4.1 Intro to Subspaces

Friday, February 15, 2019 12:35 PM

- A subset S of $\mathbb{R}^n$ is a subspace if S satisfies the following three conditions:

  (a) S contains $\vec{0}$ (Must pass through origin)

  (b) $v, u \in S$, $u+v \in S$ (be closed under addition)

  (c) $r \in \mathbb{R}$, $u \in S$, $ru \in S$ (closed under scalar multiplication)

- $S = \text{span}\{u_1, \cdots, u_m\} \in \mathbb{R}^n$, S is a subspace of $\mathbb{R}^n$

  Proof. $\vec{0} = 0u_1 + \cdots + 0u_m$. $\therefore 0 \in S$

  $v, w \in S$, $v = v_1u_1 + \cdots + v_mu_m$, $w = w_1u_1 + \cdots + w_mu_m$

  $\therefore v+w \in S$

  $r \in \mathbb{R}$, $r \cdot v \in S$.

  - S is the subspace spanned (generated) by $\{u_1, \cdots, u_m\}$

- $\mathbb{R}^n$ is a subspace of itself

  $\mathbb{R}^n$, $\{0\}$ are trivial subspaces.

- $A_{n\times m}$, then solution set of $Ax=0$ forms a subspace of $\mathbb{R}^m$.

  Proof. $x=0$, $Ax=0$

  $Ax_1=0$, $Ax_2=0 \Rightarrow A(x_1+x_2) = Ax_1+Ax_2=0$

  $Ax_1 = 0 \Rightarrow A(rx_1) = rAx_1 = 0$

  - solution set of $Ax=0$ is called null space of A, denoted by null(A)

- $T: \mathbb{R}^m \to \mathbb{R}^n$ linear trans, then kernel of T is a subspace of $\mathbb{R}^m$, range(T) is a subspace of $\mathbb{R}^n$.

  - Kernel: ker(T), the set of vectors that $T(x)=0$.

  Proof. $T(x) = Ax = 0$

- Kernel: ker(T), the set of vectors that $T(x)=0$.

Proof. $T(x) = Ax = 0$

$\therefore \ker(T) = \text{null}(A)$

$\therefore \ker(T)$ is a subspace of $\mathbb{R}^m$.

$\because \text{range}(T) = \text{span}\{a_1, \cdots, a_m\}$

$\therefore \text{range}(T)$ is a subspace of $\mathbb{R}^n$.

- Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a linear trans. T is injective if & only if $\ker(T) = \{0\}$.

if $\ker(T) \neq \{0\}$, it's clear that T can't be injective.

Then assume $v_1, v_2 \in \mathbb{R}^m$, $T(v_1) = T(v_2)$

$T(v_1) - T(v_2) = 0$

$T(v_1 - v_2) = 0$

$\because \ker(T) = \{0\}$

$\therefore v_1 - v_2 = 0$, $v_1 = v_2$

$\therefore$ T is injective $\Leftrightarrow$ $\ker(T) = 0$

- Unifying Theorem

Let $S = \{a_1, \ldots, a_n\}$ be a set of vectors in $\mathbb{R}^n$. $A = [a_1 \cdots a_n]$,

$T: \mathbb{R}^n \to \mathbb{R}^n$: $T(x) = Ax$. following're equivalent:

(a) S spans $\mathbb{R}^n$

(b) S is linearly dependent

(c) $Ax = b$ has a unique soln for $\forall b \in \mathbb{R}^n$

(d) T surjective

(e) T bijective

(f) A invertible

(g) $\ker(T) = \{0\}$

4.2 Basis & dimension

Tuesday, February 19, 2019 9:56 AM

- def A set $B = \{u_1, \cdots, u_m\}$ is a basis for a subspace S if

  (a) B spans S

  (b) B is linearly independent

  - The zero subspace $\{0\}$ is not a basis, since it's not linearly independent.

- Theorem Let $B = \{u_1, \cdots, u_m\}$ be a basis for a subspace S. For every s in S there exists a unique set of scalars $s_1, \cdots, s_m$ such that

$$s = s_1 u_1 + \cdots + s_m u_m$$

Proof. Assume exists another set of scalars $c_1, \dots, c_m$

$$s = c_1 u_1 + \cdots + c_m u_m$$

$$\therefore (s_1 - c_1) u_1 + \cdots + (s_m - c_m) u_m = 0$$

$s_i = c_i$, or u_i will not be independent.

- If A, B are equivalent matrices, then subspace spanned by rows of A is the same as the subspace spanned by the rows of B.

  e.g. Let S be subspace spanned by

$$u_1 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -6 \\ 7 \\ 5 \\ 2 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 4 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 & 1 \\ -6 & 7 & 5 & 2 \\ 4 & -3 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 2 & 3 & 1 \\ 0 & -5 & -13 & -4 \\ 0 & 5 & 13 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 2 & 3 & 1 \\ 0 & 5 & 13 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \text{basis is } \left\{ \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 13 \\ 4 \end{bmatrix} \right\}$$

$\Rightarrow$ Nonzero rows of equivalent reduced-echelon matrix gives basis.

Soln2. $A = [u_1 \ u_2 \ u_3] = \begin{bmatrix} -1 & -6 & 4 \\ 2 & 7 & -3 \\ 3 & 5 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} -1 & -6 & 4 \\ 0 & 3 & -3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -4 & 4 \\ 0 & 3 & -3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [v_1 \ v_2 \ v_3]$$

$v_2 = 2v_1 + v_3$

$v_2 = 2v_1 + v_3$

span$\{v_1, v_2, v_3\}$ = span$\{v_1, v_3\}$

$\Rightarrow$ basis $\{u_1, u_3\}$

- U, V are equivalent matrices, then any linear independence exists among columns of U also exists that of V.

- S is subspace of $\mathbb{R}^n$, then every basis of S has same number of vectors.
- def Let S be a subspace of $\mathbb{R}^n$, then dimension of S is the number of vectors in any basis of S.
  - $\{e_1, \cdots, e_n\}$ form a standard basis of $\mathbb{R}^n$.
  - Theorem. Let $\mathcal{U} = \{u_1, \cdots, u_m\}$ be a set of vectors in $S \neq \{0\}$ of $\mathbb{R}^n$
    (1) If $\mathcal{U}$ linearly independent, then either $\mathcal{U}$ is a basis or additional vectors can be added to form a basis
    (2) If $\mathcal{U}$ spans S, then eith $\mathcal{U}$ is a basis or vectors can be removed to form one.

e.g. Expand $\mathcal{U} = \left\{\begin{bmatrix}1\\1\\-2\end{bmatrix}, \begin{bmatrix}3\\2\\-4\end{bmatrix}\right\}$ to a basis for $\mathbb{R}^3$.

Write

$$A = \begin{bmatrix}1 & 3 & 1 & 0 & 0\\1 & 2 & 0 & 1 & 0\\-2 & -4 & 0 & 0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}0 & 0 & 1 & 0 & 0\\0 & 2 & 0 & 1 & 0\\-2 & 0 & 0 & 0 & 1\end{bmatrix} = B$$

$$\therefore \left\{\begin{bmatrix}1\\1\\-2\end{bmatrix}, \begin{bmatrix}3\\2\\-4\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$$

- def nullity(A) the dimencion of null space of A.
- $\dim(S) = m$, $U = \{u_1, \cdots, u_m\}$, U either spans S or linearly independent. then U is a basis of S.
- Suppose that S_1, S_2 are both subspace of $\mathbb{R}^n$, $S_1 \subset S_2$, $\Rightarrow \dim S_1 \leq \dim S_2$. and $\dim S_1 = \dim S_2$ only when $S_1 = S_2$.
- $\mathcal{U} = \{u_1, \cdots, u_m\} \in S$, $\dim S = k$.

(1) If $m < k$, $\mathcal{U}$ doesn't span S.

(2) If $m > k$, $\mathcal{U}$ doesn't independent.

THEOREM 4.18

(THE UNIFYING THEOREM, VERSION 5) Let $\mathscr{S} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a set of n vectors in $\mathbf{R}^n$, let $A = [\mathbf{a}_1 \ldots \mathbf{a}_n]$, and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be given by $T(\mathbf{x}) = A\mathbf{x}$. Then the following are equivalent:

(a) $\mathscr{S}$ spans $\mathbf{R}^n$.

(b) $\mathscr{S}$ is linearly independent.

(c) $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^n$.

(d) T is onto.

(e) T is one-to-one.

(f) A is invertible.

(g) $\ker(T) = \{\mathbf{0}\}$.

(h) $\mathscr{S}$ is a basis for $\mathbf{R}^n$.

- find the basis of the solution space of $Ax=0$

  1° Get echelon form of matrix A

  2° solve the system, get $x = v_0 + s_1 \cdot v_1 + \cdots + s_n \cdot v_n$

  3° $\{v_1, \cdots, v_n\}$ is the basis of solution space.

4.3 Row & Column Spaces

Wednesday, February 20, 2019 4:48 PM

- def $A_{n\times m}$.
  - Row space of A: the subspace of $\mathbb{R}^m$ spanned by row vectors of A. row(A).
  - Column space of A: the subspace of $\mathbb{R}^n$
- Theorem. Let A be a matrix & B an echelon form of A.

  (1) Nonzero rows of B form a basis for row(A)

  (2) The columns of A corresponding to the pivot columns of B form a basis for col(A)

  - Theorem. $\dim(\text{row}(A)) = \dim(\text{col}(A))$.

    Proof. each non-zero row corresponds to a pivot column.
- def The rank of matrix A is the dimension of row (or column) space of A, and is denoted by rank(A).
- Rank-Nullity Theorem Let A be an $n\times m$ matrix,

  $$\text{rank}(A) + \text{nullity}(A) = m$$

  Proof. $A \to B$ (echelon form)

  each non-pivot column corresponds to a free variable

e.g. find a linear-trans T that has kernel equal to span$\{x_1, x_2\}$

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

$T = Ax$.

x_1, x_2 linearly independent $\therefore \text{nullity}(A) = 2$

A must have 4 columns, because x_1, x_2 are 4×1.

$\therefore \text{rank}(A) = 4-2 = 2$ A has at least 2 rows

assume $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$

$Ax_1 = Ax_2 = 0$

$$\begin{bmatrix} a-2c+d \\ e-2g+h \end{bmatrix} = \vec{0} \quad \begin{bmatrix} b+3c+2d \\ f+3g+2h \end{bmatrix} = \vec{0}$$

$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = s_1 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} = t_1 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

take $s_1 = 0, s_2 = 1$

$s_1 = 1, s_2 = 0$

$\therefore A = \begin{bmatrix} -1 & -2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{bmatrix}$

- $A_{n \times m}$, $\vec{b} \in \mathbb{R}^n$

  (a) $Ax = \vec{b}$ consistent $\Leftrightarrow$ $\vec{b}$ is in $col(A)$

  (b) $Ax = \vec{b}$ has unique soln $\Leftrightarrow$ $\vec{b} \in col(A)$ && columns of A linearly independent.

- The Unifying Theorem

  $\mathcal{S} = \{a_1, \cdots, a_n\} \in \mathbb{R}^n$. $A = [a_1 \cdots a_n]$, $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(x) = Ax$

  $\mathcal{S}$ spans $\mathbb{R}^n$

  $\Leftrightarrow$ $\mathcal{S}$ linearly independent

  $\Leftrightarrow$ $Ax = b$ has unique soln for $\forall b \in \mathbb{R}^n$

  $\Leftrightarrow$ T surjective

  $\Leftrightarrow$ T injective

  $\Leftrightarrow$ A invertible

  $\Leftrightarrow$ $ker(T) = \{0\}$

  $\Leftrightarrow$ $\mathcal{S}$ is a basis for $\mathbb{R}^n$

  $\Leftrightarrow$ $col(A) = \mathbb{R}^n$

  $\Leftrightarrow$ $row(A) = \mathbb{R}^n$

  $\Leftrightarrow$ $rank(A) = n$.

4.4 Change of Basis

Tuesday, March 5, 2019 8:21 PM

- <u>def</u> The coordinate vector of y with respect to $\mathcal{B} = \{u_1, \cdots u_n\}$, a basis of $\mathbb{R}^n$

  $$[y]_{\mathcal{B}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

  - $U = [u_1 \cdots u_n]$ Change of basis matrix
    - $U \cdot [y]_{\mathcal{B}} = y_1 u_1 + \cdots + y_n u_n = y$
  - $[y]$ is expressed with respect to the standard basis.

- <u>Theorem</u> Let x be expressed with respect to standard basis, and let $\mathcal{B} = \{u_1, \cdots, u_n\}$ be any basis for $\mathbb{R}^n$. If $U = [u_1 \cdots u_n]$, then

  (a) $x = U[x]_{\mathcal{B}}$

  (b) $[x]_{\mathcal{B}} = U^{-1}x$

  e.g. Let $x = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$, find $[x]_{\mathcal{B}}$

  $$U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

  $$[x]_{\mathcal{B}} = U^{-1}x$$

  $$U^{-1}: \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -3 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 & 1 & -3 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

  $$\Rightarrow \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 & 1 & -3 \\ 1 & 0 & 0 & -6 & -2 & 7 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & -2 & 7 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 & 1 & -3 \end{array}\right]$$

  $$[x]_{\mathcal{B}} = \begin{bmatrix} -6 & -2 & 7 \\ 1 & 0 & -1 \\ 3 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

- Two Non-Standard Basis

  <u>Theorem</u> $\mathcal{B}_1 = \{u_1, \cdots, u_n\}$ and $\mathcal{B}_2 = \{v_1, \cdots, v_n\}$ be basis for $\mathbb{R}^n$. If $U = [u_1 \cdots u_n]$, $V = [v_1 \cdots v_n]$, then

  ① $[x]_{\mathcal{B}_2} = V^{-1}U[x]_{\mathcal{B}_1}$

② $[x]_{\mathcal{B}_1} = U^{-1} V [x]_{\mathcal{B}_2}$

$\because$ $U[x]_{\mathcal{B}_1} = V[x]_{\mathcal{B}_2} = x$ (in standard basis)

- Change of Basis in subspace

<u>Theorem</u>. S, subspace of $\mathbb{R}^n$, $\mathcal{B}_1 = \{u_1, \cdots, u_k\}$ $\mathcal{B}_2 = \{v_1, \cdots, v_k\}$

if $C = [\,[u_1]_{\mathcal{B}_2} \cdots [u_k]_{\mathcal{B}_2}\,]$, then $[x]_{\mathcal{B}_2} = C \cdot [x]_{\mathcal{B}_1}$

o $C = V^{-1} \cdot U$ when $S = \mathbb{R}^n$

o Proof. $x = x_1 u_1 + \cdots + x_k u_k \Rightarrow [x]_{\mathcal{B}_1} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$

$C[x]_{\mathcal{B}_1} = [[u_1]_{\mathcal{B}_2} \cdots [u_k]_{\mathcal{B}_2}] \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$

$= [x]_{\mathcal{B}_2}$

e.g. Let $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ -5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 3 \end{bmatrix} \right\}$, $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$ be two basis of a subspace S of $\mathbb{R}^3$. find change of basis matrix from $[x]_{\mathcal{B}_1}$ to $[x]_{\mathcal{B}_2}$ find $[x]_{\mathcal{B}_2}$ if $[x]_{\mathcal{B}_1} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Because they're subspaces, previous theorems doesn't apply

find C.

$u_1 = 3v_1 + 2v_2$, $u_2 = 2v_1 - v_2$

$[u_1]_{\mathcal{B}_2} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $[u_2]_{\mathcal{B}_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow C = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$

$\therefore [x]_{\mathcal{B}_2} = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 9-2 \\ 6+1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$

Probs

Wednesday, February 20, 2019 6:23 PM

Determine if the statement is true or false, and justify your answer.

If A and B are equivalent matrices, then col(A) = col(B).

- ○ True, by the Big Theorem.
- ○ True, by the definition of column space.
- ○ False. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
- ○ False. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- ● False. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

Determine if the statement is true or false, and justify your answer.

If A and B are equivalent matrices, then row(A) = row(B).

- ● True, by the theorem that says if A and B are equivalent matrices, then the subspace spanned by the rows of A is the same as the subspace spanned by the rows of B.
- ○ False. For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then A and B are equivalent matrices, but row(A) ≠ row(B).
- ○ False. For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then A and B are equivalent matrices, but row(A) ≠ row(B).
- ○ False. For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, then A and B are equivalent matrices, but row(A) ≠ row(B).
- ○ False. For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then A and B are equivalent matrices, but row(A) ≠ row(B).

2/2 points | Previous Answers HoltLinAlg2 4.3.023.

Suppose that A is a 13 × 8 matrix and that $T(\mathbf{x}) = A\mathbf{x}$. If T is one-to-one, then what is the dimension of the null space of A?

0

one-to-one $\Leftrightarrow$ $\ker(A) = \{0\}$

1/1 points | Previous Answers HoltLinAlg2 4.3.024.

Suppose that A is a 6 × 14 matrix and that $T(\mathbf{x}) = A\mathbf{x}$. If T is onto, then what is the dimension of the null space of A?

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onto $\Leftrightarrow$ $\forall b \in \mathbb{R}^{row} = \mathbb{R}^{\dim(row)}$

(rows independent)