T7/MATH4402/2024-25/2nd

THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4402 Analysis II

Tutorial 7 (Apr 11)

Chapter 5 (Section 5.3 to 5.5)

1. Let S be a Jordan measurable set in $\mathbb{R}^n$ and let $S \subset A$ for some closed rectangle A in $\mathbb{R}^n$.
   (a) Prove that if S has measure zero, then $v(S) = 0$.
   (b) Let $f : A \longrightarrow \mathbb{R}$ be a bounded function. Prove that if S has measure zero and f is integrable over S, then $\int_S f = 0$.

2. **(Partition of Unity (Optional))**
   Consider $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. An open covering of S^1 is given by $\{U, V\}$ where
   $$U = S^1 \setminus \{(0, 1)\} \quad \text{and} \quad V = S^1 \setminus \{(0, -1)\}.$$
   Consider the C^∞ functions $g_U : U \longrightarrow \mathbb{R}$ and $g_V : V \longrightarrow \mathbb{R}$ defined respectively by
   $$g_U(x, y) = \frac{x}{1-y} \quad \text{and} \quad g_V(x, y) = \frac{x}{1+y}.$$
   (a) Draw diagrams to illustrate how g_U and g_V maps the "punctured" unit circle onto the real line and show that $g_U(x, y) \cdot g_V(x, y) = 1$ for any $(x, y) \in U \cap V$.
   (b) It is known that we can always construct a C^∞ function $f : \mathbb{R} \longrightarrow [0, 1]$ such that
   $$\begin{cases} f(x) = 1 & \text{if } x \in [-1, 1] \\ 0 < f(x) < 1 & \text{if } x \in (-2, -1) \cup (1, 2) \\ f(x) = 0 & \text{if } |x| \geq 2 \end{cases}$$
   (Such function is usually called a *bump function*. You may read Problem 2-26 of *Calculus on Manifolds* by Spivak for further details.)
   Assuming the existence of such f and using the functions g_U and g_V, construct a partition of unity $\{\varphi_1, \varphi_2\}$ of S^1 subordinate to the open cover $\{U, V\}$.

3. **(Change of Variables)**
   Let $\pi_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the projection onto the k-th coordinate, that is,
   $$\pi_k(x_1, \ldots, x_n) = x_k.$$
   Let $S \subset \mathbb{R}^n$ be a Jordan measurable set with nonzero volume. Define the *centroid* $c(S)$ of S to be the point in $\mathbb{R}^n$ whose k-th coordinate is given by
   $$c_k(S) = \frac{1}{v(S)} \int_S \pi_k.$$

(a) Suppose that S is also symmetric with respect to the subspace $x_k = 0$, that is, $\varphi(S) = S$ where $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is given by

$$\varphi(x_1, \ldots, x_k, \ldots, x_n) = (x_1, \ldots, -x_k, \ldots, x_n).$$

Prove that $c_k(S) = 0$.

(b) Let

$$S = \{(x, y) \in \mathbb{R}^2 : y > 0 \text{ and } x^2 + y^2 < 1\}.$$

Use the polar coordinates to compute $c(S)$ in $\mathbb{R}^2$.

Chapter 6 (Section 6.1)

4. Determine which of the following are 3-tensors on $\mathbb{R}^4$:

$$\begin{aligned} T_1(x, y, z) &= 3x_1y_2z_3 - x_3y_1z_4 \\ T_2(x, y, z) &= 2x_1x_2z_3 + x_3y_4z_2 \\ T_3(x, y, z) &= 2x_1y_2z_2 - x_2y_3z_1 \end{aligned}$$

where $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$ and $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \in \mathbb{R}^4$. Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis for $\mathbb{R}^4$ and $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ its dual basis for $\mathbb{R}^{4*}$. Express those tensors T_i in terms of the basis for $\mathcal{T}^3(\mathbb{R}^{4*})$ given in *Theorem 6.3*.

5. Let $\{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$ and $\{\varphi_1, \varphi_2, \varphi_3\}$ its dual basis for $\mathbb{R}^{3*}$. Let $T \in \mathcal{T}^2(\mathbb{R}^{3*})$ be defined by $T(x, y) = x_1y_1 + x_2y_3$ for any $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$. Show that T is not a tensor product of two 1-tensors, that is, T is not of the form $f \otimes g$ for any $f, g \in \mathcal{T}^1(\mathbb{R}^{3*})$.