

Writing Sample: Undergraduate, coursework assessment

The Role of Money in Incomplete Market

A Review on Martine Quinzii's Work in General Equilibrium Theory

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1 Introduction

This paper critically reviews Martine Quinzii's work in General Equilibrium (GE) theory. The review mainly focuses on her contribution towards the role of money in GE under market incomplete conditions (GEI). Quinzii has provided valuable theorems and insights, particularly under incomplete stock market and non-neutrality of money and its role in GE.

2 An Overview of Martine Quinzii

2.1 A Brief Biography

Martine Quinzii was a leading researcher in mathematical economics. Quinzii obtained her PhD in 1988 at the University of Paris. She was a faculty member at the University of Southern California and then UC Davis before she retired. She was a Fellow of the Econometric Society since 2000, and an Economic Theory Fellow since 2011 ([2]).

Quinzii's academic career starts with her PhD thesis on the core in production economies and its decentralization by marginal cost pricing[4], which has provided rigorous proof of her previous work on the sufficient condition for a core to exist under a production economy with increasing return.[12]. She then focuses on the financial market, issues with money and the incomplete market. She and Michael Magill, her partner in research and life, offered significant contributions in such fields for more than a quarter century.[2]

With significant expertise and intellectual capability, she has made a great contribution to the General Equilibrium analysis and incomplete market. Her book, *Theory of Incomplete Market* was regarded as the best reference in such topic[2].

2.2 Overall Contribution in Economics

Quinzii's contributions were not limited to General Equilibrium theory. She had devoted her excellence to mathematical techniques and rigorous analysis of financial market studies and indivisible goods.

She provided a proof of the core existence under the condition of finitely many individuals each owning at most 1 indivisible commodity (e.g. Housing) and some dividable resources (e.g. money) under competitive market[13] and this is the first general theorem on indivisible good. Years later, she published another paper[14] on the consumption under a similar endowment bundle. Since the product is indivisible, the price is no longer continuous. Quinzii proved the price of such indivisible goods is between the social value of the object and its value in its second-best use.[14]

Quinzii's other great contributions are in the financial market. She has provided a theoretical framework to study the different effects of social welfare under nominal or index bonds[7]. After Quinzii and Magill developed their general equilibrium under an incomplete market, they have studied some specific situations under this framework. For example, in the study of the managerial compensation equilibrium model, Quinzii et al. demonstrated the market inefficiency under equilibrium due to the differences of *Market Value maximization* and *Expected utility maximization*, at the same time moral hazard accrued[11]. In their study of moral hazard under General equilibrium setting, Quinzii et al. suggested a constrained Pareto optimal under reasonable spanning assumption is the equilibrium with rational, competitive price perceptions (RCP) even under moral hazard condition[8].

3 Quinzii's Work in General Equilibrium

3.1 Background

Quinzii's work is built upon the notation of an incomplete market. In GE, a market is complete if there exists a price for each commodity at every state[10]. Incomplete market lack of Arrow-Debreu security. Therefore some interstate wealth transactions would be blocked.

3.1.1 Arrow-Debreu Security

Arrow [1] suggested a way of allocating resources across states by *security*, which can realize all types of wealth reallocation across states.

Consider the claimed commodity bundle $x_{is} = \{x_{is1}, x_{is2} \dots x_{isC}\}$, which indicates the ownership of individual i 's commodity under the state s . For any commodity c , if s happened, individual i would claim x_{isc} unit(s) of c . Such a claim has a cost of p_{sc}^- , and the monetary value of such is c under s is notated as p_{sc} [1].

The security of such a good, with the price q_s will be paid by the individual without knowing whether s would happen or not. If s happened, i will receive 1 unit of wealth, if not then receive 0 unit of wealth.[1]

Hence, under competitive circumstances, we have:

$$\begin{aligned} x_{isc} \cdot p_{sc} \cdot 1 &= x_{isc} \cdot p_{sc}^- \cdot q_s \\ p_{sc} &= p_{sc}^- \cdot q_s \end{aligned}$$

This implies per unit spend of claim under security should be overall the per unit gain of claim. If Arrow security exists, the security market can achieve optimal allocation[1].

3.1.2 Market Incomplete Model

Despite the high level of elegance and rigor of Arrow-Debreu's GE model, it was criticised for its inconsistency with the empirical evidence[10]. On this, Hart constructed his model on the market incomplete equilibrium[3], providing a solid foundation for Quinzii's work.

Quinzii and Magill have provided a summary of GEI setup[9]:

*For a two period economy, $t = (0, 1)$, there are I individuals, L types of commodities, and S possible states that could occur in the future. Consider at $t = 0$, the state of the world is 0, for all individuals i in a market are endowed with $\omega_0^i = (\omega_{1,0}^i, \omega_{2,0}^i \dots \omega_{L,0}^i)$. At state s , i will receive another endowment $\omega_s^i = (\omega_{1,s}^i, \omega_{2,s}^i \dots \omega_{L,s}^i)$ without known which state it would be. Thus, i is faced with endowment vector: $\omega^i = (\omega_0^i, \omega_1^i, \dots, \omega_S^i)$ Each individuals have well behaved utility function, that is strictly quasi-concave, monotonic and differentiable at all points. All commodities are traded in **spot market**.*

*In state s , the commodities have the value of $p_s = (p_{s,1}, p_{s,2}, \dots, p_{s,L})$, with state unknown, the price vector is $p = (p_0, p_1, \dots, p_S)$. There are J types of securities, comes with the price $q = (q_1, q_2, \dots, q_J)$. For security j in state s , it would offer payment $n_{(s,j)}$. All the entries generates a $S \times J$ payoff matrix called N . N_s is the s 'th row of such matrix. Individuals can buy or sell (buy negative amount) securities in **financial market**.*

Individual i has a decision under each state on how much to consume. In state s , i consume $x_s^i = (x_{s,1}^i, \dots, x_{s,L}^i)$, when $s = 0$, such bundle implies the the consumption when $t = 0$. The consumption bundle of individual i is $x^i = (x_0^i, x_1^i, \dots, x_S^i)$. When $t = 0$, i decides her investment portfolio $z^i = (z_1^i, z_2^i, \dots, z_J^i)$ on how many securities she would purchase.

Under this setup, the individual is facing a utility maximization problem for choosing (x^i, z^i) under the constrain:

$$\begin{cases} p_0 \cdot \omega_0^i &= p_0 \cdot x_0^i + q \cdot z^i \\ p_s \cdot \omega_s^i + N_s \cdot z^i &= p_s \cdot x_s^i \end{cases} \text{ This implies: } \begin{cases} p_0 \cdot (x_0^i - \omega_0^i) &= -q \cdot z^i \\ p_s \cdot (x_s^i - \omega_s^i) &= N_s \cdot z^i \end{cases} \text{ For all } i \text{ in } [1; I] [9]$$

This mathematics notation shows that, when $t = 0$, all individuals spend all their endowment (in monetary value), to consume and to invest. $N_s \cdot z^i$ is the reward earned from security at state s subjected portfolio bundle.

The equilibrium notion is as regular: If $(\bar{x}, \bar{\omega})$ is an equilibrium.

- Individuals Maximize utility under the constrain.
- (Spot market clearing) $\sum_{i=1}^I (\bar{x}_s^i - \bar{\omega}_s^i) = 0$ For all s .
- (Market Clearing on securities.) $\sum_{i=1}^I (\bar{z}_s^i - \bar{\omega}_s^i) = 0$ For all s .

Here the market is incomplete if $\text{Rank}(N) < S$ [9]. Indeed, consider $N \cdot z^i$, the outcome of this dot product is a vector in R^S , represent the outcome of wealth under different states. This can be interpreted as an inter-state wealth transfer.

If $\text{Rank}(N) < S$, there is less than N linearly independent vectors can be generated by $N \cdot z^i$. To span a R^S space, at least S linearly independent vectors are required. Thus, the outcome of $N \cdot z^i$ does not span R^S . This implies there exist some inter-state wealth transaction is unable to achieve with such N , in economics sense, it implies market incomplete. It was proved that this equilibrium indeed exist and is the optimal[3].

3.2 Quinzii's Contribution

Magill performed a survey, asking people the consequences of market incomplete[10]. There are 3 results standout:

- Non-neutrality of money.
- Conflicting objectives of firms.
- Intrinsic inadequacy of such decentralized system.

Quinzii's work follow closely with these aspects, here the reviewer present her work on the role of money.

3.2.1 A Pure Exchange Economy under Monetary Policy

Under Quinzii's work in 1988[6], they defined the notion of *Monetary Equilibrium*, and then show such equilibrium exist, finite and their real effect under different situations.

The Model was set up in a two period ($t = 0, 1$) pure exchange economy with the notion of money, provided by a centralized government. When $t = 0$, the economy is in state 0. For each state $s \in 0, 1, \dots, S$, there are 3 sub-periods, denoted respectively as s_1, s_2, s_3 . In s_1 , individual sell their endowment. In s_2 , individual make investment decision. In s_3 , individual spend the rest of their money. Central government perform a monetary policy $m = (m_0, m_1, \dots, m_S)$ by inject m_s unit of money into the economy at state s .

The notation is almost consistent with the previous GEI model, with I individuals, J securities, here they use 'Assets' to refer to such, L types of commodities in S states. The assets are endowed with a payoff matrix N , and have the price for purchase q .

For L commodities, there are 2 types of prices endowed. $p' = (p'_0, p'_1, \dots, p'_S)$ is the price when selling the commodities, $p = (p_0, p_1, \dots, p_S)$ is the price when buying the commodities.

When individual i is making investment decision, he can buy a portfolio $z = (z_1, z_2, \dots, z_J)$ and save z_0 in cash, used for investment or consumption in the next period.

Define $\epsilon = (-1, 1, \dots, 1)$ as a vector in \mathbf{R}^{S+1} . A new matrix W is defended as a $(S+1) \times J$ matrix. With the first row being the row vector of q , and N underneath. Consider $W \cdot z_1 + z_0 \cdot \epsilon$, it gives a $(S+1)$ column vector, indicated how much money was earned/ spent in each state.

A notion of wealth was introduced as "square product". It is defined as $p * x^i = (p_0 \cdot x_0^i, p_1 \cdot x_1^i, \dots, p_S \cdot x_S^i)$. This generates a vector, each entry of which is the monetary value of the commodity bundle in that state.

The Author made the following assumptions:

- Individuals sell all their endowments.
- Individuals have a smooth utility function that has a well defined expected value. In each state, no individual would receive 0 endowment and the total endowment is far larger than 0.
- There exist a vector $v = (v_0, v_1, \dots, v_S) \in \mathbf{R}^{S+1}$, such that for any state s , $p_s = v_s \cdot p'_s$.

3.2.2 Pure Exchange Economies' Monetary Equilibrium

[6] A monetary equilibrium is a set of choices $(\bar{x}, \bar{z}, \bar{z}_0)$ and price: $(\bar{p}', \bar{p}, \bar{q}, \bar{v})$, such that:

- $\bar{x} = \operatorname{argmax}(U^i(x^i))$ constrained under $p * \bar{x}^i - p' * \bar{\omega}^i = W \cdot z_1 + z_0 \cdot \epsilon$
- $\sum_{i=1}^I (\bar{x}^i - \omega^i)$ (Spot Market Clearing)
- $\sum_{i=1}^I \bar{z}^i = 0$ (Financial Market Clearing)
- $\bar{p}^0 \sum_{i=1}^I \bar{x}_0^i + \sum_{i=1}^I \bar{z}_0^i = M_0$ (Monetary Market Clearing initially)
- $\bar{p}^s \sum_{i=1}^I \bar{x}_s^i + \sum_{i=1}^I \bar{z}_0^i = M_s$ (Monetary Market Clearing in all states)
- $\bar{p}_s = \bar{v}_s \cdot p'_s$ (Assumption holds)

The existence of the equilibrium was proved using Kakutani Fixed Point Theorem. There are only finitely many equilibria.

Quinzii then defined the notion of *abstract equilibrium*[6]. Which is a variation of monetary equilibrium. This variation is possible since there exist a clear relationship between p' and p . The original notation of equilibrium could be rewrite. By finding the abstract equilibrium, we observed an important parameter: $v_0 = (M_0 - \sum_{i=1}^I \bar{z}_0^i)/M_0$.

All abstract equilibrium could be map back to monetary equilibrium and vice-versa by a factor β . The analytical solution of abstract equilibrium is correspond to the solution of equations consist of v_0, P, β, ω, M . From here, there is a correspondence between v_0 and monetary equilibrium[6].

3.2.3 The Neutrality of Money in Pure Exchange Economy

When $v_0 = 1$, $1 = (M_0 - \sum_{i=1}^I \bar{z}_0^i)/M_0$. This condition holds if and only if $\sum_{i=1}^I \bar{z}_0^i/M_0 = 0$, then implies $\sum_{i=1}^I \bar{z}_0^i = 0$. If $v_0 < 1$, then $\sum_{i=1}^I \bar{z}_0^i > 0$.

Since some v_0 can be found in all equilibrium, all equilibrium either have $v_0 < 1$ or $v_0 = 1$. When $v_0 = 1$, all the money earned in $t = 0$ are in exchange for good or potential future asset. In this case, money serve only as an media to transfer between goods and potential welfare. If $v_0 < 1$, there exist some agents bring money to the future period. This time, money not only serve as a media of exchange, but also a storage of value.

When some money are served as storage of value, those money introduced the income effect[6]. Those money has some real value as it can be used to purchase assets or commodities and they are relative fixed, changing M_s would have a real effect on endowment reallocation.

For an economy with no individual bring money to the future, all the money went back to the central government and is ready for another round of distribution. When government inject more money in the economy, the nominal money increase but no impact on commodity. However, the injected money also does not affect the payoff matrix N , the relative price could still matters due to the income is fixed subjected to the monetary policy from financial market. The question then become, how can a payoff matrix behave, so that there exist no real effect of money.

When money only works as a medium in transaction, if $\operatorname{Rank}(N) = S$, then the allocation is independent from money level. If not, then there is some cases of economy, money would impact the allocation[6]. Consider if $\operatorname{Rank}(N) < S$, we have already showed that such there exist some interstate wealth transaction could not be achieved. However, since the money already have no real effect on spot market, it only would be neutral if it has no real effect on financial market. This requires the market structure to transfer the inflow of money from such state to another. Since the interstate transaction might not always be viable, an incomplete market would suffer from real effect of money.

3.2.4 A Production Economy under Monetary Policy

When money is not neutral, it has a real effect on resources allocation. Quinzii extend the model to include the production side of the economy. In her study in 1989[5], money is used only as a medium of exchange. Therefore, all money has a positive interest rate r_1 . The central government performs a monetary policy $m = (m_0, m_1, \dots, m_S)$ by injecting m_s units of money into the economy at state s .

There are two types of financial instrument: there are J companies each offering their share at q_j performed the same with *asset* mentioned previously with a $S \times J$ payoff matrix D and a bond with rank K , whose payoff matrix is denoted by N . The first column of this matrix is a *risk-less bond*, allowing individuals to transfer money from $t = 0$ to $t = 1$ with the adjustment of real interest. That is the first column of N is a column vector $(1, 1, \dots, 1) \in \mathbf{R}^S$, and the price of this bond $q'_1 = \frac{1}{1+r_1}$. With such a bond and assuming $M_s \geq M_0$ if $s \in [1 : S]$, individuals will not be better-off to keep cash instead of investing such cash in the risk-less bond, therefore no incentive for saving.

In such case, an individual's investment portfolio is a vector in \mathbf{R}^{J+K} , with the first J entries denoting the purchasing decision on the share of a company, and the later K entries denoting the purchase decision on a bond. A total payoff matrix W is a $(S+1) \times (J+K)$ matrix, with the first being the negative value of the price of shares and bonds, and D and N underneath:

$$W = \begin{bmatrix} q_1 & \dots & q_J & q'_1 & \dots & q'_K \\ d_{1,1} & \dots & d_{1,J} & n_{1,1} & \dots & n_{1,K} \\ & \dots & & & \dots & \\ d_{S,1} & \dots & d_{S,J} & n_{S,1} & \dots & n_{S,K} \end{bmatrix}$$

On the production side, each company chooses a production bundle $y^i = (y_1^i, \dots, y_S^i)$, where $y_s^i = (y_{s,1}^i, \dots, y_{s,L}^i)$. Each company has a well-defined production technology, in which they are able to minimize the production cost conditioned to the output, thus the profit at each level of output is a fraction of revenue.

Firm j has the endowment η^j under each state. After selling some share, they hold ξ^j of the stock, and own ξ'' of bonds. The consumers are endowed by ω , they hold z^j of stock, and own z'' of bonds.

3.2.5 Production Economies' Monetary Equilibrium[5]

The Equilibrium is a set of actions and prices: A monetary equilibrium is a set of choices, such that:

- Consumers are maximizing their utility from consumption subject to wealth. The wealth consists of investing earnings (or loss), original endowment.
- Companies maximize their profit by choosing the optimal quantity of production and the share of ownership to put into the financial market.
- (Spot market clearing) $\sum_{i=1}^I (\bar{x}^i - \omega^i) = \sum_{j=1}^J (\bar{y}^j - \eta^j)$
- (Share and Bond market clearing) $\sum_{i=1}^I \bar{z}^i + \sum_{j=1}^J \bar{\xi}^j = (1, 1, \dots, 1) \in \mathbf{R}^J$ and $\sum_{i=1}^I \bar{z}''^i + \sum_{j=1}^J \bar{\xi}''^j = 0$
- (Monetary market clearing) $M_0 = p(\sum_{i=1}^I \omega_0^i + \sum_{j=1}^J \eta_0^j)$ and $M_s = p(\sum_{i=1}^I \omega_s^i + \sum_{j=1}^J (\eta_s^i + y_s^j))$

3.2.6 The Neutrality of Money in Production Economy

The conclusion is basically consistent with a pure exchange economy. When the financial market is complete, the money is neutral and has no real effect. When the financial market is not complete, there exists some type of allocation and monetary policy such that the money has a real effect[5].

Here Quinzii also proved that the incorrect anticipation or unanticipated monetary policy would have a real effect regardless of the completeness of the market[5]. In short, it is due to the lack of a corresponding 'Mechanism' to transfer the effect of money across states despite the fact that in a complete market this mechanism exists, individuals fail to adapt to it due to the lack of anticipation.

4 The Assessment Quinzii's Contribution

Quinzii's work is highly theoretical and mathematical, providing a rigorous proof of the stated theorem. As part of her study on GEI, her research partly reveals the nature of money as a unique 'commodity', and indeed rigorously showed the role of money can be different in different type of monetary equilibrium.

In Arrow-Debreu's work, money does not play an important role as it is assumed to be neutral in their specific assumption on the security market. Quinzii weakened this assumption and offered a generalized theory on the role of money when it fails to be neutral. Her work fills the gap in knowledge regarding the role of monetary policy in general equilibrium theory, connecting equilibrium analysis in microeconomics with macroeconomics policy.

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