14.5 Chain rule

Wednesday, January 30, 2019 9:03 AM

- $\frac{dy}{dx} = \frac{dy}{dh} \cdot \frac{dh}{dx}$

- Assume $z = z(x, y)$, and assume x is a function of t, y is a function of t. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

e.g. $z = x^3 y^2 + 7y^3 x$, where $x = \sin t$, $y = \cos t$.

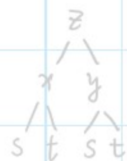
$$\frac{dz}{dt} = (3x^2y^2 + 7y^3)\cos t + (2x^3y + 21y^2x)(-\sin t)$$

$$\left.\frac{dz}{dt}\right|_{t=0} = 7$$

- $z(x, y)$, $x(s, t)$, $y(s, t)$

$$\Rightarrow \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$



- In general, $z(x_1, \cdots, x_n)$, and $x_i(t_1, \cdots, t_m)$ then

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial z}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$$

e.g. $u(x, y, z) = x^4 y + y^2 z^3$, $x = r \cdot s \cdot e^t$, $y = rs^2 e^{-t}$, $z = r^2 s \sin t$, find $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$, $t = 0$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial s}$$

$$= 4x^3 y \cdot re^t + (x^4 + 2z^3 y) \cdot 2rse^{-t} + 3z^2 \cdot y^2 \cdot (r^2 \sin t)$$

$x = 2$, $y = 2$, $z = 0$

$$\frac{\partial u}{\partial s} = 16 \times 4 \times 2 + 16 \times 2 \times 2 + 0$$

$$= 16 \times 12 = 160 + 32 = 192$$

14.6 Directional Derivatives & the Gradient Vector

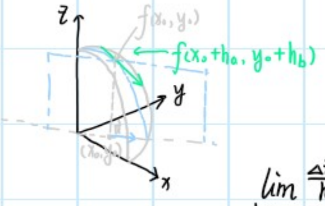
Wednesday, February 6, 2019 9:05 AM

- REM for $f(x,y)$, directional vector

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = f_x(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} \quad \text{in direction of } \hat{i}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f_y(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h} \quad \text{in direction of } \hat{j}$$

- Pick $\vec{u} = \langle a,b\rangle$, $|\vec{u}|=1$ Unit Vector!



$$\lim_{h\to 0}\frac{\Delta z}{h} = \frac{f(x_0+ha,\ y_0+ha)-f(x_0,y_0)}{h}$$

Directional derivative of $f(x,y)$ at (x_0,y_0) in the direction of $\vec{u}=\langle a,b\rangle$, unit vector is

$$D_u f(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+ha,y_0+hb)-f(x_0,y_0)}{h}$$: the slope of the tangent line on (x_0,y_0)

o Theorem $D_u f(x,y) = f_x(x,y)\cdot a + f_y(x,y)\,b$

Define $g(h) = f(x_0+ha,\ y_0+hb)$, $\vec{u}=\langle a,b\rangle$ unit vector

then $\lim_{h\to 0}\frac{g(h)-g(0)}{h} = g'(0)$

$\lim_{h\to 0}\frac{f(x_0+ha,\ y_0+hb)-f(x_0,y_0)}{h} = D_u(x_0,y_0)$

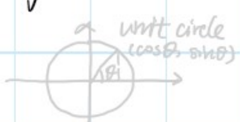
$\therefore g'(0) = D_u(x_0,y_0)$

$\frac{dg}{dh} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dh} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dh}$, $\begin{cases} x(h) = x_0+ha \\ y(h) = y_0+hb \end{cases}$

$g'(h) = f_x\cdot a + f_y\cdot b = D_u(x_0,y_0)$

o $\because \vec{u}$ is unit vector, then $\vec{u} = \langle\cos\theta, \sin\theta\rangle$ for some θ.

$$D_u f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$



e.g. Find $D_u(x,y)$ if $f(x,y) = x^3-3xy+4y^2$, $\vec{u}$ is unit vector with angle $\frac{\pi}{6}$.

what is $D_u f(1,2)$?

$\vec{u} = \langle\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\rangle = \langle\frac{\sqrt{3}}{2}, \frac{1}{2}\rangle$

$f_x = 3x^2 - 3y = 3 - 6 = -3 \qquad f_y = -3x + 8y = -3 + 16 = 13$

$D_u f(1,2) = -\frac{3\sqrt{3}}{2} + \frac{13}{2}$

- Let f be a differentiable function with two variables, $f(x,y)$. The gradient vector is $\text{grad} f = \nabla f = \langle f_x, f_y \rangle = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$ (can only be taken by a scalar field)

$D_u f(x,y) = \nabla f \cdot \vec{u} = |\nabla f| \cdot |\vec{u}| \cdot \cos\theta = |\nabla f| \cdot \cos\theta$

$= \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \cdot \vec{u}$

e.g. Find $D_u f(x,y)$, $f(x,y) = x^2y^2 - 4y^3$ at $(2,-1)$ in the direction of $\vec{v} = 2\hat{i} + 5\hat{j}$

$\vec{V} = \langle 2, 5 \rangle \Rightarrow \frac{\vec{V}}{|\vec{V}|} = \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \rangle$ Remember to convert to unit vector.

$\frac{\partial f}{\partial x} = 2xy^2 = 4$

$\frac{\partial f}{\partial y} = 2x^2y - 12y^2 = -8 - 12 = -20$

$D_u f(x,y) = \frac{8}{\sqrt{29}} - \frac{100}{\sqrt{29}} = -\frac{92}{\sqrt{29}}$

- $D_u f(x,y) = \nabla f \cdot \vec{u}$ is maximum when $\vec{u}$ is the same direction with ∇f

$= |\nabla f||\vec{u}|$

$= |\nabla f| \cdot \cos\theta$, $\theta = 0$ maximum

$= |\nabla f|$

- Theorem. Let f be a differentiable function of two (or three) variables. The max value of $D_u f(x,y) = |\nabla f|$, and it occurs when $\vec{u}$ has the same direction as ∇f

- We can describe the gradient of f as the vector that points in the same direction as the direction of fastest increase (can be used to check if conservative) or the direction of steepest ascent. $-\nabla f$ is fast decrease
    - The gradient vector is perpendicular to level curve.

- Tangent plane to level surface at $P(x_0, y_0, z_0)$

$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$

direction of normal line : ∇F

Prob

Thursday, February 7, 2019 10:12 AM

Suppose you are climbing a hill whose shape is given by the equation $z = 1500 - 0.005x^2 - 0.01y^2$, where x, y, and z are measured in meters, and you are standing at a point with coordinates (100, 120, 1306). The positive x-axis points east and the positive y-axis points north.

(a) If you walk due south, will you start to ascend or descend?

- (●) ascend
- ( ) descend

$\frac{\partial z}{\partial x} = -0.01x = -1$, $\frac{\partial z}{\partial y} = -0.02y = -\frac{12}{5}$

$\vec{u} = \langle 0, -1 \rangle$

$\nabla z = \frac{12}{5} > 0$, ascend

At what rate?

$0.02 \cdot 120$ ✓ vertical meters per horizontal meter

(b) If you walk northwest, will you start to ascend or descend?

- ( ) ascend
- (●) descend

$\vec{u} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$\nabla z = -\frac{7\sqrt{2}}{10}$

At what rate? (Round your answer to two decimal places.)

$-\frac{7\sqrt{2}}{10}$ ✓ vertical meters per horizontal meter

(c) In which direction is the slope largest?

$< -0.01 \cdot 100, -0.02 \cdot 120 >$ ✓

direction of $\langle f_x, f_y \rangle$

$\nabla z = \frac{f_x^2}{\sqrt{f_x^2+f_y^2}} + \frac{f_y^2}{\sqrt{f_x^2+f_y^2}}$

What is the rate of ascent in that direction?

$\frac{13}{5}$ ✓ vertical meters per horizontal meter

At what angle above the horizontal does the path in that direction begin? (Round your answer to two decimal places.)

68.96 ✓ °

$\tan\theta = \frac{13}{5}$ = slope.

$\theta \approx 68.96°$