Propositional Logic: Semantic Reasoning

Source: Computational Logic Lecture Notes
Stanford University

IF1221 Computational Logic 2024/2025

Informatics Engineering Study Program
School of Electrical Engineering and Informatics ITB

Contents

- Review
- Propositional Logic → Logical Entailment
  - Semantic Reasoning

Review

- Computational Logic
  - **Propositional Logic:**
    - Sintax → Simple sentence, Compound Sentence
    - Semantics → interpretation, evaluation, reverse evaluation, types of compund sentence
  - Relational Logic

Review Deduction

- In deduction, the conclusion is true whenever the premises are true.

  Premise: p

  Conclusion: $(p \vee q)$

  Premise: p

  Non-Conclusion: $(p \wedge q)$

  Premises: p, q

  Conclusion: $(p \wedge q)$

Logical Entailment

- Validity, satisfiability, unsatisfiability: property of single sentence
- Logical Reasoning: relationship between sentences
- Given sentences → other sentence is a conclusion or non-conclusion?

Example of Logical Entailment

- Price of pertamax is up only if $ is up
- $ is up only if Rp is down
- Price of pertamax is up
  - Conclusion…. Rp is down

→ logical entailment

Logical Entailment

- $\Delta \models \varphi$
  - Set of premises Δ logically entails a conclusion φ iff every interpretation that satisfies the premises also satisfies the conclusion
- Propositional Logic: **Propositional entailment**
- Example:

  $\{p\} \models (p \vee q)$

  $\{p\} \not\models (p \wedge q)$

  $\{p,q\} \models (p \wedge q)$

Logical Entailment Checking

- Semantic reasoning:
  - Truth table
  - Validity checking
  - Unsatisfiability checking
- Proof Method
  - Rules of Inference
  - Axiom schemata
  - Propositional Resolution

Truth Table

1. **Create two interpretation tables: premise, conclusion**
   Example : {p} $\models$ $(p \vee q)$?

p		p	q	$p \vee q$
T		T	T	T
T		T	F	T
F		F	T	T
F		F	F	F

2. **In table-1: eliminate all rows that do not satisfy the premises**

p		p	q	$p \vee q$
T		T	T	T
T		T	F	T
F x		F	T	T
F x		F	F	F

Truth Table (2)

3. In table-2: eliminates all rows that do not satisfy the conclusion

p		p	q	$p \vee q$
T		T	T	T
T		T	F	T
F		F	T	T
F x		**F**	**F**	**F x**

4. If the remaining rows in table-1 are subset of remaining rows of table-2, then the premises logically entail the conclusion

$\therefore$ **{p} |= ($p \vee q$)**

$\{p\} \models (p \wedge q)$?

- Truth Table

p
T
T
F ×
F ×

p	q	$p \wedge q$
T	T	T
T	F	F x
F	T	F x
F	F	F x

$\therefore \{p\} \not\models (p \wedge q)$

Truth Table (3)

- Advantage : easy to understand
  - Direct implementation of the definition of logical entailments
- Drawbacks:
  - Managing two tables
  - Solution → one table method (Validity checking and Unsatisfiability checking)

One Table Approach

- $\{\varphi_1, \varphi_2, \ldots, \varphi_n\} \models \varphi$

- Validity checking:

  $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n \rightarrow \varphi$: valid

- Unsatisfiability checking :

  $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n \wedge \sim\varphi$: unsatisfiable

$\{p\} \models (p \vee q)$?

- Validity checking: $\mathbf{p \longrightarrow (p \vee q)}$ valid ?

p	q	$p \vee q$	$p \longrightarrow (p \vee q)$
T	T	T	**T**
T	F	T	**T**
F	T	T	**T**
F	F	F	**T**

$\therefore$ $p \longrightarrow (p \vee q)$ valid → $\{p\} \models (p \vee q)$

$\{p\} \models (p \vee q)$?

- Unsatisfiability checking: $p \wedge \sim(p \vee q)$?

p	q	$p \vee q$	$p \wedge \sim(p \vee q)$
T	T	T	**F**
T	F	T	**F**
F	T	T	**F**
F	F	F	**F**

$\therefore$ $p \wedge \sim(p \vee q)$ unsatisfiable → $\{p\} \models (p \vee q)$

Example

- If Mary loves Pat, then Mary loves Quincy. If it is Monday, then Mary loves Pat or Quincy. Prove that if it is Monday, Mary loves Quincy.
- Premises: $p \rightarrow q$, $m \rightarrow p \vee q$
- Conclusion: $m \rightarrow q$
- $\{p \rightarrow q, m \rightarrow p \vee q\} \models m \rightarrow q$?

Truth Table:
$\{p \to q,\ m \to p \vee q\} \models m \to q$?

m	p	q	p→q	m→p∨q
T	T	T	T	T
T	T	F	**F**	T
T	F	T	T	T
T	F	F	T	**F**
F	T	T	T	T
F	T	F	**F**	T
F	F	T	T	T
F	F	F	T	T

m	p	q	m→q
T	T	T	T
T	T	F	**F**
T	F	T	T
T	F	F	**F**
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

One Table approach

- $\{p \rightarrow q, m \rightarrow p \vee q\} \models m \rightarrow q$
- Validity checking:
  $(p \rightarrow q) \wedge (m \rightarrow p \vee q) \rightarrow (m \rightarrow q)$ valid ?

$(p \to q) \land (m \to p \lor q) \to (m \to q)$

m	p	q	(p → q)	∧	(m	→	p ∨ q)	→	(m → q)
T	T	T	T	T		T	T	T	T
T	T	F	F	F		T	T	T	F
T	F	T	T	T		T	T	T	T
T	F	F	T	F		F	F	T	F
F	T	T	T	T		T	T	T	T
F	T	F	F	F		T	T	T	T
F	F	T	T	T		T	T	T	T
F	F	F	T	T		T	F	T	T

One Table Approach (2)

- $\{p \rightarrow q, m \rightarrow p \vee q\} \models m \rightarrow q$
- Unsatisfiability Checking:
  $(p \rightarrow q) \wedge (m \rightarrow p \vee q) \wedge \sim(m \rightarrow q)$ unsatisfiable ?

$(p \rightarrow q) \wedge (m \rightarrow p \vee q) \wedge \sim(m \rightarrow q)$

m	p	q	(p-->q)	∧	(m	-->	(p∨q))	∧	~	(m-->q)
T	T	T	T	T	T	T	T	F	F	T
T	T	F	F	F	T	T	T	F	T	F
T	F	T	T	T	T	T	T	F	F	T
T	F	F	T	F	T	F	F	F	T	F
F	T	T	T	T	F	T	T	F	F	T
F	T	F	F	F	F	T	T	F	F	T
F	F	T	T	T	F	T	T	F	F	T
F	F	F	T	T	F	T	F	F	F	T

Semantic Reasoning

- Advantage : easy to understand
- Drawbacks:
  - Number of interpretation: 2^n --> Enumeration of all interpretation is time and space consuming
  - Some logical constants are irrelevant to the conclusion → time consuming and increase checking complexity

Exercise 1:

Terdapat tiga orang yang menjadi tersangka sebuah kasus yaitu Ang, Beng, dan Cing. Ang berkata," Beng bersalah dan Cing tidak bersalah." Beng berkata," Jika Ang bersalah, maka Cing juga." Cing berkata," Saya tidak bersalah, tapi setidaknya salah satu dari Ang atau Beng yang bersalah."

- Dengan menggunakan proposisi:
- a: Ang tidak bersalah;
- b: Beng tidak bersalah;
- c: Cing tidak bersalah;
- jawablah pertanyaan berikut.

Exercise 1 (con't):

a) Tuliskan kembali tiga kalimat yang diucapkan di atas dalam representasi propositional logic. Setiap kalimat menjadi satu kalimat proposisi.
b) Jika Ang, Beng, dan Cing tidak bersalah, tentukan siapa yang berbohong dan siapa yang jujur. Gunakan tabel kebenaran untuk mendapatkan jawabannya.
c) Jika seseorang yang tidak bersalah berkata jujur, dan yang bersalah berkata bohong, siapakah yang tidak bersalah? Gunakan tabel kebenaran untuk mendapatkan jawabannya, dengan memanfaatkan operator biimplikasi ($\leftrightarrow$).

Exercise 2

Terdapat kumpulan premis (fakta) sebagai berikut. Jika anda bekerja keras, maka anda beruntung. Anda beruntung atau anda bekerja keras. Jika anda beruntung, maka anda bukan anak yang suka bermain atau anda bekerja keras (tapi tidak keduanya). Anda anak yang suka bermain. Buktikan apakah kesimpulan "Anda bekerja keras" dapat diturunkan dari kumpulan premis tersebut, dengan menggunakan ***unsatisfiability dan validity checking***. Gunakan proposisi sebagai berikut:

- p: Anda bekerja keras ; q: Anda beruntung ; r: Anda anak yang suka bermain.

Propositional Logic: Proof Method

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  - Proof Method

Review

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  - **Propositional Logic:**
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    - Logical Entailment
      - Semantic Reasoning

Proof Method

- Proof of a conclusion from set of premises:
  - Sequence of sentences terminating in conclusion in which each item is either a premise, an instance of axiom schema, or the result of applying a rule of inference to earlier items in sequence.
  - Base: Applied Rule of Inference to premises

Proof Method

- Proof of a conclusion from set of premises:
  - Sequence of sentences terminating in conclusion in which each item is either a premise, an instance of axiom schema, or the result of applying a rule of inference to earlier items in sequence.
  - Base: Applied Rule of Inference to premises

- A rule of inference:
  - a pattern of reasoning consisting of premises and conclusions.

Proof Method

- More rule of inference:

1. Modus ponens

$p \rightarrow q$

p

$\therefore q$

2. Modus tollens

$p \rightarrow q$

$\sim q$

$\therefore \sim p$

3. Disjunctive syllogism

$p \vee q$

$\sim p$

$\therefore q$

4. Simplification

$p \wedge q$

$\therefore p$

Proof Method

- More rule of inference:

5. Addition

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

6. Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

7. Hypothetical syllogism

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

8. Resolution

$$\begin{array}{l} p \vee q \\ \sim p \vee r \\ \hline \therefore q \vee r \end{array}$$

Proof Method

- More rule of inference:

1. Modus ponens

   $p \to q$

   p

   $\therefore q$

2. Modus tollens

   $p \to q$

   $\sim q$

   $\therefore \sim p$

3. Disjunctive syllogism

   $p \vee q$

   $\sim p$

   $\therefore q$

4. Simplification

   $p \wedge q$

   $\therefore p$

Proof Method

- More rule of inference:

5. Addition

p

$\therefore p \vee q$

6. Conjunction

p

q

$\therefore p \wedge q$

7. Hypothetical syllogism

$p \rightarrow q$

$q \rightarrow r$

$\therefore p \rightarrow r$

8. Resolution

$p \vee q$

$\sim p \vee r$

$\therefore q \vee r$

Rules of Replacement

- Associativity → disjunction, conjunction, equivalence
- Commutativity → disjunction, conjunction, equivalence
- Distributivity: $p \vee (r \wedge q) \leftrightarrow (p \vee q) \wedge (p \vee q)$
- Double Negation: $\neg\neg p \leftrightarrow p$
- De Morgan's Law:

  $\neg(p \wedge q) \leftrightarrow (\neg p) \vee (\neg q)$

  $\neg(p \vee q) \leftrightarrow (\neg p) \wedge (\neg q)$
- Transposition: $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
- Material Implication: $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$
- Tautology: $(p \wedge p) \leftrightarrow p$ or $(p \vee p) \leftrightarrow p$

Rules of Replacement

- Associativity $\rightarrow$ disjunction, conjunction, equivalence
- Commutativity $\rightarrow$ disjunction, conjunction, equivalence
- Distributivity: $p \vee (r \wedge q) \leftrightarrow (p \vee q) \wedge (p \vee q)$
- Double Negation: $\neg\neg p \leftrightarrow p$
- De Morgan's Law:

  $\neg(p \wedge q) \leftrightarrow (\neg p) \vee (\neg q)$

  $\neg(p \vee q) \leftrightarrow (\neg p) \wedge (\neg q)$
- Transposition: $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
- Material Implication: $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$
- Tautology: $(p \wedge p) \leftrightarrow p$ or $(p \vee p) \leftrightarrow p$

Example

Premises:

1. $p \rightarrow q$
2. $m \rightarrow p \vee q$

Conclusion: $m \rightarrow q$

$\{p \rightarrow q, m \rightarrow p \vee q\} \models m \rightarrow q$?

3. $\sim p \vee q$ Material Implication 1
4. $\sim m \vee p \vee q$ Material Implication 2
5. $\sim m \vee q$ Resolution 3,4
6. $m \rightarrow q$ Material Implication 5

Jadi dapat dibuktikan $\{p \rightarrow q, m \rightarrow p \vee q\} \models m \rightarrow q$

Proving without premises

- No premise → no place to apply rules of inference
- Facts: valid sentences → true for all interpretations
- How to prove $p \rightarrow (q \rightarrow p)$ is a valid sentence?
- Requires: rule of inference without premises
- Example: axiom schemata

Schemata

- Schema: expression satisfying the grammatical rules of our language → occurs meta-variables in the expression
  - $\varphi \rightarrow (\psi \rightarrow \varphi)$
- Instance of sentence schema: substituting the occurrences of metavariables, form legal expressions

$$
\begin{array}{lll}
p \Rightarrow (p \Rightarrow p) & p \Rightarrow (p \wedge p \Rightarrow p) & p \Rightarrow (p \vee p \Rightarrow p) \\
p \Rightarrow (q \Rightarrow p) & p \Rightarrow (p \wedge q \Rightarrow p) & p \Rightarrow (p \vee q \Rightarrow p) \\
p \Rightarrow (r \Rightarrow p) & p \Rightarrow (p \wedge r \Rightarrow p) & p \Rightarrow (p \vee r \Rightarrow p) \\
q \Rightarrow (p \Rightarrow q) & \ldots & \ldots \\
q \Rightarrow (q \Rightarrow q) & & \\
q \Rightarrow (r \Rightarrow q) & & \\
r \Rightarrow (p \Rightarrow r) & & \\
r \Rightarrow (q \Rightarrow r) & & \\
r \Rightarrow (r \Rightarrow r) & &
\end{array}
$$

Axiom

- Axiom:
  - Proposition that is believed to be true
  - base assumption for proving
  - valid proposition
- Example: $p \rightarrow (q \rightarrow p)$

Standard Axiom Schemata

- Implication Introduction (II):
  $A \rightarrow (B \rightarrow A)$

- Implication Distribution (ID):
  $A \rightarrow (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

- Contradiction Realization (CR):
  $(A \rightarrow \sim B) \rightarrow ((A \rightarrow B) \rightarrow \sim A)$
  $(\sim A \rightarrow B) \rightarrow ((\sim A \rightarrow \sim B) \rightarrow A)$

Standard Axiom Schemata (2)

- Equivalence (EQ):
  $(A\leftrightarrow B)\rightarrow(A\rightarrow B)$
  $(A\leftrightarrow B)\rightarrow(B\rightarrow A)$
  $(A\rightarrow B)\rightarrow((B\rightarrow A)\rightarrow(A\leftrightarrow B))$

- Other operators:
  $(A\leftarrow B)\leftrightarrow(B\rightarrow A)$
  $(A \vee B)\leftrightarrow(\sim A\rightarrow B)$
  $(A \wedge B)\leftrightarrow\sim(\sim A \vee \sim B)$

Proofs

- Prove all logical consequences from any set of premises
  - Standard axiom schemata
  - Modus Ponen

Example

- Whenever p is true, q is true. Whenever q is true, r is true. Prove that whenever p is true, r is true.
- Premis: $p \to q$, $q \to r$
- Konklusi: $p \to r$

1.	$p \to q$	premise
2.	$q \to r$	premise
3.	$(q \to r) \to (p \to (q \to r))$	II
4.	$(p \to (q \to r))$	Modus Ponen 2,3
5.	$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$	ID
6.	$(p \to q) \to (p \to r)$	Modus Ponen 4,5
7.	$p \to r$	Modus Ponen 1,6

Exercise Axiom Schemata

1. Premises: $p \rightarrow q$, $q \rightarrow r$
   Prove conclusion: $(p \rightarrow \sim r) \rightarrow \sim p$

II: $A \rightarrow (B \rightarrow A)$

ID: $A \rightarrow (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

CR: $(A \rightarrow \sim B) \rightarrow ((A \rightarrow B) \rightarrow \sim A)$
$(\sim A \rightarrow \sim B) \rightarrow ((\sim A \rightarrow B) \rightarrow A)$

EQ: $(A \leftrightarrow B) \rightarrow (A \rightarrow B)$
$(A \leftrightarrow B) \rightarrow (B \rightarrow A)$
$(A \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow (A \leftrightarrow B))$

O: $(A \leftarrow B) \leftrightarrow (B \rightarrow A)$
$(A \vee B) \leftrightarrow (\sim A \rightarrow B)$
$(A \wedge B) \leftrightarrow \sim(\sim A \vee \sim B)$

$\{p \rightarrow q,\ q \rightarrow r\} \models (p \rightarrow \sim r) \rightarrow \sim p$

1.	$p \rightarrow q$	premise
2.	$q \rightarrow r$	premise
3.	$(q \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))$	II
4.	$(p \rightarrow (q \rightarrow r))$	Modus Ponen 2,3
5.	$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$	ID
6.	$(p \rightarrow q) \rightarrow (p \rightarrow r)$	Modus Ponen 4,5
7.	$p \rightarrow r$	Modus Ponen 1,6
8.	$(p \rightarrow r) \rightarrow ((p \rightarrow \sim r) \rightarrow \sim p)$	CR
9.	$(p \rightarrow \sim r) \rightarrow \sim p$	Modus Ponen 7,8

Provability

- $A \vdash B$
  - A conclusion is said to be *provable* from a set of premises (written $\vdash$) if and only if there is a finite proof of the conclusion from the premises using only Modus Ponens and the Standard Axiom Schemata.
- Previous Example: $\{p \rightarrow q, q \rightarrow r\} \vdash (p \rightarrow r)$
- $(A \vdash B) \leftrightarrow (A \models B)$

Deduction Theorems

A |- (B→C) iff A∪{B} |- C

Example:

{p→q, q→r} |- (p→r)

{p→q, q→r, p} |- r

Review

- $\Delta \models \varphi$
  - Set of premises Δ logically entails a conclusion φ iff every interpretation that satisfies the premises also satisfies the conclusion
- Propositional Logic: **Propositional entailment**
- Semantic reasoning:
  - Truth table
  - Validity checking
  - Unsatisfiability checking
- Proof Method:
  - Rules of Inference
  - Axiom schemata

THANK YOU