Kalman Filter and Ensemble Kalman Filter With Application

12-Page Essay

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Abstract

This project is a self-directed exploration into the implementation and application of the Ensemble Kalman Filter (EnKF), alongside the well-known Kalman Filter discussed in Homework 8. Leveraging knowledge acquired through self-study, and from the Monte Carlo Class, the project aims to delve into both theoretical underpinnings and practical applications of the KF and EnKF. Through this endeavor, we seek to enhance our understanding of Data Assimilation techniques and their relevance in solving inverse problems. The project will include theoretical discussions, implementation examples, and real-world applications, serving as a comprehensive exercise to solidify our grasp of the EnKF methodology.

1 Introduction

The Kalman Filter [3], introduced by Rudolf E. Kalman in 1960, is an efficient recursive algorithm to estimate the state of a dynamic system from a series of noisy measurements. It operates by predicting the future state of the system and updating this prediction with new measurement data, achieving a balance between the model's predictions and the actual observations. The Kalman Filter is widely used in various fields, including navigation, signal processing, and control systems, due to its simplicity and robustness in handling linear systems and Gaussian noise.

The Ensemble Kalman Filter (EnKF) [2], developed by G. Evensen in 1994, is an extension of the Kalman Filter suitable for high-dimensional nonlinear systems. It employs an ensemble of state vectors to represent the probability distribution of the system's state, which is updated with observations in a manner similar to the Kalman Filter. The EnKF has become a popular tool in geophysical data assimilation, particularly in meteorology and oceanography, because it can efficiently handle the complexities and uncertainties of large-scale models.

2 Theoratical Background

2.1 Kalman Filter

The filtering problem aims to infer the state at a particular time j based on all the data up to that time, denoted as $Y_j := \{y_1, ..., y_j\}$. This involves determining the probability density function $\pi_j = P(v_j|Y_j)$. The process involves predicting $\widehat{\pi}_{j+1} = P(v_{j+1}|Y_j)$ and then updating it to π_{j+1} using the prediction and analysis steps. The prediction step is governed by the same Markov chain at each time step, hence $\mathcal{P}$ does not vary with j. However, the analysis step $\mathcal{A}_j$ depends on j as it considers different data at each step. The assumptions of linear dynamics and linear observation $(h(\cdot))$ allow for simplifications, ensuring that both prediction and analysis steps result in Gaussian distributions. Mathematically, Suppose we have the stochastic dynamics with transition function $\Psi(\cdot)$ and data models, let $v_0 \sim \mathcal{N}(m_0, C_0)$, $C_0, \Gamma,$ and Σ are positive definite and $v_0 \perp \{\xi_j\} \perp \{\eta_j\}$,

$$v_{j+1} = \Psi(v_j) + \xi_j, \quad \xi_j \sim \mathcal{N}(0, \Sigma) \quad i.i.d$$

$$y_{j+1} = h(v_{j+1}) + \eta_{j+1} \quad \eta_j \sim \mathcal{N}(0, \Gamma) \quad i.i.d$$

2.1.1 Prediction Step

In the prediction step, the filter uses the system's dynamic model to predict the state of the system at the next time step. This prediction is based on the previous state estimate and the system's dynamics. The predicted state and its uncertainty are represented by a mean vector and a covariance matrix, respectively.

2.1.2 Analysis Step

In the analysis step, the filter incorporates measurements from sensors to refine its state estimate. This is done by combining the predicted state with the measurement using a weighted average, where the weights are determined by the uncertainty of the prediction and the measurement. The result is a new state estimate with reduced uncertainty.

The Kalman Filter can be described mathematically by finding the analysis formulae for these means and covariance using the following equations

$$\begin{aligned} \text{Predict:} \quad & \widehat{\pi}_{j+1} = P(v_{j+1}|Y_j) = \mathcal{N}(\widehat{m}_{j+1}, \widehat{C}_{j+1}) \\ \text{Analysis:} \quad & \pi_{j+1} = P(v_{j+1}|Y_{j+1}) = \mathcal{N}(\widehat{m}_{j+1}, \widehat{C}_{j+1}) \end{aligned}$$

In the homework 8, we find the variables $\widehat{m}_{j+1}, \widehat{C}_{j+1}, \widehat{C}^{-1}_{j+1}, \widehat{C}^{-1}_{j+1} m_{j+1}$ can be expressed by our known variables by assuming C_j is positive definite and $j \in \mathbb{N} \setminus 0$

$$\widehat{m}_{j+1} = M m_j$$

$$\widehat{C}_{j+1} = M C_j M^\top + \Sigma$$

$$C^{-1}_{j+1} = \widehat{C}^{-1}_{j+1} + H^\top \Gamma^{-1} H$$

$$C^{-1}_{j+1} m_{j+1} \widehat{C}^{-1}_{j+1} M m_j + H^\top \Gamma^{-1} y_{j+1}$$

Then, the Algorithm has the form.

Algorithm 2.1 Kalman Filter Algorithm

Input: Initial distribution $\pi_0 = \mathcal{N}(m_0, C_0)$ with $m_0 \in \mathbb{R}, C_0 \in \mathbb{R}^{d \times d}$

For $j = 0, 1, \cdots J-1$, Do the prediction and analysis steps

Prediction:

$$\widehat{m}_{j+1} = M m_j$$

$$\widehat{C}_{j+1} = M C_j M^\top + \Sigma$$

Analysis:

$$m_{j+1} = \widehat{m}_{j+1} + K_{j+1} d_{j+1}$$

$$C_{j+1} = (I - K_{j+1} H) \widehat{C}_{j+1}$$

where

$$d_{j+1} = y_{j+1} - H \widehat{m}_{j+1}$$

$$S_{j+1} = H \widehat{C}_{j+1} H^\top + \Gamma$$

$$K_{j+1} = \widehat{C}_{j+1} H^\top S^{-1}_{j+1}$$

Output: Predicted distributions $\widehat{\pi}_{j+1} = \mathcal{N}(\widehat{m}_{j+1}, \widehat{C}_{j+1})$ and filtering distributions $\pi_{j+1} = \mathcal{N}(m_{j+1}, C_{j+1})$, $j = 0, 1, ..., J-1$.

2.2 Ensemble Kalman Filter

Suppose we have the dynamics with the dynamics model is nonlinear, but the observation function is linear. let $v_0 \sim \mathcal{N}(m_0, C_0)$, $C_0, \Gamma,$ and Σ are positive definite and $v_0 \perp \{\xi_j\} \perp \{\eta_j\}$,

$$v_{j+1} = \Psi(v_j) + \xi_j, \quad \xi_j \sim \mathcal{N}(0, \Sigma) \quad i.i.d$$

$$y_{j+1} = h(v_{j+1}) + \eta_{j+1} \quad \eta_j \sim \mathcal{N}(0, \Gamma) \quad i.i.d$$

The Ensemble Kalman Filter (EnKF) is essentially an approximate form of the Kalman filter. But why do we need ensemble Kalman Filter? Instead of representing the state distribution exactly, it uses a sample or "ensemble" from the distribution. This ensemble is then evolved over time and updated as new data is received. The ensemble approach acts as a form of dimension reduction, making the method computationally feasible for systems with very high dimensions [4].

The Algorithm has the form below

Algorithm 2.2 Ensemble Kalman Filter Algorithm

1: **Input**: Ensemble size N. Initial ensemble $\{v_0^{(n)}\}_{n=1}^N$. Parameter $s \in \{0, 1\}$.
2: For $j = 0, 1, \cdots J-1$, Do the prediction and analysis steps
3:
4: **Prediction:**

$$\xi_j^{(n)} \sim \mathcal{N}(0, \Sigma), \quad i.i.d \quad n = 1, \cdots, N,$$

$$\widehat{v}_{j+1}^{(n)} = \Psi(v_j^{(n)}) + \xi_j^{(n)}, \quad n = 1, \cdots, N,$$

$$\widehat{m}_{j+1} = \frac{1}{N} \sum_{n=1}^{N} \widehat{v}_{j+1}^{(n)},$$

$$\widehat{C}_{j+1} = \frac{1}{N}(v_{j+1}^{(n)} - \widehat{m}_{j+1})(v_{j+1}^{(n)} - \widehat{m}_{j+1})^\top$$

5: **Analysis:**

$$\eta_{j+1}^{(n)} \sim \mathcal{N}(0, \Gamma), \quad n = 1, \cdots, N$$

$$y_{j+1}^{(n)} = y_{j+1} + s\eta_{j+1}^{(n)} \quad n = 1, \cdots, N$$

$$v_{j+1}^{(n)} = (I - K_{j+1}H)\widehat{v}_{j+1}^{(n)} + K_{j+1}y_{j+1}^{(n)} \quad n = 1, \cdots N.$$

where

$$K_{j+1} = \widehat{C}_{j+1} H^\top S_{j+1}^{-1}$$

$$S_{j+1} = H\widehat{C}_{j+1}H^\top + \Gamma$$

6: **Output**: Ensembles $\{v_j^{(n)}\}_{n=1}^N$

3 Algorithm Implementation and Results

In this section, we will implement both the Kalman Filter and the Ensemble Kalman Filter. We will assess the difference between the true state and the estimated state generated by each method, showcasing the effectiveness of these techniques. Also, we will show the effect of choosing different noise.

3.1 Implement Kalman Filter and Ensemble Kalman Filter

First, let's define the variables that we will use later in this section.

$$m_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{Initial state estimate}$$

$$C_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{Initial estimate covariance}$$

$$M = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} \qquad \text{Dynamics matrix}$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{Observation matrix}$$

$$\xi = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix} \cdot \text{noise} \qquad \text{Process noise covariance}$$

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \text{noise} \qquad \text{Observation noise covariance}$$

3.1.1 Implementation of Kalman Filter with $t = 100$

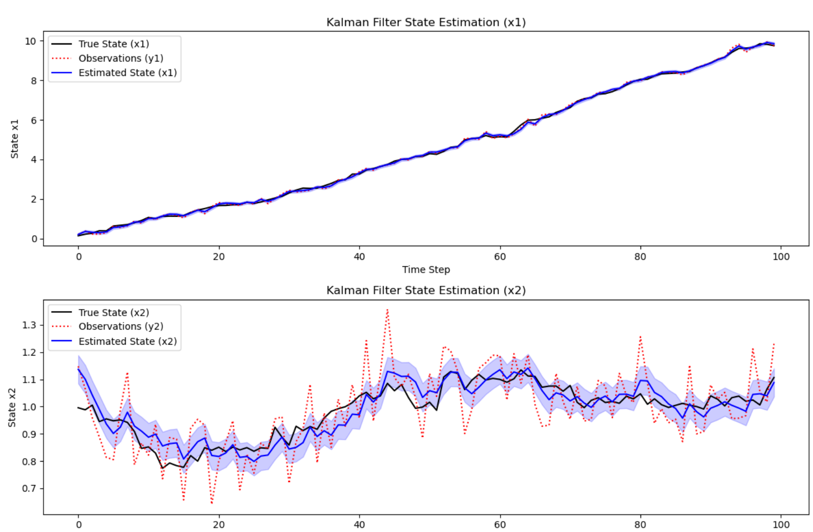

Figure 1 True State and Estimate State

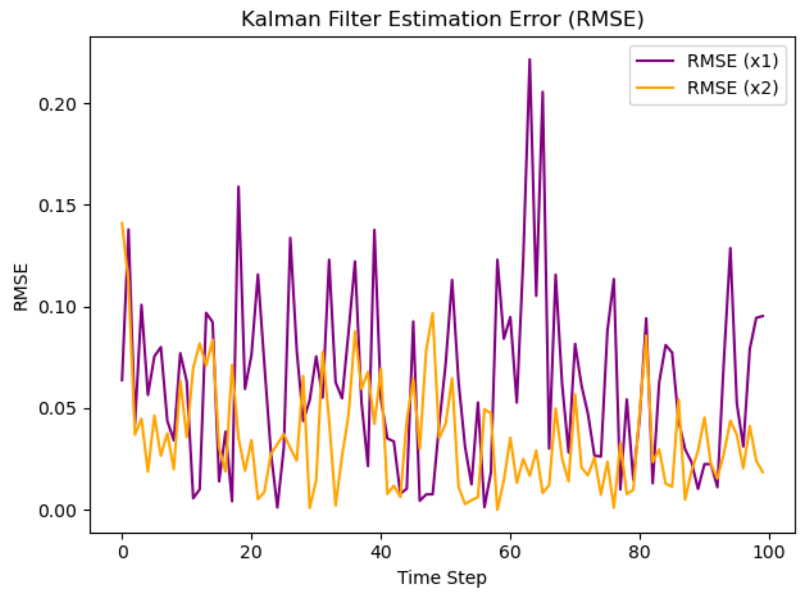

Figure 2 The error between true state and estimate state

The image 1 2 demonstrate the performance of a Kalman Filter in estimating two state variables x_1 and x_2 over 100 time steps. For x_1 , the filter's estimates closely follow the true state with narrow uncertainty bounds, indicating high confidence despite noisy observations. For x_2, although the observations are noisier and the uncertainty bounds are wider, the filter still reasonably tracks the true state. Overall, the Kalman Filter effectively estimates the state variables and quantifies uncertainty, showcasing its reliability in the presence of measurement noise.

3.1.2 Implementation of Kalman Filter with Different Noise Level

In this implementation of the Kalman Filter with varying noise terms, we assume that smaller noise terms lead to estimates that are closer to the true values. This is because smaller noise terms indicate less uncertainty or disturbance in the system, allowing the Kalman Filter to more accurately predict and update the state estimates based on observations. As the noise terms decrease in magnitude, the Kalman Filter is better able to differentiate between the true signal and the noise, resulting in improved estimation accuracy and convergence towards the true state of the system.

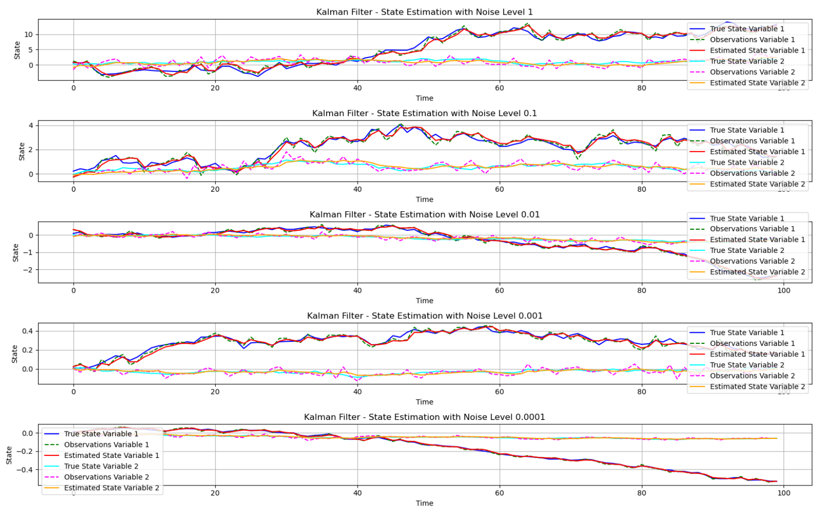

Figure 3 Different Noise Level using Kalman Filter

The image 3 consists of five subplots illustrating the performance of a Kalman Filter in estimating two state variables across different noise levels (1, 0.1, 0.01, 0.001, 0.0001). Each subplot compares the true state (solid lines) and noisy observations (dashed lines) with the Kalman Filter's estimates (solid lines) for both variables. As the noise level decreases, the estimated states increasingly align with the true states, and the filter more effectively mitigates the noise in observations. The plots demonstrate that the Kalman Filter maintains accurate state estimation across various noise levels, with performance improving as noise diminishes, showcasing its robustness and efficacy in different noise environments.

3.2 Ensemble Kalman Filter on Nonlinear Model

3.2.1 Case 1: A simple Nonlinear Dynamical Model

In this section, we introduce a dynamical system model governed by the following differential equations:

$$\frac{dx}{dt} = v$$
$$\frac{dv}{dt} = -0.1x + \sin(x)$$

where x represents the position and v represents the velocity of the system. These equations describe the evolution of the system over time. The first equation states that the rate of change of position x with respect to time t is equal to the velocity v. The second equation describes how the velocity v changes over time. It is influenced by the position x through a linear term $-0.1x$ and a sinusoidal term $\sin(x)$. This dynamical system captures the behavior of a system where the velocity is affected by both the position and a sinusoidal force. Then, we are going to define some parameters we are going to use. Let $\Delta t = 0.1$, Ensemble number be 10 and others defined above.

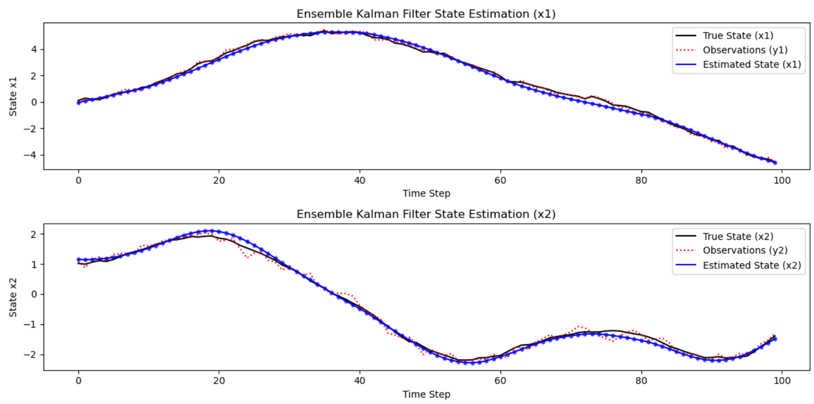

Figure 4 EnKF on the simple nonlinear case

Based on Figure 4 the true state (black solid line) and the estimated state (blue line with markers) align closely, indicating high accuracy in the EnKF's performance. Despite the noisy observations (red dotted line), the EnKF successfully filters out the noise and provides accurate state estimates, effectively capturing the system's temporal dynamics.

3.2.2 Case 2: Ensemble Kalman Filter on Lorenz 63 Model

We're transitioning from implementing the EnKF in a linear function to applying it in a nonlinear function known as the Lorenz 63 Model, as discussed in our homework 8. Here's how the Lorenz 63 Model is defined:

$$\frac{\partial v}{\partial t} = \sigma(w - v)$$

$$\frac{\partial w}{\partial t} = v(\rho - z) - w$$

$$\frac{\partial z}{\partial t} = vw - \beta z$$

with the parameter value $\sigma = 10, \rho = 28, \beta = \frac{8}{3}$. Let $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, assume the number of ensemble members is 10, and the noise level can be modified while keeping other parameters the same.

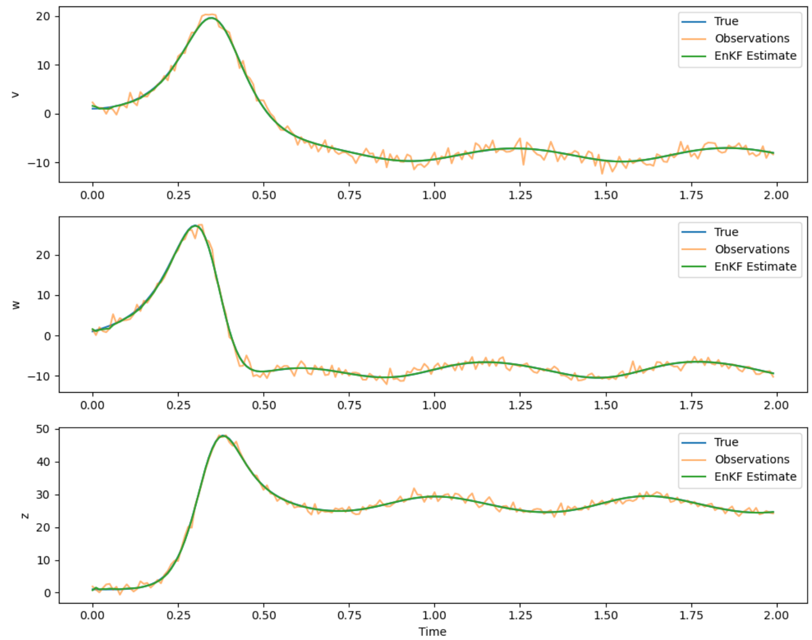

Figure 5 EnKF on Lorenz 63

Figure 5 illustrates the effectiveness of the Ensemble Kalman Filter (EnKF) in estimating the states v,w and z of the Lorenz 63 model over time. For each state, the EnKF estimates (green line) closely follow the true state values (blue line), despite the presence of noise in the observations (orange line). This demonstrates the EnKF's ability to filter out noise and accurately track the system's dynamics.

3.2.3 Implementation of EnKF with Different Noise level and Different Number of Ensembles

Implementing the Ensemble Kalman Filter (EnKF) with varying noise levels involves understanding how different magnitudes of measurement and process noise affect the performance and accuracy of the filter. Last subsection, we used the noise level at 1. Do different noise levels, similar to the Kalman Filter, provide us with markedly different results in the Ensemble Kalman Filter (EnKF)?

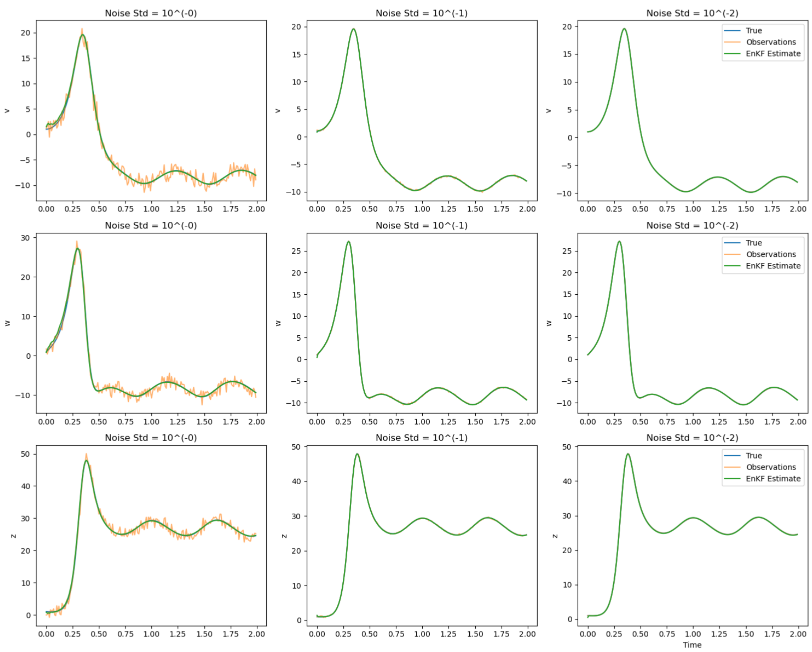

Figure 6 EnKF on Lorenz 63 with Different Noise Level

The figure 6 demonstrates the performance of the Ensemble Kalman Filter (EnKF) on the Lorenz 63 system with varying noise levels (1,0.1,0.01)for different state variables (y, w, z). Higher noise levels result in noisier observations and reduced accuracy of the EnKF estimates, while lower noise levels provide more accurate state estimates that closely follow the true state. This pattern indicates that, similar to the Kalman Filter, the EnKF's performance significantly improves with decreasing noise levels, highlighting the importance of noise magnitude in filtering accuracy.

Since our input includes the number of ensembles, it is natural to consider whether the number of ensembles has any influence on our experiment.

According to figure 7, the number of ensembles affects the accuracy and stability of the EnKF estimates. As the ensemble size increases, the EnKF estimates more closely follow the true values, indicating better performance and reduced estimation error. For smaller ensemble sizes, the estimates are noisier and deviate more from the true values, showing the influence of ensemble size on the experiment's outcome.

4 Discussion and Bibliography

This project investigated the implementation and application of the Kalman Filter (KF) and the Ensemble Kalman Filter (EnKF) to enhance the understanding of data assimilation techniques. The KF, known for its effectiveness in linear systems with Gaussian noise, demonstrated strong state estimation capabilities and uncertainty quantification. In contrast, the EnKF, tailored for high-dimensional nonlinear systems, utilized an ensemble approach to represent the state distribution, proving particularly effective in complex scenarios such as the Lorenz 63 model. Both filters were evaluated

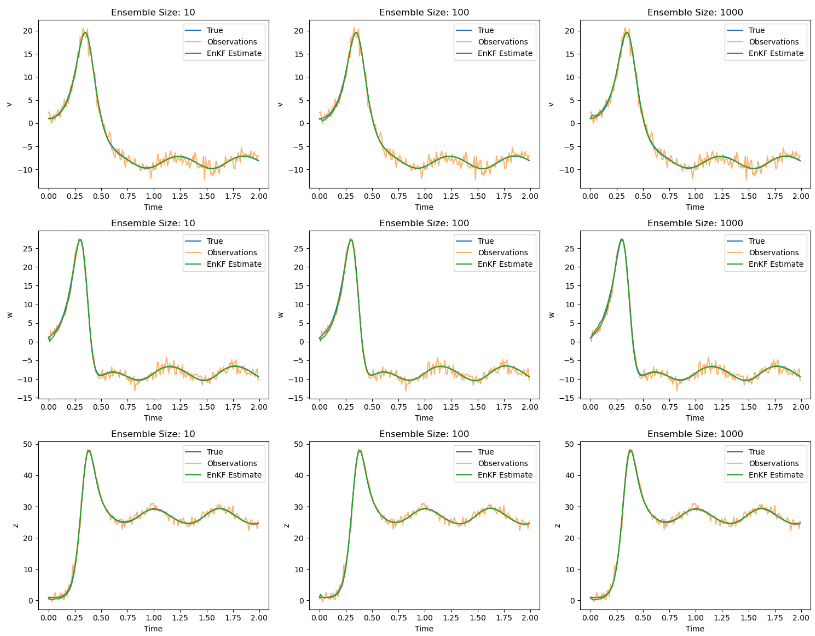

Figure 7 EnKF on Lorenz 63 Different Number of Ensembles

under various noise levels, revealing increased accuracy with lower noise and larger ensemble sizes for the EnKF. Additionally, the theoretical foundation was primarily based on [1], with practical implementation across multiple dynamic models.

References

[1] A. T. Daniel Sanz-Alonso, Andrew Stuart. *Inverse Problems and Data Assimilation*. Courier Dover Publications, 2018.

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