Linear 1st order ODEs

Wednesday, October 3, 2018 3:30 PM

- $\frac{dy}{dt} + p(t)\cdot y = g(t)$ (Linear)

  Linear first order ODE

- def of Linear DE:

  $a_0(x)\cdot y + a_1(x)y' + \cdots + a_n(x)y^{(n)} + b(x) = 0$

  is linear. (y n-step differentiable)

  (a are arbitrary functions)

- e.g. 1. $(t^2+1)y' + 2ty = t$

  $\frac{d}{dt}((t^2+1)\cdot y) = t$

  $\int \frac{d}{dt}((t^2+1)\cdot y)\,dt = \int t\,dt$

  $(t^2+1)\cdot y = \frac{1}{2}t^2 + C$

  $y = \frac{\frac{1}{2}t^2 + C}{t^2+1}$

- e.g. 2. $\frac{dy}{dt} + \frac{1}{2}y = 4$

  $\mu(t)\cdot\frac{dy}{dt} + \frac{1}{2}y\mu(t) = 4\mu(t)$ if multiply by a $\mu(t)$,

  $\therefore \frac{d\mu}{dt} = \frac{1}{2}\mu(t)$ can transform it into: $\frac{d}{dt}(\mu(t)\cdot y) = \mu\cdot\frac{dy}{dt} + \frac{d\mu}{dt}\cdot y$

  $\ln|\mu(t)| = \frac{1}{2}t + C$ μ: Integrating factor

  $\mu = Ce^{\frac{t}{2}}$ choose $C=1$, $\mu = e^{\frac{t}{2}}$

  $e^{\frac{t}{2}}\cdot y = \int 4e^{\frac{t}{2}}\,dt = 8e^{\frac{t}{2}} + C$

  $y = 8 + C\cdot e^{-\frac{t}{2}}$

- Steps: $\frac{dy}{dt} + p(t)y = g(t)$

I. Multiply by $\mu(t)$, make DE $\frac{d}{dt}(\mu\cdot y) = \frac{d\mu}{dt}\cdot y + \frac{dy}{dt}\cdot\mu$

II. Find μ

$\mu(t)p(t) = \frac{d\mu}{dt}$ $\quad \ln|\mu| = \int p(t)\,dt$

III. Integral.

$\mu = e^{\int p(t)dt}$ $\quad \frac{\int g(t)\,dt}{e^{\int p(t)dt}} = y$

- e.g. 3. $y' + ty = 0$; $y(0) = 1$

  $\frac{d\mu}{dt} = \mu\cdot t$

  $\ln|\mu| = \frac{t^2}{2} + C$

  let $C = 0$

let $C=0$

$\mu = e^{\frac{t^2}{2}}$

$e^{\frac{t^2}{2}} \cdot y = C$

$y = Ce^{-\frac{t^2}{2}}$

$\because y(0)=1 \quad \therefore C=1, \quad \therefore y = e^{-\frac{t^2}{2}}$

- Steps in general:

I. $\frac{dy}{dt} + p(t)\cdot y = g(t)$

choose: $\mu(t) = e^{\int p(t)\,dt}$

II. $\frac{d}{dt}\cdot(e^{\int p(t)\,dt}\cdot y) = g(t)e^{\int p(t)\,dt}$

$y = \frac{1}{\mu(t)}\cdot\int g(t)\mu(t)\,dt + \frac{C}{\mu(t)}$

- e.g.4. $t\cdot\frac{dy}{dt} + y = t^{-2}$; $y(1)=1$

$\frac{dy}{dt} + t^{-1}y = t^{-3}$

$\mu(t) = e^{\int t^{-1}dt} = t$

$t\cdot y = \int t^{-2}\cdot dt = -t^{-1} + C$

$y = -t^{-2} + Ct^{-1}$

$\because y(1)=1 \quad \therefore y = -t^{-2} + 2t^{-1}$

- e.g.5. $\frac{dy}{dt} + y = \cos t$, $y(0)=1$

$\mu(t) = e^{\int dt} = e^t$

$e^t\cdot y = \int e^t\cos t\,dt =$

$= e^t\cdot\sin t - \int \sin t\,d(e^t)$

$= e^t\cdot\sin t + \cos t\cdot e^t - \int \cos t\cdot e^t\,dt$

$\therefore \int e^t\cos t\,dt = \frac{e^t\cdot\sin t + \cos t e^t}{2}$

$y = \frac{\sin t + \cos t}{2} + C$

u	dv
e^t $+$	$\cos t$
e^t $-$	$\sin t$
e^t	$-\cos t$

$e^t\cdot\sin t + e^t\cdot\cos t - \int \cos t\, e^t\,dt$

- Trick : Tabular Integration

$\int x^5 \cos x \, dx$

derivative → u		dv ← integral
x^5	+	$\cos x$
$5x^4$	−	$\sin x$
$20x^3$	+	$-\cos x$
$60x^2$	−	$-\sin x$
$120x$	+	$\cos x$
120	−	$\sin x$
0		$-\cos x$

$$\int x^5 \cos x \, dx = x^5 \cdot \sin x + 5x^4 \cos x$$

$$-20x^3 \cdot \sin x - 60x^2 \cos x$$

$$+120x \sin x + 120 \cos x$$

HW3

Tuesday, October 9, 2018 12:58 AM

In an isolated environment, a disease spreads at a rate proportional to the product of the infected and non-infected populations. Let I(t) denote the number of infected individuals. Suppose that the total population is 2000, the proportionality constant is 0.0001, and that 1% of the population is infected at time t=0. Write down the intial value problem and the solution I(t).

$\frac{dI}{dt} = \boxed{0.0001I(2000-I)}$ ✓

$\frac{dI}{I(2000-I)} = 0.0001\,dt$

$\frac{A}{I} + \frac{B}{2000-I} = \frac{1}{I(2000-I)}$

$2000A = 1, \therefore A = B = \frac{1}{2000}$

I(0) = 20 ✓

- I(t) = $\boxed{\frac{2000e^{0.2t}}{99+e^{0.2t}}}$ ✓ $\boxed{\frac{2000}{1+99\cdot e^{-0.2t}}}$

$\therefore \left(\frac{1}{I} + \frac{1}{2000-I}\right)dI = 10^{-4}\times 2\times 10^{3} = 0.2\,dt$

$\ln|I| - \ln|2000-I| = 0.2t + C$

$\ln\left|\frac{I}{2000-I}\right| = 0.2t + C$

$\frac{I}{2000-I} = Ce^{0.2t}$ $\because I(0) = 20 \therefore C = \frac{1}{99}$

$I = \frac{2000}{99}e^{0.2t} - \frac{e^{0.2t}}{99}I$

$I = \frac{2000e^{0.2t}}{99+e^{0.2t}}$

The rate of change of the downward velocity of a falling object is the acceleration of gravity (10 meters/sec^2) minus the acceleration due to air resistance. Suppose that the acceleration due to air resistance is 0.2 inverse seconds times the downward velocity.

Write the initial value problem and the solution for the downward velocity for an object that is dropped (not thrown) from a great height.

$\frac{dv}{dt} = \boxed{10-0.2v}$ ✓

v(0) = 0 ✓

- v(t) = $\boxed{50-50e^{-\left(\frac{t}{5}\right)}}$ ✓

what is the terminal velocity? 50 ✓ meters/second

How long before the object reaches 90% of terminal velocity? -5*ln((✓ seconds

How far has it fallen by that time? -250*ln(0.1)-225 ✓ 350.6463 meters $t=0,\ y=250.$

A tank holds 1000 liters of water in which 100 grams of salt have been dissolved. Saltwater with a concentration of 1 grams/liter is pumped in at 10 liters/minute and the well mixed saltwater solution is pumped out at the same rate. Write initial the value problem for:

a) The mass (i.e number of grams) of salt in the tank at time t.

$\frac{dS}{dt} = \boxed{10-\frac{S}{100}}$ ✓

S(0) = 100 ✓

- The solution is: S(t) = $\boxed{1000-900e^{-\left(\frac{t}{100}\right)}}$ ✓

b) The concentration, in grams/liter, of salt in the tank at time t (this should be very easy).

$\frac{m}{V} = \frac{C}{100} = \frac{10}{10^3} - \frac{C}{100}$

$= 0.01 - \frac{C}{100}$

$\frac{dC}{dt} = \boxed{10-\frac{C}{100}}$ ✗ $\boxed{0.01-0.01C}$

C(0) = 0.1 ✓

The solution is: C(t) = $\boxed{0.01-0.009e^{t}}$ ✗ $\boxed{1-0.9e^{-0.01\cdot t}}$

Determine a differential equation that models the growth of a population of fish as a function of time in days under each of the following hypotheses:

a) The rate of population increase is proportional to the size of the population. The population increases by 2 percent per day. (Treat time in days as a continuous variable, i.e. the rate at which the population increases is .02 times the population size.)

$\frac{dP}{dt} = \boxed{0.02P}$ ✓

- b) The rate of population increase is again proportional to the size of the population with the same constant of proportionality but 5 percent of the population is harvested each day.

$\frac{dP}{dt} = \boxed{-0.03P}$ ✓

c) The ~~rate of population~~ increase is again proportional to the size of the population with the same constant of proportionality but 4000 fish are harvested each day.

$\frac{dP}{dt} = \boxed{0.02P-4000}$ ✓

d) The equation in part c) has a threshhold. What is it? 20000 ✓

The two equations below express conservation of energy and conservation of mass for water flowing from a circular hole of radius 3 centimeters at the bottom of a cylindrical tank of radius 20 centimeters. In these equations, Δ m is the mass that leaves the tank in time Δt, v is the velocity of the water flowing through the hole, and h is the height of the water in the tank at time t. g is the accelertion of gravity, which you should approximate as 1000 cm/s^2.

The first equation says that the gain in kinetic energy of the water leaving the tank equals the loss in potential energy of the water in the tank. $\Rightarrow v = \sqrt{2gh}$

$\frac{1}{2}\Delta m\, v^2 = \Delta m\, g\, h$

the sign!

The second equation says that the rate at which water leaves the tank equals the rate of decrease in the volume of water in the tank (which is conservation of mass because water has constant density).

$\pi\, 20^2 \frac{dh}{dt} = \pi\, 3^2\, v$

Derive a differential equation for the height of water in the tank.

$\frac{dh}{dt} = -\frac{3^2}{20^2}v = -\frac{9}{400}\cdot\sqrt{2000h}$

- $\frac{dh}{dt} = \boxed{-9\cdot\frac{\sqrt{2000h}}{400}}$ ✓

$h(0) = 30$

If the initial height of the water is 30 centimeters, find a formula or the solution

h(t) = $\boxed{\left(-\frac{9\sqrt{5}}{40}t+\sqrt{30}\right)^2}$ ✓

According to the model, how long does it take to empty the tank? $\boxed{\frac{40\sqrt{6}}{9}}$ ✓ seconds $h(t) = 0$

$h = w^2$

Another way to solve this differential equation is to make the substitution w = $\sqrt{h}$. What is the differential equation that w satisfies?

$\frac{dh}{dt} = 2w\cdot\frac{dw}{dt}$ $w \neq 0$

$\frac{dw}{dt} = \boxed{-9\cdot\frac{\sqrt{2000}}{800}}$ ✓

$\therefore \frac{dw}{dt} = \frac{1}{2w}\cdot\frac{dh}{dt}$

Existence & Uniqueness

Wednesday, October 10, 2018 3:28 PM

(General)

- Theorem I: $y' = f(t,y)$; $y(t_0) = y_0$. If f and $\frac{\partial f}{\partial y}$ are continuous around (t_0, y_0), then there exists a unique solution $y = \phi(t)$, satisfying the IVP.

- Theorem II (Linear): $y' + p(t)y = g(t)$; $y(t_0) = y_0$.

  $y' = \underbrace{g(t) - p(t) \cdot y}$

  If $f(x,y) = g(t) - p(t) \cdot y$ and $\frac{\partial f}{\partial y} = -p(t)$ are continuous

  $f(x,y)$ is continuous in Theorem I.

  $-p(t)$ continuous
  $\Downarrow$
  $-p(t) \cdot y$ continuous.
  $\Downarrow$
  $p(t) \cdot y$ continuous

  $f(x,y) + p(t)y$ cont
  $\parallel$
  $g(t)$ cont.

  - $\exists$ exist $\quad$! unique

    $\exists!$ $y = \phi(t)$, satisfying the IVP if and only if $p(t)$ & $g(t)$ are continuous.

- e.g.1. $\frac{dy}{dt} + t \cdot y = 1$; $y(0) = 1$

  (a) Solve IVP.

  (b) Check uniqueness.

  (a) $\frac{d\mu}{dt} = \mu t$

  $\mu = Ce^{\frac{t^2}{2}} = e^{\frac{t^2}{2}}$

  $e^{\frac{t^2}{2}} \cdot y = \int e^{\frac{t^2}{2}} dt$

  (b) $p(t) = -t$, $g(t) = 1$

  Both g, p are continuous.

  $\therefore$ There are unique solution around 0.

- e.g.2. $\frac{dy}{dt} = \frac{1}{t}$; $y(0) = 1$

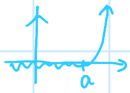
  $g(t) = \frac{1}{t}$, not continuous at $t = 0$.

• e.g.2. [illegible] dt [illegible] t , $y(0)$ [illegible]

$g(t) = \frac{1}{t}$, not continuous at $t=0$.

e.g.3. $\frac{dy}{dt} = \sqrt{2|y|}$, $y(0) = 0$

solved: $y_a(t) = \begin{cases} \frac{(t-a)^2}{2} & t \geq a \\ 0 & t \leq a \end{cases}$



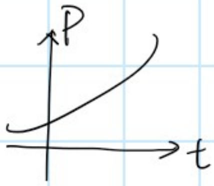
• Picard's Theorem

Population Dynamics

Wednesday, October 10, 2018 3:55 PM

- Model I:

  $\frac{dP}{dt} = rP$, $P(t) = P_0 e^{rt}$



- Model II. Logistic Equation

  $\frac{dy}{dt} = h(y) \cdot y$

  $h(y) \sim r$, when y is small — some constant

  but $h \to 0$ as y gets large

  r: intrinsic rate of growth

  K: carrying capacity

  e.g. $\frac{dy}{dt} = r(1 - \frac{y}{K})y$

  $h(y) = r \cdot (1 - \frac{y}{K})$

  $\lim_{y \to 0} h(y) = r.$

  $\lim_{y \to K} h(y) = 0.$

- Equilibrium Solutions:

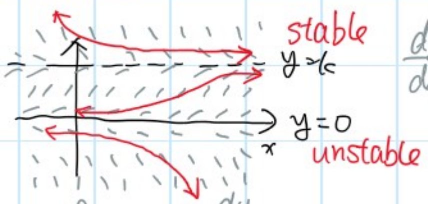
  $r(1 - \frac{y}{K})y = 0$

  $1 - \frac{y}{K} = 0$ or $y = 0$

  $y = K$ or $y = 0$

- Classification of Equilibrium Solution

  - Phase line



$\frac{dy}{dx} = r(1 - \frac{y}{K})y$

unstable

if $y<0$, $\frac{dy}{dt}<0$

if $0<y<K$, $\frac{dy}{dt}>0$

if $y>K$, $\frac{dy}{dt}<0$

Only need to draw phase line.

e.g. 2. $\frac{dy}{dt} = -r(1-\frac{y}{T})(1-\frac{y}{K})\cdot y$ T: Thredshold Population

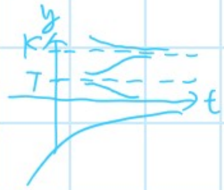
$\begin{Bmatrix} 0<T<K \\ 0<r \end{Bmatrix}$ Find & classify equilibrium solution.

$y<0$ $- + + -$ $\frac{dy}{dt}>0$

$0<y<T$ $- + + +$ $\frac{dy}{dt}<0$

$T<y<K$ $- - + +$ $\frac{dy}{dt}>0$

$y>K$ $- - - +$ $\frac{dy}{dt}<0$



$y=K$: stable

$y=T$: unstable

$y=0$: stable.

e.g. $\frac{dy}{dt} = -(y-60)^2\cdot y$

$y=0$ or $y=60$

$y<0$ $\frac{dy}{dt}>0$

$0<y<60$ $\frac{dy}{dt}<0$

$y>60$ $\frac{dy}{dt}<0$

semi-stable

stable

- Exercise:

Logistic model of fish

I. Assume that they're harvested proportional to the population, r, N, and K.

Find & classify equilibrium solution.

$\frac{dN}{dt} = r\cdot(1-\frac{N}{K})\cdot N$

$$\frac{dN}{dt} = r \cdot \left(1 - \frac{N}{K}\right) \cdot N$$

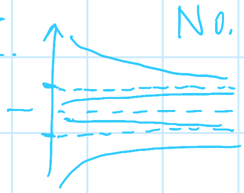
$N = 0$ or $N = K$.

- Exercise 2. Give an example of an autonomous DE with exactly 2 equilibrium solution that are both:

  I. Stable

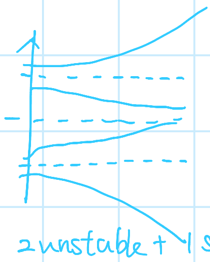
  II. Unstable

  III. Semistable

I. 

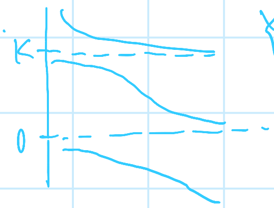
No.

2 stable + 1 unstable

II. 

No.

2 unstable + 1 stable

III. 

Yes.

$$-(K-y)^2 y^2 = \frac{dp}{dt}$$

Modeling

Friday, October 12, 2018 3:26 PM

- Types of Models:
  - Population / Radioactive Decay / Compound Interest

    $\frac{dP}{dt} = rP$. $\frac{dQ}{dt} = -kQ$ $\frac{ds}{dt} = rS$; $\frac{dy}{dt} = k \cdot y$

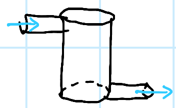
  - Population Dynamics / Logistic Equation

    $\frac{dy}{dt} = r(1 - \frac{y}{K})y$ K : carrying capacity

    r : intrinsic growth rate

  - ~~X~~ Mixture Problems (can be linear or separable)

    Solution $\frac{dQ}{dt}$ = Rate in − Rate out

    Q(t) : amount of chemical

  - Force / Motion

    $F = ma = m \cdot \frac{dv}{dt}$

    e.g. Falling object

    - no air resistance

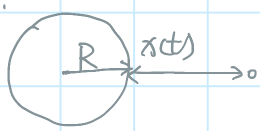
      $mg = m \cdot \frac{dv}{dt}$

    - air resistance

      (if drag ∝ velocity)

      $m \cdot \frac{dv}{dt} = mg - kv$.

    e.g.

    $F = -\frac{mR^2}{(R+x)^2}$

    $m \cdot \frac{dv}{dt} = \frac{-mR^2}{(R+x)^2}$

    $m \frac{dv}{dt} \cdot \frac{dx}{dx} = m \cdot v \cdot dv = -\frac{mR^2}{(R+x)^2} dx$

  - Infected / Healthy

    $\frac{dI}{dt} = r \cdot I \cdot H$, $H + I = P$ ← Population

    $= r \cdot I (P - I)$

  - Fish Harvesting Model.
  - Newton's Law of Cooling

    $\frac{dT}{dt} = -k(T - T_A)$

- **Schaefer Model (Fish harvesting model)**

If an undistributed population of fish grows w.r.t the DE:

$\frac{dy}{dt} = r(1-\frac{y}{K})y$. and we assume that the rate of harvesting is proportional to the population,

(a) Find a DE modeling the population whilst being harvested.

(b) Find & classify Equilibrium Solutions. $(0<E<r)$

if it's not indicating, need to clarify

↓

but population can't be negative!

(a) $\frac{dy}{dt}$ = rate in − rate out $= r(1-\frac{y}{K})y - Ey$

(b) Eq slns:

$r(1-\frac{y}{K})y - Ey = 0$

$ry(1-\frac{y}{K} - \frac{E}{r}) = 0 \Rightarrow ry(K(1-\frac{E}{r}) - y) = 0$

$y=0$ or $y = -K(\frac{E}{r}-1)$

Phase line

$K(1-\frac{E}{r})$ $0<E<r$ unstable

0 stable

- **Newton's Law of Cooling**

$\frac{dT}{dt} = -k(T - T_A)$

Assume $T(t)$ = temperature of a building

Assume $T_A(t)$ oscillates sinusoidally

$T_A(t) = T_0 + T_1\cos(\omega t)$

Assume $\omega = \frac{\pi}{12}$, 24hr cycle (t hrs)

$T_0 = 60\,°F$ $T_1 = 15°F$, $k = 0.2/hr$

(a) Solve for $T(t)$

(Bonus)

(b) Write $T(t) = \tau(t) + S(t)$

Steady State Solution ($S(t)$)

Transient Solution ($\tau(t)$)

$$\lim_{t\to\infty} \tau(t) = 0$$

(a) $T_A(t) = 60 + 15\cos\frac{\pi}{12}t$. $\quad T(\text{period}) = \frac{2\pi}{\frac{\pi}{12}} = 24$

$$\frac{dT}{dt} = -k(T - T_A) = -\frac{1}{5}T + \frac{1}{5}(60 + 15\cos\frac{\pi}{12}t)$$

$$\frac{dT}{dt} + \frac{1}{5}T(t) = \frac{1}{5}(60 + 15\cos\frac{\pi}{12}t)$$

$$\mu(t) = e^{\int\frac{1}{5}dt} = e^{\frac{1}{5}t}$$

$$T(t) = \frac{1}{e^{\frac{t}{5}}} \cdot \int e^{\frac{t}{5}}(60 + 15\cos\frac{\pi}{12}t)\,dt$$

$$= \frac{1}{e^{\frac{t}{5}}} \cdot \left(300e^{\frac{t}{5}} + \int e^{\frac{t}{5}}\, 15\cos\frac{\pi}{12}t\,dt\right)$$

$$= 300 + \frac{15}{e^{\frac{t}{5}}} \cdot \underline{\int e^{\frac{t}{5}} \cdot \cos\frac{\pi}{12}t \cdot dt}$$

$$= \frac{12}{\pi} \cdot \sin(\frac{\pi}{12}t)\, e^{\frac{t}{5}} - \frac{12}{5\pi}\int \sin\frac{\pi}{12}t \cdot e^{\frac{t}{5}}\,dt$$

$$= \frac{12}{\pi} \cdot \sin(\frac{\pi}{12}t) \cdot e^{\frac{t}{5}} - \frac{12}{5\pi}\left(\frac{12}{\pi}\cos\frac{\pi}{12}t \cdot e^{\frac{t}{5}} + \frac{12}{5\pi}\int \cos\frac{\pi}{12}t \cdot e^{\frac{t}{5}}\right)dt$$

$$= \frac{\frac{144}{5\pi^2}\cos\frac{\pi}{12}t \cdot e^{\frac{t}{5}} + \frac{12}{\pi}\sin(\frac{\pi}{12}t)\, e^{\frac{t}{5}}}{1 + \frac{144}{25\pi^2}}$$

$$= 300 + \frac{\frac{3\times 144}{\pi^2} \cdot \cos\frac{\pi}{12}t + \frac{12\times 15}{\pi}\sin(\frac{\pi}{12}t)}{1 + \frac{144}{25\pi^2}} + Ce^{-\frac{1}{5}t}$$

$\cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

(b) $T(t) = \underset{\text{Steadstate}}{\underset{\uparrow}{S(t)}} + \underset{\text{Transient}}{\underset{\uparrow}{\tau(t)}}$.

$$Ce^{-\frac{t}{5}} \to 0$$