Chapter II: Schemes

1. Sheaves

Xie, Kaitao

Exercise 2.1.1

Let A be an abelian group, and define the constant presheaf associated to A on the topological space X to be the presheaf $U \to A$ for all $U \neq \emptyset$, with restriction maps the identity. Show that the constant sheaf $\mathscr{A}$ defined in the text is the sheaf associated to this presheaf.

Solution: Denote this presheaf by A. There is a natural presheaf map $sh : A \to \mathscr{A}$. Suppose that $\mathscr{F}$ is a sheaf, and $\varphi : A \to \mathscr{F}$ is a presheaf map. If U is an open set of X and $U = \coprod U_i$, where each U_i is connected, $s \in \mathscr{A}(U)$, then $s(U_i) = \{a_i\} \subseteq A$ since s is continuous. There is only one way to define $\varphi^+ : \mathscr{A} \to \mathscr{F}$ such that $\varphi^+ \circ sh = \varphi$, namely, $\varphi^+_U(s) =$ gluing section of $\phi_{U_i}(a_i)$.

Exercise 2.1.2

(a) For any morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, show that for each point P, $(ker\ \varphi)_P = ker(\varphi_P)$ and $(im\ \varphi)_P = im(\varphi_P)$.

(b) Show that ϕ is injective (respectively, surjective) if and only if the induced map on the stalks ϕ_P is injective (respectively, surjective) for all P.

(c) Show that a sequence $... \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to ...$ of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

Solution: (a) By definition of ϕ_P we have that $\varphi_P(s, U) = (\varphi_U(s), U)$. If $(s, U) \in ker(\varphi_P)$ then $\rho^U_V(\varphi_U(s)) = 0$ for some open set V containing p and contained in U, but $\rho^U_V(\varphi_U(s)) = \varphi_V(\rho^U_V(s)) = 0$, so $\rho^U_V(s) \in ker(\varphi_V)$, and $(\rho^U_V(s), V) \in (ker\ \varphi)_P$, that is, $(s, U) \in (ker\ \varphi)_P$. All this implications can be reversed, so we have that $(ker\ \varphi)_P = ker(\varphi_P)$.

According to the definition of sheafification, one need only to verify this equation with regarding $im\ \varphi$ as a presheaf. If $(t, V) \in im(\varphi_P)$, then there is a pair $(s, U) \in \mathscr{F}_P$, such that $(\phi_U(s), U) \sim (t, V)$, so there is an open subset W of $U \cap V$, containing P such that $\rho^U_W(\varphi_U(s)) = \varphi_W(\rho^U_W(s)) = \rho^V_W(t)$. Hence $(\rho^U_W(s), W) \in (im\ \varphi)_P$, that is, $(s, U) \in (im\ \varphi)_P$. Again, all the implications can be reversed, so $(im\ \varphi)_P = im(\varphi_P)$.

(b) If φ is injective, then $ker\ \varphi = 0$, and $(ker\ \varphi)_P = ker(\varphi_P) = 0$ by (a), so φ_P is injective, and vice versa. Same for the condition of surjective.

(c) If $ker\ \varphi^i = im\ \varphi^{i-1}$, then their stalks at P are equal, so by (a) we have that $ker(\varphi^i_P) = im(\varphi^{i-1}_P)$, and we have that $\mathscr{F}^{i-1}_P \to \mathscr{F}^i_P \to \mathscr{F}^{i+1}_P$ is exact. Vice versa.

Exercise 2.1.3

(a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\varphi(t_i) = s|_{U_i}$, for all i.

(b) Give an example of a surjective morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, and an open set U such that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is not surjective.

Solution: (a) Suppose that the condition holds, then for every point P, and $(s, U) \in \mathscr{G}_P$, there is an open subset V of U, containing P, and $t \in \mathscr{F}(V)$ such that $\varphi(t) = s|_V$, so $\varphi_P(t, V) = (s, U)$, and φ_P is surjective for all point P. Hence φ is surjective.

Suppose that φ is surjective, then φ_P is surjective for every $P \in X$. Only need to show that for $P \in U$, there is an open subset V of U containing P, and $t \in \mathscr{F}(V)$, such that $\varphi(t) = s|_V$, but this is clear by the definition of φ_P.

(b) Let $X = \mathbb{C}$, $\mathcal{O}$ be the sheaf of holomorphic functions, $\mathcal{O}^\times$ be the sheaf of nowhere zero holomorphic function, and $\exp : \mathcal{O} \to \mathcal{O}^\times$ sending $f(z)$ to $e^{f(z)}$. This map is surjective since it is surjective at all stalks by basic complex functions theory (that on a simply connected set not containing 0 one can define a branch of logarithm). However, let $U = \mathbb{C} - \{0\}$, then $z \in \mathcal{O}^\times(U)$, while there is no holomorphic function $f(z)$ on U such that $e^{f(z)} = z$.

Exercise 2.1.4

(a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves such that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each U. Show that the induced map $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ of associated sheaves is injective.

(b) Use part (a) to show that if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then *im* φ can be naturally identified with a subsheaf of $\mathscr{G}$. as mentioned in the text.

Solution: (a) φ is injective so φ_P is injective, $(-)_P$ is functorial, so from the diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[d, "sh"'] & \mathscr{G} \arrow[d, "sh"'] \\ \mathscr{F}^+ \arrow[r, "\varphi^+"] & \mathscr{G}^+ \end{tikzcd}$$

we can get

$$\begin{tikzcd} \mathscr{F}_P \arrow[r, "\varphi_P"] \arrow[d, "sh_P"'] & \mathscr{G}_P \arrow[d, "sh_P"'] \\ \mathscr{F}^+_P \arrow[r, "\varphi^+_P"] & \mathscr{G}^+_P \end{tikzcd}$$

for all $P \in X$. However, the two sh_P maps are isomorphism, and φ_P is injective implies that φ_P^+ is injective for all $P \in X$, so φ^+ is injective.

(b) There is a natural presheaf map $i : im\varphi \to \mathscr{G}$ (Here $im\varphi$ is regarded as the presheaf $U \to im\varphi_U$), which is injective, so one get an injective sheaf map $i^+ : (im\varphi)^+ \to \mathscr{G}^+ = \mathscr{G}$. Hence the image sheaf of φ can be regarded as a subsheaf of $\mathscr{G}$.

Exercise 2.1.5

Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective.

Solution: Suppose that $\varphi : \mathscr{F} \to \mathscr{G}$ is an isomorphism, which means that every φ_U is an isomorphism, then φ obviously is injective ($ker\varphi_U = 0$) and surjective ($im\varphi_U = \mathscr{G}(U)$).

Suppose that φ is both injective and surjective, we only need to show that every φ_U is surjective. By Exercise 2.1.3 (a), we know that for every $s \in \mathscr{G}(U)$, we can find an open cover $\{U_i\}$ of U and $t_i \in \mathscr{F}(U_i)$ such that $\varphi_{U_i}(t_i) = s|_{U_i}$. By the injectivity of φ_{U_i}, we know that such t_i is unique for every i. We show that t_i and t_j are restricted to the same element in $\mathscr{F}(U_i \cap U_j)$. Then we can glue these t_i to get a $t \in \mathscr{F}(U)$ such that $\varphi_U(t) = s$.

Check that $\varphi_{U_i \cap U_j}(\rho^{U_i}_{U_i \cap U_j}(t_i)) = \rho^{U_i}_{U_i \cap U_j}(\varphi_{U_i}(t_i)) = s|_{U_i \cap U_j}$. Similarly, $\varphi_{U_i \cap U_j}(\rho^{U_j}_{U_i \cap U_j}(t_j)) = \rho^{U_j}_{U_i \cap U_j}(\varphi_{U_j}(t_j)) = s|_{U_i \cap U_j}$. Hence $\varphi_{U_i \cap U_j}(\rho^{U_i}_{U_i \cap U_j}(t_i)) = \varphi_{U_i \cap U_j}(\rho^{U_j}_{U_i \cap U_j}(t_j))$, but $\varphi_{U_i \cap U_j}$ is injective, so $\rho^{U_i}_{U_i \cap U_j}(t_i) = \rho^{U_j}_{U_i \cap U_j}(t_j)$. This finishes the proof.

Exercise 2.1.6

(a) Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. Show that the natural map of $\mathscr{F}$ to the quotient sheaf $\mathscr{F}/\mathscr{F}'$ is surjective, and has kernel $\mathscr{F}'$. Thus there is an exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0$$

.

(b) Conversely, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, show that $\mathscr{F}'$ is isomorphic to a subsheaf of $\mathscr{F}$, and that $\mathscr{F}''$ is isomorphic to the quotient of $\mathscr{F}$ by this subsheaf.

Solution: (a) The natural map is given by the composition $\mathscr{F} \xrightarrow{\pi} \mathscr{F}/\mathscr{F}' \xrightarrow{sh} (\mathscr{F}/\mathscr{F}')^+$. (Here $\mathscr{F}/\mathscr{F}'$ is regarded as a presheaf.) We apply the functor $(-)_P$ here to get $\mathscr{F}_P \xrightarrow{\pi_P} (\mathscr{F}/\mathscr{F}')_P \xrightarrow{sh_P} (\mathscr{F}/\mathscr{F}')^+_P$. Here sh_P is an isomorphism, $(\mathscr{F}/\mathscr{F}')_P = \mathscr{F}_P/\mathscr{F}'_P$ and π_P is the natural projection $\mathscr{F}_P \to \mathscr{F}_P/\mathscr{F}'_P$, which is surjective. Hence the natural map $\mathscr{F} \to (\mathscr{F}/\mathscr{F}')^+$ is surjective since it is surjective on all the stalks. The kernel of the induced stalk map on P is $\mathscr{F}'_P$, so the kernel of the natural map is $\mathscr{F}'$.

(b) Denote the second map by ϕ. ϕ is injective so by Exercise 2.1.4 (b), $\mathscr{F}'$ is isomorphic to $im\phi$ which can be regarded as a subsheaf of $\mathscr{F}$. From (a), we have an exact sequence $0 \to im\phi \to \mathscr{F} \to \mathscr{F}/im\phi \to 0$. Since $\mathscr{F}' \cong im\phi$, we have that $\mathscr{F}'' \cong \mathscr{F}/im\phi$.

Exercise 2.1.7

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.

(a) Show that $im\varphi \cong \mathscr{F}/ker\varphi$.

(b) Show that $coker\varphi \cong \mathscr{G}/im\varphi$.

Solution: (a) We already know that $(im\varphi)_P = im(\varphi_P)$ and that $(\mathscr{F}/ker\varphi)_P = \mathscr{F}_P/(ker\varphi)_P = \mathscr{F}_P/ker(\varphi_P)$, so $(im\varphi)_P = (\mathscr{F}/ker\varphi)_P$ since $im(\varphi_P) = \mathscr{F}_P/ker(\varphi_P)$. Hence the exact sequence $0 \to ker\ \varphi \to \mathscr{F} \to im\ \varphi \to 0$. Then Exercise 2.1.6 implies the result.

(b) As above, we only need to prove that $(coker\varphi)_P = coker(\varphi_P) = \mathscr{G}_P/im(\varphi_P)$. This is obvious from the definition.

Exercise 2.1.8

For any open subset $U \subseteq X$, show that the functor $\Gamma(U, \cdot)$ from sheaves on X to abelian groups is a left exact functor, i.e., if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ is an exact sequence of sheaves, then $0 \to \Gamma(U, \mathscr{F}') \to \Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{F}'')$ is an exact sequence of groups. The functor $\Gamma(U, \cdot)$ need not be exact; see (Ex. 1.21) below.

Solution: Denote $\mathscr{F}' \to \mathscr{F}$ by ϕ and $\mathscr{F} \to \mathscr{F}''$ by φ. We already know that the injectivity of ϕ implies the injectivity of ϕ_U for all U open in X. The only thing remained to show is that $ker(\varphi_U) \subseteq im(\phi_U)$ for every open subset U of X.

From the assumption that $ker(\varphi) = im(\phi)$, we know that $ker(\varphi_P) = im(\phi_P)$ for all $P \in X$. For $s \in ker(\varphi_U)$, $s_P \in ker(\varphi_P)$, so there exists t_P such that $\phi_P(t_P) = s_P$. In this way we obtain an open cover $\{U_i\}$ of U and $t_i \in \mathscr{F}'(U_i)$ such that $\phi_{U_i}(t_i) = s|_{U_i}$. By the injectivity of each ϕ_U, we can apply the method used in Exercise 2.1.5 to check that t_i and t_j admit on $U_i \cap U_j$ for all i and j, so we can glue all this t_i to get a $t \in \mathscr{F}'(U)$ such that $\phi_U(t) = s$. Then $ker(\varphi_U) \subseteq im(\phi_U)$.

Exercise 2.1.9

Direct Sum. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X. Show that the presheaf $U \to \mathscr{F}(U) \oplus \mathscr{G}(U)$ is a sheaf. It is called the direct sum of $\mathscr{F}$ and $\mathscr{G}$. and is denoted by $\mathscr{F} \oplus \mathscr{G}$. Show that it plays the role of direct sum and of direct product in the category of sheaves of abelian groups on X.

Solution: We denote element in $\mathscr{F}(U) \oplus \mathscr{G}(U)$ by (s,t) where $s \in \mathscr{F}(U)$ and $t \in \mathscr{G}(U)$. If $\{U_i\}$ is an open cover of U and $(s,t)|_{U_i} = 0$ for all i, then $s|_{U_i} = 0$ and $t|_{U_i} = 0$ for all i, so $s = 0$ and $t = 0$ since $\mathscr{F}$ and $\mathscr{G}$ are sheaves. Hence $(s,t) = (0,0)$. Now if we have $(s_i, t_i) \in \mathscr{F}(U_i) \oplus \mathscr{G}(U_i)$ that agree on the intersection of U_i's, then it works for all s_i's and t_i's, so we can glue these s_i to get an $s \in \mathscr{F}(U)$ and a $t \in \mathscr{G}(U)$, so a $(s,t) \in \mathscr{F}(U) \oplus \mathscr{G}(U)$. Hence $\mathscr{F} \oplus \mathscr{G}$ is indeed a sheaf.

There are two natural projections $\pi_1 : \mathscr{F} \oplus \mathscr{G} \to \mathscr{F}$, $\mathscr{F} \oplus \mathscr{G} \to \mathscr{G}$ given by $\pi_{1,U}(s,t) = s$ and $\pi_{2,U}(s,t) = t$, and two natural inclusions $i_1 : \mathscr{F} = \mathscr{F} \oplus \mathscr{G}$ and $i_2 : \mathscr{G} \to \mathscr{F} \oplus \mathscr{G}$ given by $i_{1,U}(s) = (s,0)$ and $i_{2,U}(t) = (0,t)$. If there are $\phi_1 : \mathscr{H} \to \mathscr{F}$ and $\phi_2 : \mathscr{H} \to \mathscr{G}$, then there is a unique way to define a map $\phi : \mathscr{H} \to \mathscr{F} \oplus \mathscr{G}$ such that $\phi_k = \pi_k \circ \phi$ for $k = 1,2$, namely, $\phi_U(h) = (\phi_{1,U}(h), \phi_{2,U}(h))$. Similarly, if there are $\varphi_1 : \mathscr{F} \to \mathscr{H}$, $\varphi_2 : \mathscr{G} \to \mathscr{H}$, there is a unique way to define a map $\varphi : \mathscr{F} \oplus \mathscr{G} \to \mathscr{H}$ such that $\varphi \circ i_k = \varphi_k$ for $k = 1,2$, namely, $\varphi_U(s,t) = \varphi_{1,U}(s) + \varphi_{2,U}(t)$. This shows that $\mathscr{F} \oplus \mathscr{G}$ plays the roles of direct sum and direct product of $\mathscr{F}$ and $\mathscr{G}$.

Exercise 2.1.10

Direct Limit. Let $\{\mathscr{F}_i\}$ be a direct system of sheaves and morphisms on X. We define the direct limit of the system $\{\mathscr{F}_i\}$, denoted $\varinjlim \mathscr{F}_i$, to be the sheaf associated to the presheaf $U \to \varinjlim \mathscr{F}_i(U)$. Show that this is a direct limit in the category of sheaves on X, i.e., that it has the following universal property: given a sheaf $\mathscr{G}$, and a collection of morphisms $\mathscr{F}_i \to \mathscr{G}$ compatible with the maps of the direct system, then there exists a unique map $\varinjlim \mathscr{F}_i \to \mathscr{G}$ such that for each i, the original map $\mathscr{F}_i \to \mathscr{G}$ is obtained by composing the maps $\mathscr{F}_i \to \varinjlim \mathscr{F}_i \to \mathscr{G}$.

Solution: For every open subset $U \subseteq X$, $\{\mathscr{F}_i(U)\}$ is a direct system of abelian groups, and $\varinjlim \mathscr{F}_i(U)$ is the direct limit of this direct system, so there are presheaf maps $\mathscr{F}_i \to \varinjlim \mathscr{F}_i$ (that compatible with the restriction maps). This map, composites with the sheafification map, gives the natural map $\mathscr{F}_i \to \varinjlim \mathscr{F}_i$.

Given a sheaf $\mathscr{G}$, and a collection of morphisms $\mathscr{F}_i \to \mathscr{G}$ compatible with the maps of the direct system, for every open subset $U \subseteq X$, $\mathscr{G}(U)$ is an abelian group and there are maps $\mathscr{F}_i(U) \to \mathscr{G}(U)$ that are compatible with the maps of the direct system, then there is a unique map $\varinjlim \mathscr{F}_i(U) \to \mathscr{G}(U)$ such that it has the right property by the universal property of direct limit, which gives a presheaf map. Then by the universal property of sheafification, there is a natural sheaf map $\varinjlim \mathscr{F}_i \to \mathscr{G}$. This is the map we want.

Exercise 2.1.11

Let $\{\mathscr{F}_i\}$ be a direct system of sheaves on a noetherian topological space X. In this case show that the presheaf $U \to \varinjlim \mathscr{F}_i(U)$ is already a sheaf. In particular, $\Gamma(X, \varinjlim \mathscr{F}_i) = \varinjlim \Gamma(X, \mathscr{F}_i)$.

Solution: $\varinjlim F_i(U) = \{s_i \in \mathscr{F}_i\}/\{s_i \sim s_j \Leftrightarrow s_j$ is the image of s_i by $\mathscr{F}_i(U) \to \mathscr{F}_j(U)\}$, and the restriction map is given by $s_i \to \rho_i(s_i)$. Let $\{U_\alpha\}$ be an open cover of U. From these descriptions, if a section in the direct limit sheaf over U satisfies $s|_{U_\alpha} = 0$ for all α, then $s = 0$.

Note that X is noetherian implies that every open subset U is quasi-compact, so we can find a finite subcover $\{U_{\alpha_j}\}$ of $\{U_\alpha\}$. If $s_{i,\alpha} \in \varinjlim \mathscr{F}_i(U_\alpha)$, and any two s_α and s_β admit on intersection, and if we can glue s_{i_j,α_j} then this is the section in $\varinjlim \mathscr{F}_i(U)$ we want for gluing all $s_{i,\alpha}$. This can be done in this way: since the index set $\{i\}$ is directed, and $\{i_j\}$ is finite, we can find a $k \in \{i\}$ such that $k \geq i_j$ for all j. Then we push s_{i_j,α_j} to $\mathscr{F}_k(U_\alpha)$, and glue them in the sheaf $\mathscr{F}_k$.

Exercise 2.1.12

Inverse Limit. Let $\{\mathscr{F}_i\}$ be an inverse system of sheaves on X. Show that the presheaf $U \to \varprojlim \mathscr{F}_i(U)$ is a sheaf. It is called the inverse limit of the system $\{\mathscr{F}_i\}$, and is denoted by $\varprojlim F_i$. Show that it has the universal property of an inverse limit in the category of sheaves.

Solution: $\varprojlim \mathscr{F}_i(U) = \{(s_i) \in \prod \mathscr{F}_i(U) : s_i$ is the image of s_j by $\mathscr{F}_j \to \mathscr{F}_i$ for all $i \leq j\}$. The restriction map is given by $(s_i) \to (\rho_i(s_i))$. From these descriptions, if $\{U_\alpha\}$ is an open cover of U, $s \in \varprojlim \mathscr{F}_i(U)$ and $s|_{U_\alpha} = 0$ for all α, then $s = 0$.

If $s_\alpha \in \varprojlim \mathscr{F}_i(U_\alpha)$, and any two s_α and s_β admit on intersection, then we can glue them coordinate by coordinate to get an $s \in \prod \mathscr{F}_i(U)$. Keeping in mind that each $\mathscr{F}_j \to \mathscr{F}_i$ is compatible with the restriction maps in two sheaves, we find that the coordinates of s behave well, i.e. s_i is the image of s_j by $\mathscr{F}_j \to \mathscr{F}_i$ for all $i \leq j\}$, so $s \in \varprojlim \mathscr{F}_i(U)$.

The proof of that this sheaf satisfies the universal property of inverse limit is similar to one in Exercise 2.1.10.

Exercise 2.1.13

Espace Étalé of a Presheaf. (This exercise is included only to establish the connection between our definition of a sheaf and another definition often found in the literature. See for example Godement [1, Ch. II, §1.2].) Given a presheaf $\mathscr{F}$ on X, we define a topological space $Sp\acute{e}(\mathscr{F})$. called the *espace étalé* of $\mathscr{F}$, as follows. As a set, $Sp\acute{e}(\mathscr{F}) = \cup_{P \in X} \mathscr{F}_P$. We define a projection map $\pi : Sp\acute{e}(\mathscr{F}) \to X$ by sending $s \in \mathscr{F}_P$ to P. For each open set $U \subseteq X$ and each sections $s \in \mathscr{F}(U)$, we obtain a maps: $U \to Sp\acute{e}(\mathscr{F})$ by sending $P \to s_P$, its germ at P. This map has the property that $\pi \circ \bar{s} = id_U$, in other words, it is a "section" of n over U. We now make $Sp\acute{e}(\mathscr{F})$ into a topological space by giving it the strongest topology such that all the maps $s : U \to Sp\acute{e}(\mathscr{F})$ for all U, and all $s \in \mathscr{F}(U)$, are continuous. Now show that the sheaf $\mathscr{F}^+$ associated to $\mathscr{F}$ can be described as follows: for any open set $U \subseteq X$, $\mathscr{F}^+(U)$ is the set of continuous sections of $Sp\acute{e}(\mathscr{F})$ over U. In particular, the original presheaf $\mathscr{F}$ was a sheaf if and only if for each U, $\mathscr{F}(U)$ is equal to the set of all continuous sections of $Sp\acute{e}(\mathscr{F})$ over U.

Solution: We denote the original construction of sheafification of $\mathscr{F}$ by $\mathscr{F}^{sh}$. We define a map $\mathscr{F}^{sh} \to \mathscr{F}^+$. Over U, it is given by sending $(f_P)_{P \in U}$ to $\bar{f}$ where $\bar{f}(P) = f_P$. First we should verify that $\bar{f} \in \mathscr{F}^+(U)$. Note that a basis in the topological space $Sp\acute{e}(\mathscr{F})$ is of the form $B_s = \{s_P : s \in \mathscr{F}(V)\}$, and $\bar{f}^{-1}(B_s) = \{Q : f_Q = s_Q\}$. If $P \in \{Q : f_Q = s_Q\}$, then by definition of sheafification, there is an open set $V \subseteq U$ that contains P, $s^{'} \in \mathscr{F}(V)$ such that $f_Q = s^{'}_Q$ for all $Q \in V$, so $s_P = s^{'}_P$ and there is an open set $W \subseteq V$ that contains P, such that $s|_W = s^{'}|_W$, so $s_Q = f_Q$ for all $Q \in W$. This shows that $\bar{f}^{-1}(B_s)$ is open, so $\bar{f}$ is continuous, hence in $\mathscr{F}^+(U)$.

It is clearly from the definition that this map commutes with the restriction maps. It remains to verify that this map is injective and surjective. It is clear from the definition of continuous function that this map is injective. If $\bar{f} \in \mathscr{F}^+(U)$, then we need to verify that $(\bar{f}(P))_{P \in U} \in \mathscr{F}^{sh}(U)$. This can be deduced from the fact that $\bar{f}^{-1}(B_s)$ is open. See the first paragraph.

Exercise 2.1.14

Support. Let $\mathscr{F}$ be a sheaf on X, and let $s \in \mathscr{F}(U)$ be a section over an open set U. The support of s, denoted Supp s, is defined to be $\{P \in U | s_P \neq 0\}$, where s_P denotes the germ of s in the stalk $\mathscr{F}_P$. Show that Supp s is a closed subset of U. We define the support of $\mathscr{F}$, Supp $\mathscr{F}$, to be $\{P \in X | \mathscr{F}_P \neq 0\}$. It need not be a closed subset.

Solution: Only need to prove that $\{P \in U : s_P = 0\}$ is an open subset of U. $s_P = 0$ means that there is a open subset of $V \subset U$ such that $P \in V$ and $\rho^U_V(s) = 0$, so $\forall Q \in V$, $s_Q = 0$. This shows that $\{P \in U : s_Q = 0\}$ is open.

Exercise 2.1.15

Sheaf Hom. Let $\mathscr{F}, \mathscr{G}$ be sheaves of abelian groups on X. For any open set $U \subseteq X$, show that the set $\mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of abelian group. Show that the presheaf $U \to \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf. It is called the sheaf of local morphisms of $\mathscr{F}$ into $\mathscr{G}$, "sheaf hom" for short, and is denoted $\mathscr{H}om(\mathscr{F}, \mathscr{G})$.

Solution: For $\psi, \varphi \in \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$, define $\psi + \varphi$ by $(\psi + \varphi)_V(s) = \psi_V(s) + \varphi_V(s)$, where V is an open subset of U and $s \in \mathscr{F}(V)$. This is a well-defined sheaf map since for every open subset $W \subseteq V$,

$$(\psi + \varphi)_V(\rho^V_W(s)) = \psi_W(\rho^V_W(s)) + \varphi_W(\rho^V_W(s)) = \rho^V_W(\psi_V(s)) + \rho^V_W(\varphi_V(s)) = \rho^V_W((\psi + \varphi)_V(s)).$$

Hence $\mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is an abelian group. The definition of the restriction maps is clear.

Suppose that $\{U_\alpha\}$ is an open cover of U. If $\varphi \in \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ satisfying $\varphi|_{U_\alpha} = 0$ for all α, and if V is an open subset of U, then $U_\alpha \cap V$ is an open cover of V, and for every $s \in \mathscr{F}(V)$, $\rho^V_{U_\alpha}(\varphi_V(s)) = \varphi_{U_\alpha}(\rho^V_{U_\alpha \cap V}(s)) = 0$. Since $\mathscr{F}$ is a sheaf, $\varphi_V(s) = 0$. This shows that $U \to \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ satisfies the first condition of sheaf.

Suppose that $\varphi_\alpha \in \mathrm{Hom}(\mathscr{F}|_{U_\alpha}, \mathscr{G}|_{U_\alpha})$ that admit on the intersection. We define $\varphi \in \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ by $\varphi_V(s) =$ glue all $\varphi_{\alpha, V \cap U_\alpha}(\rho^V_{V \cap U_\alpha}(s))$, where V is an open subset of U and $s \in \mathscr{F}(V)$. The gluing can be done since $\mathscr{G}$ is a sheaf, and that φ_α admit on the intersection. One can verify directly that this is a sheaf map, so $U \to \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf.

Exercise 2.1.16

Flasque Sheaves. A sheaf $\mathscr{F}$ on a topological space X is Flasque if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.

(a) Show that a constant sheaf on an irreducible topological space is flasque.

(b) If $0 \to \mathscr{F}^{'} \to \mathscr{F} \to \mathscr{F}^{''} \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}^{'}$ is flasque, then for any open set U, the sequence $0 \to \mathscr{F}^{'}(U) \to \mathscr{F}(U) \to \mathscr{F}^{''}(U) \to 0$ of abelian groups is also exact.

(c) If $0 \to \mathscr{F}^{'} \to \mathscr{F} \to \mathscr{F}^{''} \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}$ and $\mathscr{F}^{'}$ are flasque, then $\mathscr{F}^{''}$ is flasque.

(d) If $f : X \to Y$ is a continuous map, and if $\mathscr{F}$ is a flasque sheaf on X, then $f_*\mathscr{F}$ is a flasque sheaf on Y.

(e) Let $\mathscr{F}$ be any sheaf on X. We define a new sheaf $\mathscr{G}$, called the sheaf of discontinuous sections of $\mathscr{F}$ as follows. For each open set $U \subseteq X$, $\mathscr{G}(U)$ is the set of maps $s : U \to \cup_{P\in U}\mathscr{F}_P$ such that for each $P \in U$, $s(P) \in \mathscr{F}_P$. Show that $\mathscr{G}$ is a flasque sheaf, and that there is a natural injective morphism of $\mathscr{F}$ to $\mathscr{G}$.

Solution: (a) In an irreducible topological space, every open subset is irreducible and hence connected, so the constant sheaf over every non-empty open subset is just the abelian group used to define the sheaf, and the restriction maps are just the identity map, so this sheaf is flasque.

(b) (Follow Bryden R. Cais' Solution) Denote the first nontrivial map by ϕ and the second by φ. By the fact that φ is surjective, for every $s \in \mathscr{F}^{''}(U)$, there are an open cover $\{U_\alpha\}_{\alpha\in I}$ of U and $t_\alpha \in \mathscr{F}(U_\alpha)$ such that $\varphi_{U_\alpha}(t_\alpha) = s|_{U_\alpha}$. Let $S = \{(f, J) : J \subset I, f \in \mathscr{F}(\cup_{\alpha\in J}U_\alpha)$ and $\varphi(f) = s|_{\cup_{\alpha\in J}U_\alpha}\}$. Define an order on S by $(f^{'}, J^{'}) \leq (f, J)$ if $J^{'} \subseteq J$ and $f|_{\cup_{\alpha\in J^{'}}U_\alpha} = f^{'}$. S is non-empty since $(t_\alpha, \{\alpha\}) \in S$. Every ascending chain is bounded by the gluing axiom of sheaf, so there is a maximal element (f, I_0) in S.

We show that $I_0 = I$ and complete the proof. If not, then there is an $\alpha_0 \in I - I_0$. Let $V = \cup_{\alpha\in I_0}U_\alpha$ and $y = f|_{V\cap U_{\alpha_0}} - t_{\alpha_0}|_{V\cap U_{\alpha_0}}$, then $\varphi_{V\cap U_{\alpha_0}}(x) = 0$, so there is a $x \in \mathscr{F}^{'}(U_{\alpha_0} \cap V)$ that is mapped to y. $\mathscr{F}^{'}$ is flasque, so there is an $x^{'} \in \mathscr{F}(U_{\alpha_0})$ such that $x^{'}|_{V\cap U_\alpha} = x$, let $y^{'}$ denote its image in $\mathscr{F}(U_{\alpha_0})$. Define $t^{'}_{\alpha_0} = t_{\alpha_0} + y^{'}$, then we can glue f and $t^{'}_{\alpha_0}$ to get a section on $V^{'} = U_{\alpha_0} \cup V$, which is mapped to $s|_{V^{'}}$. This is a contradiction to the fact that (f, I_0) is maximal.

(c) Note that we have the following commutative diagram by (b) since $\mathscr{F}^{'}$ is flasque:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}^{'}(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{F}^{''}(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}^{'}(V) \arrow[r] & \mathscr{F}(V) \arrow[r] & \mathscr{F}^{''}(V) \arrow[r] & 0
\end{tikzcd}$$

for all $V \subseteq U$. Each horizontal sequence is exact, and the second vertical arrow is surjective, so the last vertical arrow is surjective, and $\mathscr{F}^{''}$ is flasque.

(d) If $V \subseteq U$, then $f^{-1}(V) \subseteq f^{-1}(U)$, so $f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U)) \to f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$ is surjective since $\mathscr{F}$ is flasque.

(e) The restriction map is defined by usual restriction of function, which is surjective (for example, extending by 0 outside V). The natural sheaf map is given by $f \to (s(P) = f_P)$. This is injective since $f = 0$ if and only if $f_P = 0$ for all $P \in U$.

Exercise 2.1.17

Skyscraper Sheaves. Let X be a topological space, let P be a point, and let A be an abelian group. Define a sheaf $i_P(A)$ on X as follows: $i_P(A)(U) = A$ if $P \in U$, 0 otherwise. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \{P\}^-$, and 0 elsewhere, where $\{P\}^-$ denotes the closure of the set consisting of the point P. Hence the name "skyscraper sheaf". Show that this sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\{P\}^-$, and $i : \{P\}^- \to X$ is the inclusion.

Solution: For every point $Q \in \{P\}^-$, and every open subset U that contains Q, U contains P, so $i_P(A)(U) = A$. This shows that $i_P(A)_Q = A$. If $Q \notin \{P\}^-$, then there is a neighborhood V of Q such that V does not contain P, so $i_P(A)(V) = 0$ and $i_P(A)_Q = 0$.

$i_*(A)(U) = A(i^{-1}(U))$. If U contains P then $i^{-1}(U) \cap \{P\}^- \neq \emptyset$; if not, then $i^{-1}(U) = \emptyset$, so $i_*(A) = i_P(A)$.

Exercise 2.1.18

Adjoint Property of f^{-1}. Let $f : X \to Y$ be a continuous map of topological spaces. Show that for any sheaf $\mathscr{F}$ on X there is a natural map $f^{-1}f_*\mathscr{F} \to \mathscr{F}$, and for any sheaf $\mathscr{G}$ on Y there is a natural map $\mathscr{G} \to f_*f^{-1}\mathscr{G}$. Use these maps to show that there is a natural bijection of sets, for any sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y,

$$\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) \to \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}).$$

Hence we say that f^{-1} is a left adjoint of f_* and that f_* is a right adjoint of f^{-1}.

Solution: If $f(U) \subseteq V$, then $U \subseteq f^{-1}(V)$, so the restriction map $\mathscr{F}(f^{-1}(V)) \to \mathscr{F}(U)$. They induce $f^{-1}f_*\mathscr{F}(U) = \lim\limits_{V \subseteq f(U)} \mathscr{F}(f^{-1}(V)) \to \mathscr{F}(U)$. Composing with the sheafification map, we get the map $\varphi : f^{-1}f_*\mathscr{F} \to \mathscr{F}$. Observe that $f(f^{-1}(V)) \subseteq V$, we have the "inclusion" $\mathscr{G}(V) \to f_*f^{-1}\mathscr{G}(V) = \lim\limits_{f(f^{-1}(V)) \subseteq W} \mathscr{G}(W)$. Again, composing with the sheafification map we get the map $\psi : \mathscr{G} \to f_*f^{-1}\mathscr{G}$.

Now that f_* is functorial, $h : f^{-1}\mathscr{G} \to \mathscr{F}$ gives $f^*h : f_*f^{-1}\mathscr{G} \to f_*\mathscr{F}$. Composing with $\mathscr{G} \to f_*f^{-1}\mathscr{G}$ we get a map $\mathscr{G} \to f_*\mathscr{F}$. The other direction is similar.

It remains to show that the compositions of two maps in the two directions are identities. Namely, $f^{-1}\psi \circ (f^{-1}f_*h) \circ \phi = h$ and the other direction. We just verify the first one. We may reduce the equation to every stalk, but $(f_*\alpha)_P = \alpha_P$, $(f^{-1}\beta)_Q = \beta_Q$, $\varphi_P = id_{\mathscr{F}_P}$, and $\psi_P = id_{\mathscr{G}_Q}$ for all sheaf maps α, β, together gives us the equation at all stalks.

Exercise 2.1.19

Extending a Sheaf by Zero. Let X be a topological space, let Z be a closed subset, let $i : Z \to X$ be the inclusion. let $U = X - Z$ be the complementary open subset, and let $j : U \to X$ be its inclusion.

(a) Let $\mathscr{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathscr{F})_P$ of the direct image sheaf on X is $\mathscr{F}_P$ if $P \in Z$, 0 if $P \notin Z$. Hence we call $i_*\mathscr{F}$ the sheaf obtained by extending $\mathscr{F}$ by zero outside Z. By abuse of notation we will sometimes write $\mathscr{F}$ instead of $i_*\mathscr{F}$, and say "consider $i_*\mathscr{F}$ as a sheaf on X," when we mean "consider $i_*\mathscr{F}$."

(b) Now let $\mathscr{F}$ be a sheaf on U. Let $j_!\mathscr{F}$ be the sheaf on X associated to the presheaf $V \to \mathscr{F}(V)$ if $V \subset U$, $V \to 0$ otherwise. Show that the stalk $(j_!\mathscr{F})_P$ is equal to $\mathscr{F}_P$ if $P \in U$, 0 if $P \notin U$, and show that $j_!\mathscr{F}$ is the only sheaf on X which has this property, and whose restriction to U is $\mathscr{F}$. We call $j_!\mathscr{F}$ the sheaf obtained by extending $j_!\mathscr{F}$ by zero outside U.

(c) Now let $\mathscr{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X,

$$0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0.$$

Solution: (a) If $P \in Z$, then for every open set $U \ni P$, $i_*\mathscr{F}(U) = \mathscr{F}(U \cap Z)$. Hence $(i_*\mathscr{F})_P = \mathscr{F}_P$. If $P \notin Z$, then there exists an open set, containing P, that does not intersect Z, so $i_*\mathscr{F}(U) = \mathscr{F}(U \cap Z) = \mathscr{F}(\varnothing) = 0$, and $(i_*\mathscr{F})_P = 0$.

(b) This is clear from the fact that the stalk of a presheaf is equal to the stalk of its sheafification.

(c) The presheaf used to define $j_!(\mathscr{F}|_U)$ gives raise to a presheaf map to $\mathscr{F}$, namely, identity if $V \subseteq U$, and zero inclusion otherwise. Hence there is a sheaf map $j_!(\mathscr{F}|_U) \to \mathscr{F}$. Note that if $P \in U$, the induced map at P is $j_!(\mathscr{F}|_U)_P \to \mathscr{F}_P$ is just the identity map, and if $P \notin U$, the induced map at P is $j_!(\mathscr{F}|_U)_P \to \mathscr{F}_P$ is just the zero inclusion.

By the definition in the text, $\mathscr{F}|_Z = i^{-1}\mathscr{F}$, where $i : Z \to X$ is the inclusion, and $(i^{-1}\mathscr{F})_P = \mathscr{F}_P$. By Exercise 2.1.18, there is a natural map $\mathscr{F} \to i_* i^{-1}\mathscr{F}$. The induced map at P is identity if $P \in Z$, and zero inclusion otherwise.

By above there is a sequence $0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0$, to verify that it is exact, we just verify the exactness at all stalks. However, the induced sequence is $0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0 \to 0$ at $P \in U$, and $0 \to 0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0$ at $P \in Z$. The non-trivial maps here are all identity, so at every stalk the sequence is obviously exact. Hence the sheaf sequence is exact.

Exercise 2.1.20

Subsheaf with Supports. Let Z be a closed subset of X, and let $\mathscr{F}$ be a sheaf on X. We define $\Gamma_Z(X, \mathscr{F})$ to be the subgroup of $\Gamma(X, \mathscr{F})$ consisting of all sections whose support (Ex. 1.14) is contained in Z.

(a) Show that the presheaf $V \to \Gamma_{Z\cap V}(V, \mathscr{F}|_V)$ is a sheaf. It is called the subsheaf of $\mathscr{F}$ with supports in Z, and is denoted by $\mathscr{H}_Z^0(\mathscr{F})$.

(b) Let $U = X - Z$, and let $U \to X$ be the inclusion. Show there is an exact sequence of sheaves on X

$$0 \to \mathscr{H}_Z^0(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U).$$

Furthermore, if $\mathscr{F}$ is flasque, the map $\mathscr{F} \to j_*(\mathscr{F}|_U)$ is surjective.

Solution: (a) Let $\{V_\alpha\}$ be an open cover of V. The identity axiom holds by definition. If $s_\alpha \in \Gamma_{Z\cap V_\alpha}(V_\alpha, \mathscr{F}|_{V_\alpha})$, and they admit on intersections, then we can glue them to get an $s \in \mathscr{F}(V)$. It's sufficient to show that the support of s is contained in $Z \cap V$, but this is obvious because the supports of all s_α are contained in Z.

(b) $\Gamma_{Z\cap V}(V, \mathscr{F}_V)$ is a subgroup of $\mathscr{F}(V)$, so there is a natural inclusion $\mathscr{H}_Z^0(\mathscr{F}) \to \mathscr{F}$. For an open set V, there is restriction map $\mathscr{F}(V) \to \mathscr{F}(U \cap V)$, which gives raise to a sheaf map $\varphi : \mathscr{F} \to j_*(\mathscr{F}|_U)$. $ker(\varphi_V)$ consists of the sections vanish on $U \cap V$, which means that the supports of these sections are contained in Z. Hence the exact sequence. The last statement is immediate from the definition of $\mathscr{F} \to j_*(\mathscr{F}|_U)$.

Exercise 2.1.22

Glueing Sheaves. Let X be a topological space, let $\mathfrak{U} = \{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathscr{F}_i$ on U_i, and for each i, j an isomorphism $\varphi_{ij} : \mathscr{F}_i|_{U_i\cap U_j} \to \mathscr{F}_j|_{U_i\cap U_j}$ such that (1) for each i, $\varphi_{ii} = id$, and (2) for each i, j, k, $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$. Then there exists a unique sheaf $\mathscr{F}$ on X, together with isomorphisms $\psi_i : \mathscr{F}|_{U_i} \to \mathscr{F}_i$ such that for each i, j, $\psi_j = \phi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say loosely that $\mathscr{F}$ is obtained by glueing the sheaves $\mathscr{F}_i$ via the isomorphisms φ_{ij}.

Solution: Define $\mathscr{F}$ as follow: For every open set U, $\{U \cap U_i\}_{i\in I}$ is an open cover of U, define $\mathscr{F}(U) = \{(s_i) \in \prod \mathscr{F}_i(U_i \cap U) : \phi_{ij,U_i\cap U_j\cap U}(\rho^{U_i\cap U}_{U_i\cap U_j\cap U}(s_i)) = \rho^{U_j\cap U}_{U_i\cap U_j\cap U}(s_j), \forall (i,j) \in I \times I\}$. If $U' \subseteq U$, then $U' \cap U_i \subseteq U \cap U_i$ for all i, and then we can define the restriction map by $(s_i) \to (\rho^{U_i\cap U}_{U_i\cap U'}(s_i))$. Every open cover of U induce an open cover of each U_i, so we can check the identity axiom and gluing axiom coordinate by coordinate, using the fact that every $\mathscr{F}_i$ is a sheaf.

If $U \subseteq U_i$, then given an $(s_i) \in \mathscr{F}(U)$, we can glue it to get a section in $\mathscr{F}_i(U)$. This gives ψ_i.