《量子信息基础》第二部分作业

1. (Text book* Problem 3.4)
   (a) Show that the sum of two Hermitian operators is Hermitian.

Assume $\hat{Q}$ and $\hat{S}$ are Hermitian operators, and $f(x)$ and $g(x)$ are arbitrary functions.

$\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$ and $\langle f|\hat{S}g\rangle = \langle \hat{S}f|g\rangle$. Therefore

$$\langle f|(\hat{Q}+\hat{S})g\rangle = \langle f|\hat{Q}g\rangle + \langle f|\hat{S}g\rangle = \langle \hat{Q}f|g\rangle + \langle \hat{S}f|g\rangle = \langle (\hat{Q}+\hat{S})f|g\rangle$$

So $(\hat{Q}+\hat{S})$ are Hermitian

   (b) Suppose $\hat{Q}$ is Hermitian, and α is a complex number. Under what condition (on α) is $\alpha\hat{Q}$ Hermitian?

$$\langle f|\alpha\hat{Q}g\rangle = \alpha\langle f|\hat{Q}g\rangle$$

$$\langle \alpha\hat{Q}f|g\rangle = \alpha^*\langle f|\hat{Q}g\rangle$$

If α is a real number, $\alpha\hat{Q}$ is Hermitian.

   (c) When is product of two Hermitian operators Hermitian?

$$\langle f|\hat{Q}\hat{S}g\rangle = \langle f|\hat{Q}(\hat{S}g)\rangle = \langle \hat{Q}f|\hat{S}g\rangle = \langle \hat{S}\hat{Q}f|g\rangle$$

If $\hat{Q}\hat{S}$ is Hermitian,

$$\langle f|\hat{Q}\hat{S}g\rangle = \langle \hat{Q}\hat{S}f|g\rangle$$

$\hat{Q}\hat{S}$ needs to be commutable.

   (d) Show that the position operator $(\hat{x})$ and the Hamiltonian operator $\left(\hat{H} = -\left(\frac{\hbar^2}{2m}\right)\frac{d^2}{dx^2} + V(x)\right)$ are hermitian.

$$\langle f|\hat{x}g\rangle = \int_{-\infty}^{+\infty} f^*(x)xg(x)dx = \int_{-\infty}^{+\infty} \left(xf(x)\right)^* g(x)dx = \langle \hat{x}f|g\rangle$$

$$
\begin{aligned}
\langle f|\hat{H}g\rangle &= \int_{-\infty}^{+\infty} f^*(x)\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]g(x)dx \\
&= -\frac{\hbar^2}{2m}f^*\frac{dg}{dx}\bigg|_{-\infty}^{+\infty} + \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\frac{df^*}{dx}\frac{dg}{dx}dx + \int_{-\infty}^{+\infty}[V(x)f(x)]^*g(x)dx \\
&= \frac{\hbar^2}{2m}g\frac{df^*}{dx}\bigg|_{-\infty}^{+\infty} - \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\frac{d^2f^*}{dx^2}gdx + \int_{-\infty}^{+\infty}[f(x)V(x)]^*g(x)dx \\
&= \int_{-\infty}^{+\infty}\left[-\frac{\hbar^2}{2m}\frac{d^2f}{dx^2}+V(x)f\right]^* g(x)dx = \langle \hat{H}f|g\rangle
\end{aligned}
$$

2. A Hermitian operator $\hat{A}$ has a complete orthonormal set of eigenfunctions $|\psi_n\rangle$ with associated eigenvalues α_n. Show that we can always write

$$\hat{A} = \sum_i \alpha_i |\psi_i\rangle\langle\psi_i|$$

$$\hat{A}|\psi_n\rangle = \alpha_n|\psi_n\rangle$$

$$\hat{A}|\psi_n\rangle = \sum_i \alpha_i|\psi_i\rangle\langle\psi_i|\psi_n\rangle = \alpha_n|\psi_n\rangle$$

$$\therefore\ \hat{A} = \sum_i \alpha_i|\psi_i\rangle\langle\psi_i|$$

3. An operator $\hat{Q}$ has the complete sets of Eigen wave functions $\{|a_n\rangle\}$ and $\{|b_n\rangle\}$ in the *A* and *B* representations respectively. Assuming $\{|a_n\rangle\}$ and $\{|b_n\rangle\}$ are connected by unitary transformation

$$|b_n\rangle = \hat{U}|a_n\rangle$$

prove that

$$\hat{Q}_{(B)} = \hat{U}\hat{Q}_{(A)}\hat{U}^\dagger$$

$$\langle b_n|\hat{Q}_{(B)}|b_n\rangle = \langle a_n|\hat{U}^\dagger\hat{Q}_{(B)}\hat{U}|a_n\rangle = \langle a_n|\hat{Q}_{(A)}|a_n\rangle$$

$$\hat{U}^\dagger\hat{Q}_{(B)}\hat{U} = \hat{Q}_{(A)}$$

$$\therefore \hat{Q}_{(B)} = \hat{U}\hat{Q}_{(A)}\hat{U}^\dagger$$

4. (Text book* Problem 3.16)

Show that two noncommuting operators cannot have a complete set of common eigenfunctions. Hint: Show that if $\hat{P}$ and $\hat{Q}$ have a complete set of common eigenfunctions, then $[\hat{P}, \hat{Q}]f = 0$ for any function in Hilbert space.

Assuming $\hat{P}f_n = \lambda_n f_n$ and $\hat{Q}f_n = \mu_n f_n$, and $\{f_n\}$ are a complete set of eigenfunctions

For arbitrary wavefunction

$$f = \sum_n c_n f_n$$

$$[\hat{P}, \hat{Q}]f = (\hat{P}\hat{Q} - \hat{Q}\hat{P}) \sum_n c_n f_n = \hat{P}\left(\sum_n c_n \mu_n f_n\right) - \hat{Q}\left(\sum_n c_n \lambda_n f_n\right)$$
$$= \sum_n c_n \mu_n \lambda_n f_n - \sum_n c_n \lambda_n \mu_n f_n = 0$$

Therefore, $\hat{P}\hat{Q} = \hat{Q}\hat{P}$ or $f = 0$. The former contradicts to $\hat{P}$ and $\hat{Q}$ are noncommuting. The latter contradicts to f is an arbitrary wavefunction.

5. $\widehat{D}_x(a)$ is a translation operator in one dimension. When it applies to a wavefunction
$$\widehat{D}_x(a)\psi(x) = \psi(x - a)$$
If $\hat{f}(x)$ is commutable with $\widehat{D}_x(a)$, prove $\hat{f}(x) = \hat{f}(x - a)$.

Since $\hat{f}(x)$ is commutable with $\widehat{D}_x(a)$,
$$[\hat{f}(x), \widehat{D}(a)] = 0$$
For an arbitrary wavefunction $\psi(x)$
$$\hat{f}(x)\widehat{D}_x(a)\psi(x) = \hat{f}(x)\psi(x - a) = \widehat{D}_x(a)\hat{f}(x)\psi(x) = \hat{f}(x - a)\psi(x - a)$$
Since $\psi(x - a)$ is an arbitrary wavefunction
$$\hat{f}(x) = \hat{f}(x - a)$$

* David J. Griffiths, and Darrell F. Schroeter, Introduction to Quantum Mechanics (3rd Edition), Cambridge University Press (2018).