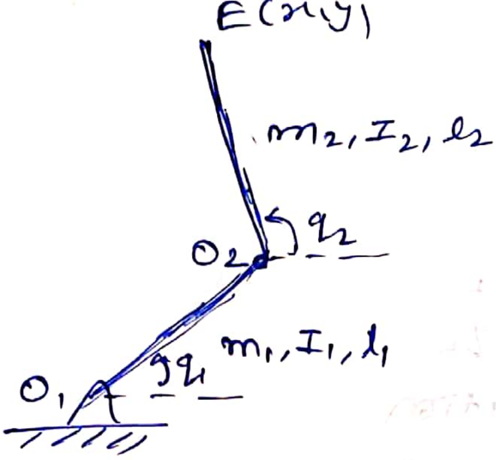
2R Manipulator



E - end effector

(x, y) - end effector position

(q_1, q_2) - Joint angles

Note - absolute angles

Assume origin at O_1

Let's us assume motors are connected to both joints O_1 & O_2 and we have the ability to control either torques τ_1 and τ_2 applied at these joints or control the angles q_1 and q_2.

Angles are sometimes θ_1 & θ_2, or ϕ_1 & ϕ_2

we will study later how (hardware, algorithm and software) and can control τ_1 & τ_2 or q_1 & q_2.

Let's consider 4 tasks

Task 1 (T1) - Given arbitrary trajectory of end effector (given x, y as function of time) make the robot follow this trajectory.

Task 2 (T2) - Given a location of a wall, make the robot touch the wall and apply a constant force against the wall.

Task 3 (T3) - Make the robot ~~behave~~ behave like a virtual spring connected to a given point (x_0, y_0).

Task 4 (T4) - Given any mechanical constraints on the angles determine the range of possible possitions of E (workspace).

~~No~~

05/08/21 Thursday

Now

$$x = l_1 \cos q_1 + l_2 \cos q_2$$

& $$y = l_1 \sin q_1 + l_2 \sin q_2$$

or using simplified notation

$$\left.\begin{aligned} x &= l_1 c q_1 + l_2 c q_2 \\ y &= l_1 s q_1 + l_2 s q_2 \end{aligned}\right\} \quad \text{--- (1)}$$

Differentating (1) we get

$$\dot{x} = -l_1 s q_1 \cdot \dot{q}_1 - l_2 \cdot s q_2 \cdot \dot{q}_2$$

$$\dot{y} = l_1 c q_1 \cdot \dot{q}_1 + l_2 \cdot c q_2 \cdot \dot{q}_2$$

$\Rightarrow$ End-Effector velocity

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 s q_1 & -l_2 s q_2 \\ l_1 c q_1 & l_2 c q_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \quad \text{--- (2)}$$

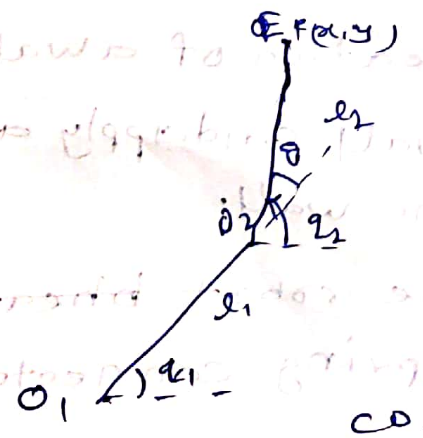
We will also need the reverse relationships

Given x & y, we need to be able to solve for q_1 & q_2 using (1)

option (1) - solve numerically

option (2) - Derive closed form expression

- Hard in general
- multiple solutions

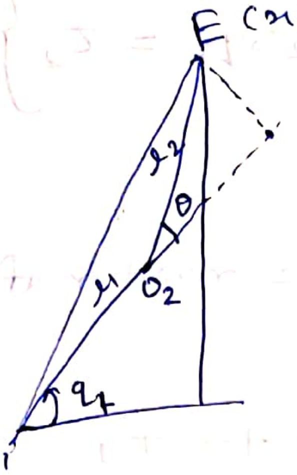


using cosine rule on triangle containing both links & O_1E

$\Delta\, O_1 O_2 E$ + switching to acute angle

$$\cos(180 - \theta) = \frac{l_1^2 + l_2^2 - (x^2+y^2)}{2 l_1 l_2}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{x^2+y^2-l_1^2-l_2^2}{2l_1l_2}\right)$$



$$\angle\beta = \angle q_1 + \angle\gamma$$

$$\angle\gamma = \angle E O_1 O_2$$

$$q_1 = \tan^{-1}(\beta) - \tan^{-1}(\gamma)$$

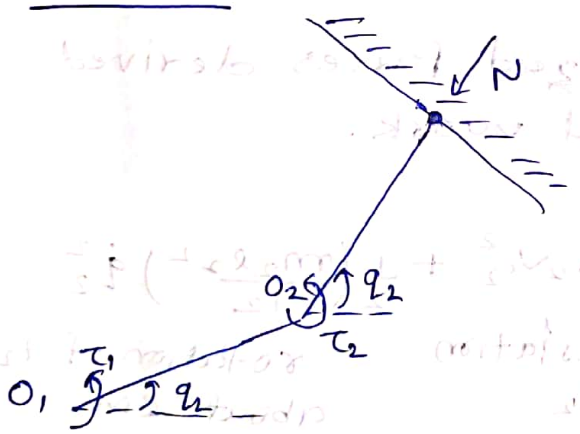
$$q_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2 \cos\theta}\right)$$

$$q_2 = q_1 + \theta$$

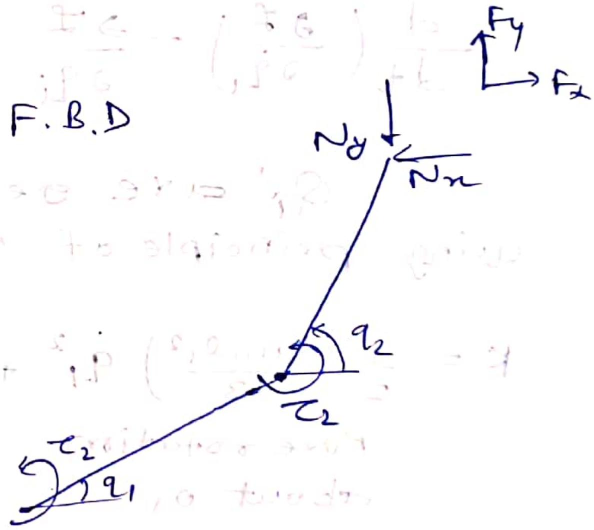
——— (3)

→ Control both motor in position control mode to achive q_1, q_2 at each time step in T1.

Task 2



F.B.D

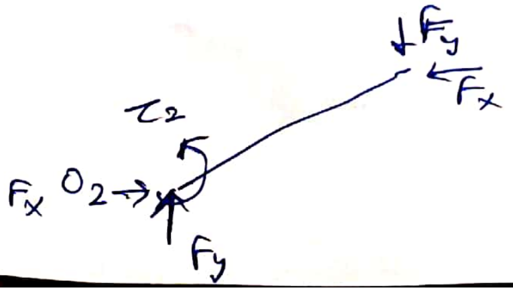


static Equilibrium

$\Rightarrow \quad \Sigma M_{O_1} = 0 \quad \& \quad \Sigma M_{O_2} = 0$

F.B.D each link separately

FBD of link 2

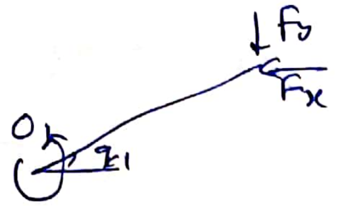


Ignor gravity

$\Sigma M_{O_2} = 0$

$\Rightarrow \quad F_y l_2 C q_2 - F_x l_2 S q_2 = \tau_2$

FBD Link 1



$$\Rightarrow \Sigma M_{O_1} = 0$$

$$\left.\begin{aligned} F_y l_1 c q_1 - F_x l_1 s q_1 &= \tau_1 \\ F_y l_2 c q_2 - F_x l_2 s q_2 &= \tau_2 \end{aligned}\right\} \quad \text{—} (4)$$

(3) along with (4) answers T_2

Apply torques τ_1 & τ_2 at the motor after reaching the wall

for T_3 and next-level answer to T_1 need to understant the dynamics of the robot

Lagrange's Equations

Lagrangian : $\mathcal{L} = K - V$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i \quad \text{———} (5)$$

Q_i' are generalized forces derived using principle of virtual work.

$$K = \frac{1}{2}\left(\frac{m_1 l_1^2}{3}\right)\dot{q}_1^2 + \frac{1}{2} m_2 v_{c_2}^2 + \frac{1}{2}\left(\frac{m_2 l_2^2}{12}\right)\dot{q}_2^2$$

Pure rotation about O_1 — translation of l_2 — rotation of l_2 about CM

$$v_{c_2}^2 = (l_1 \dot{q}_1)^2 + \left(\frac{l_2}{2}\dot{q}_2\right)^2 + 2 l_1 \dot{q}_1 \frac{l_2}{2}\dot{q}_2 \cos(q_2 - q_1)$$

let's bring back gravity

$$V = \cancel{mg}\ m_1 g \frac{l_1}{2} s q_1 + m_2 g \left(l_1 s q_1 + \frac{l_2}{2} s q_2\right)$$

$$
\left.
\begin{aligned}
&\rightarrow \frac{1}{3} m_1 l_1^2 \ddot{q}_1 + m_2 l_1^2 \ddot{q}_1 + m_2 \frac{l_1 l_2}{2} \ddot{q}_2 \cos(q_2 - q_1) \\
&\quad - m_2 \frac{l_1 l_2}{2} \dot{q}_2 (\dot{q}_2 - \dot{q}_1) \sin(q_2 - q_1) + \\
&\quad m_1 g \frac{l_1}{2} c q_2 + m_2 g l_1 c q_1 = \tau_1 \\
&\rightarrow \frac{1}{3} m_2 l_2^2 \ddot{q}_2 + m_2 \frac{l_2^2}{4} \ddot{q}_2 + m_2 \frac{l_1 l_2}{2} \ddot{q}_1 \cos(q_2 - q_1) - \\
&\quad m_2 \frac{l_1 l_2}{2} \dot{q}_1 (\dot{q}_2 - \dot{q}_1) \sin(q_2 - q_1) \\
&\quad + m_2 g \frac{l_2}{2} s q_2 = \tau_2
\end{aligned}
\right\} \quad 6
$$