Probs

Monday, February 25, 2019 8:57 PM

Give an example of each of the following. If it is not possible write "NOT POSSIBLE", and **give justification as to why**.

g) [2 pt] An $n \times n$ matrix $A \neq I_2$ such that $A^{2018} = I_2$, but $A^k \neq I_2$ for all $k < 2018$..

Take a rotation matrix for $\theta = \frac{2\pi}{2018}$, this looks like

$$C_{\frac{2\pi}{2018}} = \begin{bmatrix} \cos(\frac{2\pi}{2018}) & -\sin(\frac{2\pi}{2018}) \\ \cos(\frac{2\pi}{2018}) & sin(\frac{2\pi}{2018}) \end{bmatrix}$$

Since rotating through an angle of $\frac{2\pi}{2018}$, 2018 times will get you back where you started, $(C_{\frac{2\pi}{2018}})^{2018} = I_2$. Note that one can also just say $A = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$ where $x^{2018} = 1$.

h) [2 pt] A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that range(T)=ker(T).

Recall from class that when you look at a 2×2 matrix, the columns are reading off the image of the standard basis vectors. By rank-nullity, dim(range(T))+dim(ker(T))= 2, so range(T) and ker(T) must both be 1-dimensional. Lets pick a subspace and make it work. Picking the x or y axis will be easiest so lets pick the subspace to be span$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, the x-axis. If I want the range AND kernel to be this subspace, I must send $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, to get the kernel. Moreover, I must send the other vector to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, to get the range. The matrix that does this is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. To see it more explicitly, reaidng off from the columns of the matrix shows that e_1 will go to the zero vector, and e_2 will go to e_1 (because column 2 of the matrix is e_1. Now looking at any arbitrary vector, we can see that any vector of the form $\begin{bmatrix} a \\ 0 \end{bmatrix}$ will go to the zero vector, and any vector of the form $\begin{bmatrix} 0 \\ b \end{bmatrix}$ will go to $\begin{bmatrix} b \\ 0 \end{bmatrix}$. This means that anything on the x-axis goes to zero and anything on the y-axis goes to the x-axis, which is tha same as $ker(T) = range(T) = span\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Note: There are certainly other choices that work and just writing a matrix that works is sufficient for full credit.

Can't write $\vec{0}$, $\Rightarrow \because$ dim(ker(T)) = 2 $\neq$ dim(range(T)) = 0

3. Let $v = (1, 1, -1)$ and $L_v = \text{span}(\{v\})$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transform that is the projection onto L_v. This tells us 2 things about T:

- $T(x) = x$ if $x \in L_v$,
- $T(x) = 0$ if x is orthogonal to v (so if $x \cdot v = 0$).

There exists a matrix A such that $T(x) = Ax$. The goal of this problem is to determine A.

(a) (4 points) Give a basis for $\mathbb{R}^3$ that contains v and 2 vectors orthogonal to v. (Hint: Recall that $(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3$.)

$V \cdot \langle 1,0,-1\rangle = 0$

$V \cdot \langle 0,1,-1\rangle = 0$

-4

$\therefore \boxed{\text{basis} = \{V, \langle 1,0,-1\rangle, \langle 0,1,-1\rangle\}}$

(b) (4 points) Answer the following questions about A.

i. Give a basis for null(A).

$\text{null}(A) = \{\langle 1,0,-1\rangle, \langle 0,1,-1\rangle\}$

ii. Give a basis for col(A).

$\text{col}(A) = \{\langle 1,1,-1\rangle\}$

iii. What is the rank of A?

$\text{rank}(A) = 1$

iv. What is det(A)?

$\det(A) = 0$

(c) (4 points) What is A? You may express A as a product of matrices and their inverses.

$T(x) = Ax$

when $x = cV$,

$Ax = x$

$(A - I)x = 0$

$x_1 + x_2 - x_3 = 0$

$x_3 = x_1 + x_2$

assume $T(u_i) = V_i$ then $A = V \cdot U^{-1}$

p

$\because T\begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$

$T\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$T\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\therefore A = \begin{bmatrix}1&0&0\\1&0&0\\-1&0&0\end{bmatrix} \cdot \begin{bmatrix}1&1&0\\1&0&1\\-1&1&1\end{bmatrix}^{-1}$

5. Let A and B be equivalent matrices given by

$$A = \begin{bmatrix} 2 & 4 & -1 & -2 \\ -1 & -3 & -1 & 0 \\ 1 & 1 & 2 & 2 \\ 2 & 6 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

Let a_1, a_2, a_3, a_4 be the columns of A. Let $S = \text{span}(\{a_1, a_2\})$ and $T = \text{span}(\{a_3, a_4\})$.

(a) (2 points) What is $\dim(\text{span}(\{a_1, a_2, a_3, a_4\}))$?

$\dim(\text{span}\{a_1, a_2, a_3, a_4\}) = 3$

(b) (2 points) What is a basis for null(A)?

take $x_4 = s$, $x_3 = -s$ $x_2 = \frac{1}{2}s$ $x_1 = -\frac{1}{2}s$

$\left\{\begin{bmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ -1\\ 1\end{bmatrix}\right\}$

(c) (2 points) Denote that intersection of S and T by $S \cap T$. This is the subspace of vectors that are in span($\{a_1, a_2\}$) **and** in span($\{a_3, a_4\}$). What is $\dim(S \cap T)$?

$[a_3\ a_4] \Rightarrow \begin{bmatrix}-1&-2\\-1&0\\2&2\\2&0\end{bmatrix} \Rightarrow \begin{bmatrix}-1&0\\-1&2\\2&-2\\2&-4\end{bmatrix} \Rightarrow \begin{bmatrix}-1&0\\0&2\\1&-2\\0&-4\end{bmatrix}$

$\boxed{\dim(S \cap T) = 1}$

because $\text{rank}(A) = 3$, which means they have at least one overlap.

(d) (6 points) (Hard.) What is a basis for $S \cap T$?

$\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$

assume $v \in \text{span}\{a_1, a_2\}$,

(d) (6 points) (Hard.) What is a basis for $S \cap T$?

$\left\{ \cancel{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}} \right\}$

-6

assume $v \in \text{span}\{a_1, a_2\}$,
$\in \text{span}\{a_3, a_4\}$

$v = c_1 a_1 + c_2 a_2 = c_3 a_3 + c_4 a_4$

$\therefore c_1 a_1 + c_2 a_2 - c_3 a_3 - c_4 a_4 = 0$

$\therefore \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = c \in null(A)$

$\therefore c \in \left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 1 \end{bmatrix} \right\}$

6

$\therefore v = -\frac{1}{2} a_1 + \frac{1}{2} a_2 \Rightarrow$ plug in $c_1 = -2, c_2 = 2$

$\therefore$ basis $= \{a_1 - a_2\}$

2. Give an example of each of the following. If it is not possible, write "NOT POSSIBLE".

(a) (3 points) Give an example of 2 linear transforms $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T \circ S : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible.

Solution: Let $T(x, y, z) = (x, y)$ and $S(x, y) = (x, y, 0)$. Then $(T \circ S)(x, y) = (x, y)$ which is the identity transform which is invertible.

$\because T, S$ linear

$\therefore T = Ax$, A is 2×3

$S = By$, B is 3×2

$T \circ S = AB \cdot y$

$\because T \circ S$ invertible

$\therefore AB$ must be invertible.

assume $T \circ S = AB \cdot y = z$

$\Rightarrow y = (AB)^{-1} z$ exists.

assume $(AB)^{-1} = C$

then $AB \cdot C = I$

$\Rightarrow A(BC) = I$

A must be invertible.

$\therefore$ Impossible.

Because C is 2×2 while B 3×2.

1. (16 points) For each of the following statements, determine if they are **TRUE** or **FALSE**. If the statement if FALSE, provide a counter example. If the statement is TRUE, no justification is needed. (An intuitive explanation of why something is false is **NOT** a counterexample).

(a) (4 points) Suppose $T : \mathbb{R}^8 \to \mathbb{R}^8$ is linear. Then $\dim(\ker(T \circ T)) \geq \dim(\ker(T))$.

assume $T(x) = A^2x$

True

$Ax = 0$

$A(Ax) = 0$

$Ax \in \ker L$ or $Ax = 0$

(b) (4 points) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear and satisfies $T \circ T = 0$, then $\dim(\ker(T)) \leq \dim(\text{range}(T))$

False. $\leftarrow$ $T(x) = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$T(x) = Ax$

$T \circ T = A^2x = 0$

$A(Ax) = 0$

$A(\text{range}) = 0$

for every Ax, $x \in \mathbb{R}^3$

$\because A(Ax) = 0$ $\therefore Ax \in \ker(T)$ $\therefore \ker(T) > r$

$\therefore \dim(\ker(T)) \geq \dim(r$

(c) (4 points) Suppose A, B are $n \times m$ matrices. Then $\text{rank}(A+B) \geq \min\{\text{rank}(A), \text{rank}(B)\}$

False.

$A+B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\text{rank}(A+B) = 1$

$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\text{rank}(A) = \text{rank}(B) = 2$

(d) (4 points) Suppose $T : \cancel{\mathbb{R}^4 \to \mathbb{R}^5}$ is linear, ~~one-to-one~~ and $T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}$. Then

$T: \mathbb{R}^5 \to \mathbb{R}^4$ onto

$T \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \neq \vec{0}$.

~~False~~.

$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \cancel{=} \begin{pmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{pmatrix}$

not onto

True

$\because$ onto

$\therefore \dim(\text{range}(T)) = 4$

$\therefore \dim(\ker(T)) = 1$

$\therefore T \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \neq 0$

4. (16 points) For each of the following situations, state whether it is **Possible** or **Impossible.** For each case that it is **Possible**, give an example that meets the specifications. If it is **Impossible** no justification is needed.

rank(A) = 3

(a) (4 points) An invertible 3×3 matrix A, such that $\text{rank}(A^2) = 2$.

Impossible.

$A^2 \cdot A^{-1} \cdot A^{-1} = I$

A^2 invertible.

(b) (4 points) A 2×2 matrix A such that $\text{rank}(A) = \text{rank}(A^2)$, but $\text{rank}(A) \neq \text{rank}(A^3)$.

Impossible.

(c) (4 points) Subspaces V and W of $\mathbb{R}^3$ and vectors $\{v_1, v_2, v_3\}$, such that $\{v_1, v_2\}$ is a basis of V and $\{v_2, v_3\}$ is a basis of W, but $\{v_1, v_2, v_3\}$ is **NOT** a basis of $\mathbb{R}^3$.

V=W?

Possible.

$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

at this time $V = W = \mathbb{R}^2$

(d) (4 points) A 3×3 matrix A such that $A^2 \neq 0$ but $A^3 = 0$.

$A \neq 0, A^2 \neq 0$

Possible

$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$

$A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$