《量子信息基础》第一部分作业

1. (Text book* Problem 3.27)

   Let $\hat{Q}$ be an operator with a complete set of orthonormal eigenvectors:

$$\hat{Q}|e_n\rangle = q_n|e_n\rangle$$

   (a) Show that $\hat{Q}$ can be written in terms of its spectral decomposition:

$$\hat{Q} = \sum_n q_n|e_n\rangle\langle e_n|$$

   Hint: An operator is characterized by its action on all possible vectors, so what you must show is that

$$\hat{Q}|\alpha\rangle = \left\{\sum_n q_n|e_n\rangle\langle e_n|\right\}|\alpha\rangle$$

   for any $|\alpha\rangle$.

Define an arbitrary vector $|\alpha\rangle$

$$|\alpha\rangle = \sum_n c_n\,|e_n\rangle$$

$$\hat{Q}|\alpha\rangle = \sum_n c_n\,\hat{Q}|e_n\rangle = \sum_n \langle e_n|\alpha\rangle \cdot q_n|e_n\rangle = \left\{\sum_n q_n|e_n\rangle\langle e_n|\right\}|\alpha\rangle$$

$$\hat{Q} = \sum_n q_n|e_n\rangle\langle e_n|$$

   (b) Another way to define a function of $\hat{Q}$ is via the spectral decomposition:

$$f(\hat{Q}) = \sum_n f(q_n)|e_n\rangle\langle e_n|$$

   Show that this is equivalent to

$$e^{\hat{Q}} \equiv 1 + \hat{Q} + \frac{1}{2}\hat{Q}^2 + \frac{1}{3!}\hat{Q}^3 + \cdots$$

   when $f(\hat{Q}) = e^{\hat{Q}}$.

$f(\hat{Q}) = e^{\hat{Q}} = \sum_n e^{q_n}|e_n\rangle\langle e_n| = \sum_n\left(1 + q_n + \frac{1}{2}q_n^2 + \frac{1}{3!}q_n^3 + \cdots\right)|e_n\rangle\langle e_n| =$ $\sum_n|e_n\rangle\langle e_n| + \sum_n q_n|e_n\rangle\langle e_n| + \frac{1}{2}\sum_n q_n^2|e_n\rangle\langle e_n| + \frac{1}{3!}\sum_n q_n^3|e_n\rangle\langle e_n| = 1 + \hat{Q} + \frac{1}{2}\hat{Q}^2 + \frac{1}{3!}\hat{Q}^3 +$ $\cdots$In the EPRB experiment, state whether each of the following states is entangled (i.e. can it be factored into a product of states of the individual particles, in which case it is not entangled)

(a) $\frac{1}{\sqrt{2}}(|0_1,1_2\rangle - |0_1,0_2\rangle)$
(b) $\frac{1}{\sqrt{2}}(|0_1,1_2\rangle - |1_1,0_2\rangle)$
(c) $\frac{3}{5}|0_1,1_2\rangle + \frac{4i}{5}|1_1,1_2\rangle$
(d) $\frac{1}{2}(|0_1,0_2\rangle + |0_1,1_2\rangle + |1_1,0_2\rangle + |1_1,1_2\rangle)$

(a) $\frac{1}{\sqrt{2}}(|0_1,1_2\rangle - |0_1,0_2\rangle) = \frac{1}{\sqrt{2}}|0_1\rangle(|1_2\rangle - |0_2\rangle)$
Not entangled

(b) $\frac{1}{\sqrt{2}}(|0_1,1_2\rangle - |1_1,0_2\rangle)$
Entangled

(c) $\frac{3}{5}|0_1,1_2\rangle + \frac{4i}{5}|1_1,1_2\rangle = \left(\frac{3}{5}|0_1\rangle + \frac{4i}{5}|1_1\rangle\right)|1_2\rangle$
Not entangled

(d) $\frac{1}{2}(|0_1,0_2\rangle + |0_1,1_2\rangle + |1_1,0_2\rangle + |1_1,1_2\rangle) = \frac{1}{2}\big(|0_1\rangle(|0_2\rangle + |1_2\rangle) + |1_1\rangle(|0_2\rangle + |1_2\rangle)\big) = \frac{1}{2}(|0_1\rangle + |1_1\rangle)(|0_2\rangle + |1_2\rangle)$
Not entangled

2. Consider a Bell state $\frac{1}{\sqrt{2}}(|0_1,0_2\rangle + |1_1,1_2\rangle)$ of two photons.
   (a) Express it now on a basis of $|\theta\rangle$ and $|\theta + \pi/2\rangle$ by rotating the original basis by an angle θ. Hint: on such a basis, $|1\rangle = \sin\theta|\theta\rangle + \cos\theta\,|\theta + \pi/2\rangle$ and also a similar expression for $|0\rangle$.
   (b) In the new basis of such sate, show that the two photons will always come out of the same arm of each polarizer when two aligned polarizers are used to examine the pair of photons.

(a)

$$|1\rangle = \sin\theta|\theta\rangle + \cos\theta\,|\theta + \pi/2\rangle$$
$$|0\rangle = \cos\theta|\theta\rangle - \sin\theta\,|\theta + \pi/2\rangle$$

$$\begin{aligned}\langle 1|0\rangle &= (\langle\theta|\sin\theta + \langle\theta + \pi/2|\cos\theta)(\cos\theta|\theta\rangle - \sin\theta\,|\theta + \pi/2\rangle) \\ &= \sin\theta\cos\theta - \cos\theta\sin\theta = 0\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{\sqrt{2}}(|0_1,0_2\rangle+|1_1,1_2\rangle)\\
&\qquad=\frac{1}{\sqrt{2}}\left(\cos\theta|\theta_1\rangle-\sin\theta\left|\theta_1+\frac{\pi}{2}\right\rangle\right)\left(\cos\theta|\theta_2\rangle-\sin\theta\left|\theta_2+\frac{\pi}{2}\right\rangle\right)\\
&\qquad+\frac{1}{\sqrt{2}}\left(\sin\theta|\theta_1\rangle+\cos\theta\left|\theta_1+\frac{\pi}{2}\right\rangle\right)\left(\sin\theta|\theta_2\rangle+\cos\theta\left|\theta_2+\frac{\pi}{2}\right\rangle\right)\\
&\qquad=\frac{1}{\sqrt{2}}\cos\theta\cos\theta|\theta_1,\theta_2\rangle\cancel{-\frac{1}{\sqrt{2}}\cos\theta\sin\theta\left|\theta_1,\theta_2+\frac{\pi}{2}\right\rangle}\\
&\qquad\cancel{-\frac{1}{\sqrt{2}}\sin\theta\cos\theta\left|\theta_1+\frac{\pi}{2},\theta_2\right\rangle}+\frac{1}{\sqrt{2}}\sin\theta\sin\theta\left|\theta_1+\frac{\pi}{2},\theta_2+\frac{\pi}{2}\right\rangle\\
&\qquad+\frac{1}{\sqrt{2}}\sin\theta\sin\theta|\theta_1,\theta_2\rangle\cancel{+\frac{1}{\sqrt{2}}\sin\theta\cos\theta\left|\theta_1,\theta_2+\frac{\pi}{2}\right\rangle}\\
&\qquad\cancel{+\frac{1}{\sqrt{2}}\cos\theta\sin\theta\left|\theta_1+\frac{\pi}{2},\theta_2\right\rangle}+\frac{1}{\sqrt{2}}\cos\theta\cos\theta\left|\theta_1+\frac{\pi}{2},\theta_2+\frac{\pi}{2}\right\rangle\\
&\qquad=\frac{1}{\sqrt{2}}|\theta_1,\theta_2\rangle+\frac{1}{\sqrt{2}}\left|\theta_1+\frac{\pi}{2},\theta_2+\frac{\pi}{2}\right\rangle
\end{aligned}
$$

(b)

There are only the $|\theta_1,\theta_2\rangle$ or $\left|\theta_1+\frac{\pi}{2},\theta_2+\frac{\pi}{2}\right\rangle$ states. So the photons will only come out of the same arm of the polarizers.

* David J. Griffiths, and Darrell F. Schroeter, Introduction to Quantum Mechanics (3rd Edition), Cambridge University Press (2018).