Applied Analysis

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1 Preliminary

We only consider the notation in the Euclidean space.

$$x \in \mathbb{R}^n, x = (x_1, \ldots, x_n)$$

$$A \subset R^n = \{x \in R^n : x \in A\} \text{ and } A^c(R^n \setminus A) = \{x \in R^n : x \notin A\}$$

Given a map, $f : A \to B$

$$E \subset A \to f(E) = \{f(x) : x \in E\}$$

$$F \subset B \to f^{-1}(F) = \{x \in A : f(x) \in F\}$$

f is said injective (one to one) :

$$\forall x, y \in A, f(x) = f(y) \implies x = y$$

f is said surjective (onto):

$$\forall z \in B, \exists x \in A \text{ such that } f(x) = z$$

$$f(A) = B \text{ as sets; } Range(f) = B := f(A)$$

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Given a family $\mathcal{F} = \{A_i\}_{i \in I}$ of subsets of X. indexed by on "index set" I,

$$\bigcup_{i \in I} A_i = \bigcup \mathcal{F} = \{x \in X : \exists \text{ subsets } i \in I \text{ such that } x \in A_i\}$$

$$\bigcap_{i \in I} A_i = \bigcap \mathcal{F} = \{x \in X : \forall i \in I \text{ such that } x \in A_i\}$$

$(A_i)_{i \in \mathbb{N}}$ sequence of subsets of X:

1. (A_i) is increasing if $A_i \subset Ai + 1$ for all $i \geq 1$.
2. (A_i) is increasing if $A_{i+1} \subset Ai$ for all $i \geq 1$.

Definition.

$$\lim_{i \to \infty} \sup A_i = \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} A_i$$

$$\lim_{i \to \infty} \inf A_i = \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} A_i$$

Remark. *1.* $E_j = \bigcup_{i=j}^{\infty} A_i \longrightarrow (E_j)$ *decreasing sequence of sets*

2. $F_j = \bigcap_{i=j}^{\infty} A_i \longrightarrow (F_j)$ *increasing sequence of sets*

3.

$$\lim_{i \to \infty} \sup A_i = \{x \in X : x \textit{ belongs to infinite many of the sets } A_i\}$$

$$\lim_{i \to \infty} \inf A_i = \{x \in X : x \textit{ belongs to all } A_i \textit{ beyond a certain index } i_*\}$$

4.

$$\lim_{i \to \infty} \inf A_i \subset \lim_{i \to \infty} \sup A_i$$

Proof. $\boxed{\text{Goal : for all } i \geq 1, \exists k \geq 1 \text{ s.t. } x \in A_i}$
Let $x \in \lim_{i \to \infty} \inf A_i$ be given, then recalling that $\lim_{i \to \infty} \inf A_i = \bigcup_{j=1}^{\infty} F_j$, then for some j we have that $x \in F_j$ □

Remark.

$$A = B \iff A \subset B \textit{ and } B \subset A$$

2 Vector Space

X is a **vector space** over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $A, B \subset X, \alpha \in \mathbb{F}$

$$\longrightarrow \begin{cases} A + B = \{x + y : x \in A, y \in B\} \\ \alpha A = \{\alpha x : x \in A\} \end{cases}$$

2.1 Cauchy-Schwarz inequality.

Let $\alpha, \beta \in \mathbb{R}$

$$\alpha\beta \leq \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2$$

Proof using the fact that $(\alpha - \beta)^2 \geq 0$.
This shows that, for $\mathbf{x} = (x_1, \cdots, x_n), \mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$

$$\sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{n} \frac{1}{2}x_i^2 + \frac{1}{2}y_i^2 = \frac{1}{2}(|\mathbf{x}|^2 + |\mathbf{y}|^2)$$

This holds for $|\mathbf{x}| = |\mathbf{y}| = 1$ and $|\mathbf{x}| = 0$ or $|\mathbf{y} = 0|$.

2.2 Distance between points:

$$d(x, y) = |x - y| = (\sum_{i=1}^{n}(x_i - y_i)^2)^{1/2}$$

and the triangle inequality for $x, y, z \in \mathbb{R}^n$ is

$$|x - z| \leq |x - y| + |y - z|$$

3 Metric Space

Definition. *A* <u>*Metric Space*</u>(X, ρ) *consists of* X *nonempty set,* $\rho : X \times X \to [0, \infty)$ *such that*

1. $\rho(x, y) = \rho(y, x) \quad \forall x, y \in X$
2. $\rho(x, y) = 0 \iff x = y \quad \forall x, y \in X$, *and*
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z) \quad \forall x, y, z \in X$

Remark. (X, ρ) *metric space,* $Y \subset X \implies (Y, \rho|_{Y \times Y})$ *is also a metric space*

Definition. (X, ρ) *metric space,* $A \subset X$

$$diamA := \sup_{x,y \in A} \rho(x, y) \quad \text{if } A \neq \phi \quad (diam\phi := 0)$$

A is <u>bounded</u> if $diamA \leq \infty$ (i.e,equaivalently, A is bounded $\iff$ for all $y \in X$, the set $\{\rho(x, y) : x \in A\}$ is a bounded subset of $\mathbb{R}$)

Example. 1. $\mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n$

2. $a, b \in \mathbb{R}, C([a, b]) := \{f : [a, b] \to \mathbb{R} | f$ *is continuous on* $[a, b]\}$ *with* $\rho(f, g) = \max_{x \in [a,b]} |f(x) - g(x)|$ *for* $f, g \in C([a, b])$ *by the Extreme Value Theorem*

3. $A = \{a_1, \cdots, a_k\}, [n] = \{1, 2, \cdots, n\}$

$$X = A^{[n]} = \{\sigma : [n] \to A\} = \{(\sigma(1), \cdots, \sigma(n)) : \sigma(i) \in A \text{ for } i = 1, \cdots, n\}$$

$$\rho(\sigma, \omega) := \#\{i : \sigma(i) \neq \omega(i)\}$$

4. $(X, \rho_x), (Y, \rho_y)$ *is a metric space*

$$X \times Y = \{(u, w) : u \in X, w \in Y\} \text{ is a metric space with product metric}$$

$$\rho(u, v) = \rho_x(u_v, v_x) + \rho_y(u_y, v_y)$$

where $u = (u_x, u_y) \in X \times Y$*,*$v = (v_x, v_y) \in X \times Y$

3.1 Normed Linear Space

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Definition. *Let E be a vector space over $\mathbb{F}$. We say E is a Normed Linear Space if* $\exists$ $||\cdot|| : E \to [0, \infty)$ *such that*

1. $||x|| = 0 \iff x = 0 \in E \quad \forall x \in E$
2. $||\alpha x|| = |\alpha|||x|| \quad \forall x \in E, \alpha \in \mathbb{F}.$*, and*
3. $||x + y|| \leq ||x|| + ||y|| \quad \forall x, y \in E$

Remark. *E is a Normed Linear Space $\implies$ E is a metric space with metric* $\rho(x, y) := ||x - y||$ *for* $x, y \in E$*.*

Remark. *E is a n.l.s, $(x_n)_{n\geq 1}$ seq in E. If $x_n \to x$ in E for some $x \in E$, then* $\lim_{n\to\infty} ||x_n|| = ||x||$*, i.e $||\cdot||$ is continuous on E.*
(Idea: $||x_n|| - ||x_n - x|| \leq ||x|| \leq ||x_n|| + ||x_n - x||$*)*

Definition. *if a n.l.s E is 'complete' with respect to the above metric, it is said to be a* **Banach Space***.*

Example. $1 \leq p < \infty$

$$\ell^p(N^*) := \{(x_1, \cdots, x_n, \cdots) : x_i \in \mathbb{R} \quad \forall i, (\sum_{i=1}^{\infty} |x_i|^p)^{1/p} < \infty\}$$

$$||x||_p := (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$$

$\ell^p(N^*)$ is a n.l.s with norm $||\cdot||_p$

Lemma 3.1.1. *$\alpha, \beta \geq 0, \alpha + \beta = 1 \implies$ for all $\zeta, \eta > 0, \zeta^\alpha \eta^\beta \leq \alpha\zeta + \beta\eta$.*

Proof. if $\alpha = 0, \beta = 1$, it is obvious. Similarly for $\alpha = 1, \beta = 1$. So, we may asuume $0 < \alpha < 1$ and $0 < \beta < 1$. $\square$