《量子信息基础》2022.3.1 随堂作业：

1. <即教材*问题 1.5>

Consider the wave function

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t}$$

where A, λ, and ω are positive real constants.

(a) Normalize Ψ.

$$\int |\Psi(x,t)|^2 dx = \int \left|Ae^{-\lambda|x|}e^{-i\omega t}\right|^2 dx = |A|^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|}dx$$
$$= 2|A|^2 \int_0^{\infty} e^{-2\lambda x}dx = \frac{|A|^2}{\lambda} = 1$$
$$\therefore A = \sqrt{\lambda}$$

(b) Determine the expectation values of x and x^2.

$$\langle x\rangle = \int_{-\infty}^{+\infty} x|\Psi(x,t)|^2 dx = \int_{-\infty}^{\infty} x\left|Ae^{-\lambda|x|}\right|^2 dx = |A|^2 \int_{-\infty}^{\infty} xe^{-2\lambda|x|}dx$$
$$= \lambda \int_{-\infty}^{\infty} xe^{-2\lambda|x|}dx$$

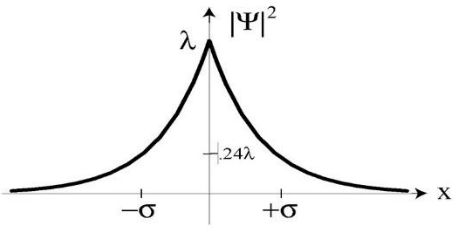
$\because xe^{-2\lambda|x|}$是奇函数　　　　$\therefore \langle x\rangle = 0$

$$\langle x^2\rangle = \int_{-\infty}^{+\infty} x^2|\Psi(x,t)|^2 dx = |A|^2 \int_{-\infty}^{\infty} x^2 e^{-2\lambda|x|}dx = 2\lambda \int_0^{\infty} x^2 e^{-2\lambda x}dx$$
$$= -\int_0^{\infty} x^2 de^{-2\lambda x} = -x^2e^{-2\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-2\lambda x}dx^2$$
$$= \int_0^{\infty} 2xe^{-2\lambda x}dx = -\frac{1}{\lambda}\int_0^{\infty} xde^{-2\lambda x}$$
$$= -\frac{1}{\lambda}xe^{-2\lambda x}\Big|_0^{\infty} + \frac{1}{\lambda}\int_0^{\infty} e^{-2\lambda x}dx = -\frac{1}{2\lambda^2}\int_0^{\infty} de^{-2\lambda x} = \frac{1}{2\lambda^2}$$

(c) Find the standard deviation of x. Sketch the graph of $|\Psi|^2$, as a function of x, and mark the points $(\langle x\rangle - \sigma)$ and $(\langle x\rangle + \sigma)$, to illustrate the sense in which σ represents the “spread” in x.

$$\sigma = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{1}{\sqrt{2}\lambda}$$

$|\Psi|^2$的图像如下图所示：



(d) What is the probability that the particle would be found outside this range (which means outside of $(\langle x\rangle - \sigma)$ and $(\langle x\rangle + \sigma)$)?

$$P = 2\int_{\sigma}^{\infty} |\Psi(x,t)|^2 dx = 2\lambda \int_{\sigma}^{\infty} e^{-2\lambda x} dx = -\int_{\sigma}^{\infty} de^{-2\lambda x} = e^{-2\lambda\sigma} = e^{-\sqrt{2}}$$
$$= 0.2431$$

2. <PPT 最后一页>

A photon propagates in the z direction and passes a linear optical polarizer which is oriented in the x direction (see figure below). The state in Figure (a) is ψ_a while the state in Figure (b) is ψ_b.

(a) Write done the formula of ψ_c, assuming the light beam is polarized with an angle of α to the x axis.

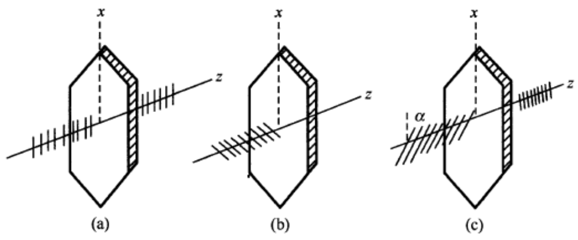
$$\psi_{\alpha} = \cos\alpha \cdot \psi_{x} + \sin\alpha \cdot \psi_{y}$$

(b) How much is the probability that a single photon could pass the polarizer in (c)?

$$P = |\cos\alpha|^2$$

(c) How does the system maintain the normalization condition?

$$|\cos\alpha|^2 + |\sin\alpha|^2 = 1$$



3. <即教材*问题 1.15>

Show that

$$\frac{d}{dt}\int_{-\infty}^{\infty} \Psi_1^* \Psi_2 dx = 0$$

for any two (normalizable) solutions to the Schrödinger equation (with the same $V(x)$), Ψ_1 and Ψ_2.

$$\frac{d}{dt}\int_{-\infty}^{\infty} \Psi_1^* \Psi_2 dx = \int_{-\infty}^{\infty} \frac{\partial \Psi_1^*}{\partial t}\Psi_2 + \Psi_1^* \frac{\partial \Psi_2}{\partial t} dx$$

复共轭函数的薛定谔方程

$$-i\hbar\frac{\partial\Psi_1^*}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi_1^*}{\partial x^2}+V\Psi_1^*$$

$$i\hbar\frac{\partial\Psi_2}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi_2}{\partial x^2}+V\Psi_2$$

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2 dx=\int_{-\infty}^{\infty}\left[\frac{\hbar}{2im}\frac{\partial^2\Psi_1^*}{\partial x^2}-\frac{V}{i\hbar}\Psi_1^*\right]\Psi_2+\Psi_1^*\left[-\frac{\hbar}{2im}\frac{\partial^2\Psi_2}{\partial x^2}+\frac{V}{i\hbar}\Psi_2\right]dx$$

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2 dx=\int_{-\infty}^{\infty}\frac{\hbar}{2im}\frac{\partial^2\Psi_1^*}{\partial x^2}\Psi_2-\Psi_1^*\frac{\hbar}{2im}\frac{\partial^2\Psi_2}{\partial x^2}dx$$

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2 dx=\frac{\hbar}{2im}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left(\frac{\partial\Psi_1^*}{\partial x}\Psi_2-\Psi_1^*\frac{\partial\Psi_2}{\partial x}\right)dx$$

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2 dx=\frac{\hbar}{2im}\left(\frac{\partial\Psi_1^*}{\partial x}\Psi_2-\Psi_1^*\frac{\partial\Psi_2}{\partial x}\right)\Bigg|_{-\infty}^{\infty}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2 dx=0$$

* David J. Griffiths, and Darrell F. Schroeter, Introduction to Quantum Mechanics (3rd Edition), Cambridge University Press (2018).