Linear Algebra MATH 325: Assignment 8

(Due in class, April 1)

Problem 1:

(i) Show that the map

$$\langle -,- \rangle \,:\, \mathbb{C}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}\,,$$

$$\langle \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \rangle \;:=\; 3a_1\bar{b}_1 + a_1\bar{b}_2 + a_2\bar{b}_1 + 2a_2\bar{b}_2$$

is an inner product on $\mathbb{C}^2$.

(ii) Give an orthogonal basis of $\mathbb{C}^2$ with respect to this inner product.

Solution. (i) It is straightforward to see that axioms **(L1)** and **(H)** hold. Let us verify **(P)**, i.e. the real number

$$3a_1\bar{a}_1 + a_1\bar{a}_2 + a_2\bar{a}_1 + 2a_2\bar{a}_2$$

is positive if at least one the complex numbers a_1, a_2 is nonzero.

Write $a_1 = u_1 + i\,u_2$ and $a_2 = v_1 + i\,v_2$. Then the above expression is of the form

$$3(u_1^2 + u_2^2) + 2(u_1v_1 + u_2v_2) + 2(v_1^2 + v_2^2).$$

Note that

$$3u_1^2 + 2u_1v_1 = 3(u_1^2 + \frac{2}{3}\,u_1v_1) = 3(u_1 + \frac{1}{3}\,v_1)^2 - \frac{1}{3}v_1^2$$

and similarly

$$3u_2^2 + 2u_2v_2 = 3(u_2^2 + \frac{2}{3}\,u_2v_2) = 3(u_2 + \frac{1}{3}v_2)^2 - \frac{1}{3}\,v_2^2.$$

Then the required expression can be written as

$$3(u_1 + \frac{1}{3}\,v_1)^2 + 3(u_2 + \frac{1}{3}\,v_2)^2 + \frac{5}{3}\,v_1^2 + \frac{5}{3}\,v_2^2.$$

Obviously, it is a positive real number if at least one of u_1, u_2, v_1, v_2 is nonzero.

(ii) Let us start with the standard basis $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and use the Gram-Schmidt process to get an orthogonal basis: We set $w_1 := e_1$, and

$$w_2 \;:=\; e_2 - \frac{\langle e_2, w_1 \rangle}{\langle w_1, w_1 \rangle} \cdot w_1 \;=\; \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{3} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}.$$

The vectors w_1, w_2 form an orthogonal basis of $\mathbb{C}^2$ with respect to our given inner product.

Problem 2: Let $(V, \langle -,- \rangle)$ be the inner product space $\mathbb{R}^4$ with respect to the usual Euclidean inner product. Let

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 6 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \text{and } w_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix},$$

and $W \subseteq V = \mathbb{R}^4$ be the subspace generated by these vectors. Give an orthogonal basis for W.

Solution. We use the Gram-Schmidt process to get an orthogonal basis v_1, v_2 and v_3 of W. We set $v_1 := w_1$,

$$v_2 := w_2 - \frac{\langle w_2, v_1\rangle}{\langle v_1, v_1\rangle} \cdot v_1 \ = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{5}{23} \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \\ 6 \end{pmatrix} = \frac{1}{23} \begin{pmatrix} 18 \\ 23 \\ 8 \\ -7 \end{pmatrix},$$

and $v_3 := w_3 - \sum_{i=1}^{2} \frac{\langle w_3, v_i\rangle}{\langle v_i, v_i\rangle} \cdot v_i$

$$= \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix} - \frac{9}{23} \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \\ 6 \end{pmatrix} - \frac{25}{966} \begin{pmatrix} 18 \\ 23 \\ 8 \\ -7 \end{pmatrix} = \frac{1}{966} \cdot \begin{pmatrix} -828 \\ 391 \\ 598 \\ -511 \end{pmatrix}.$$

Problem 3: Let V be a $\mathbb{C}$-vector space with inner product $\langle -, -\rangle$. Determine all complex numbers λ such that the map

$$(v, w) \longmapsto \lambda \cdot \langle v, w\rangle$$

is also an inner product on V?

Solution. A straightforward computation shows that $(v, w) \mapsto \lambda \cdot \langle v, w\rangle$ is linear in the first variable for all $\lambda \in \mathbb{C}$, *i.e.* satisfies **(L1)**.

Next, the complex number $\lambda \cdot \langle v, v\rangle$ is real positive for $v \neq 0$ if and only only if λ is a real number > 0 (because $\langle v, v\rangle > 0$ for all $v \neq 0$). Thus λ must be real positive in order axiom **(P)** holds.

But for any such $\lambda > 0$ axiom **(H)** is also satisfied since then $\bar{\lambda} = \lambda$ and so

$$\lambda\langle v, w\rangle \ = \ \lambda\overline{\langle w, v\rangle} \ = \ \overline{\lambda\langle w, v\rangle}\,.$$

To sum up, this map is an inner product if and only if λ is a real number > 0.

Problem 4: Let $A = \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}$. Find a 2×2-matrix S with real coefficients, such that

$$S^T \cdot A \cdot S \ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Solution. Obviously, the map

$$(v, w) \longmapsto \langle v, w\rangle \ := \ v^T \cdot \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix} \cdot w$$

is an inner product on $\mathbb{R}^2$.

Let S be an arbitrary matrix. Denote its columns by v and u. It is straightforward to check that the 11-entry of the matrix $S^T AS$ is $\langle v, v\rangle$; the 12-entry is $\langle v, u\rangle$; the 21-entry is $\langle u, v\rangle$; the 22-entry is $\langle u, u\rangle$.

Thus if the columns of a matrix S are an orthonormal basis v, u of $\mathbb{R}^2$ with respect to the inner product $\langle -, -\rangle$ given above we have $S^T \cdot A \cdot S = \mathrm{I}_2$.

To get such a basis we apply first the Gram-Schmidt process to the standard basis $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ of $\mathbb{R}^2$: We set $w_1 := e_1$, and

$$w_2 = e_2 - \frac{\langle e_2, w_1 \rangle}{\langle w_1, w_1 \rangle} \cdot w_1 = \begin{pmatrix} -\frac{1}{5} \\ 1 \end{pmatrix}.$$

The vectors w_1, w_2 are an orthogonal basis. To get an orthonormal basis we have to normalize them, *i.e.* to divide them by their norm. We get

$$v_1 := \frac{1}{\|w_1\|} \cdot w_1 = \begin{pmatrix} \frac{\sqrt{5}}{5} \\ 0 \end{pmatrix}$$

and

$$v_2 := \frac{1}{\|w_2\|} \cdot w_2 = \begin{pmatrix} -\frac{\sqrt{5}}{10} \\ \frac{\sqrt{5}}{2} \end{pmatrix}.$$

The resulting matrix is

$$S := \begin{pmatrix} \frac{\sqrt{5}}{5} & -\frac{\sqrt{5}}{10} \\ 0 & \frac{\sqrt{5}}{2} \end{pmatrix}.$$