1. (Text book* Problem 5.20)
   Find the average energy per free electron (E_{tot}/N), as a fraction of the Fermi energy.

$$\frac{E_{tot}/N}{E_F} = \frac{\hbar^2(3\pi^2 N)^{5/3}}{10\pi^2 m V^{2/3}}\frac{1}{N} \Bigg/ \frac{\hbar^2}{2m}\left(\frac{3N\pi^2}{V}\right)^{2/3} = \frac{3}{5}$$

2. Considering we have free electron gas in a rectangular area in two dimension, derive the Fermi energy and the density of energy states in two dimension. *Note: the Fermi-energy formula written on the text book was derived in three dimension. You need follow the same procedure but the result will be slightly different comparing to the three dimension case.*

   The wavefunction should have the form of

$$X(x) = A_x \sin(k_x x) + B_x \cos(k_x x)$$
$$Y(y) = A_y \sin(k_y y) + B_y \cos(k_y y)$$

   where

$$k_x \equiv \frac{\sqrt{2mE_x}}{\hbar}, k_y \equiv \frac{\sqrt{2mE_y}}{\hbar}$$

   Considering the following boundary conditions held

$$X(0) = Y(0) = 0,\ X(l_x) = Y(l_y) = 0$$

   The solutions for wavefunctions are

$$\psi_{n_x,n_y,n_z} = \sqrt{\frac{4}{l_x l_y}} \sin\left(\frac{n_x \pi}{l_x} x\right) \sin\left(\frac{n_y \pi}{l_y} y\right)$$

   The area in the k-space every state would occupy

$$\frac{\pi^2}{l_x l_y} = \frac{\pi^2}{S}$$

   For $k_F \leq \frac{\sqrt{2mE_F}}{\hbar}$

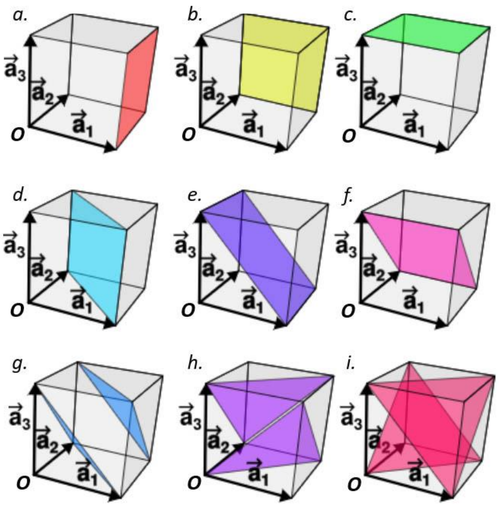
$$n(E_F) = N = 2 \cdot \frac{1}{4} \cdot \pi k_F^2 \cdot \frac{S}{\pi^2} = \frac{S}{2\pi}\left(\frac{2mE_F}{\hbar^2}\right) = \frac{SmE_F}{\pi\hbar^2}$$

   So

$$E_F = \frac{\pi N \hbar^2}{mS}$$

$$\rho(E) = \frac{dn(E)}{SdE} = \frac{m}{\pi\hbar^2}$$

3. (1) Write down the indices of the following 9 crystal planes in the cubic lattice system.



a. (1, 0, 0); *b.* (0, 1, 0); *c.* (0, 0, 1);

d. (1, 1, 0); *e.* (1, 0, 1); *f.* (0, 1, 1);

g. (1, 1, 1); *h.* ($\bar{1}$, 1, 1); *i.* (1, $\bar{1}$, 1).

(2) Write down the indices of the shortest lattice vector which starts from the point *O* and ends at the crystal planes with colors in the above figure.

a. (1, 0, 0); *b.* (0, 1, 0); *c.* (0, 0, 1);

d. (1, 1, 0); *e.* (1, 0, 1); *f.* (0, 1, 1);

g. $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$; $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$; *h.* $(-\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$; (0, 0, 0); *i.* $(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$; (0, 0, 0).

g. (1, 1, 1); *h.* ($\bar{1}$, 1, 1); *i.* (1, $\bar{1}$, 1).

4. (1) Write down the reciprocal vector $\vec{G}$ of the 1D and 2D lattice of

$$\vec{R} = u\vec{a}$$
$$\vec{R} = u\vec{a}_1 + v\vec{a}_2$$

在一维情况下，令$\vec{\boldsymbol{a}}_1 = \vec{\boldsymbol{\imath}}$，$\vec{\boldsymbol{a}}_2 = \vec{\boldsymbol{\jmath}}$，$\vec{\boldsymbol{a}}_3 = \vec{\boldsymbol{k}}$

$$\vec{b} = \frac{2\pi}{a}\vec{\boldsymbol{\imath}}, \qquad \vec{G} = u\vec{b}$$

在二维情况下，令$\vec{\boldsymbol{a}}_1 = a_1\vec{\boldsymbol{\imath}}$，$\vec{\boldsymbol{a}}_2 = a_2\vec{\boldsymbol{\jmath}}$，$\vec{\boldsymbol{a}}_3 = \vec{\boldsymbol{k}}$

$$\vec{b}_1 = \frac{2\pi}{a_1}\vec{\boldsymbol{\imath}}$$
$$\vec{b}_2 = \frac{2\pi}{a_2}\vec{\boldsymbol{\jmath}}$$
$$\vec{G} = u\vec{b}_1 + v\vec{b}_2$$

(2) Prove that in the Bloch's theorem, where $\vec{G}$ is the 1D reciprocal vector.

$$u_k(x) = \sum_h A_h e^{iGx}$$

$$G = \frac{2\pi}{a} h, \; h = 0, \; \pm 1, \; \pm 2, \; \ldots$$

Comparing to,

$$\vec{G} = u \frac{2\pi}{a} \vec{\boldsymbol{\imath}}$$

QED.

* David J. Griffiths, and Darrell F. Schroeter, Introduction to Quantum Mechanics (3rd Edition), Cambridge University Press (2018).