Rings and Modules of Fractions

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Exercise 1

Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Solution: If there exists $s \in S$ such that $sM = 0$, then $\forall m \in M$, $m/t = 0$ for every $t \in S$ since $sm = 0$. Hence $S^{-1}M = 0$.

Suppose that $S^{-1}M = 0$. Let $\{x_1, \cdots, x_n\}$ be a generating set for M. Then $x_i/1 = 0$ for every i, so there exist $s_1, \cdots, s_n$ in S such that $s_i x_i = 0$ for every i, and $s = s_1 s_2 \cdots s_n \in S$ satisfies $sM = 0$.

Exercise 2

Let I be an ideal of a ring A, and $S = 1 + I$. Show that $S^{-1}I$ is contained in the Jacobson radical of $S^{-1}A$.

Use this result and Nakayama's lemma to give a proof of (2.5) which does not depend on determinants.

Solution: We shall prove that for all $x \in S^{-1}I$ and $y \in S^{-1}A$, $1 - xy$ is a unit. xy is of the form $a/(1+b)$ where $a, b \in I$, so $1 - xy = \dfrac{1+(b-a)}{1+b}$ and $b - a \in I$. This is a unit in A since $\dfrac{1+b}{1+(b-a)}$ is its inverse.

(2.5) states that if $M = IM$, then $\exists x = 1(mod\ I)$, with $xM = 0$. If $M = IM$ then $S^{-1}M = (S^{-1}I)(S^{-1}M)$, and by Nakayama's lemma, $S^{-1}M = 0$. By Exercise 1, there is $s \in S = 1 + I$, s.t. $xM = 0$.

Exercise 3

Let A be a ring, let S and T be two multiplicatively closed subsets of A, and let U be the image of T in $S^{-1}A$. Show that the rings $(ST)^{-1}A$ and $U^{-1}(S^{-1}A)$ are isomorphic.

Solution: Define $\phi : (ST)^{-1}A \to U^{-1}(S^{-1}A)$ by $\dfrac{a}{st} \to \dfrac{a/s}{t/1}$. This is well-defined: if $\dfrac{a}{st} = \dfrac{a'}{s't'}$ in $(ST)^{-1}A$, then $u(sta' - s't'a) = 0$ for some $u \in S$. Hence $\dfrac{a't}{s'} = \dfrac{a't}{s}$ in $S^{-1}A$, and $\dfrac{a'/s'}{t'/1} = \dfrac{a/s}{t/1}$ in $U^{-1}(S^{-1}A)$. verifying that ϕ is a bijective ring homomorphism completes the proof, but is omitted.

Exercise 4

Let $f : A \to B$ be a homomorphism of rings and let S a multiplicatively closed subsets of A. Let $T = f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.

Solution: $\phi : S^{-1}B \to T^{-1}B$ which sends $\dfrac{b}{s} \to \dfrac{b}{f(s)}$ gives the isomorphism.

Exercise 5

Let A be a ring. Suppose that, for each prime ideal P, the local ring A_P has no nilpotent element $\neq 0$. Show that A has no nilpotent element $\neq 0$. If each A_P is an integral domain, is A necessarily an integral domain?

Solution: If there is $a \in A$, s.t. $a^n = 0$ but $a \neq 0$ for some n, then $\forall P$, $a/1 \in A_P$ is nilpotent, so it must be 0, i.e. $\exists t \in P$ with $ta = 0$. Let $\mathrm{Ann}(a)$ be the set of all $t \in A$ with $ta = 0$, then $\mathrm{Ann}(a) \neq (1)$. We may find a maximal ideal M with $\mathrm{Ann}(a) \subset M$. By the first sentence of the solution, in A_M, $a/1 = 0$, but this is absurd since $\mathrm{Ann}(a) \cap (A - M) = \emptyset$.

Suppose that each A_P is an integral domain, the same technique does not work to show that A is an integral domain, since if $a \in A$ is a zero-divisor, we can not claim that $a/1 \in A_P$ is 0 for all P.

Exercise 6

Let A be a ring $\neq 0$ and let Σ be the set of all multiplicatively closed subsets S of A such that $0 \notin S$. Show that Σ has maximal elements, and that $S \in \Sigma$ is maximal if and only if $A - S$ is a minimal prime ideal of A.

Solution: $\{1\} \in \Sigma$ so Σ is not empty. If $S_1 \subset S_2 \subset \cdots \subset S_k \subset \cdots$, then $\cup_{k=1}^{\infty} S_k$ is an upper bound of this chain. We apply Zorn's Lemma to see that Σ has maximal elements.

Suppose that $S \in \Sigma$ is maximal, and let $P = A - S$. Note that the smallest multiplicatively closed subset which contains a and S is $\{a^n s : s \in S\}$. If $a \in P$, then $a^n s = 0$ for some integers n and $s \in S$, and vice versa since $0 \notin S$, and S is a multiplicatively closed subset. If $a, b \in P$, then $a^n s = 0$ and $b^m t = 0$. Since a^n or b^m divides each terms of $(a-b)^{n+m}$, we have that $st(a-b)^{n+m} = 0$ so $a - b \in P$. For every $x \in A$, $s(ax)^m = (sa^m)x^m = 0$, so $ax \in P$. P is an ideal, and hence a prime ideal. If it's not minimal, then we can find $P' \subset P$ strictly, and $S \subset A - P' \in \Sigma$, which is a contradiction. Hence if $S \in \Sigma$ is maximal, then $A - S$ is a minimal prime ideal.

On the other hand, if P is a minimal prime ideal, then $A - P \in \Sigma$. If $T \supset S$ is maximal, then $A - T$ is a minimal prime ideal, which have to be P, so $T = S$ is maximal.

Exercise 7

A multiplicatively closed subset S of a ring A is said to be *saturated* if $xy \in S \;\Leftrightarrow\; x \in S$ and $y \in S$. Prove that

i) S is saturated $\Leftrightarrow$ $A - S$ is a union of prime ideals.

ii) If S is any multiplicatively closed subset of A, there is a unique smallest saturated multiplicative closed subset $\bar{S}$ containing S, and that $\bar{S}$ is the complement in A of the union of the prime ideals which do not meet S ($\bar{S}$ is called the saturation of S.)

If $S = 1 + I$, where I is an ideal of A, find $\bar{S}$.

Solution: i) Suppose that $A - S$ is a union of prime ideals, and $x \in S$, $y \in S$ implies $xy \in S$, and $xy \in S$ but $x \notin S$, then $xy \in P$ for some $P \subset A - S$, which is a contradiction.

Suppose that S is saturated. For each $a \in A - S$, we shall construct a prime ideal P_a which does not intersect S. This completes the proof. Since S is saturated, $(a) \cap S = \emptyset$. Let Σ_a be the set of all ideals which contains a but do not intersect S. $(a) \in \Sigma_a$, so Σ_a is not empty. If $I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$, then $\cup_{k=1}^{\infty} I_k$ is an upper bound of this chain. We may apply Zorn's Lemma to show that Σ_a has maximal element. We denote it by P. If $xy \in P$ but $x \notin P$, $y \notin P$, then one of x, y must lay in $A - S$, since S is multiplicatively closed. Suppose that $x \in A - S$, then $P + (x) \in \Sigma_a$ and $P \subset P + (x)$, which is a contradiction. Hence P is prime.

ii) There is nothing to prove in the first statement. If $P \cap (1 + I) \neq \emptyset$, we have $P + I = (1)$, and vice versa. Hence P does not meet S if and only if $P + I \subset (1)$. For each such P, we can choose a maximal ideal M, s.t. $P + I \subset M$. Every M is a prime ideal. In fact, $\cup P = \cup M$, where $I \subset M$.

Hence $\bar{S} = A - \cup_{I \subset M} M$.

Exercise 8

Let S, T be multiplicatively closed subsets of A, such that $S \subset T$. Let $\phi : S^{-1}A \to T^{-1}A$ be the homomorphism which maps each $a/s \in S^{-1}A$ to a/s considered as an element of $T^{-1}A$. Show that the following statements are equivalent:

i) ϕ is bijective.

ii) For each $t \in T$, $t/1$ is a unit in $S^{-1}A$.

iii) For each $t \in T$ there exists $x \in A$ such that $xt \in S$.

iv) T is contained in the saturation S (Exercise 7).

v) Every prime ideal which meets T also meets S.

Solution: i) $\Rightarrow$ ii) Suppose that ϕ is bijective. Since f is surjective, for every t in T, there is $x \in S^{-1}A$, s.t. $\phi(x) = 1/t$, then $\phi(x)\phi(t/1) = 1 = \phi(x(t/1))$. Since f is injective, this implies that $x(t/1) = 1$ and x is inverse of $t/1$.

ii) $\Rightarrow$ iii) $t/1$ is a unit in $S^{-1}A$, then there is a a/s s.t. $(t/1)(a/s) = at/s = 1/1$, which yields $s^{'}(at - s) = 0$ for some $s^{'} \in S$. This is $s^{'}at = s^{'}s \in S$. Let $x = s^{'}a \in A$ will suffice.

iii) $\Rightarrow$ v) If $t \in P \cap T$ for prime ideal P, then there exits $x \in A$ such that $xt \in S$ by assumption, and this implies that $xt \in S \cap P$.

iv) $\Leftrightarrow$ v) Trivial by Exercise 7 ii).

v) $\Rightarrow$ ii) If $t/1$ is not a unit in $S^{-1}A$, then for every $a \in A$, $at \notin S$, so $(t) \cap S = \emptyset$. Let Σ_t be the set of all ideals, which contain t but do not meet S. It's not empty since $(t) \in \Sigma_t$. If $I_1 \subset I_2 \subset \cdots \subset I_k \subset \cdots$, $\cup_{k=1}^{\infty} I_k$ is an upper bound of this chain. We apply Zorn's Lemma to see that Σ_t has a maximal element, and denote it by P. If $xy \in P$ but $x \notin P$, $y \notin P$, then one of x, y must in $A - S$ since S is multiplicatively closed. Suppose $x \in A - S$, then $P \subset P + (x) \in \Sigma_t$, which is a contradiction. Hence P is a prime ideal which meets T but does not meet S.

Above we show that ii)-v) are equivalent and i) implies them.

iii) $\Rightarrow$ i) One verify that ϕ is indeed a well-defined ring homomorphism. If $\phi(a/s) = a/s = 0$, then there exists $t \in T$, such that $at = 0$. By iii), there is an $x \in A$ such that $xt \in S$, so $xta = 0$ and $a/s = 0$ in $S^{-1}A$. This proves that ϕ is injective. For every b/t in $T^{-1}A$, we can find $x \in A$ such that $xt \in S$, then $\phi(\dfrac{xa}{xt}) = a/t$. This proves that ϕ is surjective. Hence ϕ is bijective.

Exercise 9

The set S_0 of all non-zero-divisors in A is a saturated multiplicatively closed subset of A. Hence the set D of zero-divisors in A is a union of prime ideals (see Chapter 1, Exercise 14). Show that every minimal prime ideal of A is contained in D.

The ring $S_0^{-1}A$ is the largest multiplicatively closed subset of A. Prove that

i) S_0 is the largest multiplicatively closed subset of A for which the homomorphism $A \to S_0^{-1}A$ is injective.

ii) Every element in $S_0^{-1}A$ is either a zero-divisor or a unit.

iii) Every ring in which every non-unit is a zero-divisor is equal to its total ring of fractions (i.e., $A \to S_0^{-1}A$ is bijective).

Solution: Let P be a minimal prime ideal. To show that $P \subset D$, it is sufficient to show that $S_0 = A - D \subset A - P$. By Exercise 6, $A - P$ is a maximal multiplicatively closed subset S which does not contain 0. Hence it is sufficient to show that S_0 is contained in every S. If $s_0 \in S_0 - S$, then $\{s_0^n s : s \in S,\ n \in \mathbb{N}\}$ is a multiplicative closed subset of A which contains S and do not contain 0, which contradicts to the fact that S is maximal.

i) $A \to S_0^{-1}A$ is injective, since S_0 does not contain any zero divisor. If $S_0 \subset S$, where S is a multiplicatively closed subset, then S must contain a zero-divisor, say s with $sa = 0$. Then the image of a is 0 in $S^{-1}A$, so the map $A \to S^{-1}A$ is not injective.

ii) If $a \in A$ is a zero-divisor with $ab = 0$, then $a/s \in S_0^{-1}A$ is a zero-divisor for every $s \in S_0$, since $(a/s)(b/1) = ab/s = 0$. If $a \in A$ is not a zero-divisor, then it is a unit in $S_0^{-1}A$ since $a \in S_0$.

iii) If A is a ring in which every non-unit is a zero-divisor, then S_0 consists of all the units in A, so $S_0^{-1}A$ is isomorphic to A.

Exercise 10

Let A be a ring.

i) If A is absolutely flat (Chapter 2, Exercise 27) and S is any multiplicatively closed subset of A, then $S^{-1}A$ is absolutely flat.

ii) A is absolutely flat $\Leftrightarrow$ $A_{\mathfrak{m}}$ is a field for each maximal ideal $\mathfrak{m}$.

Solution: i) If $M^{'} \to M \to M^{''}$ is an exact sequence of $S^{-1}A$-modules and N a $S^{-1}A$-module, we can regard these modules as A-modules, then $M^{'} \otimes N \to M \otimes N \to M^{''} \otimes N$ is exact as A-modules sequence since A is absolutely flat. The functor S^{-1} is exact, so $S^{-1}(M^{'} \otimes N) \to S^{-1}(M \otimes N) \to S^{-1}(M^{''} \otimes N)$ is exact as $S^{-1}A$-modules. The isomorphism of $S^{-1}A$-modules exact sequence:

$$\begin{array}{ccccc} S^{-1}(M^{'} \otimes N) & \to & S^{-1}(M \otimes N) & \to & S^{-1}(M^{''} \otimes N) \\ \downarrow & & \downarrow & & \downarrow \\ M^{'} \otimes N & \to & M \otimes N & \to & M^{''} \otimes N \end{array}$$

implies that N is flat as $S^{-1}A$-module, so $S^{-1}A$ is absolutely flat.

ii) $\Rightarrow$ If A is absolutely flat, by i), $A_{\mathfrak{m}}$ is absolutely flat. By Chapter 2, Exercise 28, being local, A_m is a field.

$\Leftarrow$ If M is an A-module, then $M_{\mathfrak{m}}$ is an $A_{\mathfrak{m}}$-module. Since $A_{\mathfrak{m}}$ is a field, $M_{\mathfrak{m}} = \bigotimes_J A_{\mathfrak{m}}^j$, i.e. a vector space over a field is the direct sum of copies of this field indexed by a set J. The direct sum commutes with tensor product, so $M_{\mathfrak{m}}$ is a flat $A_{\mathfrak{m}}$-module since $A_{\mathfrak{m}}$ is a flat $A_{\mathfrak{m}}$-module. This holds for every maximal ideal $\mathfrak{m}$ of A, so M is flat by (3.10). Hence A is absolutely flat. Note that the process also implies that every field is absolutely flat.

Exercise 11

Let A be a ring. Prove that the following are equivalent:

i) A/R is absolutely flat (R being the nilradical of A).

ii) Every prime ideal of A is maximal.

iii) Spec(A) is a T_1-space (i.e., every subset consisting of a single point is closed).

iv) Spec(A) is Hausdorff.

If these conditions are satisfied, show that Spec(A) is compact and totally disconnected (i.e. the only connected subsets of Spec(A) are those consisting of a single point).

Solution: i) $\Leftrightarrow$ ii) Every prime ideal of A is maximal $\Leftrightarrow$ For every maximal ideal $\mathfrak{m}$ of A, the only prime ideal of A/R contained in $\mathfrak{m}/R$ is (0) $\overset{(3.11.iv)}{\Longleftrightarrow}$ For every maximal ideal $\tilde{\mathfrak{m}}$ of A/R, the only prime ideal of $(A/R)_{\tilde{\mathfrak{m}}}$ is (0) $\Leftrightarrow$ $(A/R)_{\tilde{\mathfrak{m}}}$ is a field for each maximal ideal $\tilde{\mathfrak{m}}$ of A/R. $\Leftrightarrow$ A/R is absolutely flat by Exercise 10.

i),ii) $\Rightarrow$ iv) By Chapter 1, Exercise 21, Spec(A) $\cong$ Spec(A/R), so it is sufficient to show that Spec(A/R) is Hausdorff. If $\mathfrak{p}_x$ and $\mathfrak{p}_y$ is two distinct prime ideals in Spec(A/R), they are maximal by ii). There are $a \in \mathfrak{p}_x$ and $b \in \mathfrak{p}_y$ such that $(a) + (b) = (1)$. By Chapter 2, Exercise 27, there are idempotent e and f such that $(e) = (a)$ and $(f) = (b)$. Let $g = f(1-e)$, then $eg = 0$, and $g + fe = f$, so $(e, g) = (e) + (f) = (1)$. Now $e \in (a) \subset \mathfrak{p}_x$, we have that $g \notin \mathfrak{p}_x$, which is, $\mathfrak{p}_x \in X_g$. Similarly, $\mathfrak{p}_y \in X_e$. $X_g \cap X_e = X_0 = \emptyset$, so Spec($A/R$) is Hausdorff.

iv) $\Rightarrow$ iii) Every Hausdorff (T_2) space is T_1. If $x \in X$, for every $y \neq x$, find an open set U_y such that $y \in U_y$ but $x \notin U_y$, then $\{x\} = X - \cup_{y \neq x} U_y$ is closed.

iii) $\Rightarrow$ ii) For every prime ideal $\mathfrak{p}$, $\{\mathfrak{p}\}$ is closed, so $\{\mathfrak{p}\} = V(E)$ for some $E \subset A$ strictly, but E must be contained in some maximal ideals, so $\mathfrak{p}$ is maximal.

X is compact by Exercise 17 v), Chapter 1. To show that X is totally disconnected, we show that $S = \{\mathfrak{p}_x, \mathfrak{p}_y\}$ is not connected for every $\mathfrak{p}_x$ and $\mathfrak{p}_y$. In the proof of i),ii) $\Rightarrow$ iii), $X_f \cup X_g = X_1 = X$. $S \cap X_f$ and $S \cap X_g$ are two disjoint open subsets of S, with $S = (S \cap X_f) \cup (S \cap X_g)$.

Exercise 12

Let A be an integral domain and M an A-module. An element $x \in M$ is a *torsion element* of M if Ann$(x) \neq 0$, that is if x is killed by some non-zero element of A. Show that the torsion elements of M forms a submodule of M. This submodule is called the torsion submodule of M and is denoted by $T(M)$. If $T(M) = 0$, the module M is said to be torsion-free. Show that

i) If M is any A-module, then $M/T(M)$ is torsion-free.

ii) If $f : M \to N$ is a module homomorphism, then $f(T(M)) \subset T(N)$.

iii) If $0 \to M^{'} \to M \to M^{''}$ is an exact sequence, then the sequence $0 \to T(M^{'}) \to T(M) \to T(M^{''})$ is exact.

iv) If M is any A-module, then $T(M)$ is the kernel of the mapping $x \to 1 \otimes x$ of M to $K \otimes_A M$, where K is the field of fractions of A.

Solution: If $x, y \in M$ are torsion elements, with $ax = 0$, $by = 0$ for nonzero $a, b \in A$, since A is an integral domain, $ab \neq 0$, $ab(x + y) = 0$, so $a + b$ is a torsion element. For every $c \in A$, $a(cx) = c(ax) = 0$, so cx is a torsion element. These show that $T(M)$ is a submodule.

i) For every $\bar{x} \in M$ with $a\bar{x} = \overline{ax} = \bar{0}$ for some $a \in A$, $ax \in T(M)$ so $x \in T(M)$, which means that $\bar{x} = \bar{0}$.

ii) $\forall x \in T(M)$, there exists nonzero $a \in A$ with $ax = 0$, so $af(x) = f(ax) = f(0) = 0$ and $f(x) \in T(N)$.

iii) Denote the second and third arrows by f and g, respectively. (Abusing notations, f and g are used to represent the arrows on the original modules and their restriction to the torsion submodules.) Only the fact that $im\ f = ker\ g$ is new. $g \circ f = 0$ trivially. If $y \in ker\ g$, then by the exactness of $0 \to T(M^{'}) \to T(M) \to T(M^{''})$, there exists a $x \in M^{'}$, such that $f(x) = y$. Since y is a torsion element, $ay = 0$ for some nonzero $a \in A$, then $f(ax) = af(x) = ay = 0$, and since f is injective, $ax = 0$, so $x \in T(M^{'})$.

iv) $K \otimes_A M \cong S^{-1}M$ where $S = A - \{0\}$, by $a/s \otimes m \to am/s$. Compose the maps $M \to K \otimes_A M \to S^{-1}M$, then the kernel of the first map is the kernel of the composition of two maps, which takes m to $m/1$, and $m/1 = 0$ if and only if $sm = 0$ for some $s \in S$.

Exercise 13

Let S be a multiplicatively closed subset of an integral domain A. In the notation of Exercise 12, show that $T(S^{-1}M) = S^{-1}(TM)$. Deduce that the following are equivalent:

i) M is torsion-free.

ii) $M_{\mathfrak{p}}$ is torsion-free for all prime ideals $\mathfrak{p}$.

iii) $M_{\mathfrak{m}}$ is torsion-free for all maximal ideals $\mathfrak{m}$.

Solution: Once we prove that $T(S^{-1}M) = S^{-1}(TM)$, the other statement is obvious by (3.8).

If $m/s \in T(S^{-1}M)$, then there is a nonzero $a \in A$, such that $a(m/s) = (am)/s = 0/1$. Therefore there exists $t \in S$, such that $t(am) = (at)m = 0$, which implies that $m \in TM$, so $m/s \in S^{-1}(TM)$. All the directions can be reversed, so $T(S^{-1}M) = S^{-1}(TM)$.

Exercise 14

Let M be an A-module and $\mathfrak{a}$ an ideal of A. Suppose that $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \supset \mathfrak{a}$. Prove that $M = \mathfrak{a}M$.

Solution: View $M/\mathfrak{a}M$ as a $A/\mathfrak{a}$-module. We need only to show that $(M/\mathfrak{a}M)_{\bar{\mathfrak{m}}} = 0$ for all $\bar{\mathfrak{m}}$ maximal of $A/\mathfrak{a}$, and in fact $\bar{\mathfrak{m}} = \pi(\mathfrak{m})$ for $\mathfrak{m}$ maximal and contains $\mathfrak{a}$ in A. We show that $(M/\mathfrak{a}M)_{\bar{\mathfrak{m}}} \cong M_{\mathfrak{m}}/(\mathfrak{a}M)_{\mathfrak{m}}$ as $(A/\mathfrak{a})_{\bar{\mathfrak{m}}}$-modules. $M_{\mathfrak{m}}/(\mathfrak{a}M)_{\mathfrak{m}}$ may be viewed as $(A/\mathfrak{a})_{\mathfrak{m}}$-module with the scalar multiplication defined by $\frac{\bar{a}}{\bar{t}}(\overline{m/s}) = \overline{am/st}$. Define $\phi(\frac{\bar{m}}{\bar{s}}) = \overline{m/s}$ from $(M/\mathfrak{a}M)_{\bar{\mathfrak{m}}}$ to $M_{\mathfrak{m}}/(\mathfrak{a}M)_{\mathfrak{m}}$.

This is well-defined. If $\frac{\bar{m}_1}{\bar{s}_1} = \frac{\bar{m}_2}{\bar{s}_2}$, then there exists $t \in A - \mathfrak{m}$, such that $t(m_1 s_2 - m_2 s_1) \in \mathfrak{a}M$, so $\frac{t(m_1 s_2 - m_2 s_1)}{t s_1 s_2} = \frac{m_1 s_2 - m_2 s_1}{s_1 s_2} \in (\mathfrak{a}M)_{\mathfrak{m}}$, and add a bar on this implies that $\overline{m_1/s_1} = \overline{m_2/s_2}$.

This is a $(A/\mathfrak{a})_{\mathfrak{m}}$-modules isomorphism. This is not hard and omitted.

There are still some words to say. We know that $M_{\mathfrak{m}} = 0$ as an $A_{\mathfrak{m}}$-module, but does this still hold if we change the base ring? The answer is yes, since this only depends on the abelian group structure on $M_{\mathfrak{m}}$. As the same, the additive identity, i.e., 0, and the additive rules, do not change at all. Therefore we know that $M/\mathfrak{a}M = 0$, which is $M = \mathfrak{a}M$.

Exercise 15

Let A be a ring, and let F be the A-module A^n. Show that every set of n generators of F is a basis of F.

Deduce that every set of generators of F has at least n elements.

Solution: Let $x_1, x_2, ..., x_n$ be a set of generators and $e_1, e_2, ..., e_n$ the canonical basis of F. Define $\phi : F \to F$ by $\phi(e_i) = x_i$. Then ϕ is surjective. By (3.9) we may assume that A is a local ring. Let N be the kernel of ϕ and let $k = A/\mathfrak{m}$ be the residue field of A. Since k is a flat A-module, the exact sequence $0 \to N \to F \to F \to 0$ gives an exact sequence $0 \to k \otimes N \to k \otimes F \xrightarrow{1\otimes\phi} k \otimes F \to 0$. Now $k \otimes F = k^n$ is an n-dimensional vector space over k; $1 \otimes \phi$ is surjective, hence bijective, hence $k \otimes N = 0$. Also N is finitely generated, by Chapter 2, Exercise 12, hence $N = 0$ by Nakayama's lemma. Hence ϕ is an isomorphism.

By this process, if the number of a set of elements is less than n, then $N \otimes k \neq 0$ and $N \neq 0$, so this set can not be a set of generators.

Exercise 16

Let B be a flat A-algebra. Then the following conditions are equivalent:

i) $\mathfrak{a}^{ec} = \mathfrak{a}$ for all ideals $\mathfrak{a}$ of A.

ii) $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

iii) For every maximal ideal $\mathfrak{m}$ of A we have $\mathfrak{m}^e \neq (1)$.

iv) If M is any non-zero A-module, then $M_B \neq 0$.

v) For every A-module M, the mapping $x \to x \otimes 1$ of M to M_B is injective.

B is said to be *faithfully flat* over A.

Solution: Denote $A \to B$ by f.

i) $\Rightarrow$ ii) For each $\mathfrak{p}$ prime in A, $\mathfrak{p}^{ec} = \mathfrak{p}$, so it is the contraction of a prime ideal of B by (3.16), so f^* is surjective.

ii) $\Rightarrow$ iii) There exists $\mathfrak{q} \in \mathrm{Spec}(B)$, such that $f^*(\mathfrak{q}) = \mathfrak{q}^c = \mathfrak{m}$, and $\mathfrak{m} = \mathfrak{q}^{ce} \subset \mathfrak{q} \neq (1)$.

iii) $\Rightarrow$ iv) Choose $x \neq 0$ in M and let $M' = Ax$. Since $0 \to M' \to M$ is exact and B is flat over A, we have $0 \to M' \otimes B \to M \otimes B$ is exact. It is sufficient to show that $M'_B \neq 0$. The maps $A \to Ax$ is surjective, and $x \neq 0$, so $M' = Ax \cong A/\mathfrak{a}$ for some ideal $\mathfrak{a} \neq (1)$ in A.

We claim that $M'_B \cong B/\mathfrak{a}^e$ as A-modules by $\phi(\bar{a} \otimes b) = \overline{bf(a)}$ and extends by linearity, which insures that ϕ is an A-modules homomorphism, if well-defined. It is well-defined, since if $\bar{a}_1 = \bar{a}_2$, then $a_1 - a_2 \in \mathfrak{a}$, so $f(a_1 - a_2) \in \mathfrak{a}^e$ and $\overline{bf(a_1) - bf(a_2)} = \overline{bf(a_1 - a_2)} = \bar{0}$. It is surjective since $\phi(\bar{1} \otimes b) = \bar{b}$. The only thing left to show is that ϕ is injective. If $\phi(\sum \bar{a}_i \otimes b_i) = \overline{\sum b_i f(a_i)} = \bar{0}$ then

$\sum b_i f(a_i) \in \mathfrak{a}^e$, so $\sum b_i f(a_i) = \sum b_j' f(x_j)$ for $x_j \in \mathfrak{a}^e$. This implies that $\sum \bar{a}_i \otimes b_i = \bar{1} \otimes \sum b_i f(a_i) = \bar{1} \otimes \sum b_j' f(x_i) = \sum \bar{x}_j \otimes b_j = \sum \bar{0} \otimes b_j' = 0$. Here we have proved that ϕ is injective. Now $\mathfrak{a} \subset \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ and $\mathfrak{a}^e \subset \mathfrak{m}^e \neq (1)$, so $M_B' \cong B/\mathfrak{a}^e \neq 0$.

iv) $\Rightarrow$ v) Let K be the kernel of $M \to M_B$. Since B is flat over A, the sequence $0 \to K_B \to M_B \to (M_B)_B$ is exact. By Chapter 2, Exercise 13, with $N = M_B$, the mapping $M_B \to (M_B)_B$ is injective, hence $K_B = 0$ and therefore $K = 0$ by vi).

v) $\Rightarrow$ i) Take $M = A/\mathfrak{a}$, then $A/\mathfrak{a} \to (A/\mathfrak{a}) \otimes B \cong B/\mathfrak{a}^e$ is injective, so $\mathfrak{a}^{ec} \subset \mathfrak{a}$. Since $\mathfrak{a} \subset \mathfrak{a}^{ec}$ always holds, we have $\mathfrak{a} = \mathfrak{a}^{ec}$.

Exercise 17

Let $A \xrightarrow{f} B \xrightarrow{g} C$ be ring homomorphism. If $g \circ f$ is flat and g is faithfully flat, then f is flat.

Solution: If M, N are two A-modules with $M \to N$ injective. Since C is flat over A, $M \otimes_A C \to N \otimes_A C$ is injective. We have that $M \otimes_A C \cong M \otimes_A (B \otimes_B C) \cong (M_B)_C$, and similarly, $N \otimes C \cong (N_B)_C$. C is faithfully flat over B, so $M_B \to (M_B)_C$ is injective, and so is $N_B \to (N_B)_C$. We have the following commutative diagram:

$$\begin{array}{ccc} M_B & \to & N_B \\ \downarrow & & \downarrow \\ (M_B)_C & \to & (N_B)_C \end{array}$$

The maps $M_B \to (M_B)_C \to (N_B)_C$ and $N_B \to (N_B)_C$ are both injective, so $M_B \to (N_B)_C$ is injective.

Exercise 18

Let $f : A \to B$ be a flat homomorphism of rings, let $\mathfrak{q}$ be a prime ideal of B and let $\mathfrak{p} = \mathfrak{q}^c$. Then $f^* : \mathrm{Spec}(B_\mathfrak{q}) \to \mathrm{Spec}(A_\mathfrak{p})$ is surjective.

Solution: For $B_\mathfrak{p}$ is flat over $A_\mathfrak{p}$ by (3.10), and $B_\mathfrak{q}$ is a local ring of $B_\mathfrak{p}$, hence is flat over $B_\mathfrak{p}$. Hence $B_\mathfrak{q}$ is flat over $A_\mathfrak{p}$, and satisfy iii) of Exercise 16. Hence ii) of Exercise 16 holds.

Exercise 19

Let A be a ring, M an A-module. The *support* of M is defined to be the set $\mathrm{Supp}(M)$ of prime ideal $\mathfrak{p}$ of A such that $M_\mathfrak{p} \neq 0$. Prove the following results:

i) $M \neq 0 \Leftrightarrow \mathrm{Supp}(M) \neq \emptyset$.

ii) $V(\mathfrak{a}) = \mathrm{Supp}(A/\mathfrak{a})$.

iii) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$.

iv) If $M = \sum M_i$, then $\mathrm{Supp}(M) = \cup\, \mathrm{Supp}(M_i)$.

v) If M is finitely generated, then $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$ (and is therefore a closed subset of $\mathrm{Spec}(A)$).

vi) If M, N are finitely generated, then $\mathrm{Supp}(M \otimes_A N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$.

vii) If M is finitely generated and $\mathfrak{a}$ is an ideal of A, then $\mathrm{Supp}(M/\mathfrak{a}M) = V(\mathfrak{a} + \mathrm{Ann}(M))$.

viii) If $f : A \to B$ is a ring homomorphism and M is finitely generated A-module, then $\mathrm{Supp}(B \otimes_A M) = f^{*-1}(\mathrm{Supp}(M))$.

Solution: i) $M = 0$ if and only if $M_\mathfrak{p} = 0$ for every prime ideal $\mathfrak{p}$ of A by (3.8).

ii) If $\mathfrak{a} \subset \mathfrak{p}$, then there is no $s \in A - \mathfrak{p}$ such that $s \in \mathfrak{a}$, so $1/1 \neq 0$ in $(A/\mathfrak{a})_\mathfrak{p}$. All the directions can be reversed, so $V(\mathfrak{a}) = \mathrm{Supp}(A/\mathfrak{a})$.

iii) For every prime ideal $\mathfrak{p}$, $0 \to M_\mathfrak{p}' \to M_\mathfrak{p} \to M_\mathfrak{p}'' \to 0$ is exact, and if $M_\mathfrak{p} \neq 0$, then at least one of $M_\mathfrak{p}'$ and $M_\mathfrak{p}''$ is not 0 and vice versa.

iv) Trivial since $(\sum M_i)_\mathfrak{p} = \sum (M_i)_\mathfrak{p}$ by (3.4).

v) If $\mathrm{Ann}(M) \subset \mathfrak{p}$, then $M_\mathfrak{p} \neq 0$ and vice versa by Exercise 1.

vi) $(M \otimes_A N)_\mathfrak{p} \cong M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p}$ by (3.7). If $M_\mathfrak{p} \neq 0$ and $N_\mathfrak{p} \neq 0$, i.e., $\mathfrak{p} \in \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$, then $(M \otimes_A N)_\mathfrak{p} \neq 0$, i.e., $\mathfrak{p} \in \mathrm{Supp}(M \otimes_A N)$. If $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} \neq 0$, then none of $M_\mathfrak{p}$ and $N_\mathfrak{p}$ is 0 since they are finitely generated and by Chapter 2, Exercise 3.

vii) $M/\mathfrak{a}M \cong A/\mathfrak{a} \otimes_A M$, so by vi), $\mathrm{Supp}(M/\mathfrak{a}M) = \mathrm{Supp}(M) \cap \mathrm{Supp}(A/\mathfrak{a}) = V(\mathfrak{a}) \cap V(\mathrm{Ann}(M)) = V(\mathfrak{a} + \mathrm{Ann}(M))$.

viii) Let $\mathfrak{q} \in \mathrm{Spec}(B)$ and let $\mathfrak{p} = \mathfrak{q}^c$. We have that

$$(B \otimes_A M)_\mathfrak{q} \cong B_\mathfrak{q} \otimes_A M \cong (B_\mathfrak{q} \otimes_{A_\mathfrak{p}} A_\mathfrak{p}) \otimes_A M \cong B_\mathfrak{q} \otimes_{A_\mathfrak{p}} (A_\mathfrak{p} \otimes_A M) \cong B_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p},$$

so if $M_\mathfrak{p} = 0$, then $(B \otimes_A M)_\mathfrak{q} = 0$.

If $(B \otimes_A M)_\mathfrak{q} = 0$, then $B_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p} = 0$. $A_\mathfrak{p}$ is a local ring, by Chapter 2, Exercise 3, we have that $B_\mathfrak{q} = 0$ or $M_\mathfrak{p} = 0$. But $B_\mathfrak{q} \neq 0$, since it is a local ring.

Exercise 20

Let $f : A \to B$ be a ring homomorphism, $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ the associated mapping. Show that

i) Every prime ideal of A is a contracted ideal $\Leftrightarrow$ f^* is surjective.

ii) Every prime ideal of B is an extended ideal $\Rightarrow$ f^* is injective.

Is the converse of ii) true?

Solution: i) By definition.

ii) If every prime ideal of B is an extended ideal, then for $\mathfrak{q}_1$, $\mathfrak{q}_2$ two prime ideals of B, we have $\mathfrak{p}_1^e = \mathfrak{q}_1$ and $\mathfrak{p}_2^e = \mathfrak{q}_2$. If $f^*(\mathfrak{q}_1) = f^*(\mathfrak{q}_2)$, i.e. $\mathfrak{p}_1^{ec} = \mathfrak{p}_2^{ec}$, then $\mathfrak{p}_1^{ece} = \mathfrak{p}_1^e = \mathfrak{q}_1 = \mathfrak{p}_2^{ece} = \mathfrak{p}_2^e = \mathfrak{q}_2$ by (1.17). Hence f^* is injective. The converse is not generally true.

Exercise 21

i) Let A be a ring, S a multiplicatively closed subset of A, and $\phi : A \to S^{-1}A$ the canonical homomorphism. Show that $\phi^* : \mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ is a homeomorphism of $\mathrm{Spec}(S^{-1}A)$ onto its image in $X = \mathrm{Spec}(A)$. Let this image be denoted by $S^{-1}X$. In particular, if $f \in A$, the image of $\mathrm{Spec}(A_f)$ is the basic open set X_f (Chapter 1, Exercise 17).

ii) Let $f : A \to B$ be a ring homomorphism. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, and let $f^* : Y \to X$ be the mapping associated with f. Identifying $\mathrm{Spec}(S^{-1}A)$ with its canonical image $S^{-1}X$ in X, and $\mathrm{Spec}(S^{-1}B)$ ($= \mathrm{Spec}(f(S)^{-1}B)$) with its canonical image $S^{-1}Y$ in Y, show that $S^{-1}f^* : \mathrm{Spec}(S^{-1}B) \to \mathrm{Spec}(S^{-1}A)$ is the restriction of f^* to $S^{-1}Y$, and that $S^{-1}Y = f^{*-1}(S^{-1}X)$.

iii) Let $\mathfrak{a}$ be an ideal of A and let $\mathfrak{b} = \mathfrak{a}^e$ be its extension in B. Let $\bar{f} : A/\mathfrak{a} \to B/\mathfrak{b}$ be the homomorphism induced by f. If $\mathrm{Spec}(A/\mathfrak{a})$ is identified with its canonical image $V(\mathfrak{a})$ in X, and $\mathrm{Spec}(B/\mathfrak{b})$ with its image $V(\mathfrak{b})$ in Y, show that $\bar{f}^*$ is restriction of f^* to $V(\mathfrak{b})$.

iv) Let $\mathfrak{p}$ be a prime ideal of A. Take $S = A - \mathfrak{p}$ in ii) and then reduce mod $S^{-1}\mathfrak{p}$ as in iii). Deduce that the subspace $f^{*-1}(\mathfrak{p})$ of Y is naturally homeomorphic to $\mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) = \mathrm{Spec}(k(\mathfrak{p}) \otimes_A B)$, where $k(\mathfrak{p})$ is the residue field of the local ring $A_\mathfrak{p}$. $\mathrm{Spec}(k(\mathfrak{p}) \otimes_A B)$ is called the *fiber* of f^* over $\mathfrak{p}$.

Solution: i) ϕ^* is injective by (3.11) iv), and is continuous trivially. Denote $\mathrm{Spec}(S^{-1}A)$ by Y, then $\phi^*(Y_{a/s}) = X_a \cap \phi^*(Y)$, so ϕ^* is a homeomorphism from Y to its image. The image of $\mathrm{Spec}(A_f)$ is the set of all prime ideals of A which does not meet $\{1, f, f^2, f^3, \cdots\}$, by (3.11), iv), which is, X_f.

ii) This asks for $\phi_A^* \circ (S^{-1}f)^* = f^* \circ \phi_B^*$ to holds. Since $S^{-1}f \circ \phi_A = \phi_B \circ f$, this follows from the fuctority of Spec.

iii) This asks for $\pi_A^* \circ \bar{f}^* = f^* \circ \pi_B^*$ to holds. Since $\bar{f} \circ \pi_A = \pi_B \circ f$, this follows from the functority of Spec.

iv) f^*: $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ reduced to $\bar{f}_{\mathfrak{p}} : \mathrm{Spec}(B/\mathfrak{p}B_{\mathfrak{p}}) \to \mathrm{Spec}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$. $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is a field ($A_{\mathfrak{p}}$ is a local ring with $\mathfrak{p}A_{\mathfrak{p}}$ its maximal ideal), so $\mathrm{Spec}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) = \{0\}$, and $f^{*-1}(\mathfrak{p}) = \bar{f}_{\mathfrak{p}}^{*-1}(\{0\}) = \mathrm{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$. Also, $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} \cong (A/\mathfrak{p}) \otimes_A A_{\mathfrak{p}} \otimes_A B \cong (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) \otimes B \cong k(\mathfrak{p}) \otimes B$.

Exercise 22

Let A be a ring and $\mathfrak{p}$ a prime ideal of A. Then the canonical image of $\mathrm{Spec}(A_{\mathfrak{p}})$ in $\mathrm{Spec}(A)$ is equal to the intersection of all the open neighborhoods of $\mathfrak{p}$ in $\mathrm{Spec}(A)$.

Solution: By (3.11) iv), it is sufficient to show that $\{\mathfrak{q} : \mathfrak{q} \subset \mathfrak{p}\} = \cap_{\mathfrak{p}\in U} U$ in $\mathrm{Spec}(A)$. If $\mathfrak{p} \in U = X - V(E)$, then $E \not\subseteq \mathfrak{p}$. If $\mathfrak{q} \subset \mathfrak{p}$, then $E \not\subseteq \mathfrak{p}$. Hence $\subset$ holds. If $\mathfrak{q} \in \cap_{\mathfrak{p}\in U} U$, then for every $E \not\subseteq \mathfrak{p}$, we have $E \not\subseteq \mathfrak{q}$. If $\mathfrak{q} \not\subseteq \mathfrak{p}$, then let $E = \mathfrak{q} \not\subseteq \mathfrak{q}$, which is a contradiction. Hence $\supset$ holds.

Exercise 23

Let A be a ring, let $X = \mathrm{Spec}(A)$ and let U be a basic open set in X (i.e., $U = X_f$ for some $f \in A$: Chapter 1, Exercise 17).

i) If $U = X_f$, show that the ring $A(U) = A_f$ depends only on U and not on f.

ii) Let $U^{'} = X_g$ be another basic open set such that $U^{'} \subset U$. Show that there is an equation of the form $g^n = uf$ for some integers $n > 0$ and some $u \in A$, and use this to define a homomorphism $\rho : A(U) \to A(U^{'})$ (i.e., $A_f \to A_g$) by mapping a/f^m to au^m/g^{mn}. Show that ρ depends only on U and $U^{'}$. This homomorphism is called the *restriction* homomorphism.

iii) If $U = U^{'}$, then ρ is the identity map.

iv) If $U \supset U^{'} \supset U^{''}$ are basic open sets in X, show that the diagram

$$\begin{tikzcd}
\mathrm{A}(U) \arrow[rr] \arrow[dr] & & \mathrm{A}(U'') \\
& \mathrm{A}(U') \arrow[ur] &
\end{tikzcd}$$

(in which the arrows are restriction homomorphisms) is commutative.

v) Let x $(= \mathfrak{p})$ be a point of X. Show that $\varinjlim A(U) \cong A_{\mathfrak{p}}$, where $U \ni x$.

The assignment of the ring $A(U)$ to each basic open set U of X, and the restriction homomorphisms ρ, satisfying the condition iii) and iv) above, constitutes a *presheaf* of rings on the basic open sets $(X_f)_{f\in A}$. v) says that the stalk of this presheaf at $x \in X$ is the corresponding local ring $A_{\mathfrak{p}}$.

Solution: i) If $X_f = X_{f'}$, then $r(f) = r(f^{'})$ and $f^{'i} = xf$, $f^j = yf^{'}$ for some $a, b \in A$ and $i, j \in \mathbb{N}$. Hence for $a/f^n \in A_f$, $a/f^n = (ax^n)/(f^n x^n) = (ax^n)/f^{'jn} \in A_{f'}$, and vice versa. $A_f = A_{f'}$.

ii) Suppose that $g^n \notin (f)$ for any $n \in \mathbb{N}$. Let Σ be the set of ideals which contains f but not g^n for any $n \in \mathbb{N}$, then it is not empty since $(f) \in \Sigma$. If $I_1 \subset I_2 \subset I_3 \subset \cdots \subset I_k \subset \cdots$ is a chain in Σ, then $I = \cup_{k=1}^{\infty} I_k$ is an element in Σ that is larger than all of I_k. We apply Zorn's Lemma to show that Σ has a maximal element P. Here P is a prime ideal: If $xy \in P$, but $x \notin P$ and $y \notin P$, then $P + (x) \notin \Sigma$ and $P + (y) \notin \Sigma$ since P is maximal in Σ, so $g^m \in P + (x)$ and $g^n \in P + (y)$. Say, $g^m = a + bx$ and $g^n = c + dy$, then $g^{m+n} = ac + ady + bcx + bcxy \in P$, which is a contradiction. However, by this process we obtain a prime ideal which contains f but not g, which is another contradiction. Hence $g^n \in (f)$ for some $n \in \mathbb{N}$, that is, $g^n = uf$ for some $n \in \mathbb{N}$ and $u \in A$.

If $U = X_f = X_{f'}$ and $U^{'} = X_g = X_{g'}$, then we have four equalities as i), which implies that ρ is well-defined.

iii) By ii), we can just choose the same representative f for U, and use the equality $f = f$ to define ρ. It obviously gives the identity map.

iv) Suppose that U, U', U'' are X_f, X_g, X_h respectively. Then $g^m = uf$ and $h^n = vg$ for some $u, v \in A$ and $i, j \in \mathbb{N}$, and $h^{mn} = uv^m f$ by ii). Use this formula, we know that the diagram commutes.

v) We verify that $A_\mathfrak{p}$ satisfy the universal property that defines $\varinjlim A(U)$. We can define $\phi_U : A(U) \to A_\mathfrak{p}$ to be $\phi_U(a/f) = a/f$ since $\mathfrak{p} \in X_f$, which is $f \notin \mathfrak{p}$. If B is a commutative ring with $\psi_U : U \to B$ such that $\psi_U = \psi_{U'} \circ \rho_{UU'}$. Then we define $\psi : A_\mathfrak{p} \to B$ by $\psi(a/s) = \psi_{X_s}(a/s)$. ψ is well-defined. If $a/s = b/t$ then $at/st = bs/st$, and $\psi_{X_{st}}(at/st) = \psi_{X_{st}}(bs/st)$, which is $\psi_{X_s} \circ \rho_{X_{st}, X_s}(at/st) = \psi_{X_t} \circ \rho_{X_{st}, X_t}(bs/st)$, and this gives $\psi_{X_s}(a/s) = \psi_{X_t}(b/t)$. This is a homomorphism since $\psi(a_1/s_1 + a_2/s_2) = \psi(\dfrac{a_1 s_2 + a_2 s_1}{s_1 s_2}) = \psi_{X_{s_1 s_2}}(\dfrac{a_1 s_2 + a_2 s_1}{s_1 s_2})$, and $\psi_{X_{s_1 s_2}}$ is a homomorphism (similar for multiplicative and identity preserving). This is the unique way to define ψ so that the diagram commutes. Hence $\varinjlim A(U) \cong A_\mathfrak{p}$.

$A(U)$ for arbitrary U: Let $S_f = \{f^n : n \in \mathbb{N}\}$, $S_U = \cap_{X_f \subset U} S_f$, and define $A(U) = S_U^{-1} A$.

Exercise 24

Show that the presheaf of Exercise 23 has the following property. Let $(U_i)_{i \in I}$ be a covering of X by basic open sets. For each $i \in I$ let $s_i \in A(U_i)$ be such that, for each pair of indices i, j, the images of s_i and s_j in $A(U_i \cap U_j)$ are equal. Then there exists a unique $s \in A$ $(= A(X))$ whose image in $A(U_i)$ is s_i, for all $i \in I$. (This essentially implies that the presheaf is a *sheaf*.)

Solution: Since X is compact, we may choose finite X_{f_j} which cover X. Then $\cap_{i=j}^n V(f_j) = V(\sum_{j=1}^n (f_j)) = \emptyset$, so $\sum_{j=1}^n (f_j) = (1)$. For every $m \geq 1$, $\mathfrak{a} + \mathfrak{b} = (1)$ implies $\mathfrak{a}^m + \mathfrak{b} = (1)$, since $r(\mathfrak{a}^m) = \mathfrak{a}$, by (1.16). This implies that $\sum_{j=1}^n (f_j)^m = (1)$, which is $\sum_{j=1}^n a_j f_j^m = 1$ for some a_j.

We show that if such s exists, then it is unique. Denote the restriction map from X to U_i by ρ_{X, U_i}. If $t \in ker(\rho_{X, X_{f_j}})$ for every X_{f_j}. Then $t/1 = 0$ in every A_{f_j}, and $t f_j^{m_j} = 0$ for some positive integers m_j. Let $m = \max_j\{m_j\}$, then $t f_j^m = 0$ for all j and $t = t \sum_{j=1}^n a_j f_j^m = \sum_{j=1}^n a_j (t f_j^m) = 0$.

Next we show that s exists for these X_{f_j}. If $s_j \in A_{f_j}$ is given by $b_j'/f_j^{m_j}$. Let $m = \max_j\{m_j\}$, and $b_j = b_j' f^{m - m_j}$, so that the denominators are of the same power, i.e. $s_j = b_j / f_j^m$. Let $g_j = f_j^n$. On $X_{f_j} \cap X_{f_k} = X_{f_j f_k}$, $b_j / f_j^m = b_k / f_k^m$, so there is m_{jk} such that $(g_j g_k)^{m_{jk}} (b_j g_k - b_k g_j) = 0$. (1. Multiply both sides by the same power of f_j and f_k does not change the equality. 2. If $X_{f_j f_k} = \emptyset$ then $A_{f_j f_k} = 0$.) Let $p = \max\{m_{jk}\}$, then $g_j^p g_k^{p+1} b_j = g_j^{p+1} g_k^p b_k$. We have

$$g_k^p b_k = g_k^p b_k = \sum_{j=1}^n a_j g_j^{p+1} (g_k^p b_k) = g_k^{p+1} \sum_{j=1}^n a_j b_j g_j^p,$$

where a_j are chosen so that $\sum_{j=1}^n a_j g_j^{p+1} = 1$.

We take $s = \sum_{j=1}^n a_j b_j g_j^p$, and have $g_k^p b_k = g_k^{p+1} s$, so $\rho_{X, X_{f_j}}(s) = s_j$ for all j.

Now we show that, this s satisfies $\rho_{X, U_i}(s) = s_i$ for all i, but not just j as above. On $W_j = U_i \cap X_{f_j}$ are basic open sets, and by assumption $\rho_{U_i, W_j}(s_i) = \rho_{U_j, W_j}(s_j) = \rho_{X, W_j}(s) = \rho_{U_i, W_j}(\rho_{X, U_i}(s))$. Hence $\rho_{U_i, W_j}(s_i - \rho_{X, U_i}(s)) = 0$ for $j = 1, 2, \cdots, n$. But W_j covers $U_i = X_h = \mathrm{Spec}(A_h)$. Now the uniqueness claim applied to A_h and open cover W_j of X_h, and using the isomorphisms $(A_h)_{f_j/1} \cong A_{h f_j}$, we have $\rho_{X, U_i}(s) = s_i$.

Exercise 25

Let $f : A \to B$, $g : A \to C$ be ring homomorphisms and let $h : A \to B \otimes_A C$ be defined by $h(x) = f(x) \otimes g(x)$. Let X, Y, Z, T be the prime spectra of $A, B, C, B \otimes_A C$ respectively. Then $h^*(T) = f^*(Y) \cap g^*(Z)$.

Solution: Let $\mathfrak{p} \in X$, and let $k = k(\mathfrak{p})$ be the residue field at $\mathfrak{p}$. By Exercise 21, the fiber $h^{*-1}(\mathfrak{p})$ is the spectrum of $(B \otimes_A C) \otimes_A k \cong (B \otimes_A k) \otimes_k (C \otimes_A k)$. Hence $\mathfrak{p} \in h^*(T) \Leftrightarrow (B \otimes_A k) \otimes_k (C \otimes_A k) \neq 0 \Leftrightarrow B \otimes_A k \neq 0$ and $C \otimes_A k \neq 0 \Leftrightarrow \mathfrak{p} \in f^*(Y) \cap g^*(Z)$.

Exercise 26

Let $(B_\alpha, g_{\alpha\beta})$ be a direct system of rings and B the direct limit. For each α, let $f_\alpha : A \to B_\alpha$ be a ring homomorphism such that $g_{\alpha\beta} \circ f_\alpha = f_\beta$ whenever $\alpha \leq \beta$ (i.e. the B_α form a direct system of A-algebras). The f_α induce $f : A \to B$. Show that $f^*(\mathrm{Spec}(B)) = \cap_\alpha f_\alpha^*(\mathrm{Spec}(B_\alpha))$.

Solution: For $\mathfrak{p} \in \mathrm{Spec}(A)$, we show that $f^{*-1}(\mathfrak{p})$ is empty, i.e., $\mathfrak{p} \notin f^*(\mathrm{Spec}(B))$ if and only if $f_\alpha^{*-1}(\mathfrak{p})$ is empty for some α, i.e., $\mathfrak{p} \notin \cap_\alpha f_\alpha^*(\mathrm{Spec}(B_\alpha))$. By Exercise 21, $f^{*-1}(\mathfrak{p}) = \mathrm{Spec}(B \otimes_A k(\mathfrak{p}))$. The tensor products commute with direct limits, so $B \otimes_A k(\mathfrak{p}) \cong \varinjlim(B_\alpha \otimes_A k(\mathfrak{p}))$. $f^{*-1}(\mathfrak{p}) = \emptyset$ if and only if $B \otimes_A k(\mathfrak{p}) = 0$; if and only if $B_\alpha \otimes_A k(\mathfrak{p}) = 0$ for some α, i.e., $f_\alpha^{*-1}(\mathfrak{p}) = \emptyset$, by Chapter 2, Exercise 21.

Exercise 27

i) Let $f_\alpha : A \to B_\alpha$ be any family of A-algebras and let $f : A \to B$ be their tensor product over A (Chapter 2, Exercise 23). Then $f^*(\mathrm{Spec}(B)) = \cap_\alpha f_\alpha^*(\mathrm{Spec}(B_\alpha))$.

ii) Let $f_\alpha : A \to B_\alpha$ be any finite family of A-algebras and let $B = \prod_\alpha B_\alpha$. Define $f : A \to B$ by $f(x) = (f_\alpha(x))$. Then $f^*(\mathrm{Spec}(B)) = \cup_\alpha f_\alpha^*(\mathrm{Spec}(B_\alpha))$.

iii) Hence the subsets of $X = \mathrm{Spec}(A)$ of the form $f^*(\mathrm{Spec}(B))$, where $f : A \to B$ is a ring homomorphism, satisfy the axioms for closed sets in a topological space. The associated topology is the *constructible* topology on X. It is finer than the Zariski topology (i.e., there are more open sets, or equivalently more closed sets).

vi) Let X_C denote the set X endowed with the constructible topology. Show that X_C is quasi-compact.

Solution: i) By Exercise 26, $f^*(\mathrm{Spec}(B)) = \cap_J f_J^*(\mathrm{Spec}(B_J))$ where J runs through all the finite subsets of the index set, see Chapter 2, Exercise 23. By Exercise 25, we have $f_J^{*-1}(\mathrm{Spec}(B_J)) = \cap_{\alpha \in J} f_\alpha^{*-1}(\mathrm{Spec}(B_\alpha))$. Hence the equality holds.

ii) $f^{*-1}(\mathfrak{p}) = \mathrm{Spec}((\prod B_\alpha) \otimes_A k) \neq \emptyset \Leftrightarrow (\prod B_\alpha) \otimes_A k \cong \prod (B_\alpha \otimes_A k) \neq \{0\}$ for finite $\alpha \Leftrightarrow$ at least one of $B_\alpha \otimes_A k \neq \{0\} \Leftrightarrow$ at least one of $\mathrm{Spec}(B_\alpha \otimes_A k) = f_\alpha^{*-1}(\mathfrak{p}) \neq \emptyset$.

iii) $id^*(\mathrm{Spec}(A)) = \mathrm{Spec}(A)$, $p^*(\mathrm{Spec}(0)) = \emptyset$. Hence it's indeed a topology on X. For every $\mathfrak{a}$ an ideal of A, $\pi^*(\mathrm{Spec}(A/\mathfrak{a})) = V(\mathfrak{a})$, where $\pi : A \to A/\mathfrak{a}$ is the natural project map. Hence the closed sets in Zariski topology is closed in the constructible topology. The constructible topology is finer than the Zariski topology.

iv) If $U_\alpha = X - f_\alpha^*(\mathrm{Spec}(B_\alpha))$ is an open cover of X_C, then

$$\cap_\alpha f_\alpha^*(\mathrm{Spec}(B_\alpha)) = f^*(\mathrm{Spec}(B)) = \emptyset,$$

where $f : A \to B$ is their tensor product. Then $B = 0$, hence $B_J = 0$ for some finite subset J of the index set, so $\cap_{\alpha \in J} f_\alpha^*(\mathrm{Spec}(B_\alpha)) = \emptyset$ and $\{U_\alpha : \alpha \in J\}$ is a finite subcover. X_C is quasi-compact.

Exercise 28

(Continuation of Exercise 27.)

i) For each $g \in A$, the set X_g (Chapter 1, Exercise 17) is both open and closed in the constructible topology.

ii) Let $C^{'}$ denote the smallest topology on X for which the sets X_g are both open and closed, and let $X_{C'}$ denoted the set X endowed with this topology. Show that $X_{C'}$ is Hausdorff.

iii) Deduce that the identity mapping $X_C \to X_{C'}$ is a homeomorphism. Hence a subset E of X is of the form $f^*(\text{Spec}(B))$ for some $f : A \to B$ if and only if it closed in the topology $C^{'}$.

iv) The topological space X_C is compact, Hausdorff and totally disconnected.

Solution: i) By Exercise 21, i), $X_g = f^*(\text{Spec}(A_g))$, and hence is closed in constructible topology. It's open in constructible topology since constructible topology is finer than the Zariski topology (Exercise 27 iii)).

ii) If $\mathfrak{p}_x \neq \mathfrak{p}_y$ then either there is $f \in \mathfrak{p}_x - \mathfrak{p}_y$ or there is $f \in \mathfrak{p}_y - \mathfrak{p}_x$. Suppose the first case, then $\mathfrak{p}_y \in X_f$ but $\mathfrak{p}_x \in X_{C'} - X_f$. Since X_f is both open and closed, the component is open, and $X_f \cup (X_{C'} - X_f) = \emptyset$ and done.

iii) By i) and the definition of $C^{'}$, we have $C^{'} \subset C$. Hence the identity map $X_C \to X_{C'}$ is continuous bijection. A continuous bijection from a quasi-compact space to a Hausdorff space is a homeomorphism. (If $f : X \to Y$ is a continuous bijection, with X quasi-compact and Y Hausdorff, then every closed set E in X is compact so $f(E)$ is compact in Y, so $f(E)$ closed since Y is Hausdorff.)

iv) By ii), iii), and Exercise 27 iv), X_C is compact and Hausdorff. We shall show that $X_{C'}$ is totally disconnected. If $\mathfrak{p}_x \neq \mathfrak{p}_y$ and $S = \{\mathfrak{p}_x, \mathfrak{p}_y\}$, we proceed as ii) to find an X_f which contains $\mathfrak{p}_x$ but not $\mathfrak{p}_y$ (or the converse), then $S \cap X_f$ and $S \cap (X - X_f)$ are two disjoint open subsets of S and their union is S.

Exercise 29

Let $f : A \to B$ be a ring homomorphism. Show that $f^* : \text{Spec}(B) \to \text{Spec}(A)$ is a continuous closed mapping (i.e., maps closed sets to closed sets) for constructible topology.

Solution: Since C is the smallest topology where X_g, $X - X_g$ are both open (Exercise 28, iii)), they are the subbasis for X_C. We have that $f^{*-1}(Y_g) = X_{f(g)}$, and $f^{*-1}(Y - Y_g) = X - X_{f(g)}$, so f^* is continuous.

For every closed set in $\text{Spec}(B)$, it is of the form $h^*(\text{Spec}(C))$. Then $f^*(h^*(\text{Spec}(C))) = (h \circ f)^*(\text{Spec}(C))$, which is closed in $\text{Spec}(A)$ with constructible topology. Hence f^* is a continuous closed mapping.

Exercise 30

Show that the Zariski topology and the constructible topology on $\text{Spec}(A)$ are the same if and only if A/R is absolutely flat (where R is the nilradical of A).

Solution: If the Zariski topology and the constructible topology are the same, then the Zariski topology is Hausdorff (Exercise 28), hence A/R is flat (Exercise 11).

If A/R is flat, then the Zariski topology is Hausdorff and compact (Exercise 11). X_f is compact (Chapter 1, Exercise 17), so X_f is closed. Then X_f is both open and closed, hence the Zariski topology is finer than the constructible topology (Exercise 28, iii)). But the constructible topology is finer than the Zariski topology (Exercise 27, iii)), so they are the same.