LINEAR ALGEBRA

ASSIGNMENT 3

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EX 5.4

QUESTION 11

Find β-matrix for transformation $u_1 \longrightarrow Au$, when $\beta = \{b_1, b_2\}$

$$A = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

If $P = \begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

then β matrix is

$$P^{-1}AP = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}.$$

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QUESTION.12

Find B-matrix:-

$$A = \begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix}, b_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, b_1 - \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

If $P = [b_1 \quad b_2] = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$

$$P^{-1}AP = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

QUESTION.13

Define $T: R^2 \to R^2$ by $T(u) = Au$. Find a basis B for R^2 with propety that $[T]_B$ is diagnol.

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$$

Characteristic polynomial:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} -\lambda & 1 \\ -3 & 4-\lambda \end{bmatrix}$$

$$= (-\lambda)(4-\lambda) - (-3)(1)$$

$$= -4\lambda + \lambda^2 + 3$$

$$= \lambda^2 - 4\lambda + 3$$

$= (\lambda-1)(\lambda-3)$

$\therefore$ Eigen values of A are 1 and 3.

Basis for $\lambda = 1$

$$(A-I)u = \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}$$

$$= -u_1 + u_2$$

So, $u_1 = u_2$

Basis vector is $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Basis for $\lambda = 3$

$$A-3I = \begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix}$$

$$(A-3I)u = -3u_1 + u_2 = 0$$

$$3u_1 = u_2$$

$$v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

From v_1 & v_2.

$$P = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$ Basis B (diagnol matrix)

QUESTION·14

Find basis B for R^2 with property that $[T]_B$ is diagnol

$$A = \begin{bmatrix} 5 & -3 \\ -7 & 1 \end{bmatrix}.$$

Characteristic polynomial

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 5 & -3 \\ -7 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} 5-\lambda & -3 \\ -7 & 1-\lambda \end{bmatrix}$$

$$= (5-\lambda)(1-\lambda) - (-7)(-3)$$

$$= 5 - 5\lambda - \lambda + \lambda^2 - 21$$

$$= \lambda^2 - 6\lambda - 16$$

$$= (\lambda-8)(\lambda+2)$$

$\therefore$ Eigen values are 8 and -2

Basis for $\lambda = 8$

$$(A - 8I) = \begin{bmatrix} -3 & -3 \\ -7 & -7 \end{bmatrix}$$

$$(A-8I)u = 0$$

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Basis for $\lambda = -2$

$$A + 2I = \begin{bmatrix} -7 & -3 \\ -7 & 3 \end{bmatrix}$$

$$(A+2I)u = 7u_1 - 3u_2 = 0$$

$$v_2 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 3 \\ 1 & 7 \end{bmatrix}$$

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QUESTION 15

Find basis for R^2 with property that $[T]_B$ is diagonal

$$A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

Characteristic polynomial

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} 4-\lambda & -2 \\ -1 & 3-\lambda \end{bmatrix}$$

$$= (4-\lambda)(3-\lambda) - (-1)(-2)$$

$$= 12 - 4\lambda - 3\lambda + \lambda^2 - 2$$

$$= \lambda^2 - 7\lambda + 10$$

$$= (\lambda - 5)(\lambda - 2)$$

$\because$ Eigen values are $+5$ and $+2$

Basis for $\lambda = +5$

$$(A - 5I) = \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix}$$

$$(A - 5I)u = u_1 + 2u_2 = 0$$

$$u_1 = -2u_2$$

$$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Basis for $\lambda = 2$

$$(A - 2I) = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

$$(A - 2I)u = u_1 - u_2$$

$$u_1 = u_2$$

$$V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

QUESTION 16

Find basis for R^2 with property that $[T]_B$ is diagonal

$$A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}$$

Characteristic polynomial

$$\det(A - \lambda I) = \left(\begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right)$$

$$= \det \begin{bmatrix} 2-\lambda & -6 \\ -1 & 3-\lambda \end{bmatrix}$$

$$= (2-\lambda)(3-\lambda) - (-1)(-6)$$

$$= 6 - 5\lambda + \lambda^2 - 6$$

$$= \lambda^2 - 5\lambda$$

$$= \lambda(\lambda - 5) = 0$$

$\because$ Eigen values of A are 0 and 5.

Basis for $\lambda = 5$

$$A - 5I = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}$$

$$(A - 5I)u = u_1 + 2u_2 = 0$$

$$u_1 = -2u_2$$

$$V_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Basis for $\lambda = 0$

$$A - 0I = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}$$

$$(A - 0I)u = u_1 - 3u_2 = 0$$

$$u_1 = 3u_2$$

$$V_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix}$$

QUESTION 17

(a) $$Ab_1 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2b_1$$

Characteristic polynomial

$$\det(A - \lambda I) = \left(\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right)$$

$$= \det \begin{bmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{bmatrix}$$

$$= (1-\lambda)(3-\lambda) - (-1)(1)$$

$$= 3 - \lambda - 3\lambda + \lambda^2 + 1$$

$$= \lambda^2 - 4\lambda + 4$$

$$= (\lambda - 2)(\lambda - 2)$$

So Eigen values of A are 2 & 2.

Basis for $\lambda = 2$

$$A - 2I = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

Eigenspace corresponding to eigenvalue 2 is one-dimensional - Thus matrix A is not diagonalizable.

(b) If $P = [b_1 \quad b_2]$ then B matrix for T is

$$P^{-1}AP = \begin{bmatrix} -4 & 5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

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QUESTION 18

Define $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(u) = Au$ where A is 3×3 matrix with eigen values 5 and -2. Does there exist a basis B for $\mathbb{R}^3$ such that B-matrix for T is diagnol matrix?

Let us consider a matrix $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$, we have to find the basis b for R^2.

We know that

$$A = PDP^{-1}$$

$$\text{where } P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \text{ \& } D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

The columns of P, b_1 & b_2 are eigen vectors of A.

So, if there is a bases B such that $[T]_B$ is diagnol, then A is similar to diagnol matrix as mapping $u \mapsto Au$ and $u \mapsto Du$ describes the same linear transformation, related to different basis. For this to happen, A would have three linearly independent eigenvectors. However this is not necessarily the case because A has only two distinct eigenvalues.

QUESTION 19

If A is invertiable and similar to B then B is invertible and A^{-1} is similar to B^{-1}. Find invertiable Q such that $Q^{-1}A^{-1}Q = B^{-1}$.

If A is similar to B, then there exists an invertible matrix P such that $P^{-1}AP = B$. Thus B is invertible because it is product of invertible matrices. By a theorem about inverses of products,

$B^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^{-1}A^{-1}P$ which shows that A^{-1} is similar to B^{-1}.

QUESTION 20

If A is similar to B then A^2 is similar to B^2.

If $A = PBP^{-1}$, then

$$
\begin{aligned}
A^2 &= (PBP^{-1})(PBP^{-1}) \\
&= PB(P^{-1}P)BP^{-1} \\
&= PB \cdot I \cdot BP^{-1} \\
&= PB^2P^{-1}
\end{aligned}
$$

So A^2 is similar to B^2.