Monday, October 1, 2018 3:27 PM

- **Basic Models** $\frac{dy}{dt} = ay + b$
  - Falling Objects

    $F = ma = m \cdot \frac{dv}{dt} = kv - mg$
  - Population

    $\frac{dP}{dt} = rP - P_0$

    $\frac{dP}{dt} = $ rate pop increases $-$ rate pop decreases
  - Radioactive Decay

    $\frac{dQ}{dt} = -rQ$

    τ = half-life

    $\ln|Q| = -rt + C$

    $Q = Ce^{-rt}$

    $t = 0,\ Q = C,\quad Q = \frac{C}{2}$

    $\frac{C}{2} = e^{-rt}$

    $\ln\frac{\cancel{C}\,1}{2} = -rt$

    $\tau = t = -\frac{\ln\frac{\cancel{C}\,1}{2}}{r} = \frac{\ln 2}{r}$ (C the starting amount of chemical)

Eg. substitution

$\frac{dy}{dx} = \frac{y - 4x}{x - y}$

$V = \frac{y}{x}$, $y = Vx$

$\frac{dy}{dx} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}}$

$\frac{dy}{dx} = \frac{V-4}{1-V}$

$\frac{dy}{dx} = \frac{dV}{dx} \cdot x + V = \frac{V-4}{1-V}$

$\frac{dV}{dx} x = \frac{2V-5}{1-V}$

$\int \frac{1-u}{2u-5} du = \ln x$

$\int \left(\frac{A}{2u-5} + B\right) = \ln|x|$ $\quad \begin{cases} A - 5B = 1 \\ 2B = -1 \end{cases}$ $\quad \begin{cases} A = -\frac{3}{2} \\ B = -\frac{1}{2} \end{cases}$

$-\frac{3}{4}\ln|2u-5| - \frac{u}{2} = \ln|x| + C$

2 Direction Field

Wednesday, October 3, 2018 3:30 PM

- Warm-up

$\frac{dy}{dx} = \frac{3y-x}{x+y}$; $y(1)=0$

$\frac{dy}{dx} = \frac{\frac{3y}{x}-1}{1+\frac{y}{x}}$ $\quad u=\frac{y}{x} \Rightarrow y=ux$

$\frac{dy}{dx} = \frac{du}{dx}\cdot x+u$

$\frac{du}{dx}x+u = \frac{3u-1}{1+u}$

$\frac{du}{dx}x = \frac{-u^2+2u-1}{1+u} = \frac{-(u-1)^2}{1+u}$

$\frac{-(1+u)}{(u-1)^2}du = \frac{dx}{x}$ $\quad = \int u\cdot d\frac{1}{u-1}$

$\int\frac{-1}{(u-1)^2}du - \int\frac{u}{(u-1)^2}du = \ln|x|+C$

$(u-1)^{-1} + \frac{1}{u-1}\cdot u - \int\frac{1}{u-1}\cdot du$

$= \frac{u+1}{u-1} - \ln|u-1| = \ln|x|+C$

$\therefore C=0$

Solution 2.

let $v=u-1$, $\therefore u=1+v$

$\int\frac{-1}{v^2}dv - \int\frac{v+1}{v^2}dv = \ln|x|+C$

$\int\frac{-1}{v^2}dv - \int(\frac{1}{v}+\frac{1}{v^2})dv = \ln|x|+C$

$\frac{2}{v} - \ln|v| = \ln|x|+C$

$C=-2$.

there can be two Cs

- Direction Fields

- 1^{st} DE, $\frac{dy}{dx}=F(x,y)$

e.g. $\frac{dy}{dx}=x+y$; $y(1)=1$

slope at $(1,1)$

$\frac{dy}{dx}=2$ $\qquad \frac{dy}{dx}<0 \Rightarrow x+y<0$

do this for every point.

← Direction field = Vector field

- Autonomous DE : $\frac{dy}{dx}=F(y)$

Autonomous DE often has constant solution.

e.g. $\frac{dy}{dx}=y(1-y)$

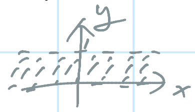
$\frac{dy}{dx}=0=y(1-y)$ $\quad (\because y=c)$

e.g. $\frac{dy}{ds} = y(1-y)$

$\frac{dy}{ds} = 0 = y(1-y) \quad (\because y = c)$

$y = 0$ or $y = 1$

- Equilibrium solution : the constant solution



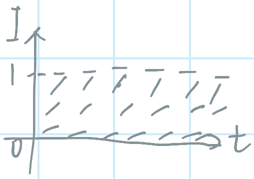
- Model : Infection

r = probability of spread.

possible encounters = % infected · % healthy

$I + H = 1$ %Infected : I.

$\frac{dI}{dt} = r \cdot I(1-I)$



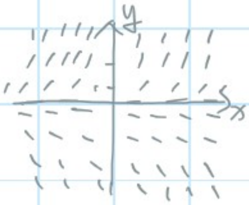
- Model : Stability of Equilibrium Solution

- Model : Movement

Euler's Method

Friday, October 5, 2018 3:29 PM

- Warm up: Sketch the direction field of $\frac{dy}{dx}=y$ & classify equilibrium solution(s) as stable / unstable



$y=0$ unstable solution

- Euler's Method

Given $\frac{dy}{dt}=F(t,y)$, $y(t_0)=y_0$

To approximate a solution:

I. Choose a step size.

II. Approximate linearly using the tangent line.

III. Repeat.

use $y=y_0+\frac{dy}{dx}(x-x_0)$

$=y_0+\frac{dy}{dx}\cdot h$ h: step size.

e.g. 1. $y'=y$; $y(0)=1$. step size: 0.5, Approximate on $[0,1]$

x	0	0.5	1
y	1	$\frac{3}{2}$	$\frac{9}{4}$
$\frac{dy}{dt}$	1	$\frac{3}{2}$	

$y=1+x$, $x=0.5$

$y=\frac{3}{2}+\frac{3}{2}(x-\frac{1}{2})$ $x=1$

e.g. 2. step size $=\frac{1}{3}$

Approximate $y'=y$; $y(0)=1$, on $[0,1]$

x	0	$\frac{1}{3}$	$\frac{2}{3}$	1
y	1	$\frac{4}{3}$	$\frac{16}{9}$	$\frac{64}{27}\approx 2.370$.
$\frac{dy}{dt}$	1	$\frac{4}{3}$	$\frac{16}{9}$	$\frac{64}{27}$

$y-1=1\cdot(x-0)$, $x=\frac{1}{3}$

dt 1   3   9   27

$y - 1 = 1 \cdot (x - 0)$, $x = \frac{1}{3}$

$y = \frac{4}{3} + \frac{4}{3}(x - \frac{1}{3})$ $x = \frac{2}{3}$

$y = \frac{16}{9} + \frac{16}{9}(x - \frac{2}{3})$ $x = 1$

e.g. 3. Step Size $= \frac{1}{4}$, approximate a solution

$\frac{dy}{dx} = x^2 + y^2$ on $[1, 2]$, $y(1) = 2$.

x	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$
y	2	$\frac{13}{4}$	$\frac{151}{32}$	
$\frac{dy}{dx}$	5	$\frac{194}{16}$	$\frac{22801}{1024} + \frac{9}{4}$	

$y = 2 + 5(x-1)$, $x = \frac{5}{4}$

$y = \frac{13}{4} + \frac{\cancel{194}^{97}}{16} \times \frac{1}{4_{2}} = \frac{108}{64} + \frac{194}{64} = \frac{\cancel{302}^{151}}{\cancel{64}_{32}}$

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