ANALOG 11.14 Q-25

EE23BTECH11207 -KAILASH.C*

QUESTION:

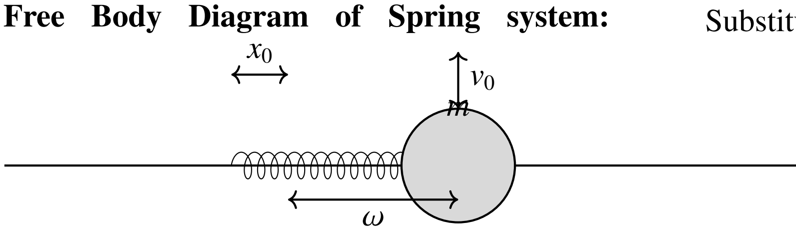
A mass attached to a spring is free to oscillate, with angular velocity ω, in a horizontal plane without friction or damping. It is pulled to a distance x_0 and pushed towards the centre with a velocity v_0 at time t = 0. Determine the amplitude of the resulting oscillations in terms of the parameters ω, x_0, and v_0 . [Hint : Start with the equation $x = a\cos(\omega t + \theta)$ and note that the initial velocity is negative.]

SOLUTION:

1) **Input Parameters:**

Symbols	Definition
x	Displacement
t	Time
v	Velocity
ω	Angular velocity
θ	Phase constant
m	Mass
a	Acceleration
k	Force constant

2) **Free Body Diagram of Spring system:**



3) From Newtons second law of Motion,

$$F = ma \tag{1}$$

We also know that From Hooke's Law,

$$F = -kx \tag{2}$$

By equation-(1) and (2):

$$ma = -kx \tag{3}$$

$$m(\frac{dx^2}{dt^2}) = -kx \tag{4}$$

$$\frac{dx^2}{dt^2} = \frac{-kx}{m} \tag{5}$$

We know that $\omega = \sqrt{\frac{k}{m}}$

$$\frac{dx^2}{dt^2} = -\omega^2 x \tag{6}$$

$$\frac{dx^2}{dt^2} + \omega^2 x = 0 \tag{7}$$

4) It is given that $\frac{dx}{dt} = -v_0$ at t=0s
By using the laplace trasnformation formula:$L(f''(t)) = s^2 L(f(t)) - sf(0) - f'(0)$ provided that $s > 0$ and By taking Laplace transformation in eq-(7),

$$(s^2 + \omega^2)X(s) - sx_0 - v_0 = 0 \tag{8}$$

$$X(s) = \frac{sx_0 + x_0'}{s^2 + \omega^2} \tag{9}$$

Let us assume that the shm equation for displacement is:

$$x = A(\cos\omega t + \theta) \tag{10}$$

Substituting eq-(10) in eq-(9):

$$X(s) = \frac{s(A\cos\theta) + x'(0)}{s^2 + \omega^2} \tag{11}$$

$$v_0 = \frac{dx_0}{dt} \tag{12}$$

$$v_0 = -A\omega\sin(\theta) \tag{13}$$

Using eq-(13) in eq-(11):

$$X(s) = \frac{s(A\cos\theta) + A\omega\sin(\theta)}{s^2 + \omega^2} \tag{14}$$

$$X(s) = \frac{A(s\cos(\theta) + \omega\sin(\theta))}{(s + i\omega)(s - i\omega)} \tag{15}$$

$$X(s) = \frac{A}{2i\omega}\left(\frac{1}{s - i\omega} - \frac{1}{s + i\omega}\right) \tag{16}$$

Using the inverse laplace transformation Formulas:
IL($\frac{1}{s-i\omega}$)=$e^{i\omega t}$ and IL($\frac{1}{s+i\omega}$)=$e^{-i\omega t}$,

We apply Inverse Laplace Transformation in eq-(16):

$$X(s) = \frac{A}{i\omega}(e^{i\omega t} - e^{-i\omega t}) \tag{17}$$

$$x(t) = \frac{A}{i\omega}\sin(\omega t) \tag{18}$$

$$x(t) = \frac{-iA\sin\omega t}{\omega} \tag{19}$$

Since $\sin(\omega t) = \sin(\omega t+\pi)$ we can write eq-(19) as:

$$x(t) = A\cos(\omega t + \theta) \tag{20}$$

Hence it is the correct displacement equation

5) Let at time t=0s, x=x_0
Substituting t=0s in equation (20):

$$x = A\cos(\omega(0) + \theta) \tag{21}$$

$$x_0 = A\cos(\theta) \tag{22}$$

$$v_0 = \frac{dx_0}{dt} \tag{23}$$

$$v_0 = -A\omega\sin(\omega(0) + \theta) \tag{24}$$

$$v_0 = -A\omega\sin(\theta) \tag{25}$$

$$-A\sin(\theta) = \frac{v_0}{\omega} \tag{26}$$

Squaring and Adding eq-(22) and eq-(25):

$$(A\cos(\theta))^2 + (A\sin(\theta)^2) = (x_0)^2 + \left(\frac{v_0}{\omega}\right)^2 \tag{27}$$

$$A^2(\cos^2(\theta) + \sin^2(\theta)) = (x_0)^2 + \left(\frac{v_o}{\omega}\right)^2 \tag{28}$$

$$A^2 = (x_0)^2 + (\frac{v_o}{\omega})^2 \tag{29}$$

Answer:
Amplitude of the resulting oscillation is: $\sqrt{(x_0)^2 + (\frac{v_0}{\omega})^2}$