T2/MATH4402/2024-25/2nd

THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4402 Analysis II

Tutorial 2 (Feb 21)

1. Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined by
$$f(x) = \langle x, Mx\rangle = x^T Mx.$$
   (a) Find a candidate for the derivative $Df(x)$ by considering the difference $f(x+h)-f(x)$ for any $x, h \in \mathbb{R}^n$.
   (b) Show by definition that the candidate in (a) is indeed the derivative $Df(x)$.

2. Let $f : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ be defined by $f(A) = A^T A$. Here we interpret $\mathbb{R}^{n\times n}$ as the $n \times n$ matrix space and all multiplications are matrix multiplications.
   (a) Find a candidate for the derivative $Df(A)$ by considering the difference $f(A+H) - f(A)$ for any $A, H \in \mathbb{R}^{n\times n}$.
   (b) Show by definition that the candidate in (a) is indeed the derivative $Df(A)$.

3. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined by $f(x) = \|x\|$, where $\|\cdot\|$ is the standard Euclidean norm.
   (a) Prove that f is NOT differentiable at 0.
   (*Hint*: Consider the function $h : \mathbb{R} \longrightarrow \mathbb{R}^n$ defined by $h(t) = (t, 0, \ldots, 0)$ and consider the composition $f \circ h$.)
   (b) Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined by $g(x) = \|x\|^2 = \langle x, x\rangle$. Use the result of **Q.1** and the Chain rule to show that f is differentiable at any $a \in \mathbb{R}^n \setminus \{0\}$. Find also the derivative $Df(a)$ at any $a \in \mathbb{R}^n \setminus \{0\}$.

In-Class Questions

4. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined by
$$f(x) = \begin{cases} \dfrac{\sqrt{1+\|x\|^2}-1}{\|x\|^2} & \text{if } x \neq 0 \\ \alpha & \text{if } x = 0. \end{cases}$$
Find all value(s) of α such that f is differentiable at 0. Find also the derivative $Df(0)$.

5. Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable at $a \in \mathbb{R}^n$. Prove that
   (a) $D(f+g)(a) = Df(a) + Dg(a)$
   (b) $D(fg)(a) = g(a)Df(a) + f(a)Dg(a)$
   (c) if $g(a) \neq 0$, then
$$D\left(\frac{1}{g}\right)(a) = \frac{-1}{(g(a))^2} Dg(a).$$