Linear Algebra MATH 325: Assignment 6

(Due in class, March 15)

Problem 1: Find a square root of the matrix:

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}.$$

Solution. We have $P_A(T) = (T-2)^2$, and so A has only one eigenvalue $\lambda = 2$. Furthermore, since A is not a scalar matrix, it is not diagonalizable. This implies that every nonzero vector is a generalized eigenvector and that the Jordan normal form $J(A)$ of A consists of one block.

Thus, if $0 \neq v$ is not an eigenvector then $\{(A - 2\,\mathrm{I}_2) \cdot v, v\}$ is a Jordan basis for A. For instance $v = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ does the job, and so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a Jordan basis.

The base change matrix S from the standard basis to the above Jordan basis is

$$S = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$

hence from $J(A) = S^{-1}AS$ we conclude that

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

We saw in class that

$$\begin{pmatrix} \sqrt{2} & \frac{\sqrt{2}}{4} \\ 0 & \sqrt{2} \end{pmatrix}$$

is a square root of $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ and so

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & \frac{\sqrt{2}}{4} \\ 0 & \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} \\ -\frac{1}{4}\sqrt{2} & \frac{3}{4}\sqrt{2} \end{pmatrix}$$

is a square root of A.

Problem 2: Find $\exp(A)$ where

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}.$$

Solution. We have

$$P_A(T) = T^2 - 5T + 6 = (T-2) \cdot (T-3),$$

and so A is diagonalizable with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$. The vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector for $\lambda_1 = 2$ and the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is one for $\lambda_2 = 3$. The base change matrix S is of the form

$$S = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}.$$

The exponential of the diagonal matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ is equal $\begin{pmatrix} e^2 & 0 \\ 0 & e^3 \end{pmatrix}$, where $e^x = \exp(x)$ is the exponential function, and so we have

$$\begin{aligned} \exp(A) &= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} e^2 & 0 \\ 0 & e^3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2e^2 - e^3 & e^2 - e^3 \\ -2e^2 + 2e^3 & -e^2 - 2e^3 \end{pmatrix}. \end{aligned}$$

Problem 3: Give a Jordan basis and the Jordan normal form for the following complex 4×4-matrix:

$$A = \begin{pmatrix} 3 & 1 & 0 & 1 \\ -1 & 1 & 2 & 0 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & -4 & -2 \end{pmatrix}.$$

Hint: the matrix A is block triangular, hence its characteristic polynomial is equal to the product of the characteristic polynomials of $\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 6 & 4 \\ -4 & -2 \end{pmatrix}$.

Solution. We have

$$P_A(T) = (T-2)^4.$$

Therefore A has only one eigenvalue $\lambda = 2$ and every nonzero vector is a generalized eigenvector. Computing the powers of $(A - 2 \cdot \mathrm{I}_4)$ one gets

$$(A - 2 \cdot \mathrm{I}_4)^3 = \begin{pmatrix} 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0.$$

It follows that a Jordan basis of A consists of one full cycle of generalized eigenvectors for A of lenght 4 and for every vector $v \in \mathbb{C}^4$, such that $(A - 2 \cdot \mathrm{I}_4)^3 \cdot v \neq 0$ the set of vectors

$$\left\{(A - 2 \cdot \mathrm{I}_4)^3 \cdot v\,,\ (A - 2 \cdot \mathrm{I}_4)^2 \cdot v\,,\ (A - 2 \cdot \mathrm{I}_4) \cdot v\,,\ v\right\}$$

is a Jordan basis. For instance $v = e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, the fourth standard basis vector, has this property, and so

$$\left\{ \begin{pmatrix} 4 \\ 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 7 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a Jordan basis for A.

The Jordan normal form of A consists of one Jordan block of size 4:

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$