6.4

Monday, December 3, 2018 3:32 PM

- Dirac $\delta_0(t) = \delta(t) = \lim_{n\to\infty} f_n$
  - Properties : $\delta(t) = 0,\ t \neq 0$

    $\int_{-\varepsilon}^{\varepsilon} \delta(t)\, dt = 1$, for any $\varepsilon > 0$
  - $\mathcal{L}\{\delta_c(t)\} = \mathcal{L}\{\delta(t-c)\} = e^{-cs}$

e.g. $2y'' + y' + 2y = \delta(t-5)$: $y(0) = 0,\ y'(0) = 0$

$(2s^2+s+2)\mathcal{L}\{y\} = e^{-5s}$

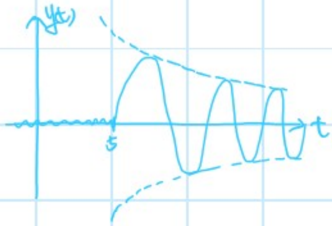
$$\mathcal{L}\{y\} = \frac{e^{-5s}}{2s^2+s+2}$$

$\Delta = 1 - 2\times 2\times 4 < 0$, irreducible.

$$\mathcal{L}\{y\} = \frac{e^{-5s}}{2}\cdot\frac{1}{(s+\frac{1}{4})^2+\frac{15}{16}}$$

$$\Rightarrow \frac{4}{\sqrt{15}}\frac{u_5}{2}\cdot e^{-\frac{1}{4}(t-5)}\cdot \sin\left(\frac{\sqrt{15}}{4}(t-5)\right)$$

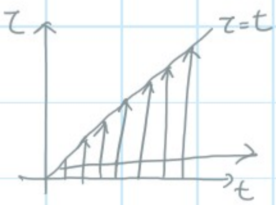
$$y = \frac{2\sqrt{15}}{15} u_5 e^{-\frac{1}{4}(t-5)}\cdot \sin\left(\frac{\sqrt{15}}{4}(t-5)\right)$$



- $\frac{d}{dt} u_c(t) = \delta_c(t) \longrightarrow \int_0^t \delta_c(\tau)\, d\tau = u_c(t)$
- Define $\underset{h(t)}{(f*g)(t)} = \int_0^t f(t-\tau)g(\tau)\cdot d\tau = \int_0^t g(t-\tau)\cdot f(\tau)\,d\tau$ a **convolution**.

$$\mathcal{L}\{h(t)\} = \int_0^\infty e^{-st}\int_0^t f(t-\tau)g(\tau)\cdot d\tau\ dt$$

$$= \int_0^\infty\int_0^t e^{-st}\cdot f(t-\tau)g(\tau)\cdot d\tau\, dt.$$



$0 < t < \infty$

$0 < \tau < t$

$\tau < t < \infty$

$0 < \tau < \infty$

$\int^\infty \int^\infty$

$$(f * u_c(t)) = \int_0^t u_c(\tau)\cdot f(t-\tau)\,d\tau$$

$$= \int_\tau^t f(t-\tau)\,d\tau$$

$$= \int_0^t f(\tau)\,dt = F(t) - F(0)$$

$(f * \delta_c(t)) = f(c)$

$$0 < \tau < \infty \qquad \int_0^\infty f(t)\,dt \quad f(t) f(c)$$

$$(f * \delta_c(t)) = f(c)$$

$$\int_0^\infty \int_\tau^\infty e^{-st} \cdot f(t-s) \cdot g(\tau) \cdot dt\,d\tau$$

$$= \int_0^\infty g(\tau) \int_\tau^\infty e^{-st} \cdot f(t-s)\, dt\, d\tau$$

$$\int_0^\tau e^{-st} \cdot 0 f(t-\tau)\,dt + \int_\tau^\infty e^{-st} f(t-\tau)\,dt$$

$$= \int_0^\infty g(\tau) \underline{\int_0^\infty e^{-st} u_\tau(t) \cdot f(t-\tau)\,dt}\,d\tau$$

$$= \int_0^\infty g(\tau) \cdot e^{-\tau s} \cdot \mathcal{L}\{f(t)\}\,d\tau$$

$$= \mathcal{L}\{g(\tau)\} \cdot \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{g(\tau)\} \cdot \mathcal{L}\{f(t)\} \qquad (f*g) = (g*f)$$

$$\mathcal{L}^{-1}\{\mathcal{L}\{g(\tau)\} \cdot \mathcal{L}\{f(t)\}\} = \mathcal{L}^{-1}\{F(s) \cdot G(s)\} = \int_0^t f(t-\tau) \cdot g(\tau)\,d\tau$$

- e.g. $y'' + 9y = 3t$; $y(0) = 0$, $y'(0) = 0$

$$(s^2+9)\mathcal{L}\{y\} = 3 \cdot \frac{1}{s^2}$$

$$\mathcal{L}\{y\} = \frac{3}{s^2(s^2+9)}$$

$$\mathcal{L}\{(f*g)(t)\} = \frac{1}{s^2} \cdot \frac{3}{s^2+9}$$

$$= t \cdot \sin 3t$$

$$y(t) = \int_0^t (t-\tau) \cdot \sin 3\tau \cdot d\tau$$

$$= -\frac{t\cos 3\tau}{3}\Big|_0^t - \int_0^t \tau \cdot \sin 3\tau\, d\tau$$

$$= -\frac{t\cos 3\tau}{3}\Big|_0^t - \left(-\frac{\tau\cos 3\tau}{3} + \frac{\sin 3\tau}{9}\right)\Big|_0^t$$

$$= \frac{\sin 3t}{9} - \frac{t}{3}$$

Convolutions

Wednesday, December 5, 2018 3:26 PM

- Warm-up: $(f*g) := \int_0^t f(t-\tau)g(\tau)\,d\tau$

I. Use Laplace to solve:

$y''+y=\cos t$; $y(0)=1$, $y'(0)=1$.

$(s^2+1)\mathcal{L}\{y\} - s - 1 = \frac{s}{s^2+1}$

$\mathcal{L}\{y\} = \frac{s}{(s^2+1)^2} + \frac{s-1}{s^2+1}$

$y = \cos t - \sin t + \mathcal{L}^{-1}\{\frac{1}{s^2+1}\cdot\frac{s}{s^2+1}\}$

$= \cos t - \sin t + \int_0^t \sin(t-\tau)\cdot\cos\tau\,d\tau$

$(\sin t\cdot\cos\tau - \cos t\cdot\sin\tau)\cdot\cos\tau$

$\int_0^t (\frac{\cos 2\tau+1}{2})\sin t - \cos t\cdot\frac{\sin 2\tau}{2}\,d\tau$

$= (\frac{\sin 2\tau}{4}+\frac{\tau}{2})\sin t + \cos t\cdot\frac{\cos 2\tau}{4}\Big|_0^t$

$= (\frac{\sin 2t}{4}+\frac{t}{2})\cdot\sin t + \frac{\cos t\cdot\cos 2t}{4} - \frac{\cos t}{4}$

$y = \cos t - \sin t + (\frac{\sin 2t}{4}+\frac{t}{2})\cdot\sin t + \frac{\cos t\cdot\cos 2t}{4} - \frac{\cos t}{4}$

- Terminology

$ay''+by'+cy=g(t)$ $\mathcal{L}\{g(t)\}=G(s)$

$(as^2+bs+c)\mathcal{L}\{s\} - (as+b)y(0) - ay'(0) = G(s)$

$\mathcal{L}\{s\} = \boxed{\frac{(as+b)y(0)+ay'(0)}{as^2+bs+c}} + \boxed{\underbrace{\boxed{\frac{1}{as^2+bs+c}}}_{H(s)}\cdot G(s)}$

$\mathcal{L}\{\varphi(t)\} = \Phi(s)$ $\mathcal{L}\{\psi(t)\} = \Psi(s)$

phi

$\Psi(s) = G(s)\cdot H(s)$ transfer function

convolution

$\psi(t) = \int_0^t h(t-\tau)g(\tau)\,d\tau$

$h(t)$ impulse function

$y(t) = \varphi(t) + \psi(t)$

solution to $ay''+by'+cy=g(t)$, $y(0)=0$, $y'(0)=0$

solution to $ay''+by'+cy=0$; $y(0)=y_0$, $y'(0)=y'_0$

II. Implicitly solve $y''+y=t^{10}$; $y(0)=0$, $y'(0)=0$.

$(s^2+1)\mathcal{L}\{y\} = \frac{10!}{s^{11}}$

II. Implicitly solve $y''+y=t^{10}$; $y(0)=0, y'(0)=0$.

$$(s^2+1)\mathcal{L}\{y\} = \frac{10!}{s^{11}}$$

$$\mathcal{L}\{y\} = \frac{10!}{s^{11}\cdot(s^2+1)}$$

$$\mathcal{L}\{y\} = \frac{1}{s^2+1}\cdot\frac{10!}{s^{11}}$$

$$= \int_0^t \sin(t-\tau)\cdot\tau^{10}\,d\tau$$

III. Implicitly solve.

$y''+y=\ln(t+1)$; $y(0)=0, y'(0)=0$

$$(s^2+1)\,\mathcal{L}\{y\} = \int e^{-st}\cdot\ln(t+1)\,dt$$

$$\mathcal{L}\{y\} = \frac{1}{s^2+1}\cdot\mathcal{L}\{\ln(t+1)\}$$

$$= \int_0^t \sin(t-\tau)\cdot\ln(\tau+1)\,d\tau$$

- Show: $(f*g)(t) = (g*f)(t)$

$$\mathcal{L}^{-1}\{F(s)G(s)\} \qquad \mathcal{L}^{-1}\{G(s)F(s)\}$$

Laplace are equal Q.E.D.

$$\int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t g(t-\tau)\cdot f(\tau)\,d\tau$$

$u=t-\tau$, $du=-d\tau$

$\tau=t-u$

$$\int_0^t f(t-\tau)g(\tau)\,d\tau = \int_t^0 -g(u)\cdot f(t-u)\,du$$

$$= \int_0^t g(u)f(t-u)\,du.$$

Review

Friday, December 7, 2018 3:31 PM

-
- §2
  - §2.1 Integrating factor $y'+p(t)y=g(t)$

    $\mu(t)=e^{\int p(t)dt}$ $y(t)=\frac{1}{\mu(t)}\int \mu(t)g(t)\cdot dt.$

  - §2.2 Separable DEs

    $\frac{dy}{dx}=f(y)\cdot g(x)$

    $\int f y dy=\int g(x)dx$

  - §2.3 Applications
    - Growth $\frac{dP}{dt}=kP$
    - Newton's Law of Cooling

      $\frac{dT}{dt}=-k(T-T_0)$
    - Mixture

      $\frac{dQ}{dt}=$ Rate In $-$ Rate Out
    - Gravity / Force

      $F=ma=m\frac{dv}{dt}$

  - §2.4 Existence/Uniqueness

    linear: $\frac{dy}{dt}+p(t)y=g(t)$; $y(t_0)=y_0$

    has a unique solution if $p(t)$ & $g(t)$ are continuous to t_0.

  - §2.5 Autonomous DEs $\frac{dy}{dt}=f(y)$

    Population Models (Logistics) $\frac{dy}{dt}=r(1-\frac{y}{k})y$ k- carrying capacities r: intrinsic rate.

    - Classify equilibrium solutions

- §3 2^{nd} order DEs
  - §3.2 Wronskian: $y_1(t)$, $y_2(t)$ are s/ns to $ay''+by'+cy=g(t)$

    Then if $W(y_1,y_2)=y_1'y_2-y_2'y_1\neq 0$ at some (all) points,

§3.2 Fundamental: $y_1(t), y_2(t)$ are sols to $ay''+by'+cy=g(t)$

Then if $W(y_1, y_2) = y_1'y_2 - y_2'y_1 \neq 0$ at some (all) points,

All slns to DE are $y(t) = C_1 y_1 + C_2 y_2$ for some constants.

∘ § 3.1, 3.3, 3.4

$ay'' + by' + cy = 0$

$\Delta = b^2 - 4ac$

$\Delta > 0$ $\quad ar^2 + br + c = 0$

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$\Delta = 0$ $\quad y = C_1 e^{rt} + C_2 t e^{rt}$

$\Delta < 0$ $\quad \lambda \pm \mu i$

$$y = e^{\lambda t}(C_1 \cos\mu t + C_2 \sin\mu t)$$

∘ § 3.5 Non-homogeneous

Guess $y_p(t)$

e.g. $y'' + y = e^t$, guess $y_p(t) = Ae^t$

however: $y'' + y = \cos t$ $\qquad y_p(t) = (A\cos t + B\sin t)t$

$y_c(t) = C_1 \cos t + C_2 \sin t$

∘ § 3.7, 3.8 Spring

$$mu'' + \gamma u' + ku = F(t)$$

(a) $\gamma = 0$, $F(t) = 0$

$\omega = \sqrt{\frac{k}{m}}$, $u(t) = C_1 \cos\omega t + C_2 \sin\omega t$

(b) $F(t) = 0$, $\gamma > 0$

■ $\Delta > 0$: $\gamma^2 - 4mk > 0$

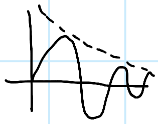
over-damped $\gamma > 2\sqrt{mk}$

■ $\Delta = 0$: $\gamma^2 = 4mk$

critically-damped $\gamma = 2\sqrt{mk}$

- $\Delta < 0$: $\gamma < 2\sqrt{mk}$

underdamped



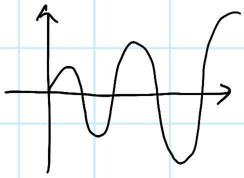
quasi-periodic

μ = quasi-frequency $T = \frac{2\pi}{\mu}$ = quasi-period

(c) $F(t) = F_0 \cos(\omega_0 t)$ $\gamma = 0$ ω_0: frequency of forcing term.

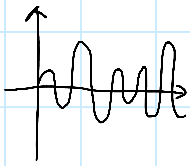
- $\omega_0 = \omega$ ω: natural frequency

Resonance



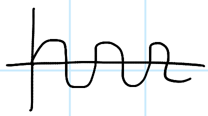
- $\omega_0 \neq \omega$

Beat



(d) $F(t) = F_0 \cos(\omega_0 t)$, $\gamma > 0$

$u(t) = \underbrace{S(t)}_{\text{Steady-state}} + \underbrace{T(t)}_{\text{Transient}}$



$\lim_{t\to\infty} T(t) = 0$

- §6 Laplace Transform

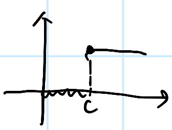
§6.1 Defin: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\,dt$

§6.2, 6.4 Solve IVPs

$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$

$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$

§6.3 Heaviside Function



$u_c(t)$

§6.5 Impulse function (Dirac delta)

$\mathcal{L}\{\delta_c(t)\} = e^{-cs}$

§6.6 Convolution

$$\mathcal{L}^{-1}\{F(s)\cdot G(s)\} = (f*g)(t)$$
$$= \int_0^t f(t-\tau)g(\tau)\,d\tau$$

- Exam:

8 qn. 1h 50 min

1. Conceptual (T/F)
2. Give the form of particular slns
3. Solve DEs
4. IVP
5. 6. Applications
7. 8. Laplace

8:30 – 10:30 a.m. Padelford C-8?

e.g. $y''+4y'+4y=e^{-2t}$

$y_p(t) = Ae^{-2t}t^2$

$y_c(t) = C_1e^{-2t}+C_2te^{-2t}$

e.g.2. $y''+4y'+4y=(t+1)e^{-2t}$

$y_p(t) = (At+B)e^{-2t}t^2$

e.g.3. $y'+p(t)y = g(t)$

Mult. $\mu'(t) = \mu(t)\cdot p(t)$

$\mu(t)\cdot\frac{dy}{dt} + p(t)\cdot y = g(t)\cdot\mu(t)$

$\frac{d}{dt}(\mu(t)\cdot y) = g(t)\cdot\mu(t)$

$y(t)\cdot\mu(t) = \int g(t)\mu(t)dt$

$y(t) = \frac{1}{\mu}\int g(t)\mu(t)\,dt.$

Final prob set

Monday, December 10, 2018 6:16 PM

5 (4pts) Find the Laplace transform,

$$\mathcal{L}\{t\delta(t-2)\}.$$

$\frac{1}{s^2}$ $\rightarrow e^{-2s}$

-3

Hint: either use #16 in the table of Laplace transforms, or use the definition of Laplace transforms directly if you remember the fact about integrating the delta function.

$$= \int_0^\infty e^{-st} t\,\delta(t-2)\,dt$$

$$= e^{-2s}\cdot \mathcal{L}\{(t-2)\}$$

$$= e^{-2s}\cdot\left(\frac{1}{s^2}+\frac{2}{s}\right)$$

Method I. $\mathcal{L}\{t\delta(t-2)\} = -\frac{d}{ds}\mathcal{L}\{\delta(t-2)\} = -\frac{d}{ds}(e^{-2s}) = 2e^{-2s}$

Method II. $\mathcal{L}\{t\delta(t-2)\} = \int_0^\infty t\delta(t-2)\cdot e^{-st}dt = \int_{-\infty}^\infty te^{-st}\delta(t-2)dt$

$$= e^{-st}\cdot t\Big|_{t=2}$$

$$= 2e^{-2s}$$

(b) (8 points) Use the substitution $v = \dfrac{y}{x}$ to solve

$$\frac{dy}{dx} = \frac{y^2+yx+x^2}{x^2}, \quad x > 0.$$

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{y}{x} + 1$$

$$\frac{dv}{dx} = -\frac{y}{x^2} + \frac{1}{x}\frac{dy}{dx}$$

$$\frac{dv}{dx}\cdot x = -v + \frac{dy}{dx}$$

$$\frac{dv}{dx}\cdot x + v = v^2+v+1$$

$$\frac{dv}{v^2+1} = \frac{dx}{x}$$

$$\tan^{-1} v = \ln x + C$$

$$v = \tan(\ln x + C)$$

$$y = x\tan(\ln x + C)$$

2. (12 points) A new homeowner just took out a loan of \$235,000 at an annual interest rate of 4.5% compounded continuously. She wants to analyze the benefits of paying a monthly mortgage of \$1,250 instead of \$1,200. Let $P(t)$ be the amount she owes after t years. Assume she makes payments continuously. How many years would it take to pay off the loan in each case? What's the total amount paid in interest in each case? How much money would she save if she paid \$1,250 each month instead of \$1,200? Write your answers to these questions in the provided table. Round your final answers to the nearest tenth of a year and nearest dollar.

-12

$\cancel{\frac{dP}{dt} = (4.5\%)^t P - M \times 12}$

$\cancel{P_n = (1+4.5\%)P_{n-1}}$

$$\frac{dP}{dt} = 0.045P - 12k,\ P(0) = 235{,}000$$

$$\frac{1}{0.045}\ln|0.045P - 12k| = t$$

$$P = \frac{Ce^{0.045t} + 12k}{0.045}$$

$\because P(0) = 235000$

$\therefore C = (0.045(235000) - 12k)e^{0.045t}$

$\because P(t) = 0$

$\therefore -12k = (0.045 \cdot 235000 - 12k)e^{0.045t}$

$$t = \frac{1}{0.045}\ln\left(\frac{-12k}{0.045 \cdot 235000 - 12}\right)$$

$k_1 = 1200,\ t_1 = 29.4593$

$k_2 = 1250,\ t_2 = 27.1284$

total $\Rightarrow$

$tot_1 = k_1 t_1 - 235000$

≈ 189214

$tot_2 = k_2 t_2 - 235000$

≈ 171926

Monthly mortgage	Years to pay off loan	Total interest paid
\$1,200	29.5	189214
\$1,250	27.1	171926
Difference	2.4	17288

3. For the following parts you DO NOT need to solve the differential equations.

(a) (6 points) Consider the initial value problem

$$\frac{dy}{dt} = (2 - e^y)(y^4 - y^2), \quad y(0) = y_0.$$

$(2-e^y)y^2(y^2-1)$

For what values of y_0 does the solution approach zero as $t \to \infty$? Justify your answer.

$\because\ y \to 0,\ t \to \infty$

$2 = e^y$

$y = \ln 2$

$y^2(y^2-1) = 0$

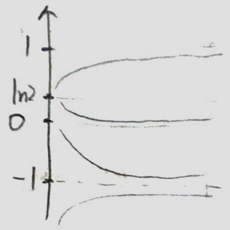
$y = 0 \quad y = \pm 1$

$-+-$

$++-$

$++-$

$+++$



Note that 0 is also included

$\boxed{y_0 \in [0, \ln 2)}$

-1

(b) (8 points) Find $\mathcal{L}^{-1}\left\{2 - \frac{4}{s^4} + \frac{s-6}{s^3+9s}e^{-4s}\right\}$.

$= -\frac{2}{3}t^3 + \mathcal{L}^{-1}\left\{2 + \frac{S-6}{S(S^2+9)}e^{-4s}\right\}$

$= -\frac{2}{3}t^3 + \mathcal{L}^{-1}\left\{2 + \left(-\frac{2}{3}\cdot\frac{1}{S} + \frac{2}{3}\cdot\frac{S}{S^2+9} + \frac{1}{S^2+9}\right)e^{-4s}\right\}$

$\frac{A}{S} + \frac{Bs+C}{S^2+9}$

$= -\frac{2}{3}t^3 + u_4\left(-\frac{2}{3} + \frac{2}{3}\cos(3(t-4)) + \sin(3(t-4))\right)$

$+ \cancel{\mathcal{L}^{-1}\{2\}}$

$2\delta(t)$

$\begin{cases} A+B=0 \\ 9A=-6 \\ C=1 \end{cases}$

$A = -\frac{2}{3}$

$B = \frac{2}{3}$

$\because \mathcal{L}^{-1}\{1\} = \delta(t)$

7. (12 points) Use the Laplace transform to solve the initial value problem

$$y' - 2y = \begin{cases} 0 & \text{if } 0 \le t < 2 \\ 4t & \text{if } 2 \le t < \infty \end{cases} \qquad y(0) = 3$$

$y' - 2y = 4t \cdot u_2$

$(s-2)\mathcal{L}\{y\} = \mathcal{L}\{4t \cdot u_2\} + 3$

$\mathcal{L}\{y\} = \boxed{\frac{1}{S-2}\mathcal{L}\{4t \cdot u_2\}} + \boxed{\frac{3}{S-2}} \to 3e^{2t}$

$$= \mathcal{L}\{e^{2t}\} \cdot \mathcal{L}\{4t \cdot u_2\}$$

$$= \int_0^t e^{2(t-\tau)} 4\tau \cdot u_2 \, d\tau$$

$$= 4u_2 \int_2^t e^{2(t-\tau)} \cdot \tau \, d\tau$$

remember the 2 here

τ	$e^{2(t-\tau)}$
1	$-\frac{e^{2(t-\tau)}}{2}$
0	$\frac{e^{2(t-\tau)}}{4}$

$$= 4u_2 \left(-\frac{\tau e^{2(t-\tau)}}{2} - \frac{e^{2(t-\tau)}}{4} \right) \Big|_2^t$$

$$= u_2 (-2\tau - 1) e^{2(t-\tau)} \Big|_2^t$$

$$= u_2(-2t-1) - u_2(-4-1)e^{2(t-2)}$$

$$= u_2(-2t - 1 + 5e^{2(t-2)})$$

$$\therefore y = u_2(-2t - 1 + 5e^{2(t-2)}) + 3e^{2t}$$

6. (5 points) Consider the differential equation

$$y'' - 3y' + 2y = (3t^2 - 4t)\cos 6t + 7e^t.$$

DO NOT SOLVE. According to the method of undetermined coefficients, what should you try for the particular solution? (Do not solve for the coefficients, just tell me the form for the solution. You should probably determine y_h first, of course. Check your work for y_h!)

$$(r-1)(r-2) = 0$$

$$r = 1, 2$$

$$y_h = C_1 e^t + C_2 e^{2t}$$

$$\therefore \boxed{y_p = (At^2 + Bt + C)(C_1 \cos 6t + C_2 \sin 6t) + Dte^t}$$

will not work because it need 6 independent coefficients.

$$y_p = (At^2 + Bt + C)\cos 6t + (Dt^2 + Et + F)\sin 6t + Gte^t$$

8. (5 points) If we have an underdamped mass-spring system described by the equation

$$y'' + \frac{1}{10000}y' + 4y = \cos \omega t$$

(so $m = 1$, $\gamma = 1/10000$, and $k = 4$), then we can write the steady state solution in "amplitude-phase" form, $y = R\cos(\omega t - \delta)$.

If the driving frequency ω is chosen so that the system achieves resonance, what (approximately) is the phase angle?

$$\omega = \omega_0 = \sqrt{\frac{k}{m}} = 2$$

guess $y_p = A\cos 2t + B\sin 2t$

$y' = -2A\sin 2t + 2B\cos 2t$

$y'' = -4A\cos 2t - 4B\sin 2t$

$$-4A\cos 2t - 4B\sin 2t - \frac{A\sin 2t}{5000} + \frac{B\cos 2t}{5000} + A\cos 2t + B\sin 2t = \cos 2t.$$

$$\begin{cases} -4A + \frac{B}{5000} + A = 1 \\ -4B + B - \frac{A}{5000} = 0 \end{cases}$$

$$A = \frac{1}{-3 - \frac{1}{15000 \times 5000}} \approx -\frac{1}{3}$$

$$B = -\frac{A}{15000} \approx 2.2 \times 10^{-5}$$

$$\tan \delta = \frac{2.2 \times 10^{-5}}{-1/3}$$

$$\boxed{\begin{aligned} \delta &= \tan^{-1}\left(\frac{2.2 \times 10^{-5}}{-1/3}\right) \\ &= -0.00378^\circ \\ &= -6.6 \times 10^{-5} \text{ rad} \end{aligned}}$$

$$\because \cos \delta = \frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \quad \sin \delta = \frac{\gamma\omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \quad \omega_0 \approx \omega \approx 2$$

$$\therefore \cos \delta \approx 0$$

$$\sin \delta \approx 1$$

$$\therefore \boxed{\delta \approx \frac{\pi}{2}}$$

1. (13pts) Consider the initial value problem

$$\frac{dy}{dx} = \frac{2x+1}{y-1}, \quad y(0) = -1.$$

Find $y(1)$.

-3

$(y-1)dy = (2x+1)dx$

$\frac{y^2}{2} - y = x^2 + x + C$

$\because y(0) = -1$

$\therefore \frac{1}{2} + 1 = C$

$C = \frac{3}{2}$

$y^2 - 2y = 2x^2 + 2x + 3$

$y = 1 \pm \sqrt{4 + 2x^2 + 2x}$

$\cancel{y(x) =}$

$\because y(0) = -1$

$\therefore y = 1 - \sqrt{4 + 2x^2 + 2x}$

tricky

$\therefore y(1):$

$\frac{y^2}{2} - y = 2 + \frac{3}{2}$

$y^2 - 2y - 7 = 0$

$\sqrt{\Delta} = \sqrt{4+28} = 4\sqrt{2}$

$\boxed{\cancel{y_1(1) = \frac{2+4\sqrt{2}}{2} = 1+2\sqrt{2}} \quad y_2(1) = 1 - 2\sqrt{2}}$

2. (12 points) A new homeowner just took out a loan of \$235,000 at an annual interest rate of 4.5% compounded continuously. She wants to analyze the benefits of paying a monthly mortgage of \$1,250 instead of \$1,200. Let $P(t)$ be the amount she owes after t years. Assume she makes payments continuously. How many years would it take to pay off the loan in each case? What's the total amount paid in interest in each case? How much money would she save if she paid \$1,250 each month instead of \$1,200? Write your answers to these questions in the provided table. Round your final answers to the nearest tenth of a year and nearest dollar.

$\cancel{\frac{dP}{dt} = (4.5\%)^t P - m \cdot 12}$ −12

$\cancel{P_n = (1+4.5\%)P_{n-1}}$

$\frac{dP}{dt} = 0.045P - 12k, \quad P(0) = 235{,}000$

$\frac{1}{0.045} \ln|0.045P - 12k| = t$

$P = \frac{Ce^{0.045t} + 12k}{0.045}$

$\because P(0) = 235000$

$\therefore C = (0.045(235000) - 12k)e^{0.045t}$

$\because P(t) = 0$

$\therefore -12k = (0.045 \cdot 235000 - 12k)e^{0.045t}$

$t = \frac{1}{0.045} \ln\left(\frac{-12k}{0.045 \cdot 235000 - 12}\right)$

$k_1 = 1200, t_1 = 29.4593$

$k_2 = 1250, t_2 = 27.1284$

total ⇒

$tot_1 = k_1 t_1 - 235000$
≈ 189214

$tot_2 = k_2 t_2 - 235000$
≈ 171926

Monthly mortgage	Years to pay off loan	Total interest paid
\$1,200	29.5	189214
\$1,250	27.1	171926
Difference	2.4	17288

Bonus (5 points) Find a function $y = f(t)$ such that

$$\int_1^t f(s)ds = f(t)^3 + 8.$$

Is this function unique? Why or why not? Justify your answer.

$F(t) - F(1) = f(t)^3 + 8$

$f(t) = 3f(t)^2 \cdot \frac{df}{dt}$

$dt = 3f(t)df$

$t + C = \frac{3}{2} f(t)^2$

$\pm\sqrt{\frac{2}{3}t + C} = f(t)$

$0 = \int_1^1 f(s)ds = f(1)^3 + 8$

$f(1) = -2$

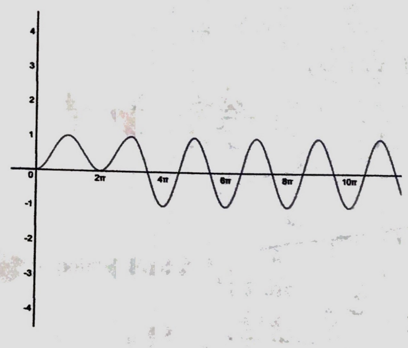
~~not unique,~~ unique.

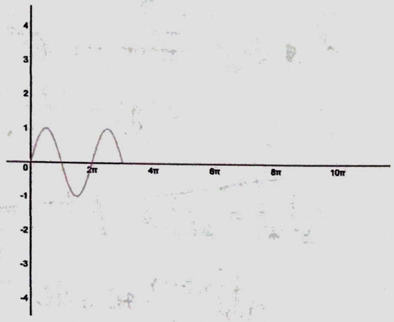
solve!

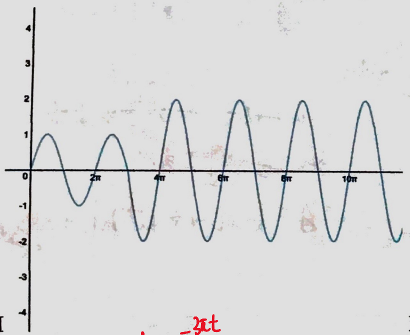
8. (8 points) Graphed below are the equations of motion for a spring-mass system modeled by $u'' + u = g(t)$, $u(0) = 0$, $u'(0) = 0$ for various forcing functions $g(t)$. For each $g(t)$ given in parts (a), (b), (c), and (d) below, write the numeral I, II, III, or IV of the corresponding graph. You do not need to show any work.

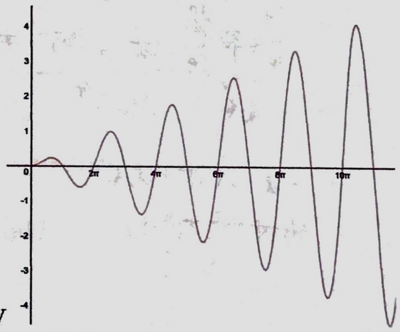
$\Delta = \sqrt{-4}$

$(s^2+1)\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$


I


II


III


IV

$\frac{1-e^{-3\pi t}}{s^2+1} = \sin(t-1) - u_3 \sin(t-3\pi)$

(a) $g(t) = \delta(t) - \delta(t - 3\pi)$ ~~I~~ III

(b) $g(t) = \frac{1}{4}\cos(t)$ IV

(c) $g(t) = \delta(t) + \delta(t - 3\pi)$ ~~III~~ II

(d) $g(t) = 1 - u_{3\pi}(t)$ ~~II~~ I

Final cheat sheet

Tuesday, December 11, 2018 1:52 PM

- Autonomous Differential Equation $\frac{dy}{ds} = f(y)$ always has constant soln/equilibrium solution

---

- First Order DE Models
  - Logistic Equation $\frac{dP}{dt} = -r(1-\frac{P}{K})(1-\frac{P}{T})P$
    r: intrinsic rate of growth; K: carrying capacity; T: Threshold population
  - Solution Problems
    $\frac{dQ}{dt}$ = rate in − rate out
    ★ Sometimes the volume is changing
  - Falling object
    $m\frac{dv}{dt} = mg - kv$

- Uniqueness: If the function p and g are continuous on an open interval $I: \alpha < I < \beta$, containing point $t = t_0$, then there exist a unique function $y = \phi(t)$ that satisfy the differential equation $y' + p(t)y = g(t)$ for each t in I and the initial condition $y(t_0) = y_0$, y arbitrary.

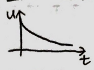
- Linear 2nd order DE $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$
  - $g(t) = 0$ homogeneous DE
  - y_c: complementary soln
  - characteristic equation $ar^2 + br + c = 0$
- Euler's formula $e^{it} = \cos t + i\sin t$
- Hyperbolic Trig
  - $\sinh(t) = \frac{e^t - e^{-t}}{2}$
    $\cosh(t) = \frac{e^t + e^{-t}}{2}$
  - Properties
    $\cos(it) = \cosh(t)$
    $\sin(it) = \sinh(t)$
  - derivatives
    $(\sinh(t))' = \cosh(t)$
    $(\cosh(t))' = \sinh(t)$
- Polar coordinate
  $a + bi = re^{i\theta}$ $a = r\cos\theta$, $b = r\sin\theta$ $r = \sqrt{a^2+b^2}$
- Wronskian / Linearity
  $y'' + p(t)y' + q(t)y = 0$, $W(y_1, y_2) = y_2 y_1' - y_1 y_2' \neq 0$
  then all solutions are of form $y(t) = C_1 y_1 + C_2 y_2$

- Solution of Homogeneous DE
  - $\Delta > 0$, $r = r_1, r_2$, $(r - r_1)(r - r_2) = 0$
    $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
  - $\Delta < 0$, $r = \lambda \pm \mu i$ $\mu = \frac{\sqrt{\Delta}}{2a}$, $\lambda = -\frac{b}{2a}$
    $y(t) = e^{\lambda t}(C_1\cos\mu t + C_2\sin\mu t)$
  - $\Delta = 0$, $(r-a)^2 = 0$
    $y(t) = C_1 e^{r_1 t} + C_2 t e^{r_1 t}$
- Solution of Non-Homogeneous DE

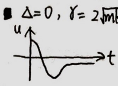
$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0 t^n + \dots + a_n$	$t^s(A_0 t^n + \dots + A_n)$
$P_n(t) \cdot e^{\alpha t}$	$t^s(A_0 t^n + \dots + A_n)e^{\alpha t}$
$P_n(t) \begin{cases} \sin\beta t \\ \cos\beta t \end{cases}$	$t^s[(A_0 t^n + \dots + A_n)\cos\beta t + (B_0 t^n + \dots + B_n)\sin\beta t]$

S: number of times 0 is a root of characteristic equation

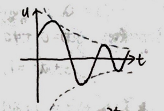
- Models of 2nd order DE
  - Spring $mu'' + \gamma u' + ku = F(t)$
    u: position from equilibrium
    γ: damping constant
    F(t): applied external force
    - $F(t) = 0$, $\Delta = \gamma^2 - 4mk$
      - $\Delta > 0$ overdamped



      - $\Delta = 0$, $\gamma = 2\sqrt{mk}$ critically damped

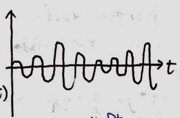


      - $\Delta < 0$, underdamped

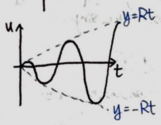


        $u(t) = e^{\lambda t}R\cos(\mu t - \delta)$
    - $F(t) \neq 0$, $\gamma = 0$
      $mu'' + ku = F_0\cos(w_0 t)$
      $y_c = C_1\cos(\sqrt{\frac{k}{m}}t) + C_2\sin(\sqrt{\frac{k}{m}}t)$
      $w = \sqrt{\frac{k}{m}}$

      - $w = q \cdot w_0$ $(q \neq 1)$
        Beat
        $u(t) = R \cdot \sin(w_1 t)\sin(w_2 t)$

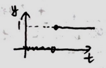


      - $w = w_0$
        Resonance
        $u(t) = Rt \cdot \sin(\mu t)$



- RLC circuits
  $Q''(t)L + Q'(t)R + Q \cdot \frac{1}{C} = E(t)$

- Laplace $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt = F(s)$
- Heaviside Function
  $u_c(t) \begin{cases} 0 & t \in (0, c) \\ 1 & t \in [c, +\infty) \end{cases}$



  - $\mathcal{L}\{u_c f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$
    $u_c f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}$
- $\mathcal{L}\{f^n(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$

- Dirac $\delta_0(t) = \delta(t) = \lim_{n\to\infty} f_n$
  - Properties $\delta(t) = 0$, $t \neq 0$
    $\int_{-\varepsilon}^{\varepsilon}\delta(t)dt = 1$, for any $\varepsilon > 0$
    - $\mathcal{L}\{\delta_c(t)\} = \mathcal{L}\{\delta(t-c)\} = e^{-cs}$

- Convolution $(f*g) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t g(t-\tau)f(\tau)d\tau$
  - $\mathcal{L}\{f*g\} = \mathcal{L}\{g(\tau)\} \cdot \mathcal{L}\{f(t)\}$
  - $\mathcal{L}^{-1}\{\mathcal{L}\{g(\tau)\} \cdot \mathcal{L}\{f(t)\}\} = \mathcal{L}^{-1}\{F(s) \cdot G(s)\} = \int_0^t f(t-\tau)g(\tau)d\tau$

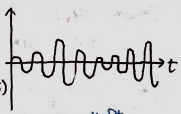
- $ay'' + by' + cy = g(t)$ $\mathcal{L}\{g(t)\} = G(s)$
  $(as^2+bs+c)\mathcal{L}\{y\} - (as+b)y(0) - ay'(0) = G(s)$
  $\mathcal{L}\{y\} = \frac{(as+b)y_0 + ay'(0)}{as^2+bs+c} + \boxed{\underbrace{\boxed{\frac{1}{as^2+bs+c}}}_{H(s)}G(s)} \dashrightarrow \Psi(s) = \boxed{H(s)} \cdot G(s)$
  H(s): transfer function
  $\psi(t) = \int_0^t h(t-\tau) \cdot g(\tau)d\tau$
  h(t): Impulse function
  $\mathcal{L}\{\psi(t)\} = \Psi(s)$

- Check
  I. use radian for angle
  II. Solving 1st DE: multiply intergrating factor on both sides
  III. Read the requirement

- $w = q \cdot w_0 \ (q \neq 1)$

Beat

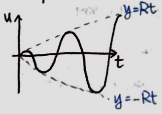
$u(t) = R\cdot \sin(w_1 t)\sin(w_2 t)$



- $w = w_0$

Resonance

$u(t) = Rt\cdot \sin(\gamma t)$

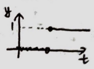


- RLC circuits

$Q''(t)L + Q'(t)R + Q\cdot\frac{1}{C} = E(t)$

- Laplace $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$
- Heaviside Function

$u_c(t) \begin{cases} 0 & t\in(0,c) \\ 1 & t\in[c,+\infty) \end{cases}$



  - $\mathcal{L}\{u_c f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$

    $u_c f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}$
- $\mathcal{L}\{f^{n}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$

- Dirac $\delta_0(t) = \delta(t) = \lim_{n\to\infty} f_n$
  - Properties $\delta(t) = 0,\ t\neq 0$

    $\int_{-\varepsilon}^{\varepsilon}\delta(t)dt = 1$, for any $\varepsilon > 0$

    - $\mathcal{L}\{\delta_c(t)\} = \mathcal{L}\{\delta(t-c)\} = e^{-cs}$
    - $\mathcal{L}^{-1}\{1\} = \delta_0(t) = \delta(t)$

- $e^{ct}f(t) = F(s-c)$

  $t^n f(t) = (-1)^n F^{(n)}(s)$

- Convolution $(f*g) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t g(t-\tau)f(\tau)d\tau$
  - $\mathcal{L}\{f*g\} = \mathcal{L}\{g(\tau)\}\cdot\mathcal{L}\{f(t)\}$
  - $\mathcal{L}^{-1}\{\mathcal{L}\{g(\tau)\}\cdot\mathcal{L}\{f(t)\}\} = \mathcal{L}^{-1}\{F(s)\cdot G(s)\} = \int_0^t f(t-\tau)g(\tau)d\tau$

- $ay'' + by' + cy = g(t)$ $\mathcal{L}\{g(t)\} = G(s)$

$(as^2+bs+c)\mathcal{L}\{y\} - (as+b)y(0) - ay'(0) = G(s)$

$\mathcal{L}\{y\} = \frac{(as+b)y_0 + ay'(0)}{as^2+bs+c} + \boxed{\underbrace{\boxed{\frac{1}{as^2+bs+c}}}_{H(s)} G(s)} \dashrightarrow \Psi(s) = \boxed{H(s)}\cdot G(s)$

transfer function

$\psi(t) = \int_0^t h(t-\tau)\cdot g(\tau)d\tau$

$h(t)$: Impulse function

$\mathcal{L}\{\psi(t)\} = \Psi(s)$

- $(\tan x)' = \sec^2 x$

$(\cot x)' = -\csc^2 x$

$(\sec x)' = \tan x\cdot\sec x$

$(\csc x)' = -\csc x\cdot\cot x$

$(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$

$(\cos^{-1}x)' = -\frac{1}{\sqrt{1-x^2}}$

$(\tan^{-1}x)' = \frac{1}{1+x^2}$

$(\cot^{-1}x)' = -\frac{1}{1+x^2}$

$(\sec^{-1}x)' = \frac{1}{|x|\sqrt{x^2-1}}$

$(\csc^{-1}x)' = -\frac{1}{|x|\sqrt{x^2-1}}$

- $\sec\theta = \frac{1}{\cos\theta}$

$1+\tan^2\theta = \sec^2\theta$

- Check

I. use radian for angle

II. Solving DE: multiply intergrating factor on both sides

III. Read the requirement

IV. Remember ~~when~~ $u_c f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}$

V. y_p = both cos & sin where there's a trig

practice

Thursday, December 13, 2018 12:34 PM

sp15m307...

Math 307 A - Spring 2015
Final Exam
June 10, 2015

Name: ______________________________

Student ID Number: ______________________________

- There are 8 pages of questions. In addition, the last page is the basic Laplace transform table. Make sure your exam contains all these pages.
- You are allowed to use a scientific calculator (**no graphing calculators and no calculators that have calculus capabilities**) and one **hand-written** 8.5 by 11 inch page of notes.
- You must show your work on all problems. The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.** Give exact answers wherever possible.
- If you need more room, use the backs of the pages and indicate to the grader that you have done so.
- Raise your hand if you have a question.
- There may be multiple versions of the exam so if you copy off a neighbor and put down the answers from another version we will know you cheated. Any student found engaging in academic misconduct will receive a score of 0 on this exam. All suspicious behavior will be reported to the student misconduct board. In such an instance, you will meet in front of a board of professors to explain your actions.
  DO NOT CHEAT OR DO ANYTHING THAT LOOKS SUSPICIOUS!
  WE WILL REPORT YOU AND YOU MAY BE EXPELLED!
- You have 110 minutes to complete the exam.

PAGE 1	13	
PAGE 2	14	
PAGE 3	10	
PAGE 4	12	
PAGE 5	13	
PAGE 6	14	
PAGE 7	12	
PAGE 8	12	
Total	100	

GOOD LUCK!

1. (13 pts)

(a) Find the general explicit solution to $ty' - 2y = t^6$.

$y' - \frac{2}{t}y = t^5$

$\frac{d\mu}{dt} = -\frac{2}{t}\cdot\mu$

$\ln\mu = -2\ln t$

$\mu = t^{-2}$

$\frac{d}{dt}(y\cdot t^{-2}) = t^3$

$y\cdot t^{-2} = -\frac{t^4}{4} + C$

$\boxed{y = -\frac{t^2}{4} + \frac{C}{t^2}}$

(b) Find the explicit solution to $y' = 4xy^2e^{2x}$ with $y(0) = 4$.

$\frac{1}{y^2}\frac{dy}{dt} = 4xe^{2x}$

$-\frac{1}{y} = 2xe^{2x} - e^{2x} + C$

x	$4e^{2x}$
1	$2e^{2x}$
0	e^{2x}

$\because y(0) = 4$

$-\frac{1}{4} = C - 1$

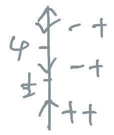
$C = \frac{3}{4}$

$\boxed{y = \frac{1}{e^{2x} - 2xe^{2x} - \frac{3}{4}}}$

2. (14 pts)

(a) Consider $y' = (1-2y)\,(y-4)^2$.

i. Determine the critical (equilibrium) points and classify each one as stable, unstable or semistable.



$y = 4$ semistable

$y = \frac{1}{2}$ stable.

ii. Let $y(t)$ be the solution that satisfies the given differential equation with the initial condition $y(1) = 3$.
Use Euler's method with $h = 0.1$ to approximate the value of $y(1.1)$.

(b) A baseball is dropped from an airplane. The mass of a baseball is about 0.2 kg. The force due to air resistance is proportional, and in opposite direction, to velocity with proportionality constant k (where $k > 0$).
Just like we did in homework, assume there are two forces acting on the ball: the force due to gravity and the force due to air resistance. (Recall: Newton's second law says $ma = F$ and the acceleration due to gravity is 9.8 meters/second^2.)

i. Give the differential equation and initial conditions for the velocity $v(t)$. (Do not solve)

$$\frac{dv}{dt} = kv - g$$

$$V(0) = 0$$

ii. The value of $\lim_{t\to\infty} v(t)$ is called the terminal velocity. For a baseball, terminal velocity is known to be about 42 meters/second. Using this fact, find the value of the proportionality constant k. (Hint: You do NOT need to solve the differential equation).

$$kv - g = 0$$

$$V = \frac{g}{k} = 42$$

$$k = \frac{9.8}{42}$$

3. (10 pts) Some cookie dough with an initial temperature of 40 degrees Fahrenheit is placed in an oven and the oven is turned on. The temperature of the oven is given by $f(t) = 350 - 280e^{-t/2}$ degrees Fahrenheit where t is in minutes. Assume the differential equation for the temperature of the cookie dough, $y(t)$, is given by

$$\frac{dy}{dt} = -\frac{1}{2}\left(y - 350 + 280e^{-t/2}\right).$$

Solve the differential equation to find the temperature of the cookie dough, $y(t)$, at time t minutes. (Hint: It's linear!)

$$\frac{dy}{dt} + \frac{y}{2} = 350 - 280e^{-\frac{t}{2}}$$

$$\frac{\mu}{2} = \frac{d\mu}{dt}$$

$$\mu = e^{\frac{t}{2}}$$

$$\frac{d}{dt}\left(y \cdot e^{\frac{t}{2}}\right) = 350e^{\frac{t}{2}} - 280$$

$$y = \frac{700e^{\frac{t}{2}} - 280t + C}{e^{\frac{t}{2}}}$$

$\because y(0) = 40$

$$\therefore \quad 40 = 700 + C$$

$$C = -660$$

$$y = \frac{700e^{\frac{t}{2}} - 280t - 660}{e^{\frac{t}{2}}}$$

4. (12 pts) A certain mass-spring system satisfies $mu'' + 3u' + u = 0$, where m is the mass of the object attached to the end of the spring. The initial conditions are $u(0) = 3$ and $u'(0) = 0$.

(a) For what masses, m, will the system exhibit (damped) oscillations?
(Your answer will be a range of values)

$$\Delta = 9 - 4m < 0$$

$$m > \frac{9}{4}$$

$$\boxed{m \in (\tfrac{9}{4}, +\infty)}$$

(b) Find the quasi-period of the solution if $m = 5$.

$m = 5 \qquad \therefore \Delta = -11$

$$\mu = \frac{\sqrt{11}}{10}$$

$$T = \frac{2\pi}{\mu} = \frac{2\pi}{\sqrt{11}} \cdot 10 = \frac{20\sqrt{11}}{11}\pi$$

(c) Find the solution $u(t)$ if $m = 2$. (Use the initial conditions.)

$2u'' + 3u' + u = 0 \qquad (2u+1)(u+1) = 0$

$$\Delta = 9 - 8 = 1$$

$$u(t) = C_1 e^{-\frac{1}{2}t} + C_2 e^{-t}$$

$\because u(0) = 3, u'(0) = 0$

$$C_1 + C_2 = 3$$

$$-\frac{1}{2}C_1 - C_2 = 0$$

$$\begin{cases} C_1 = 6 \\ C_2 = -3 \end{cases}$$

$$\therefore \boxed{u(t) = 6e^{-\frac{1}{2}t} - 3e^{-t}}$$

5. (13 pts) Give the solution to $y'' + 4y = te^{2t}$ with $y(0) = 0$ and $y'(0) = 0$.

y_c

$r^2+4=0$

$y_c = C_1\cos 2\theta + C_2\sin 2\theta$

y_p assume $y_p = (At^2+Bt+C)e^{2t} = At^2e^{2t} + Bte^{2t} + Ce^{2t}$

$y_p' = 2Ate^{2t} + 2At^2e^{2t} + 2Bte^{2t} + Be^{2t} + 2Ce^{2t}$

$= 2At^2e^{2t} + (2A+2B)te^{2t} + (B+2C)e^{2t}$

$y'' = 4At^2e^{2t} + (4A+4A+4B)te^{2t} + (2A+2B+2B+4C)e^{2t}$

$= 4At^2e^{2t} + (8A+4B)te^{2t} + (2A+4B+4C)e^{2t}$

plug in :

$$\begin{cases} 5A = 0 \\ 8A + 8B = 1 \\ 2A + 4B + 8C = 0 \end{cases}$$

$$\begin{cases} A = 0 \\ B = \frac{1}{8} \\ C = \frac{1}{16} \end{cases}$$

$\therefore y_p = \dfrac{te^t}{8} + \dfrac{e^{2t}}{16}$

$y = C_1\cos 2\theta + C_2\sin 2\theta + \frac{te^t}{8} + \frac{e^{2t}}{16}$

6. (14 pts)

(a) Give the form of a particular solution to $y'' - 4y' + 4y = 5 + te^{2t}$. (Do not solve, just give the form you would use for undetermined coefficients. Your answer will involved 'A, B, ...').

$\because y_0 = C_1 e^{2t} + C_2 t e^{2t}$ ~~$y_p = A + (Bt + C)e^{2t}$~~

$A + t^2(Bt + C)e^{2t}$

(b) Use the Laplace transform table (and step functions) to answer these questions:

i. Find the Laplace transform, $\mathcal{L}\{f(t)\}$, for the function $f(t) = \begin{cases} 3 & , 0 \le t < 6; \\ t + \cos(t-6) & , t \ge 6. \end{cases}$

ii. Find the inverse Laplace transform $\mathcal{L}^{-1}\left\{ e^{-2s}\dfrac{2}{s^3} - e^{-5s}\dfrac{4}{s-8} \right\}$.

7. (12 pts) Use the Laplace transform table (and algebra) to answer these questions:

(a) Find the Laplace transform of both sides of $y' = 3te^{4t} + 2t^3$ with $y(0) = 5$ and solve for $\mathcal{L}\{y\}$. (Don't do partial fraction and don't solve for $y(t)$, just stop when you get $\mathcal{L}\{y\}$ by itself and the other side all in terms of s.)

(b) Find the inverse Laplace transform, $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-2)} + \frac{1}{s^2+6s+13}\right\}$.

8. (12 pts) Solve $y'' + y = \begin{cases} 1 & , 0 \leq t < 3; \\ 5 & , t \geq 3. \end{cases}$ with initial conditions $y(0) = 0$, $y'(0) = 2$.

Laplace Transform Table for Final Exam - Dr. Loveless

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
t^n	$\frac{n!}{s^{n+1}}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$u_c(t)$	$\frac{e^{-cs}}{s}$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$