THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4402 Analysis II

Tutorial 6 (Mar 28)

1. (a) Show that the set of irrational numbers in $[0, 1]$ does not have measure zero in $\mathbb{R}$.
   (b) Show that any nonempty open set in $\mathbb{R}^n$ does not have measure zero in $\mathbb{R}^n$. Hence show that every measure zero set must have empty interior.

2. (a) Show that any unbounded set cannot have content zero.
   (b) Give an example of a closed set of measure zero which does not have content zero.

3. **(The Cantor Set)**
   The *Cantor Set* C is obtained from $C_0 := [0, 1]$ by successive deletion of middle thirds, and can be written as
   $$C = \bigcap_{n=0}^{\infty} C_n,$$
   where $C_1 := [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $C_2 := [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, etc. Show that C has measure zero in $\mathbb{R}$ (This gives an example of an **uncountable set** in $\mathbb{R}$ which has **measure zero**).

4. Let A be a closed rectangle in $\mathbb{R}^n$ and let $f : A \longrightarrow \mathbb{R}$ be a bounded function.
   (a) Prove that if f is integrable over A and f vanishes except on a subset C of A of measure zero, then $\int_A f = 0$ (*Theorem 5.18 (a)*).
   (b) Give an example of a bounded function $f : A \longrightarrow \mathbb{R}$ such that f vanishes except on a subset C of A of measure zero but f is not integrable over A.
   (c) Prove that if f vanishes except on a closed subset C of A of measure zero, then f is integrable over A with $\int_A f = 0$.

5. Let A be a closed rectangle in $\mathbb{R}^n$ and let $f : A \longrightarrow \mathbb{R}$ be a bounded function. Prove that if $f(x) \geq 0$ for any $x \in A$ and if f is integrable over A with $\int_A f = 0$, then f vanishes except on a subset of A of measure zero (*Theorem 5.18 (b)*).

6. Let $A = [0, 1] \times [0, 1]$ and let $a, b \in \mathbb{R}$. Define $f : A \longrightarrow \mathbb{R}$ by $f(x, y) = \begin{cases} a & \text{if } x \leq y \\ b & \text{if } x > y. \end{cases}$
   (a) Show that f is integrable over A.
   (b) Find $f_x(y)$ for each $x \in [0, 1]$ and $f_y(x)$ for each $y \in [0, 1]$. Hence find $L(x)$, $U(x)$, $L(y)$ and $U(y)$.
   (c) Use Fubini's Theorem to compute the value of $\int_A f$.