Chain Conditions

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Exercise 1

i) Let M be a Noetherian A-module and $u : M \to M$ a module homomorphism. If u is surjective, then u is an isomorphism.

ii) If M is Artinian and u is injective, then again u is an isomorphism.

Solution: i) Consider submodules chain $ker(u) \subset ker(u^2) \subset \cdots$. Since M is a Noetherian A-module, there exists n such that $ker(u^{n+1}) = ker(u^n)$, so u is injective if restricted to $im(u^n)$. But $im(u^n) = M$ since u and hence u^n are surjective.

ii) Consider submodules chain $im(u) \supset im(u^2) \supset \cdots$. Since M is Artinian, there exists n such that $im(u^n) = im(u^{n+1})$. For every $x \in M$, $u^n(x) = u^{n+1}(x^{'})$ for some $x^{'} \in M$, so $u^n(x - u(x^{'})) = 0$. Since u is injective, u^n is injective, so $x = u(x^{'})$. Hence u is surjective.

Exercise 2

Let M be an A-module. If every non-empty set of finitely generated submodules of M has a maximal element, then M is Noetherian.

Solution: If M is not Noetherian, then there exists submodules chain $M_1 \subset M_2 \subset \cdots$, where the inclusions are strict. Then for every n, there exists $x_n \in M_{n+1} - M_n$, and $(x_1) \subset (x_1, x_2) \subset \cdots$, which is a finitely generated submodules chain where the inclusions are strict. This contradicts to the fact that every non-empty set of finitely generated submodules of M has a maximal element.

Exercise 3

Let M be an A-module and let N_1, N_2 be submodules of M. If M/N_1 and M/N_2 are Noetherian, so is $M/(N_1 \cap N_2)$. Similarly with Artinian in place of Noetherian.

Solution: There is an exact sequence $0 \to N_1/(N_1 \cap N_2) \to M/(N_1 \cap N_2) \to M/N_1 \to 0$ by the third isomorphism theorem. By (6.3), we shall show that $N_1/(N_1 \cap N_2)$ is Noetherian (resp. Artinian).

$N_1/(N_1 \cap N_2) \cong (N_1 + N_2)/N_2$ by the second isomorphism theorem, but $(N_1 + N_2)/N_2$ is a submodule of M/N_2 and hence Noetherian (resp. Artinian), by (6.3).

Exercise 4

Let M be a Noetherian A-module and let $\mathfrak{a}$ be the annihilator of M in A. Prove that $A/\mathfrak{a}$ is a Noetherian ring.

If we replace "Noetherian" by "Artinian" in this result, is it still true?

Solution: M is Noetherian, so is finitely generated. Let $x_1, \cdots, x_n$ be the generators of M, and $\mathfrak{a}_i = \mathrm{Ann}(x_i)$. Then $A/\mathfrak{a}_i \cong Ax_i \subset M$, so $A/\mathfrak{a}_i$ is Noetherian. $\mathfrak{a} = \cap_{i=1}^n \mathfrak{a}_i$. By Exercise 3, $A/\mathfrak{a}$ is Noetherian. The result is not true if we replace "Noetherian" by "Artinian".

Exercise 5

A topological space X is said to be Noetherian if the open subsets of X satisfy the ascending chain condition (or, equivalently, the maximal condition). Since the closed subsets are complements of open subsets, it comes to the same thing to say that the closed subsets of X satisfy the descending chain condition (or, equivalently, the minimal condition). Show that, if X is Noetherian, then every subspace of X is Noetherian, and that X is quasi-compact.

Solution: If Y is a subspace of X, and $Y_0 \subset Y_1 \subset \cdots$ is an ascending sequence of open subsets of Y, then $Y_n = Y \cap X_n$ for some X_n open in X. $X_0 \subset X_1 \subset \cdots$ is an ascending sequence of open subsets of X, so it must be stationary since X is Noetherian, and $Y_0 \subset Y_1 \subset \cdots$ is stationary. Hence Y is Noetherian.

Let $\{U_i\}$ be an open covering of X. Let Σ be a set of all finite subsets of $\{U_i\}$. By the maximal conditions, Σ has a maximal element, says V (finite set of some U_i). If $X - \cup V \neq \emptyset$, then there exists $x \in X - \cup V$, and $U \in \{U_i\}$, such that $x \in U$. Then $V \cap \{U\}$ is a element in Σ which is larger than V, a contradiction. Hence X is quasi-compact.

Exercise 6

Prove that the following are equivalent:

i) X is Noetherian.

ii) Every open subspace of X is quasi-compact.

iii) Every subspace of X is quasi-compact.

Solution: i) $\Rightarrow$ iii) By Exercise 5, every subspace of X is Noetherian and every Noetherian space is quasi-compact.

iii) $\Rightarrow$ ii) Obvious.

ii) $\Rightarrow$ i) If $U_0 \subset U_1 \subset \cdots$ is not stationary, then $\cup_{n=0}^{\infty} U_n$ is an open subspace of X, which has an open cover $\{U_n\}$ without finite subcover, a contradiction.

Exercise 7

A Noetherian space is a finite union of irreducible closed subspaces. Hence the set of irreducible components of a Noetherian space is finite.

Solution: Suppose the converse. Consider the set Σ of closed subsets of X which are not finite unions of closed subsets of X which are not finite unions of irreducible closed subspaces, then Σ is not empty. By the d.c.c. on closed sets of X, Σ has a minimal element C. C is not the union of finitely many irreducible closed subspace, so it is not irreducible. There are open subsets U_1, U_2 of C, such that $U_1 \cap U_2 = \emptyset$, which is $(C - U_1) \cup (C - U_2) = C$. By the minimality of C, $C - U_1$ and $C - U_2$ are closed subsets which are finite unions of irreducible closed subspaces, then so is C, a contradiction.

If $X = \cup_{i=1}^{n} C_i^{'}$ where $C_i^{'}$ is a closed irreducible subspace, then $X = \cup_{i=1}^{n} C_i$, where C_i is the closed irreducible component which contains $C_i^{'}$. For any irreducible closed subset C of X, we have $C = \cup_{i=1}^{n} (C \cap C_i)$, so $\cap_{i=1}^{n} (C - C \cap C_i) = \emptyset$. But this implies that $C - C_i = \emptyset$ for some i and that $C \subset C_i$. Hence the set of irreducible components of a Noetherian space is finite.

Exercise 8

If A is a Noetherian ring then Spec(A) is a Noetherian topological space. Is the converse true?

Solution: If $V(I_1) \supset V(I_2) \supset \cdots$, then $r(I_1) \subset r(I_2) \subset \cdots$, and there exists n such that $\cdots \subset r(I_n) = r(I_{n+1}) = \cdots$ since A is Noetherian, and $\cdots \supset V(I_n) = V(I_{n+1}) = \cdots$. The closed sets of Spec(A) satisfy d.c.c., so Spec(A) is Noetherian. The converse is not generally true.

Exercise 9

Deduce from Exercise 8 that the set of minimal prime ideals in a Noetherian ring is finite.

Solution: Every minimal prime ideal corresponds to a closed irreducible component. By Exercise 7 and 8, the set of closed irreducible components is finite, so the set of minimal prime ideals in a Noetherian ring is finite.

Exercise 10

If M is a Noetherian module (over an arbitrary ring A) then $\operatorname{Supp}(M)$ is a closed Noetherian subspace of $\operatorname{Spec}(A)$.

Solution: M is a Noetherian module, so it is finitely generated, and $\operatorname{Supp}(M) = V(\operatorname{Ann}(M)) \cong \operatorname{Spec}(A/\operatorname{Ann}(M))$ by Chapter 3, Exercise 19, v) and Chapter 1, Exercise 21, iv), hence is closed. By Exercise 4, $\operatorname{Spec}(A/\operatorname{Ann}(M))$ is Noetherian.

Exercise 11

Let $f : A \to B$ be a ring homomorphism and suppose that $\operatorname{Spec}(B)$ is a Noetherian space (Exercise 5). Prove that $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a closed mapping if and only if f has the going-up property (Chapter 5, Exercise 10).

Solution: TO BE ADDED.

Exercise 12

Let A be a ring such that $\operatorname{Spec}(A)$ is a Noetherian space. Show that the set of prime ideals of A satisfies the ascending chain condition. Is the converse true?

Solution: If $P_1 \subset P_2 \subset \cdots$, then $V(P_1) \supset V(P_2) \supset \cdots$ and there exists n such that $\cdots \supset V(P_n) = V(P_{n+1}) = \cdots$ since $\operatorname{Spec}(A)$ is Noetherian. Hence $r(P_n) = r(P_{n+1}) = \cdots$ but $r(P_i) = P_i$ since P_i is prime, so the set of prime ideals of A satisfies the ascending chain condition. The converse is not true.