《量子信息基础》第 4 章第一部分：

1. In a quantum system, the Eigen energy and wavefunctions are $E_n, \psi_n, n = 0,1,2,\cdots$. When $t \leq 0$, the system is in the ground state of E_0, ψ_0. A perturbation occurs for $t \geq 0$, which is $H'(t) = Fe^{-t/\tau}$. Calculate the probabilities for the system evolves into the state of E_n, ψ_n when $t \geq 0$.

Let's define

$$F_{n0} = \langle \psi_n | F(x) | \psi_0 \rangle$$

$$c_n^{(1)}(t) = -\frac{i}{\hbar}\int_0^t H'_{n0}(t')e^{i\omega_{n0}t'}dt' = -\frac{i}{\hbar}\int_0^t F_{n0}e^{-\frac{t'}{\tau}}e^{i\omega_{n0}t'}dt' = -\frac{iF_{n0}}{\hbar}\int_0^t e^{i\omega_{n0}t'-\frac{t'}{2}}dt'$$

$$= -\frac{iF_{n0}}{\hbar}\int_0^t e^{i\omega_{n0}t'-\frac{t'}{2}}dt' = \frac{F_{n0}\left(e^{i\omega_{n0}t-t/\tau}-1\right)}{-\hbar\omega_{n0}-i\hbar/\tau}$$

$$\left|c_n^{(1)}(t)\right|^2 = \frac{|F_{n0}|^2\left(e^{i\omega_{n0}t-t/\tau}-1\right)^2}{(\hbar\omega_{n0})^2+(\hbar/\tau)^2}$$

$$t \gg \tau$$

$$\left|c_n^{(1)}(t)\right|^2 = \frac{|F_{n0}|^2}{(\hbar\omega_{n0})^2+(\hbar/\tau)^2}$$

2. (Text book* Problem 11.6)
   Solve Equation 11.17 to second order in perturbation theory, for the general case $c_a(0) = a, c_b(0) = b$.

$$\begin{cases} \dot{c}_a = -\frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}c_b \\ \dot{c}_b = -\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}c_a \end{cases}$$

Initial conditions:

$$c_a(0) = a, c_b(0) = b$$

0-order

$$c_a^0(0) = a, c_b^0(0) = b$$

First order

$$\frac{dc_a^{(1)}}{dt} = -\frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}b$$

$$c_a^{(1)}(t) = a - \frac{ib}{\hbar}\int_0^t H'_{ab}(t')e^{-i\omega_0 t'}dt'$$

$$\frac{dc_b^{(1)}}{dt} = -\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}a$$

$$c_b^{(1)}(t) = b - \frac{ia}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}dt'$$

Second order

$$\frac{dc_a^{(2)}}{dt} = -\frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}\left[b - \frac{ia}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}dt'\right]$$

$$c_a^{(2)}(t) = a - \frac{i}{\hbar}\int_0^t H'_{ab}(t')e^{-i\omega_0 t'}\left[b - \frac{ia}{\hbar}\int_0^{t'} H'_{ba}(t'')e^{i\omega_0 t''}dt''\right]dt'$$

$$\frac{dc_b^{(2)}}{dt} = -\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}\left[a - \frac{ib}{\hbar}\int_0^t H'_{ab}(t')e^{-i\omega_0 t'}dt'\right]$$

$$c_b^{(2)}(t) = b - \frac{i}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}\left[a - \frac{ib}{\hbar}\int_0^{t'} H'_{ab}(t'')e^{-i\omega_0 t''}dt''\right]dt'$$

3. (Text book* Problem 11.3)
   Solve Equation 11.17 for the case of a *time-independent* perturbation, assuming that $c_a(0) = 1$ and $c_b(0) = 0$. Check that $|c_a(t)|^2 + |c_b(t)|^2 = 1$. Comment: Ostensibly, this system oscillates between "pure ψ_a" and "some ψ_b". Doesn't this contradict my general assertion that no transitions occur for time-independent perturbations? No, but the reason is rather subtle: In this case ψ_a and ψ_b are not, and never were, eigenstates of the Hamiltonian—a measurement of the energy *never* yields E_a or E_b. In time-dependent perturbation theory we typically contemplate turning *on* the perturbation for a while, and then turning it *off* again, in order to examine the system. At the beginning, and at the end, ψ_a and ψ_b are eigenstates of the exact Hamiltonian, and only in this context does it make sense to say that the system underwent a transition from one to the other. For the present problem, then, assume that the perturbation was turned on at time $t = 0$, and off again at time t—this doesn't affect the *calculations*, but it allows for a more sensible interpretation of the result.

$$\begin{cases} \dot{c}_a = -\frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}c_b \\ \dot{c}_b = -\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}c_a \end{cases}$$

$$\ddot{c}_a = -i\omega_0\dot{c}_a - \frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}\dot{c}_b = -i\omega_0\dot{c}_a - \frac{1}{\hbar^2}H'_{ab}H'_{ba}c_a$$

$$\ddot{c}_a + i\omega_0\dot{c}_a + \frac{1}{\hbar^2}H'_{ab}H'_{ba}c_a = 0$$

Assume

$$c_a = e^{\lambda t}$$

We get

$$\lambda^2 + i\omega_0\lambda + \frac{|H'_{ab}|^2}{\hbar^2} = 0$$

$$\lambda = \frac{1}{2}\left(-i\omega_0 \pm i\sqrt{-\omega_0^2 - \frac{4}{\hbar^2}\left|H'_{ab}\right|^2}\right) = \frac{i}{2}(-\omega_0 \pm \omega)$$

where

$$\omega \equiv \sqrt{-\omega_0^2 - \frac{4}{\hbar^2}\left|H'_{ab}\right|^2}$$

The general solution is

$$c_a(t) = C_1 e^{-\frac{i}{2}(\omega_0-\omega)t} + C_2 e^{-\frac{i}{2}(\omega_0+\omega)t} = e^{-\frac{i}{2}\omega_0 t}\left[C_3\cos\left(\frac{\omega t}{2}\right) + C_4\sin\left(\frac{\omega t}{2}\right)\right]$$

$$\because\ c_a(0) = 1, \therefore C_3 = 1$$

$$c_a(t) = e^{-\frac{i}{2}\omega_0 t}\left[\cos\left(\frac{\omega t}{2}\right) + C_4\sin\left(\frac{\omega t}{2}\right)\right]$$

$$\begin{aligned}\dot{c}_a(t) &= -\frac{i\omega_0}{2}e^{-\frac{i}{2}\omega_0 t}\left[\cos\left(\frac{\omega t}{2}\right) + C_4\sin\left(\frac{\omega t}{2}\right)\right] + e^{-\frac{i}{2}\omega_0 t}\left[-\frac{\omega}{2}\sin\left(\frac{\omega t}{2}\right) + C_4\frac{\omega}{2}\cos\left(\frac{\omega t}{2}\right)\right] \\ &= e^{-\frac{i}{2}\omega_0 t}\left[\left(C_4\frac{\omega}{2} - \frac{i\omega_0}{2}\right)\cos\left(\frac{\omega t}{2}\right) - \left(C_4\frac{i\omega_0}{2} + \frac{\omega}{2}\right)\sin\left(\frac{\omega t}{2}\right)\right]\end{aligned}$$

$$c_b(t) = -\frac{\hbar}{iH'_{ab}}e^{i\omega_0 t}\dot{c}_a = -\frac{\hbar}{iH'_{ab}}e^{\frac{i}{2}\omega_0 t}\left[\left(C_4\frac{\omega}{2} - \frac{i\omega_0}{2}\right)\cos\left(\frac{\omega t}{2}\right) - \left(C_4\frac{i\omega_0}{2} + \frac{\omega}{2}\right)\sin\left(\frac{\omega t}{2}\right)\right]$$

$$\because\ c_b(0) = 1, \therefore C_4 = \frac{i\omega_0}{\omega}$$

$$c_a(t) = e^{-\frac{i}{2}\omega_0 t}\left[\cos\left(\frac{\omega t}{2}\right) + \frac{i\omega_0}{\omega}\sin\left(\frac{\omega t}{2}\right)\right]$$

$$c_b(t) = \frac{\hbar}{iH'_{ab}}e^{\frac{i}{2}\omega_0 t}\left(-\frac{\omega_0^2}{2\omega} + \frac{\omega}{2}\right)\sin\left(\frac{\omega t}{2}\right) = \frac{2H'_{ba}}{i\hbar\omega}e^{\frac{i}{2}\omega_0 t}\sin\left(\frac{\omega t}{2}\right)$$

$$\begin{aligned}|c_a(t)|^2 + |c_b(t)|^2 &= \cos^2\left(\frac{\omega t}{2}\right) + \left(\frac{\omega_0}{\omega}\right)^2\sin^2\left(\frac{\omega t}{2}\right) + \frac{4|H'_{ab}|^2}{\hbar^2\omega^2}\sin^2\left(\frac{\omega t}{2}\right) \\ &= \cos^2\left(\frac{\omega t}{2}\right) + \left(\frac{\omega_0}{\omega}\right)^2 + \frac{\omega^2 - \omega_0^2}{\omega^2}\sin^2\left(\frac{\omega t}{2}\right) = 1\end{aligned}$$

4. (Text book* Problem 11.11)
   You could derive the spontaneous emission rate (Equation 11.63) without the detour through Einstein's *A* and *B* coefficients if you knew the ground state energy density of the electromagnetic field, $\rho_0(\omega)$, for then it would simply be a case of stimulated emission (Equation 11.54). To do this honestly would require quantum electrodynamics, but if you are prepared to believe that the ground state consists of one photon in each classical mode, then the derivation is fairly simple:
   (a) To obtain the classical modes, consider an empty cubical box, of side l, with one corner at the origin. Electromagnetic fields (in vacuum) satisfy the classical wave equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) f(x, y, z, t) = 0$$

where *f* stands for any component of ***E*** or of ***B***. Show that separation of variables, and the imposition of the boundary condition *f* = 0 on all six surfaces yields the standing wave patterns

$$f_{n_x, n_y, n_z} = A\cos(\omega t)\sin\left(\frac{n_x \pi}{l}x\right)\sin\left(\frac{n_y \pi}{l}y\right)\sin\left(\frac{n_z \pi}{l}z\right)$$

with

$$\omega = \frac{\pi c}{l}\sqrt{n_x^2 + n_y^2 + n_z^2}$$

There are two modes for each triplet of positive integers $(n_x, n_y, n_z = 1,2,3, \dots)$, corresponding to the two polarization states.

--

Consider the wave function *f* can be separated into four different variables, e.g.

$$f(x, y, z, t) = X(x)Y(y)Z(z)T(t)$$

Then

$$XYZ\frac{1}{c^2}\frac{\partial^2 T}{\partial t^2} = YZT\frac{\partial^2 X}{\partial x^2} + XZT\frac{\partial^2 Y}{\partial y^2} + XYT\frac{\partial^2 Z}{\partial z^2} \tag{1}$$

Assuming

$$X(x) = A_x\, sin(k_x x) + B_x\, cos(k_x x)$$
$$Y(y) = A_y\, sin(k_y y) + B_y\, cos(k_y y)$$
$$Z(z) = A_z\, sin(k_z z) + B_z\, cos(k_z z)$$
$$T(t) = A_t\, sin(\omega t) + B_t\, cos(\omega t)$$

Since $X(0) = Y(0) = Z(0) = 0$

So

$$B_x = B_y = B_z = 0$$

Since $X(l) = Y(l) = Z(l) = 0$

$$k_x = \frac{n_x \pi}{l}$$
$$k_y = \frac{n_y \pi}{l}$$
$$k_z = \frac{n_z \pi}{l}$$
$$(n_x, n_y, n_z = 1,2,3, \dots)$$

The equation (1) evolves to be

$$A_x A_y A_z A_t \frac{\omega^2}{c^2} sin(k_x x)\, sin(k_y y)\, sin(k_z z) \sin(\omega t)$$
$$+ A_x A_y A_z B_t \frac{\omega^2}{c^2} sin(k_x x)\, sin(k_y y)\, sin(k_z z) \cos(\omega t)$$
$$= A_x A_y A_z A_t k_x^2\, sin(k_x x)\, sin(k_y y)\, sin(k_z z) \sin(\omega t)$$
$$+ A_x A_y A_z A_t k_y^2\, sin(k_x x)\, sin(k_y y)\, sin(k_z z) \sin(\omega t)$$
$$+ A_x A_y A_z A_t k_z^2\, sin(k_x x)\, sin(k_y y)\, sin(k_z z) \sin(\omega t)$$
$$+ A_x A_y A_z B_t k_x^2\, sin(k_x x)\, sin(k_y y)\, sin(k_z z) \cos(\omega t)$$
$$+ A_x A_y A_z B_t k_y^2\, sin(k_x x)\, sin(k_y y)\, sin(k_z z) \cos(\omega t)$$
$$+ A_x A_y A_z B_t k_z^2\, sin(k_x x)\, sin(k_y y)\, sin(k_z z) \cos(\omega t)$$

and

$$\omega^2 = c^2 (k_x^2 + k_y^2 + k_z^2) = \frac{\pi^2 c^2}{l^2} (n_x^2 + n_y^2 + n_z^2)$$
$$\omega = \frac{\pi c}{l} \sqrt{n_x^2 + n_y^2 + n_z^2}$$

If the normalization coefficient is A, then

$$f_{n_x, n_y, n_z} = A \cos(\omega t) \sin\left(\frac{n_x \pi}{l} x\right) \sin\left(\frac{n_y \pi}{l} y\right) \sin\left(\frac{n_z \pi}{l} z\right)$$

(b) The energy of a photon $E = h\nu = \hbar\omega$ is (Equation 4.92), so the energy in the mode (n_x, n_y, n_z) is

$$E_{n_x, n_y, n_z} = 2 \frac{\pi \hbar c}{l} \sqrt{n_x^2 + n_y^2 + n_z^2}$$

What, then, is the total energy per unit volume in the frequency range $d\omega$, if each mode gets one photon? Express your answer in the form.

$$\frac{1}{l^3} dE = \rho_0(\omega) d\omega$$

and read off $\rho_0(\omega)$. Hint: refer to Figure 5.3.

--

Considering there are two different polarization in ***E*** or ***B***

$$E_{n_x, n_y, n_z} = 2\hbar\omega = 2 \frac{\pi \hbar c}{l} \sqrt{n_x^2 + n_y^2 + n_z^2}$$

In the ω-space, each mode has a volume of

$$\frac{\pi^3 c^3}{l^3}$$

and

$$dN = 2 \cdot \frac{1}{8} \cdot 4\pi\omega^2 d\omega \cdot \frac{l^3}{\pi^3 c^3} = \frac{1}{2} \frac{\omega^2 l^3}{\pi^2 c^3} d\omega$$
$$\frac{1}{l^3} dE = \frac{1}{l^3} 2\hbar\omega dN = \frac{\hbar\omega^3}{\pi^2 c^3} d\omega$$
$$\rho_0(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}$$

(c) Use your result, together with Equation 11.54, to obtain the spontaneous emission rate. Compare Equation 11.63.

--

$$R_{b\to a}(t) \cong \frac{\pi}{3\varepsilon_0\hbar^2}|\wp|^2\rho_0(\omega) = \frac{\pi}{3\varepsilon_0\hbar^2}|\wp|^2\frac{\hbar\omega^3}{\pi^2c^3} = \frac{\omega^3}{3\pi\varepsilon_0\hbar c^3}|\wp|^2$$

* David J. Griffiths, and Darrell F. Schroeter, Introduction to Quantum Mechanics (3rd Edition), Cambridge University Press (2018).