Linear Algebra MATH 325: Assignment 7

(Due in class, March 22)

Problem 1: Find all real numbers a, b such that the map

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \longmapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \cdot \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

is an inner product on $\mathbb{R}^2$? Here T is the transpose.

Solution. It is straightforward to check that the map in question satisfies axioms **(L1)** and **(H)** of the definition of an inner product for all real numbers a, b. Hence we have only to check for which real numbers a, b also axiom **(P)** is satisfied, *i.e.* for which real numbers a, b we have

$$\| \begin{pmatrix} x \\ y \end{pmatrix} \|^2 = \langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \rangle = (x \;\; y) \cdot \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2xy + by^2 > 0$$

for all vectors $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \setminus \{0\}$.

If $a \le 0$ then $\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \|^2 = a \le 0$, and if $b \le 0$ then $\| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \|^2 = b \le 0$, and so both a and b have to be positive real numbers.

Next, the expression $ax^2 + 2xy + by^2$ can be rewritten in the form

$$ax^2 + 2xy + by^2 = y^2 \left(a \left(\frac{x}{y} \right)^2 + 2 \cdot \frac{x}{y} + b \right) = y^2(a\tilde{x}^2 + 2\tilde{x} + b)$$

where $\tilde{x} = \frac{x}{y}$. The discriminant of the quadratic polynomial $a\tilde{x}^2 + 2\tilde{x} + b$ is equal to $4 - 4ab = 4(1 - ab)$. If it is negative, i.e. if $ab > 1$, then this quadratic polynomial has no real roots (because a is positive), hence it takes positive values. If it is negative, then it has two real roots, hence it takes both positive and negative values. Thus, the axiom **(P)** holds if and only if $a, b > 0$ and $ab > 1$.

Problem 2: Which of the following two maps are inner products on $\mathbb{R}^3$?

(a)

$$\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \rangle := x_1y_1 + 2x_2y_2 + 3x_3y_3 \,.$$

(b) $\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \rangle :=$

$$x_1y_1 + 2x_1y_2 + x_1y_3 + 2x_2y_1 + x_2y_2 + x_3y_1 - x_3y_3 \,.$$

Solution. Straightforward computations show that the map (a) is an inner product. The map (b) is not an inner product since for the vector $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ we have $\langle e_3, e_3 \rangle = -1 < 0$ and so this map violates axiom **(P)**.

Problem 3: Give an orthonormal basis of the orthogonal complement of $v = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$ in the inner product space $\mathbb{R}^3$ with the usual Euclidean inner product.

Solution. A vector $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$ is orthogonal to the vector $v = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$ if and only if

$$2x_1 + 5x_3 = 0 .$$

Thus the orthogonal complement $v^\perp$ consists of all vectors of the form

$$\begin{pmatrix} -\frac{5}{2} \cdot x_3 \\ x_2 \\ x_3 \end{pmatrix} .$$

Take $x_2 = 1$ and $x_3 = 0$. Then the vector

$$y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

is contained in $v^\perp$ and it has norm 1. It remains to find a vector $z \in v^\perp$ such that z is orthogonal to y and whose norm is 1. It is straightforward to check that vectors in $v^\perp$ which are orthogonal to y are of the form

$$\begin{pmatrix} -\frac{5}{2} \cdot x_3 \\ 0 \\ x_3 \end{pmatrix} .$$

Take $x_3 = 2$. Then the vector

$$\begin{pmatrix} -5 \\ 0 \\ 2 \end{pmatrix}$$

is in the subspace $v^\perp$ in $\mathbb{R}^3$ and it is orthogonal to y. But its norm is not 1. Normalizing this vector we obtain that the vector

$$z = \frac{1}{\sqrt{29}} \cdot \begin{pmatrix} -5 \\ 0 \\ 2 \end{pmatrix}$$

has norm 1. Summarizing, two vectors $\{y, z\}$ form an orthonormal basis of the orthogonal complement $v^\perp$ of v.

Problem 4: Let $\langle -, - \rangle$ be the usual Euclidean scalar product on $\mathbb{R}^n$ and $\|v\| := \sqrt{\langle v, v \rangle}$ the associated norm. Let $v, w \in \mathbb{R}^n \setminus \{0\}$. Show that $\|v\| = \|w\|$ if and only if $v + w$ is orthogonal to $v - w$.

Solution. The vectors $v + w$ and $v - w$ are orthogonal to each other if and only if
$$\langle v + w, v - w \rangle = 0.$$
But
$$\begin{aligned} \langle v + w, v - w \rangle &= \langle v, v \rangle - \langle v, w \rangle + \langle w, v \rangle - \langle w, w \rangle \\ &= \langle v, v \rangle - \langle w, w \rangle , \end{aligned}$$
and so $\langle v + w, v - w \rangle = 0$ if and only if $\langle v, v \rangle = \langle w, w \rangle$.