Linear Algebra MATH 325: Assignment 2

Problem 1: Consider a matrix

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix}$$

and let $\alpha_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $v \mapsto A \cdot v$ be the corresponding $\mathbb{R}$-linear map. Find the matrix of the linear map α_A with respect to the basis

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

of $\mathbb{R}^2$.

Solution 1. We have

$$A \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix} \quad \text{and} \quad A \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ 6 \end{pmatrix}.$$

We express this images as linear combinations of the basis vectors v_1, v_2:

$$\begin{pmatrix} 6 \\ 7 \end{pmatrix} = \frac{11}{5} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{8}{5} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

and

$$\begin{pmatrix} 13 \\ 6 \end{pmatrix} = \frac{33}{5} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + (-\frac{1}{5}) \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Hence the matrix of α_A with respect to this basis is

$$\begin{pmatrix} \frac{11}{5} & \frac{33}{5} \\ \frac{8}{5} & -\frac{1}{5} \end{pmatrix}.$$

Solution 2. In class we proved that the matrix for the linear map α_A in basis v_1, v_2 is of the form $S^{-1}AS$ where S is the base change matrix: its first column is the vector v_1 and the second column is v_2. Thus,

$$S = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

Thus, it remains to compute the product

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

Problem 2: Let $A = \begin{pmatrix} 2 & 2 \\ 0 & 5 \end{pmatrix}$. What are the A-invariant subspaces in $\mathbb{R}^2$?

Solution. Recall that a subspace $W \subseteq \mathbb{R}^2$ is called A-invariant if $A \cdot w \in W$ for all $w \in W$. Subspaces of $\mathbb{R}^2$ can only have dimension 0, 1, or 2. The 0-dimensional subspace $W = \{0\}$ and the 2-dimensional subspace $W = \mathbb{R}^2$ are clearly A-invariant. We are left to figure out the 1-dimensional subspace which are A-invariant. In class we proved that any such subspace consists of eigenvectors of A.

Let us recall the proof of this fact. Let W be a 1-dimensional subspace and v a basis vector, *i.e.* any element of W is equal $\lambda \cdot v$ for some $\lambda \in \mathbb{R}$. This subspace W is therefore A-invariant if an only if $A \cdot v = \mu \cdot v$ for some $\mu \in \mathbb{R}$, *i.e.* if and only if v (and so all nonzero elements of W) are eigenvectors for some eigenvalue.

To sum up, to determine the 1-dimensional A-invariant subspaces we have to determine the eigenvalues and the corresponding eigenspaces.

We have $P_A(T) = (T-2)\cdot(T-5)$ and so 2 and 5 are the eigenvalues of A. We immediately check that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector for the eigenvalue 2 and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector for the eigenvalue 5. Therefore

$$W_1 := \mathbb{R} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left\{ \begin{pmatrix} r \\ 0 \end{pmatrix} \middle| r \in \mathbb{R} \right\}$$

and

$$W_2 := \mathbb{R} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \left\{ \begin{pmatrix} 2r \\ 3r \end{pmatrix} \middle| r \in \mathbb{R} \right\}$$

are the only 1-dimensional A-invariant subspaces of $\mathbb{R}^2$.

Problem 3: Give an example of two 2×2-matrices with real coefficients which have the same characteristic polynomial but which are not conjugate. (Recall that two $n \times n$-matrices A and B are called *conjugate* if there is an invertible $n \times n$-matrix S, such that $B = S^{-1} \cdot A \cdot S$.)

Solution. The matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

have both the characteristic polynomial $(T-1)^2$. If A and B were conjugate then there would exist an invertible matrix S such that $A = S^{-1}BS$. However, $BS = I_2 \cdot S = S$, hence

$$S^{-1}BS = S^{-1}S = I_2 = B \neq A.$$

Therefore A and B are not conjugate.

Problem 4: Let A be a 2×2-matrix with only one eigenvalue $\lambda = 3$. Show that $(3\cdot \mathrm{I}_2 - A)^2 = 0$. Recall that the symbol I_2 denotes the 2×2-identity matrix:

$$\mathrm{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Solution. Let us first show that A is conjugate to a matrix of the form

$$B = \begin{pmatrix} 3 & a \\ 0 & 3 \end{pmatrix}$$

where $a \in \mathbb{R}$. This is equivalent to saying that there exists a new basis v_1, v_2 of $\mathbb{R}^2$ such that $A \cdot v_1 = 3v_1$ and $A \cdot v_2 = av_1 + 3v_2$.

Since 3 is an eigenvalue of A there exists an eigenvector v_1 of A such that $A \cdot v_1 = 3v_1$. Choose any vector v_2 which is not proportional to v_1. Then $\{v_1, v_2\}$ is a basis of $\mathbb{R}^2$. We claim that this is the required basis. Indeed, we need to check only that $A \cdot v_2$ is of the form $A \cdot v_2 = av_1 + 3v_2$. Write $A \cdot v_2 = av_1 + bv_2$. We want to show that $b = 3$. Note that the linear map α_A corresponding to A in our new basis v_1, v_2 has matrix

$$\begin{pmatrix} 3 & a \\ 0 & b \end{pmatrix}.$$

It follows that b is a root of its characteristic polynomial, hence b is an eigenvalue. However we are given that 3 is the only eigenvalue for A. Therefore $b = 3$.

To sum up, there exists an invertible matrix S such that $A = S^{-1}BS$. Then we have

$$\begin{aligned} (3 \cdot I_2 - A)^2 &= (3 \cdot I_2 - S^{-1}BS)^2 \\ &= (3 \cdot S^{-1}I_2S - S^{-1}BS)^2 \\ &= S^{-1}(3 \cdot I_2 - B)^2 S \\ &= S^{-1} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}^2 S \\ &= S^{-1}O_2S = O_2. \end{aligned}$$