Linear Algebra MATH 325: Assignment 5

(Due in class, March 8)

Problem 1: Find the Jordan normal form of the following 3×3-matrix

$$A = \begin{pmatrix} -3 & 1 & 2 \\ -1 & -1 & 1 \\ -2 & 1 & 1 \end{pmatrix},$$

and find a Jordan basis for A. (Hint: -1 is an eigenvalue of A.)

Solution. We have

$$P_A(T) = T^3 + 3T^2 + 3T + 1 = (T+1)^3,$$

and so $\lambda = -1$ is the only eigenvalue of A. The corresponding generalized eigenspace is therefore the whole space $\mathbb{C}^3$.

In order to find the Jordan normal form of A we first compute $(A - (-1) \cdot \mathrm{I}_3)^2 = (A + \mathrm{I}_3)^2$:

$$(A + \mathrm{I}_3) = \begin{pmatrix} -2 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}$$

and so

$$(A + \mathrm{I}_3)^2 = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that a Jordan normal form of A consists of one block and that a Jordan basis of A consists of one full cycle of generalized eigenvectors of length 3, and hence for every $v \in \mathbb{C}^3$ with $(A + \mathrm{I}_3)^2 \cdot v \neq 0$ the set

$$\{ (A - \mathrm{I}_3)^2 \cdot v, \ (A + \mathrm{I}_3) \cdot v, \ v \}$$

is a Jordan basis for A. For instance $v = e_1$ does the job, *i.e.*

$$\left\{ \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

is a Jordan basis for A. The Jordan normal form is

$$\mathrm{J}(A) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Problem 2: Find the Jordan normal form of the following 3×3-matrix

$$B = \begin{pmatrix} 4 & 1 & 1 \\ 2 & 3 & 1 \\ -6 & -3 & -1 \end{pmatrix},$$

and find a Jordan basis for B. (Hint: 2 is an eigenvalue of B.)

Solution. We have

$$P_B(T) \;=\; T^3 - 6T^2 + 12T - 8 \;=\; (T-2)^3\,,$$

and so $\lambda = 2$ is the only eigenvalue of B and the generalized eigenspace K_2 is equal the whole space $\mathbb{C}^3$.

To find the number of Jordan blocks in the normal Jordan form of B let us compute $(B - 2\cdot \mathrm{I}_3)^2$. We have

$$B - 2\cdot \mathrm{I}_3 \;=\; \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ -6 & -3 & -3 \end{pmatrix},$$

and one computes that $(B - 2\cdot \mathrm{I}_3)^2$ is the zero matrix. Hence the Jordan normal form of B consists of blocks, i.e.

$$J(B) \;=\; \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

To get a Jordan basis for B we start with a vector $v \in \mathbb{C}^3$, such that $(B - 2\cdot \mathrm{I}_3)\cdot v \neq 0$. For instance $v = e_1$ does the job. Then

$$w := (B - 2\cdot \mathrm{I}_3)\cdot e_1 = \begin{pmatrix} 2 \\ 2 \\ -6 \end{pmatrix}$$

is an eigenvector of B and $\{w, e_1\}$ is a full cycle of generalized eigenvectors of length 2.

To extend this to a Jordan basis for B we have to find an eigenvector v of B which is linear independent of w, *i.e.* such that $\{v, w\}$ is a basis of the eigenspace $E_{\lambda=2}$. For this we have to compute the eigenspace for $\lambda = 2$, or equivalently the null space of $B - 2\cdot \mathrm{I}_3$. In other words we have to solve the system $(b - 2\cdot \mathrm{I}_3)(v) = 0$ of linear equations.

One finds that it is 2-dimensional and spanned for example by vectors

$$v = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 2 \\ 2 \\ -6 \end{pmatrix}.$$

Hence

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 2 \\ 2 \\ -6 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

is a Jordan basis for B.

Problem 3: Let A be a complex 4×4-matrix, such that $A^4 = 0$. What are the possible Jordan normal forms of A?

Solution. If $A^4 = 0$ then $\lambda = 0$ is the only possible eigenvalue. Indeed, assume that $\lambda \neq 0$ is an eigenvalue of A. Then λ is one of the diagonal entries of $J(A)$. Since $A^4 = 0$ one has $J(A)^4 = 0$. On the other side $\lambda^4 \neq 0$ is one of the diagonal entries of $J(A)$ – a contradiction.

Thus the possible Jordan normal forms are:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(four Jordan blocks of size 1; one Jordan block of size 2 and two of size 1; one Jordan block of size 3 and one of size 1; one Jordan block of size 4; and two Jordan blocks of size 2).