15.1 Iterated Integrals

Monday, January 7, 2019 8:27 AM

(Rectangular Domains

- E.g. 1. Compute $\iint_R (x-3y^2)dA$, $R=\{(x,y)\,|\,0\le x\le 2, 1\le y\le 2\}$

$= \int (\frac{x^2}{2} - 3xy^2)dy \Big|_0^2$

$= (\frac{x^2 y}{2} - xy^3)\Big|_0^2\Big|_1^2$

$= (2y-2y^3)\Big|_1^2$

$= -12$

- Fubini Theorem: If f is continuous on the rectangle $R=\{(x,y)\,|\,a\le x\le b, c\le y\le d\}$, then $\iint_R f(x,y)dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$

- e.g. 2. $\iint_R \sin x\cos y\,dA$, $R=[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]$ (x, y)

$= \int \sin x\cdot \sin y\,dx\Big|_0^{\frac{\pi}{2}}$

$= -\cos x\cdot \sin y\Big|_0^{\frac{\pi}{2}}\Big|_0^{\frac{\pi}{2}}$

$= -\sin y\Big|_0^{\frac{\pi}{2}}$

$= -1$

- Theorem. If $f(x,y)=g(x)\cdot h(y)$ is continuous on $R=[a,b]\times[c,d]$, then

$$\iint_R f(x,y)dA = \iint_R g(x)\cdot h(y)\,dA = \int_a^b g(x)\,dx\int_c^d h(y)\,dy$$

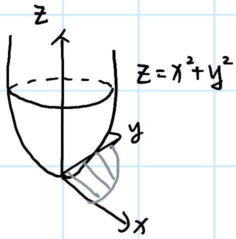
e.g. 3. $\int_0^2\int_0^{\pi} r^3\cos^2\theta\,d\theta\,dr$

$= \int_0^2 r^3\,dr\cdot\int_0^{\pi}\cos^2\theta\,d\theta$ $\qquad \cos^2\theta = \frac{1-\cos 2\theta}{2}$

$= \frac{r^4}{4}\Big|_0^2\cdot(\frac{\theta}{2}-\frac{\sin 2\theta}{4})\Big|_0^{\pi}$

$= 4\cdot\frac{\pi}{2}$

$= 2\pi$

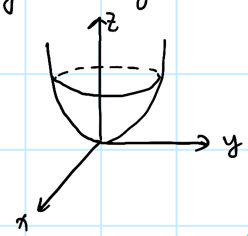

z
$z = x^2 + y^2$
y
x

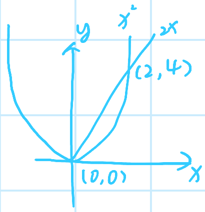
15.2 Double integrals over general regions

Monday, January 7, 2019 9:11 AM

- e.g.1. Find the volume under $z = x^2+y^2$ and above region D) bounded by

$y=2x$ & $y=x^2$





$$\int_0^2\int_{2x}^{x^2} x^2+y^2\, dy\, dx$$

$$= \int_0^2 x^2y+\frac{y^3}{3}\Big|_{2x}^{x^2} dx = \int_0^2 (x^4+\frac{x^6}{3}-2x^3-\frac{2}{3}x^3)dx$$

$$= \int_0^2 (\frac{x^6}{3}+x^4-\frac{8}{3}x^3)dx$$

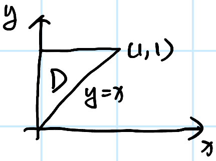
$$= (\frac{x^7}{21}+\frac{x^5}{5}-\frac{2}{3}x^4)\Big|_0^2$$

$$= \frac{128}{21}+\frac{32}{5}-\frac{32}{3}$$

$$= \frac{32}{7}+\frac{32}{5}$$

$$= \frac{32\times12}{35} = \frac{216}{35}$$

e.g. Write $\iint_D f(x,y)\, dA$ in two ways, where D is



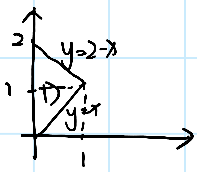
1°. $y\in[x,1]$, $x\in[0,1]$

$$\int_0^1\int_1^x f(x,y)\, dy\, dx$$

2° $x\in[0,y]$, $y\in[0,1]$

$$\int_0^1\int_0^y f(x,y)\, dx\, dy$$

Same, but let D be



$$\int_0^1\int_x^{2-x} f(x,y)\, dy\, dx$$

$y=2-x \Rightarrow x=2-y$

$$\int_1^2\int_0^{2-y} f(x,y)\, dx\, dy + \int_0^1\int_0^y f(x,y)\, dx\, dy$$

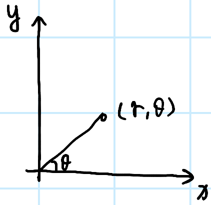
15.3 Double integrals in Polar Coordinates

Wednesday, January 9, 2019 9:07 AM

- Polar Coordinates

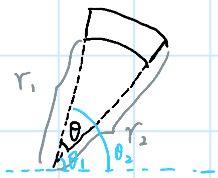
$x = r \cdot \cos\theta$

$y = r \cdot \sin\theta$



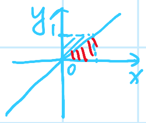
- Polar Rectangle

$R = \{(r,\theta) \mid r_1 \le r \le r_2,\ \theta_1 \le \theta \le \theta_2\}$



$dA = r \cdot dr\,d\theta$

- e.g. $\int_0^1 \int_y^1 \sqrt{5+4x^2}\,dx\,dy$



$x \in [y, 1]$

$\therefore \int_0^1 \int_0^x \sqrt{5+4x^2}\,dy\,dx$

$= \int_0^1 x\sqrt{5+4x^2}\,dx = \frac{1}{8} \cdot \frac{2}{3}(5+4x^2)^{\frac{3}{2}} \Big|_0^1$

$= \frac{1}{12} \cdot (9^{\frac{3}{2}} - 5^{\frac{3}{2}})$

15.4 Center of Mass

Friday, January 11, 2019 9:13 AM

- mass $m^2 \cdot kg/m^2 = kg$

  area · density = mass

  density $\sum \rho(x^*, y^*) \Delta A$

  - mass $m = \iint_D \rho(x,y)\,dA$ — area density
  - charges $Q = \iint_D \sigma(x,y)\,dA$ — charge density (Coulombs/m²)

-
  - def. Moment about x-axis $M_x = \iint_D y\,\rho(x,y)\,dA$
  - def. Moment about y-axis $M_y = \iint_D x \cdot \rho(x,y)\,dA$

  ⇨ def. center of mass $(\bar{x}, \bar{y})$

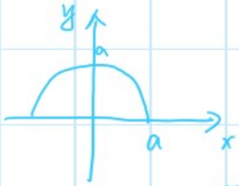
  $$\bar{x} = \frac{M_y}{m}$$

  $$\bar{y} = \frac{M_x}{m}$$

  - def. Moment of Inertia
    - about x-axis $I_x = \iint_D y^2 \rho(x,y)\,dA$
    - about y-axis $I_y = \iint_D x^2 \rho(x,y)\,dA$
    - about the origin $I_0 = \iint_D (x^2+y^2)\,\rho(x,y)\,dA$
  - Interpretation

Moment of Inertia describe how quickly the thing will rotate about x or y axis. |x|, |y| are the distance to y, x axis.

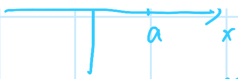
- e.g. Consider the upper half of circle $x^2+y^2=a^2$. Assume density at any point is proportional to the distance from the center of circle. Find CM.



$\rho(x,y) = k\sqrt{x^2+y^2} = kr$

$r \in [0, a]$, $\theta \in [0, \pi]$

$m = \int_0^{\pi}\int_0^{a} kr^2\,dr\,d\theta$



[illegible]

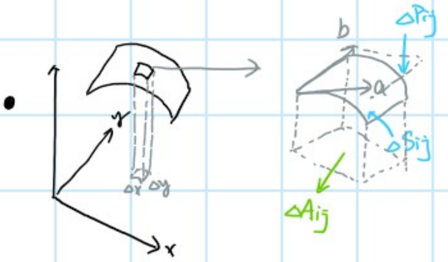
$$m = \int_0^{\pi}\int_0^{a} kr^2\,dr\,d\theta$$
$$= \int_0^{\pi} \frac{k}{3}a^3\,d\theta = \frac{k}{3}a^3\pi$$

$$M_x = k\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dy\,dx$$
$$= k\int_{-a}^{a} \frac{1}{3}(x^2+y^2)^{\frac{3}{2}}\Big|_0^{\sqrt{a^2-x^2}}\,dx$$
$$= k\int_{-a}^{a}\left(\frac{a^3}{3} - \frac{x^3}{3}\right)dx$$
$$= k\left(\frac{a^3}{3}x - \frac{x^4}{12}\right)\Big|_{-a}^{a}$$
$$= \frac{k}{2}a^4$$
$$\bar{y} = \frac{k}{2}a^4 \Big/ \frac{k}{3}a^3\pi = \frac{3a}{2\pi}$$

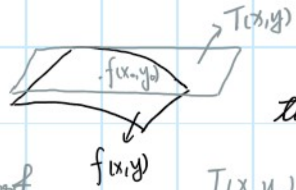
$$M_y = k\int_0^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} x\sqrt{x^2+y^2}\,dx\,dy$$
$$= k\int_0^{a} \frac{1}{3}(x^2+y^2)^{\frac{3}{2}}\Big|_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\,dy$$
$$= k\int_0^{a} \frac{a^3}{3} - \frac{a^3}{3}\,dy$$
$$= 0$$
$$\bar{x} = 0$$

15.5 surface area

Monday, January 14, 2019 9:09 AM

- 

$\Delta P_{ij} = |\vec{a} \times \vec{b}|$



tangent plane $T(x,y) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$

Proof, $T(x,y_0) = f_x(x-x_0) + f(x_0,y_0)$, $\Delta z = f_x(x_0,y_0)(x-x_0)$

$\vec{a} = \langle \Delta x, 0, \frac{\partial z}{\partial x}\cdot\Delta x \rangle$

$\vec{b} = \langle 0, \Delta y, \frac{\partial z}{\partial y}\cdot\Delta y \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ \Delta x & 0 & \frac{\partial z}{\partial x}\cdot\Delta x \\ 0 & \Delta y & \frac{\partial z}{\partial y}\cdot\Delta y \end{vmatrix} = -\frac{\partial z}{\partial x}\Delta x\cdot\Delta y\, i - \frac{\partial z}{\partial y}\Delta x\cdot\Delta y + \Delta x\cdot\Delta y$$

$$|\vec{a}\times\vec{b}| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 \Delta A_{ij}^2 + \left(\frac{\partial z}{\partial y}\right)^2 \Delta A_{ij}^2 + \Delta A_{ij}^2}$$

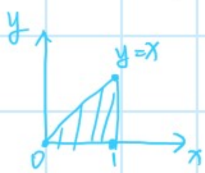
$$= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ \Delta A_{ij}$$

$$\therefore \text{Surface Area} = \sum_{i=1}^{n}\sum_{j=1}^{n} A(\Delta P_{ij}) = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} \sqrt{1+f_x^2+f_y^2}\ \Delta A_{ij}$$

$$= \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

$$\therefore\ S = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

- e.g.1. Find the area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy plane with vertices $(0,0), (1,0), (1,1)$



$\frac{\partial z}{\partial x} = 2x$

$\frac{\partial z}{\partial y} = 2$

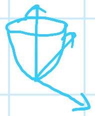
$$\int_0^1 \int_0^x \sqrt{1+4+4x^2}\ dy\,dx$$

$$= \frac{1}{8}\cdot\frac{2}{3}(5+4x^2)^{\frac{3}{2}}\Big|_0^1$$

$$= \frac{1}{12}\cdot\left(9^{\frac{3}{2}} - 5^{\frac{3}{2}}\right) = \frac{1}{12}(27-5\sqrt{5})$$

e.g.2. Area of the paraboloid $z = x^2 + y^2$ below $z = 9$

e.g. 2. Area of the paraboloid $z=x^2+y^2$ below $z=9$



$z=9,\quad x^2+y^2=9,\quad r=3$

$\frac{\partial z}{\partial x} = 2x = 2\cdot r\cdot\cos\theta,\quad \frac{\partial z}{\partial y} = 2y = 2r\cdot\sin\theta$

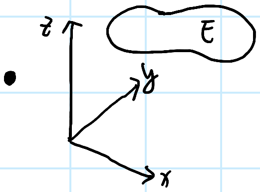
$$\int_0^{2\pi}\int_0^3 r\sqrt{1+4r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{2}{3}\cdot\frac{1}{8}(1+4r^2)^{\frac{3}{2}}\Big|_0^3\,d\theta$$

$$= \left(\frac{37^{\frac{3}{2}}}{\cancel{8}\,12} - \frac{1}{\cancel{8}\,12}\right)\cdot 2\pi$$

15.6 Triple integrals

Friday, January 18, 2019 8:45 AM

- 

$$\iiint_E f(x,y,z)\,dV$$

e.g. Evaluate $\iiint_B e^z\,dV$, where $\{(x,y,z) \mid 0\le x\le 1,\ -1\le y\le 3,\ 0\le z\le 2\}$

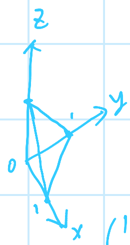
$$\int_0^2\int_{-1}^3\int_0^1 e^z\,dx\,dy\,dz$$

$$= \int_0^2 4e^z\,dz$$

$$= 4e^2-4$$

e.g.2. $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the four planes

$x=0, y=0, z=0$, and $x+y+z=1$



$x=0, y=0, z=1$ $\qquad$ $x=0,\ z=1-y$

$x=0, z=0, y=1$

$x=0, z=0, x=1$

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,dz\,dy\,dx \qquad ☆ \int_0^{1-x-y}$$

$$= \int_0^1\int_0^{1-x}\frac{(1-x-y)^2}{2}\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}\left(\frac{(1-x)^2}{2}+\frac{y^2}{2}-y(1-x)\right)dy\,dx$$

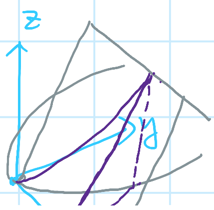
$$= \int_0^1 \frac{(1-x)^3}{2}+\frac{(1-x)^3}{6}-\frac{(1-x)^3}{2}\,dx$$

$$= \int_0^1 \frac{(1-x)^3}{6}\,dx$$

$$= -\frac{1}{4}\cdot\frac{1}{6}\cdot(1-x)^4\Big|_0^1$$

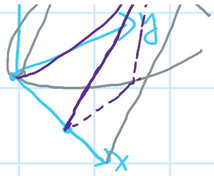
$$= \frac{1}{24}$$

e.g.3. $\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx$ as an iterated integral in a different order



$z=y \Rightarrow z=x^2$

$y=x^2 \Rightarrow x=\sqrt{y}$



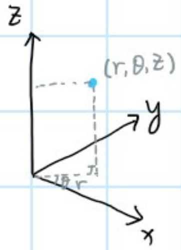
$y = x^2 \Rightarrow x = \sqrt{y}$

$$\int_0^1 \int_0^{\sqrt{y}} \int_0^{x^2} f(x,y,z)\, dz\, dy\, dx$$

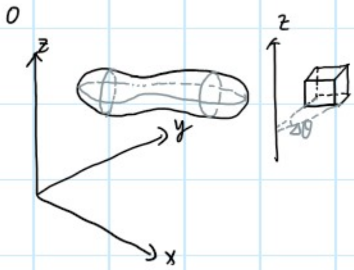
15.7, 8 Cylindrical & Spherical Coordinates

Wednesday, January 23, 2019 8:29 AM

- def: Cylindrical coordinates



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \\ z = z \end{cases}$$



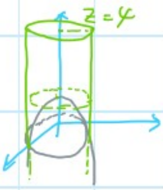
$dV = r\,dr\,d\theta\,dz$

e.g. A solid E lies in the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. The density at any point is proportional to distance from the axis of the cylinder. Find the mass of E.

$\rho = k\cdot r \quad , \quad z = 1 - r^2$

$$= \int_0^{2\pi}\int_0^1\int_{1-r^2}^{4} kr^2\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_{1-r^2}^{1} kr\cdot r\,dz\,dr\,d\theta + \int_0^{2\pi}\int_0^1\int_1^4 kr^2\,dz\,dr\,d\theta$$

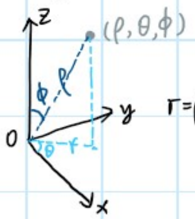


$$= k\int_0^{2\pi}\int_0^1 r^4\,dr\,d\theta + k\int_0^{2\pi}\int_0^1 3r^2\,dr\,d\theta$$

$$= \frac{k}{5}\cdot 2\pi + 2\pi k$$

$$= \cancel{\frac{5}{2}\pi k}\ \frac{12}{5}\pi k$$

- def: spherical coordinates



$r = \rho\sin\phi$
$$\begin{cases} x = \rho\cdot\sin\phi\cdot\cos\theta \\ y = \rho\cdot\sin\phi\cdot\sin\theta \\ z = \rho\cdot\cos\theta \end{cases}$$

$\phi \in [0, \pi]$

$\theta \in [0, 2\pi]$

$\rho \geq 0$

$\rho^2 = x^2 + y^2 + z^2 = z^2 + r^2$

$$\rho^2 = x^2 + y^2 + z^2 = z^2 + r^2$$

o $dV = r \cdot dz\,dr\,d\theta = \rho \cdot (\sin\phi \cdot \rho)\,d\rho\,d\phi\,d\theta$

o

e.g. Region: above $z = \sqrt{x^2+y^2}$, below $x^2+y^2+z^2 = z$, sphere.

$\rho^2 = \rho \cdot \cos\phi$ $\qquad$ $\rho \cdot \cos\phi = \rho \cdot \sin\phi \Rightarrow \phi = \frac{\pi}{4}$

$\rho = \cos\phi$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\phi} \rho^2 \cdot \sin\phi \; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{\cos^3\phi}{3} \cdot \sin\phi \; d\phi\, d\theta$$

$$= \int_0^{2\pi} -\frac{\cos^4\phi}{12}\Big|_0^{\frac{\pi}{4}} \; d\theta$$

$$= \int_0^{2\pi} -\frac{1}{4\times 12} + \frac{1}{12} \; d\theta$$

$$= \frac{\pi}{8}$$

15.9 change of variables

Friday, January 25, 2019 8:54 AM

- rem: U substitution

$$\int_0^{\pi} \sin(x^2)\cdot 2x\,dx$$

$u = x^2,\ du = 2x\,dx$

$$= \int_0^{\pi^2} \sin(u)\,du$$

- Formula $\int_a^b f(x)\,dx = \int_{x^{-1}(a)}^{x^{-1}(b)} f(x(u))\cdot x'(u)\,du$

- Change of variable in Double Integral

$$\iint_R f(x,y)\,dx\,dy = \iint_{T^{-1}(R)} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$T(u,v) = \begin{cases} x(u,v) \\ y(u,v) \end{cases}$$

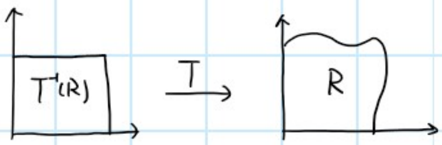
  - Define Jacobian (Matrix)

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

now written as matrix

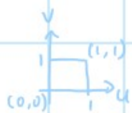
$$= \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}\right|$$

  - 

e.g. 1. Consider coordinates defined by $\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$, find the image of $S = \{(u,v) \mid 0 \le u \le 1,\ 0 \le v \le 1\}$

$T(S),\ T = \begin{cases} x(u,v) \\ y(u,v) \end{cases}$



When $v=0$, (down side)

$\begin{cases} x = u^2 \\ y = 0 \end{cases}$

When $u=0$, (left side)

$\begin{cases} x = -v^2 \end{cases}$

When $u=0$, (left side)

$$\begin{cases} x = -v^2 \\ y = 0 \end{cases}$$

When $v=1$, (upperside)

$$\begin{cases} x = u^2-1 \\ y = 2u \end{cases}$$

When $u=1$, (right side)

$$\begin{cases} x = 1-v^2 \\ y = 2v \end{cases} \Rightarrow x = 1 - \frac{y^2}{4}$$

Derivation of Jacobi

Wednesday, January 30, 2019 8:28 AM

e.g. Let $z = \sin^3 t \cos^2 t + 7\cos^3 t \cdot \sin t$, find $\frac{dz}{dt}$ for $t=0$

$$z = \sin^3 t(1-\sin^2 t) + 7\cdot\frac{\cos 2t+1}{2}\cdot\frac{\sin 2t}{2}$$

$$= \sin^3 t - \sin^5 t + \frac{7}{4}\cdot\left(\frac{\sin 4t}{2} + \sin 2t\right)$$

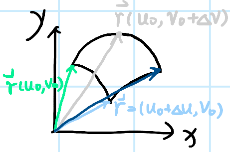
$$\frac{dz}{dt} = 3\sin^2 t\cdot\cos t - 5\sin^4 t\cdot\cos t + \frac{7}{2}\cdot\cos 4t + \frac{7}{2}\cos 2t$$

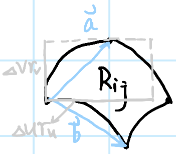
$t=0$, $\frac{dz}{dt} = 7$

- REM $\iint_R f(x,y)\,dx\,dy = \iint_{T^{-1}(R)} f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

$T(x,y) = (x(u,v), y(u,v))$

$\vec{r}(u,v) = \langle x(u,v), y(u,v)\rangle$





$\vec{a} = \vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0, v_0)$

$\vec{b} = \vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0, v_0)$

$\lim_{\Delta v\to 0}\frac{\vec{a}}{\Delta v} = \vec{r}_v$ (partial with respect to v)

$\lim_{\Delta u\to 0}\frac{\vec{b}}{\Delta u} = \vec{r}_u$

$\vec{a} \approx \Delta v\vec{r}_v$, $\vec{b}\approx \Delta u\vec{r}_u$

$|\Delta u\vec{r}_u \times \Delta v\vec{r}_v| = \Delta u\Delta v|r_u\times r_v|$ $\because \vec{r} = \langle x(u,v), y(u,v)\rangle$

$r_u = \langle\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\rangle$, $r_v = \langle\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\rangle$

$$|r_u\times r_v| = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0\end{vmatrix} = \left|\left(\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}\right)\hat{k}\right|$$

$$= \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

$$= \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

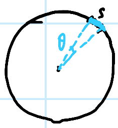
$|r_u\times r_v| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ Jacobian (absolute value)

Prob

Friday, January 11, 2019 8:30 AM

e.g. Compute $\int_0^{\pi}\int_1^2 4r^3\sin^2\theta \, dr\, d\theta$

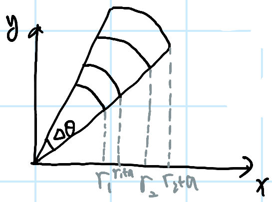
REMARK Arc length of a piece of circle



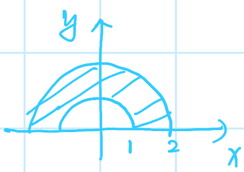
Arc length of $s = r\theta$

$2\pi r \cdot \frac{\theta}{2\pi}$

•



• e.g. evaluate $\iint_R (3x+4y^2)\,dA$, where R is the region in the upper half plane bounded between $x^2+y^2=1$ & $x^2+y^2=4$



$3x+4y^2 = 3\cdot r\cos\theta + 4r^2\sin^2\theta$

$\int_0^{\pi}\int_1^2 3r^2\cos\theta + 4r^3\sin^2\theta \, dr\, d\theta$

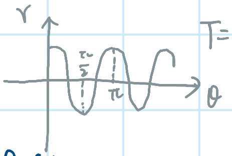
$= \int_0^{\pi} r^3\cos\theta + r^4\sin^2\theta \Big|_1^2 \, d\theta$

$= \int_0^{\pi} 7\cos\theta + 15\sin^2\theta \, d\theta$

$= 7\sin\theta + 15\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\Big|_0^{\pi}$

$= \frac{15}{2}\pi$

e.g. Compute the area of a single leaf of a four-leaved rose $r=\cos(2\theta)$



$T=\frac{2\pi}{2}=\pi$ $r=1$ when $\theta=0$

$$\cancel{\int_0^1\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta r\, d\theta\, dr} \quad \int_0^{\cos 2\theta}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r\, d\theta\, dr$$

careful!

$$= \int_0^1 \frac{\sin 2\theta}{4}\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r\,dr \qquad = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta\, d\theta$$

$$= \int_0^1 \frac{r}{2} dr$$

$$= \frac{r^2}{4}\Big|_0^1 = \frac{1}{4} \qquad = \theta + \frac{\sin 4\theta}{4}\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8}$$

☆☆☆ e.g.

Find the area of the surface.

The part of the plane $5x + 2y + z = 10$ that lies in the first octant

$5\sqrt{30}$ ✔

$z=y=0,\ x=2$

$z=0,\ y=5-\frac{5}{2}x$

$$\int_0^2\int_0^{5-\frac{5}{2}x} \sqrt{1+25+4}\, dy\, dx = \left(5\sqrt{30}x - \frac{5}{4}\sqrt{30}x^2\right)\Big|_0^2$$

$$= 5\sqrt{30}$$

$\frac{\partial z}{\partial x} = -5$

$\frac{\partial z}{\partial y} = -2$

☆☆☆☆

Evaluate the integral by changing to spherical coordinates.

$$\int_0^6\int_0^{\sqrt{36-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{72-x^2-y^2}} xy\, dz\, dy\, dx$$

$$\frac{15552\sqrt{2}}{5}\left(\frac{2}{3}-\frac{5}{6\sqrt{2}}\right)$$ ✔

$z=\sqrt{x^2+y^2} \Rightarrow \rho\cdot\cos\phi = \rho\sqrt{\sin^2\phi}$ cone $\Rightarrow \phi=\frac{\pi}{4}$

$z=\sqrt{72-x^2-y^2} \Rightarrow z^2+x^2+y^2=72$ sphere (above x-y plane)

$\rho^2=72$

$\rho=6\sqrt{2}$

$z=\sqrt{72-x^2-y^2} \Rightarrow z^2+x^2+y^2=72$ sphere (above x-y plane)

$\rho^2=72$

$\rho=6\sqrt{2}$

$\sqrt{x^2+y^2}=\sqrt{72-x^2-y^2}$: $y=\sqrt{36-x^2}$, $x\in[0,6]$, first octant

$x^2+y^2=36$ $\qquad x^2+y^2=36$, $\theta\in[0,\frac{\pi}{2}]$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{6\sqrt{2}}(\rho\sin\phi)^2\cdot\cos\theta\cdot\sin\theta\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{6\sqrt{2}}\rho^4\sin^3\phi\cdot\frac{\sin 2\theta}{2}\, d\rho\, d\phi\, d\theta$$

$$=\frac{1}{2}\cdot\frac{(6\sqrt{2})^5}{5}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\sin^3\phi\cdot\sin 2\theta\, d\phi\, d\theta$$

$$=\frac{(6\sqrt{2})^5}{10}\left(\frac{2}{3}-\frac{5}{6\sqrt{2}}\right)$$

$$\int\sin^3\phi\, d\phi$$

$$=\int\sin\phi(1-\cos^2\phi)$$

$$=\cos\phi(\cos^2\phi-1)+2\int\cos^2\phi\sin\phi$$

$$=\cos^3\phi-\cos\phi+2\left(-\cos^3\phi-2\int\cos^2\phi\sin\phi\right)$$

★★

Evaluate the integral by making an appropriate change of variables.

$\iint_R 9(x+y)\, e^{x^2-y^2}\, dA$, where R is the rectangle enclosed by the lines $x-y=0$, $x-y=7$, $x+y=0$, and $x+y=8$

assume $u=x-y$

$v=x+y$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{vmatrix}=\frac{1}{2}$$

$$\int_0^8\int_0^7 9ve^{uv}\frac{1}{2}\,du\,dv \qquad x=\frac{u+v}{2}, \quad y=\frac{v-u}{2}$$

$$=\frac{1}{2}\int_0^8 9v\boxed{\frac{e^{uv}}{v}}\Big|_0^7\, dv$$

remember to take out v when integrate.

$$=\frac{1}{2}\int_0^8(9e^{7v}-9)\,dv$$

Remember $\frac{\partial(x,y)}{\partial(u,v)}$

$$=\frac{9}{2}\left(\frac{e^{56}}{7}-8-\frac{1}{7}\right)$$

★★★★

Evaluate the integral by making an appropriate change of variables.

$\iint_R 7\sin(25x^2+64y^2)\, dA$, where R is the region in the first quadrant bounded by the ellipse $25x^2+64y^2=1$

$\frac{7(1-\cos(1))\pi}{160}$ ✔

$\{\ u=5x \Rightarrow x=\frac{u}{5}$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{1}{5} & 0\\ 0 & \frac{1}{8}\end{vmatrix}=\frac{1}{40}$$

$$\frac{7(1-\cos(1))\pi}{160}$$

$$\begin{cases} u = 5x \Rightarrow x = \frac{u}{5} \\ v = 8y \Rightarrow y = \frac{v}{8} \end{cases} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{8} \end{vmatrix} = \frac{1}{40}$$

$x \in [0, \frac{1}{5}]$, $y \in [0, \frac{1}{8}] \Rightarrow u \in [0,1]$, $v \in [0,1]$

$$\frac{1}{40}\int_0^1\int_0^1 7\sin(u^2+v^2)\,du\,dv$$

$$= \frac{7}{40}\int_0^{\frac{\pi}{2}}\int_0^1 r\sin(r^2)\,dr\,d\theta$$

$$= \frac{7(1-\cos(1))\pi}{160}$$