6.3

Wednesday, November 28, 2018 3:28 PM

- Warm-up:

e.g.1. Write the following in terms of Heaviside functions

(a) $f(t) = \begin{cases} 1 & t \in [0,1) \\ 2 & t \in [1, +\infty) \end{cases}$

(a) $f_1(t) = u_0(t)$

$f_2(t) = 1 + u_1(t)$

$f(t) = u_0(t) + u_1(t).$

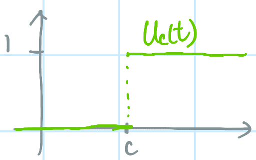
$= u_0(t) + u_1(t)$

(b) $g(t) = \begin{cases} \sin t & t < 0 \\ \cos t & t \geq 0 \end{cases}$

(b) $g_1(t) = \sin t$

$g_2(t) = \sin t - u_0(t)\sin t + u_0(t)\cdot\cos t.$

Recall $u_c(t)$



e.g.2. Solve the following IVP:

$y' = y + h(t)$; $y(0) = 0$. where

$h(t) = \begin{cases} 1, & t < 1 \\ 2, & 1 \leq t < 2 \\ \sin(t-2), & t \geq 2 \end{cases}$

$= 1 + u_1(t)(2-1) + u_2(t)(\sin(t-2) - 2)$

$s\mathcal{L}\{y\} - y(0) = \mathcal{L}\{y\} + \mathcal{L}\{1 + u_1(t) + u_2(t)(\sin t - 2)\}$

$\frac{1}{s} + e^{-s}\cdot\frac{2}{s} + e^{-2s}\cdot\frac{1}{s^2+1}$

$sY(s) - Y(s) = \underline{\frac{1}{s} + e^{-s}\cdot\frac{1}{s} - 2e^{-2s}\cdot\frac{1}{s} + e^{-2s}\cdot\frac{1}{s^2+1}}$

$u_c(t)\cdot f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}$

$\mathcal{L}\{y\} = \frac{1}{s(s-1)} + \frac{e^{-s}}{s(s-1)} - \frac{2e^{-2s}}{s(s-1)} + \frac{e^{-2s}}{(s^2+1)(s-1)}$

Partial fractions only works for distinct linear terms

$\frac{1}{s(s-1)} = -\frac{1}{s} + \frac{1}{s-1}$ (Heaviside cover-up) → cover $s-1$, $s-1=0$, $s=1$

$\frac{1}{s^2+1} = \frac{1}{2}$

$\therefore A = \frac{1}{2}$

$\frac{1}{s^2(s-1)^2}$, $\frac{1}{s^2(s^2+1)}$ ☹ $\frac{1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$

when $s=1$

the equation loses meaning

then cover $(s-1)$

plug-in s^2+1

$A = \frac{1}{2}$

$A = \frac{1}{2}$

$1 = \frac{1}{2}(s^2+1) + (Bs+C)(s-1)$

$s^2 \Rightarrow 0 = \frac{1}{2} + B$

$s \Rightarrow 0 = -B + C$

$A = \frac{1}{2}$

$s^2 \Rightarrow \; 0 = \frac{1}{2} + B$

$s \Rightarrow \; 0 = -B + C$

$1 = \frac{1}{2} - C$

$$\begin{cases} A = \frac{1}{2} \\ B = -\frac{1}{2} \\ C = \frac{1}{2} \end{cases}$$

$$\mathcal{L}\{y\} = (1 + e^{-s} - 2e^{-2s})\left(-\frac{1}{s} + \frac{1}{s-1}\right) + \left(\frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}s + \frac{1}{2}}{s^2+1}\right)e^{-2s}$$

$$y(t) = -1 + e^{t} + u_1(t)\left(-1 + e^{t-1}\right) - 2u_2(t)(-1 + e^{t-2}) + u_2(t)\cdot\left(\frac{1}{2}e^{t-2} - \frac{1}{2}\cos(t-2) - \frac{1}{2}\sin(t-2)\right)$$

- e.g. $\mathcal{L}\{u_2(t)\cdot t^2\}$

$$g(t) = f(t-2) = t^2$$

$$g(t+2) = f(t) = (t+2)^2$$

$$\Rightarrow \; = e^{-2s}\cdot \mathcal{L}\{(t+2)^2\}$$

$$= e^{-2s}\cdot \mathcal{L}\{t^2 + 4t + 4\}$$

$$= e^{-2s}\left(\frac{2!}{s^3} + 4\cdot\frac{1}{s^2} + 4\cdot\frac{1}{s}\right)$$

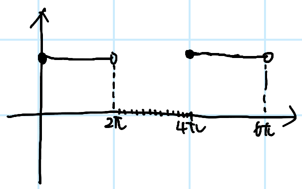
- e.g. consider IVP:

$$y'' + y = f_n(t) \qquad y(0) = 0,\; y'(0) = 0$$

$$f_n(t) = u_0(t) + u_{2\pi}(t)(-1) + \cdots + u_{2\pi n}(-1)^n$$

$$= \sum_{i=0}^{n} (-1)^i u_{2\pi i}(t)$$

Solve it & describe the function $\lim_{n\to\infty} y_n(t)$



$$s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) + \mathcal{L}\{y\} = \sum_{i=0}^{n}\left[(-1)^i + 2\pi i\right]\cdot e^{-2\pi i s}$$

$$\mathcal{L}\{y\} = \frac{\sum_{i=0}^{n}\left[(-1)^i + 2\pi i\right]e^{-2\pi i s}}{s+1}$$

$$y = \sum_{i=0}^{n}\left[(-1)^i + 2\pi i\right]e^{-(t-2\pi i)}$$

$$= \sum_{i=0}^{n}\left[(-1)^i + 2\pi i\right]e^{2\pi i - t}$$

6.2

Monday, December 3, 2018 7:56 AM

- Theorem: Suppose that f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constants K, a, and M such that $|f(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$\Rightarrow$ $f^{(n)}$ is piecewise continuous

$$\mathcal{L}\{f^{n}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

e.g. Find the solution of the differential equation

$$y'' + y = \sin 2t \qquad y(0) = 2, \ y'(0) = 1$$

$$s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) + \mathcal{L}\{f(t)\} = \frac{2}{s^2+4}$$

$$(s^2+1)\mathcal{L}\{f(t)\} = \frac{2}{s^2+4} + 2s + 1$$

$$\mathcal{L}\{f(t)\} = \frac{2s^3 + s^2 + 8s + 6}{(s^2+4)(s^2+1)}$$

$$\frac{as+b}{s^2+4} + \frac{cs+d}{s^2+1}$$

$$\begin{cases} a + c = 2 \\ b + d = 1 \\ 4c + a = 8 \\ 4b + d = 6 \end{cases}$$

$$\mathcal{L}\{f(t)\} = \frac{2s}{s^2+1} + \frac{5}{3}\cdot\frac{1}{s^2+1} - \frac{2}{3}\cdot\frac{1}{s^2+4}$$

$$y = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t.$$

HW 9

Wednesday, November 28, 2018 11:32 PM

16. 0/6 points | Previous Answers

Find the Laplace transform $Y(s)$ of the solution of the given initial value problem. Then invert to find $y(t)$. Write u_c for the Heaviside function that turns on at c, not $u_c(t)$.

$$y'' + 16y = \begin{cases} 1, & 0 \le t < 2, \\ 0, & 2 \le t < \infty; \end{cases} \quad y(0) = 3, \quad y'(0) = 6 \quad = 1 - u_2$$

only terms with t needs addition

$$s^2\mathcal{L}\{y\} - s \cdot y(0) - y'(0) + 16\mathcal{L}\{y\} = \frac{1}{s} - e^{-2s} \cdot \frac{1}{s}$$

$$\mathcal{L}\{y\} = \frac{3s+6}{s^2+16} + \frac{1-e^{-2s}}{s(s^2+16)}$$

$$Y(s) = \boxed{\frac{(1-3e^{-2s})}{s(s^2+16)} + \frac{(3s+6)}{s^2+16}} \quad \times \quad \boxed{\frac{3s+6}{s^2+16} + \frac{1-e^{-2s}}{s(s^2+16)}}$$

$$y(t) = \boxed{3\cos(4t) + \frac{3}{2}\sin(4t) + \frac{1}{16}\left(1 - \frac{1}{4}\sin(4t)\right) - \frac{1}{16}\left(-u_2 - \frac{1}{4}\sin(4(t-2))\right)} \quad \times \quad \boxed{3\cos(4t) + \frac{3}{2}\sin(4t) + \frac{1}{16}(1-\cos(4t)) - \frac{u_2}{16}(1-\cos(4(t-2)))}$$

$$y = 3\cdot\cos(4t) + \frac{3}{2}\cdot\sin(4t) + \frac{1}{16}(1-\cos(4t)) - \frac{u_2}{16}(1-\cos(4(t-2)))$$

$$\frac{A}{s} + \frac{Bs}{s^2+16} \Rightarrow \frac{1}{16}(1-\cos(4t))$$

when $\mathcal{L}\{e^{-2s}\} \Rightarrow -\frac{u_2}{16}(1-\cos(4(t-2)))$

$$\begin{cases} A+B=0 \\ 16A=1 \end{cases}$$

$$\therefore A = \frac{1}{16}, \; B = -\frac{1}{16}$$

17. 6/6 points | Previous Answers

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 16\mathcal{L}\{y\} = e^{-2s}\mathcal{L}\{e^{-3(t-2)}\}$$

Find the Laplace transform $Y(s)$ of the solution of the given initial value problem. Then invert to find $y(t)$. Write u_c for the Heaviside function that turns on at c, not $u_c(t)$.

$$\mathcal{L}\{y\} = \frac{e^{-2s-6}}{(s+3)(s^2+16)}$$

$$y'' + 16y = e^{-3t}u_2$$

$$y(0) = 0 \quad y'(0) = 0$$

$$Y(s) = \boxed{\frac{e^{-2s-6}}{(s+3)(s^2+16)}} \checkmark$$

$$y(t) = \boxed{\frac{e^{-6}}{25}u_2\left(e^{-3(t-2)} - \cos\left(4(t-2) + \frac{3}{4}\sin(4(t-2))\right)\right)} \checkmark$$

$$f(t) = u_2(t)\sin 3t$$

$$F(s) = e^{-2s}\mathcal{L}\boxed{\sin(3(t+2))} \checkmark$$

$$F(s) = \boxed{e^{-2s}\left(\cos(6)\cdot\frac{3}{s^2+9} + \sin(6)\cdot\frac{s}{s^2+9}\right)} \checkmark$$

$$\mathcal{L}\{\sin(3(t+2))\} = \mathcal{L}\{\sin 3t \cdot \cos 6 + \sin 6 \cdot \cos 3t\}$$

$$= \cos 6 \cdot \frac{3}{s^2+9} + \sin 6 \cdot \frac{s}{s^2+9}$$

Discontinuous Laplace

Friday, November 30, 2018 3:29 PM

- Warm-up:

(a) $\mathcal{L}\{u_{\frac{n-1}{n}}(t)\cdot\frac{n}{2} - u_{\frac{n+1}{n}}(t)\cdot\frac{n}{2}\}$, n is constant ≥ 1. $f_n(t)$

note that n is constant

(b) Take the limit at $n\to\infty$

(a) $= \frac{1}{s}\left(e^{\frac{1-n}{n}s}\cdot\frac{n}{2} - e^{\frac{-1-n}{n}s}\frac{n}{2}\right)$

$= \dfrac{\frac{n}{2s}\left(e^{\frac{s}{n}} - e^{-\frac{s}{n}}\right)}{e^{s}}$

$f_1(t) = \begin{cases} \frac{1}{2}, & 0\leq t<2 \\ 0, & t \end{cases}$

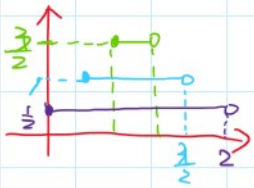
$f_2(t) = \begin{cases} \frac{2}{2}, & \frac{1}{2}\leq t<\frac{3}{2} \\ 0, & t \geq \frac{3}{2} \end{cases}$

(b) assume $x = \frac{s}{n}$, $n = \frac{s}{x}$

$\frac{1}{x}\cdot\frac{e^{x}-e^{-x}}{2}\cdot\frac{1}{e^{s}}$ $n\to+\infty$, $x\to 0$ $\int_0^\infty f_n(t)dt = 1$

$\lim_{x\to 0}\frac{\sinh(x)}{x}\cdot\frac{1}{e^{s}} = \frac{\cosh(x)}{1}\cdot\frac{1}{e^{s}} = \cosh(x)\cdot\frac{1}{e^{s}}\Big|_{x\to 0}$

(I'm so smart XD) $= \frac{e^{0}+e^{0}}{2}\cdot e^{-s} = e^{-s}$



- Impulse at $t=1$

$\delta_1(t) = \lim_{n\to\infty}\left(u_{\frac{n-1}{n}}(t)\cdot\frac{n}{2} - u_{\frac{n+1}{n}}(t)\cdot\frac{n}{2}\right)$

Properties: $\mathcal{L}\{\delta_1(t)\} = e^{-s}$

$\delta_1(t) = 0,\ t\neq 1$

$\int_{1-\varepsilon}^{1+\varepsilon}\delta_1(t)\,dt = 1$

- $y'' + y = \delta_1(t)$; $y(0) = 0$, $y'(0) = 0$

$s^2\mathcal{L}\{y\} + 0 + 0 + \mathcal{L}\{y\} = e^{-s}$

$\mathcal{L}\{y\} = \dfrac{e^{-s}}{s^2+1}$

$y = u_1(t)\cdot\sin(t-1)$