The Sum of Random Walks

Anni Hong, Kaiser Sun, Maddy Brown
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Faculty mentor: Chris Hoffman
Graduate mentor: Jacob Richey

What is a simple random walk?

Biased vs. Unbiased

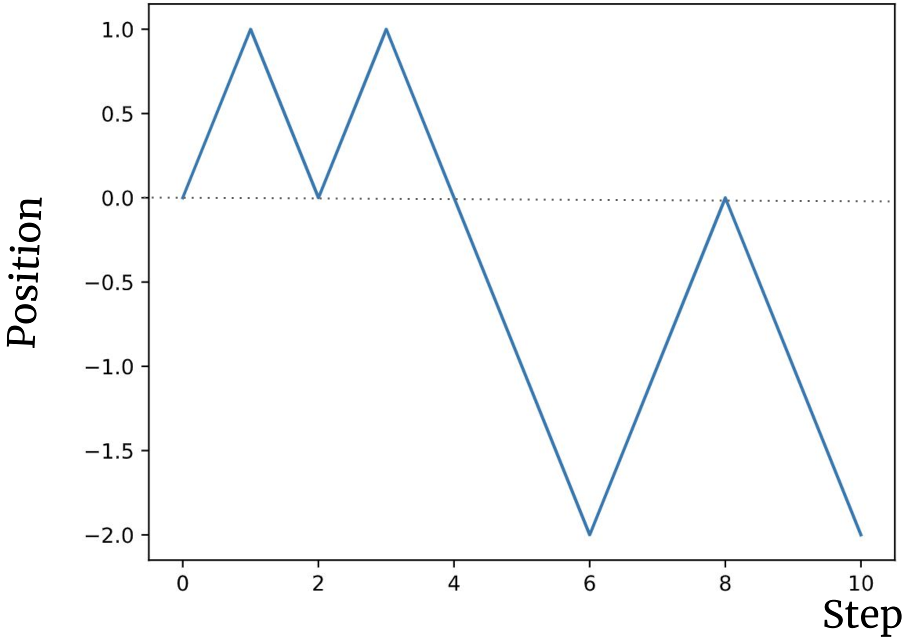
p≠q p=q=1/2

Z_j are the "steps",
equal to 1 with probability $p = \frac{1}{2}$,
and -1 with prob. $1 - p$.

Starting at $S_0 = 0$, $\{S_n\}$ is a random walk

defined by $S_n = \sum_{j=1}^{n} Z_j$.

Simple Symmetric Random Walk (n=10 steps)



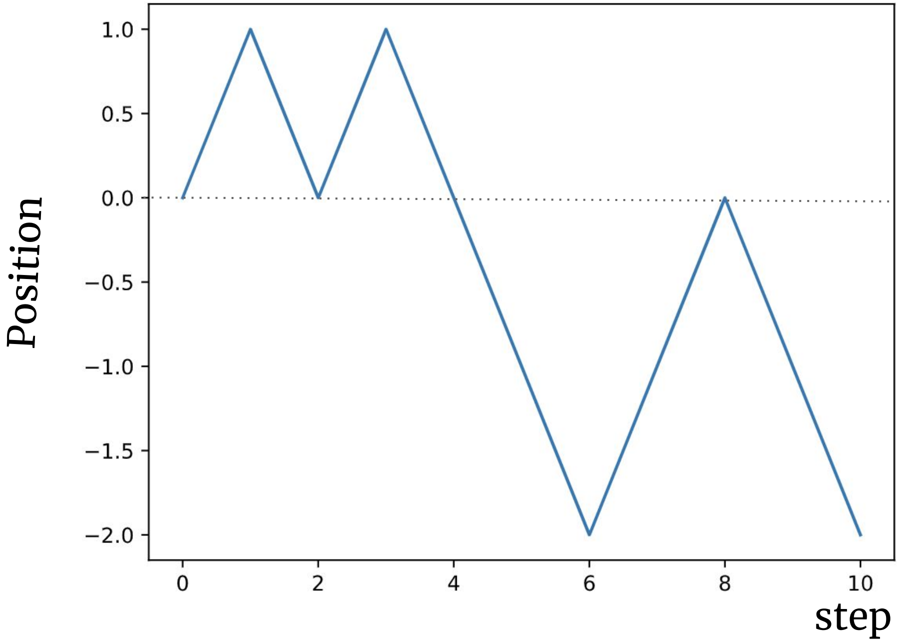
What is the argmax?

$$\operatorname*{argmax}_{0\leq x\leq n} S_x = \{x | S_x \geq S_y, \forall y \in \{0, n\}\}$$

The argmax is the step(s) where the maximum occurs.

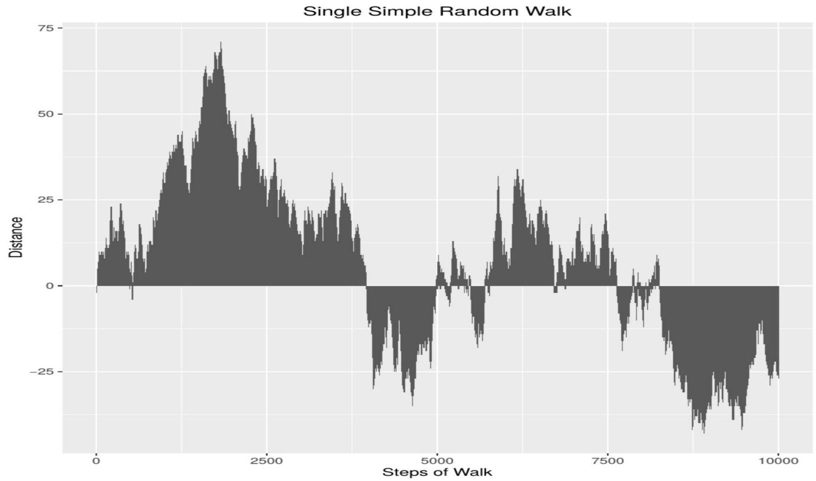
Example below: max=1, argmax={1,3}

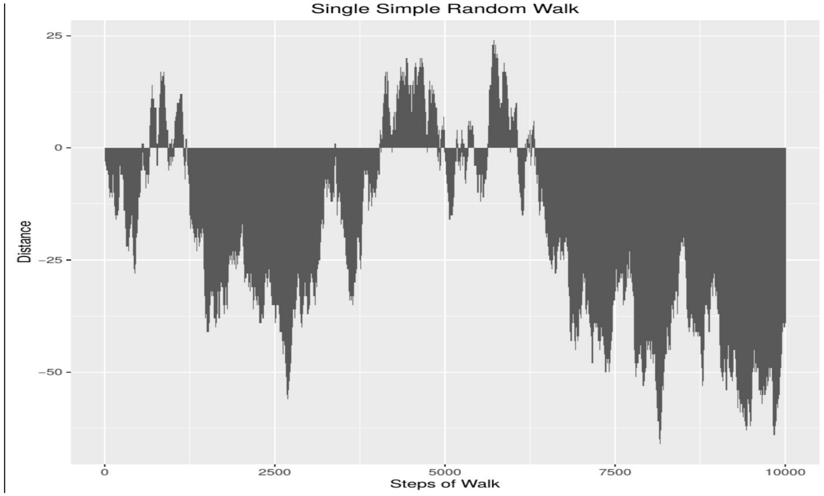
Simple Symmetric Random Walk (10 steps)



What do we expect the argmax to be for any given random walk?

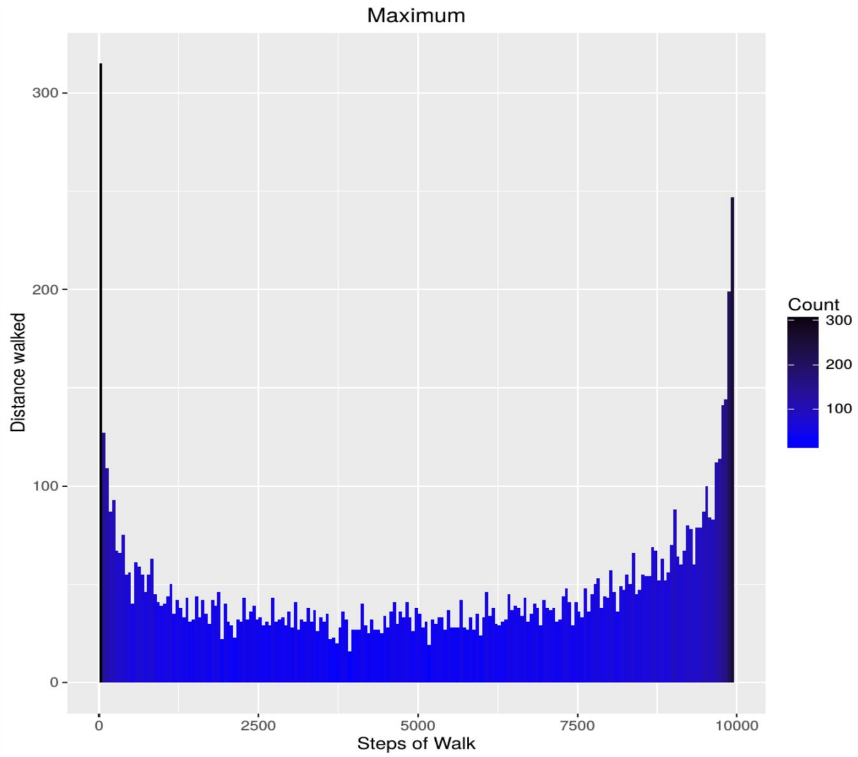
$p = q = 1/2$





What do we expect the argmax to be for any given random walk?

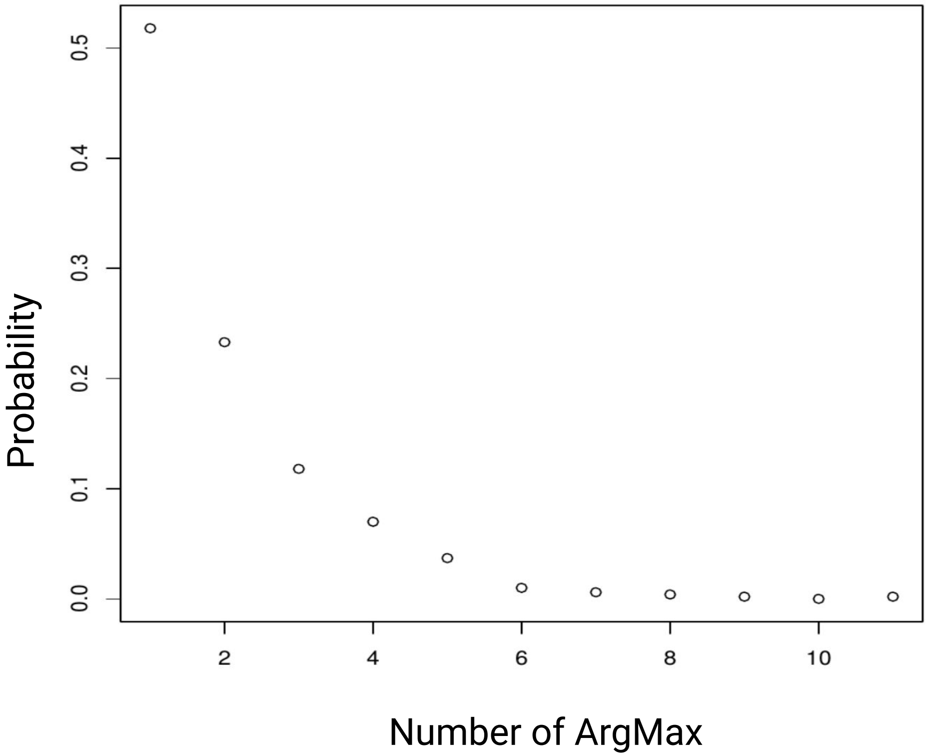
at 0 or n 😮



Number of argmax in single random walk

Symmetric case

$p = q = 1/2$

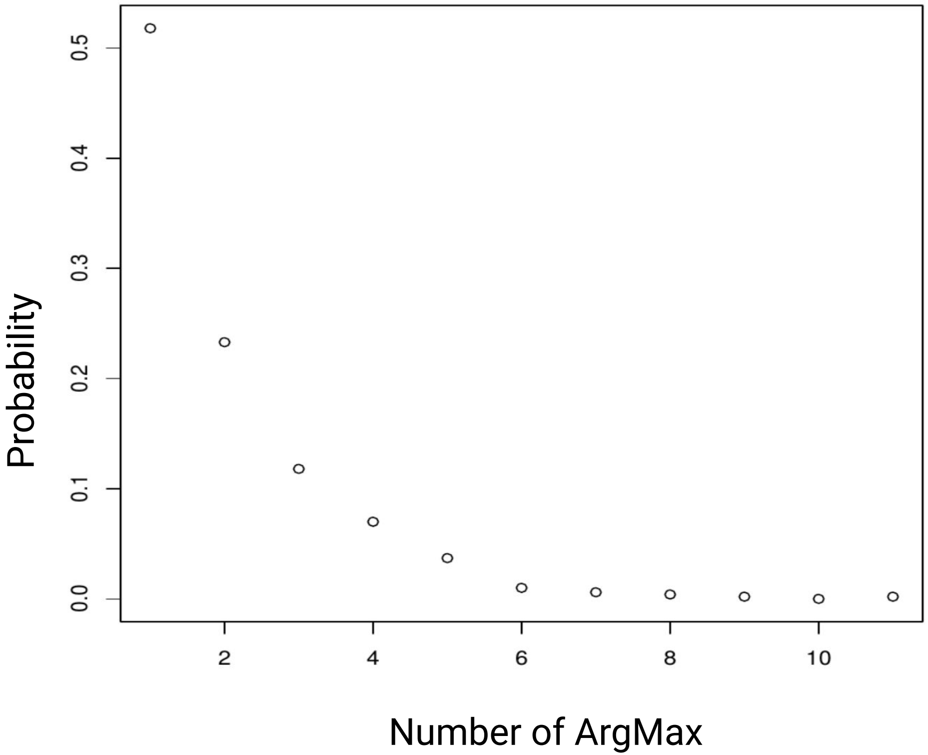


(1/2)^j ?

Number of argmax in single random walk

$$P(\#argmax = j) = (\tfrac{1}{2})^j$$

Does this hold for asymmetrical case?



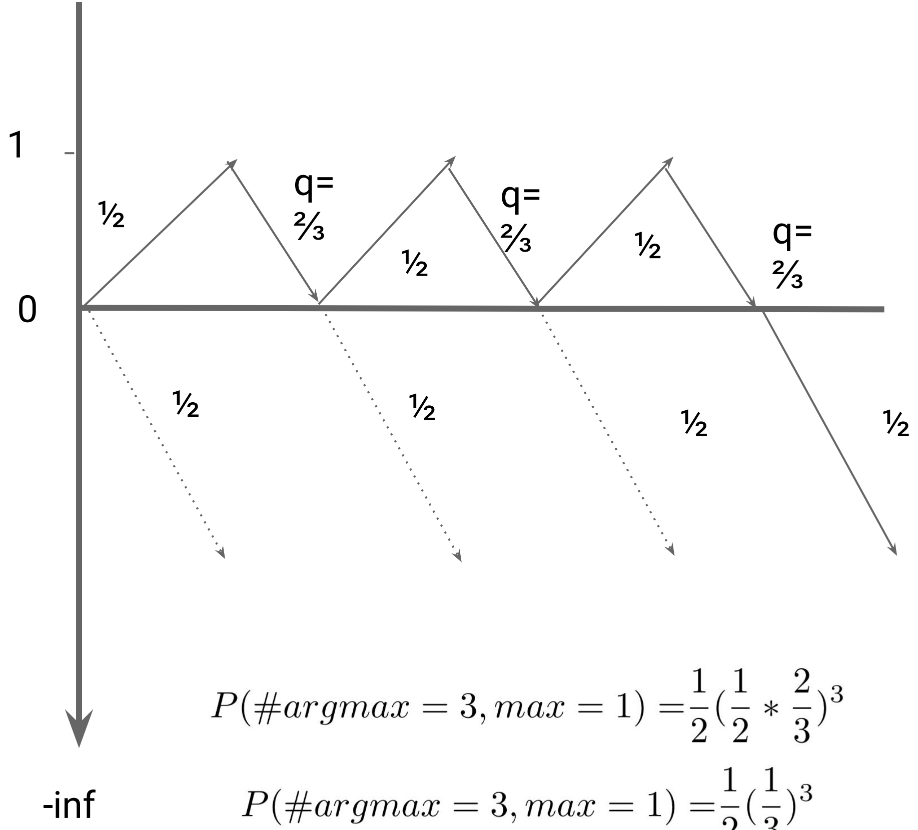
$$P = (1-p) = \frac{1}{2}$$

$$P(\#argmax = j) = (\tfrac{1}{2})^j$$

What is the Probability of getting j argmaxes?

Asymmetrical

What is the probability that the max of the walk is 1, and it returned to the max j times before diverging to -inf ? p=⅓, q=⅔



$$P(\#argmax = 3, max = 1) = \frac{1}{2}(\frac{1}{2} * \frac{2}{3})^3$$

$$P(\#argmax = 3, max = 1) = \frac{1}{2}(\frac{1}{3})^3$$

$$P(\#argmax = j, max = 1) = \frac{1}{2}(\frac{1}{3})^j$$

What is the Probability of getting j argmaxes?

General formula

Summing all the possible number of argmaxes (over j) and summing all the possible value of maxes, we got:

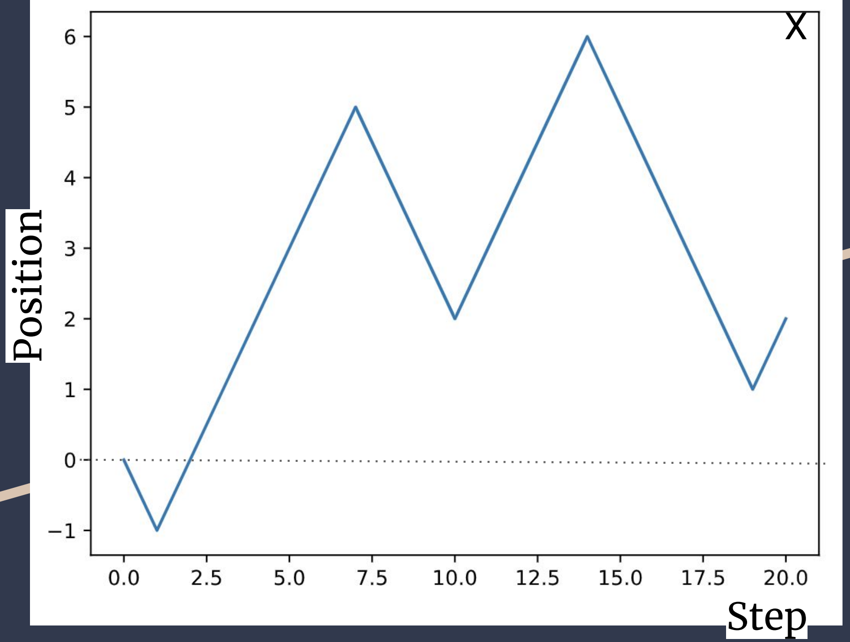
$$\boxed{Pr(\#argmaxes = j) = (1-p)p^{j-1}}$$

$$P = (1-p) = \frac{1}{2}$$

$$Pr(\#argmaxes = j) = \frac{1}{2}^{j}$$

Sums simple random walks (future research)

Simple Symmetric Random Walk (20 steps)



Maximum Likelihood Estimation (over 30,000 simulations for Y)