Linear Algebra MATH 325: Assignment 3

(Due in class, February 1)

Problem 1: Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix}$. Compute A^{-1} using the Cayley-Hamilton theorem. (Hint: compute the characteristic polynomial $P_A(T)$ of A and analyze the equality $P_A(A) = 0$.)

Solution. The characteristic polynomial of the matrix A is

$$P_A(T) \;=\; T^3 - 5T^2 + 8T - 6\,.$$

By the Cayley-Hamilton Theorem we have therefore

$$0 \;=\; P_A(A) = A^3 - 5 \cdot A^2 + 8 \cdot A - 6 \cdot \mathrm{I}_3\,,$$

and this is equivalent to the equation

$$A \cdot \frac{1}{6} \cdot \left(A^2 - 5 \cdot A + 8 \cdot \mathrm{I}_3\right) = \mathrm{I}_3\,.$$

Hence we have

$$A^{-1} \;=\; \frac{1}{6} \cdot \left(A^2 - 5 \cdot A + 8 \cdot \mathrm{I}_3\right),$$

from which we compute

$$A^{-1} \;=\; \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{2}{3} \end{pmatrix}.$$

Problem 2: Let $A = \begin{pmatrix} 2 & 9 \\ -1 & 8 \end{pmatrix}$ and $\alpha_A : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, $v \mapsto A \cdot v$, the corresponding linear map.

(i) Show that A has only one eigenvalue and that A is not diagonalizable.

(ii) Find a basis of $\mathbb{C}^2$, such that the matrix of α_A with respect to this basis is

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

where λ is the only eigenvalue of A.

Solution. (i) The characteristic polynomial of A is

$$P_A(T) \;=\; T^2 - 10T + 25 \;=\; (T-5)^2\,,$$

and so $\lambda = 5$ is the only eigenvalue of A. The vector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector for this eigenvalue, and for instance the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not an eigenvalue. Hence the dimension of the eigenspace for the eigenvalue $\lambda = 5$ is not 2. Therefore the matrix A is not diagonalizable.

(ii) We compute first the matrix of the linear map α_A with respect to the basis $v = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ (which is an eigenvector) and $w' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We have $\alpha_A(v) = A \cdot v = 5 \cdot v$ since v is an eigenvector for the eigenvalue $\lambda = 5$, and

$$\alpha_A(w') = \begin{pmatrix} 2 & 9 \\ -1 & 8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = (-1) \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 5 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and so the matrix with respect to this basis is $B' = \begin{pmatrix} 5 & -1 \\ 0 & 5 \end{pmatrix}$. We use now the formula given in class and replace the basis v, w' by

$$v \quad \text{and} \quad w = (-1)^{-1} \cdot w' = -w' = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

With respect to this basis the linear map α_A has the desired form

$$B = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}.$$

Problem 3: Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Show that $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a generalized eigenvector for the eigenvalue 1 of A.

Solution. The number 1 is the only eigenvalue of A, and

$$\lambda \cdot \mathrm{I}_3 - A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

One computes then $(\lambda \, \mathrm{I}_3 - A)^3 = 0$ and so every nonzero vector in $\mathbb{C}^3$ is a generalized eigenvector.

Problem 4: Let A be a complex nonzero $n \times n$-matrix, and $f(T)$ a complex polynomial of minimal degree ≥ 1, such that $f(A) = 0$. Show that $f(T)$ divides the characteristic polynomial of A. (Hint: apply the division algorithm.)

Solution. By the division algorithm we can write

$$P_A(T) = g(T) \cdot f(T) + r(T)$$

for some complex polynomials $g(T)$ and $r(T)$ with the degree of $r(T)$ strictly smaller than the degree of $f(T)$. By the Cayley-Hamilton Theorem we have $P_A(A) = 0$ and so

$$r(A) = P_A(A) - g(A) \cdot f(A) = 0.$$

Hence since $f(T)$ is of minimal degree ≥ 1 with this property we have $\deg r(T) \leq 0$, and so $r(T) = 0$. Hence $f(T)$ divides $P_A(T)$.