Vector

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- def. A vector is a ordered list of real numbers $u_1, \cdots, u_n$. expressed as $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ or $u = (u_1, \cdots, u_n)$
    - The set of all vectors within n entries is denoted by $\mathbb{R}^n$
- def. $u_1, u_2, \cdots, u_m$ are vectors, $c_1, \cdots, c_n$ are scalars. Then $c_1u_1 + \cdots + c_mu_m$ is a linear combination of the vectors.

- def. let $\{v_1, v_2, \cdots, v_m\}$ the collection of vectors in $\mathbb{R}^n$. The span of this set denoted by span$\{v_1, \cdots, v_m\}$, defined to be all linear combinations $x_1v_1 + \cdots + x_mv_m$.

  e.g. Span $\{(1,0,0), (0,1,0)\}$ = the x-y plane.

  e.g. span $\{(1,1,1), (1,2,3)\}$

Span

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- Span: A vector $\vec{v}$ is in span$\{u_1, \dots, u_n\}$ if there are c_1, c_2, c_3 satisfy $c_1u_1 + \cdots + c_nu_n = v$. $\Leftrightarrow$ $[u_1\ u_2 \cdots u_n \mid v]$ has a soln.

  - The zero vector is in span of any collection of vectors

e.g. is $(1,1,0)$ in the span$\{(1,2,3),(4,5,6),(5,7,9)\}$?

$$\begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 4 & 5 & 1 \\ 0 & -3 & -3 & -1 \\ 0 & -6 & -6 & -3 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 4 & 5 & 1 \\ 0 & -3 & -3 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

no solution.

- Let u_1, u_2, u_3 be vectors in $\mathbb{R}^n$, then if v_1 & v_2 are in span$\{u_1, u_2, u_3\}$ then so are cv_1 & v_1+v_2, where $c \in \mathbb{R}$.

$$v_1 = c_1u_1 + c_2u_2 + c_3u_3$$

$$v_2 = b_1u_1 + b_2u_2 + b_3u_3$$

$$\therefore cv_1 = c(c_1u_1 + c_2u_2 + c_3u_3)$$

$$v_1 + v_2 = (c_1+b_1)u_1 + (c_2+b_2)u_2 + (c_3+b_3)u_3$$

Q.E.D

- $P\begin{pmatrix}1\\1\\0\end{pmatrix}$ in span $\underline{\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}5\\7\\9\end{pmatrix}\right\}}$ linearly dependent.

  If every point is in distinct direction, then it can be reached.

  If not "distinct", then some points can't be reached.

- Theorem 2.7. Let $u_1, u_2, \cdots, u_m$ and u be vectors in $\mathbb{R}^n$. If u is in span$\{u_1, \cdots, u_m\}$, then

$\mathrm{span}\{u, u_1, \dots, u_m\} = \mathrm{span}\{u_1, \dots, u_m\}$

- Theorem 2.8. Suppose that $u_1, \dots, u_m$ are in $\mathbb{R}^n$, and let $A = [u_1, \dots, u_m] \sim B$, where B is in echelon form. Then $\mathrm{span}\{u_1, \dots, u_m\} = \mathbb{R}^n$ exactly when B has a pivot position in every row.
  - Theorem 2.9. Let $\{u_1, u_2, \dots, u_m\}$ be a set of vectors in $\mathbb{R}^n$.

$$\begin{cases} m<n & \text{this set does not span } \mathbb{R}^n \\ m \geq n & \text{this set might span } \mathbb{R}^n \text{ or might not.} \end{cases}$$

⇨ Theorem 2.11. Let $a_1, \dots, a_m$ and b be vectors in $\mathbb{R}^n$. Then the following are equivalent:

I. b is in $\mathrm{span}\{a_1, \dots, a_m\}$

II. The vector equation $x_1a_1 + \dots + x_ma_m = b$ has at least one soln.

III. The linear system corresponding to $[a_1 \dots a_m | b]$ has at least one solution.

IV. The equation $Ax = b$, has at least one soln

- def. Let $\{v_1, v_2, \dots, v_m\}$ be a set of vectors in $\mathbb{R}^n$. If the only solution to $x_1v_1 + x_2v_2 + \dots + x_mv_m = \vec{0}$ is trivial soln (zero soln) then we say that the set is linearly independent. Otherwise linearly dependent.

- Prove $\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}5\\7\\9\end{pmatrix} \right\}$ is linearly dependent.

Proof: assume there is $(x_1, x_2, x_3) \neq \vec{0}$ and is soln to

$$x_1\begin{pmatrix}1\\2\\3\end{pmatrix} + x_2\begin{pmatrix}4\\5\\6\end{pmatrix} + x_3\begin{pmatrix}5\\7\\9\end{pmatrix} = 0$$

$$\begin{pmatrix}1&4&5\\2&5&7\\3&6&9\end{pmatrix} = \begin{pmatrix}1&4&5\\0&-3&-3\\0&-6&-6\end{pmatrix} = \begin{pmatrix}1&4&5\\0&1&1\\0&0&0\end{pmatrix}$$

  - Theorem: Let $\{u_1, u_2, \dots, u_m\}$ be vectors in $\mathbb{R}^n$. Then the set is linearly dependent if and only if one of the vectors is in the span of vectors

Proof. $u_i = x_1u_1 + \dots + x_{i-1}u_{i-1} + x_{i+1}u_{i+1} + \dots + x_mu_m$

move u_1 to right,

$\therefore$ the set is linearly dependent.

- Theorem: If $\{u_1, \dots, u_m\}$ is a set of vectors in $\mathbb{R}^n$, and $m>n$. Then the set is linerly dependent.
- Theorem: Any set containing $\vec{0}$ is linearly dependent.

e.g. is $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}24\\11\\\frac{1}{2}\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}\right\}$ linearly independent?

$$\begin{pmatrix}1 & 24 & 4 & 0\\2 & 11 & 5 & 0\\3 & \frac{1}{2} & 6 & 0\end{pmatrix} = \begin{pmatrix}1 & 24 & 4 & 0\\0 & -37 & -3 & 0\\0 & -\frac{143}{2} & -6 & 0\end{pmatrix}$$

it is linearly independent.

- Homogeneous linear system $Ax=0$ (has either one soln or infinity many soln)
  - Theorem: Let $A=[a_1\ a_2 \cdots a_m]$ and $x=(x_1, \dots, x_m)$. The set $\{a_1, \dots, a_m\}$ is linearly independent if and only if the homogeneous linear system $Ax=0$ has only one trivial solution.

e.g. $\begin{pmatrix}1 & 4\\2 & 5\\3 & 6\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix}$

$= \begin{pmatrix}-1+8\\-2+10\\-3+12\end{pmatrix} = \begin{pmatrix}7\\8\\9\end{pmatrix}$

Linear Independence

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- nonhomogeneous $Ax=b$
    - its associate homogenous system $Ax=0$
- Theorem 2.19 Let x_p be a particular soln to $Ax=b$. Then all soln $x_g = x_p + x_h$, x_h: soln to the associate homogenous system.

  Proof. $Ax_g = A(x_p+x_h) = Ax_p + Ax_h = b+0$

  $\therefore x_g = x_p + x_h$

- Theorem. $a_1, \dots, a_m, b$ are vectors in $\mathbb{R}^n$. Following are equivalent:

  I. The set $\{a_1, \dots, a_m\}$ is linearly independent.

  II. vector equation $x_1 a_1 + \dots + x_m a_m = b$ has at most one soln for $\forall b$.

  III. $[a_1\ a_2\ \cdots\ a_m | b]$ has at most one soln for $\forall b$

  IV. $Ax=b$, has at most one soln for b.

- The Unifying Theorem $S = \{a_1, \dots, a_n\}$ is a set of n vectors in $\mathbb{R}^n$, $A = [a_1, \dots, a_n]$, following equivalent

  I. S spans $\mathbb{R}^n$

  II. S is linearly independent

  III. $Ax=b$ has a unique soln for all b in $\mathbb{R}^n$

prob

Friday, January 18, 2019 2:12 AM

Determine if the statement is true or false, and justify your answer.

If A is a matrix with linearly independent columns, then $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$.

- ○ False. For example, if $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $A\mathbf{x} = \mathbf{b}$ has no solution.
- ○ True. The echelon form of the augmented matrix $[A\ \mathbf{b}]$ will have at least one row of zeroes at the bottom, which means the equation has a solution.
- ○ True. The echelon form of the augmented matrix $[A\ \mathbf{b}]$ will not have a row of zeroes at the bottom, which means the equation has a solution.
- ◉ False. For example, if $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $A\mathbf{x} = \mathbf{b}$ has no solution.
- ○ True. $A\mathbf{x} = \mathbf{b}$ corresponds to $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$, and by linear independence, there exists a solution.

It did not indicate A is square

A 3x4 Matrix A, to make the general soln to $Ax = \begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix}$ is

$x = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 6 \end{pmatrix} + s\begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, s is any number. A is of reduced echlon form. x_3 is free var.

$x = \begin{pmatrix} -1-s \\ 1+2s \\ s \\ 6 \end{pmatrix}$, assume $A = \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ ← because A is in reduced echelon form.

$Ax = \begin{pmatrix} -1-s+as \\ 1+2s+bs \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix}$ $\therefore A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$s(a-1) = 0$

$a = 1.$

$s(2+b) = 0$

$b = -2.$