《量子信息基础》参考作业：

1. (Text book* Problem 5.4)
   (a) If ψ_a and ψ_b are orthogonal, and both are normalized, what is the constant A in Equation 5.17 ?
   (b) If $\psi_a = \psi_b$ (and it is normalized), what is A? (This case, of course, occurs only for bosons.)

(a) $\psi_\pm(\boldsymbol{r_1}, \boldsymbol{r_2}) = A[\psi_a(\boldsymbol{r_1})\psi_b(\boldsymbol{r_2}) \pm \psi_b(\boldsymbol{r_1})\psi_a(\boldsymbol{r_2})]$

$$\int \left|\psi_\pm(\boldsymbol{r_1}, \boldsymbol{r_2})\right|^2 d\boldsymbol{r_1} d\boldsymbol{r_2}$$
$$= |\boldsymbol{A}|^2 \int [\psi_a(\boldsymbol{r_1})\psi_b(\boldsymbol{r_2}) \pm \psi_b(\boldsymbol{r_1})\psi_a(\boldsymbol{r_2})]^*[\psi_a(\boldsymbol{r_1})\psi_b(\boldsymbol{r_2}) \pm \psi_b(\boldsymbol{r_1})\psi_a(\boldsymbol{r_2})] d\boldsymbol{r_1} d\boldsymbol{r_2}$$
$$= |\boldsymbol{A}|^2 \left[\int |\psi_a(\boldsymbol{r_1})|^2 d\boldsymbol{r_1} \int |\psi_b(\boldsymbol{r_2})|^2 d\boldsymbol{r_2} \pm \int \psi_b^*(\boldsymbol{r_1})\psi_a(\boldsymbol{r_1}) d\boldsymbol{r_1} \int \psi_a^*(\boldsymbol{r_2})\psi_b(\boldsymbol{r_2}) d\boldsymbol{r_2}\right.$$
$$\left.\pm \int \psi_a^*(\boldsymbol{r_1})\psi_b(\boldsymbol{r_1}) d\boldsymbol{r_1} \int \psi_b^*(\boldsymbol{r_2})\psi_a(\boldsymbol{r_2}) d\boldsymbol{r_2} + \int |\psi_b(\boldsymbol{r_1})|^2 d\boldsymbol{r_1} \int |\psi_a(\boldsymbol{r_2})|^2 d\boldsymbol{r_2}\right]$$
$$= |\boldsymbol{A}|^2 (1 \pm 0 \pm 0 + 1) = 2|A|^2 = 1$$

$$\mathrm{A} = \frac{1}{\sqrt{2}}$$

(b) If $\psi_a = \psi_b$

$$\psi_+(\boldsymbol{r_1}, \boldsymbol{r_2}) = 2A\psi_a(\boldsymbol{r_1})\psi_b(\boldsymbol{r_2})$$
$$\int |\psi_+(\boldsymbol{r_1}, \boldsymbol{r_2})|^2 d\boldsymbol{r_1} d\boldsymbol{r_2} = |\boldsymbol{A}|^2 \int [2\psi_a(\boldsymbol{r_1})\psi_b(\boldsymbol{r_2})]^*[2\psi_a(\boldsymbol{r_1})\psi_b(\boldsymbol{r_2})] d\boldsymbol{r_1} d\boldsymbol{r_2}$$
$$= 4|\boldsymbol{A}|^2 \int |\psi_a(\boldsymbol{r_1})|^2 d\boldsymbol{r_1} \int |\psi_b(\boldsymbol{r_2})|^2 d\boldsymbol{r_2} = 4|\boldsymbol{A}|^2 = \mathbf{1}$$
$$\mathrm{A} = \frac{1}{2}$$

2. (Text book* Problem 5.6)
   Imagine two non-interacting particles, each of mass *m*, in the infinite square well. If one is in the state ψ_n ($\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$), and the other in state $\psi_l (l \neq n)$, calculate $\langle(x_1 - x_2)^2\rangle$, assuming (a) they are distinguishable particles, (b) they are identical bosons, and (c) they are identical fermions.

The wave functions are

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$
$$\psi_l(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{l\pi}{a}x\right)$$

(a)

If the two particles are distinguishable

$$\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle_n + \langle x^2 \rangle_l - 2\langle x \rangle_n \langle x \rangle_l$$

$$\langle x^2 \rangle_n = \frac{2}{a}\int_0^a x^2 \sin^2\left(\frac{n\pi}{a}x\right) dx = \frac{2}{a}\int_0^a x^2 \frac{1-\cos(2n\pi x/a)}{2} dx$$

$$= \frac{1}{a}\int_0^a x^2 dx - \frac{1}{a}\int_0^a x^2 \cos\left(\frac{2n\pi}{a}x\right) dx = \frac{a^2}{3} - \frac{a^2}{2(n\pi)^2}$$

$$\langle x^2 \rangle_l = \frac{a^2}{3} - \frac{a^2}{2(l\pi)^2}$$

$$\langle x \rangle_n = \frac{2}{a}\int_0^a x \sin^2\left(\frac{n\pi}{a}x\right) dx = \frac{1}{a}\int_0^a x\left(1 - \cos\left(\frac{2n\pi}{a}x\right)\right) dx = \frac{a}{2}$$

$$\langle x \rangle_l = \frac{a}{2}$$

$$\therefore\ \langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle_n + \langle x^2 \rangle_l - 2\langle x \rangle_n \langle x \rangle_l = a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{l^2}\right)\right]$$

(b)

If the two particles are indistinguishable bosons

$$\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle_n + \langle x^2 \rangle_l - 2\langle x \rangle_n \langle x \rangle_l - 2|\langle x \rangle_{nl}|^2$$

$$\langle x \rangle_{nl} = \frac{2}{a}\int_0^a x \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{l\pi}{a}x\right) dx$$

$$= \frac{1}{a}\int_0^a x\left[\cos\left(\frac{(n-l)\pi}{a}x\right) - \cos\left(\frac{(n+l)\pi}{a}x\right)\right] dx$$

$$= \frac{1}{a}\left[\frac{a}{(n-l)\pi}\right]^2 \cos\left(\frac{(n-l)\pi}{a}x\right) + \frac{ax}{(n-l)\pi}\sin\left(\frac{(n-l)\pi}{a}x\right)$$

$$- \left[\frac{a}{(n+l)\pi}\right]^2 \cos\left(\frac{(n+l)\pi}{a}x\right) - \frac{ax}{(n+l)\pi}\sin\left(\frac{(n+l)\pi}{a}x\right)\Bigg|_0^a$$

$$= \frac{1}{a}\left\{\left[\frac{a}{(n-l)\pi}\right]^2 (\cos[(n-l)\pi] - 1)\right.$$

$$\left. - \left[\frac{a}{(n+l)\pi}\right]^2 (\cos[(n+l)\pi] - 1)\right\}$$

$$= \frac{a}{\pi^2}[(-1)^{n+l} - 1]\left[\frac{1}{(n-l)^2} - \frac{1}{(n+l)^2}\right]$$

$$= \begin{cases} \dfrac{-8nla}{\pi^2(n^2 - l^2)^2} & when\ n + l = 2m + 1,\ \ m\ is\ an\ integer \\ 0 & when\ n + l = 2m \end{cases}$$

$when\ n + l = 2m$

$$\langle (x_1 - x_2)^2 \rangle = a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{l^2}\right)\right]$$

$when\ n + l = 2m + 1$

$$\langle (x_1 - x_2)^2 \rangle = a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{l^2}\right)\right] - \frac{128n^2l^2a^2}{\pi^4(n^2 - l^2)^4}$$

(c)

If the two particles are indistinguishable fermions

$$\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle_n + \langle x^2 \rangle_l - 2\langle x \rangle_n \langle x \rangle_l + 2|\langle x \rangle_{nl}|^2$$

$when\ n + l = 2m$

$$\langle (x_1 - x_2)^2 \rangle = a^2 \left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{l^2}\right)\right]$$

$when\ n + l = 2m + 1$

$$\langle (x_1 - x_2)^2 \rangle = a^2 \left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{l^2}\right)\right] + \frac{128 n^2 l^2 a^2}{\pi^4 (n^2 - l^2)^4}$$

* David J. Griffiths, and Darrell F. Schroeter, Introduction to Quantum Mechanics (3rd Edition), Cambridge University Press (2018).

3. (Text book* Problem 5.26, 注意是教材的第二版)

Use the method of Lagrange multipliers to find the rectangle of largest area, with sides parallel to the axes, that can be inscribed in the ellipse $(x/a)^2 + (y/b)^2 = 1$. What is that maximum area?

The area of the rectangle shape is

$$S(x, y) = 4|xy|$$

x,y satisfy $(x/a)^2 + (y/b)^2 = 1$

We define a new function

$$G(x, y, \lambda) \equiv 4xy + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

$$\frac{\partial G}{\partial \mathrm{x}} = 4y + \frac{2\lambda x}{a^2}$$

$$\frac{\partial G}{\partial \mathrm{y}} = 4x + \frac{2\lambda y}{b^2}$$

$$\begin{cases} y = -\dfrac{\lambda x}{2a^2} \\ x = -\dfrac{\lambda y}{2b^2} = \dfrac{\lambda^2}{4a^2 b^2} x \end{cases}$$

So

$$x = 0$$

or

$$\lambda = \pm 2ab$$

The later gives the maximum area, where

$$y = \mp \frac{b}{a} x$$
$$\mathrm{x} = \frac{a}{\sqrt{2}}$$
$$\mathrm{y} = \frac{b}{\sqrt{2}}$$
$$S_{max} = 2ab$$

4. Consider a system of two particles, they have six possible states $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6$. Calculate the number of the microstates of the system, in the following conditions:
   (1) Two particles are bosons;
   (2) Two particles are fermions;
   (3) Two particles are distinguishable.

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(1)

$$N_b = C_6^1 + C_6^2 = 6 + 15 = 21$$

(2)

$$N_f = C_6^2 = 15$$

(3)

$$N_d = 6 \times 6 = 36$$

* David J. Griffiths, Introduction to Quantum Mechanics (2nd Edition), Cambridge University Press (2017).

5. Consider the Bell state $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0_1, 0_2\rangle - |1_1, 1_2\rangle)$ of two photons, and suppose now that we wish to express it not on a basis of horizontal ($|0\rangle$) and vertical ($|1\rangle$) polarized states, but instead on a basis rotated by 45°, i.e., a new basis

$$|{+45}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|{-45}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

(a) Show that, expressed on this particular basis, the resulting state is still a Bell state.

(b) Repeat part (a), but with the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0_1, 0_2\rangle + |1_1, 1_2\rangle)$. What difference do you note between the two results?

(a) Since

$$|0\rangle = \frac{1}{\sqrt{2}}(|{+45}\rangle + |{-45}\rangle)$$
$$|1\rangle = \frac{1}{\sqrt{2}}(|{+45}\rangle - |{-45}\rangle)$$

$$\begin{aligned}
|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0_1, 0_2\rangle - |1_1, 1_2\rangle) \\
&= \frac{1}{2\sqrt{2}}\big((|{+45}\rangle_1 + |{-45}\rangle_1)(|{+45}\rangle_2 + |{-45}\rangle_2) \\
&- (|{+45}\rangle_1 - |{-45}\rangle_1)(|{+45}\rangle_2 - |{-45}\rangle_2)\big) \\
&= \frac{1}{2\sqrt{2}}(\cancel{|{+45}\rangle_1|{+45}\rangle_2} + |{+45}\rangle_1|{-45}\rangle_2 + |{-45}\rangle_1|{+45}\rangle_2 \\
&+ \cancel{|{-45}\rangle_1|{-45}\rangle_2} - \cancel{|{+45}\rangle_1|{+45}\rangle_2} + |{+45}\rangle_1|{-45}\rangle_2 \\
&+ |{-45}\rangle_1|{+45}\rangle_2 - \cancel{|{-45}\rangle_1|{-45}\rangle_2}) \\
&= \frac{1}{\sqrt{2}}(|{+45}\rangle_1|{-45}\rangle_2 + |{-45}\rangle_1|{+45}\rangle_2)
\end{aligned}$$

It is still a Bell state.

(b)

$$\begin{aligned}
|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0_1, 0_2\rangle + |1_1, 1_2\rangle) \\
&= \frac{1}{2\sqrt{2}}\big((|{+45}\rangle_1 + |{-45}\rangle_1)(|{+45}\rangle_2 + |{-45}\rangle_2) \\
&+ (|{+45}\rangle_1 - |{-45}\rangle_1)(|{+45}\rangle_2 - |{-45}\rangle_2)\big) \\
&= \frac{1}{\sqrt{2}}(|{+45}\rangle_1|{+45}\rangle_2 + |{-45}\rangle_1|{-45}\rangle_2)
\end{aligned}$$

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