6.4

Monday, December 3, 2018 3:32 PM

- Dirac $\delta_0(t) = \delta(t) = \lim_{n\to\infty} f_n$
  - Properties: $\delta(t) = 0,\ t \neq 0$

    $\int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt = 1$, for any $\varepsilon > 0$
  - $\mathcal{L}\{\delta_c(t)\} = \mathcal{L}\{\delta(t-c)\} = e^{-cs}$

e.g. $2y'' + y' + 2y = \delta(t-5)$; $y(0) = 0,\ y'(0) = 0$

$(2s^2+s+2)\mathcal{L}\{y\} = e^{-5s}$

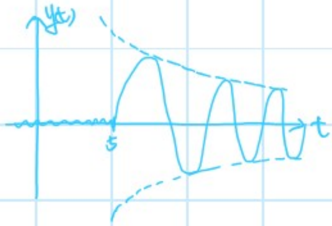
$\mathcal{L}\{y\} = \dfrac{e^{-5s}}{2s^2+s+2}$

$\Delta = 1 - 2\times2\times4 < 0$, irreducible.

$\mathcal{L}\{y\} = \dfrac{e^{-5s}}{2} \cdot \dfrac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}}$

$\Rightarrow \frac{4}{\sqrt{15}} \frac{u_5}{2} \cdot e^{-\frac{1}{4}(t-5)} \cdot \sin(\frac{\sqrt{15}}{4}(t-5))$

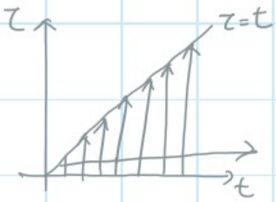
$y = \frac{2\sqrt{15}}{15} u_5 e^{-\frac{1}{4}(t-5)} \cdot \sin(\frac{\sqrt{15}}{4}(t-5))$



- $\frac{d}{dt} u_c(t) = \delta_c(t) \longrightarrow \int_0^t \delta_c(\tau)\,d\tau = u_c(t)$
- Define $\underset{h(t)}{\underset{\shortparallel}{(f*g)(t)}} = \int_0^t f(t-\tau)g(\tau)\cdot d\tau = \int_0^t g(t-\tau)\cdot f(\tau)\,d\tau$ a convolution.

$\mathcal{L}\{h(t)\} = \int_0^\infty e^{-st} \int_0^t f(t-\tau)g(\tau)\cdot d\tau\ dt$

$= \int_0^\infty \int_0^t e^{-st} \cdot f(t-\tau)g(\tau) \cdot d\tau\,dt.$



$0 < t < \infty$
$0 < \tau < t$

$\tau < t < \infty$
$0 < \tau < \infty$

$\int^\infty \int^\infty$

$(f * u_c(t)) = \int_0^t u_c(\tau) \cdot f(t-\tau)\,d\tau$

$= \int_\tau^t f(t-\tau)\,d\tau$

$= \int_0^t f(\tau)\,dt = F(t) - F(0)$

$(f * \delta_c(t)) = f(c)$

$$\int_0^\infty \int_\tau^\infty e^{-st} \cdot f(t-s) \cdot g(\tau) \cdot dt\, d\tau$$

$$(f * \delta_c(t)) = f(c)$$

$$= \int_0^\infty g(\tau) \int_\tau^\infty e^{-st} \cdot f(t-s)\, dt\, d\tau$$

$$\int_0^\tau e^{-st} \cdot 0 \cdot f(t-\tau)\, dt + \int_\tau^\infty e^{-st} f(t-\tau)\, dt$$

$$= \int_0^\infty g(\tau) \underline{\int_0^\infty e^{-st} u_\tau(t) \cdot f(t-\tau)\, dt}\, d\tau$$

$$= \int_0^\infty g(\tau) \cdot e^{-\tau s} \cdot \mathcal{L}\{f(t)\}\, d\tau$$

$$= \mathcal{L}\{g(\tau)\} \cdot \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{g(\tau)\} \cdot \mathcal{L}\{f(t)\} \qquad (f*g) = (g*f)$$

$$\mathcal{L}^{-1}\{\mathcal{L}\{g(\tau)\} \cdot \mathcal{L}\{f(t)\}\} = \mathcal{L}^{-1}\{F(s) \cdot G(s)\} = \int_0^t f(t-\tau) \cdot g(\tau)\, d\tau$$

- e.g. $y'' + 9y = 3t$; $y(0) = 0$, $y'(0) = 0$

$$(s^2+9)\mathcal{L}\{y\} = 3 \cdot \frac{1}{s^2}$$

$$\mathcal{L}\{y\} = \frac{3}{s^2(s^2+9)}$$

$$\mathcal{L}\{(f*g)(t)\} = \frac{1}{s^2} \cdot \frac{3}{s^2+9}$$

$$= t \cdot \sin 3t$$

$$y(t) = \int_0^t (t-\tau) \cdot \sin 3\tau \cdot d\tau$$

$$= -\frac{t\cos 3\tau}{3}\Big|_0^t - \int_0^t \tau \cdot \sin 3\tau\, d\tau$$

$$= -\frac{t\cos 3\tau}{3}\Big|_0^t - \left(-\frac{\tau\cos 3\tau}{3} + \frac{\sin 3\tau}{9}\right)\Big|_0^t$$

$$= \frac{\sin 3t}{9} - \frac{t}{3}$$

Convolutions

Wednesday, December 5, 2018 3:26 PM

- Warm-up: $(f*g) := \int_0^t f(t-\tau)\,g(\tau)\,d\tau$

I. Use Laplace to solve:

$$y'' + y = \cos t\ ;\quad y(0)=1,\ y'(0)=1.$$

$$(s^2+1)\mathcal{L}\{y\} - s - 1 = \frac{s}{s^2+1}$$

$$\mathcal{L}\{y\} = \frac{s}{(s^2+1)^2} + \frac{s+1}{s^2+1}$$

$$y = \cos t - \sin t + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{s}{s^2+1}\right\}$$

$$= \cos t - \sin t + \int_0^t \sin(t-\tau)\cdot\cos\tau\,d\tau$$

$$(\sin t\cdot\cos\tau - \cos t\cdot\sin\tau)\cdot\cos\tau$$

$$\int_0^t \left(\frac{\cos 2\tau + 1}{2}\right)\sin t - \cos t\cdot\frac{\sin 2\tau}{2}\,d\tau$$

$$= \left(\frac{\sin 2\tau}{4} + \frac{\tau}{2}\right)\sin t + \cos t\cdot\frac{\cos 2\tau}{4}\Big|_0^t$$

$$= \left(\frac{\sin 2t}{4} + \frac{t}{2}\right)\cdot\sin t + \frac{\cos t\cdot\cos 2t}{4} - \frac{\cos t}{4}$$

$$y = \cos t - \sin t + \left(\frac{\sin 2t}{4} + \frac{t}{2}\right)\cdot\sin t + \frac{\cos t\cdot\cos 2t}{4} - \frac{\cos t}{4}$$

- Terminology

$$ay'' + by' + cy = g(t) \qquad \mathcal{L}\{g(t)\} = G(s)$$

$$(as^2+bs+c)\mathcal{L}\{s\} - (as+b)y(0) - ay'(0) = G(s)$$

$$\mathcal{L}\{s\} = \boxed{\frac{(as+b)y(0)+ay'(0)}{as^2+bs+c}} + \boxed{\underbrace{\boxed{\frac{1}{as^2+bs+c}}}_{H(s)}\cdot G(s)}$$

$\mathcal{L}\{\varphi(t)\} = \Phi(s)$ $\mathcal{L}\{\psi(t)\} = \Psi(s)$

phi

$\Psi(s) = G(s)\cdot H(s)$ — $H(s)$: transfer function

convolution

$$\psi(t) = \int_0^t h(t-\tau)\,g(\tau)\,d\tau$$

$h(t)$ impulse function

$y(t) = \varphi(t) + \psi(t)$

$\psi(t)$: solution to $ay''+by'+cy=g(t)$; $y(0)=0,\ y'(0)=0$

$\varphi(t)$: solution to $ay''+by'+cy=0$; $y(0)=y_0,\ y'(0)=y'_0$

II. Implicitly solve $y''+y=t^{10}$; $y(0)=0,\ y'(0)=0.$

$$(s^2+1)\mathcal{L}\{y\} = \frac{10!}{s^{11}}$$

I. Implicitly solve $y''+y = t^{10}$; $y(0)=0$, $y'(0)=0$.

$$(s^2+1)\mathcal{L}\{y\} = \frac{10!}{s^{11}}$$

$$\mathcal{L}\{y\} = \frac{10!}{s^{11}\cdot(s^2+1)}$$

$$\mathcal{L}\{y\} = \frac{1}{s^2+1}\cdot\frac{10!}{s^{11}}$$

$$= \int_0^t \sin(t-\tau)\cdot\tau^{10}\,d\tau$$

III. Implicitly solve:

$y''+y=\ln(t+1)$; $y(0)=0$, $y'(0)=0$

$$(s^2+1)\,\mathcal{L}\{y\} = \int e^{-st}\cdot\ln(t+1)\,dt$$

$$\mathcal{L}\{y\} = \frac{1}{s^2+1}\cdot\mathcal{L}\{\ln(t+1)\}$$

$$= \int_0^t \sin(t-\tau)\cdot\ln(\tau+1)\,d\tau$$

- Show: $(f*g)(t) = (g*f)(t)$

$$\underbrace{\mathcal{L}^{-1}\{F(s)G(s)\} \qquad \mathcal{L}^{-1}\{G(s)F(s)\}}_{\text{Laplace are equal}}$$

Q.E.D.

$$\int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t g(t-\tau)\cdot f(\tau)\,d\tau$$

$u=t-\tau$, $du=-d\tau$

$\tau = t-u$

$$\int_0^t f(t-\tau)g(\tau)\,d\tau = \int_t^0 -g(u)\cdot f(t-u)\,du$$

$$= \int_0^t g(u)f(t-u)\,du.$$

Review

Friday, December 7, 2018 3:31 PM

-
- § 2
  - § 2.1 Integrating Factor $y' + p(t)y = g(t)$

    $\mu(t) = e^{\int p(t)dt}$ $y(t) = \frac{1}{\mu(t)} \int \mu(t) g(t) \cdot dt.$

  - § 2.2 Separable DEs

    $\frac{dy}{dx} = f(y) \cdot g(x)$

    $\int f(y) dy = \int g(x) dx$

  - § 2.3 Applications
    - Growth $\frac{dP}{dt} = kP$
    - Newton's Law of Cooling $\frac{dT}{dt} = -k(T - T_0)$
    - Mixture $\frac{dQ}{dt} =$ Rate In − Rate Out
    - Gravity / Force $F = ma = m\frac{dv}{dt}$

  - § 2.4 Existence/Uniqueness

    linear: $\frac{dy}{dt} + p(t) y = g(t)$; $y(t_0) = y_0$

    has a unique solution if $p(t)$ & $g(t)$ are continuous to t_0.

  - § 2.5 Autonomous DEs $\frac{dy}{dt} = f(y)$

    Population Models (Logistics) $\frac{dy}{dt} = r(1 - \frac{y}{k})y$ k- carrying capacities, r: intrinsic rate.

    - Classify equilibrium Solutions

- § 3 2^{nd} order DEs
  - § 3.2 Wronskian: $y_1(t)$, $y_2(t)$ are s/ns to $ay'' + by' + cy = g(t)$

    Then if $W(y_1, y_2) = y_1' y_2 - y_2' y_1 \neq 0$ at some (all) Points,

§3.2 Theorem: $y_1(t), y_2(t)$ are sols to $ay''+by'+cy=g(t)$

Then if $W(y_1, y_2) = y_1'y_2 - y_2'y_1 \neq 0$ at some (all) points,

All slns to DE are $y(t) = C_1y_1 + C_2y_2$ for some constants.

- § 3.1, 3.3, 3.4

  $ay''+by'+cy=0$

  $\Delta = b^2-4ac$

  $\Delta>0$ $\quad ar^2+br+c=0$

  $y = C_1e^{r_1t} + C_2e^{r_2t}$

  $\Delta=0$ $\quad y = C_1e^{rt} + C_2te^{rt}$

  $\Delta<0$ $\quad \lambda \pm \mu i$

  $y = e^{\lambda t}(C_1\cos\mu t + C_2\sin\mu t)$

- § 3.5 Non-homogeneous

  Guess $y_p(t)$

  e.g. $y''+y=e^t$, guess $y_p(t) = Ae^t$

  however: $y''+y=\cos t$ $\quad y_p(t) = (A\cos t + B\sin t)t$

  $y_c(t) = C_1\cos t + C_2\sin t$

- § 3.7, 3.8 Spring

  $mu''+\gamma u'+ku = F(t)$

  (a) $\gamma=0, F(t)=0$

  $\omega = \sqrt{\frac{k}{m}}$, $u(t) = C_1\cos\omega t + C_2\sin\omega t$

  (b) $F(t)=0$, $\gamma>0$

  - $\Delta>0$: $\gamma^2-4mk>0$

  over-damped $\gamma > 2\sqrt{mk}$

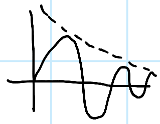
  - $\Delta=0$: $\gamma^2 = 4mk$

  critically-damped $\gamma = 2\sqrt{mk}$

■ $\Delta < 0$: $\gamma < 2\sqrt{mk}$

underdamped



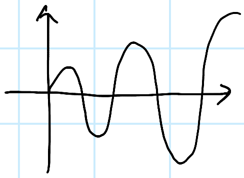
quasi-periodic

μ = quasi-frequency $T = \frac{2\pi}{\mu}$ = quasi-period

(c) $F(t) = F_0 \cos(\omega_0 t)$ $\gamma = 0$ ω_0: frequency of forcing term.

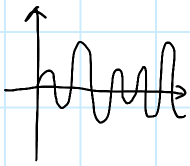
■ $\omega_0 = \omega$ ω: natural frequency

Resonance



■ $\omega_0 \neq \omega$

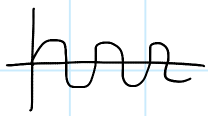
Beat



(d) $F(t) = F_0 \cos(\omega_0 t)$, $\gamma > 0$

$u(t) = S(t) + T(t)$

↑ Steady-state ↖ Transient



$\lim_{t \to \infty} T(t) = 0$

- §6 Laplace Transform

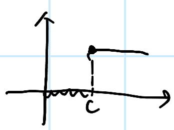
§6.1 Defin: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$

§6.2, 6.4 Solve IVPs

$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$

$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$

§6.3 Heaviside Function



$u_c(t)$

§6.5 Impulse function (Dirac delta)

$\mathcal{L}\{\delta_c(t)\} = e^{-cs}$

§6.6 Convolution

$$\mathcal{L}^{-1}\{F(s)\cdot G(s)\} = (f*g)(t)$$
$$= \int_0^t f(t-\tau)g(\tau)\,d\tau$$

- Exam:

8 qn. 1h 50 min

1. Conceptual (T/F)
2. Give the form of particular slns
3. Solve DEs
4. IVP
5.6. Applications
7.8. Laplace

8:30 – 10:30 a.m. Padelford C-8?

e.g. $y''+4y'+4y = e^{-2t}$

$y_p(t) = Ae^{-2t}t^2$

$y_c(t) = C_1e^{-2t} + C_2te^{-2t}$

e.g.2. $y''+4y'+4y = (t+1)e^{-2t}$

$y_p(t) = (At+B)e^{-2t}t^2$

e.g.3. $y'+p(t)y = g(t)$

Mult. $\mu'(t) = \mu(t)\cdot p(t)$

$\mu(t)\cdot\frac{dy}{dt} + p(t)\cdot y = g(t)\cdot\mu(t)$

$\frac{d}{dt}(\mu(t)\cdot y) = g(t)\cdot\mu(t)$

$y(t)\cdot\mu(t) = \int g(t)\mu(t)dt$

$y(t) = \frac{1}{\mu}\int g(t)\mu(t)\,dt.$

Final prob set

Monday, December 10, 2018 6:16 PM

5 (4pts) Find the Laplace transform, $\frac{1}{s^2}$ $\to e^{-2s}$

$$\mathcal{L}\{t\delta(t-2)\}.$$

Hint: either use #16 in the table of Laplace transforms, or use the definition of Laplace transforms directly if you remember the fact about integrating the delta function.

$$\cancel{= \int_0^\infty e^{-st} t\,\delta(t-2)\,dt}$$

$$\cancel{= e^{-2s}\cdot \mathcal{L}\{(t-2)\}}$$

$$\cancel{= e^{-2s}\cdot\left(\frac{1}{s^2}-\frac{2}{s}\right)}$$

Method I

$$\mathcal{L}\{t\delta(t-2)\} = -\frac{d}{ds}\mathcal{L}\{\delta(t-2)\} = -\frac{d}{ds}(e^{-2s}) = 2e^{-2s}$$

Method II.

$$\mathcal{L}\{t\delta(t-2)\} = \int_0^\infty t\delta(t-2)\cdot e^{-st}\,dt = \int_{-\infty}^\infty te^{-st}\delta(t-2)\,dt$$

$$= e^{-st}\cdot t\Big|_{t=2}$$

$$= 2e^{-2s}$$