Deteriminants

- $A_{n\times n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$. The determinant of A is

$$\det(A) = \sum_{i=1}^{n} a_{ij} \cdot C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

  - cofator $C_{ij} = (-1)^{i+j} \det(M_{ij})$
  - Minor of a_{ij} : $\det(M_{ij})$

$$M_{ij} : \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & & & \vdots \\ & a_{ij} & \cdots & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$$ (remove the coloumn j & row i

e.g. $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 0 \\ 3 & 1 & 2 \end{bmatrix}$, find $\det(A)$

$$\det(A) = \begin{matrix} -1 & 3 \\ 4 & 0 \\ 1 & 2 \end{matrix} \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 0 \\ 3 & 1 & 2 \end{bmatrix} \begin{matrix} 2 & -1 \\ 1 & 4 \\ 3 & 1 \end{matrix}$$

$= 16 - 0 + 0 - (-2) + 3 - 36$

$= -15$

$$\det(A) = 2 \cdot \begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix} + (-1)^3 \cdot 1 \cdot \begin{vmatrix} -1 & 3 \\ 1 & 2 \end{vmatrix} + (-1)^4 \cdot 3 \begin{vmatrix} -1 & 3 \\ 4 & 0 \end{vmatrix}$$

$= 2 \cdot 8 - (-2-3) - 36$

$= -15$

- Properties, $A_{n\times n}$
  - det is linear. (Multilinear)

    $\det(\gamma A^1 + \gamma B \;\; A^2 \cdots A^n) = \gamma \det(A^1 \cdots A^n) + \gamma \det(B \cdots A^n)$

  - if $A_i = A_j$, $i \neq j$, $\det(A) = 0$ (Alternating)
  - $n \geq 1$, $\det(I_n) = 1$.

    Use induction, $I_1 = 1$, $I_2 = 1 \cdots I_{n-1} = 1$, $I_n = 1$

  - $j, k \in [1, n]$, $j \neq k$, interchange column j, k. det change by sign.
  - Add a scalar multiple of one column to another, det doesn't change.
  - $\det(A^T) = \det(A)$

- $\det(A^T) = \det(A)$

- A is invertible if $\det(A) \neq 0$

- A is triangular $n \times n$, $\det(A)$ is the product of the terms along the diagonal.

- $\det(AB) = \det(A)\det(B)$, A, B are $n \times n$ matrices

- Let A be a square matrix
  - If A has a row or column of zeros, then $\det(A) = 0$.
  - If A has two identical rows or columns, then $\det(A) = 0$.

5.2 Properties of the Determinant

Thursday, February 28, 2019 6:04 PM

- Theorem 5.13. $A_{n\times n}$

  Interchange two rows of A: $\det(A) \Rightarrow -\det(A)$

  Multiply one row of A by c, $\det(A) \Rightarrow c\det(A)$

  Add a multiple of one row of A to another $\Rightarrow$ det(A) doesn't change.

- Theorem 5.12 $A_{n\times n}, B_{n\times n}$, $\det(AB) = \det(A)\det(B)$

  $\det(AB) = \det(A)\cdot\det(B) = \det(B)\det(A) = \det(BA)$

  - Theorem. $E_{n\times n}, B_{n\times n}$, E is elementary. $\det(EB) = \det(E)\det(B)$.

    Proof: 1° E is for interchange rows: $\det(E) = \det(I_n) = -1$

    $\therefore \det(EB) = \det(E)\cdot\det(B)$

    2° E is to multiply one row. $\det(E) = \det(cI_n) = c$

    $\det(E)\det(B) = c\det(EB)$

    3° E is to add scalar multiple, $\det(E) = \det(I_n)$

    $\therefore \det(E) = \det(E)\cdot\det(B)$

  - Proof. 1° A is non invertible.

    $\therefore \det(AB) = 0 = \det(A)\cdot\det(B)$

    2° A is invertible

    $\therefore A = (E_k \cdots E_1)^{-1} = E_k^{-1}\cdots E_1^{-1}$ (Because can be I after x inverse)

    $\therefore \det(AB) = \det(E_k^{-1}\cdots E_1^{-1}B)$

    $= \det(E_k^{-1})\cdot\det(E_{k-1}^{-1}\cdots E_1^{-1}B)$

    $= \cdots = \det(\overset{k}{E_k}\cdots \overset{\infty}{E_1})\cdot\det(B)$

    $= \det(A)\cdot\det(B)$

- $A_{n\times n}$, invertible. Then $\det(A^{-1}) = \dfrac{1}{\det(A)}$

  $\det(AA^{-1}) = 1 = \det(A)\cdot\det(A^{-1})$

- Let P be a partitioned $n\times n$ matrix of form

  $P = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ or $P = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$

- Let P be a partitioned n×n matrix of form

$$P = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \quad \text{or} \quad P = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$$

where A, D are square block matrices. $\det(P) = \det(A) \cdot \det(D)$