Modeling

Monday, October 15, 2018 3:37 PM

- Salt: 100 gal = Volume, 50 oz salt at tim 0-

  rate in =

  Concentration: $\frac{1}{4}(1+\frac{1}{2}\sin t)$ oz/gal

  2gal/min

  rate out: 2 gal/min

  Find: $Q(t)$ = amt of salt in tank at time t.

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

$$= \begin{pmatrix}\text{concentration}\\\text{incoming}\end{pmatrix} \cdot (\text{volume rate}) - \left(\frac{Q(t)}{\text{Volume}}\right)\begin{pmatrix}\text{Vol}\\\text{rate out}\end{pmatrix}$$

$$= \frac{1}{4}(1+\frac{1}{2}\sin t) \cdot 2 - 2 \cdot \frac{Q}{100}$$

Substitution

Wednesday, October 17, 2018 3:43 PM

- e.g.1. $\frac{dy}{dt} = 1 + t^2 - 2ty + y^2$

If there's one solution, either do
$y_1 \cdot \mu(t)$
$y_1 + \mu(t)$ y_1 is a solution
......

$y_1 = t$

$y_2 = t + \frac{1}{u(t)} \rightarrow y - t = \frac{1}{u}, \quad u = \frac{1}{y-t}$

$\frac{dy}{dt} = 1 - u^{-2}\frac{du}{dt}$

$1 - u^{-2} \cdot \frac{du}{dt} = 1 + t^2 - 2t(t + \frac{1}{u}) + (t + \frac{1}{u})^2$

$1 - u^{-2} \cdot \frac{du}{dt} = 1 + t^2 - 2t^2 - \frac{2t}{u} + t^2 + 2\frac{t}{u} + \frac{1}{u^2}$

$-u^{-2}\frac{du}{dt} = \frac{1}{u^2}$

$\frac{du}{dt} = -1$

$u = -t + C$

$\frac{1}{y-t} = -t + C$

$\frac{1}{C-t} + t = y$

- e.g.2. $\frac{dy}{dx} = \frac{2y}{x} + \cos(\frac{y}{x^2})$; $u = \frac{y}{x^2}$. (Implicit Form)

$y = u \cdot x^2$

$\frac{dy}{dx} = \frac{du}{dx} \cdot x^2 + 2x \cdot u = 2xu + \cos u$

$\frac{du}{dx} x^2 = \cos u$

$\sec u \, du = \frac{1}{x^2} dx$

$\ln|\sec u + \tan u| = -\frac{1}{x} + C$

$\sec u + \tan u = Ce^{-\frac{1}{x}}$

☆ $\frac{dy}{dx} = \frac{1}{dx/dy}$

$[f^{-1}(y)]' = \frac{1}{f'(f^{-1}(y))}$

$y = f(x) \quad f^{-1}(y) = x.$

e.g. Derive $(\tan^{-1} x)'$

$y = \tan x, \quad x = \tan^{-1} y, \quad x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{\sec^2(\tan^{-1} y)} = \frac{1}{\sec^2 x} = 1 + y^2$

$\therefore \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{\sec^2(\tan^{-1}y)} = \frac{1}{\sec^2 x} = 1+y^2$

- Complex Numbers

$i^2 = -1 \quad i = \sqrt{-1}$

e.g. $ax^2+bx+c=0$

$x = \frac{-b\pm\sqrt{\Delta}}{2a}$ $\quad \Delta = b^2-4ac$
$= \text{discriminant.}$

$\Delta > 0$ 2 real solutions

$\Delta = 0$ 1 solution

$\Delta < 0$ $x = \frac{-b\pm i\sqrt{-(b^2-4ac)}}{2a}$

e.g. $x^2+1=0$

$\Delta = 0-4 = -4$

$x = \frac{0\pm i\sqrt{4}}{2} = \frac{\pm 2i}{2} = \pm i$

Turns out

$(*)$ $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ $\longrightarrow$ Any nth poly are

There is an $x = a+bi$, $(a,b \in \mathbb{R})$ $\quad$ $(n+1)$-dimention $\mathbb{R}$ vector space.

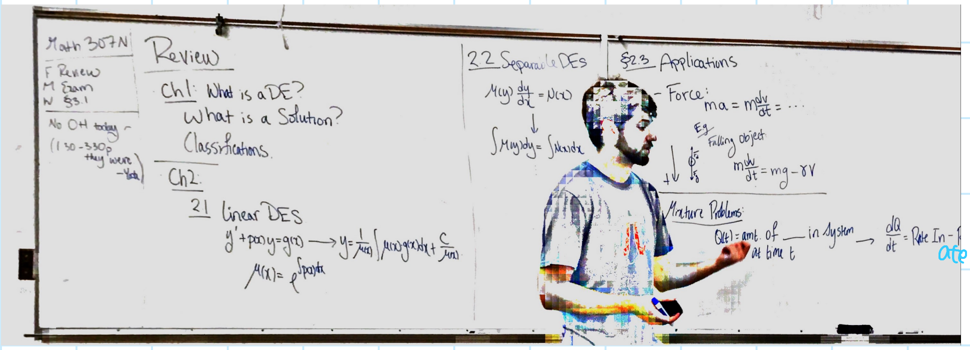
satisfying $(*)$ $\quad$ $\langle 1, x, x^2, \ldots, x^n \rangle$

Any polynomial factors:

$c\prod_i (x-a_i)\prod_j (x^2+b_j x+c_j)$

MidTerm review

Friday, October 19, 2018 3:30 PM



ate out.

- Exams:

  5 prob

  I. Concept Question

  II. DE

  III. IVP

  IV.V. word problems (Application)

MidTerm problem set

Tuesday, October 16, 2018 9:38 PM

4 ★★★★★ Math 307C Autumn 2017

(3) (10 points) Find the $\boxed{\text{general solution}}$ of the differential equation $\frac{dy}{dt} = 2t\sqrt{\frac{1-y^2}{1+t^2}}$.
Solve for y in terms of t, and simplify as much as you can.

$$\frac{dy}{\sqrt{1-y^2}} = \frac{2t}{\sqrt{1+t^2}}\,dt$$

$$\sin^{-1} y = \sqrt{1+t^2} + C$$

$$y = \sin\left(\sqrt{1+t^2} + C\right)$$

assume $y = \sin\theta$

$$\int \frac{\cos\theta}{|\cos\theta|}\,d\theta = \sqrt{1+t^2} + C$$

assume $\cos\theta \geq 0$

$$\theta = \sqrt{1+t^2} + C$$

$$\arcsin y = \sqrt{1+t^2} + C$$

also can assume $y = \cos\theta$.

$$y = \cos\left(\sqrt{1+t^2} + C\right)$$

5

★★★ (4) (10 points) Suppose a quantity $B(t)$ is governed by the first order differential equation

$$\frac{dB}{dt} = (100 - B^2)(4 - B)B,$$

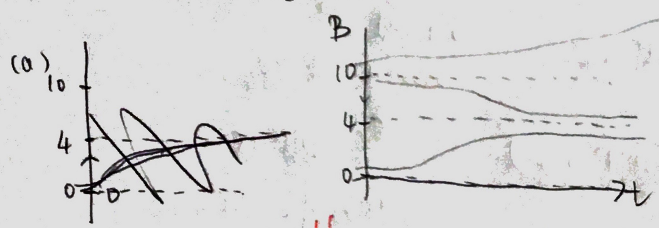
(*Do not attempt to solve this differential equation!*)

(a) (3 points) Suppose that $B(0) = 1$. Find $\lim_{t\to\infty} B(t)$.

(b) (3 points) Now suppose that $B(0) = 50$. Find $\lim_{t\to\infty} B(t)$.

(c) (4 points) Explain your answer in one or two sentences. It may help to draw a rough sketch of the direction field.

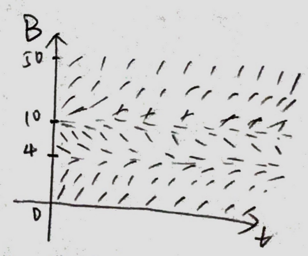
(a)



$\because B(0) = 4$, $\lim_{t\to\infty} B(t) = \infty$

(b) $B(0) = 50$

$\lim_{t\to\infty} B(t) = \infty$



(c)

(a) B is increasing and bounded above by 4

4 is stable equilibrium.

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4 is stable equilibrium.

(b) B is increasing
B gets bigger than any fixed number.

★★★★ Spring 2018

5[b]. (7pts) Max is buying a home and he plans to spend $2000 per month to pay off a 30-year mortgage. Suppose that the interest rate is 4% compounded continuously. Let $P(t)$ be the amount, in dollars, owed at time t, measured in years. Write down a differential equation for $P(t)$. (No need to show work.)

$$\frac{dP}{dt} = (4\%)P - 2000\times 12$$ −7

Winter 2015
★★★ Find the explicit solution to $\frac{dy}{dx} = x - \frac{3y}{x}$ with $y(1) = \frac{21}{5}$.
(assume $x>0$)

$\frac{dy}{dx} + \frac{3}{x}\cdot y = x$

$\mu\cdot\frac{3}{x} = \frac{d\mu}{dx}$

$\ln|\mu| = 3\ln|x|$

$\mu = e^{3\ln|x|}$ transl

$\mu = x^3$

$\therefore y\cdot x^3 = \int x^4\,dx$

$y\cdot x^3 = \frac{x^5}{5} + C$

$y = \frac{x^2}{5} + C\cdot x^{-3}$

$\frac{21}{5} = \frac{1}{5} + C$

$C = 4$

$\therefore y = \frac{x^2}{5} + 4x^{-3}$.

Remember: $e^{c\ln|x|} = x^c$

★★★ Winter 2015 Find two different solutions to $\frac{dy}{dt} = 4t\sqrt{y-5}$ with $y(0)=5$. AND specifically, and briefly, say why the uniqueness theorem doesn't apply here.

$\frac{1}{\sqrt{y-5}}\cdot\frac{dy}{dt} - 4t = 0$

$\frac{1}{\sqrt{y-5}}$ is not continuous at $y=5$.

$\therefore$ The solution is not unique.

$(y-5)^{\frac{1}{2}} = 2t^2 + C$

$\because y(0)=5$

$\therefore C = 0$

$\therefore y = 4t^4 + 5$. soln 1

$\therefore C=0$

$\therefore y=4t^4+5.$ soln 1

soln 2. $y=5.$

Constant solution!

★★ 2015 Winter.

(b) A local pond has an initial volume of 5,000 liters. Water enters the pond through a stream at 400 liters/day and water leaves the pond through a different stream at 600 liters/day. Salt, from the salting of roads, gets into the incoming stream giving it a salt concentration of 0.03 kg/liter. Initially, the pond contains no salt.
Let $y(t)$ is the amount, in kg, of salt in the pond at time t days.
Give the differential equation and initial condition for $y(t)$.
(DO NOT SOLVE, JUST SET UP)

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}$$

$$= 400 \cdot 0.03 - 600 \cancel{\frac{Y}{5000}} \quad \frac{y}{5000-200t}\cdot 600$$

$$= 12 - \frac{3}{25}Y$$

$y(0)=0$ V is changing!

6. (20 points) A rocket engine generates a constant thrust (upward force) of 90 newtons. It has a base mass of 6kg, and, initially, it carries 3kg of fuel. The fuel burns at a rate of 3 kg per second. The rocket has a coefficient of air resistance of 3 Newton seconds per meter. Assume the rocket starts from rest, and that up is the positive direction. Take the gravitational acceleration to be $-10m/sec^2$ for simplicity. **Write an initial value problem for the velocity of the rocket during the time the fuel is burning. Then solve the IVP.**

-18

$$m\frac{dv}{dt} = mg - kv$$

$$m = 6kg+3kg-3t = 9-3t, \quad t\le 1$$

$$\frac{dv}{dt} = g - \frac{k}{m}v$$

$$\boxed{\frac{dv}{dt} = -10 + \frac{1}{3-t}v, \quad v(0)=0}$$

$$\frac{dv}{dt} + \frac{v}{t-3} = -10$$

$$\frac{\mu}{t-1} = \frac{d\mu}{dt}$$

$$\ln|t-1| = \ln|\mu|$$

$$\mu = t-1$$

$$v\cdot(t-1) = \int -10(t-1)\,dt$$

$$v\cdot(t-1) = -5t^2+10t+C$$

$$v = \frac{-5t^2+10t+C}{t-1}$$

$\because v(0)=0$

$\therefore C=0$

$$\boxed{v = -\frac{5t(t-2)}{t-1}}$$

$$F = (9-3t)\frac{dv}{dt} = -3v + 90 - (9-3t)\cdot 10$$

$$\cancel{\frac{dv}{dt} = \frac{-v}{3-t} + \frac{30}{3-t} - 10}$$

$$(3-t)\frac{dv}{dt} = -v + 10t$$

$$\boxed{\frac{dv}{dt} + \frac{v}{3-t} = \frac{10t}{3-t}}$$

$$\frac{\mu}{3-t} = \frac{d\mu}{dt}$$

$$-\ln|3-t| = \ln\mu$$ transformation!

$$\mu = e^{-\ln|3-t|} = (3-t)^{-1}$$

$$(3-t)^{-1}\cdot v = \int \frac{10t}{(3-t)^2}\,dt$$

$3-t=u$, $t=3-u$

$$\int \frac{30-10u}{u^2}\,du = -\frac{30}{u} - 10\ln|u| + C$$

$$v = (3-t)\left(-\frac{30}{u} - 10\ln|u| + C\right)$$

$\because v(0)=0$

$\therefore 0 = 3\cdot(-10 - 10\ln 3 + C)$

$C = 10\ln 3 + 10$

$$v = (3-t)\left(-\frac{30}{u} - 10\ln|u| + 10\ln 3 + 10\right)$$

$$v = -30 + (3-t)(10\ln|u| + 10\ln 3 + 10)$$

- ★★★★ Autumn 2017 Solve the IVP:

- ★★★★ Autumn2017 Solve the IVP:

$$\frac{dy}{dx} = 3x^2(y^2-4), \quad y(0)=1; \quad \text{find } \lim_{x\to\infty} y(x), \lim_{x\to-\infty} y(x).$$

$$\int \frac{dy}{y^2-4} = \int 3x^2 dx$$

$$\int \frac{A}{y+2} + \frac{B}{y-2}\, dx = \int 3x^2 dx$$

$$\begin{cases} A+B=0 \\ -2A+2B=1 \end{cases} \Rightarrow \begin{cases} A=-\frac{1}{4} \\ B=\frac{1}{4} \end{cases}$$

$$-\frac{1}{4}\ln|y+2| + \frac{1}{4}\ln|y-2| = x^3+C \qquad \because y(0)=1$$

Simplify the lm!

$$\ln\left|\frac{y-2}{y+2}\right| = 4x^3+C \qquad \therefore C = \ln\left|-\frac{1}{3}\right| = -\ln 3$$

$$y-2 = e^{4x^3-\ln 3}\cdot(y+2)$$

$$-2\left(1+\frac{e^{4x^3}}{3}\right) = \left(\frac{e^{4x^3}}{3}-1\right)y$$

$$\boxed{y = \frac{2\left(1+\frac{e^{4x^3}}{3}\right)}{1-\frac{e^{4x^3}}{3}}}$$

$x\to+\infty$, $y \to \frac{2\cdot(+\infty)}{-\infty} = -2$

$x\to-\infty$, $y \to 2\cdot\frac{1}{1} = 2$,

$$\boxed{\lim_{x\to+\infty} y = -2, \quad \lim_{x\to-\infty} y = 2}$$

★★ 2018 Spring (Bonus problem) Consider the differential equation

$$y^2 + \left(\frac{dy}{dt}\right)^2 = 1$$

Find all solutions and justify your answer.

$$\left(\frac{dy}{dt}\right)^2 = 1-y^2$$

$$\frac{dy}{dt}=0, \quad y=\pm 1$$

$$\text{or} \quad \frac{dy}{dt} = \pm\sqrt{1-y^2}$$

$$\arcsin y = t$$

$$y = \sin(t+C)$$

Cheat sheet

Monday, October 22, 2018 9:46 PM

- 1^{st} order linear ODEs $\frac{dy}{dt} + p(t)y = g(t)$

  Separable 1^{st} order ODEs $M(t) = N(y) \cdot \frac{dy}{dt}$.

- Autonomous Differential Equation $\frac{dy}{dx} = F(y)$

  (always has <u>constant</u> solution)

- Integrating factor (solving 1^{st} order linear ODEs)

- Models
  - Population I.

    $$\frac{dP}{dt} = rP, \quad P(t) = P_0 e^{rt}$$

  - Logistic Equation

    $$\frac{dP}{dt} = r(1-\frac{P}{K})(1-\frac{P}{T})P$$

    intrinsic rate of growth. Threshold Population.

    carrying capacity

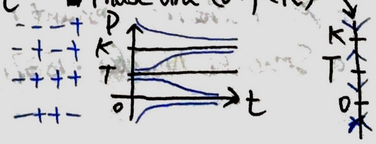
    - Equilibrium solution

      $P=0$

      $P=K$

      $P=T$

    - Phase-line $(0<T<K)$



  - Solution Problems

    $$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

    ★ Sometimes the volume is changing

- Falling Object

$$m\frac{dv}{dt} = mg - kv$$

- Planets (Gravity)

$$m\frac{dv}{dt} = -\frac{mR^2}{(R+x)^2}$$

$$\frac{dv}{dt} \cdot \frac{dx}{dx} = -\frac{mR^2}{(R+x)^2}$$

$$v\,dv = -\frac{mR^2}{(R+x)^2}\,dx$$

- Infection

$$\frac{dI}{dt} = r \cdot I(P-I)$$

- Newton's Law of cooling

$$\frac{dT}{dt} = -k(T - T_A)$$

- Compound interest

$$\frac{dS}{dt} = rS + k$$

- Uniqueness: If the function p and g are continuous on an open interval $I: \alpha < I < \beta$, containing the point $t = t_0$, then there exist a unique function $y = \phi(t)$ that satisfy the differential equation

$$y' + p(t)y = g(t)$$

for each t in I, and that also satisfy the initial condition

$y(t_0) = y_0$, y is arbitrary.

- $(\tan x)' = \sec^2 x$
  $(\cot x)' = -\csc x$
  $(\sec x)' = \tan x \cdot \sec x$
  $(\csc x)' = -\csc x \cdot \cot x$
  $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$
  $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$
  $(\arctan x)' = \frac{1}{1+x^2}$
  $(\text{arc}\cot x)' = -\frac{1}{1+x^2}$
  $(\text{arcsec} x)' = \frac{1}{|x|\sqrt{x^2-1}}$
  $(\text{arccsc} x)' = -\frac{1}{|x|\sqrt{x^2-1}}$
  $\frac{d}{dx} a^x = a^x \ln a$

- $\sec\theta = \frac{1}{\cos\theta}$
  $\csc\theta = \frac{1}{\sin\theta}$
- $1+\tan^2\theta = \sec^2\theta$
  $1+\cot^2\theta = \csc^2\theta$
- $\sin(2\theta) = 2\sin\theta \cdot \cos\theta$
  $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
  $= 2\cos^2\theta - 1$
  $= 1 - 2\sin^2\theta$
- $\sin(\alpha \pm \beta) = \sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta$
  $\cos(\alpha \pm \beta) = \cos\alpha \cdot \cos\beta \mp \sin\alpha \cdot \sin\beta$

- Check list:
  - Solving ~~L~~DE: multiply integrating factor on both sides!
  - Writing ~~ID~~ IVP:
    - Initial condition!
    - Sign!
  - Read the requirement!
  - Student Number: 1837149

- Logarithm
  $\log_a(M \cdot N) = \log_a M + \log_a N$ $\quad \log_a(M/N) = \log_a M - \log_a N$
  $\log_a M^n = n \log_a M$ $\quad \frac{\log_c b}{\log_c a} = \log_a b$.

Mid-term

Wednesday, October 24, 2018 3:31 PM

- Mean: 50/60
  Median: 53/60