MATH 325 Q1: LINEAR ALGEBRA III, PART 7

Square roots, powers, and exponentials of matrices

(a) **Square roots of diagonalizable 2×2-matrices.** Let A be a complex 2×2-matrix. A matrix B is called a square root of A if $B^2 = B \cdot B = A$. Such a matrix is not unique and in general it is not so easy do describe all such matrices. Our aim is to show how to find at least one such matrix.

Let us first start from a diagonalizable matrix

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Consider the matrix

$$B = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{b} \end{pmatrix}.$$

Then one has

$$\begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{b} \end{pmatrix}^2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

where $\sqrt{a}, \sqrt{b} \in \mathbb{C}$ are square roots of a and b, respectively.

From this observation we also get a solution if A is not diagonal but diagonalizable. Indeed, if A is diagonalizable then there is an invertible matrix S, such that $S^{-1} \cdot A \cdot S = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ for some complex numbers c, d. Since

$$\begin{pmatrix} \sqrt{c} & 0 \\ 0 & \sqrt{d} \end{pmatrix}^2 = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix},$$

we have

$$A = S \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} S^{-1} = S \begin{pmatrix} \sqrt{c} & 0 \\ 0 & \sqrt{d} \end{pmatrix}^2 S^{-1} = \left[S \cdot \begin{pmatrix} \sqrt{c} & 0 \\ 0 & \sqrt{d} \end{pmatrix} \cdot S^{-1} \right]^2.$$

Thus, the matrix

$$B = S \cdot \begin{pmatrix} \sqrt{c} & 0 \\ 0 & \sqrt{d} \end{pmatrix} \cdot S^{-1}$$

is as required.

(b) **Square roots of arbitrary 2×2-matrices.** Let A be a 2×2-matrix which is not diagonalizable. Then A has only one eigenvalue, say λ of multiplicity 2. The Jordan normal form of A is the matrix

$$\mathrm{J}(A) = \mathrm{J}(\lambda, 2) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

so that there is an invertible 2×2-matrix S such that

$$S^{-1} \cdot A \cdot S = \mathrm{J}(A).$$

Arguing as above we see that if B is a matrix, satisfying

$$B^2 = B \cdot B = \mathrm{J}(A)$$

then

$$\left(S \cdot B \cdot S^{-1}\right)^2 = A.$$

Thus we reduce our problem to finding the square root of the Jordan block $\mathrm{J}(\lambda, 2)$.

Note hat this is only possible if $\lambda \neq 0$. Indeed, the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is equivalent to the system

$$\begin{cases} a^2 + bc = 0 \\ ab + bd = b \cdot (a + d) = 1 \\ ac + dc = (a + d)c = 0 \\ bc + d^2 = 0 \end{cases}$$

From the second equation we conclude that $a + d \neq 0$. Then the third equation says us $c = 0$ which in turn forces $a = d = 0$ (using the first and last equations) – a contradiction.

So let $\lambda \neq 0$. Let us try to find a complex number z, such that

$$\begin{pmatrix} \sqrt{\lambda} & z \\ 0 & \sqrt{\lambda} \end{pmatrix}^2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

where $\sqrt{\lambda}$ is some square root of λ. We have

$$\begin{pmatrix} \sqrt{\lambda} & z \\ 0 & \sqrt{\lambda} \end{pmatrix}^2 = \begin{pmatrix} \lambda & 2 \cdot \sqrt{\lambda} \cdot z \\ 0 & \lambda \end{pmatrix}$$

and so we get as solution

$$z = \frac{1}{2} \cdot \frac{1}{\sqrt{\lambda}} = \frac{1}{2} \cdot \frac{\sqrt{\lambda}}{\lambda}.$$

We illustrate this by the following example.

Example. Let

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}.$$

We have seen before (check yourself again) that A has only one eigenvalue $\lambda = 3$ and that

$$\mathrm{J}(A) = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{2} \end{pmatrix}$$

using that the inverse of the matrix $S = \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{2} \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$. The matrix

$$B := \begin{pmatrix} \sqrt{3} & \frac{1}{6}\sqrt{3} \\ 0 & \sqrt{3} \end{pmatrix}$$

satisfies $B^2 = \mathrm{J}(A)$ and so the matrix

$$S \cdot B \cdot S^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{3} & \frac{1}{6}\sqrt{3} \\ 0 & \sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} & \frac{4}{3}\sqrt{3} \end{pmatrix}$$

is a square root of the matrix A.

(c) **The exponential of a matrix.** Let A be a $n \times n$ matrix with complex coefficients.

Definition. *The exponential* $\exp(A)$ *of the matrix* A *is the limit of the series of matrices*

$$s_n(A) := \sum_{i=0}^{n} \frac{1}{i!} \cdot A^i,$$

i.e. all sequences of coefficients converge. (Here $i! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot i$.*) We write then also*

$$\exp(A) = \sum_{i=0}^{\infty} \frac{1}{i!} \cdot A^i.$$

For instance, if $A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ is a 2×2 diagonal matrix then

$$s_n(A) \;=\; \sum_{i=0}^{n} \frac{1}{i!} \cdot \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}^i \;=\; \begin{pmatrix} \sum_{i=0}^{n} \frac{\lambda^i}{i!} & 0 \\ 0 & \sum_{i=0}^{n} \frac{\mu^i}{i!} \end{pmatrix},$$

and so we have

$$\exp(A) \;=\; \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\mu \end{pmatrix},$$

where (as usual) we have set

$$e^x = \exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!}.$$

Of course, we have to justify that the above limit exists. The above material related to square roots of matrices suggest that we have start first from Jordan normal forms.

The case of a Jordan normal form. If A and B are commuting 2×2-matrices, *i.e.* $A \cdot B = B \cdot A$, then one can show that

$$\exp(A + B) \;=\; \exp(A) \cdot \exp(B)$$

if exponentials do exist. This property can be used to compute the exponential of a matrix in Jordan normal form.

As for square roots we consider only 2×2-matrices. We have already discussed the case when the Jordan form consists of two Jordan blocks of size 1 (*i.e.* it is a diagonal matrix) above. So let us consider the remaining case

$$A \;=\; \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \;=\; \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let

$$D \;:=\; \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{and} \quad N \;:=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then one can easily checks that

$$A = D + N, \quad D \cdot N = N \cdot D,$$

so that D and N commute and that

$$N^2 = N \cdot N = 0.$$

It follows that

$$\exp(A) = \exp(D+N) = \exp(D)\cdot\exp(N)$$

and it remains to compute exponentials of D and N.

Since $N^2 = 0$ we have therefore

$$\exp(N) \;=\; \mathrm{I}_2 + N \;=\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and so

$$\exp(A) \;=\; \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \;=\; \begin{pmatrix} e^\lambda & e^\lambda \\ 0 & e^\lambda \end{pmatrix},$$

since as seen above $\exp(D) = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\lambda \end{pmatrix}$.

The exponential of a 2×2-matrix. Let now A be a 2×2-matrix and $\mathrm{J}(A)$ its Jordan normal form (this is either a diagonal matrix or a Jordan block of size 2). Then there is an invertible 2×2-matrix S, such that $\mathrm{J}(A) = S^{-1}\cdot A\cdot S$, or equivalently $S\cdot\mathrm{J}(A)\cdot S^{-1} = A$. We have then

$$s_n(A) \;=\; \sum_{i=0}^{n} \frac{1}{i!}\cdot A^i \;=\; \sum_{i=0}^{n} \frac{1}{i!}\cdot\left(S\cdot\mathrm{J}(A)\cdot S^{-1}\right)^i \;=\; S\cdot\left(\sum_{i=0}^{n} \frac{1}{i!}\cdot\mathrm{J}(A)^i\right)\cdot S^{-1}.$$

Taking the limit we get therefore

$$\exp(A) \;=\; S\cdot\exp\big(\mathrm{J}(A)\big)\cdot S^{-1}.$$

We illustrate this method by the following example.

Example. Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}$. Then $\lambda = 3$ is the only eigenvalue and we have

$$\mathrm{J}(A) \;=\; \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{2} \end{pmatrix}.$$

Since

$$\exp\big(\mathrm{J}(A)\big) \;=\; \begin{pmatrix} e^3 & e^3 \\ 0 & e^3 \end{pmatrix},$$

we finally have

$$\exp(A) \;=\; \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} e^3 & e^3 \\ 0 & e^3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \;=\; \begin{pmatrix} -e^3 & 2e^3 \\ -2e^3 & 3e^3 \end{pmatrix}.$$