16.1 vector fields

Thursday, February 14, 2019 9:30 AM

- Def. Let D be a set in $\mathbb{R}^2$, a vector field on $\mathbb{R}^2$ is a function F that assigns to each point (x,y) in D a 2-dim vector $F(x,y)$
  - def2. Let E be a subset of $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function F that assigns to each point (x,y,z) in E a 3-dim vector $F(x,y,z)$.
  - Component Functions P, Q, R

    $F(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$

    P, Q, R are scalar functions / scalar fields
    - $\vec{F}$ is continuous if & only if its component functions P, Q, R are continuous.
    - use position vector $\vec{x} = \langle x, y, z\rangle$ to replace point (x,y,z)

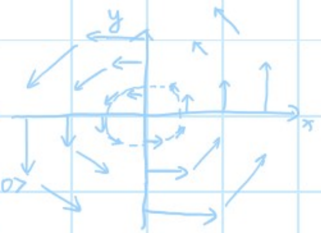
      $F(\vec{x}) = F(x,y,z)$

e.g. sketch the vector field on $\mathbb{R}^2$ defined by $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$

$x=0, y=0, \vec{F} = \langle 0,0\rangle$

$x>0, y=0, \vec{F} = \langle 0, x\rangle$

$x=0, y>0, \vec{F} = \langle -y, 0\rangle$



$|\vec{F}| = r_{circles}$

- gravitational field $F = \frac{mMG}{r^2}$

  $r = \sqrt{x^2+y^2+z^2}$, direction: $-\frac{\vec{r}}{|r|}$, since it point toward origin.

  $\therefore \vec{F} = -\frac{mMG}{|r|^3}\cdot\vec{r} = \langle -\frac{mMG}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{mMG}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{mMG}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

- $\nabla f(x,y) = f_x(x,y)\hat{i} + f_y(x,y)\hat{j}$ is a vector field.

  $\uparrow$

  Gradient Vector field.

- $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$ is a vector field.

↑

Gradient Vector field.

- A vector field $\vec{F}$ is a conservative vector field if it's the gradient of some vector function, $F = \nabla f$, then f is a potential function of F.

16.2 Line Integrals

Saturday, February 16, 2019 10:00 PM

- If f defined on a smooth curve C given by $x=x(t)$, $y=y(t)$, $t\in[a,b]$, then the **line integral** of f along C is (with respect to arc length)

$$\int_C f(x,y)\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$L=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

  - $ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$
  - Line integral with respect to x

$$\int_C f(x,y)\,dx = \int_{t=a}^{t=b} f(x(t),y(t))\cdot x'(t)\,dt$$

  - Line integral with respect to y

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

  - Line integral over vector field

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds = \int_{t=a}^{t=b} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

e.g. Evaluate $\int_C (2+x^2y)\,ds$, where C is the upper half of the unit circle $x^2+y^2=1$.

$x=\cos t$, $y=\sin t$, $t\in[0,\pi]$

$\int_0^{\pi}(2+\cos^2 t\cdot\sin t)\sqrt{1}\,dt$

$=\left(2t-\frac{\cos^3 t}{3}\right)\Big|_0^{\pi}$

$= 2\pi+\frac{1}{3}+\frac{1}{3}=2\pi+\frac{2}{3}$

- Piecewise smooth curve C: C is a union of a finite number of smooth curves $C_1, \dots, C_n$.

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds+\cdots+\int_{C_n} f(x,y)\,ds$$

e.g.2. Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y=x^2$

from (0,0) to (1,1) followed by C_2, vertical line segment from (1,1) to (1,2).

C_1: $y=x^2$

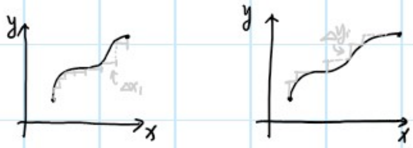
$$\int_0^1 2x\,ds = \int_0^1 2x\cdot\sqrt{1+(2x)^2}\,dx = \frac{1}{6}(1+4x^2)^{\frac{3}{2}}\Big|_0^1 = \frac{1+5^{\frac{3}{2}}}{6}$$

C_2: $x=1,\ y=y$

$$\int_1^2 2\sqrt{0+1}\,dy = 2y\Big|_1^2 = 2$$

$$\int_C 2x\,ds = \frac{1+5\sqrt{5}}{6}+2$$

- Line integral with respect to x and y



- $\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$
  - When parametrize,

    $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1,\ t\in[0,1]$, r_0 starting pt, r_1 end pt.

e.g. $\int_C y^2dx + x\,dy$, where C is the line segment from (-5,-3) to (0,2)

Must let $\frac{dx}{dt}, \frac{dy}{dt} \neq 0$

$y = x+2$

let $x=t,\ y=t+2,\ t\in[-5,0]$

$$\int_{-5}^{0}(t+2)^2\cdot 1\,dt + \int_{-5}^{0} t\cdot 1\cdot dt$$

$$= \int_{-5}^{0} t^2+5t+4\,dt$$

$$= \frac{t^3}{3}+\frac{5}{2}t^2+4t\Big|_{-5}^{0}$$

$$= \frac{125}{3}-\frac{125}{2}+20$$

$$= \frac{-125+120}{6} = -\frac{5}{6}$$

soln 2. $\because \vec{r}(t) = (1-t)\vec{r}_0+t\vec{r}_1,\ t\in[0,1]$ (Parametrize)

$r_0 = \langle -5,-3\rangle,\ r_1 = \langle 0,2\rangle$

$\vec{r}(t) = \langle -5,-3\rangle + t\cdot\langle 5,5\rangle$

$\therefore\ x(t) = 5t-5, \quad y(t) = 5t-3$

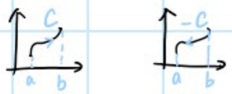
- In general, when computing line integrals, the value of the line integral is dependent on the path between two pts.

But there're also special conditions which cause the line integral to be independent of the path.

- If the starting point is changed:

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx, \quad \int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy$$

$$\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds \rightarrow \text{because } \Delta s \geq 0$$



- Line Integrals in space

  - $\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

  - $\int_C ds = \int_a^b |r'(t)|\,dt = L$

- Line Integrals of Vector fields

  - $W = \vec{F} \cdot \vec{D}$ displacement vector

  - $\vec{D} \approx \underbrace{(\Delta s_i)}_{\text{small piece of arc length}} \cdot \vec{T}(t)$ in the direction of the unit tangent vector

$$\Rightarrow W = \sum_{i=1}^{n} \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \vec{T}(x_i^*, y_i^*, z_i^*)\,\Delta s_i$$

  - $W = \int_C \vec{F} \cdot \vec{T}\,ds$, $C: r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|r'(t)|}$$

$$\therefore W = \int \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \cdot \underbrace{|\vec{r}'(t)| \cdot dt}_{|r'(t)|dt = ds}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=a}^{t=b} F(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$$

e.g. Find the work done by force field $\vec{F}(x,y) = x^2\hat{i} - xy\hat{j}$ in moving a particle along a quarter circle $\vec{r}(t) = <\underset{x}{\cos t}, \underset{y}{\sin t}>$, $t=0$ to $\frac{\pi}{2}$.

$$W = \int_0^{\frac{\pi}{2}} <\cos^2 t, -\cos t \cdot \sin t> \cdot <-\sin t, \cos t>\,dt$$

$$= \int_0^{\frac{\pi}{2}} -2\cos^2 t \cdot \sin t\,dt$$

$$= \frac{2}{3}\cos^3 t \Big|_0^{\frac{\pi}{2}}$$

$= -\frac{2}{3}$

- e.g. 2. If it's from $\frac{\pi}{2}$ to 0,

  $W = \frac{2}{3}$, sign is reversed.

- Let $\vec{F}$ be a continuous vector field defined on a smooth curve C given by $\vec{F}(r)$ $t \in [a,b]$, then the line integral of F along C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt = \int_C F \cdot T\,ds$$

- $\int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r}$ (Because T is $-T$ now)

- Connection between scalar line integrals and integrals over vector field

  assume $F = P\hat{i} + Q\hat{j} + R\hat{k}$, C: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt = \int_a^b \langle P(x(t), y(t), z(t)), Q(x(t)), \cdots), R \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle\,dt$$

$$= \int_a^b [P(x(t), y(t), z(t)) \cdot x'(t) + Q(x(t), y(t), z(t)) \cdot y'(t) + R(\cdots) \cdot z'(t)]\,dt$$

$$= \int_a^b P x'(t)\,dt + \int_a^b Q y'(t)\,dt + \int_a^b R z'(t)\,dt$$

$$= \int_C P\,dx + Q\,dy + R\,dz$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$

Note that: $\int_C y^2\,dx + x\,dy \Leftrightarrow \int_C \vec{F} \cdot d\vec{r}$, $\vec{F} = \langle y^2, x \rangle$

16.3 Fundamental Theorem for Line Integrals

Monday, February 18, 2019 2:31 PM

- Fundamental Theorem for line integrals

  Let C be a smooth curve given by the vector function $\vec{r}(t)$, $t\in[a,b]$, f be a differentiable function of two or three vars whose gradient vector ∇f is continuous on C, then $\int_C \nabla f\, d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

  $\Rightarrow$ initial point (x_1, y_1), terminal point (x_2, y_2)

  $\int_C \nabla f\, d\vec{r} = f(x_2, y_2) - f(x_1, y_1)$

  - Remark

    I. $\int_{C_1} \nabla f\, d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$ when C_1, C_2 are paths with the same initial pts and terminal pts.

    $\Leftrightarrow$ The line integral of a conservative vector field is independent of paths.

    II. Orientation/Direction of the parametrication matters.

    III. If $\vec{F} = \nabla f$ for some f, then the line integral will be the net change in f from $\vec{r}(a)$ to $\vec{r}(b)$

  - Proof

    $\nabla f(\vec{r}(t)) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

    $\vec{r}'(t) = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

$$= \int_a^b \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}\, dt$$

$$= \int_a^b \frac{d}{dt}(f(r(t)))\, dt$$

$$= f(r(b)) - f(r(a))$$

E.g. Find the work done by vector field $\vec{F}(x,y) = \langle 2xe^y, e^y x^2 - \sin y\rangle$ on a particle when moving from $(-2, 2)$ to $(3, 17)$ along the parabola $y = x^2 + 2x + 2$.

$$W = \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

find a f which $\nabla f = F$

$\frac{\partial f}{\partial x} = 2xe^y \Rightarrow f = x^2e^y + g(y)$

$\frac{\partial f}{\partial y} = e^y x^2 - \sin y \Rightarrow f = x^2e^y + \cos y + g(x)$

$\therefore f = x^2e^y + \cos y + k$, k is constant. take $k=0$.

$f(3,17) - f(-2,2) = 9e^{17} + \cos 17 - 4e^2 - \cos 2$

Thus we do not need to parametrize C

- def. $\vec{F}$ is a continuous vector field with domain D, we say that $\int_C \vec{F}\cdot d\vec{r}$ is independent of path if $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$ for any two paths C_1 and C_2 in D that have same initial & terminal pts

def A curve is closed if its terminal pt coincides with its initial pt.

i.e. $\vec{r}(a) = \vec{r}(b)$

- Theorem $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in $D \Leftrightarrow \int_C F\cdot dr = 0$ for every closed path C in D.

$C = C_1 + C_2$

$1°\ \because \int_C \vec{F} d\vec{r}$ independent

$\int_C \vec{F} d\vec{r} = \int_{C_1} \vec{F} d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$

$= \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{-C_2} \vec{F}\cdot d\vec{r}$

$= \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_1} \vec{F}\cdot d\vec{r}$

$= 0$

$2°\ \because \int_C \vec{F} d\vec{r} = 0$

$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F} d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$

$\therefore \int_{C_1} \vec{F} d\vec{r} = -\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{-C_2} \vec{F}\cdot d\vec{r}$

$\therefore \int_C \vec{F}\cdot d\vec{r}$ is independent

- def We say that D is open if $\forall$ pt $P \in D$, there's a disk with center P that lies entirely in D. (D doesn't contain boundary pts) def D is connected if any two pts in D can be joined by a path that lies in D. path-connected

- Theorem. Suppose $\vec{F}$ is a vector field that is continuous on an open, connected region D. If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \vec{F}$.

$\Rightarrow$ $\vec{F}$ is a conservative vector field on D (open connected)

$\Updownarrow$

$\int_C \vec{F}\cdot d\vec{r}$ is independent of path

$\Updownarrow$

$\int_C \vec{F}\cdot d\vec{r} = 0$ for any closed curve C.

- Theorem If $F(x,y) = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$ is a conservative vector field, where P, Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

○ Proof. If $f(x,y)$ is smooth, then $\frac{\partial f}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x}$

If F is conservative, then $\vec{F} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$

$\because \vec{F} = P\hat{i} + Q\hat{j} = \langle P(x,y), Q(x,y) \rangle$

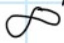
$\therefore \frac{\partial f}{\partial x \partial y} = \frac{\partial P}{\partial y}, \quad \frac{\partial f}{\partial y \partial x} = \frac{\partial Q}{\partial x}$

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

- def Simple Curve a curve that doesn't intersect itself anywhere between its end pts.

def Simply-connected region: Every simple closed curve in D encloses only pts that are in D.

$\Rightarrow$ A simply-connected region contains no hole and can't consists of two separate pieces.

 not simple closed curve  not simple, not closed curve.

 simple closed curve

- Theorem. $\vec{F} = P\hat{i} + Q\hat{j}$ a vector field on an open simply-connected region D.

Suppose that P, Q have continuous first-order partial derivatives, and

Suppose that P, Q have continuous first-order partial derivatives, and

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\vec{F}$ conservative.

- This can be used to determine whether $\vec{F}$ is a conservative vector field.

e.g. Is $F(x,y) = (x-y)\hat{i} + (x-2)\hat{j}$ conservative?

$\frac{\partial P}{\partial y} = -1 \neq \frac{\partial Q}{\partial x} = 1$

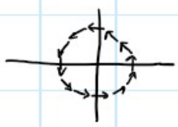
$\therefore$ not conservative.

e.g. Is $\vec{F}(x,y) = (3+2xy)\hat{i} + (x^2-3y^2)\hat{j}$ conservative?

$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$

domain of $\vec{F}$ is $\mathbb{R}^2$ $\therefore$ the domain is simply-connected.

e.g. See if conservative



Probably conservative. Not conservative.

$\Rightarrow$ can find gradient on graph. $\Rightarrow$ If there's gradient, it would be a circle

16.3 cheatsheet

Monday, February 18, 2019 4:59 PM

Lecture_1...

"Cheat Sheet"

We have the following implications:

① Fundamental Theorem of Line integrals

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

This means, if $\vec{F} = \nabla f$, then the line integral is independent of path.

"$\vec{F}$ conservative $\Rightarrow$ Independent of path"

② "Independent of path $\Leftrightarrow$ $\int_C \vec{F} \cdot d\vec{r} = 0$ on every closed loop C"

③ "Independent of path $\Rightarrow$ Conservative" $\Leftarrow$ domain D open & connected.

④ "Conservative $\Rightarrow$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$"

⑤ "$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and D open, simply-connected $\Rightarrow$ Conservative"

<u>NOTE</u> ①, ②, and ③ combined give us

"conservative $\Leftrightarrow$ independent of path $\Leftrightarrow$ $\int_C \vec{F} \cdot d\vec{r} = 0$ on every closed loop C"

<u>NOTATION</u> $X \Rightarrow Y$ "If X, then Y" $X \Leftrightarrow Y$ "X if and only if Y"

16.4 Green's Theorem

Friday, February 22, 2019 8:31 AM

- def. A positive orientation of a single closed curve C refers to a single counterclockwise traversal of C.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- Green's Theorem for not conservative things

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ have partial derivatives on an open set containing D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$\int_C \vec{F}\cdot d\vec{r}$, if it's conservative on a closed curve, then $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0$

open set containing D

o Make sure C is closed

Rem. $F(b) - F(a) = \int_a^b F'\,dx$

e.g. 1. Evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$, where C is the circle $x^2+y^2=9$.

1° Check partial in D.

2° Check if the partial function is defined on D

Notation: $\oint$ $\oint$ (sometimes different in diff books)

$\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$

$= \iint_D (7-3)\,dA = 4\iint_D dA$

$= 4\int_0^{2\pi}\int_0^3 r\,dr\,d\theta$

$= 36\pi$

e.g. 1b. SAME except clockwise orientation for C

$\int (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$

e.g. 1b. [illegible]

$$\int (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$$

$$= -\int_{-C} (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy = -36\pi$$

e.g. 2. Find the area enclosed by the eclipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$C: r(t) = \langle x(t), y(t) \rangle, \quad 0 \le t \le 2\pi$

take $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$

Note: $\vec{F} = \langle P, Q \rangle$, $P(x,y) = -y \quad \frac{\partial P}{\partial y} = -1$

$Q(x,y) = x \quad \frac{\partial Q}{\partial x} = 1$

$$A = \frac{1}{2}\oint_C P\,dx + Q\,dy = \frac{1}{2}\iint_D 1+1\,dA = \iint_D dA$$

$$\int_0^{2\pi} -\frac{1}{2} b\sin t \cdot a \cdot (-\sin t)\,dt + \frac{1}{2} a\cos t \cdot b\cos t\,dt$$

$$= \frac{ab}{2}\int_0^{2\pi} 1\,dA = \pi ab$$

o other kind of vector field

$P(x,y) = -y, \quad Q(x,y) = x$

$\frac{\partial P}{\partial y} = -1, \quad \frac{\partial Q}{\partial x} = 1$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2$

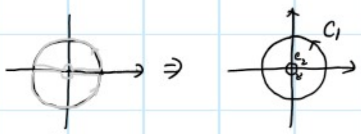
$\frac{1}{2}\oint_C P\,dx + Q\,dy = \iint_D dA$

- Extending Green's Theorem.

$\vec{F}(x,y) = \langle -\frac{K_1 y}{x^2+y^2}, \frac{K_1 x}{x^2+y^2} \rangle$

C = circle traversed counterclockwise, $x^2+y^2=1$.

Can't use Green's Theorem because $(0,0)$ is not defined.



(the horizontal two cancelled)

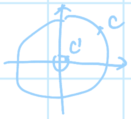
$$\oint_C P\,dx + Q\,dy = \oint_{C_1} P\,dx + Q\,dy - \oint_{C_2} P\,dx + Q\,dy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\int_C Pdx+Qdy = \int_{C_1} Pdx+Qdy + \int_{C_2} Pdx+Qdy$$

$$= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$

e.g. If $F(x,y) = \dfrac{-y\hat{i}+x\hat{j}}{x^2+y^2}$, show that $\int_C F\cdot dr = 2\pi$ for every positively oriented simple closed path that encloses the origin.



$$\int_C Pdx+Qdy + \int_{C'} Pdx+Qdy$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} dA = 0$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \oint_{C'} \vec{F}\cdot d\vec{r}$$

assume $x = a\cos t$, $y = a\sin t$, (in C')

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{C'} Pdx+Qdy = \int_0^{2\pi} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dt$$

$$= \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t) + a\cos t \cdot a\cos t}{a^2} dt$$

$$= 2\pi$$

16.5 Curl & Divergence

Monday, February 25, 2019 8:33 AM

- curl, takes a vector field, spits out a vector field

$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

$\text{curl}(\vec{F}) := \nabla \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

e.g. $\vec{F} = -y\hat{i} + x\hat{j}$. compute $\text{curl}(\vec{F})$

$\text{curl}(\vec{F}) = \nabla \times \vec{F} = (0-0)\hat{i} + (0-0)\hat{j} + (1+1)\hat{k}$

$= 2\hat{k}$

$\text{curl}(\vec{F}) = \langle 0, 0, 2 \rangle$ at every pt.

e.g. $\vec{F} = 2x\hat{i} + 2y\hat{j}$, $\text{curl}(\vec{F})$?

$$\text{curl}(\vec{F}) = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 2y & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k} = 0$$

(F is conservative, $\text{curl}(\vec{F}) = 0$)

- $\text{curl}(\vec{F}) = 0$ if $\vec{F}$ conservative.

  curl is identifying the rotation of a function $\vec{F}$.

- Remark $\vec{F} = \langle P, Q \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

$\therefore$ when $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, if it's open-simplly connected,

then it's conservative

$\Rightarrow$ Theorem. $\vec{F}$ conservative $\Rightarrow \text{curl}(\vec{F}) = \vec{0}$

$\text{curl}(\vec{F}) = \vec{0}$, D open $\boxed{\text{simply-connected}}^{\rightarrow \text{ defined in topology}} \Rightarrow \vec{F}$ conservative.

(even when $R \neq 0$)

$\Rightarrow$ $\text{curl}(\vec{F}) = \vec{0}$, for $\vec{F}$ and partials defined on $\mathbb{R}^3 \Rightarrow \vec{F}$ conservative.

e.g. $F(x,y,z) = y^2z^3\hat{i} + 2xyz^3\hat{j} + 3xy^2z^2\hat{k}$

(a) Is $\vec{F}$ conservative?

(b) find f, such that $\nabla f = \vec{F}$.

(a) $\vec{F}$ and its partial defined on $\mathbb{R}^3$

$$\text{curl}(\vec{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \langle 6xyz^2 - 6xyz^2,\ 3y^2z^2 - 3y^2z^2,\ 2yz^3 - 2yz^3 \rangle = \vec{0}$$

$\therefore \vec{F}$ is conservative.

(b) $f = xy^2z^3 + g(y) + h(z)$

$f = xy^2z^3 + i(x) + k(z)$

$f = xy^2z^3 + q(x) + p(y)$

$\therefore$ take $f(x) = xy^2z^3$.

- <u>def</u> Divergence of $\vec{F}$: "takes in a vector field, spits out a scalar (function)."

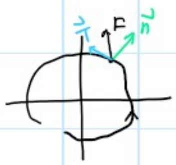
  $\text{Div}(F) := \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ $(F = \langle P, Q, R \rangle)$

  $\Rightarrow$ how much is leaving the area.

- Theorem. $\text{Div}(\text{curl}(F)) = 0$

- Green's Theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_D \underbrace{(\text{curl}(\vec{F}) \cdot \hat{k})}_{\text{times } \hat{k} \text{ to change curl}(\vec{F}) \text{ to scalar}} dA$

$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot \vec{T} ds$ $\Rightarrow$ the amount that the vector field curl around curve



$\oint_C \vec{F} \cdot \vec{n} ds = \iint_D \text{div}(\vec{F}) dA$

$\Rightarrow$ the amount that the vector field leave the curve.

16.6 Parametrization

Wednesday, February 27, 2019 8:51 AM

- 

$\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$

Parametric surface

e.g. Parametrize the surface $z = 4x^2 + 4y^2$

$x(u,v) = u$

$y(u,v) = v$

$z(u,v) = 4u^2 + 4v^2$

$\therefore r(u,v) = u\hat{i} + v\hat{j} + (4u^2 + 4v^2)\hat{k}$

e.g. Parametrize the top half of the cone $z = 5\sqrt{x^2+y^2}$

$x = u,\ y = v,\ z = 5\sqrt{u^2+v^2}$

$r(u,v) = u\hat{i} + v\hat{j} + 5\sqrt{u^2+v^2}\hat{k}$

or use (r,θ) $\quad x = \underline{r}\cos\theta,\ y = \underline{r}\sin\theta,\ z = 5r$ $\quad$ r is necessary, or z will be a constant.

$\vec{r}(r,\theta) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + 5r\hat{k}$

e.g. What surface is describe by $\vec{r}(u,v) = \cos u\,\hat{i} + \sin u\,\hat{j} + v\hat{k}$

A cylinder

$x^2 + y^2 = 1,\ z = v.$

$\Rightarrow$ equation of the cylinder $x^2 + y^2 = 1$

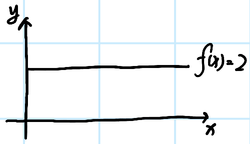
e.g. Parametrize $z^2 + y^2 = 4$ for $0 \le x \le 1$

$z = 2\cos\theta,\ y = 2\sin\theta$

$x = r$ or $\cos^2 v$

$\begin{cases} r \in [0,1] \text{ or } v \in \underline{[\frac{\pi}{2}, \pi]} \\ \underline{\theta \in [0, 2\pi]} \end{cases}$ make bounds to avoid overlap

- Surface of Revolution



Rotate $y=2$ around x-axis.

$x=x$

$y=f(x)\cdot\cos\theta,\ z=f(x)\sin\theta$

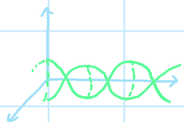
↑ surface of revolution

- Surface of rotation (about x-axis)

$$\begin{cases} x=x \\ y=f(x)\cdot\cos\theta \\ z=f(x)\sin\theta \end{cases}$$

e.g. find parametric equations for the surface generated by rotating $y=\cos x$, $x\in[0,2\pi]$, about x-axis. What does the surface look like?

$$\begin{cases} x=x \\ y=\cos x\cos\theta \\ z=\cos x\cdot\sin\theta \end{cases}$$



e.g. Parametrize $x^2+y^2+z^2=a^2$

$$\begin{cases} x=a\cos\theta\sin\varphi \\ y=a\sin\theta\cdot\sin\varphi \\ z=\cos\varphi \end{cases}$$

Use spherical coordinates.

$\theta\in[0,2\pi],\ \varphi\in[0,\pi]$

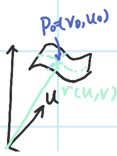
e.g. Find a vector function $\vec{r}(u,v)$ that represents a plane that passes through $P_0=(x_0,y_0,z_0)$, with position vector $\vec{r_0}=\langle x_0,y_0,z_0\rangle$, and that contains two non-parallel vectors $\vec{a}$ and $\vec{b}$.

line: $u\vec{a}+v\vec{b}=w\in$ the plane

$\therefore \langle x_0,y_0,z_0\rangle+u\vec{a}+v\vec{b}=0$

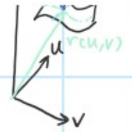
- this only needs 2 vectors

- Tangent Plane



take two derivatives along u,v directions, r_u,r_v

point: (u_0,v_0), two vectors: r_u,r_v (in tangent plane)



point: (u_0, v_0), two vectors: r_u, r_v (in tangent plane)

$r_u = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \rangle$, $r_v = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \rangle$

$\therefore$ normal vector $= r_u \times r_v$

$(r_u \times r_v) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$ tangent plane of (x, y, z)

$\vec{v} = (\vec{r} - \vec{r_0})$

e.g. Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at $(1, 1, 3)$

$r_u = \langle 2u, 0, 1 \rangle$, $r_v = \langle 0, 2v, 2 \rangle$

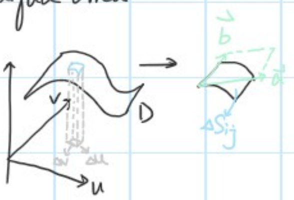
$$\vec{n} = r_u \times r_v = \begin{vmatrix} i & j & k \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = \langle -2v, -4u, 4uv \rangle$$

$\because (x_0, y_0, z_0) = (1, 1, 3)$ $\therefore u = 1, v = 1$

$\therefore \vec{n} = \langle -2, -4, 4 \rangle$

tangent plane: $-2(x-1) - 4(y-1) + 4(z-3) = 0$

- Surface Area



$|\vec{a} \times \vec{b}| = A(P_{ij})$

$= |\Delta u r_u \times \Delta v r_v|$

$= |r_u \times r_v| \Delta u \cdot \Delta v$

$\therefore A(S) = \lim_{\substack{m \to \infty \\ n \to \infty}} \sum_{i=1}^{m} \sum_{j=1}^{n} |r_u \times r_v| \cdot \Delta u \Delta v$

$A(S) = \iint_D |r_u \times r_v| \, du dv$ (for any parametric surface)

e.g. Compute SA of a sphere of radius a

$x^2 + y^2 + z^2 = a^2$

$$\begin{cases} x = a\cos\theta \sin\phi \\ y = a\sin\theta \sin\phi \\ z = a\cos\phi \end{cases}$$

remember bounds

$$\boxed{\begin{matrix} \theta \in [0, 2\pi] \\ \phi \in [0, \pi] \end{matrix}}$$

$r_\theta = \langle -a\sin\theta \sin\phi, a\cos\theta \sin\phi, 0 \rangle$

$r_\phi = \langle a\cos\theta\cos\phi, a\sin\theta\cos\phi, -a\sin\phi \rangle$

$r_\theta \times r_\phi = \begin{vmatrix} i & j & k \\ \langle & r_\theta & \rangle \end{vmatrix}$

$$r_\theta \times r_\phi = \begin{vmatrix} i & j & k \\ < & r_\theta & > \\ < & r_\phi & > \end{vmatrix}$$

$$= < a^2 \sin^2\phi \cdot \cos\theta, \quad a^2 \sin\theta \sin^2\phi, \quad a^2 \cos^2\theta \cdot \cos\phi \sin\phi + a^2 \sin^2\theta \cdot \cos\phi \sin\phi >$$

$$= < a^2 \sin^2\phi \cdot \cos\theta, \quad a^2 \sin\theta \cdot \sin^2\phi, \quad a^2 \cos\phi \sin\phi >$$

$$|r_\theta \times r_\phi| = \sqrt{a^4 \sin^4\phi + a^4 \cos^2\phi \sin^2\phi}$$

$$= a^2 \sin\phi \sqrt{\sin^2\phi + \cos^2\phi} = a^2 \sin\phi$$

$$SA(S) = \iint_D |\vec{r}_\phi \times \vec{r}_\theta| \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} a^2 \sin\phi \, d\phi \, d\theta = 2\pi a^2 \int_0^{\pi} \sin\phi \, d\phi$$

$$= 2\pi a^2 \cdot (1+1) = 4\pi a^2$$

16.7 Surface Integrals

Wednesday, March 6, 2019 9:03 AM

- Rem.
  - Integral over arc length $\int_c f(x,y)\,ds$
  - Integral over a vector field $\int_c \vec{F}\cdot d\vec{r} = \int_c \vec{F}\cdot\vec{T}\,ds$
  - Integral over surface area $\iint_D f(x,y,z)\,dS$
  - $\iint_D \vec{F}\vec{n}\cdot dS = \iint_D \vec{F}\cdot d\vec{S}$

- def $\iint_S f(x,y,z)\,dS = \int_c^d\int_a^b f(x(u,v),y(u,v),z(u,v))\,|r_u\times r_v|\,du\,dv$

  parametrize S with $\vec{r}(u,v)$, $u\in[a,b]$, $v\in[c,d]$

  - $dS = |r_u\times r_v|\,du\,dv$

e.g.1. $\iint_S x^2\,dS$, S is a sphere of radius 1.

$\vec{r}(\theta,\phi) = \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$ $(\phi\in[0,\pi], \theta\in[0,2\pi])$

$|r_\theta\times r_\phi| = \sin\phi$

$\int \cos^2\theta = \int\frac{\cos2\theta+1}{2} = \frac{\sin2\theta}{4}+\frac{\theta}{2}$

$\int_0^{2\pi}\int_0^{\pi}\cos^2\theta\sin^3\phi\,d\phi\,d\theta$

$\int\sin^3\phi\,d\phi$

$= \int_0^{2\pi}\cos^2\theta\,d\theta\cdot\int_0^{\pi}\sin^3\phi\,d\phi$

$= (1-\cos^2\phi)\sin\phi$

$= \pi\cdot(1+1 \cancel{+\ +} -\frac{1}{3}-\frac{1}{3})$

$= -\cos\phi + \frac{\cos^3\phi}{3}$

$= \frac{4\pi}{3}$

Mass of parametric surfaces with density $\rho(x,y)$

$$m = \iint_S \rho(x,y)\,dS$$

- center of mass: $\bar{x} = \frac{1}{m}\iint_S x\rho(x,y,z)\,dS$, $\bar{y} = \frac{1}{m}\iint_S y\rho(x,y,z)\,dS$, $\bar{z} = \frac{1}{m}\iint_S z\rho(x,y,z)\,dS$

- when $z = g(x,y)$, $\iint_S f(x,y)\,dS = \iint_D f(x,y,g(x,y))\boxed{\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}}\,dA$

  (plug in $|r_u\times r_v|$)

e.g. Evaluate $\iint_S y\,dS$, where S is the surface $z = x+y^2$, $x\in[0,1]$, $y\in[0,2]$

$\frac{\partial z}{\partial x} = 1$, $\frac{\partial z}{\partial y} = 2y$

$|r_u\times r_v| = \sqrt{1+1+4y^2} = \sqrt{2+4y^2}$

$\int_0^2\int_0^1 y\sqrt{2+4y^2}\,dx\,dy$

$$|r_u \times r_v| = \sqrt{1+1+4y^2} = \sqrt{2+4y^2}$$

$$\int_0^2\int_0^1 y\sqrt{2+4y^2}\,dx\,dy$$

$$= \int_0^1 \frac{1}{12}(2+4y^2)^{\frac{3}{2}}\Big|_0^2\,dx$$

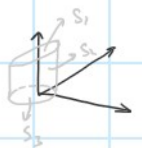
$$= \int_0^1 \frac{1}{12}\cdot\left(18^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)dx$$

$$= \frac{1}{12}\left(18^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)$$

- If a surface S is composed of a series of smaller surfaces, then

$$\iint_S f(x,y,z)\,dS = \sum_i^n \iint_{S_i} f(x,y,z)\,dS$$

such as



$S = S_1+S_2+S_3$

e.g. Evaluate $\iint_S z\,dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2+y^2=1$, whose bottom is the disk $x^2+y^2\le 1$ in the plane $z=0$ and whose top S_3 is the part of plane $z=1+x$ that lies above S_2.

S_1 : $\begin{cases} x=\cos u \\ y=\sin u \\ z=z \end{cases}$

$r_u = \langle -\sin u, \cos u, 0\rangle$

$r_z = \langle 0,0,1\rangle$

$$r_u \times r_z = \begin{vmatrix} i & j & k \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= \langle \cos u, \sin u, 0\rangle$

$\therefore |r_u \times r_z| = 1$

$$\iint_S z\,dS = \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dz\,d\theta$$

$$= \int_0^{2\pi} \frac{1+\cos^2\theta+2\cos\theta}{2}\,d\theta$$

$$= \frac{\theta}{2} + \frac{1}{2}\left(\frac{\sin 2\theta}{4}+\frac{\theta}{2}\right) + \sin\theta\Big|_0^{2\pi}$$

$$= \pi + \frac{\pi}{2} = \frac{3\pi}{2}$$

S_2 : $\begin{cases} x=u \\ y=v \\ z=0 \end{cases}$

$r_u = \langle 1,0,0\rangle$

$r_v = \langle 0,1,0\rangle$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0,0,1\rangle$$

$\therefore |r_u \times r_v| = 1$

$\therefore \iint_S z\,dS = \int_0^1\int_0^1 0\,dS = 0$

S_3 : $\because z = 1+x$

$\therefore \frac{\partial z}{\partial y} = 0,\ \frac{\partial z}{\partial x} = 1$ $D = \{(x,y) : x^2+y^2\le 1\}$

$\therefore \sqrt{1+1+1} = \sqrt{2}$

$$\int_0^{2\pi}\int_0^1 (1+r\cos\theta)\sqrt{2}\, r\, dr\, d\theta = \sqrt{2}\int_0^{2\pi} \frac{r^2}{2} + \frac{r^3}{3}\cos\theta \Big|_0^1 = \sqrt{2}\cdot\left(\frac{\theta}{2} + \frac{\sin\theta}{3}\right)\Big|_0^{2\pi}$$

$$\cancel{= \sqrt{2}\int_0^{2\pi} 1+\cos\theta\, d\theta = \sqrt{2}(2\pi) = 2\sqrt{2}\pi} \qquad = \sqrt{2}\pi$$

$\therefore S = S_1 + S_2 + S_3 = \sqrt{2}\pi + 0 + \frac{3}{2}\pi = \left(\frac{3}{2}+\sqrt{2}\right)\pi$

- Exercise 9. $\iint_S (x^2+y^2)\,dS$, S is the part of plane $z = 1+2x+3y$ that lies above the rectangle $[0,3]\times[0,2]$

$= 26$

Exercise 11. $\iint_S (x^2+y^2)\,dS$, S is the surface with vector equation $r(u,v) = \langle 2uv,\ u^2-v^2,\ u^2+v^2\rangle$, $u^2+v^2 \le 1$

$= \sqrt{2}\pi$

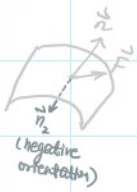
Exercise 17. $\iint_S (x^2z+y^2z)\,dS$, S is the hemisphere $x^2+y^2+z^2=4$, $z \ge 0$

- Surface integral over a vector field. (the flux of $\vec{F}$ across S)

$$\iint_S \vec{F}\cdot d\vec{S} = \iint \vec{F}\cdot\vec{n}\cdot dS = \iint \vec{F}(r(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,dA$$

find outward unit normal vector:

$\vec{n} = \frac{\vec{r}_u\times\vec{r}_v}{|r_u\times r_v|}$ $\because dS = |r_u\times r_v|\cdot du\,dv$



$\therefore \vec{n}\cdot dS = r_u\times r_v$

- if $z = g(x,y)$

$$\vec{n} = \frac{-\frac{\partial g}{\partial x}\hat{i} - \frac{\partial g}{\partial y}\hat{j} + \hat{k}}{\sqrt{1+(\frac{\partial g}{\partial x})^2+(\frac{\partial g}{\partial y})^2}} \quad (\hat{k}>0)$$

$\vec{n}$ pointing up (choose a positive orientation/outward normal)

- If the parametric surface is the graph of a function, $\vec{r}(x,y)$, $\vec{F} = P\hat{i}+Q\hat{j}+R\hat{k}$

$$\iint_D \vec{F}\cdot(\vec{r}_x\times\vec{r}_y)\,dA = \iint_D\left(-P\cdot\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)dA$$

- Some surfaces are not orientable 

- def of $\iint_S \vec{F}\cdot d\vec{S}$
  - $\vec{F} = \rho(x,y,z)\cdot\vec{v}(x,y,z)$, ρ is density of a fluid, v is the velocity
    $\Rightarrow$ rate of flow thru S.
  - $\vec{F} = \vec{E}$, electric field.

$\Rightarrow$ electric flux of $\vec{E}$ through S

- $\vec{F}$ is heat flow $\Rightarrow$

  rate of heat flow across S.

- Let $\vec{F} = -k\nabla u$, where k is the conductivity of a substance and $u(x,y,z)$ is the temperature. The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere of radius a with center at the center of the ball.

$u(x,y,z) = C(x^2+y^2+z^2)$

[illegible]

$\vec{F} = \ldots \langle 2x, 2y, 2z\rangle$

[illegible]

$= -2 \ldots 4\pi$ [illegible]

$= -8k c \pi a^3$

16.8,9 Stoke's & Divergence

Remember: Green's Theorem $\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \text{curl}\,\vec{F}\cdot\hat{k}\, dA$

$\vec{F} = \langle P, Q, 0\rangle$

k ⊥ the movement in the vector field

- Stoke's Theorem. Let S be an oriented piecewise smooth surface that is bounded by a simple closed, piecewise smooth boundry C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S, then

$$\oint_C \vec{F}\, d\vec{r} = \iint_S \text{curl}(\vec{F})\, d\vec{S} = \iint_S \text{curl}(\vec{F})\cdot\vec{n}\, dS$$

($\vec{n}$ need to go out of page when the curve is positive being seen above)

e.g. 1. Evaluate $\oint_C F\cdot d\vec{r}$, $F = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$, C is the curve of the intersection of the plane $y+z=2$ and cylinder $x^2+y^2=1$. (orient C to be clockwise when view from above

$\begin{cases} y+z=2 \\ x^2+y^2=1 \end{cases}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

$$= \langle 0, 0, 1+2y\rangle$$

Parametrize S.

$\begin{cases} x = x \\ y = y \\ z = 2-y \end{cases}$ $\quad D\ \{(x,y) : x^2+y^2 \le 1\}$

$r_x = \langle 1, 0, 0\rangle,\ r_y = \langle 0, 1, -1\rangle$

$$\begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix}$$

$\begin{cases} y=y \\ z=2-y \end{cases}$

$r_x = \langle 1,0,0\rangle, r_y = \langle 0,1,-1\rangle$

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \langle 0,1,1\rangle$

$$\iint_S \text{curl}F\,dS = \iint_D \langle 0,0,1+2y\rangle \cdot \langle 0,1,1\rangle\, dA$$
$$= \int_0^{2\pi}\int_0^1 (1+2\sin\theta) r\, dA$$
$$= \int_0^{2\pi} \frac{1}{2} + \sin\theta\, d\theta$$
$$= \pi$$

e.g. 2. Use Stoke's Theorem to compute $\iint_S \text{curl}(\vec{F})\,d\vec{S}$, where $\vec{F}(x,y,z) = xz\hat{i} + yz\hat{j} + xy\hat{k}$, and S is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above x-y plane.

when $x^2+y^2=1$, $z^2=3 \Rightarrow z=\sqrt{3}$

$$\iint_S \text{curl}(\vec{F})\,d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$$

$\vec{F}(x,y,\sqrt{3}) = \langle \sqrt{3}x, \sqrt{3}y, xy\rangle$ $\quad r'(\theta) = \langle -\sin\theta, \cos\theta, 0\rangle$

$x^2+y^2=1 \quad \therefore x=\cos\theta, y=\sin\theta \quad \therefore \vec{r} = \langle \cos\theta, \sin\theta, 0\rangle$

$$\oint_C \vec{F}\,d\vec{r} = \int_0^{2\pi} F(\vec{r}(t))\cdot r'(t)\,dt = \int_0^{2\pi} -\sin\theta\cos\theta + \sin\theta\cos\theta\, d\theta$$
$$= 0$$

- Divergence Theorem

$\oint_C \vec{F}\cdot d\vec{r} \Rightarrow \oint_C \vec{F}\cdot\vec{T}\,ds$

$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D \text{Div}(\vec{F})\,dA$

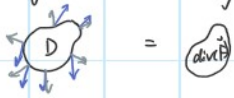
Let E be a simple solid region and S is the boundary surface of E, given with positive outward orientation.

F, a vector field, whose comp vectors has cont partial derivatives on an open region containing E. Then

$$\iint_S \vec{F}\cdot n\,dS = \iiint_E \text{div}(\vec{F})\,dV \qquad (d\vec{S} = \vec{n}\,dS)$$

$\Rightarrow$ The amount of flow (flux) leaving a region is the sum of the divergence across the space.

$\vec{T}$ —
$\vec{n}$ —



o 2^{nd} dimension : $\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D \text{div}(\vec{F})\,dA$

o Proof.

o 2 dimension: $\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D \mathrm{div}(\vec{F})\,dA$

o Proof.

$\because \ \mathrm{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

$\therefore \iiint_E \mathrm{div}(\vec{F})\,dV = \iiint_E \frac{\partial P}{\partial x}\,dV + \iiint \frac{\partial P}{\partial y}\,dV + \iiint \frac{\partial P}{\partial z}\,dV$

e.g.1. find the flux of $\vec{F} = \langle z^3 y, y, \sin x\rangle$ over unit sphere.

$\iint \vec{F}\,d\vec{S} = \iiint_{\text{sphere}} (0+1+0)\ dV$

$= \iiint dV = \frac{4}{3}\pi r^3$

• Exercise

I. Evaluate $\iint_S \vec{F}\,d\vec{S}$, $\vec{F} = \langle x, -z, y\rangle$ and S is the sphere $x^2+y^2+z^2=r^2$ in the first octant, with orientation towards origin.

Just change the order of cross product

II. $\vec{F} = \left\langle \frac{cx}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{cy}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{cz}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right\rangle$, c is a constant. Show that the flux through the sphere centered at the origin is independent of radius.

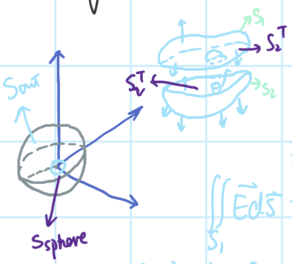
$= \cancel{2\pi c}\ 4\pi c$

3. Use divergence theorem to compute $\iint_S (2x+2y+z^2)\, d\vec{S}$ when S is the unit vector $x^2+y^2+z^2=1$.

4. $\vec{E} = \left\langle \frac{\varepsilon Q x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{\varepsilon Q y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{\varepsilon Q z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right\rangle$

(a) Show that $\operatorname{div}(\vec{E}) = 0$

★★★★ Show that the flux of E through any closed surface that contains the origin is $4\pi\varepsilon Q$ (Gaussian's law)



$\iint_{S_1} \vec{E}d\vec{S} = \iiint_E 0\, dV = 0$, $\iint_{S_2} \vec{E}d\vec{S} = \iiint_E \mathrm{div}(\vec{E})dV = 0$

$\iint_{S_1^T} \vec{E}d\vec{S} + \iint_{S_2^T} \vec{E}d\vec{S} = 0$

$\iint_{S_1} \vec{E}d\vec{S} + \iint_{S_2} \vec{E}d\vec{S} = \iint_S \vec{E}d\vec{S} + (\underbrace{\iint_{sphere} \vec{E}d\vec{S}}_{inward})$

$\therefore \iint_S \vec{E}d\vec{S} = \iint_{sphere} \vec{E}d\vec{S}$

assume $x^2+y^2+z^2=r^2$

$E = \varepsilon Q < \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} >$, $r_\phi \times r_\theta = \frac{1}{r}<x,y,z>$

$E\cdot(r_\phi \times r_\theta) = \varepsilon Q\left(\frac{x^2+y^2+z^2}{r^4}\right) = \varepsilon Q \cdot \frac{1}{r^2}$

$\iint_S \vec{F}d\vec{S} = \int_0^{2\pi}\int_0^{\pi} \varepsilon Q \cdot \sin\phi\, d\phi d\theta = \varepsilon Q \cdot \int_0^{2\pi} -\cos\phi\Big|_0^{\pi} d\theta = 4\pi\varepsilon Q$

- for surfaces don't contain origin, flux is always 0.

5. Let $\vec{F} = <\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}>$, let C be any positively oriented smooth curve containing the origin. Show $\oint_C \vec{F}d\vec{r} = 2\pi$

$\underbrace{\oint_C \vec{F}d\vec{r} - \oint_{C_1}\vec{F}dr}_{\text{can use Green's Theorem}} = \oint_{C_1}\vec{F}dr + \oint_{C_2}\vec{F}dr$

$x^2+y^2=r^2$, $x=r\cos\theta, y=r\sin\theta$

$r' = <-r\sin\theta, r\cos\theta>$

$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ ①

$\therefore \oint_C \vec{F}d\vec{r} = \oint_{C_1}\vec{F}d\vec{r}$

Cut into half:

$= \int_0^{2\pi} \frac{r^2(\sin^2\theta+\cos^2\theta)}{r^2} d\theta$

$\oint_{C_1} Pdx+Qdy = 0$

$= 2\pi$

C_2 $\oint_{C_2} Pdx+Qdy=0$

① $\vec{F}$ is not conservative. Because not simply connected, and $\oint_C \vec{F}d\vec{r} \neq 0$ for closed C.

Use Green's Theorem to compute area of triangle of vertices (0,0), (a,b), (c,d)

(Assume (0,0), (a,b), (c,d) are listed counterclockwise)

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

$+C \quad = \quad A \quad \int_{C_1} x \quad -y \quad +\frac{1}{2}\int_{C_2} \quad y- \quad x \quad \int xdy-ydx$

Prob

Sunday, February 17, 2019 2:03 PM

The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\mathbf{r} = \langle x, y, z\rangle$ is $\mathbf{F}(r) = K\mathbf{r}/|\mathbf{r}|^3$ where K is a constant. Find the work done as the particle moves along a straight line from (3, 0, 0) to (3, 3, 5).

$\frac{K}{3} - \frac{K}{\sqrt{43}}$

$\vec{r} = \langle x, y, z \rangle, \quad r = \langle 3,0,0\rangle + t\langle 0,3,5\rangle$

$\therefore |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\vec{F}(\vec{r}) = \left\langle \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right\rangle \cdot K$

find $\nabla f = F$, $F = \nabla U$

$\therefore f = -\frac{K}{\sqrt{x^2+y^2+z^2}}$

$W = f(3,3,5) - f(3,0,0) = K\left(\frac{1}{3} - \frac{1}{\sqrt{43}}\right)$

The base of a circular fence with radius 10 m is given by $x = 10\cos(t)$, $y = 10\sin(t)$. The height of the fence at position (x, y) is given by the function $h(x, y) = 5 + 0.04(x^2 - y^2)$, so the height varies from 1 m to 9 m. Suppose that 1 L of paint covers 100 m^2. Determine how much paint you will need if you paint both sides of the fence. (Round your answer to two decimal places.)

6.28 L

$ds = 10dt$

$\int_0^{2\pi} (5 + 4(\cos^2 t - \sin^2 t)) \cdot 10dt$

$= 5t \cdot 10 + 4 \cdot \frac{\sin 2\theta}{2} \cdot 10 \Big|_0^{2\pi}$

$= 100\pi$

$\because$ both sides

$\therefore S = 200\pi$

$\frac{200\pi}{100} = 2\pi \approx 6.28$ L

(a) Set up, but do not evaluate, a double integral for the area of the surface with parametric equations $x = au\cos(v)$, $y = bu\sin(v)$, $z = u^2$, $0 \le u \le 2$, $0 \le v \le 2\pi$.

(a) $\mathbf{r}_u = a\cos v\,\mathbf{i} + b\sin v\,\mathbf{j} + 2u\mathbf{k}$, $\mathbf{r}_v = -au\sin v\,\mathbf{i} + bu\cos v\,\mathbf{j} + 0\mathbf{k}$, and $\mathbf{r}_u \times \mathbf{r}_v = -2bu^2\cos v\,\mathbf{i} - 2au^2\sin v\,\mathbf{j} + abu\mathbf{k}$.

$$A(S) = \int_0^{2\pi}\int_0^2 |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv = \int_0^{2\pi}\int_0^2 \sqrt{4b^2u^4\cos^2 v + 4a^2u^4\sin^2 v + a^2b^2u^2}\,du\,dv$$

(b) Eliminate the parameters to show that the surface is an elliptic paraboloid and set up another double integral for the surface area.

(b) $x^2 = a^2u^2\cos^2 v$, $y^2 = b^2u^2\sin^2 v$, $z = u^2$ $\Rightarrow$ $x^2/a^2 + y^2/b^2 = u^2 = z$ which is an elliptic paraboloid. To find D, notice that $0 \le u \le 2$ $\Rightarrow$ $0 \le z \le 4$ $\Rightarrow$ $0 \le x^2/a^2 + y^2/b^2 \le 4$. Therefore, using this formula, we have

$$A(S) = \int_{-2a}^{2a}\int_{-b\sqrt{4-(x^2/a^2)}}^{b\sqrt{4-(x^2/a^2)}} \sqrt{1 + (2x/a^2)^2 + (2y/b^2)^2}\,dy\,dx.$$

6. (10 pts) Assume, again, the temperature at each point on the xy-plane is given by $T(x,y) = \frac{1}{3}x^2y + 5\sqrt{x^2+y^2}$ degrees Celcius. You are told that the average temperature along a curve C is given by $\frac{1}{L}\int_C T(x,y)\,ds$, where L is the total length of C.

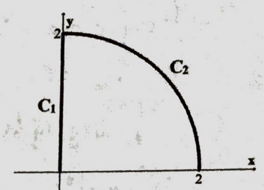
Let C be the curve consisting of a straight line segment from the origin to $(0,2)$, then one quarter of the circle $x^2+y^2=4$ from $(0,2)$ to $(2,0)$. Compute the average temperature along C. That is, compute $\frac{1}{L}\int_C T(x,y)\,ds$.

(Hint: Parameterize!)

$x = 2\cos\theta$
$y = 2\sin\theta$

$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 2$

$\theta \in [0, \frac{\pi}{2}]$



$T(x,y) = \frac{8}{3}\cos^2\theta \cdot \sin\theta + 10$

$L = 2 + \frac{2\pi \times 2}{4} = 2+\pi$

$\to C_2$

$\frac{1}{L}\int_0^{\frac{\pi}{2}} \frac{16}{3}\cos^2\theta\sin\theta + 20\ d\theta$

$= \frac{1}{2+\pi}\left(-\frac{16}{9}\cdot\cos^3\theta + 20\theta\right)$

$\boxed{\frac{10\pi+\frac{16}{9}}{2+\pi}}$

C_1: from $(0,0)$ to $(0,2)$

$x=0,\ y=2t,\ t\in[0,1]$

$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = 2$

$\frac{1}{L}\int_0^1 10t\cdot 2\,dt$

$= \frac{1}{2+\pi}\left(10t^2\Big|_0^1\right)$

$= \frac{10}{2+\pi}$

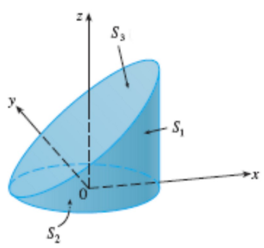
average: $\frac{10\pi + \frac{16}{9} + 10}{2+\pi}$

Find a parametric representation for the surface.

The part of the sphere $x^2 + y^2 + z^2 = 36$ that lies between the planes $z = -3$ and $z = 3$. (Enter your answer as a comma-separated list of equations. Let x, y, and z be in terms of θ and/or ϕ.)

$\boxed{x = 6\sin(\theta)\sin(\varphi), y = 6\cos(\theta)\sin(\varphi), z = 6\cos(\varphi)}$ (where $-3 < z < 3$)

$z \in [-3, 3]$ is only boundry !



EXAMPLE 3 Evaluate $\iint_S z\,dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 81$, whose bottom S_2 is the disk $x^2 + y^2 \le 81$ in the plane $z = 0$, and whose top S_3 is the part of the plane $z = 9 + x$ that lies above S_2.

SOLUTION The surface S is shown in the figure. (We have changed the usual position of the axes to get a better look at S.) For S_1 we use θ and z as parameters and write its parametric equations as

$x = 9\cos(\theta)$
$y = 9\sin(\theta)$
$z = z$

where $0 \le \theta \le 2\pi$ and $0 \le z \le 9 + x = 9 + 9\cos(\theta)$. Therefore,

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -9\sin(\theta) & 9\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \boxed{9\cos(\theta)}\ \mathbf{i} + 9\sin(\theta)\mathbf{j}$$

and $|\mathbf{r}_\theta \times \mathbf{r}_z| = 9\sqrt{(\cos(\theta))^2 + (\sin(\theta))^2} =$ $\boxed{9}$. Thus the surface integral over S_1 is

$$\iint_S z\,dS = \iint_D z\,|\mathbf{r}_\theta \times \mathbf{r}_z|\,dA$$

$$= \int_0^{2\pi}\int_0^{9+9\cos(\theta)} \boxed{9z}\ dz\,d\theta$$

$$= 729\int_0^{2\pi} \frac{1}{2}\left(\boxed{1+\cos(\theta)}\right)^2 d\theta$$

$$= \frac{729}{2}\int_0^{2\pi}\left[\boxed{1+2\cos(\theta)} + \frac{1}{2}(1+\cos(2\theta))\right]d\theta$$

$$= \frac{729}{2}\left[\frac{3}{2}\theta + 2\sin(\theta) + \boxed{\frac{\sin(2\theta)}{4}}\right]_0^{2\pi}$$

the part of the plane $z = 9 + x$ that lies above S_2.

SOLUTION The surface S is shown in the figure. (We have changed the usual position of the axes to get a better look at S.) For S_1 we use θ and z as parameters and write its parametric equations as

$$x = 9\cos(\theta)$$
$$y = 9\sin(\theta)$$
$$z = z$$

where $0 \le \theta \le 2\pi$ and $0 \le z \le 9 + x = 9 + 9\cos(\theta)$. Therefore,

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -9\sin(\theta) & 9\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \boxed{9\cos(\theta)}\ \mathbf{i} + 9\sin(\theta)\mathbf{j}$$

and $|\mathbf{r}_\theta \times \mathbf{r}_z| = 9\sqrt{(\cos(\theta))^2 + (\sin(\theta))^2} =$

$\boxed{9}$. Thus the surface integral over S_1 is

$$\iint_S z\,dS = \iint_D z\,|\mathbf{r}_\theta \times \mathbf{r}_z|\,dA$$

$$= \int_0^{2\pi}\int_0^{9+9\cos(\theta)} \boxed{9z}\ dz\,d\theta$$

$$= 729\int_0^{2\pi} \frac{1}{2}\left(\boxed{1+\cos(\theta)}\right)^2 d\theta$$

$$= \frac{729}{2}\int_0^{2\pi}\left[\boxed{1+2\cos(\theta)} + \frac{1}{2}(1+\cos(2\theta))\right]d\theta$$

$$= \frac{729}{2}\left[\frac{3}{2}\theta + 2\sin(\theta) + \boxed{\frac{\sin(2\theta)}{4}}\right]_0^{2\pi}$$

$$= \boxed{729\cdot\frac{3\pi}{2}}$$

Since S_2 lies in the plane $z = 0$, we have

$$\iint_S z\,dS = \iint_S 0\,dS = 0$$

The top surface S_3 lies above the unit disk D and is part of the plane $z = 9 + x$. So taking $g(x, y) = 9 + x$ and converting to polar coordinates, we have

$$\iint_S z\,dS = \iint_D (9+x)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= \int_0^{2\pi}\int_0^{9}(9 + r\cos(\theta))\sqrt{1+1+0}\,r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^{9}(9r + r^2\cos(\theta))\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left(\frac{729}{2} + \boxed{\frac{729}{3}\cos(\theta)}\right)d\theta$$

$$= \sqrt{2}\left[\frac{729\theta}{2} + \boxed{\frac{729}{3}\sin(\theta)}\right]_0^{2\pi}$$

$$= \boxed{729\pi\sqrt{2}}$$

Therefore

$$\iint_S z\,dS = \boxed{729\cdot\frac{3\pi}{2}} + 0 + 729\sqrt{2}\pi$$

$$= (\boxed{1093.5} + 729\sqrt{2})\pi$$

Evaluate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface S. In other words, find the flux of $\mathbf{F}$ across S. For closed surfaces, use the positive (outward) orientation.

$\mathbf{F}(x, y, z) = x\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}$
S is the part of the sphere $x^2 + y^2 + z^2 = 64$ in the first octant, with orientation toward the origin

$\boxed{-\frac{256}{3}\pi}$

Use Stokes' Theorem to evaluate $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$.

$\mathbf{F}(x, y, z) = xyz\mathbf{i} + xy\mathbf{j} + x^2yz\mathbf{k}$,
S consists of the top and four sides (but not the bottom) of the cube with vertices $(\pm3, \pm3, \pm3)$, oriented outward.

$\boxed{0}$

A particle moves along line segments from the origin to the points (1, 0, 0), (1, 4, 1), (0, 4, 1), and back to the origin under the influence of the force field

$\mathbf{F}(x, y, z) = z^2\mathbf{i} + 3xy\mathbf{j} + 2y^2\mathbf{k}$.

Find the work done.

$\oint_C \mathbf{F}\cdot d\mathbf{r} = \boxed{23}$