1 Introduction

Wednesday, September 26, 2018   3:31 PM

- Differential Equation (DE): an equation involves derivatives.
  e.g. $\frac{d^2y}{dt^2} + y = t^2 - 1$ (2nd order)
- Order of a DE: highest order derivative appeared in the DE
- Multi-variable function:

  $f(t, x) = \cos t - 3x^2 + 1 \Rightarrow f(\frac{\pi}{2}, -3) = -26$

  - partial derivative. $\frac{\partial f}{\partial t}(t, x) = -\sin t$

  $\frac{\partial f}{\partial x}(t, x) = -6x$

- Ordinary DE vs Partial DE

(ODEs) (no partials)

e.g. $F = ma$

$\Downarrow$

$\frac{dx^2}{d^2t} = a$

$F(t, y, y') = my''$

e.g. $y'' + 3y' + 2y = 0$

$y(t) = e^{-2t}$

(PDEs)

e.g. $\frac{\partial y}{\partial t} = \sin t + x$

$\rightarrow y(t, x)$

$y'' = F(t, y, y')$ ← two variable

$= G(t)$ ∵ y, y' dependent variables

e.g. $y'' + y = 0$

$y = \cos t$

$y = \sin t$

- Solution to a DE: a function that satisfies the given equation.
- Given an initial condition, is it unique?

Initial Value Problem (IVP)

- e.g. A model for fluids

  x = position, t = time, v = velocity, p = pressure

  given a position $x(x_1, x_2, x_3)$

  $u(t, x)$ $p(t, x)$

  $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x} + \frac{\partial p}{\partial x} + C$$

  solution: $u(t, x)$, $p(t, x)$

- **1st order linear ODEs**

- 1st order linear ODEs

$$y' + p(t)y = g(t)$$

Separable 1st order ODEs

$M(t) = N(y) \cdot y'$ (any ODE that can be transformed into this)

$\searrow$ $M(t)\,dt = N(y)\,dy$

e.g. $\cos x = (y+1)y'$

e.g. $y' - \frac{\cos x}{y} = 0$

$\Rightarrow y' = \frac{\cos x}{y}$

$y \cdot y' = \cos x$

- Newton's Law of Cooling:

$\frac{dT}{dt} = -k(T(t) - T_A)$ T: temperature

t: time

separable: $\frac{dT}{dt} \cdot \frac{1}{T - T_A} = -k$

$$\frac{d}{dt} \ln|T - T_A| = -k$$

$$\int \frac{d}{dt} \ln|T - T_A|\,dt = \int -k\,dt$$

$$\ln|T - T_A| = -kt + C$$

$$e^{\ln|T - T_A|} = e^{-kt} e^{C}$$

$$|T - T_A| = Ce^{-kt} \quad C \geq 0 \ (\because e^{-kt} > 0)$$

$$T - T_A = Ce^{-kt} \quad C \in R$$

$$T = Ce^{-kt} + T_A \quad (C \in R)$$

Friday, September 28, 2018 3:31 PM

- e.g.1. Consider the IVP $y'+y=0$, $y(1)=4$

$$\frac{dy}{dt} = -y$$

$$\frac{1}{y}dy = -dt$$

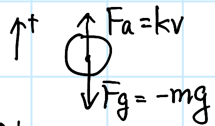
$$\ln|y| = -t+C$$

$$y = Ce^{-t} \qquad \because y(1)=4$$

$$\therefore C = 4e.$$

$$\therefore y = 4e^{-t+1}$$

- e.g.2. Falling Object



$m = 1kg$, $t = 0s$, $V = 10.8 m/s$, $F = 1N$.

!! $F = ma$

$$F = -k\cdot 10.8 - 9.8 = 1$$

$$kv - mg = m\frac{dv}{dt} \quad \text{careful!} \qquad \therefore k = 1.$$

$$\frac{1}{m}dt = \frac{1}{kv-mg}dv \qquad -10.8 = \frac{C\cdot e^0+9.8}{-1} = -C-9.8$$

$$\frac{1}{m}t + C = \frac{1}{k}\ln|kv-mg| \qquad C = \cancel{20.6}\ -1$$

$$Ce^{\frac{k}{m}t} = |kv-mg| \leftarrow C>0$$

$$v = \frac{Ce^{\frac{k}{m}t}+mg}{k}$$

$$\therefore v = \frac{\cancel{-20}e^{\frac{k}{m}t}+mg}{k} = -9.8-e^{-t}$$

e.g.3. $\frac{dy}{dt} = (y-3)\sin t$

$$\frac{dy}{y-3} = \sin t\, dt \qquad \text{(okay to do so)}$$

$$\ln|y-3| = -\cos t + C$$

$$|y-3| = Ce^{-\cos t}, \quad C>0$$

$$y = Ce^{-\cos t}+3$$

$$\frac{d(\ln|y-3|)}{dt} = \sin t \quad \text{(the correct form)}$$

$$\int \frac{d}{dt} \ln|y-3| = \sin t$$

$$\ln|y-3| = -\cos t + C$$

$$y = Ce^{-\cos t} + 3$$

e.g.4. $\frac{dr}{d\theta} = \frac{r^2}{\theta}$, $r(1) = 2$

$$\frac{dr}{d\theta} \cdot \frac{1}{r^2} = \frac{1}{\theta}$$

$$\frac{d}{d\theta}\left(-\frac{1}{r}\right) = \frac{1}{\theta}$$

$$-\frac{1}{r} = \ln|\theta| + C$$

$$r = \frac{1}{-\ln|\theta| + C}$$

$2 = \frac{1}{c} \Rightarrow C = \frac{1}{2}$ $\quad \therefore r = \frac{1}{-\ln|\theta| + \frac{1}{2}}$

• Exercise: $\frac{dy}{dt} = \frac{ay+b}{cy+d}$

$$\frac{cy+d}{ay+b} dy = dt$$

$$\frac{A}{ay+b} + B$$

$c = B \cdot a \Rightarrow B = \frac{c}{a}$

$d = A + \frac{cb}{a}$ $\quad A = d - \frac{cb}{a}$

$$\int \frac{d - \frac{cb}{a}}{ay+b} dy + \int \frac{c}{a} dy = \int dt$$

$$\frac{d - \frac{cb}{a}}{a} \ln|ay+b| + \frac{c}{a} y = t \quad \text{(implicit)}$$

long division:

$$\begin{array}{r} \frac{c}{a} \\ ay+b \overline{) cy+d} \\ -(cy + \frac{cb}{a}) \\ \hline d - \frac{cb}{a} \end{array}$$

$$\Rightarrow \int \frac{c}{a} + \frac{d - \frac{cb}{a}}{cy+d} dy$$

HW1

Saturday, September 29, 2018 10:24 PM

A newly constructed fish pond contains 1000 liters of water. Unfortunately the pond has been contaminated with 5 kg of a toxic chemical during the construction process. The pond's filtering system removes water from the pond at a rate of 200 liters/minute, removes 20% of the chemical, and returns the same volume of (the now somewhat less contaminated) water to the pond. Write a differential equation for the time (measured in minutes) evolution of:

The total mass (in kilograms) of the chemical in the pond: $\frac{dm}{dt} = -\frac{m}{25}$ ✔ $-\frac{1}{25}m$

$m = \rho V \quad \frac{dm}{dt} = 20\% \cdot \rho \cdot \frac{dV}{dt} = \frac{m}{1000} \cdot 200 \cdot 20\% = \frac{1}{25}$

$m\downarrow \therefore \frac{dm}{dt} < 0 \quad \therefore \frac{dm}{dt} = \frac{1}{25} m$

Better : $\frac{dm}{dt}$ = rate in − rate out

$= (1-20\%) \cdot \frac{m}{1000} \cdot 200 - \frac{m}{1000} \cdot 200$

$= -\frac{m}{25}$

The concentration (in kg/liter) of the chemical in the pond: $\frac{dc}{dt} = -\frac{c}{25}$ ✔ $-\frac{1}{25}c$

$c = \frac{m}{1000}$ (volume doesn't change) $\frac{dc}{dt} = \frac{dm}{dt} \cdot \frac{1}{1000} = -\frac{1}{25} \cdot \frac{m}{1000}$

$= -\frac{1}{25}c$

$\therefore \frac{dc}{dt} = -\frac{1}{25}c$

the value of c changes

The concentration (in **grams/liter**) of the chemical in the pond: $\frac{dc}{dt} = -\frac{c}{25}$ ✔ $-\frac{1}{25}c$

The concentration (in **grams/liter**) of the chemical in the pond, but with time measured in **hours**: $\frac{dc}{dt} = -\frac{12c}{5}$ ✔ $-\frac{12}{5}c$

$-\frac{1}{25}c \cdot 60 = -\frac{12}{5}c.$

Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. If we measure temperature in degrees Celsius and time in minutes, the constant of proportionality k equals 0.5. Suppose the ambient temperature $T_A(t)$ is equal to a constant 59 degrees Celsius. Write the differential equation that describes the time evolution of the temperature T of the object.

(a) $\frac{dT}{dt} = \frac{1}{2}(59 - T)$ ✔

$\frac{dT}{dt} = k(T_A - T)$

Suppose the ambient temp $T_A(t) = 59\cos(\frac{\pi}{30}t)$ degrees Celsius (time measured in minutes). Write the DE that describes the time evolution of temperature T of the object.

(b) $\frac{dT}{dt} = \frac{1}{2}\left(59\cos\left(\frac{\pi}{30}t\right) - T\right)$ ✔

If we measure time in hours the differential equation in part (b) changes. What is the new differential equation?

$T_A = 59\cos(2\pi t)$

(c) $\frac{dT}{dt} = 30(59\cos(2\pi t) - T)$ ✔

$\frac{dT}{dt} = 60 \cdot \frac{1}{2}(59\cos(2\pi t) - T)$

If we measure time in hours and we also measure temperature in degrees Fahrenheit, the differential equation in part (c) changes even more. What is the new differential equation?

(d) $\frac{dT}{dt} = 30\left(\frac{531}{5}\cos(2\pi t) + 32 - T\right)$ ✔ $30\left(\frac{531}{5}\cos(2\pi\ t) + 32 - T\right)$

T value changes.

A spherical raindrop evaporates at a rate proportional to its surface area with (positive) constant of proportionality k; i.e. the rate of change of the volume exactly equals $-k$ times the surface area. Write differential equations for each of the quantities below as a function of time. For each case the right hand side should be a function of the dependent variable and the constant k. For example, the answer to the first question should not depend on S or r.

The volume of the drop: $\frac{dV}{dt} = -k(3V)^{\left(\frac{2}{3}\right)}(4\pi)^{\left(\frac{1}{3}\right)}$ ✔

(1) $V = \frac{4}{3}\pi r^3 \quad \frac{dV}{dt} = -kS = -k \cdot 4\pi r^2$

$\therefore r = \sqrt[3]{\frac{3V}{4\pi}} \quad = -k(3V)^{\frac{2}{3}}(4\pi)^{\frac{1}{3}}$

The radius of the drop: $\frac{dr}{dt} = -k$ ✔

(2) $(\frac{4}{3}\pi r^3)' = -k4\pi r^2$

$4\pi r^2 \frac{dr}{dt} = -k4\pi r^2$

$\frac{dr}{dt} = -k$

The surface area of the drop: $\frac{dS}{dt} = -4k\sqrt{\pi S}$ ✔

(3) $S = 4\pi r^2$

$\Rightarrow r = \sqrt{\frac{S}{4\pi}} \quad \frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt}$

$= -4k\sqrt{\pi S}$