Assignment 4
Convex Optimization SS2022

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Submission: Upload your report as a single pdf–file ($\leq$ 10MB) to the TeachCenter and include your implementation (main.py) in your submission.

Deadline: June, 27^{th} 2022, 18:00

Computed Tomography Reconstruction

Let $\mathbf{b} \in \mathbb{R}^l$ be the sinogram data acquired by a computed tomography (CT) scan. For a $n = M \times M$ original image $\mathbf{x} \in \mathbb{R}^n$, the sinogram data $\mathbf{b}$ is an image of size $l = M \times L$, where L denotes the number of acquired angles. Each entry of the sinogram data represents a line integral across the original image for a certain offset and angle θ_i, see Figure 1. To account for measurement uncertainty, we assume that the observed sinogram data follows the model

$$\mathbf{b} = \mathcal{A}(\mathbf{x}) + \xi,$$

where $\xi \in \mathbb{R}^l$ is additive Gaussian noise and the linear operator $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^l$ is the Radon transform, which is given by the `pylops.signalprocessing.Radon2D` function throughout this assignment.

TV-Regularized Reconstruction (25P)

To limit the effect of the noise ξ on the reconstruction, we incorporate the total variation (TV) regularization. The resulting variational reconstruction problem reads as

$$\min_{\mathbf{x} \in \mathbb{R}^n} \left\{ E(\mathbf{x}) := \lambda \|D\mathbf{x}\|_{2,1} + \tfrac{1}{2} \|\mathcal{A}(\mathbf{x}) - \mathbf{b}\|_2^2 \right\}, \tag{2}$$

where $D \in \mathbb{R}^{2n \times n}$ is the first-order finite difference operator in x/y-direction as introduced in the previous assignment sheet and the weight parameter λ balances the trade off between regularization and data fidelity. We solve this non-smooth optimization problem by means of the primal dual hybrid gradient (PDHG) algorithm. Therefore, you need to perform the following steps:

1. Dualize the convex non-smooth function to obtain

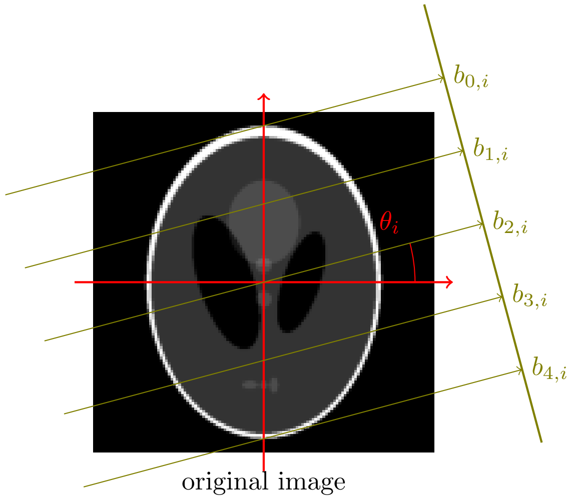
$$\min_{\mathbf{x} \in \mathbb{R}^n} \max_{\mathbf{y} \in \mathbb{R}^{2n}} \langle D\mathbf{x}, \mathbf{y} \rangle - \delta_{B_{\|\cdot\|_{2,\infty}}[0,\lambda]}(\mathbf{y}) + \tfrac{1}{2} \|\mathcal{A}(\mathbf{x}) - \mathbf{b}\|_2^2, \tag{3}$$

   where $\delta_{B_{\|\cdot\|_{2,\infty}}[0,\lambda]}(p)$ is the indicator function of the $\ell_{2,\infty}$-norm ball with radius λ. Explain all steps in your report.

2. By further dualizing the last term in (3), we arrive at the following instantiation of the PDHG algorithm:

$$\begin{aligned}
\mathbf{x}^{k+1} &= \mathbf{x}^k - \tau \left(D^T \mathbf{y}^k + \mathcal{A}^*(\mathbf{z}^k) \right) \\
\mathbf{y}^{k+1} &= \operatorname{proj}_{B_{\|\cdot\|_{2,\infty}}[0,\lambda]} \left(\mathbf{y}^k + \sigma D(2\mathbf{x}^{k+1} - \mathbf{x}^k) \right) \\
\mathbf{z}^{k+1} &= \operatorname{prox}_{\frac{\sigma}{2}\|\cdot\|_2^2} \left(\mathbf{z}^k + \sigma \left(\mathcal{A}(2\mathbf{x}^{k+1} - \mathbf{x}^k) - \mathbf{b} \right) \right),
\end{aligned}$$

   where $\mathbf{z} \in \mathbb{R}^l$ is the second dual variable due to the dualization of the quadratic ℓ^2-norm of the data fidelity term. State the resulting saddle point formulation of (3) if the quadratic ℓ^2-norm of the data fidelity term is dualized.



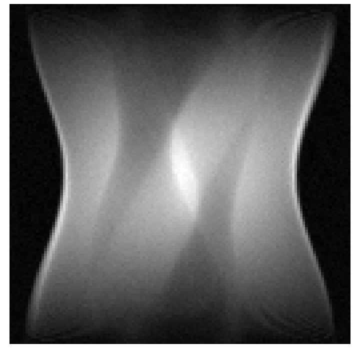


Figure 1: Illustration of the CT reconstruction problem. The measured sinogram $\mathbf{b}$ (right) is obtained by computing line integral across the original image (left) subject to additive Gaussian noise. In detail, each column of the sinogram data $\mathbf{b}$ is the column sum of the original image rotated by the angle θ_i.

Include all derivations in your report. Derive a condition for the choice of the step sizes τ and σ such that the algorithm converges and include it in your report. *Hint*: Combine all dual variables in one vector and define a corresponding operator $\mathcal{K} : \mathbb{R}^n \to \mathbb{R}^{2n+l}$, $\mathbf{x} \mapsto \begin{pmatrix} D\mathbf{x} \\ \mathcal{A}(\mathbf{x}) \end{pmatrix}$. Compare the following step size selection methods:

- Primal and dual step sizes such that
$$\tau\sigma\|\mathcal{K}\|^2 \leq 1.$$
You can use the approximation $\|\mathcal{K}\|^2 \leq \|\mathcal{A}\|^2 + \|D\|^2 \coloneqq \gamma$, and calculate $\|\mathcal{A}\|^2$ using the power method (we know $\|D\|^2 \leq 8$ from the previous assignment). We provide an implementation in 1 for convenience. A standard choice is then $\tau = \sigma = \frac{1}{\sqrt{\gamma}}$.

- 1-preconditioned step sizes such that
$$\begin{pmatrix} T^{-1} & -\mathcal{K}^* \\ -\mathcal{K} & S^{-1} \end{pmatrix} \succeq 0,$$
where $T = \operatorname{diag}(\tau_1, \dots, \tau_n)$ and $S = \operatorname{diag}(\sigma_1, \dots, \sigma_{2n+l})$. A suitable choice is
$$\tau_j = \frac{1}{(|\mathcal{K}^*|(\mathbf{1}))_j + 10^{-3}}, \quad \sigma_i = \frac{1}{(|\mathcal{K}|(\mathbf{1}))_i + 10^{-3}},$$
where $\mathbf{1}$ is associated vector whose entries are all equal to 1 and $|\cdot|$ acts element-wise on the matrix representation of $\mathcal{K}$. Note that $\mathcal{A} \geq 0$ (element-wise), so this does not have to be taken into account. The step-size multiplication in the algorithm above are then understood element-wise, where the sub-block of σ now $\in \mathbb{R}^{2n+l}$ associated with the dual variable $\mathbf{y}$ is $(\sigma_1, \dots, \sigma_{2n})^\top$ and the sub-block associated with $\mathbf{z}$ is $(\sigma_{2n+1}, \dots, \sigma_{2n+l})^\top$.

3. For $\lambda \in \{0.1, 1.0, 10.0\}$ compare the variants of the PDHG algorithm using $\mathbf{x}^0 = \mathbf{0}$, $\mathbf{y}^0 = \mathbf{0}$, and $\mathbf{z}^0 = \mathbf{0}$ as initial values. Therefore, plot for each choice of λ the primal energy E as defined in (2) for every PDHG variant and all iteration step in a `loglog`-plot.

```python
def power_iteration(A: utils.AsMatrix, max_iter: int):
    u = np.random.default_rng().random(M * M).astype(target.dtype)

    for _ in range(max_iter):
        u_ = A.T @ (A @ u)
        u_norm = np.linalg.norm(u_)
        u = u_ / u_norm

    return u_norm
```

Listing 1: Python implementation of the power method to compute the largest eigenvalue of an operator.