cheatsheet

Monday, March 18, 2019 6:39 PM

- Linear Trans : $T: \mathbb{R}^m \to \mathbb{R}^n$, LT $\Leftrightarrow$ $T(u+v)=T(u)+T(v)$
  $T(rv)=rT(v)$, r is const
  $\vec{0} \in \ker\{T\}$

  - $\text{range}(T) \subset \mathbb{R}^n$
  - $T(x)=Ax$ (T injective $\Leftrightarrow$ columns of A independent ~~injective~~ $\Leftrightarrow$ $\ker\{T\}=0$
- $f \circ g(x) = f(g(x))$, f composed with g
- subspace :
  - subset S,
    $\vec{0} \in S$; $u, v \in S$, $u+v \in S$; $r \in \mathbb{R}$, $u \in S$, $ru \in S$;
- $A_{n \times m}$, then soln set of $Ax=0$ form a subspace of $\mathbb{R}^m$
  - null space : soln set of $Ax=0$ (null(A))
- basis of subspace
  (a) B spans S (b) B linear independent
  - To find a basis : $\left\{\begin{matrix} u_1 \\ \vdots \\ u_m \end{matrix}\right\} \Rightarrow \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$ make a matrix,
    1° reduced echelon form, take non zero rows

    2° make $[u_1 \cdots u_m]$, the columns in A that corresponding to the [redacted] pivot columns in echelon form.
- $A_{n \times m}$
  - row space : subspace of $\mathbb{R}^m$
  - column space : subspace of $\mathbb{R}^n$

- $\dim(\text{row}(A)) = \dim(\text{col}(A)) = \text{rank}(A)$
- $\boxed{\text{rank}(A) + \text{nullity}(A) = m, \ A_{n\times m}}$
- The Unifying Theorem $\mathcal{S} = \{a_1, \cdots, a_n\} \in \mathbb{R}^n$, $A = [a_1 \cdots a_n]$,

  $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(x) = Ax$.

  $\Leftrightarrow$ $\mathcal{S}$ spans $\mathbb{R}^n$

  $\Leftrightarrow$ $\mathcal{S}$ is linear independent

  $\Leftrightarrow$ $\forall b \in \mathbb{R}^n$, $Ax = b$ have a unique soln

  $\Leftrightarrow$ T onto

  $\Leftrightarrow$ T one-to-one

  $\Leftrightarrow$ A invertible

  $\Leftrightarrow$ $\det(A) \neq 0$

  $\Leftrightarrow$ $\ker(T) = \{0\}$

  $\Leftrightarrow$ $\mathcal{S}$ is a basis of $\mathbb{R}^n$

  $\Leftrightarrow$ $\text{col}(A) = \mathbb{R}^n$

  $\Leftrightarrow$ $\text{row}(A) = \mathbb{R}^n$

  $\Leftrightarrow$ $\text{rank}(A) = n$

  $\Leftrightarrow$ $\lambda \neq 0$

- Change of Basis:
  - coordinate vector $\overset{[y]_B}{\nearrow}$ $\cancel{[U]}[y]_B = [u_1 \cdots u_n]\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = y$
  - change of basis matrix $U = [u_1 \cdots u_n]$, $U \cdot [y]_B = y$
  - $B_1 = \{u_1, \cdots, u_n\}$, $B_2 = \{v_1, \cdots, v_n\}$

    $[x]_{B_2} = V^{-1}U[x]_{B_1}$

    $[x]_{B_1} = U^{-1}V[x]_{B_2}$

- Change of basis in subspace

  S, subspace of $\mathbb{R}^n$, $B_1 = \{u_1, \cdots, u_k\}$, $B_2 = \{v_1, \cdots, v_k\}$

  if $C = [\,[u_1]_{B_2} \cdots [u_k]_{B_2}\,]$, then $[x]_{B_2} = C[x]_{B_1}$

- determinant
  - $A_{n\times n}$, $\det(A) = \sum_{i=1}^{n} a_{ij}C_{ij} = \sum_{j=1}^{n} a_{ij}C_{ij}$

    cofactor $C_{ij} = (-1)^{i+j}\det(M_{ij})$

    Minor of $a_{ij} = \det(M_{ij}) \Rightarrow \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & a_{ij} & & \vdots \\ a_{n1} & & a_{nj} & \cdots & a_{nn} \end{bmatrix}$ (remove col & row)

  - $\boxed{\det(A^T) = \det(A)}$
  - $\boxed{\det(AB) = \det(A)\det(B)}$
  - 1° interchange two rows of A: $\det(A) \Rightarrow -\det(A)$

    2° multiply one row A of c: $\det(A) \Rightarrow c\det(A)$

    3° add a multiple of one row of A to another $\Rightarrow \det(A)$ doesn't change
  - $\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$
  - partitioned

    $P = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ or $P = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \Rightarrow \det(P) = \det(A)\det(D)$, A, D square.

- Eigenvalue & Eigenvector
  - $A\vec{u} = \lambda\vec{u}$, $\vec{u} \neq \vec{0}$

    $c\vec{u}$ is also an ~~eigenvalue~~ eigenvector
  - eigen space: subspace of $\vec{u}$ and $\vec{0}$, with each λ

- characteristic polynomial $\det(A-\lambda I)$
  characteristic equation $\det(A-\lambda I)=0$ (find λ)

- $\boxed{\dim(\text{eigenspace of } \lambda) \le \text{multiplicity of } \lambda}$

- $A_{n\times n} = PDP^{-1}$; $P=[u_1 \cdots u_n]$, $D=\begin{bmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n\end{bmatrix}$

■ $A_{n\times n}$ diagonalizable
  ⇔ A has $\vec{u}$ s form a ~~are~~ basis for $\mathbb{R}^n$
  ⇔ $A_{n\times n}$ has only real λ,
  dim(eigenspace) = multiplicity for all λ
  ⇐ $A_{n\times n}$ has n distinct real values.

- $\prod_{i=1}^{n} \lambda_i = |A|$
  $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}$

- AB, BA has same eigenvalues

- Rotational matrix $\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$

- Check:
  I. Asked for basis of column space instead of row space!!