课后作业 3 参考答案

1. 找到一个自旋量子态$|\psi\rangle$，使得对于它，S_x和S_z的不确定性关系取等号（20）

   不确定性关系取等号时有，$\langle(\Delta S_x)^2\rangle\langle(\Delta S_z)^2\rangle = \frac{1}{4}|\langle[S_x, S_z]\rangle|^2$，设此时的自旋量子态$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$。由$\boldsymbol{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$，$\boldsymbol{S}_x{}^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$，$\boldsymbol{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$，$\boldsymbol{S}_z{}^2 = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$，$[S_x, S_z] = \frac{\hbar^2}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

   $$\langle(\Delta S_x)^2\rangle = \langle S_x{}^2\rangle - \langle S_x\rangle^2 = \langle\psi|S_x{}^2|\psi\rangle - \langle\psi|S_x|\psi\rangle^2$$

   $$= \frac{\hbar^2}{4}\begin{pmatrix} a^* & b^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} - \left[\frac{\hbar}{2}\begin{pmatrix} a^* & b^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\right]^2$$

   $$= \frac{\hbar^2}{4}(a^*a + b^*b) - \frac{\hbar^2}{4}(a^*b + b^*a)^2$$

   $$\langle(\Delta S_z)^2\rangle = \langle S_z{}^2\rangle - \langle S_z\rangle^2 = \langle\psi|S_z{}^2|\psi\rangle - \langle\psi|S_z|\psi\rangle^2$$

   $$= \frac{\hbar^2}{4}\begin{pmatrix} a^* & b^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} - \left[\frac{\hbar}{2}\begin{pmatrix} a^* & b^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\right]^2$$

   $$= \frac{\hbar^2}{4}(a^*a + b^*b) - \frac{\hbar^2}{4}(a^*a - b^*b)^2$$

   $$|\langle[S_x, S_z]\rangle|^2 = |\langle\psi|[S_x, S_z]|\psi\rangle|^2$$

   $$= \left|\frac{\hbar^2}{2}\begin{pmatrix} a^* & b^* \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\right|^2$$

   $$= \frac{\hbar^4}{4}|-a^*b + b^*a|^2$$

   不确定性关系取等号时有：

   $$[(a^*a + b^*b) - (a^*b + b^*a)^2][(a^*a + b^*b) - (a^*a - b^*b)^2]$$

   $$= |-a^*b + b^*a|^2$$

   由归一化条件知$a^*a + b^*b = 1$，上式化简为：

   $$[1 - (a^*b + b^*a)^2][1 - (a^*a - b^*b)^2] = |-a^*b + b^*a|^2$$

   满足上式的$\begin{pmatrix} a \\ b \end{pmatrix}$都可使得不确定性关系取等号，如$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2. 证明（20）

   a. $|W_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$是一个纠缠态（不能写为两个单比特态的张量积）

   b. $|W_3\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$是一个纠缠态（不能写为三个单比特态的张量积）

a. 若$|W_2\rangle$不是纠缠态，则$|W_2\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \otimes \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 \\ a_1b_2 \\ b_1a_2 \\ b_1b_2 \end{pmatrix}$

$= a_1a_2|00\rangle + a_1b_2|01\rangle + b_1a_2|10\rangle + b_1b_2|11\rangle$

于是：$\begin{cases} a_1a_2 = 0 \\ a_1b_2 = \frac{1}{\sqrt{2}} \\ b_1a_2 = \frac{1}{\sqrt{2}} \\ b_1b_2 = 0 \end{cases}$，易知该方程组无解，矛盾，所以$|W_2\rangle$是纠缠态

b. 若 $|W_3\rangle$ 不是纠缠态， 则 $|W_2\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle = \begin{pmatrix} a_1a_2 \\ a_1b_2 \\ b_1a_2 \\ b_1b_2 \end{pmatrix} \otimes \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} =$

$\begin{pmatrix} a_1a_2a_3 \\ a_1b_2a_3 \\ b_1a_2a_3 \\ b_1b_2a_3 \\ a_1a_2b_3 \\ a_1b_2b_3 \\ b_1a_2b_3 \\ b_1b_2b_3 \end{pmatrix}$，于是$\begin{cases} a_1a_2a_3 = 0 \\ a_1b_2a_3 = \frac{1}{\sqrt{3}} \\ b_1a_2a_3 = \frac{1}{\sqrt{3}} \\ b_1b_2a_3 = 0 \\ a_1a_2b_3 = \frac{1}{\sqrt{3}} \\ a_1b_2b_3 = 0 \\ b_1a_2b_3 = 0 \\ b_1b_2b_3 = 0 \end{cases}$，易知该方程组无解，矛盾，所以$|W_3\rangle$是纠缠态

3. 证明$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$和$\frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle)$是同一个态（10）

$|{+}{+}\rangle + |{-}{-}\rangle = |+\rangle \otimes |+\rangle + |-\rangle \otimes |-\rangle$

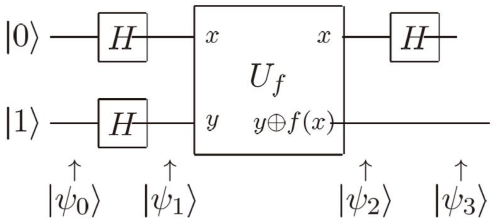
$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |00\rangle + |11\rangle$

两者为同一个态

4. 证明 Deutsch 算法输出的量子态为（20）

$$|\psi_3\rangle = \pm\, |f(0) \oplus f(1)\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

Deutsch算法



易知$|\psi_0\rangle = |0\rangle|1\rangle$， $|\psi_1\rangle = |+\rangle|-\rangle = \frac{1}{2}(|0\rangle|0\rangle - |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle)$

经过U_f操作后有：

$$|\psi_2\rangle = \frac{1}{2}(|0\rangle|f(0)\rangle - |0\rangle|1 \oplus f(0)\rangle + |1\rangle|f(1)\rangle - |1\rangle|1 \oplus f(1)\rangle)$$

Case1: f为常函数，$f(0) \oplus f(1) = 0$， $f(0) = f(1)$，此时

$|\psi_2\rangle = \frac{1}{2}(|0\rangle|f(0)\rangle - |0\rangle|1 \oplus f(0)\rangle + |1\rangle|f(1)\rangle - |1\rangle|1 \oplus f(0)\rangle)$

$= \frac{1}{2}((|0\rangle + |1\rangle)\otimes|f(0)\rangle - (|0\rangle + |1\rangle) \otimes|1 \oplus f(0)\rangle)$

$= \frac{1}{2}(|0\rangle + |1\rangle)\otimes(|f(0)\rangle - |1 \oplus f(0)\rangle)$

$= \frac{1}{\sqrt{2}}|+\rangle\otimes(|f(0)\rangle - |1 \oplus f(0)\rangle)$

经过最后一个 Hadamard 门后

$|\psi_3\rangle = \frac{1}{\sqrt{2}}|0\rangle\otimes(|f(0)\rangle - |1 \oplus f(0)\rangle)$

$= \frac{1}{\sqrt{2}}|f(0) \oplus f(1)\rangle\otimes(|f(0)\rangle - |1 \oplus f(0)\rangle)$

当$f(0) = f(1) = 1$ 时，$|\psi_3\rangle = \frac{1}{\sqrt{2}}|f(0) \oplus f(1)\rangle\otimes(|1\rangle - |0\rangle)$

当$f(0) = f(1) = 0$ 时，$|\psi_3\rangle = \frac{1}{\sqrt{2}}|f(0) \oplus f(1)\rangle\otimes(|0\rangle - |1\rangle)$

Case2: f为平衡函数，$f(0) \oplus f(1) = 1$， $f(0) \oplus 1 = f(1)$， $f(1) \oplus 1 = f(0)$

$|\psi_2\rangle = \frac{1}{2}(|0\rangle|f(0)\rangle - |0\rangle|f(1)\rangle + |1\rangle|f(1)\rangle - |1\rangle|f(0)\rangle)$

$= \frac{1}{2}((|0\rangle - |1\rangle)\otimes|f(0)\rangle - (|0\rangle - |1\rangle) \otimes|f(1)\rangle)$

$= \frac{1}{2}(|0\rangle - |1\rangle)\otimes(|f(0)\rangle - |f(1)\rangle)$

$= \frac{1}{\sqrt{2}}|-\rangle\otimes(|f(0)\rangle - |f(1)\rangle)$

经过最后一个 Hadamard 门后

$|\psi_3\rangle = \dfrac{1}{\sqrt{2}}|1\rangle\otimes(|f(0)\rangle - |f(1)\rangle)$

$= \dfrac{1}{\sqrt{2}}|f(0)\oplus f(1)\rangle(|f(0)\rangle - |f(1)\rangle)$

当$f(0)=1$，$f(1)=0$ 时，$|\psi_3\rangle = \frac{1}{\sqrt{2}}|f(0)\oplus f(1)\rangle\otimes(|1\rangle - |0\rangle)$

当$f(0)=0$，$f(1)=1$ 时，$|\psi_3\rangle = \frac{1}{\sqrt{2}}|f(0)\oplus f(1)\rangle\otimes(|0\rangle - |1\rangle)$

综上所述，$|\psi_3\rangle = \pm\, |f(0)\oplus f(1)\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$

5. 考虑一个分为左右两半的盒子，某粒子可以处于左半或右半部分，对应量子态$|L\rangle$和$|R\rangle$。则描述其位置的一般量子态为$|\psi\rangle = a|L\rangle + b|R\rangle$，其中$a$、$b$满足归一化条件。假设粒子存在一定可能从左半隧穿到右半（或反之），可以用如下哈密顿量描述：

$$H = \Delta(|R\rangle\langle L| + |L\rangle\langle R|)$$

其中Δ为实数。（30）

a. 求得该粒子的能量本征值和能量本征态

b. 若粒子在$t=0$ 时处于$|\psi, t=0\rangle = a_0|L\rangle + b_0|R\rangle$，求$t$时刻粒子所处的态

c. 若粒子在$t=0$ 时确定处于左半边，求粒子在t时刻处于右半边的概率

a. 本征方程为：$H|\psi\rangle = E_n|\psi\rangle$，在以$|L\rangle$，$|R\rangle$为正交基的坐标系中，哈密顿量的矩阵表述为：$H = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}$

本征方程变为:$\begin{pmatrix} -E_n & \Delta \\ \Delta & -E_n \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \mathbf{0}$,由其系数行列式等于 0 求得$E_n = \pm\Delta$，本征态$|+\Delta\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle)$，$|-\Delta\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|L\rangle - |R\rangle)$

b. 在本征基下，时间演化算符$U(t,0) = \begin{pmatrix} e^{\frac{-i\Delta t}{\hbar}} & 0 \\ 0 & e^{\frac{i\Delta t}{\hbar}} \end{pmatrix}$

在本征基下，$|L\rangle = \frac{1}{\sqrt{2}}(|+\Delta\rangle + |-\Delta\rangle)$，$|R\rangle = \frac{1}{\sqrt{2}}(|+\Delta\rangle - |-\Delta\rangle)$

初态$|\psi, t=0\rangle = \frac{a_0+b_0}{\sqrt{2}}|+\Delta\rangle + \frac{a_0-b_0}{\sqrt{2}}|-\Delta\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} a_0+b_0 \\ a_0-b_0 \end{pmatrix}$

任意时刻的态$|\psi, t\rangle = U(t,0)|\psi, t=0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\frac{-i\Delta t}{\hbar}}(a_0+b_0) \\ e^{\frac{i\Delta t}{\hbar}}(a_0-b_0) \end{pmatrix}$

回到$|L\rangle$，$|R\rangle$为正交基下

$$|\psi,t\rangle = \left[a_0 cos\left(\frac{\Delta t}{\hbar}\right) - ib_0 sin\left(\frac{\Delta t}{\hbar}\right)\right]|L\rangle + \left[b_0 cos\left(\frac{\Delta t}{\hbar}\right) - ia_0 sin\left(\frac{\Delta t}{\hbar}\right)\right]|R\rangle$$

c. 由题意，$a_0 = 1$，$b_0 = 0$，所以$|\psi,t\rangle = cos\left(\frac{\Delta t}{\hbar}\right)|L\rangle - isin\left(\frac{\Delta t}{\hbar}\right)|R\rangle$

处于右半边的概率为$\left|-isin\left(\frac{\Delta t}{\hbar}\right)\right|^2 = sin^2\left(\frac{\Delta t}{\hbar}\right)$