MATH 325 Q1: LINEAR ALGEBRA III, PART 5

Full cycles of generalized eigenvectors

We continue with above notation, *i.e.* $\alpha : V \to V$ is a $\mathbb{C}$-linear map, where V is a finite dimensional $\mathbb{C}$-vector space, and

$$P_\alpha(T) = \prod_{i=1}^{l} (T - \lambda_i)^{n_i}$$

is the characteristic polynomial of α, where $\lambda_1, \ldots, \lambda_l$ are all different eigenvalues of α. Our next aim is to show that every generalized eigenspace K_{λ_i} of α has a basis consisting of the so-called full cycles of generalized eigenvectors.

For brevity, we denote $\lambda := \lambda_i$ where $1 \leq i \leq l$. Let $v \neq 0$ be an arbitrary generalized eigenvector for λ. Then there exists a positive integer $m \geq 1$, such that

$$(\alpha - \lambda \cdot \mathrm{id}_V)^m(v) = 0 \quad \text{and} \quad (\alpha - \lambda \cdot \mathrm{id}_V)^{m-1}(v) \neq 0.$$

Note that this implies in particular that the vector

$$w = (\alpha - \lambda \cdot \mathrm{id}_V)^{m-1}(v)$$

is an eigenvector of α for λ.

Definition. *The* **ordered set** *of vectors*

$$\Delta = \left\{ w = (\alpha - \lambda \cdot \mathrm{id}_V)^{m-1}(v),\, (\alpha - \lambda \cdot \mathrm{id}_V)^{m-2}(v),\, \ldots ,\, (\alpha - \lambda \cdot \mathrm{id}_V)(v),\, v \right\}$$

is called a full cycle of generalized eigenvectors for the eigenvalue λ. The vector w is called the initial vector and v is called the end vector of the cycle. The number

$$m = |\Delta|,$$

which is equal to the number of elements in Δ, is called the length of the cycle.

Remark. *Note that if v is an eigenvector for λ then $m = 1$, hence the single vector v is a full cycle of generalized eigenvectors of length* 1 *with initial and end vector v.*

Remark. *Note also that the initial vector of a full cycle of generalized eigenvectors is always an eigenvector.*

Definition. *If $V = \mathbb{C}^n$ and $\alpha = \alpha_A$ for a complex $n \times n$-matrix A, then we say also that Δ is a full cycle of generalized eigenvectors for the eigenvalue λ of the matrix A.*

Example. Consider the $\mathbb{C}$-linear map $\alpha_A : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, $v \mapsto A \cdot v$, where

$$A = \begin{pmatrix} 4 & -1 \\ 1 & 6 \end{pmatrix}.$$

The characteristic polynomial of A is

$$P_A(T) = (T-4)\cdot(T-6) + 1 = T^2 - 10T + 25 = (T-5)^2.$$

Therefore A has only one eigenvalue $\lambda = 5$. By Cayley- Hamilton Theorem $(A - 5I_2)^2 = 0$ and this implies that every vector in $\mathbb{C}^2$ is a generalized eigenvector for $\lambda = 5$. Thus, the generalized eigenspace K_5 is equal the whole space $\mathbb{C}^2$.

Let $v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. This is not an eigenvector of A but

$$w = (A - 5 \cdot \mathrm{I}_2) \cdot v = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -7 \\ 7 \end{pmatrix}$$

is an eigenvector because

$$A \cdot w = \begin{pmatrix} 4 & -1 \\ 1 & 6 \end{pmatrix} \cdot \begin{pmatrix} -7 \\ 7 \end{pmatrix} = 5 \begin{pmatrix} -7 \\ 7 \end{pmatrix}.$$

The set

$$\Delta = \left\{ w = \begin{pmatrix} -7 \\ 7 \end{pmatrix}, v = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$$

is a full cycle of generalized eigenvectors for $\lambda = 5$ of length 2. Note that the vectors in this set are linear independent. This is a general fact, as the following lemma shows.

Lemma. *Let*

$\Delta = \left\{ w = (\alpha-\lambda\cdot\mathrm{id}_V)^{m-1}(v), (\alpha-\lambda\cdot\mathrm{id}_V)^{m-2}(v), \ldots, (\alpha-\lambda\cdot\mathrm{id}_V)(v), v \right\}$

be a full cycle of generalized eigenvectors for the eigenvalue λ. Then Δ is a set of linear independent vectors.

Proof. Assume that for some scalars $a_0, \ldots, a_{m-1}$ one has

$$\sum_{i=0}^{m-1} a_i \cdot (\alpha - \lambda \cdot \mathrm{id}_V)^i(v) = 0. \tag{1}$$

We prove by induction on $j \geq 0$ that $a_j = 0$. To see this for $j = 0$ we apply the linear map $(\alpha - \lambda \cdot \mathrm{id}_V)^{m-1}$ to the equation (1):

$$\sum_{i=0}^{m-1} a_i \cdot (\alpha - \lambda \cdot \mathrm{id}_V)^{i+m-1}(v) = 0.$$

But $(\alpha - \mathrm{id}_V)^m(v) = 0$ and so $(\alpha - \mathrm{id}_V)^{i+m-1}(v) = 0$ for $i \geq 1$. We get therefore

$$0 \;=\; a_0 \cdot (\alpha - \lambda \cdot \mathrm{id}_V)^{m-1}(v) \;=\; a_0 \cdot w$$

and so $a_0 = 0$. Assume now we have shown $a_0 = a_1 = \ \ldots\ a_{j-1} = 0$ for some $j \geq 1$. Then the relation (1) becomes

$$\sum_{i=j}^{m-1} a_i \cdot (\alpha - \lambda \cdot \mathrm{id}_V)^i(v) = 0,$$

and applying to this equation the linear map $(\alpha - \lambda \cdot \mathrm{id}_v)^{m-1-j}$ we get

$$\sum_{i=j}^{m-1} a_i \cdot (\alpha - \lambda \cdot \mathrm{id}_V)^{i+m-1-j}(v) \;=\; 0\,.$$

Since $(\alpha - \lambda \cdot \mathrm{id}_V)^{i+m-1-j}(v) = 0$ for $i \geq j+1$ this implies

$$0 \;=\; a_j \cdot (\alpha - \lambda \cdot \mathrm{id}_V)^{m-1}(v) \;=\; a_j \cdot v\,,$$

and so $a_j = 0$ as claimed. We are done. $\square$

Jordan blocks

Let

$$\Delta = \big\{w = (\alpha - \lambda \cdot \mathrm{id}_V)^{m-1}(v),\ (\alpha - \lambda \cdot \mathrm{id}_V)^{m-2}(v),\ \ldots\ ,\ (\alpha - \lambda \cdot \mathrm{id}_V)(v),\ v\big\}$$

be a full cycle of generalized eigenvalues for the eigenvalue λ of the linear map $\alpha : V \longrightarrow V$. Let $W = W(\Delta) \subseteq K_\lambda$ be the span of these vectors, *i.e.* the subspace generated by the vectors in Δ. By the lemma above the full cycle of generalized eigenvalues Δ is linear independent and so Δ is a basis of W.

Lemma. *The vector subspace W is α-invariant, i.e. $\alpha(u) \in W$ for all $u \in W$.*

Proof. To check this let u be an arbitrary vector in W. Then there are $a_i \in \mathbb{C}$, such that

$$u \;=\; \sum_{i=0}^{m-1} a_i \cdot (\alpha - \lambda \cdot \mathrm{id}_V)^i(v)\,,$$

and so since α is $\mathbb{C}$-linear

$$\alpha(u) \;=\; \sum_{i=0}^{m} a_i \cdot \alpha\big(\,(\alpha - \lambda \cdot \mathrm{id}_V)^i(v)\,\big)\,.$$

Hence it is enough to show that

$$\alpha\big(\,(\alpha - \lambda \cdot \mathrm{id}_V)^i(v)\,\big) \in W$$

for all $0 \le i \le m-1$. But this is the case since

$$\begin{aligned}(\alpha - \lambda\cdot \mathrm{id}_V)^{i+1}(v) &= (\alpha - \lambda\cdot \mathrm{id}_V)\big((\alpha - \lambda\cdot \mathrm{id}_V)^{i}(v)\big)\\ &= \alpha\big((\alpha - \lambda\cdot \mathrm{id}_V)^{i}(v)\big) - \lambda\cdot\big((\alpha - \lambda\cdot \mathrm{id}_V)^{i}(v)\big),\end{aligned}$$

and so

$$\alpha\big((\alpha-\lambda\cdot\mathrm{id}_V)^{i}(v)\big) = (\alpha-\lambda\cdot\mathrm{id}_V)^{i+1}(v) + \lambda\cdot\big((\alpha-\lambda\cdot\mathrm{id}_V)^{i}(v)\big) \in W\,.$$

$\square$

The above equation implies that the matrix of the restriction of α to W with respect to the basis Δ is of the form (note that the initial vector w is an eigenvalue)

$$\mathrm{J}(\lambda, m) := \begin{pmatrix} \lambda & 1 & 0 & & \dots & 0\\ 0 & \lambda & 1 & 0 & \dots & 0\\ 0 & 0 & \lambda & 1 & \dots & 0\\ \vdots & & & \ddots & \ddots & \vdots\\ 0 & & \dots & & \lambda & 1\\ 0 & & \dots & & & \lambda \end{pmatrix}.$$

Definition. *The above matrix is called a Jordan block of size m for the eigenvalue λ. This is a $m\times m$-matrix with the eigenvalue λ on the diagonal, 1's above the diagonal, and 0's elsewhere.*

Examples. The Jordan blocks of sizes $1, 2, 3$, and 4 for the eigenvalue λ are:

$$(\lambda)\,, \quad \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}, \quad \begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}\lambda & 1 & 0 & 0\\ 0 & \lambda & 1 & 0\\ 0 & 0 & \lambda & 1\\ 0 & 0 & 0 & \lambda\end{pmatrix}.$$

Jordan bases

Our next goal is to prove that there exists full cycles $\Delta_1, \dots, \Delta_s$ of generalized eigenvectors in K_λ, say

$$\Delta_j = \big\{\, w_j = (\alpha-\lambda\cdot\mathrm{id}_V)^{m_j-1}(v_j),\ (\alpha-\lambda\cdot\mathrm{id}_V)^{m_j-2}(v_j)\ \dots,\ (\alpha-\lambda\cdot\mathrm{id}_V)(v_j), v_j\big\}$$

such that

(i) $\Delta_j \cap \Delta_k = \emptyset$ for all $j \neq k$ in $\{1, \dots, s\}$, and

(ii) the ordered set of vectors $\Delta := \bigcup_{j=1}^{s} \Delta_j$ is a basis of K_λ.

We call such a basis $\Delta = \bigcup_{j=1}^{s} \Delta_j$ a *Jordan basis* for K_λ.

Assuming that Jordan basis exists we first prove the following property.

Lemma. *Let $\Delta = \bigcup_{j=1}^{s} \Delta_j$ (with Δ_j as above) be a Jordan basis for K_λ. Then $w_1, \ldots, w_s$ is a basis of the eigenspace E_λ and so in particular we have $s = \dim_{\mathbb{C}} E_\lambda$.*

Proof. By assumption Δ is a basis for K_λ and so the vectors in Δ are linear independent. In particular the vectors $w_1, \ldots, w_s \in E_\lambda$ are linear independent. (Note that these vectors are the only vectors of Δ which are in E_λ, *i.e.* $E_\lambda \cap \Delta = \{w_1, \ldots, w_s\}$.)

It is therefore enough to show that $w_1, \ldots, w_s$ span E_λ. Take an arbitrary $y \in E_\lambda$. Since Δ is a basis of $K_\lambda \supseteq E_\lambda$ there are complex numbers $a_k^{(j)}$ such that

$$y = \sum_{j=1}^{s} \sum_{k=0}^{m_j-1} a_k^{(j)} \cdot (\alpha - \lambda \cdot \mathrm{id}_V)^k(v_j).$$

Applying to this equation the linear map $\alpha - \lambda \cdot \mathrm{id}_V$ we get since y and $w_j = (\alpha - \lambda \cdot \mathrm{id}_V)^{m_j-1}(v_j)$, $j = 1, \ldots, s$, are eigenvectors

$$0 = \sum_{j=1}^{s} \sum_{k=0}^{m_j-2} a_k^{(j)} \cdot (\alpha - \lambda \cdot \mathrm{id}_V)^{k+1}(v_j)\,,$$

and so $a_k^{(j)} = 0$ for all $k \leq m_j - 2$, $1 \leq j \leq s$. Hence

$$y = \sum_{j=1}^{s} a_{m_j-1}^{(j)} \cdot (\alpha - \lambda \cdot \mathrm{id}_V)^{m_j-1}(v_j) = \sum_{j=1}^{s} a_{m_j-1}^{(j)} \cdot w_j$$

is in the span of the vectors $w_1, \ldots, w_s$. □

The matrix with respect to a Jordan basis

Assume we have such a Jordan basis $\Delta = \bigcup_{j=1}^{s} \Delta_j$ for K_λ as above. Since

$$\alpha\big((\alpha - \lambda \cdot \mathrm{id}_V)^r(v_j)\big) = \lambda \cdot (\alpha - \lambda \cdot \mathrm{id}_V)^r(v_j) + (\alpha - \lambda \cdot \mathrm{id}_V)^{r+1}(v_j)$$

the matrix of the linear map $K_\lambda \longrightarrow K_\lambda$, $v \mapsto \alpha(v)$ (*i.e.* the restriction of α to K_λ) with respect to the basis Δ is

$$\begin{pmatrix} \mathrm{J}(\lambda, m_1) & \mathrm{O}_{m_1\times m_2} & \dots & \mathrm{O}_{m_1\times m_s} \\ \mathrm{O}_{m_2\times m_1} & \mathrm{J}(\lambda, m_2) & & \mathrm{O}_{m_2\times m_s} \\ \vdots & & \ddots & \vdots \\ \mathrm{O}_{m_s\times m_1} & & \dots & \mathrm{J}(\lambda, m_s) \end{pmatrix},$$

where $\mathrm{J}(\lambda, m_j)$ is a Jordan block of size m_j for the eigenvalue λ.

Before we prove the existence of a Jordan basis we show the following uniqueness statement.

Theorem. *Let $\alpha : V \longrightarrow V$ be a $\mathbb{C}$-linear map as above and λ an eigenvalue of α with corresponding generalized eigenspace K_λ. Assume we have two families*

$$\Delta_1, \cdots, \Delta_s \quad \textit{and} \quad \Delta_1', \ldots, \Delta_s'$$

of full cycles of generalized eigenvalues where

$$\Delta_j = \left\{w_j = (\alpha-\lambda\cdot\mathrm{id}_V)^{m_j-1}(v_j),\ (\alpha-\lambda\cdot\mathrm{id}_V)^{m_j-2}(v_j),\ \ldots,\ (\alpha-\lambda\cdot\mathrm{id}_V)(v_j),\ v_j\right\}$$

and

$$\Delta_j' = \left\{w_j' = (\alpha-\lambda\cdot\mathrm{id}_V)^{m_j'-1}(v_j'),\ (\alpha-\lambda\cdot\mathrm{id}_V)^{m_j'-2}(v_j'),\ \ldots,\ (\alpha-\lambda\cdot\mathrm{id}_V)(v_j'),\ v_j'\right\}$$

for $j = 1, \ldots, s = \dim_{\mathbb{C}} E_\lambda$, such that

$$\Delta := \bigcup_{j=1}^{s} \Delta_j \qquad \textit{and} \qquad \Delta' := \bigcup_{j=1}^{s} \Delta_j'$$

are Jordan bases for K_λ. By (if necessary) renumbering the indices we can assume that

$$m_1 \geq m_2 \geq \ldots \geq m_s \qquad \textit{and} \qquad m_1' \geq m_2' \geq \ldots \geq m_s'.$$

Then

$$|\Delta_j| = m_j = m_j' = |\Delta_j'|$$

for all $k = 1, \ldots, s$. In particular, the matrices of the restriction of α to K_λ with respect to the bases $\Delta := \bigoplus_{j=1}^{s} \Delta_j$ and $\Delta' = \bigcup_{j=1}^{s'} \Delta_j'$ are the same.

Proof. We prove this by contradiction. Assume that there is some j, such that $m_j \neq m'_j$. Let $1 \leq j_0 \leq s$ be minimal with this property, *i.e.* $m_j = m'_j$ for all $1 \leq j < j_0$ and $m_{j_0} \neq m'_{j_0}$, say $m_{j_0} > m'_{j_0}$. We consider the subspace

$$U \;:=\; \mathrm{Image}(\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1} \;=\; \left\{\, (\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1}(v) \,|\, v \in K_\lambda \,\right\}$$

in K_λ. Since Δ and Δ' both generate K_λ their images

$$\Gamma \;:=\; (\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1}(\Delta) \;:=\; \left\{\, (\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1}(v) \,|\, v \in \Delta \,\right\}$$

and

$$\Gamma' \;:=\; (\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1}(\Delta') \;:=\; \left\{\, (\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1}(v') \,|\, v' \in \Delta' \,\right\}$$

generated both U. Moreover we have $\Gamma \setminus \{0\} \subset \Delta$ and $\Gamma' \setminus \{0\} \subset \Delta'$ and so $\Gamma \setminus \{0\}$ and $\Gamma' \setminus \{0\}$ are both sets of linear independent vectors and so bases of the subspace U.

We count the elements in $\Gamma \setminus \{0\}$ and $\Gamma' \setminus \{0\}$. We have $m_{j_0} > m'_{j_0} \geq m'_j$ for all $j \geq j_0$. Hence we have

$$(\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1}(\Delta'_j) \setminus \{0\} \;=\; \left\{\, (\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1}(v') \,|\, v' \in \Delta'_j \,\right\} \setminus \{0\} \;=\; \emptyset$$

for $j \geq j_0$, and for $j < j_0$ we have

$$(\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1}(\Delta'_j) \setminus \{0\} \;=\; \left\{\, (\alpha - \lambda \cdot \mathrm{id}_V)^{m'_j-1}(v'_j)\,, \ldots, (\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1}(v'_j) \,\right\}.$$

Therefore

$$\Gamma' \setminus \{0\} \;=\; \bigcup_{j=1}^{s} (\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1}(\Delta'_j) \setminus \{0\}$$

contains

$$\sum_{k=1}^{j_0-1} (m'_k - m_{j_0} + 1) = \sum_{k=1}^{j_0-1} (m_k - m_{j_0} + 1)$$

elements (the equation since by assumption $m'_k = m_k$ for $1 \leq k < j_0$).

Similarly, we have

$$(\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1}(\Delta_j) \setminus \{0\} \;=\; \left\{\, (\alpha - \lambda \cdot \mathrm{id}_V)^{m_j-1}(v_j)\,, \ldots, (\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1}(v_j) \,\right\}.$$

for all $1 \leq j < j_0$, and

$$(\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1}(\Delta_{j_0}) \setminus \{0\} \;=\; \left\{\, (\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1} \cdot v_{j_0} \,\right\}.$$

Therefore

$$\Gamma \setminus \{0\} \;=\; \bigcup_{j=1}^{s} (\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1}(\Delta_j) \setminus \{0\} \;\supseteq\; \bigcup_{j=1}^{j_0} (\alpha - \lambda \cdot \mathrm{id}_V)^{m_{j_0}-1}(\Delta_j) \setminus \{0\}$$

has at least $1+\sum_{k=1}^{j_0-1}(m_k-m_{j_0}+1)$ elements and so at least one element more than $\Gamma' \setminus \{0\}$, a contradiction. $\square$