Linear Algebra MATH 325: Assignment 4

(Due in class, February 11)

Problem 1: Determine the eigenvalues and the eigenspaces of the following two matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix},$$

and if one of them is not diagonalizable give a Jordan basis for it.

Solution. Set $A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B := \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$. We first compute the characteristic polynomials:

$$P_A(T) = T^2 + 1 = (T - i) \cdot (T + i),$$

where $i = \sqrt{-1}$ is square root of -1, and

$$P_B(T) = T^2 - 6T + 9 = (T - 3)^2.$$

Therefore A is diagonalizable with eigenvalues i and $-i$; in particular both eigenspaces of A have dimension 1 and are generated by an eigenvector. One checks a vector $\begin{pmatrix} i \\ 1 \end{pmatrix}$ is an eigenvector for i, and $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ is an eigenvector for $-i$. Hence

$$E_i = \mathbb{C} \cdot \begin{pmatrix} i \\ 1 \end{pmatrix}$$

and

$$E_{-i} = \mathbb{C} \cdot \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

The matrix B has only one eigenvalue 3 and is not diagonalizable. Indeed, assume that B is diagonalizable. Then there would be a matrix S such that

$$S^{-1}BS = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

where a, b are eigenvalues of B. But B has only one eigenvalue 3. It follows that $a = b = 3$, implying $S^{-1}BS$ is a scalar matrix. As we explained in class this forces B to be a scalar matrix as well – a contradiction.

One checks that a vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda = 3$ which generates the 1-dimensional eigenspace:

$$E_1 = \mathbb{C} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

To get a Jordan basis one first extends this eigenvector to a basis of $\mathbb{C}^2$: Set

$$v := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad w' := \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then

$$B \cdot w' \;=\; \begin{pmatrix} 2 \\ -1 \end{pmatrix} \;=\; (-1) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So in basis $\{v.w'\}$ the corresponding linear map has the matrix

$$\begin{pmatrix} 3 & -1 \\ 0 & 3 \end{pmatrix}.$$

Replacing w' by $w = -w' = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ one gets a Jordan basis v, w for B.

Problem 2: Let $A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$.

(i) Show that 1 and 2 are eigenvalues of A and that A is not diagonalizable.

(ii) Decompose the two vectors $v = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ into a sum of generalized eigenvectors.

Solution.

(i) We compute first the characteristic polynomial:

$$P_A(T) \;=\; T^3 - 5T^2 + 8T - 4 \;=\; (T-1) \cdot (T-2)^2 .$$

Hence $\lambda_1 = 1$ and $\lambda_2 = 2$ are the eigenvalues of A. The matrix A is not diagonalizable since one checks that the dimension of the eigenspace E_2 for the eigenvalue 2 is 1. (Recall that if a matrix of size $n \times n$ is diagonalizable, then on the diagonal we have its eigenvalues, say $\lambda_1, \ldots, \lambda_l$, and

$$\dim(E_{\lambda_1}) + \cdots + \dim(\mathrm{E}_{\lambda_\mathrm{l}}) = \mathrm{n}.$$

In our case we have $n = 3,\ \lambda_1 = 1,\ \ \lambda_2 = 2$ and $\dim(E_1) = 1$, $\dim(E_2) = 1$.)

(ii) In the notation of the course notes we have $f_1(T) = (T - \lambda_2)^2 = (T-2)^2$ and $f_2(T) = (T - \lambda_1) = T - 1$. We have

$$1 \;=\; (T-2)^2 \;-\; (T-3) \cdot (T-1)\,,$$

and so $\mathrm{I}_3 = (A - 2 \cdot \mathrm{I}_3)^2 - (A - 3 \cdot \mathrm{I}_3) \cdot (A - \mathrm{I}_3)$ from which we get

$$v \;=\; (A - 2 \cdot \mathrm{I}_3)^2 \cdot v - (A - 3 \cdot \mathrm{I}_3) \cdot (A - \mathrm{I}_3) \cdot v \;=\; v_1 + v_2$$

for all $v \in \mathbb{C}^3$. The vector

$$v_1 := (A - 2 \cdot \mathrm{I}_3)^2 \cdot v$$

is in the generalized eigenspace for $\lambda_1 = 1$, and the vector

$$v_2 := -(A - 3 \cdot \mathrm{I}_3) \cdot (A - \mathrm{I}_3) \cdot v$$

is in the generalized eigenspace for $\lambda_2 = 2$.

To compute the corresponding summands for the given vector v we first compute

$$(A - 2 \cdot \mathrm{I}_3)^2 = \begin{pmatrix} 1 & 0 & -2 \\ -1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$-(A - 3 \cdot \mathrm{I}_3) \cdot (A - \mathrm{I}_3) = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $v = v_1 + v_2$ with

$$v_1 = (A - 2 \cdot \mathrm{I}_3)^2 \cdot v = \begin{pmatrix} 1 & 0 & -2 \\ -1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 0 \end{pmatrix} \in K_1(A),$$

and

$$v_2 = -(A - 3 \cdot \mathrm{I}_3) \cdot (A - \mathrm{I}_3) \cdot v = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ 3 \end{pmatrix} \in K_{2(A)}.$$

Problem 3: Show that the 4×4-matrix $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ is not diagonalizable.

Solution. The characteristic polynomial of

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

is $P_A(T) = (T-2)^4$ and so 2 is the only eigenvalue of A. Hence A is diagonalizable if and only if the corresponding eigenspace E_2 is the whole space $\mathbb{C}^4$. But this is not the case since for instance $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is not an eigenvector and so dimension $E_2 \leq 3$. (In fact, the dimension of E_2 is 1 since the rank of $A - 2 \cdot \mathrm{I}_4$ is obviously 3.)

Problem 4: Recall that the trace $\mathrm{tr}(A)$ of a $n \times n$-matrix $A = (a_{ij})$ is defined as

$$\mathrm{tr}(A) := \sum_{i=1}^{n} a_{ii}.$$

Show that $\mathrm{tr}(A \cdot B) = \mathrm{tr}(B \cdot A)$ for all $n \times n$-matrices A, B.

Solution. Let $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij}) = A \cdot B$, and $D = (d_{ij}) = B \cdot A$. Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \quad \text{and} \quad d_{ij} = \sum_{k=1}^{n} b_{ik} \cdot a_{kj}\,.$$

Therefore

$$\begin{aligned} \mathrm{tr}(A \cdot B) &= \sum_{l=1}^{n} c_{ll} \;=\; \sum_{l=1}^{n} \sum_{k=1}^{n} a_{lk} \cdot b_{kl} \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} b_{kl} \cdot a_{lk} \;=\; \sum_{k=1}^{n} d_{kk} \\ &= \mathrm{tr}(B \cdot A)\,. \end{aligned}$$