Assignment 1

Convex Optimization SS2022

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Submission: Upload your solution as a single `pdf`–file ($\leq$ 10MB) to the TeachCenter.

Deadline: 11.4.2022, 17:00

Convex Sets (12P)

1. Which of the following sets are convex? Prove your results!
   (a) $\{(x, y) \in \mathbb{R}^2_+ : \frac{x}{y} \leq 1\}$
   (b) $\{(x, y) \in \mathbb{R}^2_+ : \frac{x}{y} \geq 1\}$
   (c) $\{(x, y) \in \mathbb{R}^2_+ : xy \leq 1\}$
   (d) $\{(x, y) \in \mathbb{R}^2_+ : xy \geq 1\}$
2. Show that the Euclidean ball in $\mathbb{R}^n$
$$B_{\|\cdot\|_2}(\mathbf{x}_c, r) = \{\mathbf{x} \in \mathbb{R}^n \colon \|\mathbf{x} - \mathbf{x}_c\|_2 \leq r\}$$
   is a convex set using the definition of convexity.
3. Prove that
$$\operatorname{conv}(B_{\|\cdot\|_0}(\mathbf{0}, 1) \cap B_{\|\cdot\|_\infty}(\mathbf{0}, 1)) = B_{\|\cdot\|_1}(\mathbf{0}, 1)$$
   where $\mathbf{0} = (0, 0)^\top$. Visualize the sets in a plot and include it in your submission. Side note: $\|\mathbf{x}\|_0 = \operatorname{nnz}(\mathbf{x})$ is *not* a proper norm since it is not homogeneous: $\|\alpha\mathbf{x}\|_0 = \|\mathbf{x}\|_0$.
4. Let $S \subset \mathbb{R}^n$. We define
$$S^\circ := \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{s} \rangle \leq 1 \text{ for all } \mathbf{s} \in S\}.$$
   (a) Prove that for arbitrary $S \subset \mathbb{R}^n$ the set S° is convex.
   (b) Let $\|\cdot\|$ be any norm in $\mathbb{R}^n$ and $S = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$. What is S°?
5. Prove that $\mathbb{S}^n_+$ is a convex cone.

Convex Functions (13P)

1. Are the following functions convex? Prove your results!
   (a) $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} \mapsto \exp(\exp(\exp(\|\mathbf{x}\|_2^2)))$
   (b) $f : \mathbb{R}^3 \to \mathbb{R}$, $\mathbf{x} = (x_1, x_2, x_3) \mapsto (x_1^2 + \exp(x_2^2))x_3$
   (c) $f : \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 < 1\} \to \mathbb{R}$, $\mathbf{x} \mapsto \frac{\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2}{1 - \|\mathbf{x}\|_2^2}$ for $\mathbf{A} \in \mathbb{R}^{n,n}$ and $\mathbf{b} \in \mathbb{R}^n$.
   (d) $f : \left\{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^n c_i e^{-x_i} \leq 1\right\} \to \mathbb{R}$, $\mathbf{x} \mapsto -\prod_{i=1}^n (1 - e^{-x_i})^{c_i}$ for fixed $c_1, \ldots, c_n > 0$.
2. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$. Are the following statements correct? Prove your results!
   (a) If f and g are convex, then $\mathbf{x} \mapsto (f(\mathbf{x}) + g(\mathbf{x}))^2$ is convex.

(b) If f and g are convex, then $(\mathbf{x}, \mathbf{y}) \mapsto \exp((f(\mathbf{x}) + g(\mathbf{y}))^2)$ is convex.

(c) If f and h are convex and smooth and h is strictly monotonically increasing, then $\mathbf{x} \mapsto h(f(\mathbf{x}))$ is convex.

(d) If f and g are concave and positive, then $\mathbf{x} \mapsto \sqrt{f(\mathbf{x})g(\mathbf{x})}$ is concave.

3. Suppose that $f : \mathbb{R}^n \to \mathbb{R}_+$ is convex and $g : \mathbb{R}^n \to \mathbb{R}_{++}$ is concave. Show that the function $\frac{f^2}{g}$ is convex.

4. Let $f : C \to \mathbb{R}$ be a convex function over the convex set $C \subseteq \mathbb{R}^n$. Let $\mathbf{x}_1, \mathbf{x}_3 \in C$ and $\mathbf{x}_2 \in [\mathbf{x}_1, \mathbf{x}_3]$ ly on the line connecting $\mathbf{x}_1$ and $\mathbf{x}_3$. Prove for different points $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ that

$$\frac{f(\mathbf{x}_2) - f(\mathbf{x}_1)}{\|\mathbf{x}_2 - \mathbf{x}_1\|} \leq \frac{f(\mathbf{x}_3) - f(\mathbf{x}_2)}{\|\mathbf{x}_3 - \mathbf{x}_2\|}.$$

5. Let $C \subseteq \mathbb{R}^n$ be a convex set and let $f : C \to \mathbb{R}$ be a function over C. Recall that f is strongly convex if there exists a $\sigma > 0$ such that the function $g : C \to \mathbb{R}$, $\mathbf{x} \mapsto f(\mathbf{x}) - \frac{\sigma}{2}\|\mathbf{x}\|^2$ is convex over C.

(a) Prove that f is strongly convex over C with parameter σ *if and only if*

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y}) - \frac{\sigma}{2}\alpha(1-\alpha)\|\mathbf{x} - \mathbf{y}\|^2$$

for any $\mathbf{x}, \mathbf{y} \in C$ and $\alpha \in [0, 1]$.

(b) Assume that f is continuously differentiable over C. Prove that f is strongly convex over C with parameter σ *if and only if*

$$\langle \nabla f(\mathbf{x}) - \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \geq \sigma \|\mathbf{x} - \mathbf{y}\|^2$$

for any $\mathbf{x}, \mathbf{y} \in C$.

(c) Suppose that f is twice continuously differentiable over C. Prove that f is strongly convex over C with parameter σ *if and only if* $\nabla^2 f(\mathbf{x}) \succeq \sigma I$ for any $\mathbf{x} \in C$.