# RESEARCH STATEMENT

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I study problems concerning birationality of algebraic varieties. These include rationality and stable rationality over nonclosed fields, as well as equivariant birationality of varieties over algebraically closed field. More specifically, I mainly study birational or stably-birational invariants.

I will start with a brief description of my work, my future research plans are in section 3.

## 1. Potentially stably rational del Pezzo surfaces over non-closed fields

A geometrically rational surface $S$ over a nonclosed field $k$ is $k$-birational to either a del Pezzo surface of degree $n \in [1, \ldots, 9]$ or a conic bundle (see [12]). Throughout, we assume that the set of its rational points $S(k) \neq \emptyset$. This implies $k$-rationality of $S$ when $n \in [5, \ldots, 9]$ or when the number of degenerate fibers of the conic bundle is at most 3.

Let $\Gamma_k$ be the absolute Galois group of $k$, it acts on exceptional curves and on the geometric Picard group $\mathrm{Pic}(\bar{S})$ of $S$. The surface $S$ is called *split* over $k$ if all exceptional curves are defined over $k$, and *minimal* if no blow-downs are possible over $k$, i.e., there are no $\Gamma_k$-orbits consisting of pairwise disjoint exceptional curves. A minimal del Pezzo surface of degree $\leq 4$ over $k$ is not rational (see, e.g., [24, Theorem 3.3.1]). A surface $S$ is called *stably rational* over $k$ if $S \times \mathbb{P}^m$ is birational to a projective space, over $k$, for some $m \in \mathbb{N}$. A necessary condition for stable rationality of $S$ over $k$ is

**Condition (H1)**

$$\mathrm{H}^1(\Gamma_{k'}, \mathrm{Pic}(\bar{S})) = 0, \qquad \text{for all finite extensions} \quad k'/k.$$

As a special case of a general conjecture of Colliot-Thélène and Sansuc one expects that this is also sufficient:

**Conjecture 1.** If $S$ satisfies condition (H1) then $S$ is stably rational over $k$.

Only one example of a minimal, and thus nonrational, but stably rational del Pezzo surface of degree $\leq 4$ is known at present [7, 6, 2]; in this case, the Galois group acts via the symmetric group $\mathfrak{S}_3$, the smallest nonabelian group. Finding another example is a major open problem. There are however examples of minimal del Pezzo surfaces of degrees $1 \leq n \leq 4$ and of conic bundles with at least 4 degenerate fibers, *failing* (H1) and thus not stably rational over $k$.

For $n = 3, 2$, and 1, the Galois group $\Gamma_k$ of $k$ acts on the primitive Picard group of $S$ (the orthogonal complement of the canonical class in $\mathrm{Pic}(S)$) through the Weyl group $W(\mathsf{E}_{9-n})$; for $n = 4$ and conic bundles with $n+1$ degenerate fibers through $W(\mathsf{D}_{n+1})$. These actions have been extensively studied, in connection with arithmetic applications and rationality questions, e.g., the Hasse Principle and Weak Approximation, when $k$ is a number field (see e.g., [23], [19], [29], [32], [21], [1]).

For general $k$, it is of interest to identify Galois actions potentially giving rise to minimal, stably rational surfaces, i.e., those satisfying (H1). This has been done in [19] for del Pezzo surfaces of degree 4. Our main result is a classification of the relevant actions in degrees 3, 2, and 1.

**Proposition 2.** *There are no minimal cubic surfaces satisfying Condition* (H1)*. In particular, a $k$-minimal cubic surface is not stably rational over $k$.*

**Degree 2.** In the following propositions we list the structure of Galois groups of splitting fields. For more details about the structure or orbits on the set of exceptional curves, and the stabilizers for each orbit, see [30].

**Proposition 3.** *Assume that $S$ is a minimal degree 2 del Pezzo surface over $k$ satisfying Condition* (H1)*. Then either $S$ admits a conic bundle structure over $k$ or $\Gamma_k$ acts on the primitive Picard group of $S$ via one of the following subgroups $W(\mathsf{E}_7)$, modulo conjugation:*

dP2(1) $\mathfrak{D}_7$*, the dihedral group of order 14,*
dP2(2) $\mathfrak{F}_7$*, the Frobenius group of order 42,*
dP2(3) $\mathfrak{D}_{15}$*, the dihedral group of order 30,*
dP2(4) $\mathfrak{C}_3 \rtimes \mathfrak{F}_5$.

The classification of conic bundle types is also explicit; there are 14 types, see [30]. For example, the largest appearing subgroup is isomorphic to $\mathfrak{S}_5$.

**Degree 1.**

**Proposition 4.** [30] *If $S$ is a minimal degree 1 del Pezzo surface satisfying Condition* (H1) *then $S$ is a conic bundle over $k$, and there are 10 such types.*

For more information about the orbits on the set of exceptional curves, the stabilizers for each orbit, and the `Magma` code, see

`https://cims.nyu.edu/~tschinke/papers/yuri/18h1dp/magma/`.

## 2. Equivariant birational geometry and Burnside groups

Let $G$ be a finite group, acting regularly and generically freely on a smooth projective variety over an algebraically closed field $k$, of characteristic zero. The study of such actions, up to $G$-equivariant birationality, is a classical and active area in higher-dimensional algebraic geometry (see, e.g., [28], [5], [25]). A new type of birational invariants of $G$-actions was introduced in [14]. These take values in the *Burnside group*

$$\mathrm{Burn}_n(G),$$

defined by explicit generators and relations [14, Section 4]. The invariant is computed on an appropriate birational model $X$ (standard form), where

- all stabilizers are abelian,
- a translate of an irreducible component of a locus with nontrivial stabilizer is either equal to it or is disjoint from it.

The invariant takes into account information about

- subvarieties $F \subset X$ with nontrivial (abelian) stabilizers $H$,
- the induced action on $F$ of a subgroup $Y \subseteq Z_G(H)/H$ of (the quotient of) the centralizer of $H$, and
- the representation of the abelian group $H$ in the normal bundle to $F$, i.e., an unordered sequence of characters of $H$.

Formally, the class

$$[X \circlearrowleft G] \in \mathrm{Burn}_n(G)$$

of a regular $G$-action on a smooth projective variety $X$ in standard form is written as

$$[X \circlearrowleft G] := \sum_{H \subseteq G} \sum_{F} (H, Y \circlearrowright K(F), \beta),$$

where $H$ runs over (conjugacy classes of) abelian subgroups of $G$, $F$ is a stratum whose components have generic stabilizers (conjugated to) $H$, $Y$ records the action on $F$, and $\beta$ is the collection of weights of $H$ in the normal bundle of the stratum (see [14, Definition 4.4] or [10, Section 7]). The symbols

$$(H, Y \circlearrowright K(F), \beta) \tag{2.1}$$

are generators of $\mathrm{Burn}_n(G)$, and the defining relations insure that

$$[X \circlearrowleft G] - [\tilde{X} \circlearrowleft G] = 0 \in \mathrm{Burn}_n(G),$$

for every equivariant blowup $\tilde{X} \to X$. Basic geometric operations such as restriction to subgroups $G' \subseteq G$, products of varieties, fibrations, etc. have natural realizations on the level of Burnside groups, see [17].

In particular, one is interested in the birational classification of regular $G$-actions on projective spaces $\mathbb{P}^n$, which arise as:

- the natural compactification a linear action of $G$ on $\mathbb{A}^n$,
- the action induced by a linear action of $G$ on $\mathbb{A}^{n+1}$,
- a projective linear representation of $G$.

There is an explicit combinatorial algorithm [16] to compute the class

$$[\mathbb{P}^n \circlearrowleft G] \in \mathrm{Burn}_n(G)$$

for each case above by using the formalism of De Concini-Procesi compactifications of subspace arrangements, adopted to the equivariant setting.

A purely combinatorial version $\mathcal{BC}_n(G)$ of constructions of group $\mathrm{Burn}_n(G)$ was introduced in [17]. It keeps track of the *group-theoretic* information extracted as above, while forgetting the *field-theoretic* information, i.e., the birational type of the action on irreducible components of loci with nontrivial stabilizers.

Formally, combinatorial birational invariants of $G$-actions on algebraic varieties of dimension $n$ take values in the *combinatorial Burnside group*

$$\mathcal{BC}_n(G),$$

defined via generators and relations [17, Definition 8.1]. The class

$$[X \circlearrowleft G] := \sum_H \sum_F (H, Y, \beta) \in \mathcal{BC}_n(G)$$

of a $G$-action is computed as above. Here, the symbol $(H, Y, \beta)$ is a generator of $\mathcal{BC}_n(G)$ and the defining relations reflect the invariance of the class under equivariant blowups.

When $G$ is abelian, there is a surjective homomorphism

$$\mathcal{BC}_n(G) \to \mathcal{B}_n(G),$$

a group introduced in [13, Section 1], which in turn has remarkable arithmetic properties [13], [15]. For example,

$$\mathcal{B}_n(G) \otimes \mathbb{Q} = \mathrm{H}_0(\Gamma(n, G), \mathcal{F}_n),$$

where $\Gamma(n, G) \subset \mathrm{GL}_n(\mathbb{Z})$ is a certain congruence subgroup and $\mathcal{F}_n$ is the $\mathbb{Q}$-vector space generated by characteristic functions of convex rational polyhedral cones in $\mathbb{R}^n$, modulo functions of support less than $n$ [13, Section 9]. In particular, the groups $\mathcal{B}_n(G)$ carry Hecke operators. For $n = 2$, there is a relation between $\mathcal{B}_2(G)$ and Manin symbols.

Our main result, [31, Theorem 5.2], is the construction of an isomorphism

$$\mathcal{BC}_n(G) \simeq \bigoplus_{[H,Y]} \mathcal{B}_n([H, Y]), \tag{2.2}$$

where the sum is over $G$-conjugacy classes $[H, Y]$ of pairs $(H, Y)$, with $H \subseteq G$ an abelian subgroup, $H \subseteq Y \subseteq Z_G(H)$, and

$$\mathcal{B}_n([H, Y]) \simeq \mathcal{B}_n(H)/(\mathbf{C}_{(H,Y)})$$

is the quotient by a conjugation relation which depends on the representative $(H, Y)$ of the conjugacy class of the pair.

$$(\mathbf{C}_{(H,Y)}) : \beta = \beta^g, \text{for all } g \in N_G(H) \cap N_G(Y).$$

For $G$ *abelian*, we have

$$\mathcal{B}_n([H, Y]) = \mathcal{B}_n(H), \ \text{ and } \ \mathcal{BC}_n(G) = \bigoplus_{H' \subseteq G} \bigoplus_{H'' \subseteq H'} \mathcal{B}_n(H'');$$

in particular, the groups $\mathcal{BC}_n(G)$ also carry Hecke operators, as defined in [13, Section 6] and [15, Section 3].

The combinatorial decomposition is *not* available for the geometric $\mathrm{Burn}_n(G)$, for $n \geq 3$. Clearly, the passage to the combinatorial $\mathcal{BC}_n(G)$ leads to a loss of information. On the other hand, these groups are effectively computable, and I have implemented these computations in `Magma`, for more information and `Magma` code, see

`https://kaiqi-yang1994.github.io/projects/CompBnG`.

Here are some representative examples:

**Example 5.** Recall the groups admitting *primitive* actions on $\mathbb{P}^2$: $\mathfrak{A}_5, \mathsf{ASL}_2(\mathbb{F}_3)$(the Hessian group)$, \mathsf{PSL}_2(\mathbb{F}_7), \mathfrak{A}_6$. We have:

- $G = \mathfrak{A}_5$,
$$\mathcal{BC}_2(G) = (\mathbb{Z}/2)^3, \quad \mathcal{BC}_n(G) = 0,\, n \geq 3.$$
- $G = \mathsf{ASL}_2(\mathbb{F}_3)$,
$$\mathcal{BC}_2(G) = (\mathbb{Z}/2)^7 \times \mathbb{Z}^{13},\, \mathcal{BC}_3(G) = \mathbb{Z}/2 \times \mathbb{Z},\, \mathcal{BC}_n(G) = 0, n \geq 4.$$
- $G = \mathsf{PSL}_2(\mathbb{F}_7)$,
$$\mathcal{BC}_2(G) = (\mathbb{Z}/2)^3 \times \mathbb{Z}, \quad \mathcal{BC}_3(G) = \mathbb{Z}/2, \quad \mathcal{BC}_n(G) = 0, n \geq 4.$$
- $G = \mathfrak{A}_6$,
$$\mathcal{BC}_2(G) = (\mathbb{Z}/2)^7 \times \mathbb{Z}/4 \times \mathbb{Z},\, \mathcal{BC}_3(G) = \mathbb{Z}/2 \times \mathbb{Z},\, \mathcal{BC}_n(G) = 0, n \geq 4.$$

The equivariant version of De Concini-Procesi formalism [16] on projective space are used to compute classes of linear $G$-actions,
$$[\mathbb{P}^n \circlearrowleft G] \in \mathrm{Burn}_n(G).$$
I have implemented the algorithm in `Magma`; the code is available at

`https://kaiqi-yang1994.github.io/projects/DCPonProj`.

Based on this, I was able to discover interesting examples of nonbirational actions on $\mathbb{P}^2$ and $\mathbb{P}^3$, distinguished by the Burnside invariants.

**Example 6.** Let $G$ be the extension of $C_3$ by $C_m^2$, $m \geq 3$. Consider its action on $\mathbb{P}^2$ given by generators
$$g_1 := (\zeta_m^{s_0} x_0 : x_1 : x_2), \quad g_2 := (x_0 : \zeta_m^{s_1} x_1 : x_2), \quad (x_2 : x_0 : x_1),$$
where $\zeta_m$ is a $m$-th root of unity and $s_0, s_1$ are positive integers coprime to $m$.

When $m = 5$, choosing $s_0 = 1, s_1 = 2$ and $s_0' = 3, s_1' = 4$, we obtain different linear $G$-representations $V$, $V'$, with induced faithful actions on $\mathbb{P}^2 = \mathbb{P}(V), \mathbb{P}(V')$. No previous invariants allow to distinguish these actions, up to birationality. On the other hand, from [16, Theorem 8.4, 8.5], we have

$$
\begin{aligned}
[\mathbb{P}(V) \circlearrowleft G] =& (1, G \circlearrowright k(\mathbb{P}^2), ()) \\
&+ (C_5, C_5 \circlearrowright k(t), (2)) + (C_5, C_5 \circlearrowright k(t), (3)) \\
&+ (C_5^2, 1 \circlearrowright k, ((1,0),(0,3))) + (C_5^2, 1 \circlearrowright k, ((4,0),(0,2)))
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
[\mathbb{P}(V') \circlearrowleft G] =& (1, G \circlearrowright k(\mathbb{P}^2), ()) \\
&+ (C_5, C_5 \circlearrowright k(t), (2)) + (C_5, C_5 \circlearrowright k(t), (3)) \\
&+ (C_5^2, 1 \circlearrowright k, ((3,0),(0,1))) + (C_5^2, 1 \circlearrowright k, ((2,0),(0,4)))
\end{aligned} \tag{2}
$$

where the group $C_5^2$ is generated by actions $g_1, g_2$ and the stabilizer group $C_5$ within symbols in $(1), (2)$ is generated by $g_1^{s_1} g_2^{s_0}$, fixing a projective line.

Considering the image of the difference of these classes under the natural homomorphism

$$\mathrm{Burn}_2(G) \to \mathcal{BC}_2(G) = (\mathbb{Z}/2)^2 \times (\mathbb{Z}/30)^2 \times \mathbb{Z}^{19},$$

we find that it equals

$$T_2^2 + 13T_{30}^2 + e_3 - e_7 + e_8 + 2e_{11} - e_{12} + e_{16} \neq 0 \in \mathcal{BC}_2(G),$$

where $T_2^{1,2}, T_{30}^{1,2}$ and $e_i, i = 1, \ldots, 19$ are generators for $(\mathbb{Z}/2)^2$, $(\mathbb{Z}/30)^2$, and the torsion-free part, respectively. Thus

$$[\mathbb{P}(V) \circlearrowleft G] \neq [\mathbb{P}(V') \circlearrowleft G] \in \mathrm{Burn}_2(G),$$

and we conclude that the $G$-actions are not birational.

## 3. Future plans

3.1. As mentioned in section 1, only one class of examples of minimal, and thus nonrational, but stably rational del Pezzo surfaces of degree $\leq 4$ is known [7, 6, 2]. The proof of stable rationality relies on the knowledge of explicit equations, in particular, equations for *torsors* over these surfaces. In this case, the Galois group acts via the symmetric group $\mathfrak{S}_3$, which leads to an especially simple form of the relevant torsor.

In [30], we found all Galois actions on a minimal del Pezzo surface of degree 1 and 2, satisfying condition **(H1)**. Still missing is the construction of equations realizing these actions. This is similar to the *Inverse Galois problem*, considered in [9]. Specifically, can we find minimal del Pezzo surface of degree 1 or 2, having the Galois group of splitting fields as described in proposition 3 and proposition 4? Given this, the next step is the construction of torsors. I plan to study these torsors over nonclosed fields as well as in presence of group actions.

3.2. There are 3 new groups

$$\mathcal{B}_n(G), \quad \mathcal{BC}_n(G) \text{ and } \mathrm{Burn}_n(G)$$

receiving equivariant birational invariants of some finite $G$ acting on an $n$-dimensional algebraic variety $X$. These carry a rich algebraic structure, and I plan to continue to study it. One of the key next problems is:

- The identification of the *incompressible* divisorial symbols in dimension 3, see [17]. This will draw on the classification of $G$-actions on surfaces (as in [3], [8]), as well as a detailed study of possible $H$ and $Y \subseteq Z_G(H)/H$ appearing in (2.1).

3.3. A regular $G$-action on a smooth projective variety $X$ yields a collection of locally closed subvarieties with fixed nontrivial stabilizer $H \subseteq G$. The calculation of the class

$$[X \circlearrowleft G]. \tag{3}$$

requires a *standard* model. The algorithm in [16] gives such a model, when $X = \mathbb{P}(V)$, a projectivization of a $G$-representation. I plan to

- Generalize the De Concini-Procesi subspace arrangement algorithm to general $G$-varieties, following the approach in [20] and implement it in `Magma`.
- Compute the classes (3) in representative examples.

3.4. Very little is known about the relation between Burnside-type invariants and known birational invariants of $G$-actions, such as:

(1) The existence of fixed points upon restriction to abelian subgroups [27],
(2) The determinant of weights of abelian subgroups in tangent space at fixed points[26],
(3) Amitsur subgroup [4],
(4) The group cohomology of $G$ acting on Picard group $\mathrm{Pic}(X)$ [22],
(5) The Brauer group of the quotient stack $\mathrm{Br}([X/G])$ [18].

I propose to

- Investigate to which extent the class $[X \circlearrowright G]$ captures the other birational invariants.

- Compute the unramified Brauer group of the quotient $V/G$ in terms of Burnside symbols; here $V$ is a linear representation of $G$.

3.5. New examples of nonbirational but stably birational actions on rational surfaces appeared in [11]. These were based on the analysis of torsors and of the group action on the Picard groups. Very little is known in dimension 3. I hope to extract from the systematic computations of Burnside invariants suitable candidates and to deploy the torsor formalism to find many new instances of nonlinear but stably linear actions.

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