THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4402 Analysis II

Tutorial 3 (Feb 28)

1. Prove the second part of *Lemma 3.2*, that is, prove that if the Inverse Function Theorem holds for $\hat{f}$, then it will also hold for f.

2. **(Spherical Coordinates in $\mathbb{R}^3$)**
   Let $\Omega = (0,\infty) \times \mathbb{R}^2 \subseteq \mathbb{R}^3$ and let $f : \Omega \longrightarrow \mathbb{R}^3$ be given by
   $$f(r,\varphi,\theta) = (r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi).$$
   Show that f is locally invertible everywhere on $\Omega' = (0,\infty) \times (0,\pi) \times (0,2\pi)$. Find also the derivative of the local inverse at $(0,1,0)$.

3. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by
   $$f(x,y) = (3x - y^2, 2x + y, xy + y^3) \quad \text{and} \quad g(x,y) = (2ye^{2x}, xe^y).$$
   Show that there exist a neighborhood U of $(0,1)$ and V of $(2,0)$ such that $g : U \to V$ has a continuously differentiable inverse function. Use this to compute $(f \circ g^{-1})'(2,0)$.

4. Let $f : \mathbb{R}^5 \longrightarrow \mathbb{R}^2$ be defined by
   $$f(x,y,z,u,v) = (xy^2 + xzu + yv^2 - 3,\ yzu^3 + 2xv - u^2v^2 - 2).$$
   The point $(1,1,1,1,1)$ lies in the level set $f(x,y,z,u,v) = (0,0)$. Show that there are functions $u = g_1(x,y,z)$ and $v = g_2(x,y,z)$ satisfying $g_1(1,1,1) = 1 = g_2(1,1,1)$, and $f(x,y,z,g_1(x,y,z),g_2(x,y,z)) = 0$ for points (x,y,z) near $(1,1,1)$. Find also $g'(1,1,1)$ where $g(x,y,z) = (g_1(x,y,z), g_2(x,y,z))$.

In-Class Questions

5. Consider the system
   $$\begin{cases} x^2 - 2y^2 - 3u^2 + v^2 + 9 = 0 \\ 3xy + y^2 - u^2 + v^2 + 8 = 0. \end{cases}$$
   Show that the system defines u and v implicitly as functions of (x,y) near the point $(x,y) = (2,-1)$ satisfying $u(2,-1) = 2$ and $v(2,-1) = 1$. Find also the partial derivatives of u and v with respect to x and y at $(x,y) = (2,-1)$.

6. Show that the system of equations

$$\begin{cases} (x^2 + y^2 + z^2)^3 - x + z = 0 \\ \cos(x^2 + y^4) + e^z - 2 = 0. \end{cases}$$

defines x and z implicitly as functions $x = g_1(y)$ and $z = g_2(y)$ near $y = 0$ satisfying $g_1(0) = g_2(0) = 0$. Find also $g'(0)$ where $g(y) = (g_1(y), g_2(y))$.