MATH 325 Q1: LINEAR ALGEBRA III, PART 6

Existence

We now show the existence of a Jordan basis. We argue by induction on the dimension of a vector space in question. The induction will be applied to a class of vector spaces K equipped with a linear map $\alpha : K \to K$ satisfying the property: there exists a scalar $\lambda \in \mathbb{C}$ and integer m such that $(\alpha - \lambda \cdot \mathrm{id})^m = 0$.

Fact. *We proved before that any such map α has a unique eigenvalue equal to λ. Therefore $K_\lambda = K$ implying every nonzero vector in K is a generalized eigenvector.*

Theorem. *Let $\alpha : K \to K$ be a $\mathbb{C}$-linear map satisfying $(\alpha - \lambda \cdot \mathrm{id})^m = 0$ for some $\lambda \in \mathbb{C}$ and some integer $m \geq 1$, where $\mathrm{id} = \mathrm{id}_K$ is the identity map of the finite dimensional $\mathbb{C}$-vector space K. Then there exists a Jordan basis for α in K.*

Proof. Let $E := E_\lambda = \mathrm{Ker}(\alpha - \lambda \cdot \mathrm{id})$ be the eigenspace for the only eigenvalue λ. We argue by induction on $d = \dim_{\mathbb{C}} K$. If $d = 1$ then by dimension argument we have $K = E$. Let $v \in K$ be an eigenvector for α. The set $\Delta = \Delta_1 = \{v\}$ is then a Jordan basis for α.

Let now $d \geq 2$. By the induction hypothesis there exists a Jordan basis for all linear maps $\beta : L \longrightarrow L$, such that $(\beta - \lambda \cdot \mathrm{id}_L)^s = 0$ for some integer $s \geq 1$ if $1 \leq \dim_{\mathbb{C}} L < d$. We distinguish two cases.

Case 1: $E = \mathrm{Ker}(\alpha - \lambda \cdot \mathrm{id}) = K$. Let $v_1, \ldots, v_d$ be a basis of $E = K$. Then $\Delta = \bigcup_{j=1}^{s} \{v_j\}$ is a Jordan basis for α.

Case 2: $E \neq K$. Consider a linear map

$$\alpha - \lambda \cdot \mathrm{id} : K \longrightarrow K, \quad v \to (\alpha - \lambda \cdot \mathrm{id})(v) = \alpha(v) - \lambda v.$$

let $L \subset K$ be its image. Thus, L consists of all vectors of the form

$$L = \left\{ (\alpha - \lambda \cdot \mathrm{id})(v) \,|\, v \in K \right\}.$$

This is a subspace of K of dimension

$$\dim_{\mathbb{C}} L = \dim_{\mathbb{C}} K - \dim_{\mathbb{C}} \left(\mathrm{Ker}(\alpha - \lambda \cdot \mathrm{id}) \right) = \dim_{\mathbb{C}} K - \dim_{\mathbb{C}} E < \dim_{\mathbb{C}} K = d$$

by the dimension formula.

Since $E \neq K$, the vector space L is nonzero. It is moreover α-invariant. Indeed,

$$\alpha\big((\alpha - \lambda \cdot \mathrm{id})(v)\big) = \big((\alpha - \lambda \cdot \mathrm{id})\alpha\big)(v) = \big((\alpha - \lambda \cdot \mathrm{id})(\alpha(v))\big).$$

Hence taking the restriction of α at L we have a well defined $\mathbb{C}$-linear map

$$\beta \,:\, L \longrightarrow L\,,\; x \longmapsto \beta(x) \,:=\, \alpha(x)\,,$$

which also satisfies $(\beta - \lambda \cdot \mathrm{id}_L)^m = 0$ since

$$(\beta - \lambda \cdot \mathrm{id}_L)^m(v) = (\alpha - \lambda \cdot \mathrm{id})^m(v) = 0$$

for all $v \in L$.

Let

$$F \,:=\, \mathrm{Ker}(\beta - \lambda \cdot \mathrm{id}_L) \,=\, E \cap L\,;$$

thus F consists of all eigenvectors in K which live in L. Set

$$t := \dim_{\mathbb{C}} F \leq s := \dim_{\mathbb{C}} E.$$

By our induction hypotheses there exists a Jordan basis $\Omega = \bigcup_{j=1}^{t} \Omega_j$ for β. Let $v_1, \ldots, v_t \in K$, such that

$$\begin{aligned}\Omega_j &= \big\{\, w_j = (\beta - \lambda \cdot \mathrm{id}_L)^{m_j - 1}(v_j), \ldots, (\beta - \lambda \cdot \mathrm{id}_L)(v_j) \,\big\} \\ &= \big\{\, w_j = (\alpha - \lambda \cdot \mathrm{id})^{m_j - 1}(v_j), \ldots, (\alpha - \lambda \cdot \mathrm{id})(v_j) \,\big\}\,,\end{aligned}$$

where $m_j \geq 1$ are integers, for $j = 1, \ldots, t$. We proved before that the vectors $w_1, \ldots, w_t$ are a basis for F. We extend this to a basis $w_1, \ldots, w_t, w_{t+1}, \ldots, w_s$ of E.

To get a Jordan basis for α we define now

$$\Delta_j \,:=\, \big\{\, w_j = (\alpha - \lambda \cdot \mathrm{id})^{m_j - 1}(v_j), \ldots, (\alpha - \lambda \cdot \mathrm{id})(v_j),\, v_j \,\big\}$$

for $1 \leq j \leq t$, and $\Delta_j := \{w_j\}$ for $t+1 \leq j \leq s$. These are full cycles of generalized eigenvectors for α. We claim that $\Delta := \bigcup_{j=1}^{s} \Delta_j$ is a Jordan basis for α.

We first check that $\Delta_j \cap \Delta_k = \emptyset$. We can assume $j < k$.

Subcase 1: $k \geq t + 1$. Then $\Delta_k = \{w_k\}$ but the only eigenvector in Δ_j is w_j which is $\neq w_k$ since $j \neq k$, and so $\Delta_k \cap \Delta_j = \emptyset$ in this case.

Subcase 2: $k \leq t$. We have $\Delta_j = \Omega_j \cup \{v_j\}$ and also $\Delta_k = \Omega_k \cup \{v_k\}$. Since Ω is a Jordan basis for F we have $\Omega_j \cap \Omega_k = \emptyset$ and also $(\alpha - \lambda \cdot \mathrm{id})(v_j) \neq (\alpha - \lambda \cdot \mathrm{id})(v_k)$ which implies $v_j \neq v_k$. Hence we only have to show that neither $v_k \in \Omega_j$ nor $v_j \in \Omega_k$.

If $v_k \in \Omega_j$, say $v_k = (\alpha - \lambda \cdot \mathrm{id})^c(v_j)$ for some $1 \leq c \leq m_j - 1$, then $w_k = (\alpha - \lambda \cdot \mathrm{id})^{m_k-1}(v_k) = (\alpha - \lambda \cdot \mathrm{id})^{m_k-1+c}(v_j)$ is in Ω_j which would imply $\Omega_k \cap \Omega_j$ is not empty, a contradiction. Analogously, one shows that $v_j \notin \Omega_k$.

To show that Δ is a basis we first count the number of elements in Δ. The set Δ is equal the disjoint union of the sets

$$\Omega\,,\ \{v_1, \ldots, v_t\}\,,\ \text{and } \{w_{t+1}, \ldots, w_s\}$$

and so has $|\Omega| + s = \dim_{\mathbb{C}} L + \dim_{\mathbb{C}} E = \dim_{\mathbb{C}} K$ elements. Therefore it is enough to show that the elements of Δ are linear independent. Assume we have a linear combination

$$0 \;=\; \sum_{j=1}^{t} \sum_{k=0}^{m_j-1} a_k^{(j)} \cdot (\alpha - \lambda \cdot \mathrm{id})^k(v_j) \;+\; \sum_{j=t+1}^{s} b_s \cdot w_s\,.$$

Since $(\alpha - \lambda \cdot \mathrm{id})(w_j) = 0$ for $j = 1, \ldots, s$ we get from this equation

$$0 \;=\; \sum_{j=1}^{t} \sum_{k=0}^{m_j-2} a_k^{(j)} \cdot (\alpha - \lambda \cdot \mathrm{id})^{k+1}(v_j)\,,$$

and so $a_k^{(j)} = 0$ for all $0 \leq k \leq m_j - 2$ and all $1 \leq j \leq t$ since for these indices the vectors $(\alpha - \lambda \cdot \mathrm{id})^{k+1}(v_j)$ are in the set of linear independent vectors Ω. Hence the linear dependence equation simplifies to

$$0 \;=\; \sum_{j=1}^{t} a_{m_j-1}^{(j)} \cdot w_j \;+\; \sum_{j=t+1}^{s} b_s \cdot w_s\,.$$

But the set $\{w_1, \ldots, w_t, w_{t+1}, \ldots, w_s\}$ is a basis of E and therefore also $b_j = 0$ for $j = t+1, \ldots, s$ and $a_{l_j-1}^{(j)} = 0$ for $j = 1, \ldots, t$. We are done. $\square$

Out results show that all vector spaces of the form K_λ have Jordan bases. Combining such bases for all K_λ with different λ's we have the following general result.

Theorem (Jordan normal/canonical form). *Let $\alpha : V \longrightarrow V$ be a $\mathbb{C}$-linear map with characteristic polynomial*

$$P_\alpha(T) \;=\; \prod_{i=1}^{l} (T - \lambda_i)^{n_i}\,,$$

where $\lambda_1, \ldots, \lambda_l$ are all different eigenvalues of α. Let K_{λ_1}, K_{λ_2}, …, and K_{λ_l} be the corresponding generalized eigenspaces.

Then each K_{λ_i} *has a Jordan basis*

$$\Delta^{(i)} := \bigcup_{j=1}^{s_i} \Delta_j^{(i)}$$

with

$$m_1^{(i)} = |\Delta_1^{(i)}| \geq m_2^{(i)} = |\Delta_2^{(i)}| \geq \ldots \geq m_{s_i}^{(i)} = |\Delta_{s_i}^{(i)}| \geq 1$$

for all $1 \leq i \leq l$. *The union* $\bigcup_{i=1}^{l} \Delta^{(i)}$ *is a basis of* V, *called a Jordan basis for* α, *and the matrix of* α *with respect to this basis is*

$$\mathrm{J}(\alpha) = \begin{pmatrix} A(\lambda_1) & 0 & \ldots & 0 \\ 0 & A(\lambda_2) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \ldots & A(\lambda_l) \end{pmatrix};$$

with

$$A(\lambda_i) = \begin{pmatrix} \mathrm{J}(\lambda_i, m_1^{(i)}) & 0 & \ldots & 0 \\ 0 & \mathrm{J}(\lambda_i, m_2^{(i)}) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \ldots & \mathrm{J}(\lambda_i, m_{s_i}^{(i)})\,, \end{pmatrix}$$

where $\mathrm{J}(\lambda_i, m_j^{(i)})$ *is the corresponding Jordan block for the eigenvalue* λ_i *of size* $m_j^{(i)}$. *The number* s_i *of Jordan blocks for the eigenvalue* λ_i *is equal to the dimension of the eigenspace of this eigenvalue.*

Definition. *The matrix* $\mathrm{J}(\alpha)$ *is called the* **Jordan normal form** *of* α *(note that it is unique up to permutation of the eigenvalues).*

The Jordan normal form of a matrix

Let A be a complex $n \times n$-matrix. The *Jordan normal form* $\mathrm{J}(A)$ of A is the Jordan normal form of the corresponding linear map

$$\alpha_A : \mathbb{C}^n \longrightarrow \mathbb{C}^n, \; v \longmapsto A \cdot v\,,$$

i.e. $\mathrm{J}(A) := \mathrm{J}(\alpha_A)$. The theorem above implies then that there exists an invertible matrix S, such that

$$\mathrm{J}(A) = S^{-1} \cdot A \cdot S\,.$$

Note that the column vectors of the base change matrix S constitute a Jordan basis for α_A.

Jordan normal forms for "small" matrices.

Let A be a complex $n \times n$-matrix. We consider here the Jordan normal of A for $n \leq 3$.

$n = 1$: Then $A = (a)$ for some complex number a, and A is its own Jordan normal form, consisting of a single Jordan block of size 1 for the only eigenvalue a of A.

$n = 2$: Here we have two cases. First if A is diagonalizable with eigenvalues λ_1, λ_2, where maybe $\lambda_1 = \lambda_2$, then the Jordan normal form of A is

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

and a Jordan basis consists of eigenvectors for the respective eigenvalues.

If A is not similar to a diagonal matrix then A has only one eigenvalue λ and the Jordan normal form of A is

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

In this case the generalized eigenspace of A is equal the whole space $\mathbb{C}^2$ and a Jordan basis consists of a full cycle of lenght 2.

Question. *How to get such a Jordan basis?*

Here is the answer. Since the eigenspace E_λ for the eigenvalue λ has dimension 1 (because A is not diagonalizable) there exists $v \in \mathbb{C}^2 \setminus E_\lambda$. Then $(A - \lambda \cdot \mathrm{I}_2) \cdot v$ is non zero and an eigenvector for λ. In other words $\Big\{ (A - \lambda \cdot \mathrm{I}_2) \cdot v, v \Big\}$ is a Jordan basis.

$n = 3$ Here we have essentially three cases. First A is diagonalizable with eigenvalues λ_1, λ_2 and λ_3 (not assumed to be all different). The Jordan normal form of A is then the diagonal matrix

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

and a Jordan basis consists of eigenvectors for the respective eigenvalues.

If A is not diagonalizable then there are at most two eigenvalues. Assume first that there are exactly two λ_1 and λ_2. One of them has then multiplicity 1 and the other multiplicity 2.

Up to numbering, without loss of generality we may assume that λ_2 has multiplicity 2. Thus, the characteristic polynomial of A is of the form

$$p_A(T) = (T - \lambda_1)(T - \lambda_2)^2.$$

Then the generalized eigenspace K_{λ_1} of λ_1 has dimension 1 and so is equal to its eigenspace E_{λ_1}. Also, the generalized eigenspace K_{λ_2} of λ_2 has dimension 2. The latter is not equal the eigenspace E_{λ_2} since we assume that A is not diagonalizable. Hence there is a full cycle of generalized eigenvectors of length 2 for λ_2 and the Jordan normal form of A is

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

Question: *How one can construct a Jordan basis pratically?*

Here is the answer. To get a Jordan basis one computes first an eigenvector v_1 for λ_1. To get the full cycle of generalized eigenvectors for λ_2 we have to find a nonzero vector v_2 which is in K_{λ_2} but not an eigenvector for λ_2. To find such a vector we observe first that $(A - \lambda_1 \cdot \mathrm{I}_3) \cdot v \in K_{\lambda_2}$ for all $v \in \mathbb{C}^3$. Indeed, by Cayley-Hamilton theorem we have

$$0 = (A - \lambda_1 \mathrm{I}_3)(A - \lambda_2 \mathrm{I}_3)^2 = (A - \lambda_2 \mathrm{I}_3)^2(A - \lambda_1 \mathrm{I}_3).$$

Therefore,

$$(A - \lambda_2 \cdot \mathrm{I}_3)^2(A - \lambda_1 \cdot \mathrm{I}_3) \cdot v = 0.$$

The matrix $(A-\lambda_1\cdot \mathrm{I}_3)$ has rank 2 since $\dim_{\mathbb{C}} E_{\lambda_1}=1$. Hence the dimension of the image of the linear map $\mathbb{C}^3 \longrightarrow K_{\lambda_2}$, $v\mapsto (A-\lambda_1\cdot \mathrm{I}_3)\cdot v$ is 2 and so equal to K_{λ_2}. This image is spanned by the vectors $(A-\lambda_1\cdot \mathrm{I}_3)\cdot e_i$, $i=1,2,3$, where e_i is the ith standard basis vector of $\mathbb{C}^3$. It follows that at least one of the vectors $(A-\lambda_1\cdot \mathrm{I}_3)\cdot e_i$ is not an eigenvector for λ_2, which we then choose to be our vector $v_2\in K_{\lambda_2}\setminus E_{\lambda_2}$. Then $(A-\lambda_2\cdot \mathrm{I}_3)\cdot v_2$ is an eigenvector for λ_2 and $\Big\{ (A-\lambda_2\cdot \mathrm{I}_3)\cdot v_2, v_2 \Big\}$ is a full cycle of generalized eigenvectors for λ_2 which together with the eigenvector v_1 for λ_1 is a Jordan basis for A.

The last case is that A has only one eigenvalue. Then we have two possibilities under our assumption that A is not diagonalizable. First

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

which occurs if the dimension of the eigenspace of λ is 2. This happens if and only if there is no full cycle of generalized eigenvectors of length 3, *i.e.* if and only if $(A-\lambda\cdot \mathrm{I}_3)^2=0$.

To get a Jordan basis in this case one can proceed as follows. As A is not diagonalizable the matrix $A-\lambda\cdot \mathrm{I}_3$ is not the zero matrix and so there exists $v\in\mathbb{C}^3$, such that $(A-\lambda\cdot \mathrm{I}_3)\cdot v\neq 0$. Since $(A-\lambda\cdot \mathrm{I}_3)^2=0$ this vector $(A-\lambda\cdot \mathrm{I}_3)\cdot v$ is an eigenvector for λ, which can not generate the eigenspace E_λ is this space is 2-dimensional. Hence there exists $w\in E_\lambda$ which together with $(A-\lambda\cdot \mathrm{I}_3)\cdot v$ is a basis of E_λ. Then $\Big\{w\Big\}\cup\Big\{ (A-\lambda\cdot \mathrm{I}_3)\cdot v, v \Big\}$ is a Jordan basis for A.

The second possibility is

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

in case the dimension of the eigenspace of λ is only 1. This means in particular that there is a full cycle of generalized eigenvectors of length 3 and so $(A-\lambda\cdot \mathrm{I}_3)^2\neq 0$. Here a Jordan basis is rather easy to find. Since $(A-\lambda\cdot \mathrm{I}_3)^2\neq 0$ there exists a vector v, such that $(A-\lambda\cdot \mathrm{I}_3)^2\cdot v\neq 0$. Then

$\left\{ (A - \lambda \cdot \mathrm{I}_3)^2 \cdot v, (A - \lambda \cdot \mathrm{I}_3) \cdot v, v \right\}$ is a full cycle of generalized eigenvectors for λ and so a Jordan basis for A.

Three examples

We illustrate now the 3×3-case with three concrete examples which are not diagonalizable (for the 2×2-case see Part 3 of the notes).

(i) Let

$$A = \begin{pmatrix} 3 & -3 & 1 \\ 7 & -8 & 3 \\ 15 & -18 & 7 \end{pmatrix}.$$

We compute

$$P_A(T) = T^3 - 2T^2 + T = T \cdot (T^2 - 2T + 1) = T \cdot (T-1)^2 ,$$

and so A has two different eigenvalues $\lambda_1 = 0$ which has multiplicity 1 and $\lambda_2 = 1$ which has multiplicity 2.

To get the Jordan normal form of A we have therefore to compute the dimension of the eigenspace for the eigenvalue $\lambda_2 = 1$. Solving the linear equation $A \cdot v = v$ we get that it is 1-dimensional with basis

$$v = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}.$$

Hence the Jordan normal form of A is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

To get a Jordan basis for A we choose first an eigenvector for the eigenvalue $\lambda_1 = 0$: The vector

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

is such an eigenvector. To get a full cycle of generalized eigenvectors for $\lambda_2 = 1$ we consider first the vector

$$(A - 0 \cdot \mathrm{I}_3) \cdot e_1 = A \cdot e_1 = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}.$$

This is not an eigenvector but

$$(A - \mathrm{I}_3) \cdot \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

is an eigenvector. Hence the set

$$\left\{ v_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \right\}$$

is a full cycle of generalized eigenvectors for $\lambda_2 = 1$. Thus, a Jordan basis is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \right\}.$$

(ii) Let

$$B = \begin{pmatrix} 3 & -2 & 1 \\ 2 & -2 & 2 \\ 3 & -6 & 5 \end{pmatrix}.$$

We compute the characteristic polynomial

$$P_B(T) = T^3 - 6T^2 + 12T - 8 = (T-2)^3,$$

and so B has only one eigenvalue $\lambda = 2$ of multiplicity 3. In particular the corresponding generalized eigenspace is the whole space.

To get the Jordan normal form of B we have to compute the dimension of the eigenspace of the eigenvalue $\lambda = 2$.

We have

$$B - 2 \cdot \mathrm{I}_3 = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 3 & -6 & 3 \end{pmatrix},$$

and so the first and third column of this matrix coincide and the second column is equal (-2)-times the first. Hence the rank of $(B - 2 \cdot \mathrm{I}_3) = 1$ and so $\dim_{\mathbb{C}} E_2 = 2$. We get that B has the following Jordan normal form:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Another way to see this is to observe that $(B - 2 \cdot \mathrm{I}_3)^2 = 0$ and so the maximal length of a full cycle of generalized eigenvectors is 2.

To get a Jordan basis we first choose a vector $v \in \mathbb{C}^3$, such that $(B - 2 \cdot \mathrm{I}_3) \cdot v \neq 0$. For instance $v = e_1$ does the job. Then

$$v := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (B - 2 \cdot \mathrm{I}_3) \cdot e_1$$

is an eigenvector and

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

is a full cycle of generalized eigenvectors. To extend this full cycle to a Jordan basis we need an eigenvector w which is linear independent of v. For instance

$$w := \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

has this property. We get the Jordan basis

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

(ii) Let

$$C = \begin{pmatrix} 0 & 5 & 2 \\ -1 & 8 & 3 \\ 2 & -14 & -5 \end{pmatrix}.$$

We compute the characteristic polynomial

$$P_C(T) = T^3 - 3T^2 + 3T - 1 = (T-1)^3,$$

and so C has only one eigenvalue $\lambda = 1$ of multiplicity 3. In particular the corresponding generalized eigenspace is the whole space. We compute now

$$(C - \mathrm{I}_3)^2 = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & -4 & -2 \end{pmatrix},$$

and so $(C - \mathrm{I}_3)^2 \neq 0$. Hence a Jordan basis for C consists of a full cycle of generalized eigenvectors of lenght 3 and for every $v \in \mathbb{C}^3$, such that $(C - \mathrm{I}_3)^2 v \neq 0$, the set

$$\left\{(C - \mathrm{I}_3)^2 \cdot v, (C - \mathrm{I}_3) \cdot v, v\right\}$$

is a full cycle of generalized eigenvectors and so a Jordan basis. For instance $v = e_3$ does the job, *i.e.* we have the Jordan basis

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

and the Jordan normal form of C is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark. For a complex $n \times n$-matrix with $n \geq 4$ it is in general not enough to know the eigenvalues and their multiplicities, and the dimensions of the respective eigenspaces to compute the Jordan normal form.

For instance, let $n = 4$. If there are at least two different eigenvalues then going through all cases we see that these data determine the Jordan normal form. However if the 4×4-matrix has only one eigenvalue λ this is not the case. The matrices

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

both have only one eigenvalue λ of multiplicity 4, and the dimensions of the corresponding eigenspace E_λ is in both cases 2. But their Jordan normal forms (these matrices are already in Jordan normal form) are obviously different. There are in both cases two Jordan blocks, one of them has size 3 and the other size 1 for the matrix on the left hand side, but both blocks have size 2 for the matrix on the right hand side.