3.2/3.3

Monday, October 29, 2018 3:33 PM

- Warm-up.

(a) $y'' + 2y' - 3y = 0$

assume $e^{rt} = y$

$e^{rt}(r^2 + 2r - 3) = 0$

$e^{rt}(r-1)(r+3) = 0$

$r = 1,\ r = -3$

$y = C_1 e^{t} + C_2 e^{-3t}$

(b) $y'' - 2y' + 3y = 0$

$e^{rt}(r^2 - 2r + 3) = 0$

$r = \frac{2 \pm \sqrt{-8}}{2} = 1 \pm \sqrt{2}\, i$

$y = C_1 e^{1+\sqrt{2}i} + C_2 e^{1-\sqrt{2}i}$

$= C_1 e \cos(\sqrt{2}t) + \cancel{i} C_2 e \sin(\sqrt{2}t)$

i is contained in C_2.

- Wronskian

e.g. $y'' - 2y' + y = 0$

$(r-1)^2 = 0$

$\Delta = 0,\ r = e$

Given two solutions, $y_1(t)$ and $y_2(t)$, the general soln

$y(t) = C_1 y_1 + C_2 y_2$ exactly when $W(y_1, y_2) = y_1 y_2' - y_2 y_1' \neq 0$

for any value.

- e.g. $y_1 = e^{t},\ y_2 = e^{t+1}$

$\therefore y_1' = e^{t},\ y_2' = e^{t+1}$

$y_1 y_2' - y_2 y_1' = 0$

$\therefore y_1' = 0, y_2' = 0$

$y_1 y_2' - y_2 y_1' = 0$

$\therefore$ couldn't find $y = c_1 y_1 + c_2 y_2$ satisfy the general solution

$\Rightarrow$ e.g. $y'' - 2y' + y = 0$

$y_1(t) = e^t$

guess $y(t) = v(t) e^t$

$y = ve^t, \quad y' = v'e^t + ve^t$

$y'' = v''e^t + v'e^t + v'e^t + ve^t$

plug in: $\underline{v''e^t = 0}$

$\underline{\because e^t \neq 0 \quad \therefore v'' = 0}$

$\therefore \underline{v = at + b}$

assume $y_2(t) = (at+b)e^t, \; y_1(t) = e^t$

$y = C_1 e^t + C_2 (at+b) e^t$

$\Rightarrow C_1 e^t + C_2 t e^t$ (combine constants)

Check: $y_1' = e^t \quad y_2' = e^t + te^t$

$y_1 y_2' = e^{2t} + te^{2t}$

$y_2 y_1' = te^{2t}$

$y_1 y_2' - y_2 y_1' = e^{2t} \neq 0$

- This is true in general:

$y'' + 2ay' + a^2 y = 0$

$(r+a)^2 = 0$

$\therefore y(t) = C_1 e^{-at} + C_2 t e^{-at}$

$\Delta > 0$: $(r - r_1)(r - r_2) = 0$

$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$\Delta>0$: $(r-r_1)(r-r_2)=0$

$$y(t)=C_1e^{r_1t}+C_2e^{r_2t}$$

$\Delta<0$: $r=\lambda\pm\mu i$

$$y(t)=e^{\lambda t}(C_1\cos(\mu t)+C_2\sin(\mu t))$$

$\Delta=0$: $(r-a)^2=0$

$$y(t)=C_1e^{at}+C_2te^{at}$$

e.g. 1. show if $a,b,c>0$,

$ay''+by'+cy=0$, any soln to the DE which has $\lim_{t\to\infty}y(t)=0$.

$$ar^2+br+c=0$$

$$r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Case I. $\Delta<0$

$$y(t)=e^{-\frac{b}{2a}}\left(C_1\cos\left(\frac{\sqrt{4ac-b^2}}{2a}\right)+C_2\sin\left(\frac{\sqrt{4ac-b^2}}{2a}\right)\right)$$

$\because a,b>0$ $\therefore -\frac{b}{2a}<0$

$\therefore y(t)\to0$ when $t\to\infty$

Case 2. $\Delta>0$

$$y(t)=C_1e^{\frac{-b-\sqrt{b^2-4ac}}{2a}}+C_2e^{\frac{-b+\sqrt{b^2-4ac}}{2a}}$$

$\to 0$

$\because \sqrt{b^2-4ac}<\sqrt{b^2}<b$

$\therefore \frac{-b+\sqrt{b^2-4ac}}{2a}<0$

$\therefore y(t)\to0.$

Case 3. $(r-\frac{b}{2a})^2=0$ $(\Delta=0)$

$$y=C_1e^{-\frac{b}{2a}t}+C_2te^{-\frac{b}{2a}t}$$

$t\to\infty$, $e^{-\frac{b}{2a}t}\to0$

$\Rightarrow te^{-\frac{b}{2a}t}=\frac{t}{e^{\frac{b}{2a}}}$ $\Rightarrow$ L'Hôpital Rule

$\Rightarrow \lim_{t\to\infty}te^{-\frac{b}{2a}}=\frac{1}{\frac{b}{2a}e^{\frac{b}{2a}}}=0$

$\therefore y\to0.$

Linear Algebra with Application

Jeffrey Holt

3.5 Method of Undetermined coefficients

Wednesday, October 31, 2018 3:38 PM

- e.g. $y'' - y = e^{2t}$

guess $y_p(t) = Ae^{2t}$

$\therefore y_p'(t) = 2Ae^{2t} \quad y_p''(t) = 4Ae^{2t}$

$\therefore 4Ae^{2t} - Ae^{2t} = e^{2t}$

$4A - A = 1$

$A = \frac{1}{3}$

A particular soln of the DE is $\frac{1}{3}e^{2t}$

Homogeneous: $y'' - y = 0$

$e^{rt}(r^2 - 1) = 0$

$r = \pm 1$ → complementary solution

$\therefore y_c(t) = C_1e^{-t} + C_2e^{t}$

$\therefore \underline{y = C_1e^{-t} + C_2e^{t} + \frac{1}{3}e^{2t}}$

↓

soln of nonhomogeneous

→ $y_p + y_c$ is all solutions to the original nonhomogeneous DE.

Proof: $(y_p + y_c)'' - (y_p + y_c)$

$= \underbrace{y_c'' - y_c}_{=0} + \underbrace{y_p'' - y_p}_{=g(t)} = g(t)$

- Steps to solve nonhomogeneous DE:

① Find y_c (complementary solution)

② guess (a) $y = e^{ct}$ or $\sin t$ or $\cos t$ or $a_n t^n + \cdots a_1 t + a_0$

(b) solve for coefficient

③ $y(t) = y_p + y_c$

e.g. 2. $y'' - y = \sin 3t$

$y'' - y = 0$

$y''-y=\sin 3t$

$y''-y=0$

$\therefore y_c = C_1e^{-t}+C_2e^{t}$

guess $y_p(t) = \underline{A\sin 3t}$ if there's y', guess $y_p(t) = A\sin 3t + B\cos 3t$

$y'' = -9A\sin 3t$

$y = A\sin 3t$

$\therefore -9A\sin 3t - A\sin 3t = \sin 3t$

$-10A = 1$

$A = -\frac{1}{10}$

$y_p = -\frac{1}{10}\sin 3t$

$y = -\frac{1}{10}\sin 3t + C_1e^{-t} + C_2e^{t}$.

e.g. $y''-y=t^2-1$

$y_c = C_1e^{t}+C_2e^{-t}$

guess $y = At^2+Bt+C$

$y''=2A$

$-At^2-Bt+2A-C=t^2-1$

$\therefore \begin{cases} -At^2=t^2 \\ 2A-Bt-C=-1 \Rightarrow 2A-C=-1 \end{cases}$

$A=-1,\ B=0,\ C=-1$

$\therefore y_p = -t^2-1$

$\therefore y = C_1e^{t}+C_2e^{-t}-t^2-1$

e.g. $y''-3y'-4y=e^{2t}\cdot\cos 3t$

If don't know what to guess, take derivative of $g(t)$ and find its feature.

$y''-3y'-4y=0$

$e^{rt}(r^2-3r-4)=0$

$(r-4)(r+1)=0$

$r=4,\ r=-1$

$y_c = C_1e^{4t}+C_2e^{-t}$

guess $y_p = Ae^{2t}\cdot\cos 3t + Be^{2t}\cdot\sin 3t$

$y_c = C_1 e + C_2 e$

guess $y_p = Ae^{2t}\cdot\cos 3t + Be^{2t}\cdot\sin 3t$

$y_p' = A(-3e^{2t}\sin 3t + 2e^{2t}\cos 3t) + B(2e^{2t}\sin 3t + 3e^{2t}\cos 3t)$

$= (2B - 3A)e^{2t}\sin 3t + (2A+3B)e^{2t}\cos 3t$

$y_p'' = 2(2B-3A)e^{2t}\cdot\sin 3t + 3(2B-3A)e^{2t}\cdot\cos 3t$

$+ 2(2A+3B)e^{2t}\cos 3t - 3(2A+3B)e^{2t}\cdot\sin 3t$.

eg. $y'' - y = e^{-t}$

guess $y = Ae^{-t}$

$\Rightarrow$ guess is included in y_c.

y_{guess} is linearly independent to y_c.

If guess $y = tAe^{-t}$

$y' = Ae^{-t} - tAe^{-t}$

$y'' = -Ae^{-t} - Ae^{-t} + tAe^{-t}$

$= -2Ae^{-t} - tAe^{-t}$

$-2Ae^{-t} - t\,Ae^{-t} - Ae^{-t} + tAe^{-t} = e^{-t}$

$-3A = 1$

$A = -\frac{1}{3}$

$\therefore y_p = -\frac{1}{3}te^{-t}$

$\therefore y = C_1e^{t} + C_2e^{-t} - \frac{1}{3}te^{-t}$

3.5

Friday, November 2, 2018 3:29 PM

- Warm-up

$y'' + 4y = 3\sin 2t$; $y(0) = 2$, $y'(0) = -1$

$(r^2 + 4)e^{rt} = 0$

$r = \pm 2i$

$\therefore y_c = C_1\cos 2t + C_2\sin 2t$

guess $y_p = (A\cos 2t + B\sin 2t)t$

$y_p' = -2At\sin 2t + A\cos 2t + B\sin 2t + 2Bt\cos 2t$

$y_p'' = -4At\cos 2t - 2A\sin 2t - 2A\sin 2t - 2B\cos 2t$

$+ 2B\cos 2t - 4Bt\sin 2t$

$= -4A\sin 2t + 4B\cos 2t - 4At\cos 2t - 4Bt\sin 2t$

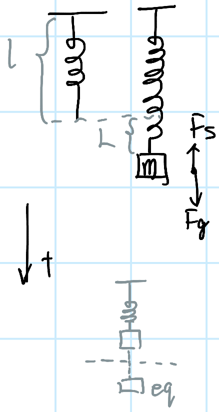
$y'' + 4y = -4A\sin 2t + 4B\cos 2t = 3\sin 2t$

$\therefore \begin{cases} -4A = 3 \\ 4B = 0 \end{cases}$ $\therefore A = -\frac{3}{4}$

$\therefore y_p = -\frac{3}{4}t\cos t$

$\therefore y = C_1\sin 2t + C_2\cos 2t - \frac{3}{4}t\cos t$

- Spring



$F_s = kL$

$F = mg - kL \Rightarrow \underset{\text{(equilibrium)}}{mg = kL}$

(*) $F = F_g + F_s + F_d + F(t)$; $u(t)$ = position from equilibrium ($F=0$)

$F_g = mg$

$F_s = -k(L + \underline{u})$ → u 为 "+" when spring goes above.

$F_d = -\gamma\frac{du}{dt}$ drag

* $m\frac{d^2u}{dt^2} = mg - k(L+u) - \gamma\frac{du}{dt} + F(t)$

$$\times\ m\frac{d^2u}{dt^2} = mg - k(L+u) - \gamma\frac{du}{dt} + F(t)$$

$$m\frac{d^2u}{dt^2} + \gamma\frac{du}{dt} + ku = \underbrace{F(t)}_{\text{additional driving force.}}$$

$\lim_{t\to\infty} u(t) = 0,\ F_d \neq 0,\ F(t) = 0$

e.g. 4lb weight stretches a spring 2inches, If you pull it 6 inches downward and release. In a viscous medium the mass experiences a force due to drag of 6 lb when traveling 3 ft/s. Give position at time t.

$2 \text{ inch} = \frac{1}{6}\text{ft}.\quad m = \frac{1}{8}\text{slug}$

$F = F_g + F_s + F_d \qquad \gamma = \frac{6}{3} = 2.$

$F_s = k \cdot L = 4,\quad k \cdot \frac{1}{6} = 4 \Rightarrow k = 24$

$\frac{1}{8} \cdot \frac{d^2u}{dt^2} + 2\frac{du}{dt} + 24u = 0 \quad u(0) = 6.\ u'(0) = 0$

$e^{rt}(\frac{1}{8}r^2 + 2r + 24) = 0$

$x \text{ lb} = \text{mass} \cdot 32\ \text{ft/s}^2$

12 inch = 1 ft

5280ft = 1mi

$mu'' + \gamma u' + ku = 0$

$mr^2 + \gamma r + k = 0$

$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$ ← Overdamped

$\Delta = \gamma^2 - 4mk > 0$

$\gamma^2 > 4mk$

$\gamma > 2\sqrt{mk}\ (m, k > 0)$

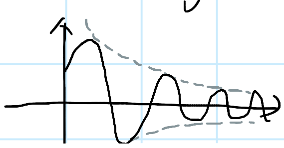
$u(t) = C_1e^{r_1t} + C_2e^{r_2t} \quad (r_1, r_2 < 0)$

$\Delta = 0,\ \gamma = 2\sqrt{mk}$ ← Critically damped

$u(t) = C_1e^{-\frac{\gamma}{2m}t} + tC_2e^{-\frac{\gamma}{2m}t}$

$\Delta < 0,\ \gamma < 2\sqrt{mk}$ ← Underdamped

$u(t) = e^{-\frac{\gamma}{2m}t}(C_1\cos\mu t + C_2\sin\mu t)$



$u(t) = e \quad (c_1 \cos \mu t + c_2 \sin \mu t)$

$\downarrow$

$R\cos(\mu t - \delta)$

$(\cos(A-B) = \cos A \cdot \cos B - \sin A \cdot \sin B)$

HW6

Sunday, November 4, 2018 1:35 PM

5/5 points | Previous Answers

Consider the differential equation:

$y'' + 4y = f(x)$

$y'' + 4y = 0 \Rightarrow y = C_1\cos(2t) + C_2\sin(2t)$

For each choice of the forcing function $f(x)$, indicate the form of the particular solution, without solving for the constants. Example: $f(x) = x^2 \quad y(x) = A\cdot x^2 + B\cdot x + C$

$f(x) = xe^{3x} \qquad y(x) = A\cdot x\cdot e^{3x} + Be^{3x}$

$f(x) = \cos 5x \qquad y(x) = A\cdot\cos(5x) + B\cdot\sin(5x)$

$f(x) = x\sin 6x \qquad y(x) = A\cdot x\cdot\cos(6x) + B\cdot x\cdot\sin(6x) + C\cdot\cos(6x) + D\cdot\sin(6x)$

$f(x) = \sin 2x \qquad y(x) = A\cdot x\cdot\sin(2x) + B\cdot x\cdot\cos(2x)$

the constant is "2" same as soln. so have to have a x.

$f(x) = e^{2x}\cos 2x \qquad y(x) = A\cdot e^{2x}\cdot\cos(2x) + B\cdot e^{2x}\cdot\sin(2x)$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

The solution to the initial value problem below takes on different forms for different ranges of the parameter ζ, which is called the dimensionless damping coefficient. In each case, write the solution in the suggested form.

$u'' + 14\zeta u' + 49u = 0$

$u(0) = 0 \qquad u'(0) = 1$

If $\zeta > 1$,

$$u_\zeta(t) = e^{(-7\zeta t)}\frac{\sinh(7\sqrt{\zeta^2-1}\,t)}{7\sqrt{\zeta^2-1}}$$

1° $\zeta > 1$

$\sqrt{\Delta} = \sqrt{14^2\zeta^2 - 49\times4} = 14\sqrt{\zeta^2-1}$

$y = C_1e^{(-7\zeta+\frac{\sqrt{\Delta}}{2})t} + C_2e^{(-7\zeta-\frac{\sqrt{\Delta}}{2})t}$

$\because u(0) = 0,\ u'(0) = 1$

$\therefore C_1 + C_2 = 0,\quad (-7\zeta+\frac{\sqrt{\Delta}}{2})C_1 + (-7\zeta-\frac{\sqrt{\Delta}}{2})C_2 = 1$

$\therefore C_1 = \frac{1}{\sqrt{\Delta}}$

$C_2 = -\frac{1}{\sqrt{\Delta}}$

$\therefore u_\zeta(t) = e^{-7\zeta t}\cdot\frac{\sinh(7\sqrt{\zeta^2-1}\cdot t)}{7\sqrt{\zeta^2-1}}$

$\zeta \to 1^+$

Compute the limit as $\zeta \to 1^+$:

$\lim_{\zeta\to1^+} u_\zeta = te^{-7t}$

If $\zeta < 1$,

$$u_\zeta(t) = e^{(-7\zeta t)}\frac{\sin(7\sqrt{1-\zeta^2}\,t)}{7\sqrt{1-\zeta^2}}$$

2° $\zeta < 0$

$\sqrt{\Delta} = 14\sqrt{1-\zeta^2}\,i$

$y = e^{-7\zeta t}(C_1\cos(\frac{\sqrt{\Delta}}{2}t) + C_2\sin(\frac{\sqrt{\Delta}}{2}t))$

$\because u(0) = 0,\ u'(0) = 1$

$\therefore C_1 = 0,\quad \frac{\sqrt{\Delta}}{2}\cdot C_2\cos(\frac{\sqrt{\Delta}}{2}t)\cdot e^{-7\zeta t} = 1$

$C_2 = \frac{2}{\sqrt{\Delta}} = \frac{1}{7\sqrt{1-\zeta^2}}$

Compute the limit as $\zeta \to 1^-$:

$\lim_{\zeta\to1^-} u_\zeta = te^{-7t}$

Find the general solution to the differential equation in the case $\zeta = 1$.

$u_1(t) = C_1e^{-7t} + C_2t\cdot e^{-7t}$

Now find the solution to the initial value problem when $\zeta = 1$.

$u_1(t) = t\cdot e^{-7t}$

$$\lim_{\zeta\to1^+}\left\{e^{-7\zeta t}\cdot\frac{\sinh(7\sqrt{\zeta^2-1}\,t)}{7\sqrt{\zeta^2-1}}\right\}$$

let $z = 7\sqrt{\zeta^2-1}$

consider make form zt

$$= \lim_{\zeta\to1^+} e^{-7\zeta t}\cdot t\cdot\frac{\sinh(zt)}{zt}$$

$$= \lim_{\zeta \to 1^+} e \cdot t \cdot \frac{\sinh(zt)}{zt}$$

form zt

$$= e^{-7t} t \cdot \lim_{\zeta \to 1^+} \frac{e^{zt} - e^{zt}}{2zt}$$

$$= e^{-7t} t \cdot \lim_{\zeta \to 1^+} \frac{2ze^{zt}}{2z} = e^{-7t} \cdot t$$