Chapter 3. The Inverse Function Theorem and the Implicit Function Theorem

The Inverse Function Theorem and the Implicit Function Theorem are two of the most widely used theorems in Analysis. The Inverse Function Theorem tells us when we can locally invert a function, whereas the Implicit Function Theorem tells us when a function is given implicitly as a function of other variables. The key principle of both results is similar – we linearize the problem at a point by studying the derivative of the function. Subject to a nondegeneracy condition on the derivative, we are able to retrieve local information of the original function.

3.1. The Inverse Function Theorem

The Inverse Function Theorem asserts that a function is locally invertible if its linearization is invertible. Therefore, local bijectivity of the function is ensured by the invertibility of its linearization.

Recall that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuously differentiable in an open neighborhood of $a \in \mathbb{R}$ and if $f'(a) \neq 0$, then there is an open neighborhood V of a in which f' does not change sign and so f is either strictly increasing or strictly decreasing there. In particular, the inverse f^{-1} of f exists on the image $f(V)$. Furthermore, we know that f^{-1} is differentiable on $f(V)$ with

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \quad \text{for any } y \in f(V).$$

In this section we shall extend this result to the case of functions of several variables.

Theorem 3.1 (Inverse Function Theorem) *Let $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and let $a \in \Omega$. If $f \in C^1(\Omega)$ and $\det f'(a) \neq 0$, then there exist an open neighborhood V of a in Ω and an open neighborhood W of $f(a) \in \mathbb{R}^n$ such that*

(a) $f|_V : V \longrightarrow W$ *is bijective;*

(b) $f^{-1} := (f|_V)^{-1} : W \longrightarrow V$ *is of class C^1 on W; and*

(c) $Df^{-1}(y) = [Df(f^{-1}(y))]^{-1}$ *for any $y \in W$, or equivalently,*

$$(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1} \quad \textit{for any } y \in W.$$

Remarks:

(a) The condition that the determinant of the Jacobian matrix $f'(a)$ is nonzero, or equivalently, $Df(a)$ is invertible, is called the *nondegeneracy condition*. We have seen in *Corollary 2.7 of Chapter 2* that this condition is a necessary condition for a differentiable function f to admit a differentiable (local) inverse. The Inverse Function Theorem guarantees that, if f is of class C^1 as well, then the nondegeneracy condition is also sufficient.

(b) In case $f'(a)$ is singular, that is, $\det f'(a) = 0$, the inverse function f^{-1} may or may not exist around $f(a)$. For instance, let $f(x) = x^2$. Then $f'(0) = 0$ and the inverse does not exist around 0. Let $g(x) = x^3$. Then $g'(0) = 0$ but the inverse still exists around 0.

(c) The Inverse Function Theorem asserts only a local invertibility. Even if $f'(a)$ is nonsingular everywhere on Ω, we cannot assert global invertibility. For example, let $\Omega = (0, \infty) \times \mathbb{R} \subseteq \mathbb{R}^2$ and let $f : \Omega \longrightarrow \mathbb{R}^2$ be given by $f(r, \theta) = (r\cos\theta, r\sin\theta)$. Then f is a continuously differentiable function whose Jacobian matrix is nonsingular everywhere on Ω. However, it is clear that f is not bijective, for instance, all points $(r, \theta + 2k\pi)$, $k \in \mathbb{Z}$, have the same image under f.

(d) We say that $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a *local diffeomorphism* at $a \in \Omega$ if there is an open neighborhood V of a in Ω and an open neighborhood W of $f(a) \in \mathbb{R}^n$ such that $f|_V : V \longrightarrow W$ is bijective, and both $f|_V$ and $(f|_V)^{-1}$ are of class C^1. With this notion, we can conclude from the Inverse Function Theorem that f is a local diffeomorphism at a.

We shall begin the proof of Theorem 3.1 by making a simplification.

Lemma 3.2 *Let $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and let $a \in \Omega$. If $f \in C^1(\Omega)$ and $\det f'(a) \neq 0$, then there exists a function $\hat{f} : \Omega \longrightarrow \mathbb{R}^n$ such that $\hat{f} \in C^1(\Omega)$ and $\hat{f}'(a) = Id$. Furthermore, if Theorem 3.1 holds for $\hat{f}$, then it will also hold for f.*

It then follows from Lemma 3.2 that we may assume without loss of generality that $f'(a) = Id$ in Theorem 3.1.

Next we show that f is locally injective.

Theorem 3.3 *Let $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and let $a \in \Omega$. If $f \in C^1(\Omega)$ and $f'(a) = Id$, then there exists an open neighborhood V of a such that f is injective on V.*

To complete the proof of Theorem 3.1, it remains to show

1. The set $W = f(V)$ is open in $\mathbb{R}^n$, where V is the open neighborhood constructed in Theorem 3.3. Here we shall need to apply the *Contraction Mapping Principle*.
2. The inverse function $g = f^{-1} : W \longrightarrow V$ is differentiable on W.
3. The inverse function $g = f^{-1} : W \longrightarrow V$ is of class C^1 on W.

Example 3.4 Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $f(x, y) = (x^3 + xy + y^3, x^2 - y^2)$. Prove that there exists an inverse of f defined on a neighborhood of $f(1, 1) = (3, 0)$. Find also its derivative at $(3, 0)$ and estimate $f^{-1}(3.1, -0.2)$.

Solution. Since

$$\begin{aligned} D_1 f_1(x, y) &= 3x^2 + y, \qquad & D_2 f_1(x, y) &= x + 3y^2, \\ D_1 f_2(x, y) &= 2x, & D_2 f_2(x, y) &= -2y, \end{aligned}$$

we see that f is continuously differentiable everywhere, and hence we have

$$f'(x,y) = \begin{pmatrix} 3x^2+y & x+3y^2 \\ 2x & -2y \end{pmatrix} \quad \text{and} \quad f'(1,1) = \begin{pmatrix} 4 & 4 \\ 2 & -2 \end{pmatrix}.$$

Since $f'(1,1)$ is nonsingular, by the Inverse Function Theorem, there exist an open neighborhood V of $(1,1)$ and an open neighborhood W of $(3,0)$ such that $f|_V : V \longrightarrow W$ has a continuously differentiable inverse function. Furthermore,

$$(f^{-1})'(3,0) = \begin{pmatrix} 4 & 4 \\ 2 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} 1/8 & 1/4 \\ 1/8 & -1/4 \end{pmatrix},$$

and $f^{-1}(3.1,-0.2)$ can be approximated by

$$f^{-1}(3.1,-0.2) \approx f^{-1}(3,0) + (f^{-1})'(3,0)\begin{pmatrix} 0.1 \\ -0.2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/8 & 1/4 \\ 1/8 & -1/4 \end{pmatrix}\begin{pmatrix} 0.1 \\ -0.2 \end{pmatrix} = \begin{pmatrix} 77/80 \\ 17/16 \end{pmatrix}.$$

$\square$

3.2. The Implicit Function Theorem

Consider $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$, which is just the unit circle in the plane. Can we find a function $y = g(x)$ such that $x^2 + g(x)^2 = 1$? Obviously, in this example we cannot find one single function to describe the whole unit circle in this way. However, we can do it locally, that is, in a neighborhood of a point $(a,b) \in S^1$, as long as $b \neq 0$. In this example we can find $g(x)$ explicitly: $g(x) = \sqrt{1-x^2}$ if $b > 0$ and $g(x) = -\sqrt{1-x^2}$ if $b < 0$ for any $-1 < x < 1$. Note that if $b = 0$, we cannot find such a function $y = g(x)$.

Now we set $f(x,y) = x^2+y^2-1$. Then the unit circle S^1 is the solution set of $f(x,y) = 0$. Thus the above suggests that for any (a,b) with $f(a,b) = 0$, if we assume that $b \neq 0$, then there is an open neighborhood A of a, an open neighborhood B of b, and a C^1 function $g : A \longrightarrow B$ such that $f(x,g(x)) = 0$ for all $x \in A$. Such a function $y = g(x)$ is said to be an *implicit function* defined by f near the point (a,b).

In general, we are looking for a condition under which a variable can be written as a function of the other. To approach this problem, we first look for a necessary condition. Supposing that the equation $f(x,y) = 0$ indeed determines y as a differentiable function of x, say $y = g(x)$. Then we have $f(x,g(x)) = 0$. Differentiating this with respect to x yields

$$\frac{\partial f}{\partial x}(x,g(x)) + \frac{\partial f}{\partial y}(x,g(x)) \cdot g'(x) = 0.$$

Thus we see that in order to find $g'(x)$ at $x = a$, it is necessary to assume that $\frac{\partial f}{\partial y}(a,g(a)) \neq 0$. It turns out that this condition is exactly what we are looking for, that is, if $f(x,y)$ has the property that $\frac{\partial f}{\partial y} \neq 0$ at a point (a,b) with $f(a,b) = 0$, then f determines y as a function of x near $x = a$, and this function is differentiable. The *Implicit Function Theorem* is the generalization of the above situation to more variables. Notice that in the unit circle example, the condition $\frac{\partial f}{\partial y}(a, g(a)) \neq 0$ is equivalent to $b \neq 0$.

To formulate the Implicit Function Theorem, we first introduce some convenient notation.

1. Write a point $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^{n+m}$;
2. Let $f : \Omega \subseteq \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$ be of class C^1 on Ω. Write

$$f'(x, y) = (f'_x(x, y)\ f'_y(x, y)) \in \mathbb{R}^{m\times(n+m)},$$

   where

$$f'_x(x, y) = \big(D_j f_i(x, y)\big)_{1\leq i\leq m, 1\leq j\leq n} \in \mathbb{R}^{m\times n} \quad \text{and} \quad f'_y(x, y) = \big(D_{n+j} f_i(x, y)\big)_{1\leq i,j\leq m} \in \mathbb{R}^{m\times m}.$$

Theorem 3.5 (Implicit Function Theorem) *Let $f : \Omega \subseteq \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$ be of class C^1 on Ω and let $(a, b) \in \Omega$ such that $f(a, b) = 0$. If $\det f'_y(a, b) \neq 0$, then there is an open neighborhood A of a in $\mathbb{R}^n$, an open neighborhood B of b in $\mathbb{R}^m$, with $A \times B \subseteq \Omega$, and a unique C^1 function $g : A \longrightarrow B$ such that $f(x, g(x)) = 0$ for all $x \in A$.*

Remarks:

(a) The key idea of the proof is to construct a function F to which we can apply the Inverse Function Theorem. Define $F : \Omega \longrightarrow \mathbb{R}^{n+m}$ by

$$F(x, y) = (x, f(x, y)).$$

Then it can be shown that

$$\det F'(a, b) = \det f'_y(a, b) \neq 0$$

and hence we can apply the Inverse Function Theorem to F.

(b) There is nothing special about the last m variables – we can certainly apply the same argument to the problem of writing any m variables in terms of the other n variables.

(c) Let g be the implicit function guaranteed by the Implicit Function Theorem. Define $G : A \longrightarrow \mathbb{R}^m$ by $G(x) = f(x, g(x))$, which is equal to 0 for all $x \in A$. The chain rule then gives

$$\begin{aligned} 0 = G'(x) = f'(x, g(x)) \cdot \begin{pmatrix} I_n \\ g'(x) \end{pmatrix} &= \big(f'_x(x, g(x))\ f'_y(x, g(x))\big) \cdot \begin{pmatrix} I_n \\ g'(x) \end{pmatrix} \\ &= f'_x(x, g(x)) + f'_y(x, g(x)) \cdot g'(x) \end{aligned}$$

Thus if $f'_y(x, g(x))$ is nonsingular at the point $(a, g(a))$, then the derivative of g at a is given by

$$g'(a) = -\big(f'_y(a, g(a))\big)^{-1} \cdot f'_x(a, g(a)).$$

Example 3.6 Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined by

$$f(x,y,z) = (x^3 + y^3 + z^3 - 7,\ xy + yz + zx + 2).$$

The point $(2,-1,0)$ lies in the level set $f(x,y,z) = (0,0)$. Show that there are functions $y = g_1(x)$ and $z = g_2(x)$ satisfying $g_1(2) = -1$ and $g_2(2) = 0$, and $f(x, g_1(x), g_2(x)) = (0,0)$ for points (x,y,z) near $(2,-1,0)$. Find also $g_1'(2)$ and $g_2'(2)$.

Solution. It is clear that f is of class C^1 on $\mathbb{R}^3$. The Jacobian matrix of f is

$$f'(x,y,z) = \begin{pmatrix} 3x^2 & 3y^2 & 3z^2 \\ y+z & x+z & x+y \end{pmatrix}.$$

Hence, at the point $(2,-1,0)$, we have

$$f'_{(y,z)}(2,-1,0) = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} \quad \text{with} \quad \det f'_{(y,z)}(2,-1,0) = 3 \neq 0.$$

The Implicit Function Theorem implies that there exist open neighborhoods $A \subseteq \mathbb{R}$ of 2 and $B \subseteq \mathbb{R}^2$ of $(-1,0)$, and a continuously differentiable function $g : A \longrightarrow B$ with $g(2) = (g_1(2), g_2(2)) = (-1,0)$ such that

$$f(x, g_1(x), g_2(x)) = (0,0)$$

for all $x \in A$. Furthermore, the derivative of g at $x = 2$ is

$$g'(2) = -\big(f'_{(y,z)}(2, g(2))\big)^{-1} \cdot f'_x(2, g(2)) = -\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 12 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \end{pmatrix}.$$

It follows that $g_1'(2) = -4$ and $g_2'(2) = 9$. □

Example 3.7 Consider the system

$$\begin{cases} x^2 + uy + e^v = 0 \\ 2x + u^2 - uv = 5. \end{cases}$$

Show that the system defines u and v implicitly as functions of (x,y) near the point $(x,y) = (2,5)$ satisfying $u(2,5) = -1$ and $v(2,5) = 0$. Find also the partial derivatives of u and v with respect to x and y at $(x,y) = (2,5)$.

Solution. Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be defined by

$$f(x,y,u,v) = (x^2 + uy + e^v,\ 2x + u^2 - uv - 5).$$

Note that $f(2,5,-1,0) = (0,0)$. The Jacobian matrix of f is

$$f'(x,y,u,v) = \begin{pmatrix} 2x & u & y & e^v \\ 2 & 0 & 2u - v & -u \end{pmatrix}.$$

Hence, at the point $(2,5,-1,0)$, we have

$$f'_{(u,v)}(2,5,-1,0) = \begin{pmatrix} 5 & 1 \\ -2 & 1 \end{pmatrix} \quad \text{with} \quad \det f'_{(u,v)}(2,5,-1,0) = 7 \neq 0.$$

The Implicit Function Theorem implies that there exist open neighborhoods $A \subseteq \mathbb{R}^2$ of $(2,5)$ and $B \subseteq \mathbb{R}^2$ of $(-1,0)$, and a continuously differentiable function $g : A \longrightarrow B$ with $g(2,5) = (g_1(2,5), g_2(2,5)) = (-1,0)$ such that

$$f(x, y, g_1(x,y), g_2(x,y)) = (0,0)$$

for all $(x,y) \in A$. Thus the functions $u(x,y) = g_1(x,y)$ and $v(x,y) = g_2(x,y)$ satisfy the requirements. Furthermore, the derivative of g at $(x,y) = (2,5)$ is

$$g'(2,5) = -\big(f'_{(u,v)}(2,5,g(2,5))\big)^{-1} \cdot f'_{(x,y)}(2,5,g(2,5)) = -\begin{pmatrix} 5 & 1 \\ -2 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix} = -\frac{1}{7}\begin{pmatrix} 2 & -1 \\ 18 & -2 \end{pmatrix}.$$

It follows that

$$\frac{\partial u}{\partial x}(2,5) = -\frac{2}{7}, \quad \frac{\partial u}{\partial y}(2,5) = \frac{1}{7}, \quad \frac{\partial v}{\partial x}(2,5) = -\frac{18}{7} \quad \text{and} \quad \frac{\partial v}{\partial y}(2,5) = \frac{2}{7}. \qquad \square$$

It is interesting to note that the Inverse Function Theorem can be deduced from the Implicit Function Theorem and hence they are equivalent. To see this, let $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and let $a \in \Omega$. Suppose that $f \in C^1(\Omega)$ and $\det f'(a) \neq 0$. Define a function $F : \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$ by

$$F(x,y) = x - f(y).$$

Then F is also C^1 on $\mathbb{R}^n \times \Omega$ and $F(f(a), a) = f(a) - f(a) = 0$. Furthermore,

$$F'_y(x,y) = \big(D_{n+j} F_i(x,y)\big)_{1 \leq i,j \leq n} = \big(-D_j f_i(y)\big)_{1 \leq i,j \leq n}$$

and hence

$$\det F'_y(f(a), a) = (-1)^n \det f'(a) \neq 0.$$

By the Implicit Function Theorem, there exists an open neighborhood A of $f(a)$ in $\mathbb{R}^n$, an open neighborhood B of a in Ω, with $A \times B \subseteq \mathbb{R}^n \times \Omega$, and a unique C^1 function $g : A \longrightarrow B$ such that

$$F(x, g(x)) = 0 \quad \text{for all } x \in A.$$

But this just means $x = f(g(x))$ for any $x \in A$. Thus if we set $V = g(A)$ (which is open as $g(A) = f^{-1}(A)$ and f is continuous) and set $W = A$, then $f|_V : V \longrightarrow W$ has inverse $g : W \longrightarrow V$ which is of class C^1.