Linear Algebra MATH 325: Solution 9

(It is for practice, not for marking)

Problem 1: Let $i = \sqrt{-1} \in \mathbb{C}$. Recall that every complex number can be written

$$z = r \cdot e^{i \cdot \alpha} = r \cdot (\cos(\alpha) + i \cdot \sin(\alpha))$$

for some real number $r \geq 0$ and some $\alpha \in \mathbb{R}$. For which r, α is the $\mathbb{C}$-linear map

$$\ell_z : \mathbb{C} \longrightarrow \mathbb{C}, \; w \longmapsto z \cdot w$$

hermitian, for which unitary?

Solution. We view $\mathbb{C}$ as a 1-dimensional vector space with the standard basis $e_1 = 1$. One has $\ell_z(1) = z = z \cdot 1$. Hence the matrix of ℓ_z is the 1×1-matrix $(z) = (r \cdot e^{i\,\alpha})$. This matrix is hermitian if an only if $(z) = (z)^*$. But $(z)^* = (\bar{z})$, where $^-$ denotes complex conjugation, and so ℓ_z is hermitian if and only if $z = \bar{z}$. This is the case for every $r \geq 0$ and $\alpha = n \cdot \pi$ for some $n \in \mathbb{Z}$.

The map ℓ_z is unitary if and only if $(z) \cdot (z)^* = (1)$, *i.e.* if and only if $|z|^2 = z \cdot \bar{z} = 1$, and this is the case if and only if $r = 1$ and for every $\alpha \in \mathbb{R}$.

Problem 2: Find a unitary 3×3-matrix U with first column $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$.

Solution. The columns of such a matrix are an orthonormal basis of $\mathbb{C}^3$ which contain the given vector as first column. Hence we have to find an orthonormal basis of $\mathbb{C}^3$ (with respect to the usual Euclidean inner product) which contains $w = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$. For this we first extend this vector to a basis of $\mathbb{C}^3$ by the second and third standard basis vector $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and then apply the Gram-Schmidt process to this basis: We set $w_1 := w$, then

$$w_2 := e_2 - \frac{\langle e_2, w_1 \rangle}{\langle w_1, w_1 \rangle} \cdot w_1 = \begin{pmatrix} -\frac{2}{9} \\ \frac{5}{9} \\ -\frac{4}{9} \end{pmatrix},$$

and

$$w_3 := e_3 - \frac{\langle e_3, w_1 \rangle}{\langle w_1, w_1 \rangle} \cdot w_1 - \frac{\langle e_3, w_2 \rangle}{\langle w_2, w_2 \rangle} \cdot w_2 = \begin{pmatrix} -\frac{2}{5} \\ 0 \\ \frac{1}{5} \end{pmatrix}.$$

Finally to get an orthonormal basis we have to normalize the vectors w_1, w_2, and w_3, i.e. to divide by their norms. We get then

$$v_1 = \frac{1}{\|w_1\|} \cdot w_1 \;=\; \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \; v_2 = \frac{1}{\|w_2\|} \cdot w_2 \;=\; \begin{pmatrix} -\frac{2}{15} \cdot \sqrt{5} \\ \frac{1}{3} \cdot \sqrt{5} \\ -\frac{4}{15} \cdot \sqrt{5} \end{pmatrix},$$

and

$$v_3 = \frac{1}{\|w_3\|} \cdot w_3 \;=\; \begin{pmatrix} -\frac{2}{5} \cdot \sqrt{5} \\ 0 \\ \frac{1}{5} \cdot \sqrt{5} \end{pmatrix}.$$

Hence an unitary matrix having $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$ as first column is:

$$U \;:=\; \begin{pmatrix} \frac{1}{3} & -\frac{2}{15} \cdot \sqrt{5} & -\frac{2}{5} \cdot \sqrt{5} \\ \frac{2}{3} & \frac{1}{3} \cdot \sqrt{5} & 0 \\ \frac{2}{3} & -\frac{4}{15} \cdot \sqrt{5} & \frac{1}{5} \cdot \sqrt{5} \end{pmatrix}.$$

Problem 3: Let

$$\langle -, - \rangle \;:\; \mathbb{R}^n \times \mathbb{R}^n \;\longrightarrow\; \mathbb{R}^n$$

be an inner product on $\mathbb{R}^n$. Show that there exists an invertible $n \times n$-matrix A, such that

(a) $A^T = A$, and

(b) $\langle v, w \rangle \;=\; v^T \cdot A \cdot w$ for all $v, w \in \mathbb{R}^n$.

Solution. Denote by $e_1, \ldots, e_n$ the standard basis of $\mathbb{R}^n$. We claim that the matrix $A = (a_{ij})$ with $a_{ij} = \langle e_i, e_j \rangle$ (i row index, j column index) has the required properties.

Since $\langle e_i, e_j \rangle = \langle e_j, e_i \rangle$ by axiom **(H)** the matrix A is symmetric, *i.e.* we have $A = A^T$. To show that $v^T \cdot A \cdot w = \langle v, w \rangle$ for all $v, w \in \mathbb{R}^n$ it is enough to check $e_i^T \cdot A \cdot e_j = \langle e_i, e_j \rangle$ for all i, j. Indeed, given $v, w \in \mathbb{R}^n$ there exists scalars $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$, such that

$$v \;=\; \sum_{i=1}^{n} a_i e_i \quad \text{and} \quad w \;=\; \sum_{i=1}^{n} b_i e_i \,.$$

Then using **(L1)** and **(L2)** we compute

$$\langle v, w \rangle \;=\; \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \langle e_i, e_j \rangle \,.$$

Assuming $e_i^T \cdot A \cdot e_j = \langle e_i, e_j \rangle$ for all i, j this gives

$$\begin{aligned}\langle v, w \rangle &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \, e_i^T \cdot A \cdot e_j \\ &= \Big(\sum_{i=1}^{n} a_i e_i^T \Big) \cdot A \cdot \Big(\sum_{j=1}^{n} b_j e_j \Big) \\ &= v^T \cdot A \cdot w \,,\end{aligned}$$

where the equality in the middle is by the linearity of matrix multiplication.

To prove the equality $e_i^T \cdot A \cdot e_j = \langle e_i, e_j \rangle$ note first that

$$A \cdot e_j \;=\; \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} ,$$

and so

$$e_i^T \cdot A \cdot e_j \;=\; a_{ij} \;=\; \langle e_i, e_j \rangle \,.$$

Finally we have to show that A is invertible. For every vector $v \neq 0$ in $\mathbb{R}^n$ we have by axiom **(P)**

$$0 \;<\; \langle v, v \rangle \;=\; v^T \cdot A \cdot v \,.$$

Hence for every $v \neq 0$ we have $A \cdot v \neq 0$. But this means that the kernel of A is $\{0\}$ and so A is invertible.

Problem 4: Let $\langle -, - \rangle$ be the usual Euclidean inner product on $\mathbb{R}^2$. Does there exists a real 2×2-matrix $A \neq \mathrm{I}_2$, such that

$$\langle A \cdot v, A \cdot w \rangle \;=\; \langle v, w \rangle$$

for all $v, w \in \mathbb{R}^2$? If such a matrix A exists, what is $\det A$?

Solution. We have

$$\langle Av, Aw \rangle \;=\; (Av)^T \cdot (Av) \;=\; v^T \cdot (A^T \cdot A) \cdot w \,.$$

Hence if $A^T \cdot A = \mathrm{I}_2$ we have $\langle Av, Aw \rangle = \langle v, w \rangle$ for all $v, w \in \mathbb{R}^2$. For instance

$$A \;=\; \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

satisfies $A^T \cdot A = \mathrm{I}_2$. Hence such a matrix $A \neq \mathrm{I}_2$ exists.

On the other hand if

$$v^T \cdot w \;=\; \langle v, w \rangle \;=\; \langle Av, Aw \rangle \;=\; v^T \cdot (A^T \cdot A) \cdot w$$

for all $v, w \in \mathbb{R}^2$ we have in particular

$$\langle e_i, e_j \rangle \;=\; e_i^T \cdot (A^T \cdot A) \cdot e_j \,,$$

for all $i, j \in \{1, 2\}$, where e_1, e_2 are the standard basis vectors of $\mathbb{R}^2$. Hence the ij-entry c_{ij} of $A^T \cdot A$ has to be equal $\langle e_i, e_j \rangle$, i.e. $c_{ij} = 1$ if $i = j$ and $= 0$ otherwise. Hence such a matrix A satisfies $A^T \cdot A = \mathrm{I}_2$. This implies

$$1 \;=\; \det(A^T \cdot A) \;=\; \det(A^T) \cdot \det(A) \;=\; \big(\det(A)\big)^2$$

(the latter equality since $\det(A) = \det(A^T)$). It follows $\det(A) = \pm 1$.