Linear Transformation

Tuesday, January 22, 2019 10:23 PM

- def. A function $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation if for all v, w in $\mathbb{R}^m$, then

  $T(v+w) = T(v) + T(w)$

  $T(rv) = rT(v)$

  $\Leftrightarrow \quad T(rv+w) = rT(v) + T(w)$

  $\mathbb{R}^m$ is domain of T, $\mathbb{R}^n$ is the codomain of T.

  e.g. $T: \mathbb{R}^3 \to \mathbb{R}^2$ by $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ z \end{pmatrix}$

  $$T\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\right) + T\left(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = \begin{pmatrix} x_1+y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2+y_2 \\ z_2 \end{pmatrix}$$

  $$= \begin{pmatrix} x_1+x_2+y_1+y_2 \\ z_1+z_2 \end{pmatrix}$$

  $$r\,T\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\right) = r\begin{pmatrix} x_1+y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} r(x_1+y_1) \\ rz_1 \end{pmatrix}$$

- $\forall\, u \in \mathbb{R}^m$, $T(u)$ is called the image of u.

  The set of all $T(u)$, $u \in \mathbb{R}^m$, $T(u) \in \mathbb{R}^n$ is called range (image) of T.

  $\text{range}(T) \subset \mathbb{R}^n$

- Theorem. A $m \times n$ $T(x) = Ax$ is a linear transformation of $\mathbb{R}^m \to \mathbb{R}^n$

  Proof. $T(x_1+x_2) = A(x_1+x_2) = Ax_1 + Ax_2 = T(x_1) + T(x_2)$

  $rT(x) = rAx = Arx = T(rx)$

- If $T: \mathbb{R}^m \to \mathbb{R}^n$ is linear, then knowing where

  $[T(e_1)\ T(e_2) \cdots T(e_m)] = A$

  e.g. $T: \mathbb{R}^2 \to \mathbb{R}^3$ is linear & $T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 0 \\ 8 \end{pmatrix}$,

  what is $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$

  $T\left(\begin{pmatrix} x \\ 0 \end{pmatrix}\right) = \begin{pmatrix} x \\ x \\ 2x \end{pmatrix}$, $T\left(\begin{pmatrix} 0 \\ y \end{pmatrix}\right) = \begin{pmatrix} -y \\ 0 \\ 8y \end{pmatrix}$

what is $T(\begin{pmatrix} x \\ y \end{pmatrix})$

$$T(\begin{pmatrix} x \\ 0 \end{pmatrix}) = \begin{pmatrix} x \\ x \\ 2x \end{pmatrix}, \quad T(\begin{pmatrix} 0 \\ y \end{pmatrix}) = \begin{pmatrix} -y \\ 0 \\ 0 \end{pmatrix}$$

$$T(\begin{pmatrix} x \\ y \end{pmatrix}) = T(\begin{pmatrix} x \\ 0 \end{pmatrix}) + T(\begin{pmatrix} 0 \\ y \end{pmatrix}) = \begin{pmatrix} x-y \\ x \\ 2x \end{pmatrix}$$

- Theorem. $A_{m \times n} = [a_1\ a_2 \cdots a_m]$, $T: \mathbb{R}^m \to \mathbb{R}^n$ with $T(x) = Ax$ is a linear transformation, then
  (1) $w \in \text{range}(T)$ if and only if $Ax = w$ is consistent.
  (2) $\text{range}(T) = \text{span}\{a_1, \cdots, a_m\}$
  - Linear transformations are equivalent to matrices.

eg. differentiation is a linear map.

$$f' + g' = (f+g)'$$
$$(cf)' = cf'$$

Integral is a linear map.

- $T: \mathbb{R}^m \to \mathbb{R}^n$ is linear transformation
  - T is one-to-one (injective) if for each $w \in \mathbb{R}^n$, there's only one (at most one) $v \in \mathbb{R}^m$ satisfies $T(v) = w$. $\Leftrightarrow$ $\text{Ker}\,T = \{0\}$ $\Leftrightarrow$ the only soln to $T = 0$ is $x = 0$
  - T is onto (surjective) if for $\forall\, w \in \mathbb{R}^n$, there exists at least one $v \in \mathbb{R}^m$ such that $T(v) = w$

injective $\Leftrightarrow$ $\text{Ker}\,T = \{0\}$

Proof: 1° T is injective.

$\therefore$ there's only one soln to $T(x) = 0$.

$\because$ T is linear map

$\therefore$ $T(0) = 0$, $\therefore$ $\text{Ker}\,T = \{0\}$.

2° $\text{Ker}\,T = \{0\}$

assume $u \in \mathbb{R}^m$, $v \in \mathbb{R}^m$, $T(u) = T(v)$

2 $\operatorname{Ker} T = \{0\}$

assume $u \in \mathbb{R}^m$, $v \in \mathbb{R}^m$, $T(u) = T(v)$

$\therefore T(u) - T(v) = 0$

$T(u-v) = 0$, $\because \operatorname{Ker} T = \{0\}$

$\therefore u = v$

o then if codomain of g is a part of the domain of f, then

$f \circ g(x) = f(g(x))$, f composed with g

$$x \to \boxed{g} \to \boxed{f} \to f(g(x))$$

- Injective transformation composed with injective is injective
- Surjective transformation composed with surjective is surjective

o Bijective $\begin{cases} \text{injective} \\ \text{surjective} \end{cases}$

o By analogy, questions about linear map can often be answered by dimension questions.

e.g. $T: \mathbb{R}^2 \to \mathbb{R}^2$ (1) Is $T \circ S$ possible?

$S: \mathbb{R}^2 \to \mathbb{R}^3$ $\mathbb{R}^2 \to \underset{\mathbb{R}^2}{\mathbb{R}^3} \to \mathbb{R}^2$ impossible

(2) $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$, $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$; $S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\3\end{pmatrix}$

$T(x) = Ax = \begin{pmatrix}1 & 2\\1 & 0\end{pmatrix}$

$S \circ T\begin{pmatrix}1\\0\end{pmatrix} = S\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\3\end{pmatrix}$ $S \circ T\begin{pmatrix}0\\1\end{pmatrix} = S\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}2\\2\\0\end{pmatrix}$

$S \circ T(x) = \begin{pmatrix}1 & 2\\1 & 2\\3 & 0\end{pmatrix}$

m: number of vars in x

- Theorem. $A_{m \times n}$, $T: \mathbb{R}^m \to \mathbb{R}^n$, $T(x) = Ax$. $m \times n$: row x column

(1) T is injective if & only if columns of A linearly independent

(2) If $A \sim B$ and B is in echelon form, then T is one to one if and

only if B has a pivot position in every column.

(3) if $n<m$, T is not one to one.

Proof. columns independent $\Rightarrow x_1 \cdots x_m$ independent

$\therefore$ soln is trivial

(4) T is surjective if and only if the columns of A $\operatorname{span}\{a_1 \cdots a_m\} = \mathbb{R}^n$

(5) If $A \sim B$ is in echelon form, then T is onto if and only if B has a pivot position in every row.

(6) If $n>m$, then T is not onto.

Proof: $\operatorname{range}(T) = \operatorname{span}\{a_1 \cdots a_m\}$

$\because \operatorname{span}\{a_1 \cdots a_m\} = \mathbb{R}^n$

$\therefore \operatorname{range}(T) = \mathbb{R}^n$ $\therefore$ surjective.

- Theorem. $T: \mathbb{R}^m \to \mathbb{R}^n$, T is a linear transformation if & only if $T(x) = Ax$ for some matrix A.

3.2 Matrix algebra

Friday, January 25, 2019 5:17 PM

- Multiplication $A_{r\times c}$, $B_{c\times m} = (v_1, \dots, v_m)$

$r\times m$ $AB = (Av_1\ Av_2 \cdots Av_n)$

- Claim: $T(x) = Bx$, $S(y) = Ay$, $T \circ S = BAy$

e.g. $S\left[\begin{pmatrix}1\\0\end{pmatrix}\right] = \begin{pmatrix}1\\1\\2\end{pmatrix}$ $\quad S\left[\begin{pmatrix}x_1\\x_2\end{pmatrix}\right] = \begin{pmatrix}1&0\\1&1\\2&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$

$S\left[\begin{pmatrix}0\\1\end{pmatrix}\right] = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$T\left[\begin{pmatrix}1\\0\\0\end{pmatrix}\right] = \begin{pmatrix}1\\1\\0\end{pmatrix}$

$T\left[\begin{pmatrix}0\\1\\0\end{pmatrix}\right] = \begin{pmatrix}2\\0\\3\end{pmatrix}$ $\quad T\left[\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right] = \begin{pmatrix}1&2&0\\1&0&1\\0&3&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$T\left[\begin{pmatrix}0\\0\\1\end{pmatrix}\right] = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$T \circ S(x) = \begin{pmatrix}1&2&0\\1&0&1\\0&3&0\end{pmatrix}\begin{pmatrix}1&0\\1&1\\2&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$

- $T: \mathbb{R}^n \to \mathbb{R}^n$ is surjective if & only if injective, vice versa.

3.3 Inverses

Wednesday, February 6, 2019 10:32 AM

- A linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ ($m=n$) is invertible if T is one-to-one and onto. When T invertible, $T^{-1}: \mathbb{R}^n \to \mathbb{R}^m$ defined by

  $T^{-1}(y) = x$ if & only if $T(x) = y$.

  $\Rightarrow T(T^{-1}(y)) = y$

  - $T: \mathbb{R}^m \to \mathbb{R}^n$ is linear transformation.
    - T is invertible, then $m=n$.
    - T is invertible, T^{-1} is also a linear transformation

  ① Prof. 1° $\because$ T is onto, $m \geq n$;

  T is one-to-one, $m \leq n$.

  $\therefore m=n$.

  ② $T(x_1) = y_1$, $T(x_2) = y_2$, $T(x_1+x_2) = y_1+y_2$

  $T^{-1}(y_1) + T^{-1}(y_2) = x_1+x_2 = T^{-1}(y_1+y_2)$

  Note: $m=n$ does not guarantee that T will have inverse.

- An $n \times n$ matrix A is invertible if there exist B_{nn}, $AB = I_n$. $B = A^{-1}$
  - A is invertible if $T(x) = Ax$ is invertible
  - A is invertible, $\exists B$, $AB = BA = I$, B is unique.

  1° $AB = BA = I$

  $AB = I \Rightarrow BAB = B$

  $(BAB - B) = 0$

  $(BA - I)B = 0 \Rightarrow BA = I$

  2° Assume there exists C, $AB = AC = I$

  $AB - AC = A(B-C) = 0$, $\because A \neq 0$

  $\therefore B = C$

- def: If an $n\times n$ matrix invertible, then A^{-1} is an inverse of A, $n\times n$.

  $A\cdot A^{-1} = A^{-1}A = I_n$,

  - Nonsingular $\Leftrightarrow$ Invertible. $\Leftrightarrow$ $\det(A) \neq 0$

    Singular $\Leftrightarrow$ Not invertible. $\Leftrightarrow$ $\det(A) = 0$

- Let A, B invertible $n\times n$ matrices, C, D $n\times m$ matrices

  (a) A^{-1} invertible, $(A^{-1})^{-1} = A$

  (b) A, B invertible, $(AB)^{-1} = B^{-1}A^{-1}$

  (c) If $AC = AD$, then $C = D$.

  (d) If $AC = 0_{nm}$, then $C = 0_{nm}$.

  (a) $\because AA^{-1} = A^{-1}A = I$

  $\therefore (A^{-1})^{-1} = A$

  (b) $(AB)^{-1}\cdot(AB) = I$

  $(AB)^{-1}\cdot(AB)\cdot B^{-1} = B^{-1}$

  $(AB)^{-1}\cdot(AB)(B^{-1}A^{-1}) = B^{-1}A^{-1}$

  $(AB)^{-1} = B^{-1}A^{-1}$

  (c) $AC - AD = 0 \Rightarrow A(C-D) = 0 \Rightarrow C = D$

  (d) $\because c$ $\therefore d$.

- A is invertible if and only if A is row equivalent to identity matrix.

  - row equivalent: can be tranform into by elementary row operation.

  - Calculate A^{-1}

    $A = [a_1 \cdots a_n]$, $B = [b_1 \cdots b_n]$, $I_n = [e_1 \cdots e_n]$

    $\because AB = I$ $\therefore Ab_1 = e_1, \cdots, Ab_n = e_n$

    consider b_i as soln to a LS

    $\therefore [a_1 \cdots a_n \mid e_1 \cdots e_n] \Rightarrow [A \mid I]$

    do elementary row operations $[A \mid I_n]$ to $[I_n \mid A^{-1}]$

e.g. find A^{-1}, $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & -1 & 2 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & -1 & 2 & -1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

- Let A be $n \times n$ matrix. Then the following are equivalent.

  (a) A is invertible.

  (b) $Ax = b$ has a unique soln for all b, given by $x = A^{-1}b$

  (c) $Ax = 0$ has only trivial soln.

  $T(x) = Ax$ is injective & surjective

  $\therefore$ b, c stands.

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $ad \neq bc$

  $\therefore A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

  $$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d-\frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

  $$\vdots$$

  $$\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ 0 & 1 & \end{array}\right]$$

- The Unifying Theorem

  Let $\mathcal{S} = \{a_1, \dots, a_n\}$ be a set of n vectors in $\mathbb{R}^n$, let $A = [a_1 \cdots a_n]$, $T: \mathbb{R}^n \to \mathbb{R}^n$ be $T(x) = Ax$. Then those're equivalent:

(a) $\mathcal{S}$ spans $\mathbb{R}^n$

(b) $\mathcal{S}$ is linearly independent

(c) $Ax = b$ has a unique soln for all b in $\mathbb{R}^n$

(d) T is onto

(e) T is one-to-one

(f) A is invertible.

probs

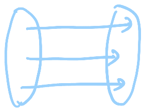
Thursday, January 24, 2019 9:01 PM

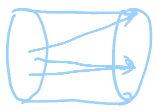
Determine if the statement is true or false, and justify your answer.

If $T_1(\mathbf{x})$ and $T_2(\mathbf{x})$ are onto linear transformations from $\mathbf{R}^n$ to $\mathbf{R}^m$, then so is $W(\mathbf{x}) = T_1(\mathbf{x}) + T_2(\mathbf{x})$.

- ○ True, by the definition of linear transformation.
- ○ True, by the definition of linear transformation and the definition of onto.
- ○ False. Consider $T_2(\mathbf{x}) = T_1(\mathbf{x})$, where T_1 is onto.
- ○ False. Consider $T_2(\mathbf{x}) = -T_1(\mathbf{x})$, where T_1 is one-to-one.
- ◉ False. Consider $T_2(\mathbf{x}) = -T_1(\mathbf{x})$, where T_1 is onto. ✔

$T_2(x) = -T_1(x)$, $T_1(x)$ onto : $W(x) = 0$

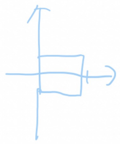




justification is needed if it is not possible.

(d) [2 pt] A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, given by $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$, which reflects the unit square about the x-axis. (Note: Take the unit square to lie in the first quadrant. Giving the matrix of T, if it exists, is a sufficient answer).

$$\cancel{T = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + \frac{x_2}{2} \\ x_1 + \frac{x_2}{2} \end{pmatrix}}$$



$T(e_1) = e_1$

$T(e_2) = -e_2$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$