem by solving $f(x,z) = 1+z$ subject to $x^2+z^2 = 2$.

32. $f(x,y,z) = x^2+y^2+z^2,\ g(x,y,z) = x-y = 1,\ h(x,y,z) = y^2-z^2 = 1 \Rightarrow \nabla f = \langle 2x, 2y, 2z\rangle$, $\lambda\nabla g = \langle \lambda, -\lambda, 0\rangle$, and $\mu\nabla h = \langle 0, 2\mu y, -2\mu z\rangle$. Then $2x = \lambda$, $2y = -\lambda + 2\mu y$, and $2z = -2\mu z \Rightarrow z = 0$ or $\mu = -1$. If $z = 0$, then $y^2-z^2 = 1$ implies $y^2 = 1 \Rightarrow y = \pm 1$. If $y = 1$, $x-y = 1$ implies $x = 2$, and if $y = -1$ we have $x = 0$, so possible points are $(2,1,0)$ and $(0,-1,0)$. If $\mu = -1$ then $2y = -\lambda + 2\mu y$ implies $4y = -\lambda$, but $\lambda = 2x$ so $4y = -2x \Rightarrow x = -2y$ and $x-y=1$ implies $-3y = 1 \Rightarrow y = -\frac{1}{3}$. But then $y^2-z^2 = 1$ implies $z^2 = -\frac{8}{9}$, an impossibility. Thus, the maximum value of f subject to the constraints is $f(2,1,0) = 5$ and the minimum is $f(0,-1,0) = 1$.

Note: Since $x-y=1 \Rightarrow x = y+1$ is one of the constraints we could have solved the problem by solving $f(y,z) = (y+1)^2+y^2+z^2$ subject to $y^2-z^2 = 1$.

33. $f(x,y,z) = yz+xy,\ g(x,y,z) = xy = 1,\ h(x,y,z) = y^2+z^2 = 1 \Rightarrow \nabla f = \langle y, x+z, y\rangle$, $\lambda\nabla g = \langle \lambda y, \lambda x, 0\rangle$, $\mu\nabla h = \langle 0, 2\mu y, 2\mu z\rangle$. Then $y = \lambda y$ implies $\lambda = 1$ [$y \neq 0$ since $g(x,y,z) = 1$], $x+z = \lambda x + 2\mu y$ and $y = 2\mu z$. Thus, $\mu = z/(2y) = y/(2y)$ or $y^2 = z^2$, and so $y^2+z^2 = 1$ implies $y = \pm\frac{1}{\sqrt{2}}$, $z = \pm\frac{1}{\sqrt{2}}$. Then $xy = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Hence the maximum of f subject to the constraints is $f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \frac{3}{2}$ and the minimum is $f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$.

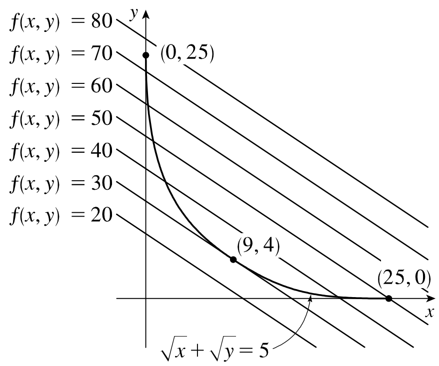
Note: Since $xy = 1$ is one of the constraints we could have solved the problem by solving $f(y,z) = yz+1$ subject to $y^2+z^2 = 1$.

34. (a) $f(x,y) = 2x+3y,\ g(x,y) = \sqrt{x}+\sqrt{y} = 5 \Rightarrow \nabla f = \langle 2,3\rangle = \lambda\nabla g = \lambda\left\langle \dfrac{1}{2\sqrt{x}}, \dfrac{1}{2\sqrt{y}}\right\rangle$. Then

$2 = \dfrac{\lambda}{2\sqrt{x}}$ and $3 = \dfrac{\lambda}{2\sqrt{y}}$ so $4\sqrt{x} = \lambda = 6\sqrt{y} \Rightarrow \sqrt{y} = \frac{2}{3}\sqrt{x}$. With $\sqrt{x}+\sqrt{y} = 5$ we have $\sqrt{x}+\frac{2}{3}\sqrt{x} = 5 \Rightarrow$ $\sqrt{x} = 3 \Rightarrow x = 9$. Substituting into $\sqrt{y} = \frac{2}{3}\sqrt{x}$ gives $\sqrt{y} = 2$ or $y = 4$. Thus, the only possible extreme value subject to the constraint is $f(9,4) = 30$. (The question remains whether this is indeed the maximum of f.)

(b) $f(25, 0) = 50$ which is larger than the result of part (a).

(c)



We can see from the level curves of f that the maximum occurs at the left endpoint $(0, 25)$ of the constraint curve g. The maximum value is $f(0, 25) = 75$.

(d) Here ∇g does not exist if $x = 0$ or $y = 0$, so the method will not locate any associated points. Also, the method of Lagrange multipliers identifies points where the level curves of f share a common tangent line with the constraint curve g. This normally does not occur at an endpoint, although an absolute maximum or minimum may occur there.

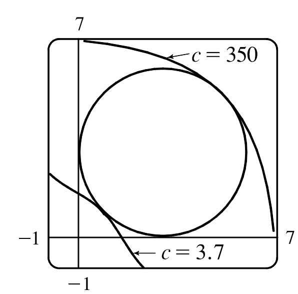
(e) Here $f(9, 4)$ is the absolute *minimum* of f subject to g.

35. (a) $f(x, y) = x$, $g(x, y) = y^2 + x^4 - x^3 = 0 \quad \Rightarrow \quad \nabla f = \langle 1, 0 \rangle = \lambda \nabla g = \lambda \left\langle 4x^3 - 3x^2, 2y \right\rangle$. Then $1 = \lambda(4x^3 - 3x^2)$ **(1)** and $0 = 2\lambda y$ **(2)**. We have $\lambda \neq 0$ from **(1)**, so **(2)** gives $y = 0$. Then, from the constraint equation, $x^4 - x^3 = 0 \quad \Rightarrow \quad x^3(x - 1) = 0 \quad \Rightarrow \quad x = 0$ or $x = 1$. But $x = 0$ contradicts **(1)**, so the only possible extreme value subject to the constraint is $f(1, 0) = 1$. (The question remains whether this is indeed the minimum of f.)

(b) The constraint is $y^2 + x^4 - x^3 = 0 \quad \Leftrightarrow \quad y^2 = x^3 - x^4$. The left side is nonnegative, so we must have $x^3 - x^4 \geq 0$ which is true only for $0 \leq x \leq 1$. Therefore, the minimum possible value for $f(x, y) = x$ is 0 which occurs for $x = y = 0$. However, $\lambda \nabla g(0, 0) = \lambda \langle 0 - 0, 0 \rangle = \langle 0, 0 \rangle$ and $\nabla f(0, 0) = \langle 1, 0 \rangle$, so $\nabla f(0, 0) \neq \lambda \nabla g(0, 0)$ for all values of λ.

(c) Here $\nabla g(0, 0) = \mathbf{0}$, but the method of Lagrange multipliers requires that $\nabla g \neq \mathbf{0}$ everywhere on the constraint curve.

36. (a) The graphs of $f(x, y) = 3.7$ and $f(x, y) = 350$ seem to be tangent to the circle, and so 3.7 and 350 are the approximate minimum and maximum values of the function $f(x, y) = x^3 + y^3 + 3xy$ subject to the constraint $(x - 3)^2 + (y - 3)^2 = 9$.



(b) Let $g(x, y) = (x - 3)^2 + (y - 3)^2$. We calculate $f_x(x, y) = 3x^2 + 3y$, $f_y(x, y) = 3y^2 + 3x$, $g_x(x, y) = 2x - 6$, and $g_y(x, y) = 2y - 6$, and use a CAS to search for solutions to the equations $g(x, y) = (x - 3)^2 + (y - 3)^2 = 9$, $f_x = \lambda g_x$, and $f_y = \lambda g_y$. The solutions are $(x, y) = \left(3 - \frac{3}{2}\sqrt{2}, 3 - \frac{3}{2}\sqrt{2}\right) \approx (0.879, 0.879)$ and $(x, y) = \left(3 + \frac{3}{2}\sqrt{2}, 3 + \frac{3}{2}\sqrt{2}\right) \approx (5.121, 5.121)$. These give $f\left(3 - \frac{3}{2}\sqrt{2}, 3 - \frac{3}{2}\sqrt{2}\right) = \frac{351}{2} - \frac{243}{2}\sqrt{2} \approx 3.673$ and $f\left(3 + \frac{3}{2}\sqrt{2}, 3 + \frac{3}{2}\sqrt{2}\right) = \frac{351}{2} + \frac{243}{2}\sqrt{2} \approx 347.33$, in accordance with part (a).

37. $P(L,K) = bL^{\alpha}K^{1-\alpha},\ g(L,K) = mL + nK = p \quad\Rightarrow\quad \nabla P = \left\langle \alpha bL^{\alpha-1}K^{1-\alpha}, (1-\alpha)bL^{\alpha}K^{-\alpha}\right\rangle,\ \lambda\nabla g = \langle \lambda m, \lambda n\rangle.$

Then $\alpha b(K/L)^{1-\alpha} = \lambda m$ and $(1-\alpha)b(L/K)^{\alpha} = \lambda n$ and $mL + nK = p$, so $\alpha b(K/L)^{1-\alpha}/m = (1-\alpha)b(L/K)^{\alpha}/n$ or $n\alpha/[m(1-\alpha)] = (L/K)^{\alpha}(L/K)^{1-\alpha}$ or $L = Kn\alpha/[m(1-\alpha)]$. Substituting into $mL + nK = p$ gives $K = (1-\alpha)p/n$ and $L = \alpha p/m$ for the maximum production.

38. $C(L,K) = mL + nK,\ g(L,K) = bL^{\alpha}K^{1-\alpha} = Q \quad\Rightarrow\quad \nabla C = \langle m,n\rangle,\ \lambda\nabla g = \left\langle \lambda\alpha bL^{\alpha-1}K^{1-\alpha}, \lambda(1-\alpha)bL^{\alpha}K^{-\alpha}\right\rangle.$

Then $\dfrac{m}{\alpha b}\left(\dfrac{L}{K}\right)^{1-\alpha} = \dfrac{n}{(1-\alpha)b}\left(\dfrac{K}{L}\right)^{\alpha}$ and $bL^{\alpha}K^{1-\alpha} = Q \quad\Rightarrow\quad \dfrac{n\alpha}{m(1-\alpha)} = \left(\dfrac{L}{K}\right)^{1-\alpha}\left(\dfrac{L}{K}\right)^{\alpha} \quad\Rightarrow$

$L = \dfrac{Kn\alpha}{m(1-\alpha)}$ and so $b\left[\dfrac{Kn\alpha}{m(1-\alpha)}\right]^{\alpha}K^{1-\alpha} = Q$. Hence $K = \dfrac{Q}{b\left(n\alpha/[m(1-\alpha)]\right)^{\alpha}} = \dfrac{Qm^{\alpha}(1-\alpha)^{\alpha}}{bn^{\alpha}\alpha^{\alpha}}$

and $L = \dfrac{Qm^{\alpha-1}(1-\alpha)^{\alpha-1}}{bn^{\alpha-1}\alpha^{\alpha-1}} = \dfrac{Qn^{1-\alpha}\alpha^{1-\alpha}}{bm^{1-\alpha}(1-\alpha)^{1-\alpha}}$ minimizes cost.

39. Let the sides of the rectangle be x and y. Then $f(x,y) = xy$, $g(x,y) = 2x + 2y = p \quad\Rightarrow\quad \nabla f(x,y) = \langle y,x\rangle$, $\lambda\nabla g = \langle 2\lambda, 2\lambda\rangle$. Then $\lambda = \frac{1}{2}y = \frac{1}{2}x$ implies $x = y$ and the rectangle with maximum area is a square with side length $\frac{1}{4}p$.

40. We maximize $A^2 = f(x,y,z) = s(s-x)(s-y)(s-z)$ subject to $g(x,y,z) = x + y + z$. Then $\nabla f = \langle -s(s-y)(s-z), -s(s-x)(s-z), -s(s-x)(s-y)\rangle$, $\lambda\nabla g = \langle \lambda,\lambda,\lambda\rangle$. Thus, $(s-y)(s-z) = (s-x)(s-z)$ **(1)**, and $(s-x)(s-z) = (s-x)(s-y)$ **(2)**. **(1)** implies $x = y$ while **(2)** implies $y = z$, so $x = y = z = p/3$ and the triangle with maximum area is equilateral.

41. The distance from $(2,0,-3)$ to a point (x,y,z) on the plane is $d = \sqrt{(x-2)^2 + y^2 + (z+3)^2}$, so we seek to minimize $d^2 = f(x,y,z) = (x-2)^2 + y^2 + (z+3)^2$ subject to the constraint that (x,y,z) lies on the plane $x + y + z = 1$, that is, that $g(x,y,z) = x + y + z = 1$. Then $\nabla f = \lambda\nabla g \quad\Rightarrow\quad \langle 2(x-2), 2y, 2(z+3)\rangle = \langle \lambda,\lambda,\lambda\rangle$, so $x = (\lambda+4)/2$, $y = \lambda/2$, $z = (\lambda-6)/2$. Substituting into the constraint equation gives $\dfrac{\lambda+4}{2} + \dfrac{\lambda}{2} + \dfrac{\lambda-6}{2} = 1 \quad\Rightarrow\quad 3\lambda - 2 = 2 \quad\Rightarrow$ $\lambda = \frac{4}{3}$, so $x = \frac{8}{3}$, $y = \frac{2}{3}$, and $z = -\frac{7}{3}$. This must correspond to a minimum, so the shortest distance is $d = \sqrt{\left(\frac{8}{3}-2\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{7}{3}+3\right)^2} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$.

42. The distance from $(0,1,1)$ to a point (x,y,z) on the plane is $d = \sqrt{x^2 + (y-1)^2 + (z-1)^2}$, so we minimize $d^2 = f(x,y,z) = x^2 + (y-1)^2 + (z-1)^2$ subject to the constraint that (x,y,z) lies on the plane $x - 2y + 3z = 6$, that is, $g(x,y,z) = x - 2y + 3z = 6$. Then $\nabla f = \lambda\nabla g \quad\Rightarrow\quad \langle 2x, 2(y-1), 2(z-1)\rangle = \langle \lambda, -2\lambda, 3\lambda\rangle$, so $x = \lambda/2$, $y = 1 - \lambda$, $z = (3\lambda+2)/2$. Substituting into the constraint equation gives $\dfrac{\lambda}{2} - 2(1-\lambda) + 3\cdot\dfrac{3\lambda+2}{2} = 6 \quad\Rightarrow\quad \lambda = \frac{5}{7}$, so $x = \frac{5}{14}$, $y = \frac{2}{7}$, and $z = \frac{29}{14}$. This must correspond to a minimum, so the point on the plane closest to the point $(0,1,1)$ is $\left(\frac{5}{14}, \frac{2}{7}, \frac{29}{14}\right)$.

43. Let $f(x,y,z) = d^2 = (x-4)^2 + (y-2)^2 + z^2$. Then we want to minimize f subject to the constraint $g(x,y,z) = x^2 + y^2 - z^2 = 0$. $\nabla f = \lambda \nabla g \Rightarrow \langle 2(x-4), 2(y-2), 2z \rangle = \langle 2\lambda x, 2\lambda y, -2\lambda z \rangle$, so $x - 4 = \lambda x$, $y - 2 = \lambda y$, and $z = -\lambda z$. From the last equation we have $z + \lambda z = 0 \Rightarrow z(1+\lambda) = 0$, so either $z = 0$ or $\lambda = -1$. But from the constraint equation we have $z = 0 \Rightarrow x^2 + y^2 = 0 \Rightarrow x = y = 0$ which is not possible from the first two equations. So $\lambda = -1$ and $x - 4 = \lambda x \Rightarrow x = 2$, $y - 2 = \lambda y \Rightarrow y = 1$, and $x^2 + y^2 - z^2 = 0 \Rightarrow$ $4 + 1 - z^2 = 0 \Rightarrow z = \pm\sqrt{5}$. This must correspond to a minimum, so the points on the cone closest to $(4,2,0)$ are $(2, 1, \pm\sqrt{5})$.

44. Let $f(x,y,z) = d^2 = x^2 + y^2 + z^2$. Then we want to minimize f subject to the constraint $g(x,y,z) = y^2 - xz = 9$. $\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 2y, 2z \rangle = \langle -\lambda z, 2\lambda y, -\lambda x \rangle$, so $2x = -\lambda z$, $y = \lambda y$, and $2z = -\lambda x$. If $x = 0$, then the last equation implies $z = 0$, and from the constraint $y^2 - xz = 9$ we have $y = \pm 3$. If $x \neq 0$, then the first and third equations give $\lambda = -2x/z = -2z/x \Rightarrow x^2 = z^2$. From the second equation we have $y = 0$ or $\lambda = 1$. If $y = 0$ then $y^2 - xz = 9 \Rightarrow$ $z = -9/x$ and $x^2 = z^2 \Rightarrow x^2 = 81/x^2 \Rightarrow x = \pm 3$. Since $z = -9/x$, $x = 3 \Rightarrow z = -3$ and $x = -3 \Rightarrow$ $z = 3$. If $\lambda = 1$, then $2x = -z$ and $2z = -x$ which implies $x = z = 0$, contradicting the assumption that $x \neq 0$. Thus, the possible points are $(0, \pm 3, 0)$, $(3, 0, -3)$, $(-3, 0, 3)$. We have $f(0, \pm 3, 0) = 9$ and $f(3, 0, -3) = f(-3, 0, 3) = 18$, so the points on the surface that are closest to the origin are $(0, \pm 3, 0)$.

45. Maximize $f(x,y,z) = xyz$ subject to $g(x,y,z) = x + y + z = 100$. $\nabla f = \lambda \nabla g \Rightarrow \langle yz, xz, xy \rangle = \lambda \langle 1, 1, 1 \rangle$. Then $\lambda = yz = xz = xy$ implies $x = y = z = \frac{100}{3}$.

46. Minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to $g(x,y,z) = x + y + z = 12$ with $x > 0$, $y > 0$, $z > 0$. Then $\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 2y, 2z \rangle = \lambda \langle 1, 1, 1 \rangle \Rightarrow 2x = \lambda$, $2y = \lambda$, $2z = \lambda \Rightarrow x = y = z$, so $x + y + z = 12 \Rightarrow$ $3x = 12 \Rightarrow x = 4 = y = z$. By comparing nearby values we can confirm that this gives a minimum and not a maximum. Thus, the three numbers are 4, 4, and 4.

47. If the dimensions are $2x$, $2y$, and $2z$, then maximize $f(x,y,z) = (2x)(2y)(2z) = 8xyz$ subject to $g(x,y,z) = x^2 + y^2 + z^2 = r^2$ ($x > 0$, $y > 0$, $z > 0$). Then $\nabla f = \lambda \nabla g \Rightarrow \langle 8yz, 8xz, 8xy \rangle = \lambda \langle 2x, 2y, 2z \rangle \Rightarrow$ $8yz = 2\lambda x$, $8xz = 2\lambda y$, and $8xy = 2\lambda z$, so $\lambda = \dfrac{4yz}{x} = \dfrac{4xz}{y} = \dfrac{4xy}{z}$. This gives $x^2 z = y^2 z \Rightarrow x^2 = y^2$ (since $z \neq 0$) and $xy^2 = xz^2 \Rightarrow z^2 = y^2$, so $x^2 = y^2 = z^2 \Rightarrow x = y = z$, and substituting into the constraint equation gives $3x^2 = r^2 \Rightarrow x = r/\sqrt{3} = y = z$. Thus the largest volume of such a box is

$$f\left(\tfrac{r}{\sqrt{3}}, \tfrac{r}{\sqrt{3}}, \tfrac{r}{\sqrt{3}}\right) = 8\left(\tfrac{r}{\sqrt{3}}\right)\left(\tfrac{r}{\sqrt{3}}\right)\left(\tfrac{r}{\sqrt{3}}\right) = \frac{8}{3\sqrt{3}} r^3.$$

48. If the dimensions of the box are x, y, and z, then minimize $f(x,y,z) = 2xy + 2xz + 2yz$ subject to $g(x,y,z) = xyz = 1000$ ($x > 0$, $y > 0$, $z > 0$). Then $\nabla f = \lambda \nabla g \Rightarrow$

$\langle 2y+2z, 2x+2z, 2x+2y\rangle = \lambda\,\langle yz, xz, xy\rangle \;\Rightarrow\; 2y+2z=\lambda yz$, $2x+2z=\lambda xz$, $2x+2y=\lambda xy$. Solving for λ in each equation gives $\lambda = \dfrac{2}{z}+\dfrac{2}{y}=\dfrac{2}{z}+\dfrac{2}{x}=\dfrac{2}{y}+\dfrac{2}{x} \;\Rightarrow\; x=y=z$. From $xyz=1000$ we have $x^3=1000 \;\Rightarrow\; x=10$ and the dimensions of the box are $x=y=z=10$ cm.

49. Maximize $f(x,y,z)=xyz$ subject to $g(x,y,z)=x+2y+3z=6$. $\nabla f=\lambda\nabla g \;\Rightarrow\; \langle yz,xz,xy\rangle=\lambda\,\langle 1,2,3\rangle$. Then $\lambda=yz=\frac{1}{2}xz=\frac{1}{3}xy$ implies $x=2y$, $z=\frac{2}{3}y$. But $2y+2y+2y=6$ so $y=1$, $x=2$, $z=\frac{2}{3}$ and the volume is $V=\frac{4}{3}$.

50. Maximize $f(x,y,z)=xyz$ subject to $g(x,y,z)=xy+yz+xz=32$. $\nabla f=\lambda\nabla g \;\Rightarrow$ $\langle yz,xz,xy\rangle=\lambda\,\langle y+z,x+z,x+y\rangle$. Then $\lambda(y+z)=yz$ **(1)**, $\lambda(x+z)=xz$ **(2)**, and $\lambda(x+y)=xy$ **(3)**. And **(1)** minus **(2)** implies $\lambda(y-x)=z(y-x)$ so $x=y$ or $\lambda=z$. If $\lambda=z$, then **(1)** implies $z(y+z)=yz$ or $z=0$ which is false. Thus $x=y$. Similarly, **(2)** minus **(3)** implies $\lambda(z-y)=x(z-y)$ so $y=z$ or $\lambda=x$. As above, $\lambda\neq x$, so $x=y=z$ and $3x^2=32$ or $x=y=z=\frac{8}{\sqrt{6}}$ cm.

51. Maximize $f(x,y,z)=xyz$ subject to $g(x,y,z)=4(x+y+z)=c$. $\nabla f=\lambda\nabla g \;\Rightarrow\; \langle yz,xz,xy\rangle=\lambda\,\langle 4,4,4\rangle$. Then $yz=4\lambda$, $xz=4\lambda$, and $xy=4\lambda$. Multiplying by x, y, and z, respectively, gives us $xyz=4\lambda x=4\lambda y=4\lambda z$, so $x=y=z$. Substituting y and z for x in g gives us $4(3x)=c \;\Rightarrow\; x=y=z=\frac{1}{12}c$ are the dimensions of the cube giving the maximum volume.

52. Maximize $C(x,y,z)=5xy+2xz+2yz$ subject to $g(x,y,z)=xyz=V$. $\nabla C=\lambda\nabla g \;\Rightarrow$ $\langle 5y+2z,5x+2z,2x+2y\rangle=\lambda\,\langle yz,xz,xy\rangle$. Then $\lambda yz=5y+2z$ **(1)**, $\lambda xz=5x+2z$ **(2)**, $\lambda xy=2(x+y)$ **(3)**, and $xyz=V$ **(4)**. Now **(1)** $-$ **(2)** implies $\lambda z(y-x)=5(y-x)$, so $x=y$ or $\lambda=5/z$, but z can't be 0, so $x=y$. Then twice **(2)** minus five times **(3)** together with $x=y$ implies $\lambda y(2x-5y)=2(2z-5y)$ which gives $z=\frac{5}{2}y$ [again $\lambda\neq 2/y$ or else **(3)** implies $y=0$]. Hence $\frac{5}{2}y^3=V$ and the dimensions which minimize cost are $x=y=\sqrt[3]{\frac{2}{5}V}$ units, $z=V^{1/3}\left(\frac{5}{2}\right)^{2/3}$ units.

53. If the dimensions of the box are given by x, y, and z, then we need to find the maximum value of $f(x,y,z)=xyz$ $[x,y,z>0]$ subject to the constraint $L=\sqrt{x^2+y^2+z^2}$ or $g(x,y,z)=x^2+y^2+z^2=L^2$. $\nabla f=\lambda\nabla g \;\Rightarrow$ $\langle yz,xz,xy\rangle=\lambda\langle 2x,2y,2z\rangle$, so $yz=2\lambda x \;\Rightarrow\; \lambda=\dfrac{yz}{2x}$, $xz=2\lambda y \;\Rightarrow\; \lambda=\dfrac{xz}{2y}$, and $xy=2\lambda z \;\Rightarrow\; \lambda=\dfrac{xy}{2z}$. Thus $\lambda=\dfrac{yz}{2x}=\dfrac{xz}{2y} \;\Rightarrow\; x^2=y^2$ [since $z\neq 0$] $\Rightarrow\; x=y$ and $\lambda=\dfrac{yz}{2x}=\dfrac{xy}{2z} \;\Rightarrow\; x=z$ [since $y\neq 0$]. Substituting into the constraint equation gives $x^2+x^2+x^2=L^2 \;\Rightarrow\; x^2=L^2/3 \;\Rightarrow\; x=L/\sqrt{3}=y=z$ and the maximum volume is $\left(L/\sqrt{3}\right)^3=L^3/\left(3\sqrt{3}\right)$.

54. Let the dimensions of the box be x, y, and z. Then we wish to maximize $f(x,y,z) = xyz$ subject to $g(x,y,z) = 2x + 2y + z = 108 \Rightarrow \nabla f = \langle yz, xz, xy \rangle$ and $\lambda \nabla g = \langle 2\lambda, 2\lambda, \lambda \rangle$. Now $yz = 2\lambda$, $xz = 2\lambda$, and $xy = \lambda$, with $x \neq 0$, $y \neq 0$, $z \neq 0$. Then $yz = xz \Rightarrow y = x$ **(1)** and $2xy = xz \Rightarrow 2y = z$ **(2)**. Substituting **(1)** and **(2)** into the constraint, we get $2y + 2y + 2y = 108 \Rightarrow y = x = 18$, and hence $z = 36$. Thus, the dimensions of the box that will give the largest volume and still meet USPS guidelines are 18 in by 18 in by 36 in.

55. If r and h are the radius and the height of the silo, respectively, we need to maximize $V(r,h) = \pi r^2 h + \frac{2}{3}\pi r^3$ subject to $g(r,h) = 2\pi rh + \pi r^2 + (4\pi r^2)/2 = 2\pi rh + 3\pi r^2 = S$. Then $\nabla V = \lambda \nabla g \Rightarrow$ $\langle 2\pi rh + 2\pi r^2, \pi r^2 \rangle = \langle 2\lambda\pi h + 6\lambda\pi r, 2\lambda\pi r \rangle$, so the three equations are $2\pi rh + 2\pi r^2 = 2\lambda\pi h + 6\lambda\pi r$, $\pi r^2 = 2\lambda\pi r$, and $2\pi rh + 3\pi r^2 = S$. The second equation implies $r = 2\lambda$ $[r \neq 0]$. Substituting $r = 2\lambda$ into the first equation gives $2\pi(2\lambda)h + 2\pi(2\lambda)^2 = 2\lambda\pi h + 6\lambda\pi(2\lambda) \Rightarrow 4\pi\lambda h + 8\pi\lambda^2 = 2\lambda\pi h + 12\pi\lambda^2 \Rightarrow 2\pi\lambda h = 4\pi\lambda^2 \Rightarrow h = 2\lambda$. Thus, $r = 2\lambda = h$, and the volume of the silo is maximized, subject to a given surface area, when the radius and height are equal.

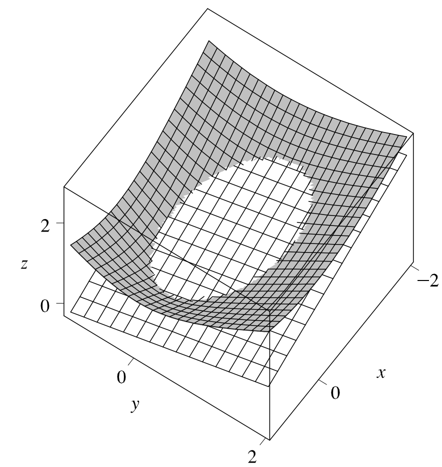
56. Let the dimensions of the box be x, y, and z, so its volume is $f(x,y,z) = xyz$, its surface area is $2xy + 2yz + 2xz = 1500$ and its total edge length is $4x + 4y + 4z = 200$. We find the extreme values of $f(x,y,z)$ subject to the constraints $g(x,y,z) = xy + yz + xz = 750$ and $h(x,y,z) = x + y + z = 50$. Then $\nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda(y+z), \lambda(x+z), \lambda(x+y) \rangle + \langle \mu, \mu, \mu \rangle$. So $yz = \lambda(y+z) + \mu$ **(1)**, $xz = \lambda(x+z) + \mu$ **(2)**, and $xy = \lambda(x+y) + \mu$ **(3)**. Notice that the box can't be a cube or else $x = y = z = \frac{50}{3}$ but then $xy + yz + xz = \frac{2500}{3} \neq 750$. Assume x is the distinct side, that is, $x \neq y$, $x \neq z$. Then **(1)** minus **(2)** implies $z(y - x) = \lambda(y - x)$ or $\lambda = z$, and **(1)** minus **(3)** implies $y(z - x) = \lambda(z - x)$ or $\lambda = y$. So $y = z = \lambda$ and $x + y + z = 50$ implies $x = 50 - 2\lambda$; also $xy + yz + xz = 750$ implies $x(2\lambda) + \lambda^2 = 750$. Hence $50 - 2\lambda = \dfrac{750 - \lambda^2}{2\lambda}$ or $3\lambda^2 - 100\lambda + 750 = 0$ and $\lambda = \dfrac{50 \pm 5\sqrt{10}}{3}$, giving the points $\left(\frac{1}{3}\left(50 \mp 10\sqrt{10}\right), \frac{1}{3}\left(50 \pm 5\sqrt{10}\right), \frac{1}{3}\left(50 \pm 5\sqrt{10}\right)\right)$.
Thus, the minimum of f is $f\left(\frac{1}{3}\left(50 - 10\sqrt{3}\right), \frac{1}{3}\left(50 + 5\sqrt{10}\right), \frac{1}{3}\left(50 + 5\sqrt{10}\right)\right) = \frac{1}{27}\left(87{,}500 - 2500\sqrt{10}\right)$, and its maximum is $f\left(\frac{1}{3}\left(50 + 10\sqrt{10}\right), \frac{1}{3}\left(50 - 5\sqrt{10}\right), \frac{1}{3}\left(50 - 5\sqrt{10}\right)\right) = \frac{1}{27}\left(87{,}500 + 2500\sqrt{10}\right)$.
Note: If either y or z is the distinct side, then symmetry gives the same result.

57. We need to find the extreme values of $f(x,y,z) = x^2 + y^2 + z^2$ subject to the two constraints $g(x,y,z) = x + y + 2z = 2$ and $h(x,y,z) = x^2 + y^2 - z = 0$. $\nabla f = \langle 2x, 2y, 2z \rangle$, $\lambda \nabla g = \langle \lambda, \lambda, 2\lambda \rangle$ and $\mu \nabla h = \langle 2\mu x, 2\mu y, -\mu \rangle$. Thus, we need $2x = \lambda + 2\mu x$ **(1)**, $2y = \lambda + 2\mu y$ **(2)**, $2z = 2\lambda - \mu$ **(3)**, $x + y + 2z = 2$ **(4)**, and $x^2 + y^2 - z = 0$ **(5)**.
From **(1)** and **(2)**, $2(x - y) = 2\mu(x - y)$, so if $x \neq y$, $\mu = 1$. Putting this in **(3)** gives $2z = 2\lambda - 1$ or $\lambda = z + \frac{1}{2}$, but putting $\mu = 1$ into **(1)** says $\lambda = 0$. Hence $z + \frac{1}{2} = 0$ or $z = -\frac{1}{2}$. Then **(4)** and **(5)** become $x + y - 3 = 0$ and $x^2 + y^2 + \frac{1}{2} = 0$. The

last equation cannot be true, so this case gives no solution. So we must have $x = y$. Then **(4)** and **(5)** become $2x + 2z = 2$ and $2x^2 - z = 0$ which imply $z = 1 - x$ and $z = 2x^2$. Thus $2x^2 = 1 - x$ or $2x^2 + x - 1 = (2x-1)(x+1) = 0$ so $x = \frac{1}{2}$ or $x = -1$. The two points to check are $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(-1, -1, 2)$: $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4}$ and $f(-1, -1, 2) = 6$. Thus, $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the point on the ellipse nearest the origin and $(-1, -1, 2)$ is the one farthest from the origin.

58. (a) After plotting $z = \sqrt{x^2 + y^2}$, the top half of the cone, and the plane $z = (5 - 4x + 3y)/8$ we see the ellipse formed by the intersection of the surfaces. The ellipse can be plotted explicitly using cylindrical coordinates (see Section 15.7): The cone is given by $z = r$, and the plane is $4r\cos\theta - 3r\sin\theta + 8z = 5$. Substituting $z = r$ into the plane equation gives $4r\cos\theta - 3r\sin\theta + 8r = 5 \quad \Rightarrow \quad r = \dfrac{5}{4\cos\theta - 3\sin\theta + 8}$.

Since $z = r$ on the ellipse, parametric equations (in cylindrical coordinates) are $\theta = t,\ r = z = \dfrac{5}{4\cos t - 3\sin t + 8},\ 0 \le t \le 2\pi$.



(b) We need to find the extreme values of $f(x, y, z) = z$ subject to the two constraints $g(x, y, z) = 4x - 3y + 8z = 5$ and $h(x, y, z) = x^2 + y^2 - z^2 = 0$.

$\nabla f = \lambda \nabla g + \mu \nabla h \quad \Rightarrow \quad \langle 0, 0, 1 \rangle = \lambda \langle 4, -3, 8 \rangle + \mu \langle 2x, 2y, -2z \rangle$, so we need $4\lambda + 2\mu x = 0 \quad \Rightarrow \quad x = -\frac{2\lambda}{\mu}$ **(1)**, $-3\lambda + 2\mu y = 0 \quad \Rightarrow \quad y = \frac{3\lambda}{2\mu}$ **(2)**, $8\lambda - 2\mu z = 1 \quad \Rightarrow \quad z = \frac{8\lambda - 1}{2\mu}$ **(3)**, $4x - 3y + 8z = 5$ **(4)**, and $x^2 + y^2 = z^2$ **(5)**. [Note that $\mu \neq 0$, else $\lambda = 0$ from **(1)**, but substitution into **(3)** gives a contradiction.]

Substituting **(1)**, **(2)**, and **(3)** into **(4)** gives $4\left(-\frac{2\lambda}{\mu}\right) - 3\left(\frac{3\lambda}{2\mu}\right) + 8\left(\frac{8\lambda - 1}{2\mu}\right) = 5 \quad \Rightarrow \quad \mu = \frac{39\lambda - 8}{10}$ and into **(5)** gives $\left(-\frac{2\lambda}{\mu}\right)^2 + \left(\frac{3\lambda}{2\mu}\right)^2 = \left(\frac{8\lambda - 1}{2\mu}\right)^2 \quad \Rightarrow \quad 16\lambda^2 + 9\lambda^2 = (8\lambda - 1)^2 \quad \Rightarrow \quad 39\lambda^2 - 16\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = \frac{1}{13}$ or $\lambda = \frac{1}{3}$.

If $\lambda = \frac{1}{13}$ then $\mu = -\frac{1}{2}$ and $x = \frac{4}{13}$, $y = -\frac{3}{13}$, $z = \frac{5}{13}$. If $\lambda = \frac{1}{3}$ then $\mu = \frac{1}{2}$ and $x = -\frac{4}{3}$, $y = 1$, $z = \frac{5}{3}$. Thus, the highest point on the ellipse is $\left(-\frac{4}{3}, 1, \frac{5}{3}\right)$ and the lowest point is $\left(\frac{4}{13}, -\frac{3}{13}, \frac{5}{13}\right)$.

59. $f(x, y, z) = ye^{x-z},\ g(x, y, z) = 9x^2 + 4y^2 + 36z^2 = 36,\ h(x, y, z) = xy + yz = 1$.

$\nabla f = \lambda \nabla g + \mu \nabla h \quad \Rightarrow \quad \left\langle ye^{x-z}, e^{x-z}, -ye^{x-z} \right\rangle = \lambda \langle 18x, 8y, 72z \rangle + \mu \langle y, x+z, y \rangle$, so $ye^{x-z} = 18\lambda x + \mu y$, $e^{x-z} = 8\lambda y + \mu(x+z)$, $-ye^{x-z} = 72\lambda z + \mu y$, $9x^2 + 4y^2 + 36z^2 = 36$, $xy + yz = 1$. Using a CAS to solve these 5 equations simultaneously for x, y, z, λ, and μ (in Maple, use the `allvalues` command), we get 4 real-valued solutions:

$x \approx 0.222444,\quad y \approx -2.157012,\quad z \approx -0.686049,\quad \lambda \approx -0.200401,\quad \mu \approx 2.108584$

$x \approx -1.951921,\quad y \approx -0.545867,\quad z \approx 0.119973,\quad \lambda \approx 0.003141,\quad \mu \approx -0.076238$

$x \approx 0.155142,\quad y \approx 0.904622,\quad z \approx 0.950293,\quad \lambda \approx -0.012447,\quad \mu \approx 0.489938$

$x \approx 1.138731,\quad y \approx 1.768057,\quad z \approx -0.573138,\quad \lambda \approx 0.317141,\quad \mu \approx 1.862675$

[continued]

Substituting these values into f gives $f(0.222444, -2.157012, -0.686049) \approx -5.3506$, $f(-1.951921, -0.545867, 0.119973) \approx -0.0688$, $f(0.155142, 0.904622, 0.950293) \approx 0.4084$, $f(1.138731, 1.768057, -0.573138) \approx 9.7938$. Thus, the maximum is approximately 9.7938, and the minimum is approximately -5.3506.

60. $f(x, y, z) = x + y + z,\ g(x, y, z) = x^2 - y^2 - z = 0,\ h(x, y, z) = x^2 + z^2 = 4.$

$\nabla f = \lambda \nabla g + \mu \nabla h \quad \Rightarrow \quad \langle 1, 1, 1 \rangle = \lambda \langle 2x, -2y, -1 \rangle + \mu \langle 2x, 0, 2z \rangle$, so $1 = 2\lambda x + 2\mu x$, $1 = -2\lambda y$, $1 = -\lambda + 2\mu z$, $x^2 - y^2 = z$, $x^2 + z^2 = 4$. Using a CAS to solve these 5 equations simultaneously for x, y, z, λ, and μ, we get 4 real-valued solutions:

$$x \approx -1.652878, \quad y \approx -1.964194, \quad z \approx -1.126052, \quad \lambda \approx 0.254557, \quad \mu \approx -0.557060$$
$$x \approx -1.502800, \quad y \approx 0.968872, \quad z \approx 1.319694, \quad \lambda \approx -0.516064, \quad \mu \approx 0.183352$$
$$x \approx -0.992513, \quad y \approx 1.649677, \quad z \approx -1.736352, \quad \lambda \approx -0.303090, \quad \mu \approx -0.200682$$
$$x \approx 1.895178, \quad y \approx 1.718347, \quad z \approx 0.638984, \quad \lambda \approx -0.290977, \quad \mu \approx 0.554805$$

Substituting these values into f gives $f(-1.652878, -1.964194, -1.126052) \approx -4.7431$, $f(-1.502800, 0.968872, 1.319694) \approx 0.7858$, $f(-0.992513, 1.649677, -1.736352) \approx -1.0792$, $f(1.895178, 1.718347, 0.638984) \approx 4.2525$. Thus, the maximum is approximately 4.2525, and the minimum is approximately -4.7431.

61. $f(x, y) = 3x^2 + y^2$, $g(x, y) = x^2 + y^2 - 4y = 0$. Then $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 6x, 2y \rangle = \langle 2\lambda x, \lambda(2y - 4) \rangle$, so the three equations are $6x = 2\lambda x$, $2y = \lambda(2y - 4)$, and $x^2 + y^2 - 4y = 0$. The first equation implies $x = 0$ or $\lambda = 3$. If $x = 0$, the third equation implies $y = 0$ or $y = 4$. If $\lambda = 3$, the second equation implies $y = 3$ and substitution into the third equation gives $x = \pm\sqrt{3}$. Thus, f has possible extreme values at $(0, 0)$, $(0, 4)$, $\left(\pm\sqrt{3}, 3\right)$. Evaluating f at these points we see that the maximum value is $f\left(\pm\sqrt{3}, 3\right) = 18$ and the minimum value is $f(0, 0) = 0$.

The minimum value of f occurs at $(0, 0)$. Substituting $y = 0$ into the second equation gives $2(0) = \lambda(2(0) - 4)$, which is true only if $\lambda = 0$. Thus, the minimum value corresponds to $\lambda = 0$.

62. (a) Let $f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum\limits_{i=1}^{n} x_i y_i$, $g(x_1, \ldots, x_n) = \sum\limits_{i=1}^{n} x_i^2$, and $h(x_1, \ldots, x_n) = \sum\limits_{i=1}^{n} y_i^2$. Then

$\nabla f = \nabla \sum\limits_{i=1}^{n} x_i y_i = \langle y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_n \rangle$, $\nabla g = \nabla \sum\limits_{i=1}^{n} x_i^2 = \langle 2x_1, 2x_2, \ldots, 2x_n, 0, 0, \ldots, 0 \rangle$ and

$\nabla h = \nabla \sum\limits_{i=1}^{n} y_i^2 = \langle 0, 0, \ldots, 0, 2y_1, 2y_2, \ldots, 2y_n \rangle$. So $\nabla f = \lambda \nabla g + \mu \nabla h \quad \Leftrightarrow \quad y_i = 2\lambda x_i$ and $x_i = 2\mu y_i$,

$1 \le i \le n$. Then $1 = \sum\limits_{i=1}^{n} y_i^2 = \sum\limits_{i=1}^{n} 4\lambda^2 x_i^2 = 4\lambda^2 \sum\limits_{i=1}^{n} x_i^2 = 4\lambda^2 \quad \Rightarrow \quad \lambda = \pm\frac{1}{2}$. If $\lambda = \frac{1}{2}$, then $y_i = 2\left(\frac{1}{2}\right) x_i = x_i$,

$1 \le i \le n$. Thus, $\sum\limits_{i=1}^{n} x_i y_i = \sum\limits_{i=1}^{n} x_i^2 = 1$. Similarly, if $\lambda = -\frac{1}{2}$, we get $y_i = -x_i$ and $\sum\limits_{i=1}^{n} x_i y_i = -1$. Similarly, we get

$\mu = \pm\frac{1}{2}$ giving $y_i = \pm x_i$, $1 \le i \le n$, and $\sum\limits_{i=1}^{n} x_i y_i = \pm 1$. Thus, the maximum value of $\sum\limits_{i=1}^{n} x_i y_i$ is 1.

(b) Here we assume $\sum\limits_{i=1}^{n} a_i^2 \neq 0$ and $\sum\limits_{i=1}^{n} b_i^2 \neq 0$. (If $\sum\limits_{i=1}^{n} a_i^2 = 0$, then each $a_i = 0$ and so the inequality is trivially true.)

$x_i = \dfrac{a_i}{\sqrt{\sum a_j^2}} \quad \Rightarrow \quad \sum x_i^2 = \dfrac{\sum a_i^2}{\sum a_j^2} = 1$, and $y_i = \dfrac{b_i}{\sqrt{\sum b_j^2}} \quad \Rightarrow \quad \sum y_i^2 = \dfrac{\sum b_i^2}{\sum b_j^2} = 1$. Therefore, from part (a),

$\sum x_i y_i = \sum \dfrac{a_i b_i}{\sqrt{\sum a_j^2}\sqrt{\sum b_j^2}} \leq 1 \quad \Leftrightarrow \quad \sum a_i b_i \leq \sqrt{\sum a_j^2}\sqrt{\sum b_j^2}.$

63. (a) We wish to maximize $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$ subject to

$g(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n = c$ and $x_i > 0$.

$\nabla f = \left\langle \frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_2 \cdots x_n),\, \frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 x_3 \cdots x_n),\, \ldots,\, \frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 \cdots x_{n-1}) \right\rangle$

and $\lambda \nabla g = \langle \lambda, \lambda, \ldots, \lambda \rangle$, so we need to solve the system of equations

$$\begin{aligned}
\tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_2 \cdots x_n) = \lambda \quad &\Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_1 \\
\tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 x_3 \cdots x_n) = \lambda \quad &\Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_2 \\
&\vdots \\
\tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 \cdots x_{n-1}) = \lambda \quad &\Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_n
\end{aligned}$$

This implies $n\lambda x_1 = n\lambda x_2 = \cdots = n\lambda x_n$. Note $\lambda \neq 0$, otherwise we can't have all $x_i > 0$. Thus, $x_1 = x_2 = \cdots = x_n$.

But $x_1 + x_2 + \cdots + x_n = c \quad \Rightarrow \quad nx_1 = c \quad \Rightarrow \quad x_1 = \dfrac{c}{n} = x_2 = x_3 = \cdots = x_n$. Then the only point where f can have an extreme value is $\left(\dfrac{c}{n}, \dfrac{c}{n}, \ldots, \dfrac{c}{n}\right)$. Since we can choose values for $(x_1, x_2, \ldots, x_n)$ that make f as close to zero (but not equal) as we like, f has no minimum value. Thus, the maximum value is

$f\left(\dfrac{c}{n}, \dfrac{c}{n}, \ldots, \dfrac{c}{n}\right) = \sqrt[n]{\dfrac{c}{n} \cdot \dfrac{c}{n} \cdot \cdots \cdot \dfrac{c}{n}} = \dfrac{c}{n}.$

(b) From part (a), $\dfrac{c}{n}$ is the maximum value of f. Thus, $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n} \leq \dfrac{c}{n}$. But

$x_1 + x_2 + \cdots + x_n = c$, so $\sqrt[n]{x_1 x_2 \cdots x_n} \leq \dfrac{x_1 + x_2 + \cdots + x_n}{n}$. These two means are equal when f attains its maximum value $\dfrac{c}{n}$, but this can occur only at the point $\left(\dfrac{c}{n}, \dfrac{c}{n}, \ldots, \dfrac{c}{n}\right)$ we found in part (a). So the means are equal only when $x_1 = x_2 = x_3 = \cdots = x_n = \dfrac{c}{n}$.

APPLIED PROJECT Rocket Science

1. Initially the rocket engine has mass $M_r = M_1$ and payload mass $P = M_2 + M_3 + A$. Then the change in velocity resulting from the first stage is $\Delta V_1 = -c \ln\left(1 - \dfrac{(1-S)M_1}{M_2 + M_3 + A + M_1}\right)$. After the first stage is jettisoned we can consider the rocket engine to have mass $M_r = M_2$ and the payload to have mass $P = M_3 + A$. The resulting change in velocity from the second stage is $\Delta V_2 = -c\ln\left(1 - \dfrac{(1-S)M_2}{M_3 + A + M_2}\right)$. When only the third stage remains, we have $M_r = M_3$ and $P = A$, so the resulting change in velocity is $\Delta V_3 = -c\ln\left(1 - \dfrac{(1-S)M_3}{A + M_3}\right)$. Since the rocket started from rest, the final velocity attained is

$$\begin{aligned}
v_f &= \Delta V_1 + \Delta V_2 + \Delta V_3 \\
&= -c\ln\left(1 - \frac{(1-S)M_1}{M_2 + M_3 + A + M_1}\right) + (-c)\ln\left(1 - \frac{(1-S)M_2}{M_3 + A + M_2}\right) + (-c)\ln\left(1 - \frac{(1-S)M_3}{A + M_3}\right) \\
&= -c\left[\ln\left(\frac{M_1 + M_2 + M_3 + A - (1-S)M_1}{M_1 + M_2 + M_3 + A}\right) + \ln\left(\frac{M_2 + M_3 + A - (1-S)M_2}{M_2 + M_3 + A}\right)\right. \\
&\qquad\qquad\left. + \ln\left(\frac{M_3 + A - (1-S)M_3}{M_3 + A}\right)\right] \\
&= c\left[\ln\left(\frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}\right) + \ln\left(\frac{M_2 + M_3 + A}{SM_2 + M_3 + A}\right) + \ln\left(\frac{M_3 + A}{SM_3 + A}\right)\right]
\end{aligned}$$

2. Define $N_1 = \dfrac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}$, $N_2 = \dfrac{M_2 + M_3 + A}{SM_2 + M_3 + A}$, and $N_3 = \dfrac{M_3 + A}{SM_3 + A}$. Then

$$\begin{aligned}
\frac{(1-S)N_1}{1 - SN_1} &= \frac{(1-S)\,\dfrac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}}{1 - S\,\dfrac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}} = \frac{(1-S)(M_1 + M_2 + M_3 + A)}{SM_1 + M_2 + M_3 + A - S(M_1 + M_2 + M_3 + A)} \\
&= \frac{(1-S)(M_1 + M_2 + M_3 + A)}{(1-S)(M_2 + M_3 + A)} = \frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A}
\end{aligned}$$

as desired.

Similarly,

$$\frac{(1-S)N_2}{1 - SN_2} = \frac{(1-S)(M_2 + M_3 + A)}{SM_2 + M_3 + A - S(M_2 + M_3 + A)} = \frac{(1-S)(M_2 + M_3 + A)}{(1-S)(M_3 + A)} = \frac{M_2 + M_3 + A}{M_3 + A}$$

and
$$\frac{(1-S)N_3}{1 - SN_3} = \frac{(1-S)(M_3 + A)}{SM_3 + A - S(M_3 + A)} = \frac{(1-S)(M_3 + A)}{(1-S)(A)} = \frac{M_3 + A}{A}$$

Then

$$\begin{aligned}
\frac{M + A}{A} &= \frac{M_1 + M_2 + M_3 + A}{A} = \frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A} \cdot \frac{M_2 + M_3 + A}{M_3 + A} \cdot \frac{M_3 + A}{A} \\
&= \frac{(1-S)N_1}{1 - SN_1} \cdot \frac{(1-S)N_2}{1 - SN_2} \cdot \frac{(1-S)N_3}{1 - SN_3} = \frac{(1-S)^3 N_1 N_2 N_3}{(1 - SN_1)(1 - SN_2)(1 - SN_3)}
\end{aligned}$$

3. Since $A > 0$, $M + A$ and consequently $\dfrac{M + A}{A}$ is minimized for the same values as M. $\ln x$ is a strictly increasing function, so $\ln\left(\dfrac{M + A}{A}\right)$ must give a minimum for the same values as $\dfrac{M + A}{A}$ and hence M. We then wish to minimize $\ln\left(\dfrac{M + A}{A}\right)$ subject to the constraint $c\,(\ln N_1 + \ln N_2 + \ln N_3) = v_f$. From Problem 2,

$$\ln\left(\frac{M+A}{A}\right) = \ln\left(\frac{(1-S)^3\, N_1N_2N_3}{(1-SN_1)\,(1-SN_2)\,(1-SN_3)}\right)$$
$$= 3\ln(1-S) + \ln N_1 + \ln N_2 + \ln N_3 - \ln(1-SN_1) - \ln(1-SN_2) - \ln(1-SN_3)$$

Using the method of Lagrange multipliers, we need to solve $\nabla\left[\ln\left(\dfrac{M+A}{A}\right)\right] = \lambda\nabla[c(\ln N_1 + \ln N_2 + \ln N_3)]$ with $c(\ln N_1 + \ln N_2 + \ln N_3) = v_f$ in terms of N_1, N_2, and N_3. The resulting system is

$$\frac{1}{N_1} + \frac{S}{1-SN_1} = \lambda\,\frac{c}{N_1} \qquad \frac{1}{N_2} + \frac{S}{1-SN_2} = \lambda\,\frac{c}{N_2} \qquad \frac{1}{N_3} + \frac{S}{1-SN_3} = \lambda\,\frac{c}{N_3}$$

$$c\,(\ln N_1 + \ln N_2 + \ln N_3) = v_f$$

One approach to solving the system is isolating $c\lambda$ in the first three equations which gives

$$1 + \frac{SN_1}{1-SN_1} = c\lambda = 1 + \frac{SN_2}{1-SN_2} = 1 + \frac{SN_3}{1-SN_3} \quad\Rightarrow\quad \frac{N_1}{1-SN_1} = \frac{N_2}{1-SN_2} = \frac{N_3}{1-SN_3} \quad\Rightarrow$$

$N_1 = N_2 = N_3$ (Verify!). This says the fourth equation can be expressed as $c(\ln N_1 + \ln N_1 + \ln N_1) = v_f \quad\Rightarrow$ $3c\ln N_1 = v_f \quad\Rightarrow\quad \ln N_1 = \dfrac{v_f}{3c}$. Thus, the minimum mass M of the rocket engine is attained for $N_1 = N_2 = N_3 = e^{v_f/(3c)}$.

4. Using the previous results, $\dfrac{M+A}{A} = \dfrac{(1-S)^3N_1N_2N_3}{(1-SN_1)(1-SN_2)(1-SN_3)} = \dfrac{(1-S)^3\left[e^{v_f/(3c)}\right]^3}{\left[1-Se^{v_f/(3c)}\right]^3} = \dfrac{(1-S)^3e^{v_f/c}}{\left[1-Se^{v_f/(3c)}\right]^3}$.

Then $M = \dfrac{A(1-S)^3e^{v_f/c}}{\left[1-Se^{v_f/(3c)}\right]^3} - A$.

5. (a) From Problem 4, $M = \dfrac{A(1-0.2)^3e^{(28{,}000/9600)}}{\left(1-0.2e^{[28{,}000/(3\cdot 9600)]}\right)^3} - A \approx 90.4A - A = 89.4A$.

(b) First, $N_3 = \dfrac{M_3+A}{SM_3+A} \quad\Rightarrow\quad e^{[28{,}000/(3\cdot 9600)]} = \dfrac{M_3+A}{0.2M_3+A} \quad\Rightarrow\quad M_3 = \dfrac{A(1-e^{35/36})}{0.2e^{35/36}-1} \approx 3.49A$.

Then $N_2 = \dfrac{M_2+M_3+A}{SM_2+M_3+A} = \dfrac{M_2+3.49A+A}{0.2M_2+3.49A+A} \quad\Rightarrow\quad M_2 = \dfrac{4.49A(1-e^{35/36})}{0.2e^{35/36}-1} \approx 15.67A$ and

$$N_3 = \frac{M_1+M_2+M_3+A}{SM_1+M_2+M_3+A} = \frac{M_1+15.67A+3.49A+A}{0.2M_1+15.67A+3.49A+A} \quad\Rightarrow\quad M_1 = \frac{20.16A(1-e^{35/36})}{0.2e^{35/36}-1} \approx 70.36A.$$

6. As in Problem 5, $N_3 = \dfrac{M_3+A}{SM_3+A} \quad\Rightarrow\quad e^{39{,}700/(3\cdot 9600)} = \dfrac{M_3+A}{0.2M_3+A} \quad\Rightarrow\quad M_3 = \dfrac{A(1-e^{397/288})}{0.2e^{397/288}-1} \approx 14.4A$,

$$N_2 = \frac{M_2+M_3+A}{SM_2+M_3+A} = \frac{M_2+14.4A+A}{0.2M_2+14.4A+A} \quad\Rightarrow\quad M_2 = \frac{15.4A(1-e^{397/288})}{0.2e^{247/180}-1} \approx 222A, \text{ and}$$

$$N_3 = \frac{M_1+M_2+M_3+A}{SM_1+M_2+M_3+A} = \frac{M_1+222A+14.4A+A}{0.2M_1+222A+14.4A+A} \quad\Rightarrow\quad M_1 = \frac{237.4A(1-e^{397/288})}{0.2e^{397/288}-1} \approx 3417A.$$

Here $A = 200$, so the mass of each stage of the rocket engine is approximately $M_1 = 3417(200) = 683{,}400$ kg, $M_2 = 222(200) = 44{,}000$ kg, and $M_3 = 14.4(200) = 288$ kg.

APPLIED PROJECT Hydro-Turbine Optimization

1. We wish to maximize the total energy production for a given total flow, so we can say Q_T is fixed and we want to maximize $KW_1 + KW_2 + KW_3$. Notice each KW_i has a constant factor $(170 - 0.002Q_T^2)$, so to simplify the computations we can equivalently maximize

$$\begin{aligned} f(Q_1, Q_2, Q_3) &= \frac{KW_1 + KW_2 + KW_3}{170 - 0.002Q_T^2} \\ &= (-18.89 + 4.5097Q_1 - 0.0509Q_1^2) + (-24.51 + 4.7957Q_2 - 0.0585Q_2^2) \\ &\quad +(-27.02 + 4.8734Q_3 - 0.0479Q_3^2) \end{aligned}$$

subject to constraint $g(Q_1, Q_2, Q_3) = Q_1 + Q_2 + Q_3 = Q_T$. So first we find the values of Q_1, Q_2, Q_3 where $\nabla f(Q_1, Q_2, Q_3) = \lambda \nabla g(Q_1, Q_2, Q_3)$ and $Q_1 + Q_2 + Q_3 = Q_T$, which is equivalent to solving the system

$$\begin{aligned} 4.5097 - 2 \cdot (0.0509)Q_1 &= \lambda \\ 4.7957 - 2 \cdot (0.0585)Q_2 &= \lambda \\ 4.8734 - 2 \cdot (0.0479)Q_3 &= \lambda \\ Q_1 + Q_2 + Q_3 &= Q_T \end{aligned}$$

Comparing the first and third equation, we have

$$\begin{aligned} 4.5097 - 2 \cdot (0.0509)Q_1 &= 4.8734 - 2 \cdot (0.0479)Q_3 \implies Q_1 = -3.5727 + 0.9411Q_3 \\ 4.7957 - 2 \cdot (0.0585)Q_2 &= 4.8734 - 2 \cdot (0.0479)Q_3 \implies Q_2 = -0.6641 + 0.8188Q_3 \end{aligned}$$

Substituting into $Q_1 + Q_2 + Q_3 = Q_T$ gives So combine above results, we could get

$$(-3.5727 + 0.9411Q_3) + (-0.6641 + 0.8188Q_3) + Q_3 = Q_T \implies Q_3 = 0.3623Q_T + 1.5351$$

Then we could get

$$\begin{aligned} Q_1 &= 0.3410Q_T - 2.1280 \\ Q_2 &= 0\ 2967Q_T + 0.5928 \end{aligned}$$

As long as we maintain $7 \leq Q_1 \leq 31$, $7 \leq Q_2 \leq 31$, and $7 \leq Q_3 \leq 35$, we can reason from the nature of the functions KW_i that these values give a maximum of f, hence a maximum energy production, not a minimum.

2. From Problem 1, the value of Q_1 that maximizes energy production is $Q_1 = 0.3410Q_T - 2.128$, but since $7 \leq Q_1 \leq 31$, we must have

$$\begin{aligned} 7 \leq 0.3410Q_T - 2.128 \leq 31 &\implies 9.128 \leq 0.3410Q_T \leq 33.128 \\ &\implies 26.7683 \leq Q_T \leq 97.1496 \end{aligned}$$

Similarly, since $7 \leq Q_2 \leq 31$, we must have

$$\begin{aligned} 7 \leq 0.2967Q_T + 0.5928 \leq 31 &\implies 6.4072 \leq 0.2967Q_T \leq 30.4072 \\ &\implies 21.5949 \leq Q_T \leq 102.4847 \end{aligned}$$

and since $7 \leq Q_3 \leq 35$, we must have

$$\begin{aligned} 7 \leq 0.3623Q_T + 1.5351 \leq 35 &\implies 5.4649 \leq 0.3623Q_T \leq 33.4649 \\ &\implies 15.0839 \leq Q_T \leq 92.3679 \end{aligned}$$

Consolidating these results, we see that the values from Problem 1 are applicable only for $26.7683 \leq Q_T \leq 92.3679$

3. If $Q_T = 70$, the results from Problem 1 show that the maximum energy production occurs for

$$\begin{aligned} Q_1 &= 0.3410Q_T - 2.1280 = 0.3410 \cdot (70) - 2.1280 = 21.7420 \\ Q_2 &= 0.2967Q_T + 0.5928 = 0.2967 \cdot (70) + 0.5928 = 21.3618 \\ Q_3 &= 0.3623Q_T + 1.5351 = 0.3623 \cdot (70) + 1.5351 = 26.8961 \end{aligned}$$

The energy produced for these values is $KW_1 + KW_2 + KW_3 \approx 8826.8159 + 8208.5974 + 11{,}118.6144 \approx 28{,}154.0277$. We compute the energy production for a nearby distribution, $Q_1 = 21, Q_2 = 21$, and $Q_3 = 28$:

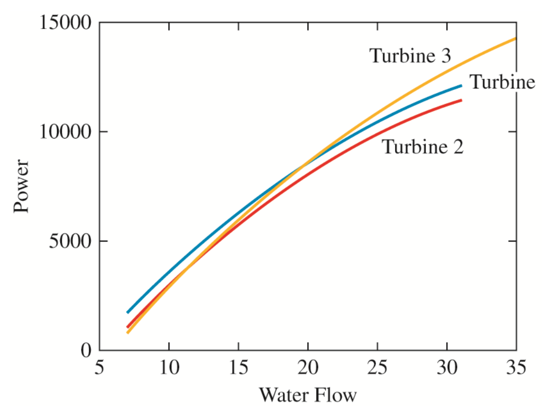
$$KW_1 + KW_2 + KW_3 \approx 28{,}139.0659$$

For another example, we take $Q_1 = 22, Q_2 = 22$, and $Q_3 = 26$:

$$KW_1 + KW_2 + KW_3 \approx 28{,}143.4874$$

These distributions are both close to the distribution from Problem 1 and both give slightly lower energy productions, suggesting that $Q_1 = 21.7420, Q_2 = 21.3618$ and $Q_3 = 26.8961$

4. First, we graph each power function in its domain if all of the flow is directed to that turbine (so $Q_i = Q_T$). If we use only one turbine, the graph indicates that for a water flow of 30 m^3/s, Turbine 3 produces the most power, approximately 12,795 kW. In comparison if we use all three turbines, the results of problem 1 with $Q_T = 30$ give $Q_1 = 8.102, Q_2 = 9.4938$ and $Q_3 = 12.4041$, resulting in a total energy production of $KW_1 + KW_2 + KW_3 \approx 9438.2465$ kW. Here, using only one turbine produces significantly more energy! If the flow is only 17 m^3/s, we do not have the option of using all three turbines, as the domain restrictions require a minimum of 7 m^3/s in each turbine. We can use just one turbine, then, and from the graph Turbine 1 produces the most energy for a water flow of 17 m^3/s.



5. If we examine the graph from Problem 4, we see that for water flows above approximately 13 m^3/s, Turbine 2 produces the least amount of power. Therefore, it seems reasonable to assume that we should distribute the incoming flow of 40 m^3/s between Turbines 1 and 3. (This can be verified by computing the power produced with the other pairs of turbines for comparison.) So now we wish to maximize $KW_1 + KW_3$ subject to the constraint $Q_1 + Q_3 = Q_T$ where $Q_T = 40$.

As in Problem 1, we can equivalently maximize

$$\begin{aligned} f(Q_1, Q_2) &= \frac{KW_1 + KW_3}{170 - 0.002Q_T^2} \\ &= (-18.89 + 4.5097Q_1 - 0.0509Q_1^2) + (-27.02 + 4.8734Q_3 - 0.0479Q_3^2) \end{aligned}$$

subject to the constraint $g(Q_1, Q_3) = Q_1 + Q_3 = Q_T$.

Then we solve $\nabla f(Q_1, Q_3) = \lambda \nabla g(Q_1, Q_3) \implies 4.5097 - 2(0.0509)Q_1 = \lambda$ and $4.8734 - 2(0.0479)Q_3 = \lambda$, thus $4.5097 - 2(0.0509)Q_1 = 4.8734 - 2(0.0479)Q_3 \implies Q_1 = -3.5727 + 0.9411Q_3$. Substituting into $Q_1 + Q_3 = Q_T$

gives $-3.5727 + 0.9411Q_3 + Q_3 = 40 \implies Q_3 = 22.4474$, and then $Q_1 = Q_T - Q_3 = 17.5526$. So we should apportion approximately $17.5526\ \mathrm{m}^3/\mathrm{s}$ to Turbine 1 and the remaining $22.4474\ \mathrm{m}^3/\mathrm{s}$ to Turbine 3. The resulting energy production is $KW_1 + KW_3 \approx 7436.7759 + 9714.2704 = 17{,}151.0463$ kW. (We can verify that this is indeed a maximum energy production by checking nearby distributions.) In comparison, if we use all three turbines with $Q_T = 40$, we get $Q_1 = 11.5120, Q_2 = 12.4608$ and $Q_3 = 16.0271$, resulting in a total energy production of $KW_1 + KW_2 + KW_3 \approx$ 15,216.7394 kW. Clearly, for this flow level it is beneficial to use only two turbines.

6. Note that an incoming flow of $96\ \mathrm{m}^3/\mathrm{s}$ is not within the domain we established in Problem 2, so we cannot simply use our previous work to give the optimal distribution. We will need to use all three turbines, due to the capacity limitations of each individual turbine, but $96\,\mathrm{m}^3/\mathrm{s}$ is less than the maximum combined capacity of $97\ \mathrm{m}^3/\mathrm{s}$, so we still must decide how to distribute the flows. From the graph in Problem 4, Turbine 3 produces the most power for the higher flows, so it seems reasonable to use Turbine 3 at its maximum capacity of $35\,\mathrm{m}^3/\mathrm{s}$ and distribute the remaining $61\ \mathrm{m}^3/\mathrm{s}$ flow between Turbine 1 and 2. We can again use the technique of Lagrange multipliers to determine the optimal distribution. Following the procedure we used in Problem 5, we wish to maximize $KW_1 + KW_2$ subject to the constraint $Q_1 + Q_2 = Q_T$ where $Q_T = 61$. We can equivalently maximize

$$\begin{aligned} f(Q_1, Q_2) &= \frac{KW_1 + KW_2}{170 - 0.002Q_T^2} \\ &= (-18.89 + 4.5097Q_1 - 0.0509Q_1^2) + (-24.51 + 4.7957Q_2 - 0.0585Q_2^2) \end{aligned}$$

subject to the constraint $g(Q_1, Q_2) = Q_1 + Q_2 = Q_T$. Then we solve $\nabla f(Q_1, Q_2) = \lambda \nabla g(Q_1, Q_2) \implies$ $4.5097 - 2(0.0509)Q_1 = \lambda$ and $4.7957 - 2(0.0585)Q_2 = \lambda$, thus

$$4.5097 - 2(0.0509)Q_1 = 4.7957 - 2(0.0585)Q_2 \implies Q_1 = -2.8094 + 1.1493Q_2$$

Substituting into $Q_1 + Q_2 = Q_T$ gives $-2.8094 + 1.1493Q_2 + Q_2 = 61 \implies Q_2 \approx 29.6885$, and then $Q_1 \approx 31.3115$. This value for Q_1 is larger than the allowable maximum flow to Turbine 1, but the result indicates that the flow to Turbine 1 should be maximized. Thus, we should recommend that the company apportion the maximum allowable flows to Turbines 1 and 3, $31\ \mathrm{m}^3/\mathrm{s}$ and $35\ \mathrm{m}^3/\mathrm{s}$, and the remaining $30\ \mathrm{m}^3/\mathrm{s}$ to Turbine 2. Checking nearby distributions within the domain verifies that we have indeed found the optimal distribution.

14 Review

TRUE-FALSE QUIZ

1. True. $f_y(a,b) = \lim\limits_{h \to 0} \dfrac{f(a, b+h) - f(a,b)}{h}$ from Equation 14.3.3. Let $h = y - b$. As $h \to 0$, $y \to b$. Then by substituting, we get $f_y(a,b) = \lim\limits_{y \to b} \dfrac{f(a,y) - f(a,b)}{y - b}$.

2. False. If there were such a function, then $f_{xy} = 2y$ and $f_{yx} = 1$. So $f_{xy} \neq f_{yx}$, which contradicts Clairaut's Theorem.

3. False. $f_{xy} = \dfrac{\partial^2 f}{\partial y\, \partial x}$.

4. True. From Equation 14.6.14 we get $D_{\mathbf{k}}\, f(x,y,z) = \nabla f(x,y,z) \cdot \langle 0,0,1 \rangle = f_z(x,y,z)$.

5. False. See Example 14.2.3.

6. False. See Exercise 14.4.54(a).

7. True. If f has a local minimum and f is differentiable at (a,b) then by Theorem 14.7.2, $f_x(a,b)=0$ and $f_y(a,b)=0$, so $\nabla f(a,b)=\langle f_x(a,b),f_y(a,b)\rangle=\langle 0,0\rangle=\mathbf{0}$.

8. False. If f is not continuous at $(2,5)$, then we can have $\lim\limits_{(x,y)\to(2,5)} f(x,y)\neq f(2,5)$. (See Example 14.2.8.)

9. False. $\nabla f(x,y)=\langle 0,1/y\rangle$.

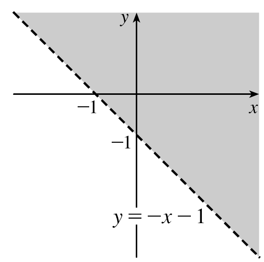
10. True. This is equivalent to part (c) of the Second Derivatives Test (14.7.3).

11. True. $\nabla f=\langle \cos x,\cos y\rangle$, so $|\nabla f|=\sqrt{\cos^2 x+\cos^2 y}$. But $|\cos\theta|\leq 1$, so $|\nabla f|\leq\sqrt{2}$. Now $D_{\mathbf{u}}\,f(x,y)=\nabla f\cdot\mathbf{u}=|\nabla f|\,|\mathbf{u}|\cos\theta$, but $\mathbf{u}$ is a unit vector, so $|D_{\mathbf{u}}\,f(x,y)|\leq\sqrt{2}\cdot 1\cdot 1=\sqrt{2}$.

12. False. See Exercise 14.7.41.

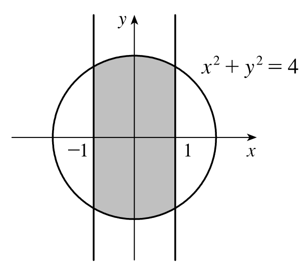
EXERCISES

1. $f(x,y)=\ln(x+y+1)$ is defined only when $x+y+1>0 \quad\Leftrightarrow\quad y>-x-1$, so the domain of f is $\{(x,y)\mid y>-x-1\}$, all those points above the line $y=-x-1$.

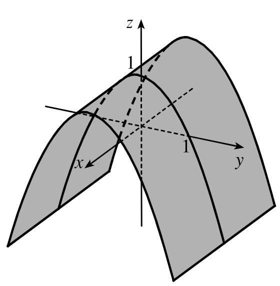


2. $f(x,y)=\sqrt{4-x^2-y^2}+\sqrt{1-x^2}$.

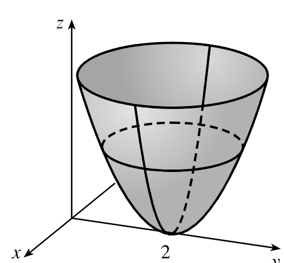
$\sqrt{4-x^2-y^2}$ is defined only when $4-x^2-y^2\geq 0 \quad\Leftrightarrow\quad x^2+y^2\leq 4$, and $\sqrt{1-x^2}$ is defined only when $1-x^2\geq 0 \quad\Leftrightarrow\quad -1\leq x\leq 1$, so the domain of f is $\left\{(x,y)\mid -1\leq x\leq 1, -\sqrt{4-x^2}\leq y\leq\sqrt{4-x^2}\,\right\}$, which consists of those points on or inside the circle $x^2+y^2=4$ for $-1\leq x\leq 1$.



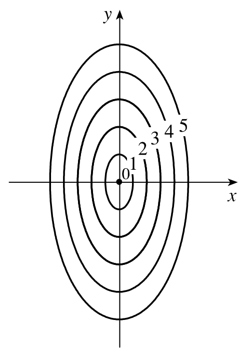
3. $z=f(x,y)=1-y^2$, a parabolic cylinder



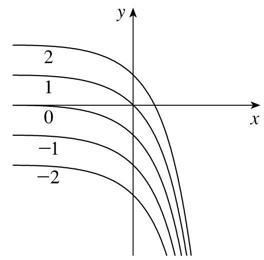
4. $z=f(x,y)=x^2+(y-2)^2$, a circular paraboloid with vertex $(0,2,0)$ and axis parallel to the z-axis



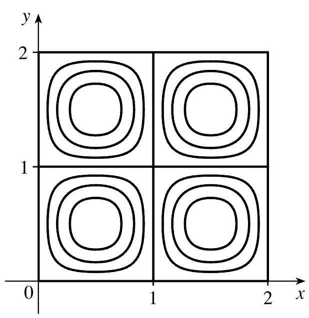
5. The level curves are $\sqrt{4x^2 + y^2} = k$ or $4x^2 + y^2 = k^2$, $k \geq 0$, a family of ellipses.



6. The level curves are $e^x + y = k$ or $y = -e^x + k$, a family of exponential curves.



7.



8. (a) The point $(3, 2)$ lies partway between the level curves with z-values 50 and 60, and it appears that $(3, 2)$ is about the same distance from either level curve. So we estimate that $f(3, 2) \approx 55$.

(b) At the point $(3, 2)$, if we fix y at $y = 2$ and allow x to vary, the level curves indicate that the z-values decrease as x increases, so $f_x(3, 2)$ is negative. In other words, if we start at $(3, 2)$ and move right (in the positive x-direction), the contours show that our path along the surface $z = f(x, y)$ is descending.

(c) Both $f_y(2, 1)$ and $f_y(2, 2)$ are positive, because if we start from either point and move in the positive y-direction, the contour map indicates that the path is ascending. But the level curves are closer together in the y-direction at $(2, 1)$ than at $(2, 2)$, so the path is steeper (the z-values increase more rapidly) at $(2, 1)$ and hence $f_y(2, 1) > f_y(2, 2)$.

9. f is a rational function, so it is continuous on its domain. Since f is defined at $(1, 1)$, we use direct substitution to evaluate the limit: $\displaystyle\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2 + 2y^2} = \frac{2(1)(1)}{1^2 + 2(1)^2} = \frac{2}{3}$.

10. $f(x, y) = \dfrac{2xy}{x^2 + 2y^2}$. As $(x, y) \to (0, 0)$ along the x-axis, $f(x, 0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x, y) \to 0$ along this line. As $(x, y) \to (0, 0)$ along the line $x = y$, $f(x, x) = 2x^2/(3x^2) = \frac{2}{3}$, so $f(x, y) \to \frac{2}{3}$. Thus, the limit doesn't exist.

11. (a) $T_x(6, 4) = \displaystyle\lim_{h\to 0} \frac{T(6 + h, 4) - T(6, 4)}{h}$, so we can approximate $T_x(6, 4)$ by considering $h = \pm 2$ and using the values given in the table: $T_x(6, 4) \approx \dfrac{T(8, 4) - T(6, 4)}{2} = \dfrac{86 - 80}{2} = 3$,

$T_x(6,4) \approx \dfrac{T(4,4)-T(6,4)}{-2} = \dfrac{72-80}{-2} = 4$. Averaging these values, we estimate $T_x(6,4)$ to be approximately $3.5°\text{C/m}$. Similarly, $T_y(6,4) = \lim\limits_{h\to 0} \dfrac{T(6,4+h)-T(6,4)}{h}$, which we can approximate with $h = \pm 2$:

$T_y(6,4) \approx \dfrac{T(6,6)-T(6,4)}{2} = \dfrac{75-80}{2} = -2.5$, $T_y(6,4) \approx \dfrac{T(6,2)-T(6,4)}{-2} = \dfrac{87-80}{-2} = -3.5$. Averaging these values, we estimate $T_y(6,4)$ to be approximately $-3.0°\text{C/m}$.

(b) Here $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, so by Equation 14.6.9, $D_{\mathbf{u}}\, T(6,4) = \nabla T(6,4) \cdot \mathbf{u} = T_x(6,4)\, \frac{1}{\sqrt{2}} + T_y(6,4)\, \frac{1}{\sqrt{2}}$. Using our estimates from part (a), we have $D_{\mathbf{u}}\, T(6,4) \approx (3.5)\, \frac{1}{\sqrt{2}} + (-3.0)\, \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} \approx 0.35$. This means that as we move through the point $(6,4)$ in the direction of $\mathbf{u}$, the temperature increases at a rate of approximately $0.35°\text{C/m}$.

Alternatively, we can use Definition 14.6.2: $D_{\mathbf{u}}\, T(6,4) = \lim\limits_{h\to 0} \dfrac{T\left(6 + h\,\frac{1}{\sqrt{2}}, 4 + h\,\frac{1}{\sqrt{2}}\right) - T(6,4)}{h}$,

which we can estimate with $h = \pm 2\sqrt{2}$. Then $D_{\mathbf{u}}\, T(6,4) \approx \dfrac{T(8,6)-T(6,4)}{2\sqrt{2}} = \dfrac{80-80}{2\sqrt{2}} = 0$,

$D_{\mathbf{u}}\, T(6,4) \approx \dfrac{T(4,2)-T(6,4)}{-2\sqrt{2}} = \dfrac{74-80}{-2\sqrt{2}} = \dfrac{3}{\sqrt{2}}$. Averaging these values, we have $D_{\mathbf{u}}\, T(6,4) \approx \frac{3}{2\sqrt{2}} \approx 1.1°\text{C/m}$.

(c) $T_{xy}(x,y) = \dfrac{\partial}{\partial y}\,[T_x(x,y)] = \lim\limits_{h\to 0} \dfrac{T_x(x,y+h) - T_x(x,y)}{h}$, so $T_{xy}(6,4) = \lim\limits_{h\to 0} \dfrac{T_x(6,4+h) - T_x(6,4)}{h}$ which we can estimate with $h = \pm 2$. We have $T_x(6,4) \approx 3.5$ from part (a), but we will also need values for $T_x(6,6)$ and $T_x(6,2)$. If we use $h = \pm 2$ and the values given in the table, we have

$T_x(6,6) \approx \dfrac{T(8,6)-T(6,6)}{2} = \dfrac{80-75}{2} = 2.5$, $T_x(6,6) \approx \dfrac{T(4,6)-T(6,6)}{-2} = \dfrac{68-75}{-2} = 3.5$.

Averaging these values, we estimate $T_x(6,6) \approx 3.0$. Similarly,

$T_x(6,2) \approx \dfrac{T(8,2)-T_x(6,2)}{2} = \dfrac{90-87}{2} = 1.5$, $T_x(6,2) \approx \dfrac{T(4,2)-T(6,2)}{-2} = \dfrac{74-87}{-2} = 6.5$.

Averaging these values, we estimate $T_x(6,2) \approx 4.0$. Finally, we estimate $T_{xy}(6,4)$:

$T_{xy}(6,4) \approx \dfrac{T_x(6,6)-T_x(6,4)}{2} = \dfrac{3.0-3.5}{2} = -0.25$, $T_{xy}(6,4) \approx \dfrac{T_x(6,2)-T_x(6,4)}{-2} = \dfrac{4.0-3.5}{-2} = -0.25$.

Averaging these values, we have $T_{xy}(6,4) \approx -0.25$.

12. From the table, $T(6,4) = 80$, and from Exercise 11 we estimated $T_x(6,4) \approx 3.5$ and $T_y(6,4) \approx -3.0$. The linear approximation then is

$$T(x,y) \approx T(6,4) + T_x(6,4)(x-6) + T_y(6,4)(y-4) \approx 80 + 3.5(x-6) - 3(y-4) = 3.5x - 3y + 71$$

Thus at the point $(5, 3.8)$, we can use the linear approximation to estimate $T(5, 3.8) \approx 3.5(5) - 3(3.8) + 71 \approx 77.1°\text{C}$.

13. $f(x,y) = (5y^3 + 2x^2y)^8 \quad \Rightarrow \quad f_x = 8(5y^3 + 2x^2y)^7(4xy) = 32xy(5y^3 + 2x^2y)^7$,

$f_y = 8(5y^3 + 2x^2y)^7(15y^2 + 2x^2) = (16x^2 + 120y^2)(5y^3 + 2x^2y)^7$

14. $g(u,v) = \dfrac{u+2v}{u^2+v^2} \quad\Rightarrow\quad g_u = \dfrac{(u^2+v^2)(1)-(u+2v)(2u)}{(u^2+v^2)^2} = \dfrac{v^2-u^2-4uv}{(u^2+v^2)^2}$,

$$g_v = \frac{(u^2+v^2)(2)-(u+2v)(2v)}{(u^2+v^2)^2} = \frac{2u^2-2v^2-2uv}{(u^2+v^2)^2}$$

15. $F(\alpha,\beta) = \alpha^2\ln(\alpha^2+\beta^2) \quad\Rightarrow\quad F_\alpha = \alpha^2 \cdot \dfrac{1}{\alpha^2+\beta^2}\,(2\alpha) + \ln(\alpha^2+\beta^2)\cdot 2\alpha = \dfrac{2\alpha^3}{\alpha^2+\beta^2} + 2\alpha\ln(\alpha^2+\beta^2)$,

$$F_\beta = \alpha^2 \cdot \frac{1}{\alpha^2+\beta^2}\,(2\beta) = \frac{2\alpha^2\beta}{\alpha^2+\beta^2}$$

16. $G(x,y,z) = e^{xz}\sin(y/z) \quad\Rightarrow\quad G_x = ze^{xz}\sin(y/z),\ G_y = e^{xz}\cos(y/z)(1/z) = (e^{xz}/z)\cos(y/z)$,

$G_z = e^{xz}\cdot\cos(y/z)(-y/z^2) + \sin(y/z)\cdot xe^{xz} = e^{xz}\left[x\sin(y/z) - (y/z^2)\cos(y/z)\right]$

17. $S(u,v,w) = u\arctan(v\sqrt{w}\,) \quad\Rightarrow\quad S_u = \arctan(v\sqrt{w}\,),\ S_v = u\cdot\dfrac{1}{1+(v\sqrt{w})^2}\,(\sqrt{w}\,) = \dfrac{u\sqrt{w}}{1+v^2w}$,

$$S_w = u\cdot\frac{1}{1+(v\sqrt{w})^2}\left(v\cdot\tfrac{1}{2}w^{-1/2}\right) = \frac{uv}{2\sqrt{w}\,(1+v^2w)}$$

18. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S-35) + 0.016D \quad\Rightarrow$

$\partial C/\partial T = 4.6 - 0.11T + 0.00087T^2 - 0.01(S-35)$, $\partial C/\partial S = 1.34 - 0.01T$, and $\partial C/\partial D = 0.016$. When $T = 10$, $S = 35$, and $D = 100$ we have $\partial C/\partial T = 4.6 - 0.11(10) + 0.00087(10)^2 - 0.01(35-35) \approx 3.587$, thus in 10°C water with salinity 35 parts per thousand and a depth of 100 m, the speed of sound increases by about 3.59 m/s for every degree Celsius that the water temperature rises. Similarly, $\partial C/\partial S = 1.34 - 0.01(10) = 1.24$, so the speed of sound increases by about 1.24 m/s for every part per thousand the salinity of the water increases. $\partial C/\partial D = 0.016$, so the speed of sound increases by about 0.016 m/s for every meter that the depth is increased.

19. $f(x,y) = 4x^3 - xy^2 \quad\Rightarrow\quad f_x = 12x^2 - y^2,\ f_y = -2xy,\ f_{xx} = 24x,\ f_{yy} = -2x,\ f_{xy} = f_{yx} = -2y$

20. $z = xe^{-2y} \quad\Rightarrow\quad z_x = e^{-2y},\ z_y = -2xe^{-2y},\ z_{xx} = 0,\ z_{yy} = 4xe^{-2y},\ z_{xy} = z_{yx} = -2e^{-2y}$

21. $f(x,y,z) = x^ky^lz^m \quad\Rightarrow\quad f_x = kx^{k-1}y^lz^m,\ f_y = lx^ky^{l-1}z^m,\ f_z = mx^ky^lz^{m-1},\ f_{xx} = k(k-1)x^{k-2}y^lz^m$,

$f_{yy} = l(l-1)x^ky^{l-2}z^m,\ f_{zz} = m(m-1)x^ky^lz^{m-2},\ f_{xy} = f_{yx} = klx^{k-1}y^{l-1}z^m, f_{xz} = f_{zx} = kmx^{k-1}y^lz^{m-1}$,

$f_{yz} = f_{zy} = lmx^ky^{l-1}z^{m-1}$

22. $v = r\cos(s+2t) \quad\Rightarrow\quad v_r = \cos(s+2t),\ v_s = -r\sin(s+2t),\ v_t = -2r\sin(s+2t),\ v_{rr} = 0,\ v_{ss} = -r\cos(s+2t)$,

$v_{tt} = -4r\cos(s+2t),\ v_{rs} = v_{sr} = -\sin(s+2t),\ v_{rt} = v_{tr} = -2\sin(s+2t),\ v_{st} = v_{ts} = -2r\cos(s+2t)$

23. $z = xy + xe^{y/x} \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = y - \dfrac{y}{x}e^{y/x} + e^{y/x},\ \dfrac{\partial z}{\partial y} = x + e^{y/x}$ and

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(y - \frac{y}{x}e^{y/x} + e^{y/x}\right) + y\left(x + e^{y/x}\right) = xy - ye^{y/x} + xe^{y/x} + xy + ye^{y/x} = xy + xy + xe^{y/x} = xy + z.$$

24. $z = \sin(x + \sin t) \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = \cos(x + \sin t),\ \dfrac{\partial z}{\partial t} = \cos(x + \sin t)\cos t,$

$\dfrac{\partial^2 z}{\partial x \partial t} = -\sin(x + \sin t)\cos t,\ \dfrac{\partial^2 z}{\partial x^2} = -\sin(x + \sin t)$ and

$$\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x \partial t} = \cos(x + \sin t)\left[-\sin(x + \sin t)\cos t\right] = \cos(x + \sin t)(\cos t)\left[-\sin(x + \sin t)\right] = \frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial x^2}.$$

25. (a) $z = 3x^2 - y^2 + 2x$, $(1, -2, 1)$. $z_x = 6x + 2 \quad \Rightarrow \quad z_x(1, -2) = 8$ and $z_y = -2y \quad \Rightarrow \quad z_y(1, -2) = 4$, so an equation of the tangent plane is $z - 1 = 8(x - 1) + 4(y + 2)$, or $z = 8x + 4y + 1$.

(b) A normal vector to the tangent plane (and the surface) at $(1, -2, 1)$ is $\langle 8, 4, -1 \rangle$. Then parametric equations for the normal line there are $x = 1 + 8t$, $y = -2 + 4t$, $z = 1 - t$, and symmetric equations are $\dfrac{x-1}{8} = \dfrac{y+2}{4} = \dfrac{z-1}{-1}$.

26. (a) $z = e^x \cos y$, $(0, 0, 1)$. $z_x = e^x \cos y \quad \Rightarrow \quad z_x(0, 0) = 1$ and $z_y = -e^x \sin y \quad \Rightarrow \quad z_y(0, 0) = 0$, so an equation of the tangent plane is $z - 1 = 1(x - 0) + 0(y - 0)$, or $z = x + 1$.

(b) A normal vector to the tangent plane (and the surface) at $(0, 0, 1)$ is $\langle 1, 0, -1 \rangle$. Then parametric equations for the normal line there are $x = t$, $y = 0$, $z = 1 - t$, and symmetric equations are $x = 1 - z$, $y = 0$.

27. (a) Let $F(x, y, z) = x^2 + 2y^2 - 3z^2$. Then $F_x = 2x$, $F_y = 4y$, $F_z = -6z$, so $F_x(2, -1, 1) = 4$, $F_y(2, -1, 1) = -4$, $F_z(2, -1, 1) = -6$. From Equation 14.6.19, an equation of the tangent plane is $4(x - 2) - 4(y + 1) - 6(z - 1) = 0$ or, equivalently, $2x - 2y - 3z = 3$.

(b) From Equations 14.6.20, symmetric equations for the normal line are $\dfrac{x-2}{4} = \dfrac{y+1}{-4} = \dfrac{z-1}{-6}$. Parametric equations are $x = 2 + 4t$, $y = -1 - 4t$, $z = 1 - 6t$.

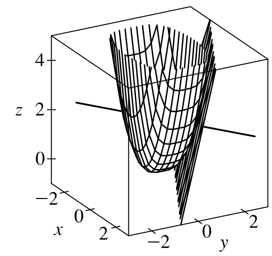
28. (a) Let $F(x, y, z) = xy + yz + zx$. Then $F_x = y + z$, $F_y = x + z$, $F_z = x + y$, so $F_x(1, 1, 1) = F_y(1, 1, 1) = F_z(1, 1, 1) = 2$. From Equation 14.6.19, an equation of the tangent plane is $2(x - 1) + 2(y - 1) + 2(z - 1) = 0$ or, equivalently, $x + y + z = 3$.

(b) From Equations 14.6.20, symmetric equations for the normal line are $\dfrac{x-1}{2} = \dfrac{y-1}{2} = \dfrac{z-1}{2}$ or, equivalently, $x = y = z$. Parametric equations are $x = 1 + 2t$, $y = 1 + 2t$, $z = 1 + 2t$.

29. (a) Let $F(x, y, z) = x + 2y + 3z - \sin(xyz)$. Then $F_x = 1 - yz\cos(xyz)$, $F_y = 2 - xz\cos(xyz)$, $F_z = 3 - xy\cos(xyz)$, so $F_x(2, -1, 0) = 1$, $F_y(2, -1, 0) = 2$, $F_z(2, -1, 0) = 5$. From Equation 14.6.19, an equation of the tangent plane is $1(x - 2) + 2(y + 1) + 5(z - 0) = 0$ or $x + 2y + 5z = 0$.

(b) From Equations 14.6.20, symmetric equations for the normal line are $\dfrac{x-2}{1} = \dfrac{y+1}{2} = \dfrac{z}{5}$ or $x - 2 = \dfrac{y+1}{2} = \dfrac{z}{5}$. Parametric equations are $x = 2 + t$, $y = -1 + 2t$, $z = 5t$.

30. Let $f(x,y) = x^2 + y^4$. Then $f_x(x,y) = 2x$ and $f_y(x,y) = 4y^3$, so $f_x(1,1) = 2$, $f_y(1,1) = 4$ and an equation of the tangent plane is $z - 2 = 2(x-1) + 4(y-1)$ or $2x + 4y - z = 4$. A normal vector to the tangent plane is $\langle 2, 4, -1 \rangle$ so the normal line is given by $\dfrac{x-1}{2} = \dfrac{y-1}{4} = \dfrac{z-2}{-1}$ or $x = 1 + 2t$, $y = 1 + 4t$, $z = 2 - t$.



31. The hyperboloid is a level surface of the function $F(x,y,z) = x^2 + 4y^2 - z^2$, so a normal vector to the surface at (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0) = \langle 2x_0, 8y_0, -2z_0 \rangle$. A normal vector for the plane $2x + 2y + z = 5$ is $\langle 2, 2, 1 \rangle$. For the planes to be parallel, we need the normal vectors to be parallel, so $\langle 2x_0, 8y_0, -2z_0 \rangle = k \langle 2, 2, 1 \rangle$, or $x_0 = k$, $y_0 = \frac{1}{4}k$, and $z_0 = -\frac{1}{2}k$. But $x_0^2 + 4y_0^2 - z_0^2 = 4 \quad \Rightarrow \quad k^2 + \frac{1}{4}k^2 - \frac{1}{4}k^2 = 4 \quad \Rightarrow \quad k^2 = 4 \quad \Rightarrow \quad k = \pm 2$. So there are two such points: $\left(2, \frac{1}{2}, -1\right)$ and $\left(-2, -\frac{1}{2}, 1\right)$.

32. $u = \ln(1 + se^{2t}) \quad \Rightarrow \quad du = \dfrac{\partial u}{\partial s}\,ds + \dfrac{\partial u}{\partial t}\,dt = \dfrac{e^{2t}}{1 + se^{2t}}\,ds + \dfrac{2se^{2t}}{1 + se^{2t}}\,dt$

33. $f(x,y,z) = x^3\sqrt{y^2 + z^2} \quad \Rightarrow \quad f_x(x,y,z) = 3x^2\sqrt{y^2 + z^2}$, $f_y(x,y,z) = \dfrac{yx^3}{\sqrt{y^2 + z^2}}$, $f_z(x,y,z) = \dfrac{zx^3}{\sqrt{y^2 + z^2}}$,

so $f(2,3,4) = 8(5) = 40$, $f_x(2,3,4) = 3(4)\sqrt{25} = 60$, $f_y(2,3,4) = \frac{3(8)}{\sqrt{25}} = \frac{24}{5}$, and $f_z(2,3,4) = \frac{4(8)}{\sqrt{25}} = \frac{32}{5}$. Then the linear approximation of f at $(2,3,4)$ is

$$\begin{aligned} f(x,y,z) &\approx f(2,3,4) + f_x(2,3,4)(x-2) + f_y(2,3,4)(y-3) + f_z(2,3,4)(z-4) \\ &= 40 + 60(x-2) + \tfrac{24}{5}(y-3) + \tfrac{32}{5}(z-4) = 60x + \tfrac{24}{5}y + \tfrac{32}{5}z - 120 \end{aligned}$$

Then $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2} = f(1.98, 3.01, 3.97) \approx 60(1.98) + \frac{24}{5}(3.01) + \frac{32}{5}(3.97) - 120 = 38.656$.

34. (a) $dA = \dfrac{\partial A}{\partial x}\,dx + \dfrac{\partial A}{\partial y}\,dy = \frac{1}{2}y\,dx + \frac{1}{2}x\,dy$ and $|\Delta x| \leq 0.002$, $|\Delta y| \leq 0.002$. Thus, the maximum error in the calculated area is about $dA = \frac{1}{2}(12)(0.002) + \frac{1}{2}(5)(0.002) = 0.017\text{ m}^2$ or 170 cm^2.

(b) $z = \sqrt{x^2 + y^2}$, $dz = \dfrac{x}{\sqrt{x^2 + y^2}}\,dx + \dfrac{y}{\sqrt{x^2 + y^2}}\,dy$ and $|\Delta x| \leq 0.002$, $|\Delta y| \leq 0.002$. Thus, the maximum error in the calculated hypotenuse length, with $x = 5$, $y = 12$, and $z = \sqrt{5^2 + 12^2} = 13$, is about $dz = \frac{5}{13}(0.002) + \frac{12}{13}(0.002) = \frac{0.17}{65} \approx 0.0026$ m or 0.26 cm.

35. $u = x^2y^3 + z^4 \quad \Rightarrow \quad \dfrac{du}{dp} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dp} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dp} + \dfrac{\partial u}{\partial z}\dfrac{dz}{dp} = 2xy^3(1 + 6p) + 3x^2y^2(pe^p + e^p) + 4z^3(p\cos p + \sin p)$

36. $v = x^2\sin y + ye^{xy} \quad \Rightarrow \quad \dfrac{\partial v}{\partial s} = \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial s} = \left(2x\sin y + y^2e^{xy}\right)(1) + \left(x^2\cos y + xye^{xy} + e^{xy}\right)(t)$.

$s = 0, t = 1 \quad \Rightarrow \quad x = 2, y = 0$, so $\dfrac{\partial v}{\partial s} = 0 + (4 + 1)(1) = 5$.

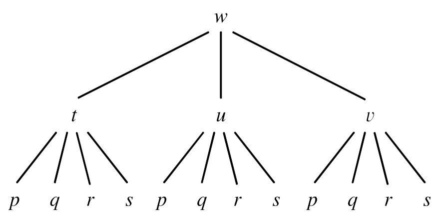
$\dfrac{\partial v}{\partial t} = \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial t} = \left(2x\sin y + y^2e^{xy}\right)(2) + \left(x^2\cos y + xye^{xy} + e^{xy}\right)(s) = 0 + 0 = 0.$

37. By the Chain Rule, $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$. When $s = 1$ and $t = 2$, $x = g(1,2) = 3$ and $y = h(1,2) = 6$, so

$\dfrac{\partial z}{\partial s} = f_x(3,6)g_s(1,2) + f_y\,(3,6)\,h_s(1,2) = (7)(-1) + (8)(-5) = -47$. Similarly, $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$, so

$\dfrac{\partial z}{\partial t} = f_x(3,6)g_t(1,2) + f_y\,(3,6)\,h_t(1,2) = (7)(4) + (8)(10) = 108$.

38.



Using the tree diagram as a guide, we have

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial p} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p} \qquad \frac{\partial w}{\partial q} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial q} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial q}$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial r} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial r} \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial s}$$

39. $z = y + f(x^2 - y^2) \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = 2xf'(x^2 - y^2),\quad \dfrac{\partial z}{\partial y} = 1 - 2yf'(x^2 - y^2) \quad \left[\text{where } f' = \dfrac{df}{d(x^2 - y^2)}\right]$. Then

$y\,\dfrac{\partial z}{\partial x} + x\,\dfrac{\partial z}{\partial y} = 2xyf'(x^2 - y^2) + x - 2xyf'(x^2 - y^2) = x.$

40. $A = \frac{1}{2}xy\sin\theta$ [Formula 6 in Appendix D], $dx/dt = 3$, $dy/dt = -2$, $d\theta/dt = 0.05$, and

$\dfrac{dA}{dt} = \dfrac{1}{2}\left[(y\sin\theta)\,\dfrac{dx}{dt} + (x\sin\theta)\,\dfrac{dy}{dt} + (xy\cos\theta)\,\dfrac{d\theta}{dt}\right]$. So when $x = 40$, $y = 50$ and $\theta = \frac{\pi}{6}$,

$\dfrac{dA}{dt} = \dfrac{1}{2}\big[(25)(3) + (20)(-2) + \big(1000\sqrt{3}\,\big)(0.05)\big] = \dfrac{35 + 50\sqrt{3}}{2} \approx 60.8\ \text{in}^2/\text{s}.$

41. $z = f(u,v)$, $u = xy$, and $v = y/x \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\,y + \dfrac{\partial z}{\partial v}\,\dfrac{-y}{x^2}$ and

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= y\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{2y}{x^3}\frac{\partial z}{\partial v} + \frac{-y}{x^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) = \frac{2y}{x^3}\frac{\partial z}{\partial v} + y\left(\frac{\partial^2 z}{\partial u^2}\,y + \frac{\partial^2 z}{\partial v\,\partial u}\,\frac{-y}{x^2}\right) + \frac{-y}{x^2}\left(\frac{\partial^2 z}{\partial v^2}\,\frac{-y}{x^2} + \frac{\partial^2 z}{\partial u\,\partial v}\,y\right)\\ &= \frac{2y}{x^3}\frac{\partial z}{\partial v} + y^2\,\frac{\partial^2 z}{\partial u^2} - \frac{2y^2}{x^2}\frac{\partial^2 z}{\partial u\,\partial v} + \frac{y^2}{x^4}\frac{\partial^2 z}{\partial v^2}\end{aligned}$$

Also, $\dfrac{\partial z}{\partial y} = x\,\dfrac{\partial z}{\partial u} + \dfrac{1}{x}\dfrac{\partial z}{\partial v}$ and

$$\frac{\partial^2 z}{\partial y^2} = x\,\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{x}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) = x\left(\frac{\partial^2 z}{\partial u^2}x + \frac{\partial^2 z}{\partial v\,\partial u}\frac{1}{x}\right) + \frac{1}{x}\left(\frac{\partial^2 z}{\partial v^2}\frac{1}{x} + \frac{\partial^2 z}{\partial u\,\partial v}\,x\right) = x^2\,\frac{\partial^2 z}{\partial u^2} + 2\,\frac{\partial^2 z}{\partial u\,\partial v} + \frac{1}{x^2}\frac{\partial^2 z}{\partial v^2}$$

Thus,

$$\begin{aligned}x^2\,\frac{\partial^2 z}{\partial x^2} - y^2\,\frac{\partial^2 z}{\partial y^2} &= \frac{2y}{x}\frac{\partial z}{\partial v} + x^2y^2\,\frac{\partial^2 z}{\partial u^2} - 2y^2\,\frac{\partial^2 z}{\partial u\,\partial v} + \frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2} - x^2y^2\,\frac{\partial^2 z}{\partial u^2} - 2y^2\,\frac{\partial^2 z}{\partial u\,\partial v} - \frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2}\\ &= \frac{2y}{x}\frac{\partial z}{\partial v} - 4y^2\,\frac{\partial^2 z}{\partial u\,\partial v} = 2v\,\frac{\partial z}{\partial v} - 4uv\,\frac{\partial^2 z}{\partial u\,\partial v}\end{aligned}$$

since $y = xv = \dfrac{uv}{y}$ or $y^2 = uv$.

42. $\cos(xyz) = 1 + x^2y^2 + z^2$, so let $F(x, y, z) = 1 + x^2y^2 + z^2 - \cos(xyz) = 0$. Then by Equations 14.5.6 we have $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{2xy^2 + \sin(xyz) \cdot yz}{2z + \sin(xyz) \cdot xy} = -\dfrac{2xy^2 + yz\sin(xyz)}{2z + xy\sin(xyz)}$,

$\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{2x^2y + \sin(xyz) \cdot xz}{2z + \sin(xyz) \cdot xy} = -\dfrac{2x^2y + xz\sin(xyz)}{2z + xy\sin(xyz)}$.

43. $f(x, y, z) = x^2e^{yz^2} \quad \Rightarrow \quad \nabla f = \langle f_x, f_y, f_z \rangle = \left\langle 2xe^{yz^2}, x^2e^{yz^2} \cdot z^2, x^2e^{yz^2} \cdot 2yz \right\rangle = \left\langle 2xe^{yz^2}, x^2z^2e^{yz^2}, 2x^2yze^{yz^2} \right\rangle$

44. (a) By Theorem 14.6.15, the maximum value of the directional derivative occurs when $\mathbf{u}$ has the same direction as the gradient vector.

(b) It is a minimum when $\mathbf{u}$ is in the direction opposite to that of the gradient vector (that is, $\mathbf{u}$ is in the direction of $-\nabla f$), since $D_{\mathbf{u}}\, f = |\nabla f| \cos\theta$ (see the proof of Theorem 14.6.15) has a minimum when $\theta = \pi$.

(c) The directional derivative is 0 when $\mathbf{u}$ is perpendicular to the gradient vector, since then $D_{\mathbf{u}}\, f = \nabla f \cdot \mathbf{u} = 0$.

(d) The directional derivative is half of its maximum value when $D_{\mathbf{u}}\, f = |\nabla f| \cos\theta = \frac{1}{2}\,|\nabla f| \quad \Leftrightarrow \quad \cos\theta = \frac{1}{2} \quad \Leftrightarrow$ $\theta = \frac{\pi}{3}$.

45. $f(x, y) = x^2e^{-y} \quad \Rightarrow \quad \nabla f = \langle 2xe^{-y}, -x^2e^{-y} \rangle$, $\nabla f(-2, 0) = \langle -4, -4 \rangle$. The direction is given by $\langle 2 - (-2), -3 - 0 \rangle = \langle 4, -3 \rangle$, so $\mathbf{u} = \frac{1}{\sqrt{4^2 + (-3)^2}}\langle 4, -3 \rangle = \frac{1}{5}\langle 4, -3 \rangle$. Thus,

$D_{\mathbf{u}}\, f(-2, 0) = \nabla f(-2, 0) \cdot \mathbf{u} = \langle -4, -4 \rangle \cdot \frac{1}{5}\langle 4, -3 \rangle = \frac{1}{5}(-16 + 12) = -\frac{4}{5}$.

46. $f(x, y, z) = x^2y + x\sqrt{1 + z} \quad \Rightarrow \quad \nabla f = \left\langle 2xy + \sqrt{1 + z}, x^2, x/\big(2\sqrt{1 + z}\big) \right\rangle$, $\nabla f(1, 2, 3) = \left\langle 6, 1, \frac{1}{4} \right\rangle$. The direction is given by $2\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$, so $\mathbf{u} = \frac{1}{\sqrt{2^2 + 1^2 + (-2)^2}}\,\langle 2, 1, -2 \rangle = \frac{1}{3}\,\langle 2, 1, -2 \rangle$. Thus,

$D_{\mathbf{u}}\, f(1, 2, 3) = \nabla f(1, 2, 3) \cdot \mathbf{u} = \left\langle 6, 1, \frac{1}{4} \right\rangle \cdot \frac{1}{3}\,\langle 2, 1, -2 \rangle = \frac{1}{3}\left(12 + 1 - \frac{1}{2}\right) = \frac{25}{6}$.

47. $f(x, y) = x^2y + \sqrt{y} \quad \Rightarrow \quad \nabla f = \left\langle 2xy, x^2 + 1/\left(2\sqrt{y}\right) \right\rangle$, $\nabla f(2, 1) = \left\langle 4, \frac{9}{2} \right\rangle$. Thus, the maximum rate of change of f at $(2, 1)$ is $|\nabla f(2, 1)| = \left|\left\langle 4, \frac{9}{2} \right\rangle\right| = \frac{\sqrt{145}}{2}$ in the direction $\left\langle 4, \frac{9}{2} \right\rangle$.

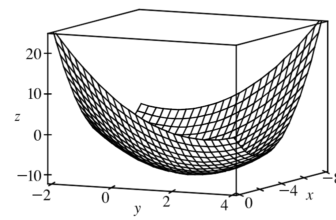
48. $f(x, y, z) = ze^{xy}$. $\nabla f = \langle zye^{xy}, zxe^{xy}, e^{xy} \rangle$, $\nabla f(0, 1, 2) = \langle 2, 0, 1 \rangle$ is the direction of most rapid increase while the rate is $|\langle 2, 0, 1 \rangle| = \sqrt{5}$.

49. First we draw a line passing through Homestead and the eye of the hurricane. We can approximate the directional derivative at Homestead in the direction of the eye of the hurricane by the average rate of change of wind speed between the points where this line intersects the contour lines closest to Homestead. In the direction of the eye of the hurricane, the wind speed changes from 80 to 90 km/h. We estimate the distance between these two points to be approximately 14 kilometers, so the rate of change of wind speed in the direction given is approximately $\frac{90 - 80}{14} = \frac{5}{7} = 0.714$ km/h/km.

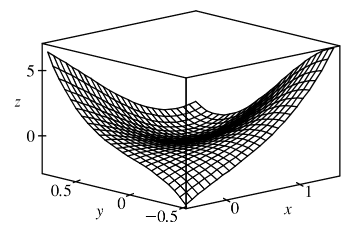
50. The surfaces are $f(x, y, z) = z - 2x^2 + y^2 = 0$ and $g(x, y, z) = z - 4 = 0$. The tangent line is perpendicular to both ∇f and ∇g at $(-2, 2, 4)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ is therefore parallel to the line. $\nabla f(x, y, z) = \langle -4x, 2y, 1\rangle \Rightarrow$ $\nabla f(-2, 2, 4) = \langle 8, 4, 1\rangle$, $\nabla g(x, y, z) = \langle 0, 0, 1\rangle \Rightarrow \nabla g\langle -2, 2, 4\rangle = \langle 0, 0, 1\rangle$. Hence

$\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4\,\mathbf{i} - 8\,\mathbf{j}$. Thus, parametric equations are: $x = -2 + 4t,\ y = 2 - 8t,\ z = 4$.

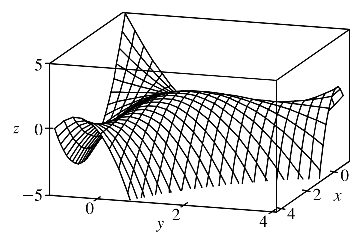
51. $f(x, y) = x^2 - xy + y^2 + 9x - 6y + 10 \Rightarrow f_x = 2x - y + 9$, $f_y = -x + 2y - 6,\ f_{xx} = 2 = f_{yy},\ f_{xy} = -1$. Then $f_x = 0$ and $f_y = 0$ imply $y = 1$, $x = -4$. Thus, the only critical point is $(-4, 1)$ and $f_{xx}(-4, 1) > 0$, $D(-4, 1) = 3 > 0$, so $f(-4, 1) = -11$ is a local minimum.



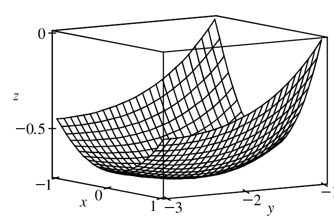
52. $f(x, y) = x^3 - 6xy + 8y^3 \Rightarrow f_x = 3x^2 - 6y,\ f_y = -6x + 24y^2,\ f_{xx} = 6x$, $f_{yy} = 48y$, $f_{xy} = -6$. Then $f_x = 0$ implies $y = x^2/2$, substituting into $f_y = 0$ implies $6x\left(x^3 - 1\right) = 0$, so the critical points are $(0, 0)$, $\left(1, \frac{1}{2}\right)$. $D(0, 0) = -36 < 0$ so $(0, 0)$ is a saddle point while $f_{xx}\left(1, \frac{1}{2}\right) = 6 > 0$ and $D\left(1, \frac{1}{2}\right) = 108 > 0$ so $f\left(1, \frac{1}{2}\right) = -1$ is a local minimum.



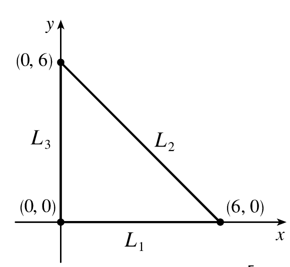
53. $f(x, y) = 3xy - x^2y - xy^2 \Rightarrow f_x = 3y - 2xy - y^2,\ f_y = 3x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x,\ f_{xy} = 3 - 2x - 2y$. Then $f_x = 0$ implies $y(3 - 2x - y) = 0$ so $y = 0$ or $y = 3 - 2x$. Substituting into $f_y = 0$ implies $x(3 - x) = 0$ or $3x(-1 + x) = 0$. Hence the critical points are $(0, 0)$, $(3, 0)$, $(0, 3)$ and $(1, 1)$. $D(0, 0) = D(3, 0) = D(0, 3) = -9 < 0$ so $(0, 0)$, $(3, 0)$, and $(0, 3)$ are saddle points. $D(1, 1) = 3 > 0$ and $f_{xx}(1, 1) = -2 < 0$, so $f(1, 1) = 1$ is a local maximum.



54. $f(x, y) = (x^2 + y)e^{y/2} \Rightarrow f_x = 2xe^{y/2}$, $f_y = e^{y/2}(2 + x^2 + y)/2$, $f_{xx} = 2e^{y/2}$, $f_{yy} = e^{y/2}(4 + x^2 + y)/4$, $f_{xy} = xe^{y/2}$. Then $f_x = 0$ implies $x = 0$, so $f_y = 0$ implies $y = -2$. But $f_{xx}(0, -2) > 0$, $D(0, -2) = e^{-2} - 0 > 0$ so $f(0, -2) = -2/e$ is a local minimum.



55. $f(x, y) = 4xy^2 - x^2y^2 - xy^3$. First, solve inside D. Here $f_x = 4y^2 - 2xy^2 - y^3$, $f_y = 8xy - 2x^2y - 3xy^2$. Then $f_x = 0$ implies $y = 0$ or $y = 4 - 2x$, but $y = 0$ isn't inside D. Substituting $y = 4 - 2x$ into $f_y = 0$ implies $x = 0$, $x = 2$ or $x = 1$, but $x = 0$ isn't inside D, and when $x = 2$, $y = 0$ but $(2, 0)$ isn't inside D. Thus, the only critical point inside D is $(1, 2)$ and $f(1, 2) = 4$. Secondly, we consider the boundary of D.

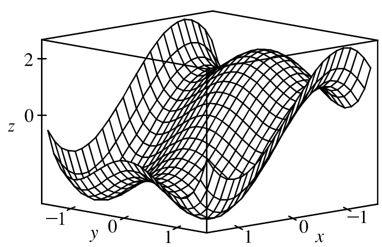


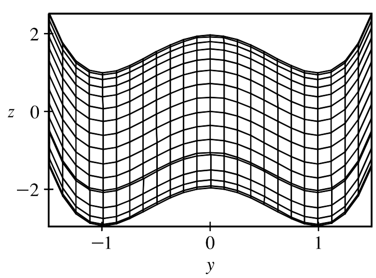
[continued]

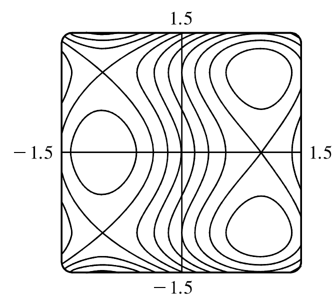
On L_1: $f(x,0) = 0$ and so $f = 0$ on L_1. On L_2: $x = -y + 6$ and $f(-y+6, y) = y^2(6-y)(-2) = -2(6y^2 - y^3)$ which has critical points at $y = 0$ and $y = 4$. Then $f(6,0) = 0$ while $f(2,4) = -64$. On L_3: $f(0,y) = 0$, so $f = 0$ on L_3. Thus, on D the absolute maximum of f is $f(1,2) = 4$ while the absolute minimum is $f(2,4) = -64$.

56. $f(x,y) = e^{-x^2-y^2}(x^2 + 2y^2)$. Inside D: $f_x = 2xe^{-x^2-y^2}(1 - x^2 - 2y^2) = 0$ implies $x = 0$ or $x^2 + 2y^2 = 1$. Then if $x = 0$, $f_y = 2ye^{-x^2-y^2}(2 - x^2 - 2y^2) = 0$ implies $y = 0$ or $2 - 2y^2 = 0$ giving the critical points $(0,0)$, $(0,\pm 1)$. If $x^2 + 2y^2 = 1$, then $f_y = 0$ implies $y = 0$ giving the critical points $(\pm 1, 0)$. Now $f(0,0) = 0$, $f(\pm 1, 0) = e^{-1}$ and $f(0,\pm 1) = 2e^{-1}$. On the boundary of D: $x^2 + y^2 = 4$, so $f(x,y) = e^{-4}(4 + y^2)$ and f is smallest when $y = 0$ and largest when $y^2 = 4$. But $f(\pm 2, 0) = 4e^{-4}$, $f(0,\pm 2) = 8e^{-4}$. Thus, on D the absolute maximum of f is $f(0,\pm 1) = 2e^{-1}$ and the absolute minimum is $f(0,0) = 0$.

57. $f(x,y) = x^3 - 3x + y^4 - 2y^2$



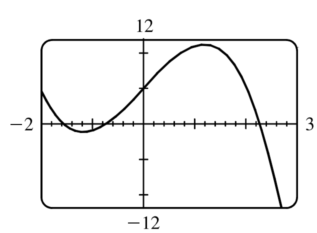


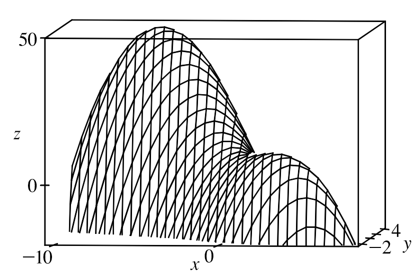


From the graphs, it appears that f has a local maximum $f(-1,0) \approx 2$, local minimums $f(1,\pm 1) \approx -3$, and saddle points at $(-1,\pm 1)$ and $(1,0)$.

To find the exact quantities, we calculate $f_x = 3x^2 - 3 = 0 \quad \Leftrightarrow \quad x = \pm 1$ and $f_y = 4y^3 - 4y = 0 \quad \Leftrightarrow$ $y = 0, \pm 1$, giving the critical points estimated above. Also, $f_{xx} = 6x$, $f_{xy} = 0$, $f_{yy} = 12y^2 - 4$, so using the Second Derivatives Test, $D(-1,0) = 24 > 0$ and $f_{xx}(-1,0) = -6 < 0$ indicating a local maximum $f(-1,0) = 2$; $D(1,\pm 1) = 48 > 0$ and $f_{xx}(1,\pm 1) = 6 > 0$ indicating local minimums $f(1,\pm 1) = -3$; and $D(-1,\pm 1) = -48$ and $D(1,0) = -24$, indicating saddle points at $(-1,\pm 1)$ and $(1,0)$.

58. $f(x,y) = 12 + 10y - 2x^2 - 8xy - y^4 \quad \Rightarrow \quad f_x(x,y) = -4x - 8y$, $f_y(x,y) = 10 - 8x - 4y^3$. Now $f_x(x,y) = 0 \quad \Rightarrow$ $x = -2x$, and substituting this into $f_y(x,y) = 0$ gives $10 + 16y - 4y^3 = 0 \quad \Leftrightarrow \quad 5 + 8y - 2y^3 = 0$.





From the first graph, we see that this is true when $y \approx -1.542$, -0.717, or 2.260. (Alternatively, we could have found the solutions to $f_x = f_y = 0$ using a CAS.) So to three decimal places, the critical points are $(3.085, -1.542)$, $(1.434, -0.717)$, and $(-4.519, 2.260)$. Now in order to use the Second Derivatives Test, we calculate $f_{xx} = -4$, $f_{xy} = -8$, $f_{yy} = -12y^2$, and $D = 48y^2 - 64$. So since $D(3.085, -1.542) > 0$, $D(1.434, -0.717) < 0$, and $D(-4.519, 2.260) > 0$, and f_{xx} is always negative, $f(x, y)$ has local maximums $f(-4.519, 2.260) \approx 49.373$ and $f(3.085, -1.542) \approx 9.948$, and a saddle point at approximately $(1.434, -0.717)$. The highest point on the graph is approximately $(-4.519, 2.260, 49.373)$.

59. $f(x, y) = x^2y$, $g(x, y) = x^2 + y^2 = 1 \quad \Rightarrow \quad \nabla f = \langle 2xy, x^2 \rangle = \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2xy = 2\lambda x$ implies $x = 0$ or $y = \lambda$. If $x = 0$, then $x^2 + y^2 = 1$ gives $y = \pm 1$ and we have possible points $(0, \pm 1)$ where $f(0, \pm 1) = 0$. If $y = \lambda$, then $x^2 = 2\lambda y$ implies $x^2 = 2y^2$ and substitution into $x^2 + y^2 = 1$ gives $3y^2 = 1 \quad \Rightarrow \quad y = \pm \frac{1}{\sqrt{3}}$ and $x = \pm \sqrt{\frac{2}{3}}$. The corresponding possible points are $\left(\pm \sqrt{\frac{2}{3}}, \pm \frac{1}{\sqrt{3}}\right)$. The absolute maximum is $f\left(\pm \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$ while the absolute minimum is $f\left(\pm \sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$.

60. $f(x, y) = 1/x + 1/y$, $g(x, y) = 1/x^2 + 1/y^2 = 1 \quad \Rightarrow \quad \nabla f = \left\langle -x^{-2}, -y^{-2} \right\rangle = \lambda \nabla g = \left\langle -2\lambda x^{-3}, -2\lambda y^{-3} \right\rangle$. Then $-x^{-2} = -2\lambda x^3$ or $x = 2\lambda$ and $-y^{-2} = -2\lambda y^{-3}$ or $y = 2\lambda$. Thus, $x = y$, so $1/x^2 + 1/y^2 = 2/x^2 = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $\left(\pm\sqrt{2}, \pm\sqrt{2}\right)$. The absolute maximum of f subject to $x^{-2} + y^{-2} = 1$ is then $f\left(\sqrt{2}, \sqrt{2}\right) = \sqrt{2}$ and the absolute minimum is $f\left(-\sqrt{2}, -\sqrt{2}\right) = -\sqrt{2}$.

61. $f(x, y, z) = xyz$, $g(x, y, z) = x^2 + y^2 + z^2 = 3$. $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$. If any of x, y, or z is zero, then $x = y = z = 0$ which contradicts $x^2 + y^2 + z^2 = 3$. Then $\lambda = \dfrac{yz}{2x} = \dfrac{xz}{2y} = \dfrac{xy}{2z} \quad \Rightarrow \quad 2y^2z = 2x^2z \quad \Rightarrow$ $y^2 = x^2$, and similarly $2yz^2 = 2x^2y \quad \Rightarrow \quad z^2 = x^2$. Substituting into the constraint equation gives $x^2 + x^2 + x^2 = 3 \quad \Rightarrow$ $x^2 = 1 = y^2 = z^2$. Thus, the possible points are $(1, 1, \pm 1)$, $(1, -1, \pm 1)$, $(-1, 1, \pm 1)$, $(-1, -1, \pm 1)$. The absolute maximum is $f(1, 1, 1) = f(1, -1, -1) = f(-1, 1, -1) = f(-1, -1, 1) = 1$, and the absolute minimum is $f(1, 1, -1) = f(1, -1, 1) = f(-1, 1, 1) = f(-1, -1, -1) = -1$.

62. $f(x, y, z) = x^2 + 2y^2 + 3z^2$, $g(x, y, z) = x + y + z = 1$, $h(x, y, z) = x - y + 2z = 2 \quad \Rightarrow$ $\nabla f = \langle 2x, 4y, 6z \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda + \mu, \lambda - \mu, \lambda + 2\mu \rangle$ and $2x = \lambda + \mu$ **(1)**, $4y = \lambda - \mu$ **(2)**, $6z = \lambda + 2\mu$ **(3)**, $x + y + z = 1$ **(4)**, $x - y + 2z = 2$ **(5)**. Then six times **(1)** plus three times **(2)** plus two times **(3)** implies $12(x + y + z) = 11\lambda + 7\mu$, so **(4)** gives $11\lambda + 7\mu = 12$. Also, six times **(1)** minus three times **(2)** plus four times **(3)** implies $12(x - y + 2z) = 7\lambda + 17\mu$, so **(5)** gives $7\lambda + 17\mu = 24$. Solving $11\lambda + 7\mu = 12$, $7\lambda + 17\mu = 24$ simultaneously gives $\lambda = \frac{6}{23}$, $\mu = \frac{30}{23}$. Substituting into **(1)**, **(2)**, and **(3)** implies $x = \frac{18}{23}$, $y = -\frac{6}{23}$, $z = \frac{11}{23}$ giving only one point. Then $f\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}\right) = \frac{33}{23}$. Now since $(0, 0, 1)$ satisfies both constraints and $f(0, 0, 1) = 3 > \frac{33}{23}$, $f\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}\right) = \frac{33}{23}$ is an absolute minimum, and there is no absolute maximum.

63. $f(x,y,z)=x^2+y^2+z^2,\ g(x,y,z)=xy^2z^3=2 \quad\Rightarrow\quad \nabla f=\langle 2x,2y,2z\rangle=\lambda\nabla g=\langle \lambda y^2z^3,2\lambda xyz^3,3\lambda xy^2z^2\rangle$.

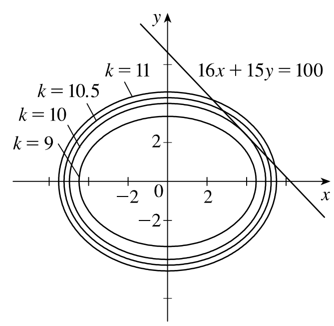
Since $xy^2z^3=2$, $x\neq 0$, $y\neq 0$ and $z\neq 0$, so $2x=\lambda y^2z^3$ **(1)**, $\quad 1=\lambda xz^3$ **(2)**, $\quad 2=3\lambda xy^2z$ **(3)**. Then **(2)** and **(3)** imply $\dfrac{1}{xz^3}=\dfrac{2}{3xy^2z}$ or $y^2=\frac{2}{3}z^2$ so $y=\pm z\sqrt{\frac{2}{3}}$. Similarly, **(1)** and **(3)** imply $\dfrac{2x}{y^2z^3}=\dfrac{2}{3xy^2z}$ or $3x^2=z^2$ so $x=\pm\frac{1}{\sqrt{3}}z$. But $xy^2z^3=2$ so x and z must have the same sign, that is, $x=\frac{1}{\sqrt{3}}z$. Thus, $g(x,y,z)=2$ implies $\frac{1}{\sqrt{3}}z\left(\frac{2}{3}z^2\right)z^3=2$ or $z=\pm 3^{1/4}$ and the possible points are $(\pm 3^{-1/4},3^{-1/4}\sqrt{2},\pm 3^{1/4})$, $(\pm 3^{-1/4},-3^{-1/4}\sqrt{2},\pm 3^{1/4})$. However, at each of these points f takes on the same value, $2\sqrt{3}$. But $(2,1,1)$ also satisfies $g(x,y,z)=2$ and $f(2,1,1)=6>2\sqrt{3}$. Thus, f has an absolute minimum value of $2\sqrt{3}$ and no absolute maximum subject to the constraint $xy^2z^3=2$.

Alternate solution: $g(x,y,z)=xy^2z^3=2$ implies $y^2=\dfrac{2}{xz^3}$, so minimize $f(x,z)=x^2+\dfrac{2}{xz^3}+z^2$. Then $f_x=2x-\dfrac{2}{x^2z^3}$, $f_z=-\dfrac{6}{xz^4}+2z$, $f_{xx}=2+\dfrac{4}{x^3z^3}$, $f_{zz}=\dfrac{24}{xz^5}+2$ and $f_{xz}=\dfrac{6}{x^2z^4}$. Now $f_x=0$ implies $2x^3z^3-2=0$ or $z=1/x$. Substituting into $f_y=0$ implies $-6x^3+2x^{-1}=0$ or $x=\frac{1}{\sqrt[4]{3}}$, so the two critical points are $\left(\pm\frac{1}{\sqrt[4]{3}},\pm\sqrt[4]{3}\right)$. Then $D\left(\pm\frac{1}{\sqrt[4]{3}},\pm\sqrt[4]{3}\right)=(2+4)\left(2+\frac{24}{3}\right)-\left(\frac{6}{\sqrt{3}}\right)^2>0$ and $f_{xx}\left(\pm\frac{1}{\sqrt[4]{3}},\pm\sqrt[4]{3}\right)=6>0$, so each point is a minimum. Finally, $y^2=\dfrac{2}{xz^3}$, so the four points closest to the origin are $\left(\pm\frac{1}{\sqrt[4]{3}},\frac{\sqrt{2}}{\sqrt[4]{3}},\pm\sqrt[4]{3}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}},-\frac{\sqrt{2}}{\sqrt[4]{3}},\pm\sqrt[4]{3}\right)$.

64. (a) The distance from a point (x,y) to the point $(-3,0)$ is $\sqrt{(x+3)^2+y^2}$. The distance from (x,y) to $(3,0)$ is $\sqrt{(x-3)^2+y^2}$. Then the function that gives the sum of the distances is

$f(x,y)=\sqrt{(x+3)^2+y^2}+\sqrt{(x-3)^2+y^2}$.

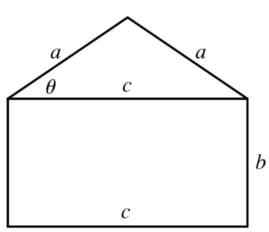
The graph shows several curves of $f(x,y)=k$ for $k=9$, 10, 10.5, and 11. The smallest value of k that $g(x,y)=16x+15y$ intersects appears to be 10, so the point that minimizes the distance is approximately $(x,y)=(4,2.4)$.



(b) $\nabla f=\left\langle \dfrac{x+3}{\sqrt{(x+3)^2+y^2}}+\dfrac{x-3}{\sqrt{(x-3)^2+y^2}},\dfrac{y}{\sqrt{(x+3)^2+y^2}}+\dfrac{y}{\sqrt{(x-3)^2+y^2}}\right\rangle$. $\nabla g=\langle 16,15\rangle$. Now

$\nabla f(4,2.4)=\left\langle\frac{7}{7.4}+\frac{1}{2.6},\frac{2.4}{7.4}+\frac{2.4}{2.6}\right\rangle=\left\langle\frac{640}{481},\frac{600}{481}\right\rangle$ and $\nabla g(4,2.4)=\langle 16,15\rangle$. ∇f and ∇g are parallel if $\nabla f=\lambda\nabla g$ for some value of λ, which is true when $\lambda=\frac{40}{481}$. Thus, ∇f and ∇g are parallel.

65.



The area of the triangle is $\frac{1}{2}ca\sin\theta$ [Formula 6 in Appendix D] and the area of the rectangle is bc. Thus, the area of the whole object is $f(a,b,c)=\frac{1}{2}ca\sin\theta+bc$. The perimeter of the object is $g(a,b,c)=2a+2b+c=P$. To simplify $\sin\theta$ in terms of a, b, and c notice that $a^2\sin^2\theta+\left(\frac{1}{2}c\right)^2=a^2 \quad\Rightarrow$

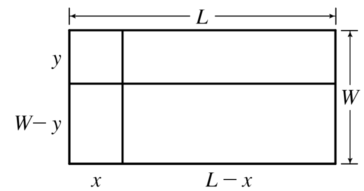
$\sin\theta = \dfrac{1}{2a}\sqrt{4a^2 - c^2}$. Thus, $f(a, b, c) = \dfrac{c}{4}\sqrt{4a^2 - c^2} + bc$. (Instead of using θ, we could just have used the Pythagorean Theorem.) As a result, by Lagrange's method, we must find a, b, c, and λ by solving $\nabla f = \lambda \nabla g$ which gives the following equations: $ca(4a^2 - c^2)^{-1/2} = 2\lambda$ **(1)**, $c = 2\lambda$ **(2)**, $\frac{1}{4}(4a^2 - c^2)^{1/2} - \frac{1}{4}c^2(4a^2 - c^2)^{-1/2} + b = \lambda$ **(3)**, and $2a + 2b + c = P$ **(4)**. From **(2)**, $\lambda = \frac{1}{2}c$ and so **(1)** produces $ca(4a^2 - c^2)^{-1/2} = c \;\Rightarrow\; (4a^2 - c^2)^{1/2} = a \;\Rightarrow\; 4a^2 - c^2 = a^2 \;\Rightarrow\; c = \sqrt{3}\,a$ **(5)**. Similarly, since $\left(4a^2 - c^2\right)^{1/2} = a$ and $\lambda = \frac{1}{2}c$, **(3)** gives $\dfrac{a}{4} - \dfrac{c^2}{4a} + b = \dfrac{c}{2}$, so from **(5)**, $\dfrac{a}{4} - \dfrac{3a}{4} + b = \dfrac{\sqrt{3}\,a}{2} \;\Rightarrow\; -\dfrac{a}{2} - \dfrac{\sqrt{3}\,a}{2} = -b \;\Rightarrow\; b = \dfrac{a}{2}\left(1 + \sqrt{3}\right)$ **(6)**. Substituting **(5)** and **(6)** into **(4)** we get:

$2a + a\left(1 + \sqrt{3}\right) + \sqrt{3}\,a = P \;\Rightarrow\; 3a + 2\sqrt{3}\,a = P \;\Rightarrow\; a = \dfrac{P}{3 + 2\sqrt{3}} = \dfrac{2\sqrt{3} - 3}{3}P$ and thus

$b = \dfrac{\left(2\sqrt{3} - 3\right)\left(1 + \sqrt{3}\right)}{6}P = \dfrac{3 - \sqrt{3}}{6}P$ and $c = \left(2 - \sqrt{3}\right)P$.

□ PROBLEMS PLUS

1. The areas of the smaller rectangles are $A_1 = xy$, $A_2 = (L - x)y$, $A_3 = (L - x)(W - y)$, $A_4 = x(W - y)$. For $0 \le x \le L$, $0 \le y \le W$, let



$$\begin{aligned} f(x,y) &= A_1^2 + A_2^2 + A_3^2 + A_4^2 \\ &= x^2y^2 + (L-x)^2y^2 + (L-x)^2(W-y)^2 + x^2(W-y)^2 \\ &= [x^2 + (L-x)^2][y^2 + (W-y)^2] \end{aligned}$$

Then we need to find the maximum and minimum values of $f(x, y)$. Here

$f_x(x,y) = [2x - 2(L-x)][y^2 + (W-y)^2] = 0 \quad \Rightarrow \quad 4x - 2L = 0$ or $x = \frac{1}{2}L$, and

$f_y(x,y) = [x^2 + (L-x)^2][2y - 2(W-y)] = 0 \quad \Rightarrow \quad 4y - 2W = 0$ or $y = W/2$. Also,

$f_{xx} = 4[y^2 + (W-y)^2]$, $f_{yy} = 4[x^2 + (L-x)^2]$, and $f_{xy} = (4x - 2L)(4y - 2W)$. Then

$D = 16[y^2 + (W-y)^2][x^2 + (L-x)^2] - (4x - 2L)^2(4y - 2W)^2$. Thus, when $x = \frac{1}{2}L$ and $y = \frac{1}{2}W$, $D > 0$ and

$f_{xx} = 2W^2 > 0$. Thus, a minimum of f occurs at $\left(\frac{1}{2}L, \frac{1}{2}W\right)$ and this minimum value is $f\left(\frac{1}{2}L, \frac{1}{2}W\right) = \frac{1}{4}L^2W^2$.

There are no other critical points, so the maximum must occur on the boundary. Now along the width of the rectangle let

$g(y) = f(0,y) = f(L,y) = L^2[y^2 + (W-y)^2]$, $0 \le y \le W$. Then $g'(y) = L^2[2y - 2(W-y)] = 0 \quad \Leftrightarrow \quad y = \frac{1}{2}W$.

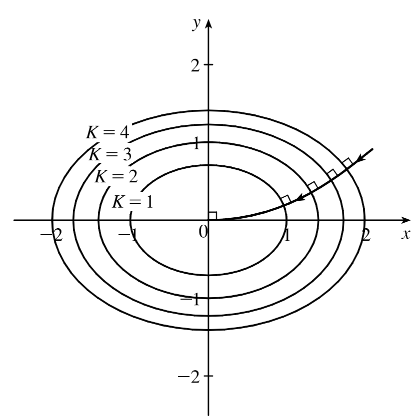
And $g\left(\frac{1}{2}\right) = \frac{1}{2}L^2W^2$. Checking the endpoints, we get $g(0) = g(W) = L^2W^2$. Along the length of the rectangle let

$h(x) = f(x,0) = f(x,W) = W^2[x^2 + (L-x)^2]$, $0 \le x \le L$. By symmetry, $h'(x) = 0 \quad \Leftrightarrow \quad x = \frac{1}{2}L$ and

$h\left(\frac{1}{2}L\right) = \frac{1}{2}L^2W^2$. At the endpoints we have $h(0) = h(L) = L^2W^2$. Therefore, L^2W^2 is the maximum value of f.

This maximum value of f occurs when the "cutting" lines correspond to sides of the rectangle.

2. (a) The level curves of the function $C(x,y) = e^{-(x^2+2y^2)/10^4}$ are the curves $e^{-(x^2+2y^2)/10^4} = k$ (k is a positive constant). This equation is equivalent to $x^2 + 2y^2 = K \quad \Rightarrow \quad \dfrac{x^2}{\left(\sqrt{K}\right)^2} + \dfrac{y^2}{\left(\sqrt{K/2}\right)^2} = 1$, where $K = -10^4 \ln k$, a family of ellipses. We sketch level curves for $K = 1$, 2, 3, and 4. If the shark always swims in the direction of maximum increase of blood concentration, its direction at any point would coincide with the gradient vector. Then we know the shark's path is perpendicular to the level curves it intersects. We sketch one example of such a path.



(b) $\nabla C = -\frac{2}{10^4} e^{-(x^2+2y^2)/10^4}(x\,\mathbf{i} + 2y\,\mathbf{j})$. And ∇C points in the direction of most rapid increase in concentration; that is, ∇C is tangent to the most rapid increase curve. If $r(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ is a parametrization of the most rapid increase curve, then $\dfrac{d\mathbf{r}}{dt} = \dfrac{dx}{dt}\,\mathbf{i} + \dfrac{dy}{dt}\,\mathbf{j}$ is tangent to the curve, so $\dfrac{d\mathbf{r}}{dt} = \lambda \nabla C \;\Rightarrow\; \dfrac{dx}{dt} = \lambda\left[-\dfrac{2}{10^4}\,e^{-(x^2+2y^2)/10^4}\right]x$ and $\dfrac{dy}{dt} = \lambda\left[-\dfrac{2}{10^4}e^{-(x^2+2y^2)/10^4}\right](2y)$. Therefore, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = 2\,\dfrac{y}{x} \;\Rightarrow\; \dfrac{dy}{y} = 2\,\dfrac{dx}{x} \;\Rightarrow\; \ln|y| = 2\ln|x|$ so that $y = kx^2$ for some constant k. But $y(x_0) = y_0 \;\Rightarrow\; y_0 = kx_0^2 \;\Rightarrow\; k = y_0/x_0^2$ ($x_0 = 0 \;\Rightarrow\; y_0 = 0 \;\Rightarrow$ the shark is already at the origin, so we can assume $x_0 \neq 0$.) Therefore, the path the shark will follow is along the parabola $y = y_0(x/x_0)^2$.

3. (a) The area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$, where h is the height (the distance between the two parallel sides) and b_1, b_2 are the lengths of the bases (the parallel sides). From the figure in the text, we see that $h = x\sin\theta$, $b_1 = w - 2x$, and $b_2 = w - 2x + 2x\cos\theta$. Therefore, the cross-sectional area of the rain gutter is

$$\begin{aligned} A(x,\theta) &= \tfrac{1}{2}x\sin\theta\,[(w-2x) + (w - 2x + 2x\cos\theta)] = (x\sin\theta)(w - 2x + x\cos\theta) \\ &= wx\sin\theta - 2x^2\sin\theta + x^2\sin\theta\cos\theta,\;\; 0 < x \le \tfrac{1}{2}w, 0 < \theta \le \tfrac{\pi}{2} \end{aligned}$$

We look for the critical points of A: $\partial A/\partial x = w\sin\theta - 4x\sin\theta + 2x\sin\theta\cos\theta$ and $\partial A/\partial\theta = wx\cos\theta - 2x^2\cos\theta + x^2(\cos^2\theta - \sin^2\theta)$, so $\partial A/\partial x = 0 \;\Leftrightarrow\; \sin\theta\,(w - 4x + 2x\cos\theta) = 0 \;\Leftrightarrow\; \cos\theta = \dfrac{4x - w}{2x} = 2 - \dfrac{w}{2x}$ $\;(0 < \theta \le \frac{\pi}{2} \;\Rightarrow\; \sin\theta > 0)$. If, in addition, $\partial A/\partial\theta = 0$, then

$$\begin{aligned} 0 &= wx\cos\theta - 2x^2\cos\theta + x^2(2\cos^2\theta - 1) \\ &= wx\left(2 - \frac{w}{2x}\right) - 2x^2\left(2 - \frac{w}{2x}\right) + x^2\left[2\left(2 - \frac{w}{2x}\right)^2 - 1\right] \\ &= 2wx - \tfrac{1}{2}w^2 - 4x^2 + wx + x^2\left[8 - \frac{4w}{x} + \frac{w^2}{2x^2} - 1\right] = -wx + 3x^2 = x(3x - w) \end{aligned}$$

Since $x > 0$, we must have $x = \frac{1}{3}w$, in which case $\cos\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{3}$, $\sin\theta = \frac{\sqrt{3}}{2}$, $k = \frac{\sqrt{3}}{6}w$, $b_1 = \frac{1}{3}w$, $b_2 = \frac{2}{3}w$, and $A = \frac{\sqrt{3}}{12}w^2$. As in Example 14.7.6, we can argue from the physical nature of this problem that we have found a local maximum of A. Now checking the boundary of A, let $g(\theta) = A(w/2, \theta) = \frac{1}{2}w^2\sin\theta - \frac{1}{2}w^2\sin\theta + \frac{1}{4}w^2\sin\theta\cos\theta = \frac{1}{8}w^2\sin 2\theta$, $0 < \theta \le \frac{\pi}{2}$. Clearly, g is maximized when $\sin 2\theta = 1$ in which case $A = \frac{1}{8}w^2$. Also, along the line $\theta = \frac{\pi}{2}$, let $h(x) = A\left(x, \frac{\pi}{2}\right) = wx - 2x^2$, $\;0 < x < \frac{1}{2}w \;\Rightarrow\; h'(x) = w - 4x = 0 \;\Leftrightarrow\; x = \frac{1}{4}w$, and $h\left(\frac{1}{4}w\right) = w\left(\frac{1}{4}w\right) - 2\left(\frac{1}{4}w\right)^2 = \frac{1}{8}w^2$. Since $\frac{1}{8}w^2 < \frac{\sqrt{3}}{12}\,w^2$, we conclude that the local maximum found earlier was an absolute maximum; that is, the base and the sides are of equal length.

(b) If the metal were bent into a semicircular gutter of radius r, we would have $w = \pi r$ and $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{w}{\pi}\right)^2 = \frac{w^2}{2\pi}$.

Since $\frac{w^2}{2\pi} > \frac{\sqrt{3}\,w^2}{12}$, it *would* be better to bend the metal into a gutter with a semicircular cross-section.

4. Since $(x+y+z)^r/(x^2+y^2+z^2)$ is a rational function with domain $\{(x,y,z) \mid (x,y,z) \neq (0,0,0)\}$, f is continuous on $\mathbb{R}^3$ if and only if $\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z) = f(0,0,0) = 0$. Recall that $(a+b)^2 \leq 2a^2 + 2b^2$ and a double application of this inequality to $(x+y+z)^2$ gives $(x+y+z)^2 \leq 4x^2 + 4y^2 + 2z^2 \leq 4(x^2+y^2+z^2)$. Now for each r,

$$|(x+y+z)^r| = \left(|x+y+z|^2\right)^{r/2} = \left[(x+y+z)^2\right]^{r/2} \leq \left[4(x^2+y^2+z^2)\right]^{r/2} = 2^r(x^2+y^2+z^2)^{r/2}$$

for $(x,y,z) \neq (0,0,0)$. Thus,

$$|f(x,y,z) - 0| = \left|\frac{(x+y+z)^r}{x^2+y^2+z^2}\right| = \frac{|(x+y+z)^r|}{x^2+y^2+z^2} \leq 2^r\frac{(x^2+y^2+z^2)^{r/2}}{x^2+y^2+z^2} = 2^r(x^2+y^2+z^2)^{(r/2)-1}$$

for $(x,y,z) \neq (0,0,0)$. Thus, if $(r/2) - 1 > 0$, that is $r > 2$, then $2^r(x^2+y^2+z^2)^{(r/2)-1} \to 0$ as $(x,y,z) \to (0,0,0)$ and so $\lim\limits_{(x,y,z)\to(0,0,0)} (x+y+z)^r/(x^2+y^2+z^2) = 0$. Hence for $r > 2$, f is continuous on $\mathbb{R}^3$. Now if $r \leq 2$, then as $(x,y,z) \to (0,0,0)$ along the x-axis, $f(x,0,0) = x^r/x^2 = x^{r-2}$ for $x \neq 0$. So when $r = 2$, $f(x,y,z) \to 1 \neq 0$ as $(x,y,z) \to (0,0,0)$ along the x-axis and when $r < 2$ the limit of $f(x,y,z)$ as $(x,y,z) \to (0,0,0)$ along the x-axis doesn't exist and thus can't be zero. Hence for $r \leq 2$ f isn't continuous at $(0,0,0)$ and thus is not continuous on $\mathbb{R}^3$.

5. Let $g(x,y) = xf\left(\frac{y}{x}\right)$. Then $g_x(x,y) = f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right)$ and $g_y(x,y) = xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = f'\left(\frac{y}{x}\right)$. Thus, the tangent plane at (x_0, y_0, z_0) on the surface has equation

$$z - x_0 f\left(\frac{y_0}{x_0}\right) = \left[f\left(\frac{y_0}{x_0}\right) - y_0x_0^{-1}f'\left(\frac{y_0}{x_0}\right)\right](x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0) \quad \Rightarrow$$

$\left[f\left(\frac{y_0}{x_0}\right) - y_0x_0^{-1}f'\left(\frac{y_0}{x_0}\right)\right]x + \left[f'\left(\frac{y_0}{x_0}\right)\right]y - z = 0$. But any plane whose equation is of the form $ax + by + cz = 0$ passes through the origin. Thus, the origin is the common point of intersection.

6. (a) At $(x_1, y_1, 0)$ the equations of the tangent planes to $z = f(x,y)$ and $z = g(x,y)$ are

$$P_1\colon\ z - f(x_1,y_1) = f_x(x_1,y_1)(x - x_1) + f_y(x_1,y_1)(y - y_1)$$

and

$$P_2\colon\ z - g(x_1,y_1) = g_x(x_1,y_1)(x - x_1) + g_y(x_1,y_1)(y - y_1)$$

respectively. P_1 intersects the xy-plane in the line given by $f_x(x_1,y_1)(x-x_1) + f_y(x_1,y_1)(y-y_1) = -f(x_1,y_1)$, $z = 0$; and P_2 intersects the xy-plane in the line given by $g_x(x_1,y_1)(x-x_1) + g_y(x_1,y_1)(y-y_1) = -g(x_1,y_1)$, $z = 0$. The point $(x_2, y_2, 0)$ is the point of intersection of these two lines, since $(x_2, y_2, 0)$ is the point where the line of intersection of the two tangent planes intersects the xy-plane. Thus (x_2, y_2) is the solution of the simultaneous equations

$$f_x(x_1, y_1)(x_2 - x_1) + f_y(x_1, y_1)(y_2 - y_1) = -f(x_1, y_1)$$

and

$$g_x(x_1, y_1)(x_2 - x_1) + g_y(x_1, y_1)(y_2 - y_1) = -g(x_1, y_1)$$

For simplicity, rewrite $f_x(x_1, y_1)$ as f_x and similarly for f_y, g_x, g_y, f and g and solve the equations $(f_x)(x_2 - x_1) + (f_y)(y_2 - y_1) = -f$ and $(g_x)(x_2 - x_1) + (g_y)(y_2 - y_1) = -g$ simultaneously for $(x_2 - x_1)$ and $(y_2 - y_1)$. Then $y_2 - y_1 = \dfrac{gf_x - fg_x}{g_x f_y - f_x g_y}$ or $y_2 = y_1 - \dfrac{gf_x - fg_x}{f_x g_y - g_x f_y}$ and $(f_x)(x_2 - x_1) + \dfrac{(f_y)(gf_x - fg_x)}{g_x f_y - f_x g_y} = -f$ so $x_2 - x_1 = \dfrac{-f - [(f_y)(gf_x - fg_x)/(g_x f_y - f_x g_y)]}{f_x} = \dfrac{fg_y - f_y g}{g_x f_y - f_x g_y}$. Hence $x_2 = x_1 - \dfrac{fg_y - f_y g}{f_x g_y - g_x f_y}$.

(b) Let $f(x, y) = x^x + y^y - 1000$ and $g(x, y) = x^y + y^x - 100$. Then we wish to solve the system of equations $f(x, y) = 0$, $g(x, y) = 0$. Recall $\dfrac{d}{dx}[x^x] = x^x(1 + \ln x)$ (differentiate logarithmically), so $f_x(x, y) = x^x(1 + \ln x)$, $f_y(x, y) = y^y(1 + \ln y)$, $g_x(x, y) = yx^{y-1} + y^x \ln y$, and $g_y(x, y) = x^y \ln x + xy^{x-1}$. Looking at the graph, we estimate the first point of intersection of the curves, and thus the solution to the system, to be approximately $(2.5, 4.5)$. Then following the method of part (a), $x_1 = 2.5$, $y_1 = 4.5$ and

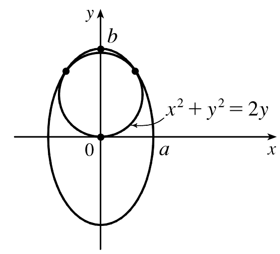
$$x_2 = 2.5 - \frac{f(2.5, 4.5)\, g_y(2.5, 4.5) - f_y(2.5, 4.5)\, g(2.5, 4.5)}{f_x(2.5, 4.5)\, g_y(2.5, 4.5) - f_y(2.5, 4.5)\, g_x(2.5, 4.5)} \approx 2.447674117$$

$$y_2 = 4.5 - \frac{f_x(2.5, 4.5)\, g(2.5, 4.5) - f(2.5, 4.5)\, g_x(2.5, 4.5)}{f_x(2.5, 4.5)\, g_y(2.5, 4.5) - f_y(2.5, 4.5)\, g_x(2.5, 4.5)} \approx 4.555657467$$

Continuing this procedure, we arrive at the following values. (If you use a CAS, you may need to increase its computational precision.)

$x_1 = 2.5$	$y_1 = 4.5$
$x_2 = 2.447674117$	$y_2 = 4.555657467$
$x_3 = 2.449614877$	$y_3 = 4.551969333$
$x_4 = 2.449624628$	$y_4 = 4.551951420$
$x_5 = 2.449624628$	$y_5 = 4.551951420$

Thus, to six decimal places, the point of intersection is $(2.449625, 4.551951)$. The second point of intersection can be found similarly, or, by symmetry it is approximately $(4.551951, 2.449625)$.



7. Since we are minimizing the area of the ellipse, and the circle lies above the x-axis, the ellipse will intersect the circle for only one value of y. This y-value must satisfy both the equation of the circle and the equation of the ellipse. Now $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad x^2 = \frac{a^2}{b^2}\left(b^2 - y^2\right)$. Substituting into the equation of the circle gives $\frac{a^2}{b^2}\left(b^2 - y^2\right) + y^2 - 2y = 0 \quad \Rightarrow \quad \left(\frac{b^2 - a^2}{b^2}\right)y^2 - 2y + a^2 = 0$.

In order for there to be only one solution to this quadratic equation, the discriminant must be 0, so $4 - 4a^2\,\frac{b^2 - a^2}{b^2} = 0 \quad \Rightarrow$ $b^2 - a^2b^2 + a^4 = 0$. The area of the ellipse is $A(a,b) = \pi ab$, and we minimize this function subject to the constraint $g(a,b) = b^2 - a^2b^2 + a^4 = 0$.

Now $\nabla A = \lambda \nabla g \quad \Leftrightarrow \quad \pi b = \lambda(4a^3 - 2ab^2),\ \pi a = \lambda(2b - 2ba^2) \quad \Rightarrow \quad \lambda = \frac{\pi b}{2a(2a^2 - b^2)}$ **(1)**, $\lambda = \frac{\pi a}{2b(1 - a^2)}$ **(2)**, $b^2 - a^2b^2 + a^4 = 0$ **(3)**. Comparing **(1)** and **(2)** gives $\frac{\pi b}{2a(2a^2 - b^2)} = \frac{\pi a}{2b(1 - a^2)} \quad \Rightarrow$ $2\pi b^2 = 4\pi a^4 \quad \Leftrightarrow \quad a^2 = \frac{1}{\sqrt{2}}\,b$. Substitute this into **(3)** to get $b = \frac{3}{\sqrt{2}} \quad \Rightarrow \quad a = \sqrt{\frac{3}{2}}$.

8. Let $\mathbf{u} = \langle a, b, c \rangle$ and $\mathbf{v} = \langle x, y, 1 \rangle$, so $|\mathbf{u}| = \sqrt{a^2 + b^2 + c^2}$, $|\mathbf{v}| = \sqrt{x^2 + y^2 + 1}$, and $\mathbf{u} \cdot \mathbf{v} = ax + by + c$. Then by the Cauchy-Schwarz Inequality, $|\mathbf{u} \cdot \mathbf{v}| \le |\mathbf{u}|\ |\mathbf{v}| \quad \Rightarrow \quad |ax + by + c| \le \sqrt{a^2 + b^2 + c^2}\,\sqrt{x^2 + y^2 + 1}$. Squaring both sides, we have $(ax + by + c)^2 \le \left(a^2 + b^2 + c^2\right)\left(x^2 + y^2 + 1\right) \quad \Rightarrow \quad \frac{(ax + by + c)^2}{x^2 + y^2 + 1} \le a^2 + b^2 + c^2$ (since $x^2 + y^2 + 1 > 0$). Thus $f(x,y) = \frac{(ax + by + c)^2}{x^2 + y^2 + 1} \le a^2 + b^2 + c^2$. We have equality if $(ax + by + c)^2 = \left(a^2 + b^2 + c^2\right)\left(x^2 + y^2 + 1\right)$ or equivalently, $c^2\left[(a/c)x + (b/c)y + 1\right]^2 = c^2\left[(a/c)^2 + (b/c)^2 + 1\right]\left(x^2 + y^2 + 1\right)$ which is true when $x = a/c$ and $y = b/c$. Thus, the maximum value of f is $f(a/c, b/c) = a^2 + b^2 + c^2$.