6.1 Eigenvalues and Eigenvectors

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- def: $A_{n\times n}$ non-zero vector $\vec{u}$, scalar λ, which satisfy $A\vec{u}=\lambda\vec{u}$, then $\vec{u}$ is an eigenvector (EVec) of A, and λ is an eigenvalue (EVal) of A.
  - Notice: $\vec{u}\neq\vec{0}$, while λ can be 0
- $A_{n\times n}$, $\vec{u}$ is an eigenvector of A, λ is the corresponding eigenvalue, for $c\neq 0$, c is scalar, $c\vec{u}$ is also an eigenvector of A, with eigenvalue λ.

  Proof. $A\vec{u}=\lambda\vec{u}$

  $A\cdot c\vec{u} = cA\vec{u} = c\lambda\vec{u} = \lambda(c\vec{u})$
- def $A_{n\times n}$ with EVal λ, the subspace of all EVec associated with λ, together with $\vec{0}$, called the eigenspace of λ.
  - Theorem. S: the set of all eigenvectors associated with λ, with $\vec{0}$, then S is a subspace of $\mathbb{R}^n$.

    $A\cdot\vec{0}=\lambda\vec{0}$; $u_1, u_2 \in S$, $u_1, u_2 \neq 0$

    $Au_1 + Au_2 = \lambda u_1 + \lambda u_2 \Rightarrow \lambda(u_1+u_2) = A(u_1+u_2)$

    $cu_1 \in S$ $\therefore$ S is a subspace of $\mathbb{R}^n$.
  - Each distinct EVec has its own associated eigenspace.
- Theorem. $A_{n\times n}$, λ is an EVal $\Leftrightarrow$ $\det(A-\lambda I_n)=0$

  $A\vec{u}=\lambda\vec{u} \Rightarrow (A-\lambda I_n)\vec{u}=0$

  assume $\det(A-\lambda I_n)\neq 0$, then $A-\lambda I_n$ invertible,

  $I_n\cdot\vec{u} = (A-\lambda I_n)^{-1}\cdot\vec{0} = 0 = \vec{u}$ conflict.

  then $\det(A-\lambda I_n)=0$.
  - Characteristic Polynomial: $\det(A-\lambda I) = (t-\lambda_1)\cdots(t-\lambda_n)$

    Characteristic Equation $\det(A-\lambda I)=0$
- def The multiplicity of an eigenvalue is equal to its factor's exponent.

  $-\lambda(\lambda-2)^2$: $\lambda=0$ has multiplicity 1, $\lambda=2$ has that of 2.

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  - Polynomial $P(x)$, a root α of $P(x)=0$ has multiplicity if $(x-\alpha)Q(x)=P(x)$, $Q(\alpha)\neq 0$. The multiplicity is the number of times a root is repeated.

- Theorem. $A_{n\times n}$, dim(Eigenspace of λ) $\boxed{\leq}$ multiplicity of λ.

- $A_{n\times n}$, Evals of A doesn't contain $\lambda=0 \Leftrightarrow \det(A)\neq 0$

- Types of Probs:
  - known Evals, find Evecs.

    $A\vec{u}=\lambda\vec{u} \Rightarrow (A-\lambda I)\vec{u}=0$ solve the homogeneous LS.
  - known Evecs, find Evals.

    Simply take $A\vec{u}$, then $A\vec{u}=\lambda\vec{u}$
  - find Evals and Evecs.

    Solve $\det(A-\lambda I)=0$, get λs.

    then find $\vec{u}$.

6.2 Diagonalization

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- $A_{n\times n}$ **diagonalizable**, if there exist $n\times n$ matrices D, P, D diagonal & P invertible, such that $A = PDP^{-1}$
  - Suppose A has n linearly independent eigenvectors $u_1, u_2, \cdots, u_n$, with corresponding eigenvalues $\lambda_1, \cdots, \lambda_n$, then

    $P = [u_1 \cdots u_n]$, $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$

- $A_{n\times n}$ diagonalizable $\Leftrightarrow$ A has eigenvectors that form a ~~basis~~ for $\mathbb{R}^n$.
  - if $\{\lambda_1, \cdots, \lambda_k\}$ are distinct eigenvalues of a matrix A, then any set of associated eigenvectors $\{u_1, \cdots, u_k\}$ are linearly independent.

    assume $\exists\, u_i$ that make the eigenvectors not independent.

    $\therefore$ $u_i = c_1u_1 + \cdots + c_{i-1}u_{i-1} + c_{i+1}u_{i+1} + \cdots + c_ku_k$, coeff are unique.

    $Au_i = \lambda_i u_i$

    $\therefore Ac_1u_1 + \cdots + Ac_ku_k = c_1\lambda_1u_1 + \cdots + c_k\lambda_ku_k = \lambda_i u_i$

    $\therefore u_i = c_1\frac{\lambda_1}{\lambda_i}u_1 + \cdots + c_k\frac{\lambda_k}{\lambda_i}u_k$ is another vector.

    $\therefore u_1, \cdots, u_i$ independent.

- Suppose that $A_{n\times n}$ has only real eigenvalues, A is diagonalizable

  $\Leftrightarrow$ dim(eigenspace) = multiplicity of the corresponding eigenvalue.

  $\sum$ multiplicity $= n$

- If $A_{n\times n}$ with n distinct real eigenvalues, then A is diagonalizable.

  because distince, $\therefore$ each eigenvector has a multiplicity of 1.

Prob

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Suppose that A is a square matrix with characteristic polynomial $(\lambda - 6)^2(\lambda - 4)^3(\lambda + 1)$.

(a) What are the dimensions of A? (Give n such that the dimensions are $n \times n$.)

$n =$ 6 ✔

$2+3+1$

(b) What are the eigenvalues of A? (Enter your answers as a comma-separated list.)

$\lambda =$ $6,4,-1$ ✔

(c) Is A invertible?

- (•) Yes
- ( ) No

✔

$\lambda \neq 0$

(d) What is the largest possible dimension for an eigenspace of A?

3 ✔

the more $a_{ij} - \lambda = 0$, the more dimension of eigenspace

$\dim = 6 - 3 = 3$