ANALOG 11.14 Q-25

EE23BTECH11207 -KAILASH.C*

QUESTION:

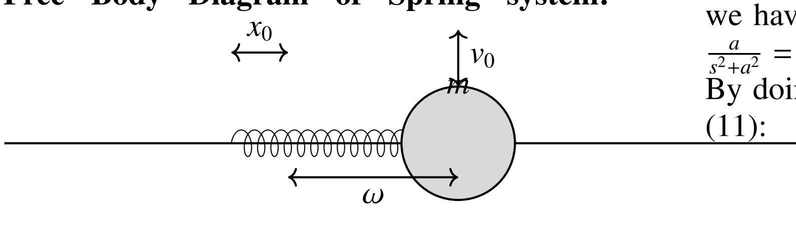
A mass attached to a spring is free to oscillate, with angular velocity ω, in a horizontal plane without friction or damping. It is pulled to a distance x_0 and pushed towards the centre with a velocity v_0 at time t = 0. Determine the amplitude of the resulting oscillations in terms of the parameters ω, x_0, and v_0 . [Hint : Start with the equation $x = a\cos(\omega t + \theta)$ and note that the initial velocity is negative.]

SOLUTION:

1) **Input Parameters:**

Symbols	Definition
x	Displacement
t	Time
v	Velocity
ω	Angular velocity
θ	Phase difference
m	Mass
a	Acceleration
k	Force constant
F_r	Restoring Force

2) **Free Body Diagram of Spring system:**



3) By Hooke's Law:

$$ma = -kx \tag{1}$$

$$m\frac{d^2x}{dt^2} = -kx \tag{2}$$

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{3}$$

4) Applying Laplace transformation on eq-(3):

$$mL[x"] + kL[x] = 0 \tag{4}$$

Using the laplace transformation formula:$L[x"] = s^2X(s) - sx(t = 0) - x'(t = 0)$ in eq-(4):

$$m[s^2X(s) - sx(0) - x'(0)] + kX(s) = 0 \tag{5}$$

At boundary Conditions:
$v(t = 0) = 0$ and $x(t = 0) = x_0$:

$$ms^2X(s) - msx_0 + kX(s) = 0 \tag{6}$$

$$X(s)[ms^2 + k] = msx_0 \tag{7}$$

$$(X(s)[ms^2 + k]).\mu(t) = msx_0 \tag{8}$$

Laplace transformation of $\mu(t)$ is $\frac{1}{s}$

$$X(s) = x_0\frac{ms}{s(ms^2 + k)} \tag{9}$$

$$X(s) = x_0\frac{s}{s(s^2 + \frac{k}{m})} \tag{10}$$

We know that $\sqrt{\frac{k}{m}} = \omega$:

$$X(s) = x_0\frac{s}{s(s^2 + \omega^2)} \tag{11}$$

5) By the inverse laplace transformation formulas we have:
$\frac{a}{s^2+a^2} = cos(at)$ and $\frac{1}{s} = 1$
By doing inverse laplace transformation in eq-(11):

$$x(t) = x_0L^{-1}\left[\frac{a}{s^2 + a^2}\right] \tag{12}$$

$$x(t) = x_0\cos\omega t \tag{13}$$

Eq-(13) represents the displacement equation From mean position.Amplitude is defined as maximum displacement from mean position i.e $x_0 = A$.
At a given random time the mass system has a phase difference from the mean position:

$$x(t) = A\cos(\omega t + \theta) \tag{14}$$

Eq-(14) represents the S.H.M Displacement equation.

6) Let at time t=0s, x=x_0
   Substituting t=0s in equation (13):

$$x = A\cos(\omega(0) + \theta) \tag{15}$$
$$x_0 = A\cos(\theta) \tag{16}$$
$$v_0 = \frac{dx_0}{dt} \tag{17}$$
$$v_0 = -A\omega\sin(\omega(0) + \theta) \tag{18}$$
$$v_0 = -A\omega\sin(\theta) \tag{19}$$
$$-A\sin(\theta) = \frac{v_0}{\omega} \tag{20}$$

   Squaring and Adding eq-(16) and eq-(20):

$$(A\cos(\theta))^2 + (A\sin(\theta)^2) = (x_0)^2 + \left(\frac{v_0}{\omega}\right)^2 \tag{21}$$
$$A^2(\cos^2(\theta) + \sin^2(\theta)) = (x_0)^2 + \left(\frac{v_o}{\omega}\right)^2 \tag{22}$$

$$A^2 = (x_0)^2 + (\frac{v_o}{\omega})^2 \tag{23}$$

Answer:

Amplitude of the resulting oscillation is: $\sqrt{(x_0)^2 + (\frac{v_0}{\omega})^2}$

Plot of Amplitude vs x_0