Linear Algebra MATH 325: Assignment 1

Problem 1: Let $z_1 = 2 - 5\,i$ and $z_2 = -1 - i$. Compute

$$z_1 - z_1 z_2; \quad (z_1 + 3z_2)^2; \quad i\,z_2 - z_1^2.$$

Solution. (a) One has

$$z_1 z_2 = (2 - 5\,i)(-1 - i) = (-2 - 5) + (-2 + 5)\,i = -7 + 3\,i,$$

hence

$$z_1 - z_1 z_2 = (2 - 5\,i) - (-7 + 3\,i) = 9 - 8\,i.$$

(b) Next,

$$z_1 + 3z_2 = (2 - 5\,i) + 3(-1 - i) = (2 - 5\,i) + (-3 - 3\,i) = -1 - 8\,i,$$

hence

$$(z_1 + 3z_2)^2 = (-1 - 8\,i)(-1 - 8\,i) = (1 - 64) + (8 + 8)\,i = -63 + 16\,i.$$

(c) Lastly,

$$i\,z_2 = i\,(-1 - i) = 1 - i,$$

$$z_1^2 = (2 - 5\,i)(2 - 5\,i) = (4 - 25) + (-10 - 10)\,i = -21 - 20\,i$$

and this implies

$$i\,z_2 - z_1^2 = (1 - i) - (-21 - 20\,i) = 22 + 19\,i.$$

Problem 2: Use polar forms of the complex numbers

$$z_1 = 1 + \sqrt{3}\,i \quad \text{and} \quad z_2 = \sqrt{3} + i$$

to compute $z_1 \cdot z_2$ and $\frac{z_1}{z_2}$.

Solution. Polar forms of these numbers are

$$z_1 = 2(\cos\frac{\pi}{3} + i\,\sin\frac{\pi}{3}) \quad \text{and} \quad z_2 = 2(\cos\frac{\pi}{6} + i\,\sin\frac{\pi}{6}).$$

It follows that

$$z_1 z_2 = 4\left[\cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right) + i\,\sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right)\right] = 4\left[\cos\frac{\pi}{2} + i\,\sin\frac{\pi}{2}\right] = 4\,i.$$

Similarly,

$$\frac{z_1}{z_2} = 1 \cdot \left[\cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + i\,\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right] = \cos\frac{\pi}{6} + i\,\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}\,i.$$

Problem 3: Compute AB and BA for the matrices A and B where

$$A = \begin{pmatrix} 1+i & 2i \\ 2 & 3i \end{pmatrix}, \qquad B = \begin{pmatrix} -i & 3 \\ 2+i & 4i \end{pmatrix}.$$

Solution. One has

$$AB = \begin{pmatrix} 1+i & 2i \\ 2 & 3i \end{pmatrix} \begin{pmatrix} -i & 3 \\ 2+i & 4i \end{pmatrix} = \begin{pmatrix} -1+3i & -5+3i \\ -3+4i & -6 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} -i & 3 \\ 2+i & 4i \end{pmatrix} \begin{pmatrix} 1+i & 2i \\ 2 & 3i \end{pmatrix} = \begin{pmatrix} 7-i & 2+9i \\ 1+11i & -14+4i \end{pmatrix}.$$

Problem 4: Find all eigenvalues for the following matrix

$$A = \begin{pmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{pmatrix}.$$

(It might help to know that -7 is an eigenvalue.)

Solution. Recall that the eigenvalues of our matrix A are roots of the characteristic polynomial $\det(\lambda\, I_3 - A)$. One has

$$\det(\lambda\, I_3 - A) = (\lambda-2)^3+27+216-18(\lambda-2)-18(\lambda-2)-18(\lambda-2) = (\lambda-2)^3-54(\lambda-2)+243.$$

For brevity we denote $t := \lambda - 2$, so that we need to find all solutions of the equation

$$t^3 - 54t + 243 = 0.$$

One root is given: $t = -7 - 2 = -9$. Therefore the polynomial $t^3 - 54t + 243$ is divisible by $t + 9$. Applying division algorithm one finds that

$$t^3 - 54t + 243 = (t + 9)(t^2 - 9t + 27)$$

It follows that the solutions of the above equation are $t_1 = -9$ and two solutions of the equation

$$t^2 - 9t + 27 = 0.$$

The solutions are

$$t_{2,3} = \frac{9}{2} \pm \frac{3\sqrt{3}}{2}\, i.$$

Coming back to λ we obtain that the eigenvalues are

$$\lambda_1 = -7, \quad \lambda_2 = t_2 + 2 = \frac{13}{2} + \frac{3\sqrt{3}}{2}\, i, \quad \lambda_3 = t_3 + 2 = \frac{13}{2} - \frac{3\sqrt{3}}{2}\, i.$$

Problem 5: Describe the set of complex numbers $z = a + b\, i$ such that $a^2 + b^2 = 1$. Show that if z_1, z_2 are such numbers then so are $z_1 \cdot z_2$ and $\frac{z_1}{z_2}$.

Solution. If $a^2 + b^2 = 1$ then the absolute value of the complex number $z = a + b\, i$ is 1. Thus, all such points live on the circle of radius 1. From formulas in class we conclude that the absolute values of $z_1 z_2$ and $\frac{z_1}{z_2}$ are also 1, hence they also live on the circle of radius 1.