THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4402 Analysis II

Tutorial 1 (Feb 14)

Chapter 1

1. Prove that the union of finitely many closed sets is closed. What about arbitrary unions?

2. Let $A \subseteq \mathbb{R}^n$. A subset $K \subseteq A$ is said to be *closed in A* if there exists a closed set $C \subseteq \mathbb{R}^n$ such that $K = C \cap A$.

   Suppose that $A \subseteq \mathbb{R}^n$ is compact. Prove that if $K \subseteq A$ is closed in A, then K is compact.

3. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation.

   (a) Prove that the set $\{\|T(x)\| : \|x\| \leq 1\}$ is a bounded subset of $\mathbb{R}$ (here it is understood that the same symbol $\|\cdot\|$ is being used for the standard norms in $\mathbb{R}^n$ and $\mathbb{R}^m$).

   (b) Define $\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\}$. Prove that $\|\cdot\|$ is a norm on the vector space of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ (you may use without proof the properties of supremum).

   (c) Prove that $\|T(x)\| \leq \|T\|\|x\|$ for any $x \in \mathbb{R}^n$ (here it is understood that the same symbol $\|\cdot\|$ is being used for the norm of linear transformations introduced in (b), and the standard norms in $\mathbb{R}^n$ and $\mathbb{R}^m$). Hence prove that T is continuous on $\mathbb{R}^n$.

In-Class Questions

4. Let (X, d) be a metric space. A subset A of X is said to be *totally bounded* if for any $\epsilon > 0$, there exist finitely many points $x_1, x_2, \ldots, x_k$ in A such that

$$A \subset \bigcup_{i=1}^{k} B(x_i, \epsilon).$$

   (a) Prove that if A is compact, then A is totally bounded.

   (b) Prove that if $\overline{A}$, the closure of A, is compact, then A is totally bounded.

   (c) Let X be the Euclidean space $\mathbb{R}^n$ with the standard metric d. Prove that any bounded subset of $(\mathbb{R}^n, d)$ is totally bounded.

   (d) Give an example of a metric space (X, d) and a bounded subset of it which is not totally bounded.

Chapter 2

5. Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \langle x, Mx \rangle = x^T Mx.$$

   (a) Find a candidate for the derivative $Df(x)$ by considering the difference $f(x+h)-f(x)$ for any $x, h \in \mathbb{R}^n$.

   (b) Show by definition that the candidate in (a) is indeed the derivative $Df(x)$.

6. Let $f : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ be defined by $f(A) = A^T A$. Here we interpret $\mathbb{R}^{n\times n}$ as the $n \times n$ matrix space and all multiplications are matrix multiplications.

   (a) Find a candidate for the derivative $Df(A)$ by considering the difference $f(A+H) - f(A)$ for any $A, H \in \mathbb{R}^{n\times n}$.

   (b) Show by definition that the candidate in (a) is indeed the derivative $Df(A)$.