Modules

Xie, Kaitao

Exercise 1

Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

Solution: m, n are coprime, so there are integers u, v, such that $um + vn = 1$. For every $x \otimes y \in (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$, $x \otimes y = (um + vn)x \otimes y = umx \otimes y + x \otimes vny = 0 \otimes y + x \otimes 0 = 0$. In fact, $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$, where d is the greatest common divisor of m and n.

Exercise 2

Let A be a ring, I an ideal, M an A-Module. Show that $(A/I) \otimes_A M$ is isomorphic to M/IM.

Solution: Tensor the exact sequence

$$0 \to I \to A \to A/I \to 0$$

to get

$$I \otimes_A M \xrightarrow{i} A \otimes_A M \xrightarrow{p} (A/I) \otimes M \to 0.$$

Since $A \otimes_A M \cong M$, and $ker(p) = im(i)$ is sent to IM by this isomorphism, $(A/I) \otimes_A M \cong M/IM$.

Exercise 3

Let A be a local ring, M and N finitely generated A-modules. Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$.

Solution: Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by Exercise 2. By Nakayama's Lemma, $M_k = 0$ implies $M = 0$.

If $M \otimes_A N = 0$ then $(M \otimes_A N)_k = 0$. $0 = (M \otimes_A N)_k = M \otimes_A N \otimes_A k \cong M \otimes_A [k \otimes_k (N \otimes_A k)] \cong (M \otimes_A k) \otimes_k (N \otimes_A k) = M_k \otimes_k N_k$ by (2.15). M_k and N_k are finite dimensional vector space, so the dimension of the tensor product is the product of dimensions of M and N, and one of them must be 0. This implies that $M = 0$ or $N = 0$.

Exercise 4

Let $M_i (i \in I)$ be a family of A-modules, and let M be their direct sum. Prove that M is flat $\Leftrightarrow$ each M_i is flat.

Solution: It in fact requires to show that $N \otimes \bigoplus_i M_i \cong \bigoplus_i (N \otimes M_i)$ for every A-module N. The bilinear map $N \times \bigoplus_i M_i \to \bigoplus_i (N \otimes M_i)$ that sends $(n, \sum_{j=1}^k m_j)$ to $\sum_{j=1}^k n \otimes m_j$ gives raise to the isomorphism.

Exercise 5

Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra.

Solution: This is a immediate result of Exercise 4 since $A[x] \cong \bigoplus_{n \in \mathbb{N}} A$ as an A-module.

Exercise 6

For any A-module, let $M[x]$ denote the set of all polynomials in x with coefficients in M, that is to say expressions of the form $m_0 + m_1x + \cdots + m_rx^r$ with $m_i \in M$. Defining the product of an element of $A[x]$ and an element of $M[x]$ in the obvious way, show that $M[x]$ is an $A[x]$-module, and that $M[x] \cong A[x] \otimes_A M$.

Solution: $M[x]$ is itself a abelian group. $1 \cdot (m_0 + m_1x + \cdots + m_rx^r) = m_0 + m_1x + \cdots + m_rx^r$, and if $f = \sum_{i=0}^{n} a_ix^i$ and $g = \sum_{j=0}^{k} b_jx^j$, then $fg = \sum_{t=0}^{n+k} x^t \sum_{i+j=t} a_ib_j$ and $fg \cdot \sum_{l=0}^{r} m_lx^l = \sum_{p=0}^{n+m+r} x^p \sum_{t+l=p} c_tm_l = \sum_{p=0}^{n+k+r} x^p \sum_{i+j+l=p} a_ib_jm_l = f \cdot (g \cdot \sum_{l=0}^{r} m_lx^l)$, where $c_t = \sum_{i+j=t} a_ib_j$. The two distributive laws follow trivially. $M[x]$ is an $A[x]$-module.

Define $\phi : A[x] \otimes_A M \to M[x]$ by $\phi(f \otimes_A m) = f \cdot m$ and extend by linearity. This is a ring homomorphism, and obviously surjective since we have that $\phi(1 \otimes_A m_0 + x \otimes_A m_1 + \cdots + x^r \otimes_A m_r) = m_0 + m_1x + \cdots + m_rx^r$. If $\phi(\sum_i b_i(\sum_j a_{ij}x^j) \otimes_A m_i) = \sum_i b_i \sum_j a_{ij}m_ix^j = \sum_j x^j \sum_i a_{ij}b_im_i = 0$, then $\sum_i a_{ij}b_im_i = 0$ for all j. Then we have that $\sum_i b_i(\sum_j a_{ij}x^j) \otimes_A m_i = \sum_j x^j \otimes_A (\sum_i a_{ij}b_im_i) = 0$. Hence ϕ is injective, and ϕ is an isomorphism.

Exercise 7

Let $\mathfrak{p}$ be a prime ideal in A. Show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. If $\mathfrak{m}$ is maximal ideal in A, is $\mathfrak{m}[x]$ a maximal ideal in $A[x]$?

Solution: $A[x]/\mathfrak{p}[x] \cong (A/\mathfrak{p})[x]$, which is an integral domain. See Chapter 1 Exercise 2. iii).

Let $A = \mathbb{C}$, and $\mathfrak{m} = (0)$. Then $\mathfrak{m}$ is a maximal ideal in A, but $\mathfrak{m}[x] = (0)$ is not a maximal ideal in $\mathbb{C}[x]$.

Exercise 8

i)If M and N are flat A-modules, then so is $M \otimes_A N$.

ii)If B is flat A-algebra and N is a flat B-module, then N is flat as an A-module.

Solution: i) If $P^{'} \to P \to P^{''}$ is exact, then $P^{'} \otimes M \to P \otimes M \to P^{''} \otimes M$ is exact by the flatness of M, and $(P^{'} \otimes M) \otimes N \to (P \otimes M) \otimes N \to (P^{''} \otimes M) \otimes N$ is exact by the flatness of N. By the associative law of tensor product, we have that $P^{'} \otimes (M \otimes N) \to P \otimes (M \otimes N) \to P^{''} \otimes (M \otimes N)$ is exact. Hence $M \otimes N$ is flat.

ii) First tensor the exact sequence over A by B and view all the modules as B-modules. Then tensor the exact sequence over B by N. The exactness still holds. Use the isomorphism $(M \otimes_A B) \otimes_B N \cong M \otimes_A (B \otimes_B N)$ by (2.15) and $B \otimes_A N \cong N$ as A-modules.

Exercise 9

Let $0 \to M^{'} \to M \to M^{''} \to 0$ be an exact sequence of A-modules. If $M^{'}$ and $M^{''}$ are finitely generated, then so is M.

Solution: Denote the second arrow by f and the third arrow by g. Let $u_1, \cdots, u_n$ be generators of $M^{''}$. Since g is surjective, we can pick one v_i from each $g^{-1}(u_i)$. Let $e_1, \cdots, e_m$ be generators of $M^{'}$. We show that $f(e_1), \cdots, f(e_m)$ and $v_1, \cdots, v_n$ generate M. $\forall m \in M$, $g(m) = \sum_{i=1}^{n} a_iu_i = g(\sum_{i=1}^{n} a_iv_i)$ for some $a_i \in A$. Then $m - \sum_{i=1}^{n} a_iv_i \in ker(g) = im(f)$, and $m - \sum_{i=1}^{n} a_iv_i = f(\sum_{j=1}^{m} b_je_j) = \sum_{j=1}^{m} b_jf(e_j)$ for some $b_j \in A$. Hence M is finitely generated.

Exercise 10

Let A be a ring, I an ideal contained in the Jacobson radical of A; let M be an A-module and N a finitely generated A-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/IM \to N/IN$ is surjective, then u is surjective.

Solution: Denote the induced map by $\bar{u}$. Since $\bar{u}$ is surjective, $\forall n \in N$, there is an $m \in M$ such that $\bar{u}(\bar{m}) = \overline{u(m)} = \bar{n}$. This implies that $N = IN + u(M)$, and $I(N/u(M)) = (IN + u(M))/u(M) = N/u(M)$. By Nakayama's lemma, $N/u(M) = 0$ and u is surjective.

Exercise 11

Let A be a ring $\neq 0$. Show that $A^m \cong A^n \Rightarrow m = n$. If $\phi : A^m \to A^n$ is surjective, then $m \geq n$. If $\phi : A^m \to A^n$ is injective, is it always the case that $m \leq n$?

Solution: Let $\mathfrak{m}$ be a maximal ideal of A and let $\phi : A^m \to A^n$ be an isomorphism. Then $(A/\mathfrak{m}) \otimes_A A^m$ and $(A/\mathfrak{m}) \otimes_A A^n$ are vector spaces over $A/\mathfrak{m}$ and $1 \otimes \phi : (A/\mathfrak{m}) \otimes A^m \to (A/\mathfrak{m}) \otimes A^n$ is an isomorphism between A-modules for $1 \otimes \phi^{-1}$ is the inverse map. Since $1 \otimes \phi$ is also a homomorphism between vector spaces over $A/\mathfrak{m}$, it's a isomorphism between vector space. Hence $m = n$.

Since the tensor product functor by $A/\mathfrak{m}$ is right exact, the same technique can be use to show that if $\phi : A^m \to A^n$ is surjective, then $m \geq n$.

If ϕ is injective, then $m \leq n$ indeed. Suppose that $m > n$, we show that ϕ is not injective. Composite with inclusion to view ϕ as a map $\psi = i \circ \phi : A^m \to A^m$, with last $m - n$ coordinates being 0, and define $\pi : A^m \to A$ be the projection to the last coordinate. Then $\pi \circ \psi = 0$. By (2.4), $\psi^n + \cdots + a_1\psi + a_0 id_{A^m} = 0$ for some $a_i \in A$, and we choose this coefficients such that n minimal. Composite with π we know that $a_0 = 0$, so $\psi \circ (\phi^{n-1} + a_{n-1}\phi^{n-1} + \cdots + a_1 id_{A^m}) = 0$. By the minimality of n, $\psi = 0$. Hence ϕ is not injective.

Exercise 12

Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective homomorphism. Show that $ker(\phi)$ is finitely generated.

Solution: Let $e_i = (0, \cdots, 1, \cdots, 0)$, where the ith coordinate is 1 and others are 0, be a basis of A^n and choose $u_i \in M$ such that $\phi(u_i) = e_i$, $1 \leq i \leq n$. We first show that $M = ker(\phi) \oplus N$, where N is the submodule of M generated by $u_1, \cdots, u_n$. $\forall m \in M$, $\phi(m) \in A^n$ and $\phi(m) = \sum_{i=1}^n a_i e_i = \phi(\sum_{i=1}^n a_i u_i)$ for some $a_i \in A^n$. Hence $m - \sum_{i=1}^n a_i u_i \in ker(\phi)$, and $M = ker(\phi) + N$. If $m' \in ker(\phi) \cap N$, then $m' = \sum_{i=1}^n b_i u_i$ for some $b_i \in A$, and we have that $\phi(m') = (b_1, b_2, \cdots, b_n) = 0$, which implies that $b_i = 0$ for all i, so $m' = 0$. Hence $M = ker(\phi) \oplus N$, which is equivalent to $M/N \cong ker(\phi)$. Since M is finitely generated, so is $ker(\phi)$.

Exercise 13

Let $f : A \to B$ be a ring homomorphism, and N be a B-module. Regarding N as an A-module by restriction of scalars, form the B-module $N_B = B \otimes_A N$. Show that the homomorphism $g : N \to N_B$ which maps y to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_B.

Solution: Define $p : N_B \to N$ by $p(b \otimes y) = by$, then $p \circ g = id_N$ implies that g is injective. $N_B/ker(p) = im(p)$ and $im(p) \cong im(g)$, so $N_B = ker(p) \oplus im(g)$.

Exercise 14

(*Direct limits*) A partially ordered set I is said to be *directed* set if for each pair i, j in I there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

Let A be a ring, let I be a directed set and let $(M_i)_{i \in I}$ be a family of A-modules indexed by I. For each pair i, j in I such that $i \leq j$, let $\mu_{ij} : M_i \to M_j$ be an A-homomorphism, and suppose that the following axioms are satisfied:

(1)μ_{ij} is the identity mapping of M_i, for all $i \in I$;

(2)$\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leq j \leq k$.

Then the modules M_i and homomorphisms μ_{ij} are said to form a *direct system* $\mathbf{M} = (M_i, \mu_{ij})$ over the directed set I.

We shall construct an A-module M called the *direct limit* of the direct system $\mathbf{M}$. Let C be the direct sum of the M_i, and identify each module M_i with its canonical image C. Let D be the submodule of C generated by all elements of the form $x_i - \mu_{ij}(x_i)$ where $i \leq j$ and $x_i \in M_i$. Let $M = C/D$, let $\mu : C \to M$ be the projection and let μ_i be the restriction of μ to M_i.

The module M, or more correctly the pair consisting of M and the family of homomorphism $\mu_i : M_i \to M$, is called the *direct limit* of the direct system $\mathbf{M}$, and is written $\varinjlim M_i$. From the construction it is clear that $\mu_i = \mu_j \circ \mu_{ij}$ whenever $i \leq j$.

Solution: The only thing to show is that $\mu_i = \mu_j \circ \mu_{ij}$ whenever $i \leq j$, but $\forall x_i \in M_i$, $\mu_j \circ \mu_{ij}(x_i) = \mu_j(\mu_{ij}(x_i)) = \overline{\mu_{ij}(x_i)} = \bar{x}_i = \mu_i(x_i)$.

Exercise 15

In the situation of Exercise 14, show that every element of M can be written in the form $\mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$.

Show also that if $\mu_i(x_i) = 0$ then there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$ in M_j.

Solution: By definition, an element of M can be written as $\mu_{i_1}(x_{i_1}) + \cdots + \mu_{i_n}(x_{i_n})$. Since I is directed, $\exists i \in I$, s.t. $i_1 \leq i, \cdots, i_n \leq i$. Let $x_i = \mu_{i_1 i}(x_{i_1}) + \cdots + \mu_{i_n i}(x_{i_n})$ and

$$\mu_i(x_i) = \mu_i(\mu_{i_1 i}(x_{i_1}) + \cdots + \mu_{i_n i}(x_{i_n})) = \mu_{i_1}(x_{i_1}) + \cdots + \mu_{i_n}(x_{i_n}).$$

Suppose that $\mu_i(x_i) = 0$. If $x_i = 0$ then $\mu_{ii}(x_i) = 0$ will satisfied. If not, x_i is a linear combination of $x_k - \mu_{kj}(x_k)$, but without loss of generality, we suppose that $x_i = x_k - \mu_{kj}(x_k) \in C$ for some $j, k \in I$ and $x_k \in C$, which is $x_i - x_k + \mu_{kj}(x_k) = 0$. Other cases can be dealt with similarly. C is the direct sum of all M_i, so it have to be $x_k = x_i$ (which also asks $k = i$) and $\mu_{kj}(x_k) = 0$, or $x_k = 0$ and $\mu_{kj}(x_j) = -x_i$. The first case is what we want, and the second case implies $x_i = 0$, which we have already done.

Exercise 16

Show that the direct limit is characterized (up to isomorphism) by the following property. Let N be an A-module and for each $i \in I$ let $\alpha_i : M_i \to N$ be an A-module homomorphism such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$. Then there exists a unique homomorphism $\alpha : M \to N$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in I$.

Solution: This is a universal property, and it determines a unique object up to isomorphism, so we only need to show that the construction of M satisfies this property.

By Exercise 15, we need only to define α on the elements of the form $\mu_i(x_i)$, but $\alpha_i = \alpha \circ \mu_i$ determines that $\alpha(\mu_i(x_i)) = \alpha_i(x_i)$. Now it's sufficient to show that α is an A-homomorphism.

$\alpha(\mu_i(x_i) + \mu_j(x_j)) = \alpha(\mu_k(\mu_{ik}(x_i) + \mu_{jk}(x_j))) = \alpha_k(\mu_{ik}(x_i) + \mu_{jk}(x_j)) = \alpha_k(\mu_{ik}(x_i)) + \alpha_k(\mu_{jk}(x_j)) = \alpha_i(x_i) + \alpha_j(x_j) = \alpha(\mu_i(x_i)) + \alpha(\mu_j(x_j))$, for chosen k such that $i \leq k$ and $j \leq k$. Hence α is additive. Since $\alpha(b\mu_i(x_i)) = \alpha(\mu_i(bx_i)) = \alpha_i(bx_i) = b\alpha_i(x_i) = b\alpha(\mu_i x_i)$, $\forall b \in A$ we have shown that α is an A-homomorphism.

Exercise 17

Let $(M_i)_{i \in I}$ be a family of submodules of an A-module, such that for each pair of indices i, j in I there exists $k \in I$ such that $M_i + M_j \subset M_k$. Define $i \leq j$ to mean $M_i \subset M_j$ and let $\mu_{ij} : M_i \to M_j$ be the embedding of M_i in M_j. Show that

$$\varinjlim M_i = \sum M_i = \cup M_i$$

.

In particular, any A-module is the direct limit of its finitely generated submodules.

Solution: Define $\mu_i : M_i \to \sum M_i$ be the embedding of M_i in $\sum M_i$, then $\mu_i = \mu_i \circ \mu_{ij}$ whenever $i \leq j$. We show that $\sum M_i$ with μ_i satisfies the universal property stated in Exercise 16.

Let N be an A-module and for each $i \in I$ let $\alpha_i : M_i \to N$ be an A-module homomorphism such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$. The only way to define α is that $\alpha(\sum_i x_i) = \sum_i \alpha_i(x_i)$, and $\alpha_i = \alpha \circ \mu_i$. The fact that α is an A-homomorphism is trivial.

Next we show that $M = \sum M_i = \cup M_i$. $\cup M_i \subset \sum M_i$ trivially. It's sufficient to show that $\cup M_i$ is a module. It's sufficient to show that $\cup M_i$ is closed under addition and scalar multiplication. If $x, y \in M$, then $x \in M_i$ and $y \in M_j$ for some $i, j \in I$. By assumption, $\exists k \in I$, s.t. $M_i + M_j \subset M_k$. Hence $x + y \in M_k \subset M$. $\forall a \in A$, $ax \in M_i \subset M$.

The family of finitely generated submodules of a module N satisfies the assumption, and it's sufficient to show that $\sum N_i$ is the whole module N, but $\forall x \in N$, x generates one of the N_i. Hence $x \in \sum N_i$.

Exercise 18

Let $\mathbf{M} = (M_i, \mu_{ij})$, $\mathbf{N} = (N_i, \nu_{ij})$ be direct systems of A-modules over the same directed set. Let M, N be the direct limits and $\mu_i : M_i \to M$, $\nu_i : N_i \to N$ the associated homomorphisms.

A homomorphism $\mathbf{\Phi} : \mathbf{M} \to \mathbf{N}$ is by definition a family of A-module homomorphisms $\phi_i : M_i \to N_i$ such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ whenever $i \leq j$. Show that $\mathbf{\Phi}$ defines a unique homomorphism $\phi = \varinjlim \phi_i : M \to N$ such that $\phi \circ \mu_i = \nu_i \circ \phi_i$ for all $i \in I$.

Solution: $\phi \circ \mu_i = \nu_i \circ \phi_i$ for all $i \in I$ uniquely determines ϕ. It remains to show that ϕ is an A-homomorphism. See the proof of Exercise 16.

Exercise 19

A sequence of direct systems and homomorphisms $\mathbf{M} \to \mathbf{N} \to \mathbf{P}$ is exact if the corresponding sequence of modules and module homomorphisms is exact for each $i \in I$. Show that the sequence $M \to N \to P$ of direct limits is then exact.

Solution: Denote that $\mathbf{M} = (M_i, \mu_i)$, $\mathbf{N} = (N_i, \nu_i)$, $\mathbf{P} = (P_i, \sigma_i)$, the left arrow by f and the right arrow by g. We compute that

$$g(f(\mu_i(x_i))) = g(\nu_i(f_i(x_i))) = \sigma_i(g_i f_i(x_i)) = \sigma_i(0) = 0.$$

Hence $im(f) \subset ker(g)$.

$\forall \nu_i(y_i) \in ker(g)$, $g(\nu_i(y_i)) = \sigma_i(g_i(y_i)) = 0$. By Exercise 15, there exists $j \geq i$ such that $\sigma_{ij}(g_i(y_i)) = g_j(\nu_{ij}(y_i)) = 0$. By the exactness at index j, $\exists x_j \in M_j$, s.t. $f_j(x_j) = \nu_{ij}(y_i)$. Hence $\nu_j(f_j(x_j)) = \nu_j(\nu_{ij}(y_i))$, which is $f(\mu_j(x_j)) = \nu_i(y_i)$. Hence $ker(g) \subset im(f)$, and the sequence is exact.

Exercise 20

(*Tensor products commute with direct limits*) Keeping the same notation as in Exercise 14, let N be any A-module. Then $(M_i \otimes N, \mu_i \otimes 1)$ is a direct system; let $P = \varinjlim (M_i \otimes N)$ be its direct limit. For each $i \in I$ we have a homomorphism $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$, hence by Exercise 16 a homomorphism $\psi : P \to M \otimes N$. Show that ψ is an isomorphism, so that $\varinjlim (M_i \otimes N) \cong (\varinjlim M_i) \otimes N$.

Solution: Let $g_i : M_i \times N \to M_i \otimes N$ be the canonical bilinear mapping, i.e. $g_i(x_i, y) = x_i \otimes y$. Define $g : M \times N \to P$ by $g \circ (\mu_i \times 1) = (\mu_i \otimes 1) \circ g_i$. g is uniquely determined by this property since every element in M is of the form $\mu_i(x_i)$ with $x_i \in M_i$, by Exercise 15.

In fact, g is A-bilinear. (Same technique used in the proof of Exercise 19.) It gives raise to an A-homomorphism $\phi : M \otimes N \to P$, defined by $\psi(\mu_i(x_i) \otimes y) = (\mu_i \otimes 1)(x_i \otimes y)$. It is the inverse of ψ.

Exercise 21

Let $(A_i)_{i\in I}$ be a family of rings indexed by a directed set I, and for each pair $i \leq j$ in I let $\alpha_{ij} : A_j \to A_i$ be a ring homomorphism, satisfying conditions (1) and (2) of Exercise 14. Regarding each A_i as a $\mathbb{Z}$-module we can then form the direct limit $A = \varinjlim A_i$. Show that A inherits a ring structure from the A_i so that the mappings $A_i \to A$ are ring homomorphisms. The ring A is the direct limit of the system (A_i, α_{ij}).

If $A = 0$ prove that $A_i = 0$ for some $i \in I$.

Solution: Define the multiplication in A by $\alpha_i(x_i) \cdot \alpha_j(x_j) = \alpha_k(\alpha_{ik}(x_i)\alpha_{jk}(x_j))$, where k is chosen so that $i \leq k$, $j \leq k$. This is well-defined since, if we choose another k' with $i \leq k'$ and $j \leq k'$, then we can choose l s.t. $k \leq l$ and $k' \leq l$, and $\alpha_k(\alpha_{ik}(x_i)\alpha_{jk}(x_j)) = \alpha_l(\alpha_{il}(x_i)\alpha_{jl}(x_j)) = \alpha_{k'}(\alpha_{ik'}(x_i)\alpha_{jk'}(x_j))$. The identity of A is simply $\mu_i(1)$.

We shall show that the multiplication is associative, commutative and satisfy the distributive law, but the proof is trivial and we skip it. By this definition, $\alpha_i(x_i)\alpha_i(x_i') = \alpha_i(\alpha_{ii}(x_i)\alpha_{ii}(x_i')) = \alpha_i(x_i x_i')$. Hence α_i is a ring homomorphism.

If $A = 0$, then $\alpha_j(1) = 0$ for all α_j. By Exercise 15 we may fix j and choose $i \geq j$ s.t. $\alpha_{ji}(1) = 0$. But a ring homomorphism sends identity to identity, so in A_i, $1 = 0$, and $A_i = 0$.

Exercise 22

Let (A_i, α_{ij}) be a direct system of ring and let R_i be the nilradical of A_i. Show that $\varinjlim R_i$ is the nilradical of $\varinjlim A_i$.

Show that if each A_i is an integral domain, then $\varinjlim A_i$ is an integral domain.

Solution: A ring homomorphism sends a nilpotent element to a nilpotent element, so α_{ij} can be restrict to a $\mathbb{Z}$-module homomorphism $R_i \to R_j$, and $\varinjlim R_i$ is defined. $\varinjlim R_i$ contained in the nilradical of A clearly.

If $\alpha_i(x_i)$ is nilpotent, then $(\alpha_i(x_i))^n = \alpha_i(x_i^n) = 0$ for some n. Then there exists some $j \geq i$, s.t. $\alpha_{ij}(x_i^n) = (\alpha_{ij}(x_i))^n = 0$ by Exercise 15, and this implies $\alpha_{ij}(x_i) \in R_j$. Since $\alpha_j(\alpha_{ij}(x_i)) = \alpha_i(x_i)$, we claim that $\varinjlim R_i$ is the nilradical of $\varinjlim A_i$.

Suppose that each A_i is an integral domain. If $\alpha_i(x_i)$ and $\alpha_j(x_j)$ satisfy

$$\alpha_i(x_i) \cdot \alpha_j(x_j) = \alpha_k(\alpha_{ik}(x_i)\alpha_{jk}(x_j)) = 0.$$

Again by Exercise 15, $\exists l \geq k$, s.t. $\alpha_{kl}\alpha_k(\alpha_{ik}(x_i)\alpha_{jk}(x_j)) = \alpha_{il}(x_i)\alpha_{jl}(x_j) = 0$. Since A_l is an integral domain, $\alpha_{il}(x_i) = 0$ or $\alpha_{jl}(x_j) = 0$. Suppose that $\alpha_{il}(x_i) = 0$, then $\alpha_l(\alpha_{il}(x_i)) = \alpha_i(x_i) = 0$. Hence A is an integral domain.

Exercise 23

Let $(B_\lambda)_{\lambda\in\Lambda}$ be a family of A-algebras. For each finite subset of Λ let B_J denote the tensor product (over A) of the B_λ for $\lambda \in J$. If J' is another finite subset of Λ and $J \subset J'$, there is a canonical A-algebra structure for which the homomorphisms $B_J \to B$ are A-algebra homomorphisms. The A-algebra B is the tensor product of the family $(B_\lambda)_{\lambda\in\Lambda}$.

Solution: The statement in Exercise 21 works here with a little justification.

Flatness and Tor: **TO BE ADDED.**