総合演習課題 0：小課題

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1 問 0-1

2 問 0-2

以下に示す (1) 式を変形し、講義スライド p.11 の (2) 式になることを示す。

$$
\begin{aligned}
\frac{\partial v_x}{\partial t} + \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}\right) v_x &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v_x \quad \cdots① \\
\frac{\partial v_x}{\partial t} + \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}\right) v_y &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v_y \quad \cdots② \\
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0 \quad \cdots③
\end{aligned} \tag{1}
$$

i $\frac{\partial}{\partial x}$② $-$ $\frac{\partial}{\partial y}$① とする。

$$
\frac{\partial}{\partial t}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) + \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}\right)\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) = \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)
$$

ここで、$\omega = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \quad \cdots$④ とすると、

$$
\begin{aligned}
-\frac{\partial \omega}{\partial t} - \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}\right)\omega &= -\nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\omega \\
\frac{\partial \omega}{\partial t} + \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}\right)\omega &= \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\omega
\end{aligned} \tag{2.1}
$$

ii ④式に、$v_x = \frac{\partial \psi}{\partial y}, v_y = -\frac{\partial \psi}{\partial x}$ を代入して、

$$
\begin{aligned}
\omega &= \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) \\
&= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi
\end{aligned} \tag{2.2}
$$

iii $\frac{\partial}{\partial x}$① $+$ $\frac{\partial}{\partial y}$② とする。

$$\frac{\partial}{\partial t}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)+\frac{\partial}{\partial x}\left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}\right)v_x+\frac{\partial}{\partial y}\left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}\right)v_y$$
$$=-\frac{1}{\rho}\left(\frac{\partial}{\partial x}\frac{\partial P}{\partial x}+\frac{\partial}{\partial y}\frac{\partial P}{\partial y}\right)+\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)$$

ここで、③ 式より、$\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)=0$ であるため、

$$-\frac{1}{\rho}\left(\frac{\partial}{\partial x}\frac{\partial P}{\partial x}+\frac{\partial}{\partial y}\frac{\partial P}{\partial y}\right)=\frac{\partial}{\partial x}\left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}\right)v_x+\frac{\partial}{\partial y}\left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}\right)v_y$$
$$-\frac{1}{\rho}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)P=2\left[\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2-\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2}\right]$$
$$\therefore \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)P=2\rho\left[\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2}-\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\right] \tag{2.3}$$

以上式 (2.1)〜式 (2.3) から、$v_x=\frac{\partial\psi}{\partial y}, v_y=-\frac{\partial\psi}{\partial x}$ と置いたとき、式 (2) が導かれた。