MATH 325 Q1: LINEAR ALGEBRA III, PART 4

Decomposition of a vector into a sum of generalized eigenvectors

We use two facts about complex polynomials mentioned in Part 3 to prove the following result.

Theorem *Let V be a vector space over $\mathbb{C}$ and let $\alpha : V \to V$ be a linear map. Then any vector v can be written as a sum $v = v_1 + \cdots + v_l$ of generalized eigenvectors v_i of α.*

Proof. Let

$$P_\alpha(T) \;=\; \prod_{i=1}^{l} (T - \lambda_i)^{n_i}$$

be the characteristic polynomial of α, where $\lambda_1, \ldots, \lambda_l$ are all different eigenvalues of α. Let $v \in V$. By the Cayley-Hamilton Theorem we know that $P_\alpha(\alpha)(v) = 0$.

Case 1: α has only one eigenvalue, say $\lambda = \lambda_1$. Then $P_\alpha(T) = (T - \lambda)^n$, where $n = n_1 = \dim_{\mathbb{C}} V$ and so

$$0 = P_\alpha(\alpha)(v) \;=\; (\alpha - \lambda \cdot \mathrm{id}_V)^n(v).$$

It follows that every $v \in V$ is in the generalized eigenspace K_λ, hence $K_\lambda = V$ and we get the trivial decomposition $v = v$.

Case 2: Assume now that α has at least two eigenvalues, *i.e.* $l \geq 2$. Fix an integer $1 \leq i \leq l$. Then

$$\begin{aligned} 0 = P_\alpha(\alpha)(v) \;&=\; \Big[\prod_{j=1}^{l} (\alpha - \lambda_j \cdot \mathrm{id}_V)^{n_j}\Big](v) \\ &= (\alpha - \lambda_i \, \mathrm{id}_V)^{n_i}\Big(\big[\prod_{j \neq i} (\alpha - \lambda_j \cdot \mathrm{id}_V)^{n_j}\big](v)\Big). \end{aligned}$$

Therefore, the vector

$$v_i' = \Big[\prod_{j \neq i} (\alpha - \lambda_j \cdot \mathrm{I}_n)^{n_j}\Big](v)$$

is either 0, or a generalized eigenvector for λ_i. In both cases $v_i' \in K_\lambda(\alpha)$.

Since $\lambda_r \neq \lambda_s$ for $r \neq s$ the polynomials

$$f_i(T) := \prod_{j\neq i}(T-\lambda_j)^{n_j}\,,\; i=1,\dots,l$$

do not have a common root. Therefore their *g.c.d.* is equal to 1 and this implies there are polynomials $h_1(T),\dots,h_l(T)$, such that

$$1 = \sum_{i=1}^{l} h_i(T)\cdot f_i(T)\,. \tag{1}$$

We have all tools to prove the required decomposition $v = v_1+\cdots+v_l$.

As we saw above the vector

$$v_i' = \Big[\prod_{j\neq i}(\alpha-\lambda_j\cdot \mathrm{I}_n)^{n_j}\Big](v) = f_i(\alpha)(v)$$

lies in the generalized eigenspace K_{λ_i} for all $1\le i\le l$, and so is

$$v_i = h_i(\alpha)\big(f_i(\alpha)(v)\big)$$

(because K_{λ_i} is α-invariant). By (1) we have

$$v = \mathrm{id}_V(v) = \sum_{i=1}^{l} h_i(\alpha)\big(f_i(\alpha)(v)\big) = \sum v_i.$$

with $v_i \in K_{\lambda_i}$. $\square$

Examples.

(i) Let A be a 2×2-matrix with two different eigenvalues λ_1,λ_2. Then, as we proved before, A is diagonalizable and

$$P_A(T) = (T-\lambda_1)\cdot(T-\lambda_2).$$

In above notation we have

$$f_1(T) = (T-\lambda_2) \quad\text{and}\quad f_2(T) = (T-\lambda_1)\,.$$

We have then

$$\frac{1}{\lambda_2-\lambda_1}\cdot(T-\lambda_1) + \frac{-1}{\lambda_2-\lambda_1}\cdot(T-\lambda_2) = 1\,.$$

Then for a vector v in $\mathbb{C}^2$ we get the decomposition

$$v = \frac{-1}{\lambda_2-\lambda_1}\cdot(A-\lambda_2\cdot \mathrm{I}_2)\cdot v + \frac{1}{\lambda_2-\lambda_1}\cdot(A-\lambda_1\cdot \mathrm{I}_2)\cdot v = v_1+v_2$$

with (again in above notation)

$$v_1 := \frac{-1}{\lambda_2-\lambda_1}\cdot(A-\lambda_2\cdot \mathrm{I}_2)\cdot v$$

in $E_{\lambda_1} \subseteq K_{\lambda_1}$ and

$$v_2 := \frac{1}{\lambda_2 - \lambda_1} \cdot (A - \lambda_1 \cdot \mathrm{I}_2) \cdot v$$

in $E_{\lambda_2} \subseteq K_{\lambda_2}$ (in this case we have actually $E_{\lambda_i} = K_{\lambda_i}$ for $i = 1, 2$).

(ii) Consider the 3×3-matrix

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 3 & -2 & 5 \\ 2 & -2 & 4 \end{pmatrix}.$$

We compute the characteristic polynomial

$$P_A(T) = T^3 - 4T^2 + 5T - 2 = (T-1)^2 \cdot (T-2),$$

and so A has two different eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. The polynomials $f_i(T)$ are then (in above notation) $f_1(T) = T - 2$ and $f_2(T) = (T-1)^2$. Then $(-T) \cdot (T-2) + (T-1)^2 = 1$ and so (inserting A for T) we have

$$\mathrm{I}_3 = (-A) \cdot (A - 2 \cdot \mathrm{I}_3) + (A - \mathrm{I}_3)^2,$$

which implies that a vector $v \in \mathbb{C}^3$ has the decomposition $v = v_1 + v_2$ with

$$v_1 := (-A) \cdot (A - 2\,\mathrm{I}_3) \cdot v$$

in K_{λ_1} and

$$v_2 := (A - \mathrm{I}_3)^2 \cdot v$$

in K_{λ_2}. For instance we get the following decomposition for the vector $v = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$: We compute first

$$(-A) \cdot (A - 2\,\mathrm{I}_3) = \begin{pmatrix} -1 & 2 & -3 \\ -4 & 5 & -6 \\ -2 & 2 & -2 \end{pmatrix} \text{ and } (A - \mathrm{I}_2)^2 = \begin{pmatrix} 2 & -2 & 3 \\ 4 & -4 & 6 \\ 2 & -2 & 3 \end{pmatrix}.$$

Hence

$$v_1 = (-A) \cdot (A - 2 \cdot \mathrm{I}_3) \cdot v = \begin{pmatrix} -1 & 2 & -3 \\ -4 & 5 & -6 \\ -2 & 2 & -2 \end{pmatrix} \cdot v = \begin{pmatrix} -3 \\ -9 \\ -4 \end{pmatrix}$$

and

$$v_2 = (A - \mathrm{I}_3)^2 \cdot v = \begin{pmatrix} 2 & -2 & 3 \\ 4 & -4 & 6 \\ 2 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 4 \end{pmatrix}.$$

Uniqueness

Our next aim is now to show that the decomposition of a vector into a sum of generalized eigenvectors for the various eigenvalues is unique, *i.e.* if

$$\sum_{i=1}^{l} v_i = v = \sum_{i=1}^{l} w_i$$

with $v_i, w_i \in K_{\lambda_i}$ then $v_i = w_i$ for all $1 \leq i \leq l$. To this end we first show the following result which also plays some role later on.

Lemma. *Let λ be an eigenvalue of α and $\nu \neq \lambda$ another complex number. Then the $\mathbb{C}$-linear map*

$$(\alpha - \nu \cdot \mathrm{id}_V)^m : K_\lambda \longrightarrow K_\lambda, \; v \longmapsto (\alpha - \nu \cdot \mathrm{id}_V)^m(v)$$

is an isomorphism for all integers $m \geq 0$.

Proof. Since a $\mathbb{C}$-linear map of a finite dimensional $\mathbb{C}$-vector space into itself is an isomorphism if and only if its kernel is trivial, we have only to prove that for any $v \neq 0$ in K_λ we have $(\alpha - \nu \cdot \mathrm{id}_V)^m(v) \neq 0$. Since $v \in K_\lambda$ there is some $d \geq 1$, such that $(\alpha - \lambda \cdot \mathrm{id}_V)^d(v) = 0$.

On the other hand since $\lambda \neq \nu$ the polynomials $(T - \lambda)^d$ and $(T - \nu)^m$ have no common root, their *g.c.d.* is equal to 1, hence there are polynomials $h(T)$ and $k(T)$, such that

$$1 = h(T) \cdot (T - \nu)^m + k(T) \cdot (T - \lambda)^d.$$

Substituting α instead of T we have

$$\begin{aligned} v &= \mathrm{id}_V(v) \\ &= \big[h(\alpha) \cdot (\alpha - \nu \cdot \mathrm{id}_V)^m + k(\alpha) \cdot (\alpha - \lambda \cdot \mathrm{id}_V)^d\big](v) \\ &= h(\alpha)\Big((\alpha - \nu \cdot \mathrm{id}_V)^m(v) \Big) + k(\alpha)\Big((\alpha - \lambda \cdot \mathrm{id}_V)^d(v) \Big) \\ &= h(\alpha)\Big((\alpha - \nu \cdot \mathrm{id}_V)^m(v) \Big), \end{aligned}$$

the last equation since $(\alpha - \lambda \cdot \mathrm{id}_V)^d(v) = 0$. Therefore

$$(\alpha - \nu \cdot \mathrm{id}_V)^m(v) \neq v$$

(because $v \neq 0$). $\square$

By induction we then get the following

Corollary. *Let λ be an eigenvalue of α and $\nu_1, \ldots, \nu_r$ complex numbers, which are all $\neq \lambda$. Then the $\mathbb{C}$-linear map*

$$K_\lambda \longrightarrow K_\lambda\,,\ v \longmapsto \Big[\prod_{i=1}^{r}(\alpha - \nu_i \cdot \mathrm{id}_V)^{m_i}\Big](v)$$

is an isomorphism for all integers $m_1, \ldots, m_r \geq 0$.

We now pass to the following result which summarizes all our previous discussions.

Theorem. *Let $\alpha : V \to V$ be a linear map with characteristic polynomial*

$$P_\alpha(T) = \prod_{i=1}^{l}(T - \lambda_i)^{n_i},$$

where $\lambda_1, \ldots, \lambda_l$ are all different eigenvalues of α. Then:

(i) *If $v \in K_{\lambda_i}$ for some $1 \leq i \leq l$ we have*

$$(\alpha - \lambda_i \cdot \mathrm{id}_V)^{n_i}(v) \ = 0\,.$$

(ii) *Every vector $v \in V$ can be uniquely written as a sum*

$$v \ = \ v_1 + v_2 + \ \ldots \ + v_l$$

with $v_i \in K_{\lambda_i}$ for $i = 1, \ldots, l$.

Proof. (i) Fix an integer $1 \leq i \leq l$. By the corollary above

$$\Big[\prod_{j\neq i}(\alpha - \lambda_j \cdot \mathrm{id}_V)^{n_j}\Big](w) \neq \ 0$$

for all $w \neq 0$ in K_{λ_i}. Hence if $w = (\alpha - \lambda_i \cdot \mathrm{id}_V)^{n_i}(v) \neq 0$ also

$$\begin{aligned} 0 &\neq \Big[\prod_{j\neq i}(\alpha - \lambda_j \cdot \mathrm{id}_V)^{n_j}\Big]\Big(\ (\alpha - \lambda_i \cdot \mathrm{id}_V)^{n_i}(v)\Big) \\ &= \Big[\prod_{j=1}^{l}(\alpha - \lambda_j \cdot \mathrm{id}_V)^{\mu_j}\Big](v) \\ &= P_\alpha(\alpha)(v)\,. \end{aligned}$$

So the assumption $(\alpha - \lambda_i \cdot \mathrm{id}_V)^{n_i}(v) \neq 0$ leads to $P_\alpha(\alpha)(v) \neq 0$, which contradicts the Cayley-Hamilton Theorem. Thus,

$$(\alpha - \lambda_i \cdot \mathrm{id}_V)^{\mu_i}(v) = 0.$$

(ii) The existence of the sum decomposition has been already proven. It remains to show uniqueness. Assume

$$\sum_{i=1}^{l} v_i \;=\; v \;=\; \sum_{i=1}^{l} w_i$$

with $v_i, w_i \in K_{\lambda_i}$ for all $1 \leq i \leq l$.

We fix $1 \leq i \leq l$. From the above equation we get

$$v_i - w_i \;=\; \sum_{j \neq i} (w_j - v_j)\,.$$

By (i) we have

$$(\alpha - \lambda_j \cdot \mathrm{id}_V)^{n_j}(w_j - v_j) = 0$$

for all $1 \leq j \leq l$. Therefore using the fact that polynomials in α commute with each other we have

$$\Big[\prod_{k \neq i} (\alpha - \lambda_k \cdot \mathrm{id}_V)^{n_k}\Big](w_j - v_j) = 0$$

for all $j \neq i$. Hence

$$\Big[\prod_{k \neq i} (\alpha - \lambda_k \cdot \mathrm{id}_V)^{n_k}\Big]\Big(\sum_{j \neq i} (w_j - v_j)\Big) \;=\; \sum_{j \neq i} \Big[\prod_{k \neq i} (\alpha - \lambda_k \cdot \mathrm{id}_V)^{n_k}\Big](w_j - v_j) \;=\; 0\,,$$

and so also

$$\Big[\prod_{k \neq i} (\alpha - \lambda_k \cdot \mathrm{id}_V)^{n_k}\Big](v_i - w_i) \;=\; 0\,.$$

But by the corollary above the linear map $\Big[\prod_{k \neq i} (\alpha - \lambda_k \cdot \mathrm{id}_V)^{n_k}\Big]$ is an isomorphism on K_{λ_i} and so $v_i - w_i$ has to be zero, *i.e.* $v_i = w_i$. We are done. $\square$

The dimensions of the subspaces K_{λ_i}

Let α be as in the theorem with characteristic polynomial

$$P_\alpha(T) = \prod_{i=1}^{l} (T - \lambda_i)^{n_i}.$$

Let $(v_j^{(i)})_{1 \leq j \leq m_i}$ be a basis of K_{λ_i} for $i = 1, \ldots, l$, and B_i the matrix of the restriction of $\alpha : V \to V$ to the α-invariant subspace K_{λ_i} with respect to this basis.

Question: What are the eigenvalues of B_i?

We proved above that for all complex numbers $\nu \neq \lambda_i$ the $\mathbb{C}$-linear map

$$K_{\lambda_i} \longrightarrow K_{\lambda_i}\,, \;\; x \longmapsto (\alpha - \nu \cdot \mathrm{id}_V)(x)$$

is an isomorphism, and so in particular $(\alpha - \nu \cdot \mathrm{id}_V)(x) \neq 0$ for all $x \in K_{\lambda_i} \setminus \{0\}$, or equivalently $\alpha(x) \neq \nu \cdot x$ for all $x \neq 0$ in K_{λ_i} and all $\nu \neq \lambda_i$. But B_i is the matrix of the $\mathbb{C}$-linear map $K_{\lambda_i} \longrightarrow K_{\lambda_i}$, $x \mapsto \alpha(x)$ with respect to the basis $(v_j^{(i)})_{1 \leq j \leq n_i}$ and so λ_i is the only eigenvalue of B_i. Hence we have

$$P_{B_i}(T) = (T - \lambda_i)^{m_i}\,, \tag{2}$$

where $m_i = \dim_{\mathbb{C}} K_{\lambda_i}$, for all $i = 1, \ldots, l$.

We need the following fact.

Lemma *The union of the ordered bases $(v_j^{(i)})_{1 \leq j \leq n_i}$ is a basis of V.*

Proof. Indeed, to prove that these vectors are linearly independent assume that

$$\sum_{i=1}^{l} \Big(\sum_{j=1}^{m_i} a_{ij} \cdot v_j^{(i)} \Big) \;=\; 0.$$

Then by uniqueness part of the theorem we have $\sum\limits_{j=1}^{m_i} a_{ij} \cdot v_j^{(i)} = 0$ for all $i = 1, \ldots, l$, and so since the vectors $v_1^{(i)}, \ldots, v_{m_i}^{(i)}$ are linear independent also $a_{ij} = 0$ for all i, j.

By the existence part of the theorem we see that these vectors generate V. Hence their union is a basis. □

The matrix of α with respect to this basis is then

$$\begin{pmatrix} B_1 & \mathrm{O}_{n_1 \times n_2} & \ldots & \mathrm{O}_{n_1 \times n_l} \\ \mathrm{O}_{n_2 \times n_1} & B_2 & & \mathrm{O}_{n_2 \times n_l} \\ \vdots & & \ddots & \vdots \\ \mathrm{O}_{n_l \times n_1} & & \ldots & B_l \end{pmatrix},$$

and so we have

$$P_\alpha(T) \;=\; \prod_{i=1}^{l} P_{B_i}(T) \;=\; \prod_{i=1}^{l} (T - \lambda_i)^{m_i}.$$

But we also have

$$P_\alpha(T) = \prod_{i=1}^{l} (T - \lambda_i)^{n_i}$$

and so we conclude:

Corollary. *Let $\alpha : V \longrightarrow V$ be a $\mathbb{C}$-linear map with characteristic polynomial $P_\alpha(T) = \prod_{i=1}^{l}(T - \lambda_i)^{n_i}$, where $\lambda_1, \ldots, \lambda_l$ are all different eigenvalues of α, then $n_i = \dim_{\mathbb{C}} K_{\lambda_i}$ for all $1 \leq i \leq l$.*