Probs

Thursday, March 14, 2019 9:08 PM

(5) In this problem, $\vec{v}_1$ and $\vec{v}_2$ are vectors in $\mathbb{R}^n$ with

$$\vec{v}_1 \cdot \vec{v}_1 = 3, \quad \vec{v}_2 \cdot \vec{v}_2 = 5, \quad \text{and} \quad \vec{v}_1 \cdot \vec{v}_2 = 3.$$

(a) [5 points] Find the values of a so that $2\vec{v}_1 + a\vec{v}_2$ is orthogonal to $a\vec{v}_1 + 3\vec{v}_2$.

$(2v_1 + av_2) \cdot (av_1 + 3v_2) = 0$

$2av_1^2 + 6v_1v_2 + a^2v_1v_2 + 3av_2^2 = 0$

$6a + 18 + 3a^2 + 15a = 0$

$3a^2 + 21a + 18 = 0$

$a^2 + 7a + 6 = 0$

$(a+1)(a+6) = 0$

$\boxed{a = -1, -6}$

(b) [5 points] Find vectors $\vec{v}_1, \vec{v}_2$ in $\mathbb{R}^2$ satisfying the equations above.

$V_1 = \langle x_1, y_1 \rangle \quad V_2 = \langle x_2, y_2 \rangle$

$$\begin{cases} x_1^2 + y_1^2 = 3 \\ x_2^2 + y_2^2 = 5 \\ x_1x_2 + y_1y_2 = 3 \end{cases}$$

let $x_1 = 0,\ y_1 = \sqrt{3}$

$$\begin{cases} \sqrt{3}y_2 = 3 \\ x_2^2 + y_2^2 = 5 \end{cases}$$

$\therefore y_2 = \sqrt{3},\ x_2 = \sqrt{2}$

$V_1 = \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix},\ V_2 = \begin{bmatrix} \sqrt{2} \\ \sqrt{3} \end{bmatrix}$

(c) [5 points] *Either* give an example of $\vec{v}_1, \vec{v}_2$ (in any $\mathbb{R}^n$) that are linearly dependent, *or* explain why they must be independent.

$V_1 \cdot V_1 - V_1 \cdot V_2 = 0$

$V_1 \cdot (V_1 - V_2) = 0$

V_1 orthogonal to $V_1 - V_2$

$\therefore$ independent.

consider $av_1 + bv_2 = 0$

$(av_1 + bv_2) \cdot v_1 = 3a + 3b = 0$

$(av_1 + bv_2) \cdot v_2 = 5a + 3b = 0$

$a, b = 0$

independent.

(6) A scientist, Kaiser (your name here), is preparing an *influenza levinsonia* antibody shaped like a parallelogram. A 5-dimensional parallelogram.

(a) [10 points] The volume is given by det(A), where $A = \begin{bmatrix} 2 & 0 & 0 & 1 & 3 \\ 0 & 1 & -7 & 5 & -2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & s & 0 \\ 0 & 0 & 3 & 3 & 3 \end{bmatrix}$.

Find s so that $\det(A) = 2$.

$$\det(A) = S \cdot 1 \cdot \begin{vmatrix} 2 & 0 & 0 & 3 \\ 0 & 1 & -7 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 3 \end{vmatrix} = S \cdot \begin{vmatrix} 2 & 0 & 0 & 3 \\ 0 & 1 & -7 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{vmatrix} = \cancel{-6S} = 2$$

$2 \times (3s+6)$

$+2 \times \Big| \; : \; \Big|$

$\boxed{S = -\frac{1}{3}}$

$$\det(A) = 2 \times 1 \times \begin{vmatrix} 1 & 1 & 2 \\ 2 & s & 0 \\ 3 & 3 & 3 \end{vmatrix} = 2 \times (-3s+6) = 2$$

$S = \frac{5}{3}$

-10

(b) [5 points] Oops, your microscope settings were backwards. The volume is actually $\det(A^{-1})$. Find s so that $\det(A^{-1}) = 2$.

$\det(A^{-1}) = 2 \quad \therefore \det(A) = \frac{1}{2}$

$\therefore \cancel{-6S = \frac{1}{2}}$

$\cancel{\boxed{S = -\frac{1}{12}}}$

$2 \cdot (-3s+6) = \frac{1}{2}$

$S = \frac{23}{12}$

-2.5

☆☆☆☆

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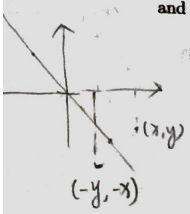
(7) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transform defined by the following properties:

- $T(0,0,1) = (0,0,0)$. $\to z = 0$
- If v is in the xy-plane, then v is reflected across the $x+y=0$ plane. $y = -x$

There is a matrix A such that $T(x) = Ax$. The goal of this problem is to understand A.

(a) (3 points) Find a basis $\{u, v, w\}$ where the action of T is well-understood. Give also $T(u), T(v)$, and $T(w)$.



T: exchange (y,x) if $z=0$

$T(x) = \begin{bmatrix} 0 & -1 & c \\ -1 & 0 & d \\ a & b & 0 \end{bmatrix} x$

$\{u, v, w\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$T(u) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ ✗. $T(v) = \begin{bmatrix} -1 \\ \cancel{0} \\ 0 \end{bmatrix}$. $T(w) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$V = [1,1,0]$, $T(v) = (-1,-1,0)$

$\boxed{W = [1,-1,0], T(w) = (1,-1,0)}$

when it's on the line!!

(b) (3 points) Find the eigenvalues of A and a basis for each eigenspace of A. (Think geometrically.)

$A(x,y,0) = (-y,-x,0)$

$A(0,0,1) = (0,0,0)$

$(A+I)(x,y,0) = 0$

$\lambda = 0, \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\lambda_1 = -1, \cancel{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}} \quad \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\lambda_3 = 1, \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

(c) (3 points) What is A? You may express it as product of matrices and their inverses.

$A = [\,1 \; 0 \; 0 \,][\,-1 \; 0 \; 0\,][\,1 \; 0 \; 0\,] \quad A = [\,0 \; 1 \; 1\,][\,0 \; 0 \; 0\,][\,0 \; 1 \; 1\,]^{-1}$

(c) (3 points) What is A? You may express it as product of matrices and their inverses.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \cancel{0} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \cancel{0} \\ \cancel{0} & 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(d) (3 points) What is A^2? Give it explicitly as a single matrix. (Think geometrically.)

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$