3.7 3.8

Monday, November 5, 2018 3:34 PM

- Warm-up:

A mass of 100g stretches a spring 5cm. If it begins from equilibrium with v 10cm/s ↓, give its position at t. (no damping)

$u(0)=0$; $u'(0)=10$ cm/s

$u''m = mg - k(L+u) - \gamma u'$

$u''m + \gamma u' + ku = 0$ ∵ no damping ∴ $\gamma = 0$

$m = 0.1$ kg, $L = 5$ cm $= 0.05$ m

$u'(0) = 0.1$ m/s

$F_s = k \cdot L = mg$

$k = \frac{1}{0.05} = 20$

$0.1u'' + 20u = 0$

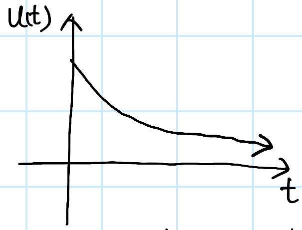
$r^2 + 200 = 0$

$r = \pm 10\sqrt{2}\, i$

- $mu'' + \gamma u' + ku = 0$

$\Delta = \gamma^2 - 4mk$

Case I. $\Delta > 0 \iff \gamma > \sqrt{2mk}$



overdamped

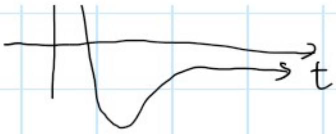
$u(t) = C_1 e^{r_1 t} + C_2 e^{-r_2 t}$

Case II. $\Delta = 0 \iff \gamma = \sqrt{2mk}$

$u(t) = C_1 e^{-\frac{\gamma}{2m}t} + C_2 t e^{-\frac{\gamma}{2m}t}$

critically damped

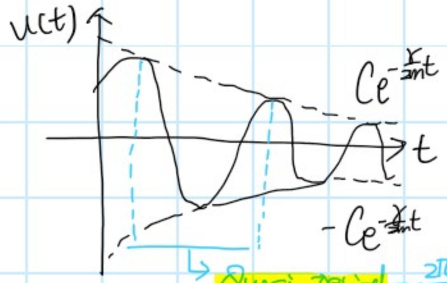
critically damped



Case III. $\Delta < 0 \quad 0 < \gamma < \sqrt{4mk}$

$$u(t) = e^{-\frac{\gamma}{2m}t}\underbrace{(C_1\cos\mu t + C_2\sin\mu t)}_{R\cos(\mu t-\delta)\ (*)}$$

$$\mu = \frac{\sqrt{4mk-\gamma^2}}{2m}$$



Underdamped.

Quasi-period $= \frac{2\pi}{\mu}$

Proof of (*): Quasi-frequency: μ

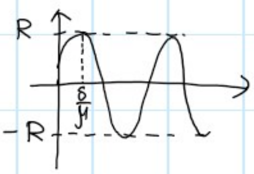
$$\cos(A-B) = \cos A\cdot\cos B + \sin A\cdot\sin B$$

$$R\cos(\mu t-\delta) = R\cos\delta\cdot\cos\mu t + R\sin\delta\cdot\cos\mu t$$

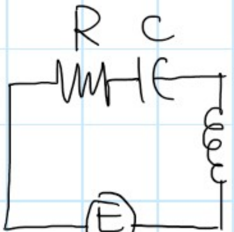
$$\begin{cases} C_1 = R\cos\delta \\ C_2 = R\sin\delta \end{cases}$$

$$\Rightarrow \begin{cases} R^2 = C_1^2 + C_2^2 \\ \tan\delta = \frac{C_2}{C_1} \end{cases}$$

$R\cdot\cos(\mu t-\delta)$:



- Circuits



I(t) current

Q(t) charge

$\frac{dQ}{dt} = I(t)$

Induced Voltage

$Q(t)$ charge

$\frac{dQ}{dt} = I(t)$

$V_R = IR$

$V_L = l \cdot \frac{dI}{dt}$, $V_c = \frac{Q}{c}$

$E(t) = I(t)R + \frac{Q}{C} + L \cdot \frac{dI}{dt}$

$Q''L + Q'R + Q \cdot \frac{1}{c} = E(t)$

$e^{rt}(Lr^2 + Rr + \frac{1}{c}) = E(t)$ (When $E(t) = 0$)

Case I. $\Delta > 0$ $R > \sqrt{\frac{4L}{c}}$

Case II. $\Delta = 0$ $R = 2\sqrt{\frac{L}{c}}$ critically damped circuit.

Case III. $\Delta < 0$ $0 < R < 2\sqrt{\frac{L}{c}}$

§ 3.8

e.g. $u'' + u' + \frac{5}{4}u = 3\cos t$ $u(0) = 2$, $u'(0) = 3$.

$(r^2 + r + \frac{5}{4}) = 0$

$\sqrt{\Delta} = \sqrt{1-5} = 2i$

$r = \frac{-2 \pm 2i}{2} = -1 \pm i$

$y_c = e^{-t}(C_1\cos t + C_2\sin t)$

guess $y_p = A\sin t + B\cos t$

$\vdots$

$\begin{cases} B = \frac{12}{17} \\ A = \frac{48}{17} \end{cases}$

$\therefore u(t) = e^{-\frac{t}{2}}(C_1\cos t + C_2\sin t) + \frac{48}{17}\sin t + \frac{12}{17}\cos t$

$\therefore u(t) = \underbrace{e^{-\frac{t}{2}}(\frac{22}{17}\cos t + \frac{14}{17}\sin t)}_{\lim_{t\to\infty} T(t) = 0} + \underbrace{\frac{48}{17}\sin t + \frac{12}{17}\cos t}_{S(t)}$

Transient Solution Steady State Solution

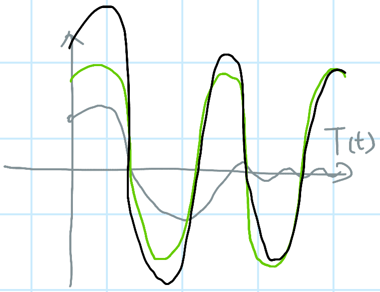
$\lim_{t\to\infty} I(t) = 0$

Transient Solution

$S(t)$

Steady State Solution

$R\cdot\cos(\gamma t - \delta)$



3.8 continued

Wednesday, November 7, 2018 3:28 PM

- Warm-ups: A 1kg weight is attatched to a spring with a spring constant of 4N/m, it is immersed in a medium that exerts a force of 6 N when the object is traveling at 2m/s. Assume it starts from rest at the equilibrium position. If an outside force of $-2\cos t$ is applied at time t, what is the steadystate of the system?

$u(0)=0,\ u'(0)=0$

$u''\cdot m+\gamma u'+4u=-2\cos t$

$\gamma\cdot 2=6 \Rightarrow \gamma=3.$

$u''+3u'+4u=-2\cos t.$

$\sqrt{\Delta}=\sqrt{9-16}=\sqrt{7}\,i$

$u_c(t)=e^{-\frac{3}{2}t}\cdot\left(A\cos\left(\frac{\sqrt{7}}{2}t\right)+B\sin\left(\frac{\sqrt{7}}{2}t\right)\right)$

$u_p(t)=A\cos t+B\sin t$

$u_p'=-A\sin t+B\cos t$

$u_p''=-A\cos t-B\sin t$

$-A\cos t-B\sin t-3A\sin t+3B\cos t+4A\cos t+4B\sin t=-2\cos t$

$(3A+3B)\cos t+(3B-3A)\sin t=2\cos t.$

$\begin{cases}3A+3B=-2\\ 3B=3A\end{cases}$

$\therefore\ A=-\frac{1}{3}=B$

$u_p(t)=-\frac{1}{3}\cos t-\frac{1}{3}\sin t$ (Steady State)

$\therefore\ u(t)=e^{-\frac{3}{2}t}\left(A\cos\left(\frac{\sqrt{7}}{2}t\right)+B\sin\left(\frac{\sqrt{3}}{2}t\right)\right)-\frac{1}{3}\cos t-\frac{1}{3}\sin t.$

$\because\ u(0)=0,\ u'(0)=0$

$A-\frac{1}{3}=0\Rightarrow A=\frac{1}{3}$

careful!!

careful!!

- Resonance
  - with damping

$$\underbrace{mu''+\gamma u'+ku}=F_0\cos(w_0 t)$$

$$u_c(t)=e^{\lambda t}\underbrace{(c_1\cos\mu t+c_2\sin\mu t)}_{\mu=q w_0}$$

$$R\cdot\cos(wt-\delta)$$

$$\frac{R}{\left(\frac{F_0}{k}\right)}=\frac{F_0}{\sqrt{m^2(w_0^2-w^2)+\gamma^2w^2}}$$

when γ is small

$$\approx\frac{F_0}{\gamma w}\left(1+\frac{\gamma^2}{8mk}\right)\approx\frac{F_0}{\gamma w}$$

when $\gamma\to 0^+$ $\quad \frac{R}{\left(\frac{F_0}{k}\right)}\to\infty$

$\gamma\to\infty$ $\quad \frac{R}{\left(\frac{F_0}{k}\right)}\to 0$

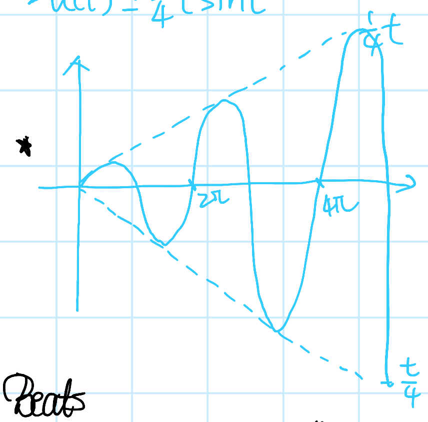
$w_0\approx w_{max}$

  - $\gamma=0,\ u''+u=\frac{1}{2}\cos t.\ u(0)=0,\ u'(0)=0$

$u_c(t)=c_1\cos t+c_2\sin t$

$u_p(t)=(A\cos t+B\sin t)t$

$u(t)=\frac{1}{4}t\sin t$



- Beats

e.g. $u''+u=\frac{1}{2}\cos\left(\frac{4}{5}t\right)$

$u_c(t)=c_1\cos t+c_2\sin t$

$u_p(t)=A\cos\frac{4}{5}t+B\sin\frac{4}{5}t$

$\frac{9}{25}A=\frac{1}{2}\Rightarrow A=\frac{25}{18}$

$B=0\quad u_p(t)=\frac{25}{18}\cos\frac{4}{5}t$

$25 \quad A = 2 \ , \ A = 18$

$B = 0 \qquad u_p(t) = \frac{25}{18} \cos \frac{4}{5} t$

$\therefore \ u(t) = C_1 \cos t + C_2 \sin t + \frac{25}{18} \cos \frac{4}{5} t$

$C_1 = -\frac{25}{18} \ . \ C_2 = 0$

$u(t) = \frac{25}{18} \left(\cos \frac{4}{5} t - \cos t \right)$

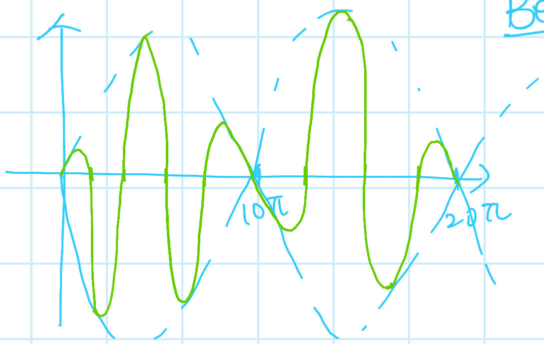
$$\boxed{\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)}$$

$u(t) = 2 \cdot \frac{25}{18} \left(\sin \frac{9}{10} t \cdot \sin \frac{1}{10} t \right) = \frac{25}{9} \sin \frac{9}{10} t \cdot \sin \frac{t}{10}$

$T_1 = \frac{2\pi}{\frac{9}{10}} = \frac{20}{9} \pi$

$T_2 = \frac{2\pi}{\frac{1}{10}} = 20\pi$

Beat



Midterm problems set

Wednesday, November 7, 2018 9:42 PM

★★★ Autumn2017

(3) **(10 points)** Suppose that you are designing a new shock absorber for a small automobile. The automobile has a mass of 800 kilograms and the combined effect of the springs in the suspension system is that of a spring constant of 160000 Newtons/meter.

(a) **(5 points)** Before a damping mechanism is installed in the automobile, when it hits a bump it will bounce up and down. What is the period of the oscillations when it hits a bump?

(b) **(5 points)** Your job is to design a damping mechanism which eliminates oscillations when the automobile hits a bump. What is the minimum value of the effective damping constant that you need?

(a) $u''m + ku = 0$

$u'' \cdot 800 + 1.6\times10^5 u = 0$

$e^{rt}\, 800(r^2 + 200) = 0$

$\therefore \mu = 10\sqrt{2}$

$u(t) = C_1\cos(\mu t) + C_2\sin(\mu t)$

$T = \frac{2\pi}{10\sqrt{2}} = \boxed{\frac{\sqrt{2}}{10}\pi}$

(b) $\Delta = \gamma^2 - 4mk$

when $\gamma^2 \geq 4mk$

$\gamma \geq \sqrt{32\times1.6\times10^7}$

$\gamma \geq 8\times2\times10^3\sqrt{2}$

$\boxed{\gamma \geq 16\times10^3\sqrt{2}}$

minimum: $16\times10^3\sqrt{2}$.

$y'' + \gamma y' + 20y = 0$

$r^2 + \gamma r + 20 = 0$

$(r + \frac{\gamma}{2})^2 = \frac{\gamma^2}{4} - 20$

$\therefore \frac{\gamma^2}{4} - 20 \geq 0$

$\gamma \geq 4\sqrt{5}$

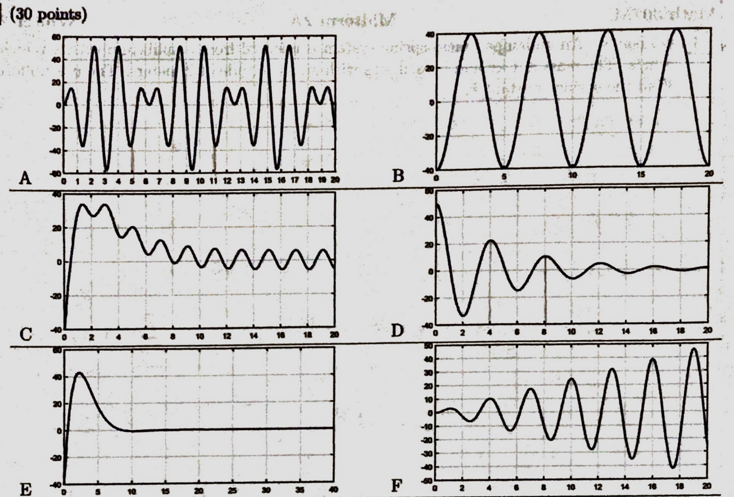
damping coefficient $= 800\gamma$

$\therefore \geq 3200\sqrt{5}$

★★★★★ Winter 2018

Math 307M **Midterm 2A** **Winter 2018**

5 **(30 points)**



Which graph shows a solution of a forced damped mass-spring system? What is the forcing frequency? A C ~~$\frac{2\pi}{\mu} = 6 \Rightarrow \mu = \frac{\pi}{3}$~~ Resonance

Which graph is a solution to an unforced overdamped mass-spring system? E The spring first become very long then stay at equilibrium.

Which graph is a solution to an unforced underdamped mass-spring system? What is the natural quasi-frequency? $\frac{2\pi}{\mu} = 4$ $\mu = \frac{\pi}{2}$ The amplitude is decreasing.
D

Which graph is a solution to an unforced overdamped mass-spring system?

E The spring first become very long then stay at equilibrium.

Which graph is a solution to an unforced underdamped mass-spring system? What is the natural quasi-frequency? $\frac{2\pi}{\mu}=4$ $\boxed{\mu=\frac{\pi}{2}}$ The amplitute is decreasing.

D

Which graph is a solution to a system that is unforced and undamped? What is the natural frequency? B $\frac{2\pi}{w}=5$ $\boxed{w=\frac{2}{5}\pi}$ Do a sinuoual motion

Which two graphs show solutions to forced and undamped mass-spring systems? C At first it stretch a lot, then do sinuoual with almost a fixed amplitute.
What phenomena does each exhibit?
Write a plausible formula for each of these two solutions. *I don't care if you write sine when it shold be cosine, but try to get the basic form and the frequencies approximately right. don't worry about amplitude or phase.*

$u(t)=60\sin 4t\cdot\sin\frac{t}{2}$

A: $u(t)=A\cos\mu_1 t\cdot\cos\mu_2 t$ (Beats)

B: $u(t)=A\cos(\mu t)$

C: $u(t)=Ae^{\lambda t}\cos(\mu t)+B\cos(kt)$

D: $u(t)=A\cos(\mu t-\delta)$

E: \

F: Resonance

$y(t)=t\sin\frac{2\pi}{3}t$

5. (10 total points + 3 bonus points) A $\frac{1}{4}$ kg mass is placed on a flat frictionless surface and attached to a horizontal spring. It takes 4 N of force to move the mass 36 cm to the right of its equilibrium position. The mass starts at rest in its equilibrium position. Starting at time $t=0$ seconds a horizontal force of $0.41\cos(7t)$ Newtons acts on the mass. Friction in this problem is negligible.

(a) (3 points) Formulate an initial value problem that describes the position of the mass at time t.

$36\text{cm}=0.36\text{m}$

$k=\frac{4}{0.36}=\frac{100}{9}$

$mu''+ku=F(t)$

$\frac{u''}{4}+\frac{100}{9}u=0.41\cos(7t)$

$u(0)=0.36,\ u'(0)=0$

(b) (5 points) Solve the above initial value problem to find the position of the mass at time t. You may use known formulae to save time, but be sure to indicate if you are quoting a formula you've seen in class.

y_c: $\mu=\sqrt{\frac{k}{m}}=\sqrt{\frac{100}{9}\times 4}=\frac{20}{3}$

$y_c=C_1\cos(\frac{20}{3}t)+C_2\sin(\frac{20}{3}t)$

y_p assume $y_p=A\cos 7t+B\sin 7t$

$u'=-7A\sin 7t+7B\cos 7t$

$u''=-49A\cos 7t-49B\sin 7t$

plug in:

$(\frac{100}{9}-\frac{49}{4})A\cos 7t+(\frac{100}{9}-\frac{49}{4})B\sin 7t$

$\begin{cases}(\frac{100}{9}-\frac{49}{4})A=0.41\\ B=0\end{cases}$

$A=-0.36$

$\therefore y_p=-0.36\cdot\cos 7t$

$\because u(0)=0.36,\ u'(0)=0$

$\therefore \begin{cases}C_1=0.72\\ \frac{20}{3}C_2=0\end{cases}$

$C_2=0$

$\therefore \boxed{y=0.72\cos\frac{20}{3}t-0.36\cos 7t}$

$=0.36(2\cos\frac{20t}{3}-\cos 7t)$

$y=R\sin(w_1 t)\cdot\sin(w_2 t)$

$y=\left[\frac{2F_0}{m(w_0^2-w^2)}\cdot\sin(\frac{1}{2}(w_0-w)t)\right]\sin(\frac{1}{2}(w_0+w)t)$

where $w_0=\frac{\sqrt{k}}{\sqrt{m}}=\frac{20}{3}$

$y=\frac{2\times 0.41}{\frac{1}{4}((\frac{20}{3})^2-0.49)}\sin(\frac{1}{2}(\frac{20}{3}-7)t)\cdot\sin(\frac{1}{2}(\frac{20}{3}+7)t)$

$=-\frac{18}{25}\sin(-\frac{1}{6}t)\sin\frac{41}{6}t$

$=\frac{18}{25}\sin\frac{1}{6}t\sin\frac{41}{6}t$

[NB: Question continued on the next page!]

(c) (2 points) What is the maximum distance the mass achieves from its equilibrium position?

$\because u(t) = R\cos(\omega t + \delta)$

$\therefore U_{max} = R = \cancel{\sqrt{0.36^2 + 0.72^2} = 0.805}$ m

0.72

$R = \frac{18}{25} = 0.72$ m

(d) (Bonus: 3 points) Estimate the maximum amount of kinetic energy that the mass will have.

$\frac{1}{2}mv^2$

$u(t) = R\omega = 0.805 \times \frac{20}{3} = 5.37$

$\because y = \frac{18}{25}\sin\frac{1}{6}t \sin\frac{41}{6}t$

$\mu = \frac{41}{6}$, ~~then~~ $\cancel{\sin\frac{1}{6}t}$

$\frac{41}{6}$

then $\sin\frac{t}{6}$ should be 1 to reach max

$y \approx \frac{18}{25}\sin\frac{41}{6}t$

$v = y' = \frac{18}{25} \times \frac{6}{41} = \frac{123}{25}$

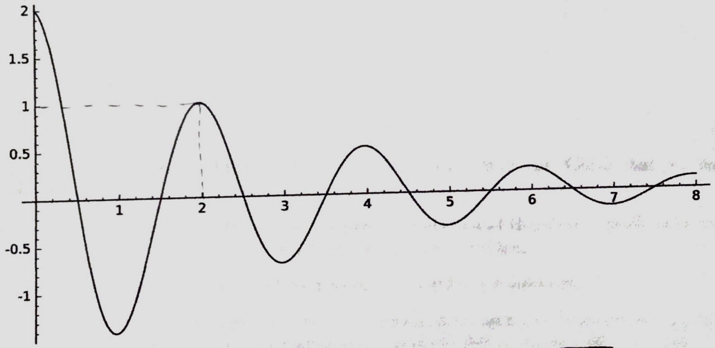
$E = \frac{1}{2}mv^2 = \frac{15129}{3000}$ J

★★★ Spring 2018

4. (14pts). You have a spring in a damping media, but you don't know the spring constant nor the damping constant. To find that out, you decide to attach a mass of 1 kg to the spring and plot the motion of this unforced damped spring-mass system. The graph below is a plot of the displacement of the mass at any time t. Write down the differential equation governing its motion.

Explain your reasoning to get full credit. Note: you should write down actual (estimated) numbers based on what you gather from the graph, not just a symbolic equation.

The next page is blank in case you need more space to work.



$m \cdot u'' + \gamma u' + ku = 0$ $\because$ unforced, $\mu = \frac{\sqrt{4mk - \gamma^2}}{2}$

$\Rightarrow u(t) = e^{-\frac{\gamma}{2m}t}\left(C_1\cos\left(\frac{\sqrt{4mk-\gamma^2}}{2}t\right) + C_2\sin\left(\frac{\sqrt{4mk-\gamma^2}}{2}t\right)\right)$

$$\Rightarrow \quad u(t) = e^{-\frac{\gamma}{2m}t}\left(C_1 \cos\left(\frac{\sqrt{4mk-\gamma^2}}{2} t\right) + C_2 \sin\left(\frac{\sqrt{4mk-\gamma^2}}{2} t\right)\right)$$

period $T \approx 2.$

$$\therefore \frac{2\pi}{\mu} = 2\pi \cdot \frac{2}{\sqrt{4mk-\gamma^2}} = 2$$

$$4mk - \gamma^2 = 4\pi^2, \quad \mu = \pi$$

$t = 0, \; C_1 = 2$

$$t = 2, \quad e^{-\frac{\gamma}{m}} \cdot 2 \cdot \cos(2\pi) = 1$$

$$e^{-\gamma} = \frac{1}{2}$$

$$\gamma = -\ln\left|\frac{1}{2}\right|$$

$$\gamma = \ln 2$$

$$4mk = 4\pi^2 + (\ln 2)^2$$

$$k = \pi^2 + \frac{(\ln 2)^2}{4}$$

-1

$$\therefore \boxed{u'' + \ln 2 \, u' + \left(\pi^2 + \frac{(\ln 2)^2}{4}\right) u = 0}$$

Mid term review

Friday, November 9, 2018 3:33 PM

- Wronskian/Linearity

  $y'' + p(t)y' + q(t)y = 0$ (*)

  $y_1(t)$ and $y_2(t)$ are solutions.

  and if $W(y_1, y_2) = y_2 y_1' - y_2' y_1 \neq 0$, then all the slns to (*) are of the form $y(t) = C_1 y_1 + C_2 y_2$

- Non-homogenous
  - $ay'' + by' + cy = f(t)$

    $y_1(t)$, $y_2(t)$ are slns, then $y_1(t) - y_2(t)$ is a sln to $ay'' + by' + cy = 0$. All solns are in the form $y = y_c + y_t$.
  - solve:

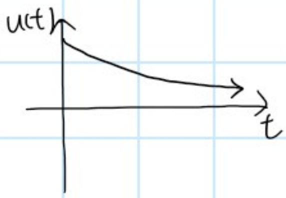
    SI. Find $y_c(t)$

    SII. Guess $y_p(t)$

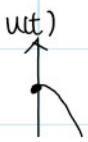
    SIII. Solve for constants.

- Springs

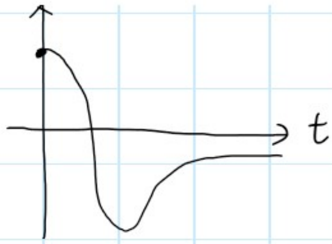
  $mu'' + \gamma u' + ku = F(t)$
  - $F(t) = 0$, $\Delta = r^2 - 4mk$
    - $\Delta > 0$

      overdamped
    - $\Delta = 0$, $\gamma = 2\sqrt{mk}$

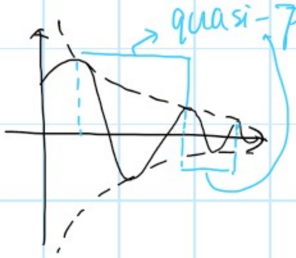
      critically damped.



critically damped.

- $\Delta < 0$

quasi-period. $T = \frac{2\pi}{\mu}$



$u(t) = e^{\lambda t} R\cos(\mu t - \delta)$

- $F(t) \neq 0$

$\gamma = 0, \quad mu'' + ku = F_0 \cos(\omega_0 t)$

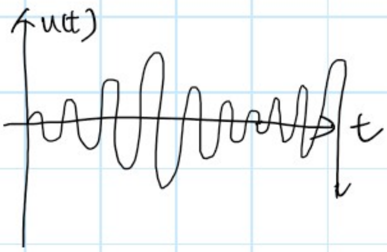
$r = \pm i\sqrt{\frac{k}{m}}$

$u_c(t) = C_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$

$\omega = \sqrt{\frac{k}{m}}$

- $\omega = q \cdot \omega_0$ $(q \neq 1)$

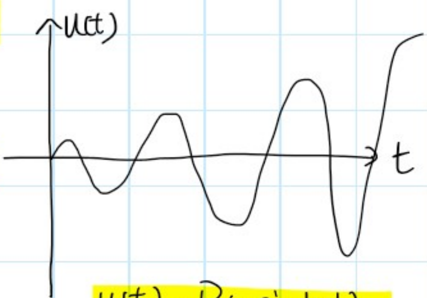
Beat



$u(t) = R\sin(\omega_1 t) \cdot \sin(\omega_2 t)$

$\omega = \omega_0$

Resonance



$u(t) = Rt \cdot \sin(\mu t)$

★ Not: $u(t) = S(t) + T(t)$ $\qquad \lim_{t \to \infty} T(t) = 0$

$= u_c(t) + u_p(t)$

- unit.

- Exam 2.

1. Conceptual
2. General Sln
3. IVP
4.5. Word Prob

(Spring / Circuits)

- Linear 2^{nd} order DEs

$$\frac{d^2y}{dt^2} + p(t)\cdot\frac{dy}{dt} + q(t)y = g(t)$$

- $g(t)=0$ homogeneous DE
- y_c : complementary solution
- characteristic equation $ar^2+br+c=0$

- Euler's formula $e^{it} = \cos t + i\sin t$
- Hyperbolic Trig
  - $\sinh(t) = \frac{e^t - e^{-t}}{2}$
    $\cosh(t) = \frac{e^t + e^{-t}}{2}$
  - Properties
    $\cos(it) = \cosh(t)$
    $\sin(it) = \sin h(t)$

- Polar coordinate

$a+bi = re^{i\theta}$ $\quad a = r\cos\theta$

$= r(\cos\theta + i\sin\theta)$ $\quad b = r\sin\theta$

$r = \sqrt{a^2+b^2}$

- Wronskian / Linearity

$$y'' + p(t)y' + q(t)y = 0 \quad (*)$$

$y_1(t), y_2(t)$ are solutions if $W(y_1,y_2) = y_2y_1' - y_1y_2' \neq 0$, then all the solutions to $(*)$ are of the form $y(t) = c_1y_1 + c_2y_2$

- Homogeneous DE
  - $\Delta > 0$: $r = r_1$, $r = r_2$ $(r_0 - r_1)(r - r_2) = 0$
    $y(t) = c_1e^{r_1t} + c_2e^{r_2t}$
  - $\Delta < 0$: $r = \lambda \pm \mu i$ $\quad \mu = \frac{\sqrt{\Delta}}{2a}$
    $y(t) = e^{\lambda t}(c_1\cos(\mu t) + c_2\sin(\mu t))$
  - $\Delta = 0$: $(r-a)^2 = 0$
    $y(t) = c_1e^{rt} + c_2te^{rt}$

- Solution of Non-homogeneous DE.
  - Method of Undetermined Coefficients
    - When don't know what to guess, take derivative.
    - Find particular solution

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)$
$P_n(t) \cdot e^{\alpha t}$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t) \begin{cases} \sin\beta t \\ \cos\beta t \end{cases}$	$t^s[(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t} \cdot \cos\beta t + (B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t} \sin\beta t]$

s: number of times 0 is a root of the characteristic equation.

$\alpha + \beta i$ / α: a root of the characteristic equation

- If $g(t)$ is included in the y_c
  - guess $y_p = t \cdot g(t)$

$(\tan x)' = \sec^2 x$

$(\cot x)' = \csc^2 x$

$(\sec x)' = -\csc x$

$(\csc x) = -\csc x \cdot \cot x.$

$\frac{d}{dx} a^x = a^x \ln a$

$\sec\theta = \frac{1}{\cos\theta}$

$\csc\theta = \frac{1}{\sin\theta}$

- Check:

I. use radian for angle

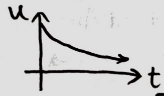
II. 1837149

- Spring

$$mu'' + \gamma u' + ku = F(t)$$

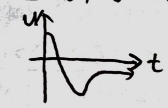
u: position from equilibrium
γ: damping constant
$F(t)$: applied external force

- $F(t) = 0$, $\Delta = \gamma^2 - 4mk$
  - $\Delta > 0$



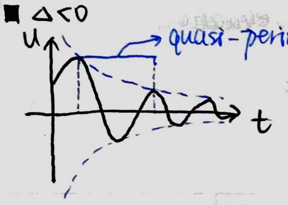
overdamped

  - $\Delta = 0$, $\gamma = 2\sqrt{mk}$



critically damped

  - $\Delta < 0$



$$u(t) = e^{\lambda t} R\cos(\mu t - \delta)$$

underdamped

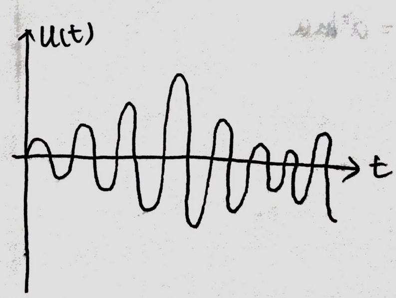
- $F(t) \neq 0$, $\gamma = 0$

$$mu'' + ku = F_0 \cos(w_0 t)$$

$$y_c = C_1\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) + C_2\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)$$

$$w = \sqrt{\tfrac{k}{m}}$$

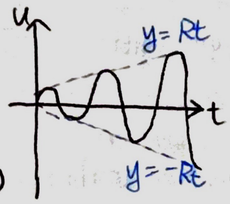
  - $w = q \cdot w_0$
    $(q \neq 1)$
    Beat



$$u(t) = R \cdot \sin(w_1 t)\sin(w_2 t)$$

■ $\omega = \omega_0$

Resonance



$u(t) = Rt\cdot\sin(\gamma t)$

- RLC circuits

$$Q''(t)L + Q'(t)R + Q\cdot\frac{1}{C} = E(t)$$

$$I = Q'(t), \quad V_L = L\cdot\frac{dI}{dt}, \quad V = \frac{Q}{C}$$

- Transient Solution

$\lim_{t\to\infty} T(t) = 0.$

Steady State Solution

$R\cdot\cos(\gamma t - \delta)$

- Superposition

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

- $\sin(\alpha\pm\beta) = \sin\alpha\cdot\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha\pm\beta) = \cancel{\sin\alpha}\ \cos\alpha\cdot\cos\beta \mp \sin\alpha\cdot\sin\beta$

- $\sin(2\theta) = 2\sin\theta\cdot\cos\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$= 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta$

$g = 32\,ft/s^2 = 9.8\,m/s^2$

12 in = 1ft

- $\log_a(M\pm N) = \log_a M \pm \log_a N$

$\log_a M^n = n\log_a M$

$$\frac{\log_c b}{\log_c a} = \log_a b$$

HW7

Sunday, November 11, 2018 11:30 PM

$mu'' + \gamma u' + ku = 0$

6/6 points | Previous Answers My Note

$mu'' + ku = 0 \; (\gamma = 0)$

A 5 kg mass is attached to a spring with spring constant 3 Nt/m.

$\mu = \frac{\sqrt{4km}}{2m} = \sqrt{\frac{k}{m}} \approx 0.7746$

What is the frequency of the simple harmonic motion? 0.7746 ✔ radians/second

What is the period? 8.1152 ✔ seconds $T = \frac{2\pi}{\mu} = 8.11$

Suppose the mass is displaced 0.2 meters from its equilibrium position and released from rest. What is the amplitude of the motion? 0.2 ✔ meters

assume $u(t) = A\cos(\omega t)$ $t=0,\; A = u(0)$ $u(0) = 2.$

Suppose the mass is released from the equilibrium position with an initial velocity of 0.6 meters/sec. What is the amplitude of the motion? 0.7746 ✔ meters

assume $u(t) = A\sin(\mu t)$ $u'(t) = \mu A\cos(\mu t) = 0.6$ $A = 0.7746$

Suppose the mass is is displaced 0.2 meters from the equilibrium position and released with an initial velocity of 0.6 meters/sec.
What is the amplitude of the motion? 0.8 ✔ meters
What is the maximum velocity? 0.6197 ✔ m/s

assume $u(t) = A\cos(\mu t + \delta)$

$u(0) = A\cos(\delta) = 0.2$

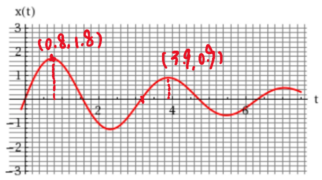
$V_{max} = A\mu \cdot 1 = 0.6197$

$u'(0) = -A\mu \cdot \sin(\delta) = 0.6$

$A\sin\delta = -\frac{0.6}{0.7746} = -0.7746$

$\therefore \sqrt{(A\sin\delta)^2 + (A\cos\delta)^2} = A = 0.8$

The graph below shows the motion of an unforced damped harmonic oscillator:



Engineers often describe damped harmonic motion with the formula

$x(t) = R\, e^{-\zeta\omega_n t}\sin(\omega_d t)$

because both ζ and ω_d can be measured in a straightforward way. There is no phase shift ϕ because we have chosen an initial time t = 0, to be a zero of x(t).

If you measure the times and displacements, (t_1, x_1) and (t_2, x_2), at two consecutive peaks, then,

$T = t_2 - t_1$ is called the quasi-period, and

$\omega_d = \frac{2\pi}{t_2 - t_1}$ is the damped natural frequency or quasi-frequency

$\Delta = \ln(\frac{x_1}{x_2})$ is called the logarithmic decrement and

$\zeta = \frac{\Delta}{\sqrt{\Delta^2 + 4\pi^2}} \approx \frac{\Delta}{2\pi}$ is called the damping ratio.

$\Delta = \ln\frac{0.8}{3.9} = -1.584$

$\zeta = 0.2522$

$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}}$ is the (undamped) natural frequency.

$-\frac{\square}{2} = -\zeta\omega_n$

$\square = 2\zeta\omega_n$

Zoom in on the graph above and measure (t_1, x_1) and (t_2, x_2), and combine with these formulas to find:

ω_d = 1.997 ✔ $\omega_d = \frac{2\pi}{3.1} \approx 2.03.$

ζ = 0.11 ✔

ω_n = 2.01 ✔

Now write the differential equation that x(t) satisfies in the form

x'' + 0.4422 ✔ x' + 3.939 ✔ $x = 0$

$\sqrt{4x - \square^2} = \omega_d$

Finally, use the original formula and your measured values of (t_1, x_1) to estimate R and the initial conditions.

R = 2 ✔ the first peak

$x(0)$ = 0 ✔ $x'(0)$ = 4 ✔

Midterm review sheet

Monday, November 12, 2018 2:21 AM

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Exam 2 Review

This review sheet contains this cover page (a checklist of topics from Chapters 3). Following by all the review material posted pertaining to chapter 3 (all combined into one file).

Chapter 3: Second Order Equations

- 3.2: Linearity/Fundamental Sets. If $y_1(t)$ and $y_2(t)$ are solutions to $y'' + p(t)y' + q(t)y = 0$ and the Wronskian $W(y_1, y_2)$ is not zero at the initial conditions, then there is a unique solution of the form $y = c_1y_1(t) + c_2y_2(t)$.

- 3.1, 3.3, 3.4: Homogeneous Equations. Solve $ar^2 + br + c = 0$.
Then $y = c_1e^{r_1t} + c_2e^{r_2t}$, $y = c_1e^{rt} + c_2te^{rt}$, or $y = c_1e^{\lambda t}\cos(\omega t) + c_2e^{\lambda t}\sin(\omega t)$ depending on roots.

- 3.4: Reduction of Order: Given one solution $y_1(t)$, write $y(t) = u(t)y_1(t)$ and substitute into the differential equation. Then solve for $u(t)$. The general solution is $y(t) = u(t)y_1(t)$.

- 3.5: Nonhomogeneous Equations. Key observation: If $y(t)$ and $Y(t)$ are any two solutions, then $y(t) - Y(t)$ is a solutions to the corresponding homogeneous equation. Thus, every solution will have the form: $y(t) = c_1y_1(t) + c_2y_2(t) + Y(t)$.
Step 1: Find a fundamental set of solutions to the corresponding homogeneous equation.
Step 2: Find a particular solution to the given equation using **undetermined coefficients**.

- 3.7, 3.8 Set Up:
For mass-spring systems: A spring hangs down from the ceiling. A mass is attached to the spring and it comes to rest at a distance of L from natural length (this is called the resting position or equilibrium position and it is when $u = 0$). The mass is pulled to an initial displacement of $u(0)$ and set into motion with an initial velocity of $u'(0)$. Let $u(t)$ be the displacement from rest.
By discussing the forces, we derived the second order system: $mu'' + \gamma u' + ku = F(t)$, where
$F(t)$ = external forcing function
m = 'the mass of the object', we know $w = mg$ and $m = \frac{w}{g}$;
γ = 'the damping constant', we know $F_d = -\gamma u'$ and $\gamma = -\frac{F_d}{u'}$;
k = 'the spring constant', we know $w = mg = kL$, so $k = \frac{w}{L} = \frac{mg}{L}$.
If you are worried about units, all you needed in the homework was:
$g = 32 \text{ ft/s}^2 = 9.8 \text{ m/s}^2$, 100 cm = 1m, 12 in = 1 ft,
and these are the only conversions you'll need to know for my exam.

  For an RLC circuit: Let $Q(t)$ be the total charge on the capacitor in coulumbs (C).
We have: $LQ'' + RQ' + \frac{1}{C}Q = E(t)$, where
$E(t)$ is the impressed voltages in volts (V);
R is the resistance in ohms (Ω);
C is the capacitance in farads (F);
L is the inductance in henrys (H).

- 3.7 Analysis: 'Free Vibrations' ($F(t) = 0$)

  1. The $F(t) = 0$ and $\gamma = 0$ case: $u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) = R\cos(\omega_0 t - \delta)$. Thus, the solution is a cosine wave with the following properties:
     The **natural frequency** is $\omega_0 = \sqrt{k/m}$ radians/second; The **period** is $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{m/k}$ seconds/wave; The **amplitude** is $\boxed{R = \sqrt{c_1^2 + c_2^2}}$; The **phase angle** is δ which is the starting angle.
  2. The $F(t) = 0$ and $\gamma > 0$ case:
     $\gamma > 2\sqrt{km} \Rightarrow y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, with both roots negative (overdamped).
     $\gamma = 2\sqrt{km} \Rightarrow y = c_1 e^{rt} + c_2 t e^{rt}$, with one negative root (critically damped).
     $\gamma < 2\sqrt{km} \Rightarrow y = e^{\lambda t}(c_1\cos(\mu t) + c_2\sin(\mu t)) = Re^{\lambda t}\cos(\mu t - \delta)$.
     In this last case, we say the **quasi frequency** is $\mu = \frac{\sqrt{4mk - \gamma^2}}{2m}$ radians/second; The **quasi period** is $T = \frac{2\pi}{\mu} = 2\pi\frac{2m}{\sqrt{4mk - \gamma^2}}$ seconds/wave; The 'amplitude' is not constant, it is given by $Re^{\lambda t}$ which will always go to zero as $t \to \infty$ (for all damped cases).

- 3.8 Analysis: 'Force Vibrations.' Consider the forcing function $F(t) = F_0\cos(\omega t)$.

  1. The $\gamma = 0$ case:
     Homogeneous solution: $u_c(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$ where $\omega_0 = \sqrt{k/m}$.
     Particular solution:
     - $\omega \neq \omega_0 \Rightarrow U(t) = A\cos(\omega t) + B\sin(\omega t) = \frac{F_0}{m(w_0^2 - w^2)}\cos(\omega t)$
     - $\omega = \omega_0 \Rightarrow U(t) = At\cos(\omega t) + Bt\sin(\omega t) = \frac{F_0}{2m\omega_0}t\sin(\omega_0 t)$ (Resonance!)
  2. The $\gamma > 0$ case:
     Homogeneous solutions: See discussion in 3.7.
     Particular solution: $U(t) = A\cos(\omega t) - B\sin(\omega t) = R\cos(\omega t - \delta)$.
     Thus, the general solution for undamped forced vibrations will always have the form
     $u(t) = (c_1 u_1(t) + c_2 u_2(t)) - (A\cos(\omega t) + B\sin(\omega t)) = u_c(t) + U(t)$.
  3. The function $u_c(t)$ is called the **transient solution** (it dies out).
     The particular solution $U(t) = A\cos(\omega t) + B\sin(\omega t)$ is called the **steady state solution**, or **forced response**.
  4. If damping is very small (i.e. if γ is close to zero), then the maximum amplitude occurs when $\omega \approx \omega_0$. In which case the amplitude will be about $\frac{F_0}{\gamma\omega_0}$ which can be quite large (and it gets larger the closer γ gets to zero). This phenomenon is known as **resonance**.

- Other skills:

  - Solving two-by-two systems (when solving for initial conditions).
  - Working with complex numbers (when we used Euler's formula).
  - Working with cosine and sine (when we wrote it as one wave).

3.2: Linearity and the Wronskian

This section contains various theorems about existence and uniqueness for second order linear systems. In lecture, we emphasized linearity and the Wronskian (Theorems 3.2.2, 3.2.3, and 3.2.4). For now, I want you to only worry about these theorems (you should read the others for your own interest).

For these theorems, we are talking about **homogeneous** linear equations. Many of the theorems apply to any situation of the form $y'' + p(t)y' + q(t)y = 0$. Our immediately applications of these theorems (in 3.1, 3.3, and 3.4) will be concerned with the simpler case of constant coefficients ($ay'' + by' + cy = 0$), but the theorems hold in the general linear case as well.

Linearity/Superposition Theorem:
In general, if $y = y_1(t)$ and $y = y_2(t)$ are two solutions to $y'' + p(t)y' + q(t)y = 0$,
then $y = c_1y_1(t) + c_2y_2(t)$ is also a solution for any constants c_1 and c_2.

Notes about linearity/superposition:

1. In other words, the theorem says that a **linear combination** of any two solutions is also a solution. We say $c_1y_1(t) + c_2y_2(t)$ is a linear combinations of y_1 and y_2.

2. You can quickly prove this as follows:
   Since $y_1(t)$ is a solution, you must have $y_1'' + p(t)y_1' + q(t)y_1 = 0$.
   Since $y_2(t)$ is a solution, you must have $y_2'' + p(t)y_2' + q(t)y_2 = 0$.
   Now consider $y = c_1y_1(t) + c_2y_2(t)$. Taking derivatives we see that:
   $$\begin{aligned} y'' + p(t)y' + q(t)y &= (c_1y_1'' + c_2y_2'') + p(t)(c_1y_1' + c_2y_2') + q(t)(c_1y_1 + c_2y_2) \\ &= c_1(y_1'' + p(t)y_1' + q(t)y_1) + c_2(y_2'' + p(t)y_2' + q(t)y_2) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0 \end{aligned}$$
   Thus, for any numbers c_1 and c_2 the function $y = c_1y_1(t) + c_2y_2(t)$ is also a solution!

3. For example: if $y_1(t) = e^{3t}$ and $y_2(t) = e^{-2t}$ are solutions to $y'' - y' - 6 = 0$,
   then $y(t) = c_1e^{3t} + c_2e^{-2t}$ is a solution for any numbers c_1 and c_2.

4. Another example: if $y_1(t) = e^{-4t}$ and $y_2(t) = te^{-4t}$ are solutions to $y'' - 8y' + 16 = 0$,
   then $y(t) = c_1e^{-4t} + c_2te^{-4t}$ is a solution for any numbers c_1 and c_2.

5. And another example: if $y_1(t) = \sin(7t)$ and $y_2(t) = \cos(7t)$ are solutions to $y'' + 49y = 0$,
   then $y(t) = c_1\sin(7t) + c_2\cos(7t)$ is a solution for any numbers c_1 and c_2.

6. Yet another example: if $y_1(t) = t$ and $y_2(t) = t\ln(t)$ are solutions to $t^2y'' - ty' + y = 0$,
   then $y(t) = c_1t + c_2t\ln(t)$ is a solutions for any numbers c_1 and c_2.

Wronskian:
Once you have two solutions and you have written $y(t) = c_1y_1(t) + c_2y_2(t)$, then we need to think about our initial conditions. First note that $y'(t) = c_1y_1'(t) + c_2y_2'(t)$.

Given **initial conditions**: $y(t_0) = y_0$ and $y'(t_0) = y_0'$. Substituting gives:

$$\begin{aligned} y(t_0) = y_0 &\quad\Rightarrow\quad c_1y_1(t_0) + c_2y_2(t_0) = y_0 \\ y(t_0) = y_0 &\quad\Rightarrow\quad c_1y_1'(t_0) + c_2y_2'(t_0) = y_0' \end{aligned}$$

This is a linear system of equation. See my review on two-by-two linear systems! From that discussion, you know that this has a **unique solution** for c_1 and c_2 if

$$\text{Wronskian determinant } = W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_1'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) \neq 0$$

In other words, if t_0 is a value where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$, then there is a unique solution for c_1 and c_2.

Wronskian Fundmental Set of Solutions Theorem:
If $y = y_1(t)$ and $y = y_2(t)$ are solutions to $y'' + p(t)y' + q(t)y = 0$
AND if $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$ for all valid values of t,
then we say y_1 and y_2 form a fundamental set of solutions.
In which case, **no matter the initial conditions** a unique solution for c_1 and c_2 will exist in the form $y = c_1y_1(t)+c_2y_2(t)$. In other words, if $W \neq 0$ for $y_1(t)$ and $y_2(t)$, then the solution $y = c_1y_1(t)+c_2y_2(t)$ is the **general solution** (meaning it contains all solutions).

Examples

1. For example: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-2t}$ are solutions to $y'' - y' - 6 = 0$,
   and $W = \begin{vmatrix} e^{3t} & e^{-2t} \\ 3e^{3t} & -2e^{-2t} \end{vmatrix} = -5e^t \neq 0$.
   Thus, e^{3t} and e^{-2t} form a fundmental set of solutions.
   Thus, ALL solutions are in the form $y(t) = c_1e^{3t} + c_2e^{-2t}$ for some numbers c_1 and c_2.

2. And another example: $y_1(t) = \sin(t) - \cos(t)$ and $y_2(t) = \cos(t) - \sin(t)$ are solutions to $y'' + y = 0$,
   but $W = \begin{vmatrix} \sin(t) - \cos(t) & \cos(t) - \sin(t) \\ \cos(t) + \sin(t) & -\sin(t) - \cos(t) \end{vmatrix} = 0$ (it takes some expanding to check this).
   Thus, y_1 and y_2 do NOT form a fundamental set of solutions. The general answer CANNOT be written in the form $y = c_1(\sin(t) - \cos(t)) + c_2(\cos(t) - \sin(t))$. This is happening because $y_2(t) = -y_1(t)$, so the 'two' given solutions are actually multiples of each other!

3. The last example again: $y_1(t) = \sin(t)$ and $y_2(t) = \cos(t)$ are solutions to $y'' + y = 0$,
   and $W = \begin{vmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{vmatrix} = -\sin^2(t) - \cos^2(t) = -1 \neq 0$.
   Thus, y_1 and y_2 form a fundamental set of solutions.
   Thus, ALL solutions are in the form $y(t) = c_1\sin(t) + c_2\cos(t)$ for some numbers c_1 and c_2.

3.1: Homogeneous Constant Coefficient 2nd Order

Some Observations and Motivations:

1. For equations of the form $ay'' + by' + cy = 0$, we are looking for a function that 'cancels' with itself if you take its first and second derivatives and add up $ay'' + by' + cy$. This means that the derivatives of y will have to look similar to y in some way. (You should be thinking of functions like $y = ke^{rt}$, $y = k\cos(rt)$ and $y = k\sin(rt)$).

2. In section 3.1, we are going to try to see if we can find solutions of the form $y = e^{rt}$ for some constant r. If $y = e^{rt}$ is a solution, then that means it works in the differential equation. Taking derivatives (using the chain rule), you get $y = e^{rt}$, $y' = re^{rt}$, and $y'' = r^2e^{rt}$. And if you substitute these into the differential equation you get

$$ay'' + by' + cy = 0 \quad \text{which becomes} \quad ar^2e^{rt} + bre^{rt} + ce^{rt} = e^{rt}(ar^2 + br + c) = 0.$$

3. We are looking for a function $y = e^{rt}$ that makes this true for all values of t. Since e^{rt} is never zero, we are looking for values of r that make $ar^2 + br + c = 0$.

4. You already do have some experience with second order equations. Consider $\frac{d^2y}{dt^2} = -9.8$. This is second order but it doesn't involve y' or y, so you can integrate twice to get $y = -4.9t^2 + c_1t + c_2$. Notice that you get **two constants** of integration. We will see in section 3.2 that is true in general for second order equations, we will get two constants in our general solutions.

Definitions and Two Real Roots Method:

1. For the equation $ay'' + by' + cy = 0$, we define the **characteristic equation** to be $ar^2 + br + c = 0$.

2. The **roots** of the characteristic equation are the solutions $r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$. There are three cases:
   - if $b^2 - 4ac > 0$, then you get two real roots. (Section 3.1 is about this case)
   - if $b^2 - 4ac = 0$, then you get one (repeated) root. (Section 3.4)
   - if $b^2 - 4ac < 0$, then you get no real roots, but two complex (imaginary) roots. (Section 3.3)

3. If there are two real roots, r_1 and r_2, then that means $y_1(x) = e^{r_1x}$ and $y_2(x) = e^{r_2x}$ are both solutions. All other solutions can be written in the form

$$y = c_1e^{r_1x} + c_2e^{r_2x},$$

for some constants c_1 and c_2. We call this the general solution.
We will discuss the 'why' all solutions are in this form in section 3.2.

Examples:

1. Give the general solution to $y'' - 7y' + 10y = 0$.

   Solution: The equation $r^2 - 7r + 10 = (r-5)(r-2) = 0$ has roots $r_1 = 2$ and $r_2 = 5$.
   The general solution is $y = c_1 e^{2t} + c_2 e^{5t}$.

2. Give the general solution to $y'' + 4y' = 0$.

   Solution: The equation $r^2 + 4r = r(r+4) = 0$ has roots $r_1 = -4$ and $r_2 = 0$.
   The general solution is $y = c_1 e^{-4t} + c_2$.

Examples with initial conditions:

1. Solve $y'' - 9y = 0$ with $y(0) = 2$ and $y'(0) = -12$.

   Solution: The equation $r^2 - 9 = (r+3)(r-3) = 0$ has roots $r_1 = -3$ and $r_2 = 3$.
   The general solution is $y = c_1 e^{-3t} + c_2 e^{3t}$. Note that $y' = -3c_1 e^{-3t} + 3c_2 e^{3t}$.
   Substituting in the initial condition gives

   $$\begin{aligned} y(0) = 2 \quad &\Rightarrow \quad c_1 + c_2 = 2 \\ y'(0) = -12 \quad &\Rightarrow \quad -3c_1 + 3c_2 = -12 \Rightarrow -c_1 + c_2 = -4 \end{aligned}$$

   Note that we divided equation (ii) by 3. Now we combine and simplify. Adding the equations gives $2c_2 = -2$, so $c_2 = -1$. And using either equation gives $c_1 = 3$.
   Thus, the solution is $y(t) = 3e^{-3t} - e^{3t}$.

2. Solve $y'' - 4y' - 5y = 0$ with $y(0) = 7$ and $y'(0) = 1$.

   Solution: The equation $r^2 - 4r - 5 = (r+1)(r-5) = 0$ has roots $r_1 = -1$ and $r_2 = 5$.
   The general solution is $y = c_1 e^{-t} + c_2 e^{5t}$. Note that $y' = -c_1 e^{-t} + 5c_2 e^{5t}$.
   Substituting in the initial condition gives

   $$\begin{aligned} y(0) = 7 \quad &\Rightarrow \quad c_1 + c_2 = 7 \\ y'(0) = 1 \quad &\Rightarrow \quad -c_1 + 5c_2 = 1 \end{aligned}$$

   Now we combine and simplify. Adding the equations gives $6c_2 = 8$, so $c_2 = \frac{4}{3}$. And using either equation gives $c_1 = 7 - \frac{4}{3} = \frac{17}{3}$. Thus, the solution is $y(t) = \dfrac{17}{3}e^{-t} + \dfrac{4}{3}e^{5t}$.

3.3: Homogeneous Constant Coefficient 2nd Order (Complex Roots)

Before I discuss the motivation of this method, let me give away the 'punchline'. In other words, let me show how easy it is to solve these problems once you know the general result, then we'll discuss the theoretical underpinnings:

Solutions for the Complex Root Case:
If $ar^2 + br + c = 0$ has complex roots $r = \lambda \pm \omega i$, then the general solution to $ay'' + by' + cy = 0$ is given by

$$y(t) = e^{\lambda t}\left(c_1 \cos(\omega t) + c_2 \sin(\omega t)\right).$$

Examples:

1. Give the general solution to $y'' + 3y' + \frac{10}{4}y = 0$.

   Solution: The equation $r^2 + 3r + \frac{10}{4} = 0$ has roots $r = \frac{-3\pm\sqrt{9-10}}{2} = -\frac{3}{2} \pm \frac{1}{2}i = \lambda \pm \omega i$.
   The general solution is $y = e^{-\frac{3}{2}t}\left(c_1 \cos\left(\frac{1}{2}t\right) + c_2 \sin\left(\frac{1}{2}t\right)\right)$.

2. Give the general solution to $y'' - 4y' + 6y = 0$.

   Solution: The equation $r^2 - 4r + 6 = 0$ has roots $r = \frac{4\pm\sqrt{-8}}{2} = 2 \pm \sqrt{2}i = \lambda \pm \omega i$.
   The general solution is $y = e^{2t}\left(c_1 \cos\left(\sqrt{2}t\right) + c_2 \sin\left(\sqrt{2}t\right)\right)$.

Examples with initial conditions:

1. Solve $y'' + 25y = 0$ with $y(0) = 2$ and $y'(0) = 3$.

   Solution: The equation $r^2 + 25 = 0$ has roots $r_1 = 0 \pm 5i = \lambda \pm \omega i$.
   The general solution is $y = c_1 \cos(5t) + c_2 \sin(5t)$.
   Note that $y' = -5c_1 \sin(5t) + 5c_2 \cos(5t)$.

$$\begin{array}{lcl} y(0) = 3 & \Rightarrow & c_1 + 0 = 2 \Rightarrow c_1 = 2 \\ y'(0) = 10 & \Rightarrow & 0 + 5c_2 = 3 \Rightarrow c_2 = \frac{3}{5} \end{array}$$

   Thus, the solution is $y(t) = 2\cos(5t) + \frac{3}{5}\sin(5t)$.

2. Solve $y'' - 4y' + \frac{25}{4}y = 0$ with $y(0) = -1$ and $y'(0) = 4$.

   Solution: The equation $r^2 - 4r + \frac{25}{4} = 0$ has roots $r = \frac{4\pm\sqrt{-9}}{2} = 2 \pm \frac{3}{2}i$.
   The general solution is $y = e^{2t}\left(c_1 \cos\left(\frac{3}{2}t\right) + c_2 \sin\left(\frac{3}{2}t\right)\right)$
   Note that $y' = 2e^{2t}\left(c_1 \cos\left(\frac{3}{2}t\right) + c_2 \sin\left(\frac{3}{2}t\right)\right) + e^{2t}\left(-\frac{3}{2}c_1 \sin\left(\frac{3}{2}t\right) + \frac{3}{2}c_2 \cos\left(\frac{3}{2}t\right)\right)$.
   Substituting in the initial condition gives

$$\begin{array}{lcl} y(0) = -1 & \Rightarrow & c_1 + 0 = -1 \Rightarrow c_1 = -1 \\ y'(0) = 4 & \Rightarrow & 2(c_1 + 0) + \left(0 + \frac{3}{2}c_2\right) = 4 \Rightarrow \frac{3}{2}c_2 = 6 \Rightarrow c_2 = 4 \end{array}$$

   Thus, the solution is $y(t) = e^{2t}\left(-\cos\left(\frac{3}{2}t\right) + 4\sin\left(\frac{3}{2}t\right)\right)$.

Some Observations and Experiments:

1. Consider $y'' + y = 0$. By guess and check, we can see that $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$ are two solutions. You can verify this by taking derivatives. From what we discussed in section 3.2, we know that $y(t) = c_1 \cos(t) + c_2 \sin(t)$ is the general solution (notice that the Wronskian is never zero).
   Now compare this to the characteristic equation: $r^2 + 1 = 0$ has roots $r_1 = -i$ and $r_2 = i$. In this case, $\lambda = 0$ and $\omega = 1$. So we see in this example that there seems to be some connection between complex roots and solutions that involve Sine and Cosine.

2. Let's explore more: Consider $y'' + 9y = 0$. Again by guess and check, notice that $y_1(t) = \cos(3t)$ and $y_2(t) = \sin(3t)$ are solutions. Thus, the general solution is $y(t) = c_1 \cos(3t) + c_2 \sin(3t)$.
   Comparing the characteristic equation: $r^2 + 9 = 0$ has roots $r = \pm 3i$. In this case, $\lambda = 0$ and $\omega = 3$. Notice the connection between the number 3 and the coefficients inside the trig functions.

3. Now consider $y'' + 2y' + 17y = 0$. Guess and check is harder here, so let's go straight to the characteristic equation: $r^2 + 2r + 17 = 0$ has roots $r = \frac{-2 \pm \sqrt{4-68}}{2} = -1 \pm 4i$. Based on what we saw in the last two examples, we might guess that our solutions will involve $\cos(4t)$ and $\sin(4t)$. If we treat the real part of the root the same way we treat real roots, then we also might guess that our solutions will involve e^{-t}. You can check that $y_1(t) = e^{-t} \cos(4t)$ and $y_2(t) = e^{-t} \sin(4t)$ are indeed solutions (compute y' and y'') and you can check that the Wronskian is not zero.

4. See the next page, so a derivation that isn't guess and check.

Euler's Formula and Derivation of the Solution

1. In section 3.1 (for real roots), we wrote all our solutions as combinations of $e^{r_1 t}$ and $e^{r_2 t}$. From our observations on the previous page, it would be nice to define $e^{\omega i}$ so that it somehow gave answers involving Cosines and Sines. In addition, using Taylor series, in my review of complex numbers (read that review sheet for more details), we saw that the following expressions are the same

$$e^{\omega i} = \cos(\omega t) + i\sin(\omega t).$$

   This is all coming together nicely. We will use this definition and it will give answers in the form we are seeing in our examples!

2. If you start with $ay'' + by' + cy = 0$ and get a characteristic equation $ar^2 + br + c = 0$ that has the complex roots $r_1 = \lambda + \omega i$ and $r_2 = \lambda - \omega i$, then, using the same method from 3.1 along with Euler's formula, you get the following:

$$y(t) = a_1 e^{r_1 t} + a_2 e^{r_2 t} = a_1 e^{\lambda t + \omega t i} + a_2 e^{\lambda t - \omega t i} \tag{1}$$

$$= a_1 e^{\lambda t} e^{\omega t i} + a_2 e^{\lambda t} e^{-\omega t i} = e^{\lambda t}\left(a_1 e^{\omega t i} + a_2 e^{-\omega t i}\right) \tag{2}$$

$$= e^{\lambda t}\left(a_1 \cos(\omega t) + a_1 i \sin(\omega t) + a_2 \cos(-\omega t) + a_2 i \sin(-\omega t)\right) \tag{3}$$

$$= e^{\lambda t}\left(a_1 \cos(\omega t) + a_1 i \sin(\omega t) + a_2 \cos(\omega t) - a_2 i \sin(\omega t)\right) \tag{4}$$

$$= e^{\lambda t}\left((a_1 + a_2)\cos(\omega t) + (a_1 i - a_2 i)\sin(\omega t)\right) \tag{5}$$

$$= e^{\lambda t}\left(c_1 \cos(\omega t) + c_2 \sin(\omega t)\right) \tag{6}$$

Note: In going from lines (3) to (4), we use the fact that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$ which are well known facts that always hold for these functions. These identities say that $\cos(x)$ is symmetric about the y-axis (*i.e.* it is an 'even' function) and that $\sin(x)$ gives the same graph if you reflect across the y-axis, then reflect across the x-axis (*i.e.* it is an 'odd' function).

Also note that in line (6), we are writing $c_1 = a_1 + a_2$ and $c_2 = a_1 i - a_2 i$. In this course, we will only give initial conditions that involve real numbers, so c_1 and c_2 will always be real numbers, even if you left the i in the general answer (which is fine if you do that), when you plug in the initial conditions and solve you would also find that the numbers in front of $\cos(\omega t)$ and $\sin(\omega t)$ are always real numbers in this class. (Ask me about this in office hours and I can show you what I mean).

3.4: Homogeneous Constant Coefficient 2nd Order (Repeated Roots)

Just like in my 3.3 review, let me give away the 'punchline'. In other words, let me show how easy it is to solve these problems once you know the general result, then we'll discuss the theoretical underpinnings:

Solutions for the One Real Root Case:
If $ar^2 + br + c = 0$ has only one real roots r, then the general solution to $ay'' + by' + cy = 0$ is given by

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

Examples:

1. Give the general solution to $y'' + 10y' + 25y = 0$.

   Solution: The equation $r^2 + 10r + 25 = (r+5)^2 = 0$ has only one root $r = -5$.
   The general solution is $y = c_1 e^{-5t} + c_2 t e^{-5t}$.

2. Give the general solution to $y'' - 6y' + 9y = 0$.

   Solution: The equation $r^2 - 6r + 9 = (r-3)^2 = 0$ has only one root $r = 3$.
   The general solution is $y = c_1 e^{3t} + c_2 t e^{3t}$.

Examples with initial conditions:

1. Solve $y'' + 4y' + 4 = 0$ with $y(0) = 2$ and $y'(0) = 5$.

   Solution: The equation $r^2 + 4r + 4 = (r+2)^2 = 0$ has only one root $r = -2$.
   The general solution is $y = c_1 e^{-2t} + c_2 t e^{-2t}$.
   Note that $y' = -2c_1 e^{-2t} + c_2(e^{-2t} - 2te^{-2t})$.

   $$\begin{aligned} y(0) = 2 \quad &\Rightarrow \quad c_1 + 0 = 2 \Rightarrow c_1 = 2 \\ y'(0) = 5 \quad &\Rightarrow \quad -2c_1 + c_2 = 5 \Rightarrow c_2 = 9 \end{aligned}$$

   Thus, the solution is $y(t) = 2e^{-2t} + 9te^{-2t}$.

2. Solve $y'' - 8y' + 16y = 0$ with $y(0) = -3$ and $y'(0) = 1$.

   Solution: The equation $r^2 - 8r + 16 = (r-4)^2 = 0$ has only one root $r = 4$.
   The general solution is $y = c_1 e^{4t} + c_2 t e^{4t}$.
   Note that $y' = 4c_1 e^{4t} + c_2(e^{4t} + 4te^{4t})$.

   $$\begin{aligned} y(0) = -3 \quad &\Rightarrow \quad c_1 + 0 = -3 \Rightarrow c_1 = -3 \\ y'(0) = 4 \quad &\Rightarrow \quad 4c_1 + c_2 = 1 \Rightarrow c_2 = 13 \end{aligned}$$

   Thus, the solution is $y(t) = -3e^{4t} + 13te^{4t}$.

Observations and Motivation:
As we discussed in class, we started with one solution $y_1(t) = e^{rt}$ and we needed to find another. We made the educated guess that a second solution might have the form $y(t) = u(t)e^{rt}$. By differentiating and substituting, we found that this indeed gave a solution when $u(t) = t$. This is called the method of reduction of order. This is a **general method** that works for linear questions (even for higher order). It takes one known solution and attempts to find other solutions. Here is a more general discussion:

Method of Reduction of Order:
If $y = y_1(t)$ is one known solution to $y'' + p(t)y' + q(t)y = 0$, then the method of reduction attempts to find another solution as follow:

1. Write $y = u(t)y_1(t)$. You will attempt to find $u(t)$.

2. Find $y' = u'(t)y_1(t) + u(t)y_1'(t)$ and $y'' = u''(t)y_1(t) + 2u'(t)y_1'(t) + u(t)y_1''(t)$.

3. Substitute into $y'' + p(t)y' + q(t)y = 0$ and simplify.

4. You now will have an equation in the form $y_1(t)u'' + (2y_1'(t) + p(t)y_1(t))u' = 0$. Note that if you write $v(t) = u'(t)$, then this equation is the first order equation $y_1(t)\frac{dv}{dt} + (2y_1'(t) + p(t)y_1(t))v = 0$. Solve this first order equation! From this get $u'(t)$.

5. Integrate to get $u(t)$. This will involve constants of integration. For any choice of those constants, the following with be a solution: $y_2(t) = u(t)y_1(t)$. (We look for a solution that is indeed different from the first).

Side Note: With a bit of general work with integrating factors and some simplification, you can find that the solution of $y_1(t)\frac{dv}{dt} + (2y_1'(t) + p(t)y_1(t))v = 0$ will look like $v(t) = \frac{1}{y_1(t)}e^{-\int p(t)\,dt}$ is a solution to this first order equation. Since $u'(t) = v(t)$, that means that $u(t) = \int v(t)\,dt = \int \frac{1}{y_1(t)}e^{-\int p(t)\,dt}\,dt$. This is a compact integral formula for the final form of $u(t)$. But for the problems we do, it will be just as easy to follow the procedure above.

Examples of Reduction of Order are on the next page.

Examples:

1. First, let's redo the example of a repeated root:
   Assume you want to solve $y'' + 10y' + 25y = 0$ and you know one solution is $y_1(t) = e^{-5t}$.

   Solution:

   (a) Let $y = u(t)e^{-5t} = ue^{-5t}$,

   (b) Then $y' = u'e^{-5t} - 5ue^{-5t} = (u' - 5u)e^{-5t}$ and
   $y'' = (u'' - 5u')e^{-5t} - 5(u' - 5u)e^{-5t} = (u'' - 10u' + 25u)e^{-5t}$

   (c) Substiting gives $y'' + 10y' + 25y = (u'' - 10u' + 25u)e^{-5t} + 10(u' - 5u)e^{-5t} + 25ue^{-5t} = 0$,
   which simplifies to $u'' - 10u' + 25u + 10u' - 15u + 25u = u'' = 0$
   (Note: The u's will always cancel here! In this case, the u' also cancelled, but that won't always happen)

   (d) Letting $v = u'$, we see that we are looking at a first order equation $v' = 0$, which has solution $u'(t) = v(t) = a_1$ (a constant).
   Integrating again we get $u(t) = a_1 t + a_2$.

   (e) Thus, any answer in the form $y = u(t)e^{-5t} = a_1 te^{-5t} + a_2 e^{-5t}$ is also a solution. We can see a 'new' solution here is $y_2(t) = te^{-5t}$.

   The general answer is $y = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{-5t} + c_2 te^{-5t}$.

2. Another example:
   Assume you need to solve the differential equation $t^2 y'' - 6ty' + 12y = 0$. After some experimentation, you find one solution is $y_1(t) = t^3$. Find the general solution.

   Solution:

   (a) Let $y = ut^3$,

   (b) Then $y' = u't^3 + 3ut^2$ and
   $y'' = u''t^3 + 3u't^2 + 3u't^2 + 6ut = u''t^3 + 6u't^2 + 6ut$

   (c) Substiting gives $t^2(u''t^3 + 6u't^2 + 6ut) - 6t(u't^3 + 3ut^2) + 12ut^3 = 0$,
   which expands to $t^3u'' + 2t^2u' - u't^3 - ut^2 - 4u't^2 - 4ut + t^2u + 4tu = 0$.
   which simplies to $t^5u'' = 0$, so we again need $u'' = 0$.

   (d) Again we get $u(t) = a_1 t + a_2$.

   (e) Thus, any answer in the form $y = u(t)t^3 = (a_1 t + a_2)t^3 = a_1 t^4 + a_2 t^3$ is also a solution. We can see a 'new' solution here is $y_2(t) = t^4$.

   The general answer is $y = c_1 y_1(t) + c_2 y_2(t) = c_1 t^3 + c_2 t^4$.

3. Another ‘messier’ example:
Assume you need to solve the differential equation $t^2y'' - t(t+4)y' + (t+4)y = 0$. After some experimentation, you find one solution is $y_1(t) = t$. Find the general solution.

Solution:

(a) Let $y = ut$,

(b) Then $y' = u't + u$ and $y'' = u''t + u' + u' = u''t + 2u'$

(c) Substiting gives $t^2(u''t + 2u') - t(t+4)(u't + u) + (t+4)ut = 0$,
which expands to $t^3u'' + 2t^2u' - u't^3 - ut^2 - 4u't^2 - 4ut + t^2u + 4tu = 0$.
which simplies to $t^3u'' - (t^3 + 2t^2)u' = 0$.
Dividing by t^3 we see we need to solve $u'' - \left(1 + \frac{2}{t}\right) u' = 0$.

(d) Letting $v = u'$, we see that we are looking at a first order equation $v' - \left(1 + \frac{2}{t}\right) v = 0$, which has solution $u'(t) = v(t) = a_1 t^2 e^t$ (do this by integrating factors or separation!)
Integrating again (by parts twice) we get $u(t) = a_1 e^t(t^2 - 2t + 2) + a_2$.

(e) Thus, any answer in the form:
$y = u(t)t = (a_1e^t(t^2 - 2t + 2) + a_2)t = a_1te^t(t^2 - 2t + 2) + a_2t$ is also a solution.
We can see a ‘new’ solution here is $y_2(t) = te^t(t^2 - 2t + 2) = e^t(t^3 - 2t^2 + 2t)$.

The general answer is $y = c_1y_1(t) + c_2y_2(t) = c_1t + c_2e^t(t^3 - 2t^2 + 2t)$.

4. A third order example!
Assume you need to solve the **third order** differential equation $y''' - 7y' + 6y = 0$. After some experimentation, you find one solution is $y_1(t) = e^t$. Find the general solution (you need three different solution).

Solution:

(a) Let $y = ue^t$,

(b) Then $y' = u'e^t + ue^t = (u' + u)e^t$, $y'' = (u'' + u')e^t + (u' + u)e^t = (u'' + 2u' + u)e^t$, and
$y''' = (u''' + 2u'' + u')e^t + (u'' + 2u' + u)e^t = (u''' + 3u'' + 3u' + u)e^t$

(c) Substiting gives $(u''' + 3u'' + 3u' + u)e^t - 7(u' + u)e^t + 6ue^t = 0$,
which expands to $u''' + 3u'' + 3u' + u - 7u' - 7u + 6u = 0$.
which simplies to $u''' + 3u'' - 4u' = 0$. (Note: that u is gone)

(d) Letting $v = u'$, we see that we are looking at a **second order** equation $v'' + 3v' - 4v = 0$ (we have reduce the order).
Using our current methods, we can solve this by getting the characteristic equation $r^2 + 3r - 4 = (r+4)(r-1) = 0$ so the solution is $u'(t) = v(t) = a_1e^{-4t} + a_2e^t$.
Integrating again gives $u(t) = \frac{a_1}{-4}e^{-4t} + a_2e^t + a_3$ (let's redefine $a_1 = a_1/(-4)$ from here on out since it is just constant)

(e) Thus, any answer in the form:
$y = u(t)e^t = (a_1e^{-4t} + a_2e^t + a_3)e^t = a_1e^{-3t} + a_2e^{2t} + a_3e^t$ is also a solution.
We can see two ‘new’ solutions here are $y_2(t) = e^{-3t}$ and $y_3(t) = e^{2t}$.

The general answer is $y = c_1y_1(t) + c_2y_2(t) + c_3y_3(t) = c_1e^t + c_2e^{-3t} + c_3e^{2t}$.

3.1, 3.3, 3.4: Homogeneous Constant Coefficient 2nd Order

Given $ay'' + by' + cy = 0$, $y(t_0) = y_0$ and $y'(t_0) = y'_0$.

Step 1: Write the characteristic equation $ar^2 + br + c = 0$ and find the roots $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Step 2: Write your answer in the appropriate form:

1. If $b^2 - 4ac > 0$, then there are two real roots r_1 and r_2 and the general solution is
$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$
2. If $b^2 - 4ac = 0$, then there is one real root r and the general solution is
$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$
3. If $b^2 - 4ac < 0$, then there are two complex roots $r = \lambda \pm \omega i$ and the general solution is
$$y(t) = e^{\lambda t}\left(c_1 \cos(\omega t) + c_2 \sin(\omega t)\right).$$

Step 3: Use initial conditions

1. Find $y'(t)$.
2. Plug in $y(t_0) = y_0$.
3. Plug in $y'(t_0) = y'_0$.
4. Combine and solve for c_1 and c_2.

Several quick examples (answers on back):

1. Solve $y'' + 2y' + y = 0$.
2. Solve $y'' - 10y' + 24y = 0$.
3. Solve $y'' + 5y = 0$.
4. Solve $y'' - 3y' = 0$.
5. Solve $y'' + 12y' + 36y = 0$.
6. Solve $y'' + y' + y = 0$.

Several quick examples (answers on back):

1. $r^2 + 2r + 1 = (r+1)^2 = 0$:
   $y(t) = c_1 e^{-t} + c_2 t e^{-t}$.
2. $r^2 - 10r + 24 = (r-6)(r-4) = 0$:
   $y(t) = c_1 e^{6t} + c_2 e^{4t}$.
3. $r^2 + 5 = 0$, $r = \pm\sqrt{5}i$:
   $y(t) = c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)$.
4. $r^2 - 3r = r(r-3) = 0$:
   $y(t) = c_1 + c_2 e^{3t}$
5. $r^2 + 12r + 36 = (r+6)^2 = 0$:
   $y(t) = c_1 e^{-6t} + c_2 t e^{-6t}$
6. $r^2 + r + 1 = 0$, $r = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$:
   $y(t) = e^{-t/2}\left(c_1 \cos(\sqrt{3}t/2) + c_2 \sin(\sqrt{3}t/2)\right)$

3.5: Non-homogeneous Constant Coefficient Second Order (Undetermined Coefficients)

Given $ay'' + by' + cy = g(t)$, $y(t_0) = y_0$ and $y'(t_0) = y_0'$.

Step 1: Find the general solution of the homogeneous equation $ay'' + by' + cy = 0$.
(Write and solve the characteristic equation, then use methods from 3.1, 3.3, and 3.4).
At this point, you'll have two independent solutions to the homogeneous equation: $y_1(t)$ and $y_2(t)$.

Step 2: From the table below, identify the likely form of the answer of a **particular solution**, $Y(t)$, to $ay'' + by' + cy = g(t)$.

Table of Particular Solution Forms

$g(t)$	e^{rt}	$\sin(\omega t)$ or $\cos(\omega t)$	C	t	t^2	t^3
$Y(t)$	Ae^{rt}	$A\cos(\omega t) + B\sin(\omega t)$	A	$At + B$	$At^2 + Bt + C$	$At^3 + Bt^2 + Ct + D$

First some notes on the use of this table:

- If $g(t)$ is a sum/difference of these problems, then so is $Y(t)$.
  For example, if $g(t) = e^{4t} + \sin(5t)$, then try $Y(t) = Ae^{4t} + B\cos(5t) + C\sin(5t)$.
- If $g(t)$ is a product of these problems, then so is $Y(t)$.
  For example, if $g(t) = t^2e^{5t}$, then try $Y(t) = (At^2 + Bt + C)e^{5t}$.
- **Important:** How to adjusting for homogeneous solutions
  Consider a particular term of $g(t)$. If the table suggests you use the form $Y(t)$ for this term, but $Y(t)$ contains a homogeneous solution, then you need to multiply by t (and if that still constains a homogeneous solution, then multiple by t^2 instead).

  For example, $g(t) = te^{2t}$, then you would initially guess the form $Y(t) = (At + B)e^{2t}$. But if the homogeneous solutions are $y_1(t) = e^{2t}$ and $y_2(t) = e^{5t}$, then Be^{2t} is a multiple of a homogeneous solution. So you use the form: $Y(t) = t(At + B)e^{2t} = (At^2 + Bt)e^{2t}$.

  For another example, if $g(t) = te^{7t}$, then you would initially guess the form $Y(t) = (At + B)e^{7t}$. But if the homogeneous solutions are $y_1(t) = e^{7t}$ and $y_2(t) = te^{7t}$, then Be^{7t} AND Ate^{7t} are both multiples of a homogeneous solution. So you use the form: $Y(t) = t^2(At + B)e^{7t} = (At^3 + Bt^2)e^{7t}$

Step 3: Compute $Y'(t)$ and $Y''(t)$. Substitute $Y(t)$, $Y'(t)$ and $Y''(t)$ into $ay'' + by' + cy = g(t)$.

Step 4: Solve for the coefficients and write your general solution:

$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t)$$

Step 5: Use the initial conditions and solve for c_1 and c_2.

Here are some problems to practice identifying the correct form.
In line, you are given $g(t)$ as well as independent homogeneous solutions $y_1(t)$, and $y_2(t)$.
Give the form of the particular solution, $Y(t)$ (solutions below).

1.	$ay'' + by' + cy = e^{2t}$	$y_1(t) = \cos(t)$	$y_2(t) = \sin(t)$
2.	$ay'' + by' + cy = \cos(3t)$	$y_1(t) = e^{3t}$	$y_2(t) = e^{-t}$
3.	$ay'' + by' + cy = e^{4t}$	$y_1(t) = e^{4t}$	$y_2(t) = e^{-2t}$
4.	$ay'' + by' + cy = t$	$y_1(t) = e^{6t}$	$y_2(t) = te^{6t}$
5.	$ay'' + by' + cy = e^{3t}$	$y_1(t) = e^{3t}$	$y_2(t) = te^{3t}$
6.	$ay'' + by' + cy = e^{t}\sin(5t)$	$y_1(t) = e^{-t}$	$y_2(t) = e^{6t}$
7.	$ay'' + by' + cy = \sin(t) + t$	$y_1(t) = e^{-2t}\cos(4t)$	$y_2(t) = e^{-2t}\sin(4t)$
8.	$ay'' + by' + cy = \cos(2t)$	$y_1(t) = \cos(2t)$	$y_2(t) = \sin(2t)$
9.	$ay'' + by' + cy = 5 + e^{2t}$	$y_1(t) = e^{3t}$	$y_2(t) = e^{-6t}$
10.	$ay'' + by' + cy = te^{2t}\cos(5t)$	$y_1(t) = e^{t}$	$y_2(t) = te^{t}$

Solutions

1. $Y(t) = Ae^{2t}$.
2. $Y(t) = A\cos(3t) + B\sin(3t)$.
3. $Y(t) = Ate^{4t}$.
4. $Y(t) = At + B$.
5. $Y(t) = At^2e^{3t}$.
6. $Y(t) = e^t(A\cos(5t) + B\sin(5t))$.
7. $Y(t) = A\cos(t) + B\sin(t) + Ct + D$.
8. $Y(t) = At\cos(2t) + Bt\sin(2t)$.
9. $Y(t) = A + Be^{2t}$.
10. $Y(t) = (At + B)e^{2t}\cos(5t) + (Ct + D)e^{2t}\sin(5t)$.

Examples:

1. Give the general solution to $y'' + 10y' + 21y = 5e^{2t}$.

   Solution:

   (a) *Solve Homogeneous*:
   The equation $r^2 + 10r + 21 = (r+3)(r+7) = 0$ has the roots $r_1 = -3$ and $r_2 = -7$.
   So $y_1(t) = e^{-3t}$ and $y_2(t) = e^{-7t}$

   (b) *Particular Solution Form*:
   $Y(t) = Ae^{2t}$

   (c) *Substitute*:
   $Y'(t) = 2Ae^{2t}$ and $Y''(t) = 4Ae^{2t}$. Substituting gives
   $4Ae^{2t} + 10(2Ae^{2t}) + 21(Ae^{2t}) = 5e^{2t} \Rightarrow 45Ae^{2t} = 5e^{2t}$. Thus, $A = \frac{5}{45} = \frac{1}{9}$.

   (d) *General Solution*:
   $y(t) = c_1e^{-3t} + c_2e^{-7t} + \frac{1}{9}e^{2t}$.

2. Give the general solution to $y'' - 2y' + y = 6t$.

   Solution:

   (a) *Solve Homogeneous*:
   The equation $r^2 - 2r + 1 = (r-1)^2 = 0$ has the one root $r = 1$.
   So $y_1(t) = e^t$ and $y_2(t) = te^t$.

   (b) *Particular Solution Form*:
   $Y(t) = At + B$

   (c) *Substitute*:
   $Y'(t) = A$ and $Y''(t) = 0$. Substituting gives
   $(0) - 2(A) + (At + B) = 5t \Rightarrow At + (B - 2A) = 6t$. Thus, $A = 6$ and $B - 2A = 0$. So $B = 12$

   (d) *General Solution*:
   $y(t) = c_1e^t + c_2te^t + 6t + 12$.

3. Give the general solution to $y'' + 4y = \cos(t)$.

Solution:

(a) *Solve Homogeneous*:
The equation $r^2 + 4 = 0$ has the roots $r = \pm 2i$.
So $y_1(t) = \cos(2t)$ and $y_2(t) = \sin(2t)$.

(b) *Particular Solution Form*:
$Y(t) = A\cos(t) + B\sin(t)$

(c) *Substitute*:
$Y'(t) = -A\sin(t) + B\cos(t)$ and $Y''(t) = -A\cos(t) - B\sin(t)$. Substituting gives $(-A\cos(t) - B\sin(t)) + 4(A\cos(t) + B\sin(t)) = \cos(t) \Rightarrow 3A\cos(t) + 3B\sin(t) = \cos(t)$.
Thus, $A = \frac{1}{3}$ and $B = 0$.

(d) *General Solution*:
$y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{3}\cos(t)$.

4. Give the general solution to $y'' - 5y' = 3e^{5t}$.

Solution:

(a) *Solve Homogeneous*:
The equation $r^2 - 5r = 0$ has the roots $r_1 = 0$, $r_2 = 5$.
So $y_1(t) = 1$ and $y_2(t) = e^{5t}$.

(b) *Particular Solution Form*:
$Y(t) = Ate^{5t}$ (because $y_2(t) = e^{5t}$).

(c) *Substitute*:
$Y'(t) = Ae^{5t} + 5Ate^{5t} = A(1+5t)e^{5t}$ and $Y''(t) = 5Ae^{5t} + 5A(1+5t)e^{5t} = A(10+25t)e^{5t}$.
Substituting gives
$A(10+25t)e^{5t} - 5A(1+5t)e^{5t} = 3e^{5t} \Rightarrow 5Ae^{5t} = 3e^{5t}$. Thus, $A = \frac{3}{5}$.

(d) *General Solution*:
$y(t) = c_1 + c_2e^{5t} + \frac{3}{5}te^{5t}$.

5. Give the general solution to $y'' - 3y' + 3y = 3t + e^{-2t}$.

Solution:

(a) *Solve Homogeneous*:
The equation $r^2 - 3r + 3 = 0$ has the roots $r = \frac{3 \pm \sqrt{9-12}}{2} = \frac{3}{2} \pm \frac{\sqrt{3}}{2}i$.
So $y_1(t) = e^{3t/2}\cos(\sqrt{3}t/2)$ and $y_2(t) = e^{3t/2}\sin(\sqrt{3}t/2)$.

(b) *Particular Solution Form*:
$Y(t) = At + B + Ce^{-2t}$.

(c) *Substitute*:
$Y'(t) = A - 2Ce^{5t}$ and $Y''(t) = 4Ce^{5t}$. Substituting gives
$4Ce^{-5t} - 3(A - 2Ce^{-2t}) + 3(At + B + Ce^{-2t}) = 3t + e^{-2t} \Rightarrow 3At + (-3A + 3B) + (4C + 6C + 3C)e^{5t} = 3t + e^{-2t}$. Thus, $3A = 3$, $-3A + 3B = 0$ and $13C = 1$. So $A = 1$, $B = 1$, and $C = \frac{1}{13}$

(d) *General Solution*:
$y(t) = c_1 e^{3t/2}\cos(\sqrt{3}t/2) + c_2 e^{3t/2}\sin(\sqrt{3}t/2) + t + 1 + \frac{1}{13}e^{-2t}$.

6. Give the general solution to $y'' - 9y = (5t^2 - 1)e^t$.

Solution:

(a) *Solve Homogeneous*:
The equation $r^2 - 9 = 0$ has the roots $r = \pm 3$.
So $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$.

(b) *Particular Solution Form*:
$Y(t) = (At^2 + Bt + C)e^t$

(c) *Substitute*:
$Y'(t) = (2At + B)e^t + (At^2 + Bt + C)e^t = (At^2 + (2A + B)t + (B + C))e^t$ and
$Y''(t) = (2At + (2A + B))e^t + (At^2 + (2A + B)t + (B + C))e^t = (At^2 + (4A + B)t + (2A + 2B + C))e^t$.
Substituting gives
$(At^2 + (4A + B)t + (2A + 2B + C))e^t - 9(At^2 + Bt + C)e^t = (5t^2 - 1)e^t$
$\Rightarrow -8At^2 + (4A - 8B)t + (2A + 2B - 8C) = 5t^2 - 1$.
Thus, $-8A = 5$, $4A - 8B = 0$ and $2A + 2B - 8C = -1$. So $A = -\frac{5}{8}$, $B = \frac{1}{2}A = -\frac{5}{16}$, and $C = \frac{2A+2B+1}{8} = -\frac{5}{32} - \frac{5}{64} + \frac{1}{8} = -\frac{7}{64}$.

(d) *General Solution*:
$y(t) = c_1 e^{3t} + c_2 e^{-3t} + (-\frac{5}{8}t^2 - \frac{5}{16}t - \frac{7}{64})e^t$.

Chapter 3: Summary of Second Order Solving Methods

We only discussed solution methods for **linear** second order equations.

Constant Coefficient Methods: To solve an equation of the form: $ay'' + by' + cy = g(t)$.

Homogeneous (when $\mathbf{g(t) = 0}$): Solve $ar^2 + br + c = 0$ to get $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$b^2 - 4ac > 0$	Two real roots: r_1 and r_2	General Solution: $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.
$b^2 - 4ac = 0$	Repeated root: r	General Solution: $y(t) = c_1 e^{rt} + c_2 t e^{rt}$.
$b^2 - 4ac < 0$	Complex roots: $r = \lambda \pm \omega i$	General Solution: $y(t) = c_1 e^{\lambda t}\cos(\omega t) + c_2 e^{\lambda t}\sin(\omega t)$.

Nonhomogeneous (when $\mathbf{g(t) \neq 0}$):

1. Solve the corresponding homogeneous equation and get independent solutions $y_1(t)$ and $y_2(t)$.
2. Find *any* particular solution, $Y(t)$, to $ay'' + by' + cy = g(t)$.
   - Option 1: If $g(t)$ is a product or sum of polynomials, exponentials, sines or cosines, then use **undetermined coefficients**.
   - Option 2: If $g(t)$ involves some function other than those mentioned above, then use **reduction of order** (or more generally, variation of parameters).
3. General Solution: $y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$.

Nonconstant Coefficient Methods: To solve an equation of the form: $y'' + p(t)y' + q(t)y = g(t)$.
Homogeneous (when $\mathbf{g(t) = 0}$):

1. Option 1: If the equation can be written as $P(x)y'' + Q(x)y' + R(x)y = 0$, then we say it is **exact** when $P''(x) - Q'(x) + R(x) = 0$. In 3.2/41-45, you see how to solve these.
   (a) Let $f(x) = Q(x) - P'(x)$. Note: $P(x)y'' + Q(x)y' + R(x) = 0$ is the same as $\frac{d}{dx}(P'(x)y') + \frac{d}{dx}(f(x)y) = 0$.
   (b) Integrate both sides to get $P'(x)y' + f(x)y = c_1$. Solve this 1st order equation.
2. Option 2: Change the variable. The only examples we saw were **Euler equations** which take the form: $t^2 y'' + \alpha t y' + \beta y = 0$. In 3.3/34-41, you see how to solve these
   (a) Making the change of variable $x = \ln(t)$ leads to $y'' + (\alpha - 1)y' + \beta y = 0$.
   (b) Solve this constant coefficient equation (using methods above).
   (c) This gives a solution equation $y = y(x)$. Now replace x with $\ln(t)$.

Nonhomogeneous (when $\mathbf{g(t) \neq 0}$): To solve an equation of the form: $y'' + p(t)y' + q(t)y = g(t)$.

1. Solve the corresponding homogeneous equation and get a solution $y = y_1(t)$ (if possible, find a second independent solution as well $y_2(t)$).
2. Use **reduction of order**,
   (a) Write $y = u(t)y_1(t)$. And compute y' and y''
   (b) Plug in and try to solve for $u(t)$. (not always possible)
   (c) Then $y = u(t)y_1(t)$ will be a general solution.
3. Or use variation of parameters from section 3.6 (you are not expected to know this for the exam).
4. General Solution: $y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$

3.7 and 3.8: Mechanical and Electrical Vibrations Application Descriptions

In this sheet, we discuss the set up of two applications of second order constant homogeneous equations.

Application 1: Oscillating Spring (See the first figure in section 3.7)
A spring is attached to the ceiling and allowed to hang downward.
Let l be the natural length of a spring with no mass attached.
Let L be the distance beyond natural length it is stretched when an object of mass of m kg is attached.
In other words, $l + L$ is the distance from the ceiling when the object is at rest.
Let $u(t)$ be the displacement of the spring from rest (with positive downward) at time t.
We will move the object to a starting displacement $u(0) = u_0$ and push it with an initial velocity $u'(0) = v_0$ and study the resulting motion.
Forces:

- $F_g = w = mg$. (Force to to gravity)
  Another name for this is the 'weight'. It is always downward which we are calling positive.
- $F_s = -k(L + u(t))$. (Force due to the spring, *i.e.* restoring force)
  This is 'Hooke's Law' which says that the force is proportional to the distance from natural position. In this case $L + u$ is the distance from natural position. Note that if $L + u$ is positive, then this force will be negative (upward) and if $L + u$ is negative this force will be positive (downward).
- $F_d = -\gamma u'(t)$. (Force due to damping, *i.e.* friction force)
  This is one model for friction that assumes that the friction force it proportional to velocity and in the positive direction. Note that if $u'(t)$ is positive, then F_d is negative and if $u'(t)$ is negative, then F_d is positive. We used the same model earlier in the term for air resistance.
- $F_e = F(t) =$ 'some external force'.
  This can be any function (typically periodic) that describes an external force for any time t.
- Special Note: When the object is at rest (in other words when it is sitting with $u(0) = 0$ and $u'(0) = 0$) all the forces will add to zero. Which means that $mg - kL = 0$. Thus, in this situation we always have
  $$w = mg = kL.$$

Newton's second law says that '(mass)(acceleration) = force', so we have:

$$mu''(t) = mg - k(L + u(t)) - \gamma u'(t) + F(t) = mg - kL - ku(t) - \gamma u'(t) + F(t)$$

Thus,

$$mu'' + \gamma u' + ku = F(t).$$

Note:

- $m =$ 'the mass of the object':
  From above $w = mg$ and $m = \frac{w}{g}$.
- $\gamma =$ 'the damping constant' = 'the proportionality constant in the friction force'
  From above $F_d = -\gamma u'$ and $\gamma = -\frac{F_d}{u'}$.
- $k =$ 'the spring constant' = 'the proportionality constant in the spring force'
  From above $w = mg = kL$, so $k = \frac{w}{L} = \frac{mg}{L}$.

Comment about units:
In, US standard units the unit pounds (lbs) is a force unit. **Pounds (lbs) is NOT a mass unit**. Pounds is already weight, w, you don't need to multiply by gravity. However, in metric units the unit kilograms (kg) is a mass unit (it is NOT force unit), so you do have to multiply by, $g = 9.8$, in order to get the force unit of Newtons. Let me summarize the important unit facts below:

Type	Metric	US Standard
$m =$ Mass	kg	slugs (not commonly used)
$g =$ Accel. due to gravity on Earth	9.8 m/s^2	32 ft/s^2
$w = mg =$ Weight (Force)	N = Newtons	pounds = lbs
$u(t) =$ displacement	$m =$ meters	$ft =$ feet

Example:

1. A mass weighing 3 kg stretches a spring 60 cm (0.06 meters) beyond natural length.
   The force due to resistance is 8 N when the upward velocity is 2 m/s (*i.e.* when $u' = -2$).
   The mass is given an initial displacement of 20 cm (0.02 meters) and is released (*i.e.* the initial velocity is zero). Assume there is no external forcing. Set up the differential equation and initial conditions for u.

   Solution:
   You are given $m = 3$ kg, $L = 0.06$ m, and we know $g = 9.8$ m/s^2.
   At rest we know $mg = kL$. Thus, $k = \frac{w}{L} = \frac{mg}{L} = \frac{3 \cdot 9.8}{0.06} = 490$ N/m.
   We also are told that $F_d = -\gamma u' = 8$ N when $u' = -2$ m/s. Thus, $\gamma = -\frac{F_d}{u'} = -\frac{8}{-2} = 4$ N·s/m.
   Therefore, $3u'' + 4u' + 490u = 0$, with $u(0) = 0.02$ and $u'(0) = 0$.

2. A mass weighing 8 lbs stretches a spring 2 in ($\frac{1}{6}$ ft) beyond natural length.
   The force due to resistance is 3 lbs when the upward velocity is 1 ft/s (*i.e.* when $u' = -1$).
   The mass is given an initial displacement of 6 in and an initial upward velocity of 2 ft/s.
   Assume there is no external forcing. Set up the differential equation and initial conditions for u.

   Solution:
   You are given $w = mg = 8$ lbs, $L = \frac{1}{6}$ ft, and we know $g = 32$ ft/s^2.
   Thus, $m = \frac{w}{g} = \frac{8}{32} = \frac{1}{4}$ lbs$\cdot$ s^2/ft (slugs).
   At rest we know $mg = kL$. Thus, $k = \frac{w}{L} = \frac{8}{1/6} = 48$ lbs/ft.
   We also are told that $F_d = -\gamma u' = 3$ lbs when $u' = -1$ ft/s. Thus, $\gamma = -\frac{F_d}{u'} = -\frac{3}{-1} = 3$ lbs·s/ft.
   Therefore, $\frac{1}{4}u'' + 3u' + 48u = 0$, with $u(0) = \frac{1}{2}$ and $u'(0) = -2$.

Application 2: Electrical Vibrations (see the last figure in section 3.7)
Consider the flow of electricity through a series circuit containing a resistor, an inductor, and a capacitor (called an RLC circuit). The total charge on the capacity at time, t, is $Q = Q(t)$ in coulombs (C). We also define $I = I(t) = Q'(t)$ to be the current in the circuit at time, t, in ampheres (A). Our goal will be to find the function $Q(t)$.
First, let me define some constants, variables and units.

Definitions and Kirchoff's circuit laws:

- Kirchoff's second law states: *In a closed circuit the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit*
- We will let $E = E(t)$ be the impressed voltage in volts (V), which is the incoming voltage to the circuit.
- Laws of electricity:
  1. The voltage drop across the resister is proportional to the current.
     We write $RI = RQ'$, where R is the proportionality constant due to resistance.
     We call R the resistance with the unit ohms (Ω).
  2. The voltage drop across the capacitor is proportional to the total charge.
     Convention is to write $\frac{1}{C}Q$, where $\frac{1}{C}$ is proportionality constant due to the capacitor.
     We call C the capacitance with the unit farads (F).
  3. The voltage drop across the inductor is proportional to the derivative of the current.
     We write $LI' = LQ''$, where L is the proportionality constant due to the inductor.
     We call L the inductance with the unit henrys (H).
- The units are related as follows: $V = \Omega \cdot A = \frac{C}{F}$, and $\Omega = \frac{H}{s}$

Putting these laws together, we have

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

Example:

1. A series circuit has a capacitor of 0.00003 F, a resister of 200 Ω, and an inductor of 0.6 H. There is no impressed voltage. The initial charge on the capacitor is 0.0001 C and there is no initial current. Set up the differential equation and initial conditions for the charge $Q(t)$.

   Solution: You are given $C = 0.00003$, $R = 200$, $L = 0.6$, and $E(t) = 0$.
   Therefore, $0.6Q'' + 200Q' + 0.00003Q = 0$, with $Q(0) = 0.0001$ and $Q'(0) = 0$.

2. A series circuit has a capacitor of 0.0002 F and an inductor of 1.5 H (and no resistor). There is no impressed voltage. The initial charge on the capacitor is 0.005 C and there is no initial current. Set up the differential equation and initial conditions for the charge $Q(t)$.

   Solution:You are given $C = 0.0002$, $R = 0$, $L = 1.5$, and $E(t) = 0$.
   Therefore, $1.5Q'' + 0.0002Q = 0$, with $Q(0) = 0.005$ and $Q'(0) = 0$.

3.7: Analyzing Mechanical and Electrical Vibrations (Free Vibrations)

This review just discusses analysis of these applications. For the set up, read the 3.7 and 3.8 applications review. In 3.7 we are considering, 'free vibrations' which means there is no forcing. In other words, we are considering the homogeneous equation with $F(t) = 0$.

For an object attached to a spring that is not being forced, we found that the displacement from rest, $u(t)$, at time t satisfies:

$$mu'' + \gamma u' + ku = 0,$$

where m is the mass, γ is the damping (friction) constant, and k is the spring constant (all these constants are positive).

We will analyze different cases:

Undamped Free Vibrations: (The $\gamma = 0$ case)
If we assume there is no friction (or that the friction is small enough to be negligible), then we are taking $\gamma = 0$. In which case we get:

$$mu'' + ku = 0.$$

The roots of $mr^2 + k = 0$ are $r = \pm i\sqrt{k/m}$, so the general solution is

$$u(t) = c_1 \cos(\sqrt{k/m}\, t) + c_2 \sin(\sqrt{k/m}\, t).$$

Using the facts from my review sheet on waves (namely, $R = \sqrt{c_1^2 + c_2^2}$, $c_1 = R\cos(\delta)$, and $c_2 = R\sin(\delta)$), we can rewrite this in the form

$$u(t) = R\cos(\omega_0 t - \delta),$$

where $\omega_0 = \sqrt{k/m}$.
Thus, the solution is a cosine wave with the following properties:

- The **natural frequency** is $\omega_0 = \sqrt{k/m}$ radians/second.
- The **period** (or **wavelength**) is $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{m/k}$ seconds/wave (this is the time from peak-to-peak or valley-to-valley).
- The **amplitude** is $R = \sqrt{c_1^2 + c_2^2}$, which will depend on initial conditions.
- The **phase angle** is δ which is the starting angle, which also depends on initial conditions.

Damped Free Vibrations: (The $\gamma > 0$ case)
If $\gamma > 0$, then we have

$$mu'' + \gamma u' + ku = 0.$$

The roots of $mr^2 + \gamma r + k = 0$ are $r = -\frac{\gamma}{2m} \pm \frac{1}{2m}\sqrt{\gamma^2 - 4mk}$. Three different things can happen here:

1. If $\gamma^2 - 4km > 0$, then there are two real roots that are both negative.
   The solution looks like $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.
   The condition simplifies to $\gamma > 2\sqrt{km}$.
   In this case we say the systems is **overdamped**.

2. If $\gamma^2 - 4km = 0$, then there is one repeated root that is negative.
   The solution looks like $y = c_1 e^{rt} + c_2 t e^{rt}$.
   The condition simplifies to $\gamma = 2\sqrt{km}$.
   In this case we say the systems is **critically damped**.

3. If $\gamma^2 - 4km < 0$, then there are two complex roots with $\lambda = -\frac{\gamma}{2m}$ and $\omega = \mu = \frac{\sqrt{4mk-\gamma^2}}{2m}$.
   The solution looks like $y = e^{\lambda t}(c_1 \cos(\mu t) + c_2 \sin(\mu t))$.
   The condition simplifies to $\gamma < 2\sqrt{km}$. In this case, we get oscillations where the amplitude goes to zero. We can analyze the wave part of this last case like we did before.
   The expression $c_1 \cos(\mu t) + c_2 \sin(\mu t)$ can be rewritten as $R\cos(\mu t - \delta)$,
   where $R = \sqrt{c^1 + c_2^2}$, $c_1 = R\cos(\delta)$ and $c_2 = R\sin(\delta)$.
   Thus, in this case, the general answer can be written as

   $$u(t) = Re^{\lambda t}\cos(\mu t - \delta),$$

   where

   - The **quasi frequency** is $\mu = \frac{\sqrt{4mk-\gamma^2}}{2m}$ radians/second.
   - The **quasi period** is $T = \frac{2\pi}{\mu} = 2\pi\frac{2m}{\sqrt{4mk-\gamma^2}}$ seconds/wave.
   - The **amplitude** is $Re^{\lambda t}$, which will always go to zero as $t \to \infty$.

   Note: If the damping is small, then γ is close to zero. Notice that the formulas above for quasi frequency and quasi period become the same as the frequency and period when $\gamma = 0$. So we get similar frequencies and periods between small damping and no damping.

3.8: Analyzing Mechanical and Electrical Vibrations (Forced Vibrations)

In section 3.8 we are considering 'forced vibrations'. In other words, we are considering the nonhomogeneous equation with $F(t) \neq 0$ in this section. One of the most common/natural situations is a forcing function that is oscillating. We have shown how to write waves in the standard way $R\cos(\omega t - \delta)$. Thus, to keep the algebra and analysis simple, we will focus only on forcing functions of the form

$$F(t) = F_0 \cos(\omega t)$$

For the situation of forcing in the mass-spring system, the displacement from rest, $u(t)$, at time t satisfies:

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t),$$

where m is the mass, γ is the damping (friction) constant, and k is the spring constant (all these constants are positive).

Undamped Forced Vibrations: (The $\gamma = 0$ case)
If we assume there is no friction, then we are taking $\gamma = 0$. In which case we get:

$$mu'' + ku = F_0 \cos(\omega t).$$

As we noted in section 3.7, the homogeneous solution has the form $u_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$ where $\omega_0 = \sqrt{k/m}$. The particular solution will have the form $U(t) = A\cos(\omega t) + B\sin(\omega t)$ or the form $U(t) = At\cos(\omega t) + Bt\sin(\omega t)$ depending on whether $\omega \neq \omega_0$ or $\omega = \omega_0$. (Remember our undetermined coefficient discussion if you don't know why).

1. If $\omega \neq \omega_0$, then it turns out a particular solution has the form $U(t) = \frac{F_0}{m(w_0^2 - w^2)} \cos(\omega t)$.
   In this case, the general solution is

   $$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(w_0^2 - w^2)} \cos(\omega t)$$

   In electronics, this situation is used in what is called amplitude modulation. See a nice picture of phenomenon in Figure 3.8.7 of the book.

2. If $\omega = \omega_0$, then it turns out a particular solution has the form $U(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)$.
   In this case, the general solution is

   $$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)$$

   The function $t\sin(\omega_0 t)$ in the solution is **unbounded**! The amplitude of the wave keeps growing. There is never zero damping so this is a bit unrealistic, but this does illustrate that when $\omega \approx \omega_0$ and damping is very small, then the amplitude can get very large. This phenomenon is called **resonance**. We will discuss this again on the next page in the more general case.
   See a nice picture of this phenomenon in Figure 3.8.8 in the book.

Damped Forced Vibrations: (The $\gamma > 0$ case)
If $\gamma > 0$, then we have

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t).$$

The homogeneous solution has the form $u_c(t) = c_1 u_1(t) + c_2 u_2(t)$ where $u_1(t)$ and $u_2(t)$ are determined as we did in section 3.7. (Remember $\gamma > 2\sqrt{mk}$ gives overdamped, $\gamma = 2\sqrt{mk}$ gives critically damped, and $\gamma < 2\sqrt{mk}$ gives oscillations with decreasing amplitudes).
In all these cases when $\gamma \neq 0$, the particular solution will take the form $U(t) = A\cos(\omega t) + B\sin(\omega t)$. Thus, the general solution for undamped forced vibrations will always have the form

$$u(t) = (c_1 u_1(t) + c_2 u_2(t)) + (A\cos(\omega t) + B\sin(\omega t)) = u_c(t) + U(t)$$

Notes:

- The homogeneous solution, in this case, goes to zero as $t \to \infty$. That is, $\lim_{t\to\infty} u_c(t) = 0$.
- Since $u_c(t)$ dies out, we call it the **transient solution**. The transient solution allows us to meet the initial conditions, but in the long run the damping causes the transient solution to die out and the forcing takes over. The particular solution $U(t) = A\cos(\omega t) + B\sin(\omega t)$ is called the **steady state solution**, or **forced response**.
- Through substitution and lengthy algebra, you can find the coefficients in the particular solution. For analysis it is convenient to write the solution in the wave form $U(t) = R\cos(\omega t - \delta)$.
  We get $R = \frac{F_0}{\Delta}$, $\cos(\delta) = \frac{m(w_0^2 - w^2)}{\Delta}$, and $\sin(\delta) = \frac{\gamma\omega}{\Delta}$,
  where $\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$ and $\omega_0 = \sqrt{k/m}$.
  This is messy, but the first observation is that the steady state solution has frequency ω (which is the same as the forcing function). The second observation, with some work, is that the formula for amplitude, R, of the steady state solution can be rewritten as
  $$R = \frac{F_0}{k}\left(\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{\gamma^2}{mk}\frac{\omega^2}{\omega_0^2}\right)^{-1/2}.$$
  Note: as $\omega \to 0$, $R \to \frac{F_0}{k}$, and as $\omega \to \infty$, $R \to 0$. In terms of ω, the maximum value of this function occurs when $\omega_{\text{max}} = \sqrt{\omega_0^2 - \gamma^2/(2m^2)}$ and at this value you get $R_{\text{max}} = \frac{F_0}{\gamma\omega_0\sqrt{1-\gamma^2/(4mk)}}$.

  Thus, if damping is very small (i.e. if γ is close to zero), then the maximum amplitude occurs when $\omega \approx \omega_0$. In which case the amplitude will be about $\frac{F_0}{\gamma\omega_0}$ which can be quite large (and it gets larger the closer γ gets to zero). This phenomenon is known as **resonance**.

  Some Side Comments: Resonance is something you have to worry about with designing buildings and bridges (you don't want the wind, or the wrong pattern of traffic, to cause resonance that makes your bridge oscillate so much that it collapses).
  When designing a RLC circuit resonance is what you want. For example, if the incoming (forcing) voltage comes from a weak signal you are getting on your car antennae, then you might want to be able to adjust the circuit frequency, ω_0, to match the incoming frequency (you design the circuit so that resistance, or inductance, or capatacitance can be adjusted with a dial). If you get these two frequencies close, then you can get resonance which will lead to a solution like the incoming signal but with a much higher amplitude. These concepts are essential in the sending and receiving of radio transmissions.

3.7 and 3.8: Vibrations Handout

Mass-Spring Systems:
An object is placed on a spring. If $u(t)$ is the displacement from rest, then we say

$$mu'' + \gamma u' + ku = F(t),$$

where m is the mass of the object, γ is the damping constant, k is the spring constant, and $F(t)$ is an external forcing function. In deriving the application, we learned various facts including: $w = mg$, $mg - kL = 0$, $F_s = k(L+u)$, and $F_d = -\gamma u'(t)$, where $g = 9.8$ m/s^2 $= 32$ ft/s^2 and L is the distance the spring is stretched beyone natural length when it is at rest.

RLC circuits:
If R, C, and L are the resistance, capacitance and inductance in a circuit and $E(t)$ is the impressed voltage (incoming forcing function), then we have

$$LQ'' + RQ' + \frac{1}{C}Q = E(t),$$

where $Q(t)$ is the charge on the capacitor at time t.

Note: This is not a physics or electronics class. You really don't have to know hardly anything about forces or electronics to do well on this material. You just have to put in the numbers and solve second order systems (like we have been doing for the last two weeks). The point of this material is to expose you to some important applications of second order equations so that you have a physical relationship between what we are getting in the solutions and what we are seeing in the application.

Summary of Analysis:
Note: Here I state everything in terms of the mass-spring system, but, if you replace $m = L$, $\gamma = R$, $k = \frac{1}{C}$, and $F(t) = E(t)$, then the analysis is the same for the circuit application.

No Forcing: $F(t) = 0$.

1. $\gamma = 0 \Rightarrow$ No Damping: Solution looks like $u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) = R\cos(\omega_0 t - \delta)$.
   - Natural frequency: $\omega_0 = \sqrt{k/m}$ radians/second.
   - Period: $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{m/k}$ seconds/wave.
   - Amplitude: $R = \sqrt{c_1^2 + c_2^2}$.
2. $\gamma \geq 2\sqrt{mk} \Rightarrow$ No Vibrations: No imaginary roots, only negative real roots.
   $\gamma = 2\sqrt{mk} \Rightarrow$ **Critically Damped** and $\gamma > 2\sqrt{mk} \Rightarrow$ **Overdamped**
3. $0 < \gamma < 2\sqrt{mk} \Rightarrow$ Damped Vibrations:
   Solutions looks like $u(t) = e^{\lambda t}(c_1\cos(\mu t) + c_2\sin(\mu t)) = Re^{\lambda t}\cos(\mu t - \delta)$.
   - Quasi-frequency: $\mu = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}$ radians/second.
   - Quasi-period: $T = \frac{2\pi}{\mu}$ seconds/wave.
   - Amplitude: $Re^{\lambda t} = \sqrt{c_1^2 + c_2^2}e^{\lambda t}$, which goes to zero as $t \to \infty$.

Forcing: $F(t) \neq 0$.
As we saw in 3.5 and 3.6, we need to find the homogeneous solution and a particular solution.
For mass-spring, we primarily considered forcing functions of the form $F(t) = F_0 \cos(\omega t)$.

1. $\gamma = 0 \Rightarrow$ No Damping: Find homogenous solutions (see 'no forcing'). It will have a natural frequency of ω_0. The particular solution depends on ω and ω_0.
   - If $\omega \neq \omega_0$, then a particular solution looks like $U(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$
   - If $\omega = \omega_0$, then a particular solution looks like $U(t) = \frac{F_0}{2m\omega_0} t \sin(\omega t)$. (Resonance!)
2. $\gamma > 0$ Damping: Find homogenous solutions (see 'no forcing').
   If $\gamma < 2\sqrt{mk}$, then label μ as the quasi-frequency. If $\gamma < 2\sqrt{mk}$, then the solution will always look like:
   $$u(t) = u_c(t) + U(t) = c_1 e^{\lambda t} \cos(\mu\, t) + c_2 e^{\lambda t} \sin(\mu\, t) + A\cos(\omega t) + B\sin(\omega t).$$
   In all cases where $\gamma > 0$ the homogeneous solution, $u_c(t)$, goes to zero as $t \to 0$. We say the homogeneous solution, $u_c(t)$, is the **transient solution** and the particular solution, $U(t)$, is the **steady state solution** (or forced response).
   - With some considerable algebra, you can get general messy formulas for A and B (see book or review sheet).
   - Amplitude of Steady State solution: $R = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}}$.
     This depends on ω.
     Applitude is maximized when $\omega = \omega_{max} = \omega_0 \sqrt{1 - \frac{\gamma^2}{2mk}} \approx \omega_0$. (if γ is close to zero)
     At this value of ω, you get $R = R_{max} = \frac{F_0}{\gamma\omega_0} \frac{1}{\sqrt{1 - \frac{\gamma^2}{4mk}}}$.
     So if γ is close to zero, then the maximum amplitude of the steady state response occurs when ω is close to ω_0 (Resonance).

Skills Review: Trigonometry and Waves

The following review discusses some trigonometry, specifically facts related to waves.

Introduction and Basic Facts:
Consider functions of the form $y(t) = A\cos(\omega t - \delta)$. Our book likes to express waves in this standard form. The graph of this function looks like a wave which is oscillating about the t-axis. Here are several important facts about this wave:

- A = 'the amplitude' = 'the distance from the middle of the wave to the highest point'
- ω = 'angular frequency' = 'how many radians between $t = 0$ and $t = 1$'.
- $\omega = 2\pi f$, where f = 'the frequency' = 'the number of full waves between $t = 0$ and $t = 1$'
- $\omega = \frac{2\pi}{T}$ or, in other words,
  $T = \frac{1}{f}$ = 'the period (or wavelength)' = 'distance on the t-axis between peaks'
- δ = 'phase (or phase shift)' = 'the *starting* angle that corresponds to $t = 0$'.

A full example with a picture is on the next page.

Converting into Standard Form:
In this class, we often will have solutions involving expressions of the form $y = A\cos(\mu t) + B\sin(\mu t)$. In order to write this in the form above, you need the trig identity:

$$y = R\cos(\omega t - \delta) = R\cos(\delta)\cos(\omega t) + R\sin(\delta)\sin(\omega t).$$

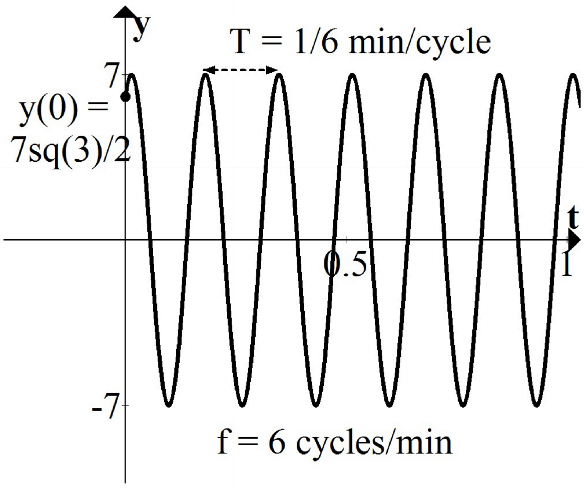
Setting this equal to $y = A\cos(\mu t) + B\sin(\mu t)$, we conclude that $\omega = \mu$, $A = R\cos(\delta)$, and $B = R\sin(\delta)$. And from these relationships we can conclude that $R^2 = A^2 + B^2$. Therefore, we get

$$R = \sqrt{A^2 + B^2}\ , \quad A = R\cos(\delta)\ , \quad \text{and} \quad B = R\sin(\delta)$$

Example: Consider $y = \frac{7\sqrt{3}}{2}\cos(12\pi t) + \frac{7}{2}\sin(12\pi t)$.
To write in the standard form above, we want $R = \sqrt{\left(\frac{7\sqrt{3}}{2}\right)^2 + \left(\frac{7}{2}\right)^2} = \sqrt{\frac{49(3+1)}{4}} = 7$.
We also want $\frac{7\sqrt{3}}{2} = 7\cos(\delta)$ and $\frac{7}{2} = 7\sin(\delta)$ which gives $\delta = \frac{\pi}{6}$.
Therefore, we get $y = 7\cos\left(12\pi t - \frac{\pi}{6}\right)$. A graph of this function is on the next page.

For example: Consider $y(t) = 7\cos\left(12\pi t - \frac{\pi}{6}\right)$. Let's say t is in minutes just to give some units. A picture is provided below.

1. $A = 7$ is the amplitude. So this wave oscillates between $y = -7$ and $y = 7$.
2. $\omega = 12\pi$ radians per minute. In other words, every minute we will add all radians from 0 to 12π.
3. $f = \frac{\omega}{2\pi} = 6$ waves per minute. In other words, every minute there will be 6 full waves (a full wave is peak-to-peak, or valley-to-valley).
4. $T = \frac{1}{f} = \frac{1}{6}$ minutes per wave. In other words, it take $\frac{1}{6}$ minute (*i.e.* 10 seconds) to complete complete one full wave.
5. $\delta = \frac{\pi}{6}$ is the 'starting angle'. In other words, when $t = 0$, the wave starts at $y(0) = 7\cos(-\frac{\pi}{6}) = 7\sqrt{3}/2$. From here the wave will go up (because this is what the Cosine wave does after $-\pi/6$) and it will complete one wave in 1/6 minute (10 seconds). After these 10 seconds, it will be back to the value of $y(1/6) = 7\sqrt{3}/2$ and the wave will continue in this way.



Skills Review: Solving Two-by-Two Systems

In this course, you will often have to solve a two-by-two system of linear equations that looks like

$$ax_1 + bx_2 = P;$$
$$cx_1 + dx_2 = Q,$$

where a, b, c, d, P, and Q are all numbers and you are solving for x_1 and x_2. Here is a reminder of your goals and your tools for solving such equations.

1. The goal is to **combine** the two equations into one equation that has only one variable so that you can solve for that variable.
2. Your two main combining tools:
   - Add or Subtract the two equations from each other. This is valid because if you add equal things to equal things you get equal things! Note that you can also multiply or divide both sides of any equation by a number (in order to set up a situation where adding/subtracting will lead to cancellation).
   - Substitute! Solve for one variable in the first equation and substitute into the second.
3. Once you have solved for one variable, you can substitute back into one (or both) of the original equations to find the other variable. As a check on your work, you should plug into both equations.

Basic example: Solve the system $\begin{array}{ll} \text{(i)} & 2x_1 + x_2 = 5 \\ \text{(ii)} & x_1 - x_2 = 4 \end{array}$

- *Solution 1: Combining by Adding/Subtracting*
  Notice the cancellation that will happen if we add!
  Adding corresponding sides of (i) and (ii) gives a combined equation of $3x_1 = 9$. Thus, $x_1 = 3$.
  Substituting back into (i) gives $2(3) + x_2 = 5$, so $x_2 = -1$.
  Substituting back into (ii) gives $(3) - x_2 = 4$, so $x_2 = -1$.
  Thus, the only solution is $x_1 = 3$ and $x_2 = -1$.
- *Solution 2: Substituting*
  Solving for x_2 in the first equation, we can rewrite equation (i) as $x_2 = 5 - 2x_1$.
  Substituting into (ii), we get a combined equation of $x_1 - (5 - 2x_1) = 4$ which simplifies to $3x_1 - 5 = 4$. Solving gives $3x_1 = 9$, so $x_1 = 3$.
  Substituting back into our simplified version of (i) gives $x_2 = 5 - 2(3) = -1$.
  Substituting back into (ii) gives $(3) - x_2 = 4$, so $x_2 = -1$.
  Thus, the only solution is $x_1 = 3$ and $x_2 = -1$.

The first method is sometimes faster, but it requires some cleverness. The second method always takes the same amount of time and requires no cleverness. That's it, now you can solve linear 2-by-2 systems!

Here is another one to try on your own:

Example: Solve the system $\begin{array}{ll} \text{(i)} & 2x_1 + 2x_2 = 6 \\ \text{(ii)} & 3x_1 - x_2 = 2 \end{array}$

Comments about the solution: You can either start by dividing the first equation by 2, then adding. Or just solve for x_1 or x_2 in the first equation and substituting into the second. Both will work. The answer you should get is $x_1 = \frac{5}{4}$ and $x_2 = \frac{7}{4}$.

Some very important theoretic comments about two-by-two systems
There are three things that can happen in a two-by-two system:

1. UNIQUE solution: The most 'likely' situation (*i.e.* if you randomly pick numbers for coefficients you probably get a system with a unique solution). See two examples on the last page.

2. NO solution: Happens if the 'left-hand side' of the second equation is a multiple of the first, but the 'right-hand side' is not the same multiple. For example: $\begin{array}{ll} \text{(i)} & x_1 - 2x_2 = 10; \\ \text{(ii)} & 3x_1 - 6x_2 = 50. \end{array}$ In this example, (i) $x_1 - 2x_2 = 10$ and (ii) $3(x_1 - 2x_2) = 50$ can't happen because 50 is NOT 3 times 10. There is NO solution.

3. INFINITELY many solutions: This happens if both sides are the same multiple of each other. For example: $\begin{array}{ll} \text{(i)} & x_1 - 2x_2 = 10; \\ \text{(ii)} & 3x_1 - 6x_2 = 30. \end{array}$ Notice that both sides of equation (ii) are exactly 3 times equation (i). In fact, equations (i) and (ii) are two different ways to write the exact same equation. Thus, all solutions will satisfy $x_1 = 10+2x_2$. For example, one solution is $x_1 = 10$, $x_2 = 0$, another is $x_1 = 12$, $x_2 = 1$, another is $x_1 = 14$, $x_2 = 2$, and so on ...

The Determinant:
For a system of the form $\begin{array}{l} ax_1 + bx_2 = P; \\ cx_1 + dx_2 = Q, \end{array}$, we define the two-by-two **determinant** by

$$\text{determinant} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Note: For a two-by-two system if the **determinant is zero**, then the 'left-hand sides' are multiples of each other. For example, the system $\begin{array}{ll} \text{(i)} & x_1 - 2x_2 = 10 \\ \text{(ii)} & 3x_1 - 6x_2 = 30 \end{array}$ has a determinant of $(1)(-6)-(-2)(3) = 0$.

Existence and Uniqueness Theorem for Linear Systems:
From what we have already said, we can summarize

1. if $ad - bc \neq 0$, then the system has a **unique solution**.

2. if $ad - bc = 0$, then the system will have no solution or infinitely many solutions (depending on the values of P and Q).

Cramer's Rule: (Just for your interest, not required)
If you combined and solved the general system $\begin{array}{l} ax_1 + bx_2 = P; \\ cx_1 + dx_2 = Q, \end{array}$ you would find that if there is a unique answer then it is always is equal to

$$x_1 = \frac{Pd - bQ}{ad - bc} = \frac{\begin{vmatrix} P & b \\ Q & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad x_2 = \frac{aQ - Pc}{ad - bc} = \frac{\begin{vmatrix} a & P \\ c & Q \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

You can use Cramer's rule to solve if you wish, but it is usually just as fast to combine and solve. To learn facts about larger systems (3-by-3 and 4-by-4), then you have to take a course in linear algebra (Math 308). Examples of Cramer's rule are on the next page:

1. Solve the system $\begin{array}{ll} \text{(i)} & 2x_1 + x_2 = 5 \\ \text{(ii)} & x_1 - x_2 = 4 \end{array}$

$$x_1 = \frac{\begin{vmatrix} 5 & 1 \\ 4 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{-9}{-3} = 2 \ , \ x_2 = \frac{\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{3}{3} = 1.$$

This is the same example from the first page of this review (notice the solutions match).

2. Solve the system $\begin{array}{ll} \text{(i)} & 2x_1 + 2x_2 = 6 \\ \text{(ii)} & 3x_1 - x_2 = 2 \end{array}$

$$x_1 = \frac{\begin{vmatrix} 6 & 2 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 2 \\ 3 & -1 \end{vmatrix}} = \frac{-10}{-8} = \frac{5}{4} \ , \ x_2 = \frac{\begin{vmatrix} 2 & 6 \\ 3 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 2 \\ 3 & -1 \end{vmatrix}} = \frac{-14}{-10} = \frac{7}{5}.$$

This was the second example from the first page of this review (notice the solutions match).

2. Solve the system $\begin{array}{ll} \text{(i)} & 5x_1 + 7x_2 = 10 \\ \text{(ii)} & 2x_1 - 6x_2 = 8 \end{array}$

$$x_1 = \frac{\begin{vmatrix} 10 & 7 \\ 8 & -6 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & -6 \end{vmatrix}} = \frac{-116}{-48} = \frac{29}{12} \ , \ x_2 = \frac{\begin{vmatrix} 5 & 10 \\ 2 & 8 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & -6 \end{vmatrix}} = \frac{20}{-48} = -\frac{5}{12}.$$

Skills Review: Complex Numbers

The following three pages give a quick introduction to complex numbers. The first page introduces basic arithmetic, the second page introduces Euler's formula, and the third page gives a graphical interpretation of complex numbers.

Introduction:
We define i to be a symbol that satisfies $i^2 = -1$. In other words, we think of i as a solution to $x^2 = -1$. The symbol i is called the **imaginary unit**.
Terminology:

- A **complex number** is any number that is written in the form $a + bi$ where a and b are real numbers.
- If $z = a + bi$ is a complex number, we say $\mathrm{Re}(z) = a$ is the **real part** of the complex number and we say $\mathrm{Im}(z) = b$ is the **imaginary part** of the complex number.

Basic Arithmetic:

1. We define all the same arithmetic properties. In other words, do arithmetic like you have always done. Just always replace i^2 by -1.
2. Here are several examples:
   - Adding Example: $(2 - 4i) + (10 + 7i) = 12 + 3i$.
   - Subtracting Example: $(-1 + 3i) - (4 - 5i) = -5 + 8i$.
   - Multiplying Example: $(3 + 2i)(5 - i) = 15 + 10i - 2i - 2i^2 = 17 + 8i$.
   - Powers Example: $i^3 = i^2 i = -i$, $i^4 = i^2 i^2 = (-1)(-1) = 1$, ...
3. Dividing: In order to divide you need to use the concept of the **conjugate**.
   The conjugate of $a + bi$ is $a - bi$.
   If you multiply a complex number by its conjugate you get $(a+bi)(a-bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2$. When you divide by a complex number you should multiply the top and bottom by the conjugate of the denominator.
   Example: Simplify $\dfrac{4+i}{2-3i}$
   Multiplying top and bottom by $2 + 3i$ gives $\dfrac{(4+i)(2+3i)}{4+9} = \dfrac{8+2i+12i-3}{13} = \dfrac{5}{13} + \dfrac{14}{13}i$.
4. Other Powers: $e^{a+bi} = e^a e^{bi}$, $2^{a+bi} = 2^a 2^{bi} = 2^a e^{b\ln(2)i}$. For what do to with e^{bi} see the next page.

Solving Polynomial Equations (For your own interest):
Every solution to a polynomial equation is a real number or a complex number. (This is a part of what is called the fundamental theorem of algebra).
For example, $x^3 + 4x = 0$ has 3 solutions. Solving gives $x(x^2 + 2) = 0$, so $x = 0$ or $x = -2i$ or $x = 2i$.
Another example, if you ask Mathematica to solve $x^6 - 3x^2 + x = -10$, you get the six complex solutions:
$x \approx -1.26 - 0.53i$, $x \approx -1.26 + 0.53i$, $x \approx -0.03 - 1.62i$,
$x \approx -0.03 + 1.62i$, $x \approx 1.29 - 0.61i$, $x \approx 1.29 + 0.61i$.
(Notice the 6th power and the 6 solutions, that is not a coincidence, it is another part of the fundamental theorem of algebra).
As you see, complex numbers play a fundamental role in studying solutions to equations in algebra.

Euler's Formula

Euler's formula defines $e^{bi} = \cos(b) + i\sin(b)$.

For example:

- $e^{\frac{\pi}{6}i} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i$.
- $e^{5-\frac{\pi}{2}i} = e^5 e^{-\frac{\pi}{2}i} = e^5\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = e^5(0-i) = -e^5 i$
- $e^{1+\frac{\pi}{4}i} = ee^{\frac{\pi}{4}i} = e\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = e\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \frac{\sqrt{2}e}{2} + \frac{\sqrt{2}e}{2}i$
- $e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1$

This definition may seem odd at first, but, after you study Taylor series (in Math 126), you see that these do indeed give the same function. For those of you that have seen Taylor series, here is the Taylor series derivation of Euler's formula.

1. The Taylor series for e^z based at 0 is $e^z = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}z^n = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$.
2. The Taylor series for $\sin z$ based at 0 is $\sin(z) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}z^{2n+1} = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots$.
3. The Taylor series for $\cos z$ based at 0 is $\cos(z) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}z^{2n} = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \cdots$.
4. Recognize that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, $i^7 = -i$, $i^8 = 1$, ...
5. Now consider e^{bi}:

$$\begin{aligned}
e^{bi} &= 1 + bi + \frac{1}{2!}b^2i^2 + \frac{1}{3!}b^3i^3 + \frac{1}{4!}b^4i^4 + \frac{1}{5!}b^5i^5 + \frac{1}{6!}b^6i^6 + \cdots \\
&= 1 + bi - \frac{1}{2!}b^2 - \frac{1}{3!}b^3i + \frac{1}{4!}b^4 + \frac{1}{5!}b^5i - \frac{1}{6!}b^6 + \cdots \\
&= (1 - \frac{1}{2!}b^2 + \frac{1}{4!}b^4 - \cdots) + i(b - \frac{1}{3!}b^3 + \frac{1}{5!}b^5 + \cdots) \\
&= \cos(b) + i\sin(b)
\end{aligned}$$

Geometric Interpretations of Complex Numbers

Complex numbers $a + bi$ are often plotted on the xy-plane, where we take $x = a$ and $y = b$. When we plot complex numbers in this way, we say the xy-plane is the **complex plane**. We say the x-axis is the **real axis** and the y-axis is the **complex axis**.

1. Polar Coordinates: Consider a point (x, y) in the plane. Draw a line segment from the origin to the point (x, y). Label the length of this line segment r. Label the angle the line makes with the positive x-axis with the symbol θ. Using basic facts from trigonometry you get

$$x = r\cos(\theta)\ ,\ y = r\sin(\theta)\ ,\ x^2 + y^2 = r^2\ ,\ \tan(\theta) = \frac{y}{x}$$

   When we think of points in the plane in terms of r and θ, we say we are using polar coordinates.

2. Now, assume $a + bi$ is a complex number and write $a = r\cos(\theta)$ and $b = r\sin(\theta)$. Then we have

$$a + bi = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta)) = re^{\theta i}$$

3. Multiplying a complex number by $e^{\theta i}$ gives a new complex number that has been rotated counterclockwise by the angle θ.
   Here are several examples:

   - Consider the point $(x, y) = (0, 5)$ written as the complex number $z = 0 + 5i$.
     Multiplying by $e^{\frac{\pi}{2}i} = \cos(\pi/2) + i\sin(\pi/2) = i$ leads to counterclockwise rotation by 90 degree. Here is the multiplication $(0 + 5i)i = -5 + 0i$ which gives the new point $(-5, 0)$.
   - Consider the point $(x, y) = (2, 1)$ written as the complex number $z = 2 + i$.
     Multiplying by $e^{\frac{\pi}{i}} = \cos(\pi) + i\sin(\pi) = -1$ leads to a counterclockwise rotation by 180 degrees. Here is the multiplication $(2 + i)(-1) = -2 - i$ which gives the new point $(-2, -1)$.
   - Consider the point $(x, y) = (-3, 4)$ written as the complex number $z = -3 + 4i$.
     Multiplying by $e^{\frac{\pi}{4}i} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ leads to a counterclockwise rotation by 45 degrees.
     Here is the multiplication $(-3+4i)\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = (\frac{-3\sqrt{2}}{2} - \frac{4\sqrt{2}}{2}) + (\frac{-3\sqrt{2}}{2} + \frac{4\sqrt{2}}{2})i = \frac{-7\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ which gives the new point $\left(\frac{-7\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Midterm

Friday, November 16, 2018 3:36 PM

Problem 1 (12 points) Give an example for each of the following, or say that it is not possible. (No explanation is needed if you state that it is not possible.)

a.) Give an example of two different functions whose Wronskian is zero.

$W = y_1 y_2' - y_2 y_1'$

$\boxed{\begin{cases} y_1 = 3 \\ y_2 = 5 \end{cases}} \Rightarrow W = 0$

b.) Give an example of an equilibrium solution to the differential equation $y'' + y = \cos(t)$.

$r^2 + 1 = 0$

$\lambda = 0, \mu = \frac{\sqrt{-0+4}}{2} = 1$

assume $y_p =$

$y_c = C_1 \cos t + C_2 \sin t$

$y = C_1 \cos t + C_2 \sin t - \frac{1}{2} t \sin t$

-3

$\boxed{y_{equi} = \frac{1}{2} t \sin t}$

equilibrium solution doesn't change over time.

$y = C \Rightarrow 0 = \cos t$. there's no eq sln.

c.) Give an example of a second-order, homogeneous differential equation where $4e^{3t}\cos(2t)$ is a solution.

$y' = -8e^{3t}\sin(2t) + 12e^{3t}\cos 2t$

$y'' = -24e^{3t}\sin 2t - 16e^{3t}\cos(2t) + 36e^{3t}\cos 2t - 24e^{3t}\sin(2t)$

$= -48e^{3t}\sin(2t) + 20e^{3t}\cos 2t$

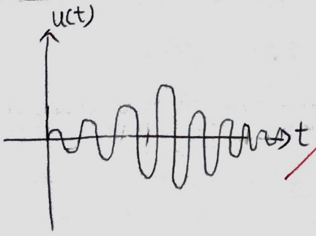
$\boxed{y'' - 6y' + 13y = 0}$

$20 - 12 \times 6$

$= 20 - 72 = -52$

$52/4 = 13$

d.) Sketch or describe the motion of a spring with position given by $u(t) = 5\sin(t)\sin(2t)$.



it's a beat.

2

Problem 2 (12 points) Give the general solution for the following differential equations:

a. $f''(t) = \sin(t)$

y_c: $r^2 = 0$

$r = 0$

$y_c = C_1 + C_2 t$

3

y_p: assume $y_p = A\sin t$

$y_p' = A\cos t$

$y_p'' = -A\sin t$

$\therefore A = -1$

$\therefore \boxed{y = C_1 + C_2 t - \sin t}$

b. $f''(t) + f'(t) = \sin(t)$

y_c: $r^2 + r = 0$

$r(r+1) = 0$

$r = 0, r = -1$

$y_c = C_1 + C_2 e^{-t}$

plug-in: $(-A-B)\sin t + (A-B)\cos t) = \sin t$

$\begin{cases} -A-B = 1 \\ A = B \end{cases}$

$\therefore A = B = -\frac{1}{2}$

3

y_p: assume $y_p = A\sin t + B\cos t$

$\therefore y_p' = A\cos t - B\sin t$

$\therefore \boxed{y = C_1 + C_2 e^{-t} - \frac{1}{2}\sin t - \frac{1}{2}\cos t}$

y_p: assume $y_p = A\sin t + B\cos t$

$\therefore\ y_p' = A\cos t - B\sin t$

$y_p'' = -A\sin t - B\cos t$

$\therefore A = B = -\frac{1}{2}$

$\therefore\ \boxed{y = C_1 + C_2 e^{-t} - \frac{1}{2}\sin t - \frac{1}{2}\cos t}$

c. $f''(t) + f(t) = \sin(t)$

y_c: $r^2 + 1 = 0$

$\mu = \frac{\sqrt{-(-4)}}{2} = 1,\ \lambda = 0$ 3

$\therefore\ y_c = C_1\sin(t) + C_2\cos t$

assume $y_p = (A\sin t + B\cos t)t$

$y_p' = A\sin t + B\cos t + At\cos t - Bt\sin t$

$y_p'' = A\cos t - B\sin t + A\cos t - At\sin t$
$= 2A\cos t - 2B\sin t - Bt\sin t - Bt\cos t - At\sin t - Bt\cos t$

plug-in:

$2A\cos t - 2B\sin t = \sin t$

$\therefore \begin{cases} -2B = 1 \\ A = 0 \end{cases}$

$B = -\frac{1}{2}$

$\boxed{y = C_1\sin t + C_2\cos t - \frac{1}{2}t\cos t}$

d. Are all solutions to each of the given differential equations bounded? (Recall: A bounded solution means that the the function does not attain arbitrarily large ~~positive/negative~~ values):

(i.) $f''(t) + f(t) = \cos(t)$ (Yes/No)

(ii.) $f''(t) + f'(t) = \cos(t)$ (Yes/No) ✗ 2

−1

$r(r+1) = 0$

$r = 0,\ r = -1$

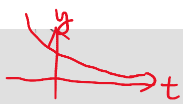
$C_1e^0 + C_2e^{-t}$

$\begin{cases} A - B = 1 \\ -A - B = 0 \end{cases}$ 3

$A = -B$

$B = -\frac{1}{2}$

$A = \frac{1}{2}$



Problem 3 (12 points) Find the solution to

$$y'' + 6y' + 13y = 0;\ y(0) = -1,\ y'(0) = -1.$$

Write your final answer in the amplitude-phase form: $y(t) = e^{rt}R\cos(\omega t - \delta)$.

$r^2 + 6r + 13 = 0$

$\lambda = -\frac{6}{2} = -3$

$\mu = \frac{\sqrt{13\times 4 - 6^2}}{2} = \frac{\sqrt{52 - 36}}{2} = 2$

$y(t) = e^{-3t}(C_1\cos 2t + C_2\sin 2t)$

$\because\ y(0) = -1,\ y'(0) = -1$

$\therefore \begin{cases} C_1 = -1 \\ -3C_1 + 2C_2 = -1 \end{cases}$

−2

$\therefore\ C_1 = -1$
$C_2 = -2$

$\therefore\ y(t) = e^{-3t}(-\cos 2t - 2\sin 2t)$

$R = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$

$\tan\delta = \frac{-2}{-1} = 2$ range of $\tan^{-1}$!!!

$\therefore\ \boxed{y(t) = e^{-3t}\cdot\sqrt{5}\cos(2t - \cancel{\arctan 2}) \approx e^{-3t}\cdot\sqrt{5}\cos(2t - 1.107)}$

$\delta = \tan^{-1}2 + \pi$

Problem 4 (12 points) Recall that an RLC circuit can be modeled by the 2nd order differential equation

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t),$$

where R, L and C are constants denoting the resistance (ohms Ω), inductance (henry H), and capacitance (farad F) of the circuit; $Q(t)$ is the charge (coulomb C); and $E(t)$ is some imposed voltage (volt V), t in seconds.

a. You have a circuit where $L = 1H, R = 0\Omega$, and $C = 1/16F$, and there is no external voltage (i.e. $E(t) = 0$). There is residual charge in the wire, i.e. $Q(0) > 0$, resulting in the charge oscillating about zero. Assume that $Q'(0) = 0$.
Give the time when $Q(t) = 0$ for the second time.

$Q'' + 16Q = 0$

$r^2 + 16 = 0$

$\lambda = 0, \mu = \frac{\sqrt{16\times4}}{2} = 4$

$Q(t) = C_1\cos4t + C_2\sin4t$

$\because Q'(0) = 0 \quad \therefore 4C_2\cos4t = 0$

$C_2 = 0$

$\therefore Q(t) = C_1\cos4t.$

$4t = \frac{\pi}{2}$ $\quad Q(t) = 0.$

$4t = \frac{3}{2}\pi$

$\boxed{t = \frac{3}{8}\pi}$ $Q(t) = 0$ second time

b. What is the smallest resistance that needs be added to the circuit in part (a.) so that the charge no longer experiences oscillation?

$\Delta \geq 0 \quad \Delta = R^2 - \frac{4L}{C} \geq 0$

$R \geq \sqrt{\frac{4L}{C}} = 2\sqrt{\frac{L}{C}} = 2\times4 = 8$

$\boxed{R = 8\Omega}$ is the smallest.

c. Propose the form for a particular solution to the differential equation in part (a.) if instead of having no external voltage, $E(t)$ is given by $E(t) = 12\cos(4t)$.
(e.g. Ae^t, $At + B$, etc.)

$$\boxed{y_p = (A\cos(4t) + B\sin(4t))t}$$

Problem 5 (12 points) For Christmas, officemates Caleb and Kevin gave each other gifts. Kevin received a box of red and green springs, and Caleb received a box of brightly colored weights of varying masses. As a fun afternoon activity, the pair have decided to create Christmas decorations from the different spring-mass pairs.

a.) To begin, Kevin sets up a spring-mass system, and finds that a 2-kilogram mass stretches a spring 4.9 meters to its equilibrium position. Kevin lifts the mass up one meter and drops it at time $t = 0$. Assume that the air exerts a damping force of 1 N when the mass has a velocity of 1 m/s. Find $u(t)$ the position of the spring at time t.

$\downarrow +$

$u(0) = -1, \; u'(0) = 0$ $\qquad k = \frac{2\times9.8\,N}{4.9\,m} = 4\,N/m$

$\gamma = \frac{1N}{1m/s} = 1\;Ns/m$

$2u'' + u' + 4u = 0$ $\qquad \therefore u(t) = e^{-\frac{t}{4}}(C_1\cos\frac{\sqrt{31}}{4}t + C_2\sin\frac{\sqrt{31}}{4}t)$

$2r^2 + r + 4 = 0$ $\qquad \because u(0) = -1, \; u'(0) = 0$

$\lambda = -\frac{1}{4}$ $\qquad \therefore C_1 = -1$

$\mu = \frac{\sqrt{4\times8-1}}{4} = \frac{\sqrt{31}}{4}$ $\qquad -\frac{1}{4}\cdot C_1 + \frac{\sqrt{31}}{4}C_2 = 0$

$C_2 = -\frac{1}{4}\times\frac{4}{\sqrt{31}} = -\frac{\sqrt{31}}{31}$

$\therefore u(t) = e^{-\frac{t}{4}}(-\cos\frac{\sqrt{31}}{4}t - \frac{\sqrt{31}}{31}\sin(\frac{\sqrt{31}}{4}t))$

b.) Meanwhile, Caleb setup a separate system in a vacuum chamber (a previous Christmas gift) and determined the position at time t to be $u(t) = 5\cos(t) - 5\sin(t)$, where $u(t)$ being positive corresponds to the spring being longer than the equilibrium length. When does this spring attain its maximal length for the first time? How far is the spring from equilibrium at this time?

$u(t) = 5\cos t - 5\sin t$

$u(t) = R\cos(wt - \delta)$

$R = \sqrt{50} = 5\sqrt{2}$

$\tan\delta = -1$ $\qquad -2$

$\delta = \arctan -1 = \frac{3}{4}\pi$

$u(t) = 5\sqrt{2}\cos(t - \frac{3}{4}\pi)$ $\qquad t - \frac{3}{4}\pi = -\frac{\pi}{2}$

$\delta = \arctan -1 = \frac{3}{4}\pi$

$u(t) = 5\sqrt{2}\cos(t - \frac{3}{4}\pi)$

$t - \frac{3}{4}\pi = -\frac{\pi}{2}$

$t = \frac{\pi}{4}$

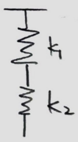
$\boxed{\text{max length} = 5\sqrt{2} \qquad \cancel{t = \frac{3}{4}\pi}}$

6

careful!

c.) (Bonus* - No Partial Credit) Note: This question is a bit more difficult than the rest of the exam, and is worth significantly fewer (optional) points. I would only suggest attempting this if you have already completed the rest of the exam.

Feeling a bit daring, Caleb and Kevin then attach the first spring to the end of the second to form a longer "Super Spring," and the two make guesses as to how the new spring's spring constant will relate to those of the smaller springs that comprise it.
Let k_1 be the spring constant of the first spring and k_2 that of the second. Write the spring constant for the composite spring, k, using the other two constants, and give an explanation for the relation.
(For example, you can say, $k = k_1/k_2$, because 'Blah blah blah...', or $k = 1/2(k_1 + k_2)$ because 'Blah BLAH blah...'. If you do not give an explanation, you will not receive any credit.)



$k \cdot (\Delta x_1 + \Delta x_2) = mg$

$k_1 \Delta x_1 + k_2 \Delta x_2 = \cancel{mg}$ 2m

$mg = k_1 \Delta x_1 \Rightarrow k_1 = \frac{mg}{\Delta x_1}$

$mg = k_2 \Delta x_2 \Rightarrow k_2 = \frac{mg}{\Delta x_2}$

$k = \frac{mg}{\Delta x_1 + \Delta x_2}$

$mu'' + \gamma u' + ku = 0$

$k_1 u_1 + k_2 u_2 = k(u_1 + u_2) \Rightarrow$

$u_1 +$

$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$

$\boxed{k = \frac{k_1 k_2}{k_1 + k_2}}$

$\frac{1}{k} = \frac{\Delta x_1 + \Delta x_2}{mg}$

~~Near~~ Very Good Attempt!
So close!

Laplace Transform

Friday, November 16, 2018 3:51 PM

- Define the Laplace of a function $f(t)$ to be
  $$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

§6.1 Laplace Transforms

§6.2 Take the Laplace of DE

§6.3 Laplace of discontinuous f

§6.4 DEs with driving force discontinuous

§6.5 Impulse

- Tactic: $\mathcal{L}\{\text{Diff. Eq}\} \to$ Algebra
  $\to$ Solve algebra
  $\to \mathcal{L}^{-1}\{-\}$

e.g.1 $\mathcal{L}\{1\}$

$$= \int_0^\infty e^{-st}\,dt$$
$$= \lim_{A\to\infty} \int_0^A e^{-st}\,dt$$
$$= \lim_{A\to\infty} -\frac{e^{-sA}}{s} + \frac{1}{s}$$
$$= -\frac{e^{-st}}{s}\Big|_0^\infty$$
$$= \frac{1}{s}, \; s>0$$

note: if $s=-1$
$\lim_{A\to\infty} -\frac{e^{-sA}}{s} + \frac{1}{s} = \infty$
so assume that $s>0$

e.g.2. $\mathcal{L}\{t\}$

$$= \int_0^\infty e^{-st}\cdot t\,dt$$
$$= \lim_{A\to\infty}\int_0^A t\cdot e^{-st}\,dt$$
$$= \lim_{A\to\infty} -\frac{Ae^{-sA}}{s} - \frac{e^{-sA}}{s^2} + \frac{1}{s^2}$$
$$= 0 - 0 + \frac{1}{s^2}$$
$$\therefore \mathcal{L}\{t\} = \frac{1}{s^2}, \; s>0$$

t		e^{-st}
1	$+$	$-\frac{e^{-st}}{s}$
0	$-$	$\frac{e^{-st}}{s^2}$

$\Rightarrow -t\frac{e^{-st}}{s} - \frac{e^{-st}}{s^2}$

$s>0$

$-\frac{A\cdot e^{-sA}}{s} = -\frac{A}{s\cdot e^{sA}} = \frac{1}{\underbrace{s^2 e^{sA}}_{\to +\infty}} \to 0$

{function in t} time
$\downarrow$
$\mathcal{L}$
$\downarrow$
{function in s} frequency

e.g. $\mathcal{L}\{\sin(2t)\}$

$$= \int_0^\infty e^{-st}\sin 2t\,dt$$
$$= \lim_{A\to\infty}\int_0^A e^{-st}\sin 2t\,dt$$

$\int e^{-st}\sin 2t\,dt = -\frac{\cos 2t}{2}\cdot e^{-st} - \int \frac{s\cos 2t}{2}\cdot e^{-st}\,dt$
$= -\frac{e^{-st}\cos 2t}{2} - \left(\frac{s}{4}e^{-st}\sin 2t + \int \frac{s^2}{4}e^{-st}\cdot\sin 2t\,dt\right)$
$= \frac{1}{1+\frac{s^2}{4}}\cdot\left(\frac{-e^{-st}\cos 2t}{2} - \frac{se^{-st}\sin 2t}{4}\right)$

$\lim_{A\to\infty}\int_0$

$$= \frac{2}{4+s^2}$$

$$= \frac{1}{1+\frac{s^2}{4}} \cdot \left(\frac{-e^{-st}\cos 2t}{2} - \frac{se^{-st}\cdot\sin 2t}{4} \right)$$

$$= \frac{1}{1+\frac{s^2}{4}} \left(\frac{-2e^{-st}\cdot\cos 2t - se^{-st}\cdot\sin 2t}{4} \right)$$

$$= \frac{2}{4+s^2} \cdot \left(-e^{-st}\cdot\cos 2t - \frac{s}{2}\cdot e^{-st}\cdot\sin 2t \right)$$