7.1 Hilbert space of identical particles

Gibbs paradox (section 4.5) : divided by $N!$ $\xleftrightarrow{\text{contradict}}$ CM implicitly treat particles as distinct

QM: the identity of particles appears at the level of allowed states in Hilbert space.

e.g., prob. for finding two identical particles at $\vec{x}_1$ and $\vec{x}_2$

$$|\Psi(\vec{x}_1,\vec{x}_2)|^2 = |\Psi(\vec{x}_2,\vec{x}_1)|^2 \Rightarrow \begin{cases} |\psi(1,2)\rangle = +|\psi(2,1)\rangle \\ |\psi(1,2)\rangle = -|\psi(2,1)\rangle \end{cases}$$

Hilbert space to describe identical particles : obey certain symmerties.

* exchange operator P_{12} : $P_{12}^2 = I$ (identity)

$P_{12}^2 = \Sigma\Lambda\Sigma^+\Sigma\Lambda\Sigma^+ = \Sigma\Lambda^2\Sigma^+ = I$: squared eigenvalues are unity

$\Rightarrow e^{i\phi}$ only if $\phi = 0$ or π

$\Rightarrow$ circumvented by anyons described by multivalued wave funcs with fractional statistics

N identical particles system

$N!$ permutations of P, forming a group S_N

E.g., $P(1\ 2\ 3\ 4) = (3\ 2\ 4\ 1)$

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

* Any permutation $\Rightarrow$ a sequence of two particle exchanges

Parity of a permutation

$$(-1)^P \equiv \begin{cases} +1 & \text{if } P \text{ invovles an even number of changes, } (1\,2\,3) \to (2\,3\,1) \\ -1 & \text{if } P \text{ invovles an odd number of changes, } (1\,2\,3) \to (2\,1\,3) \end{cases}$$

* connect the initial and final locations $\Rightarrow$ each intersection contributes (-1)

Assume

(1) single-value wave func

(2) equal prob. under exchange

(3) restrict the rep. sym. or anti-sym.

permutation on an N-particle quantum state

$\Rightarrow$ representation of the permutation group in Hilbert space

Two types of identical particles

(1) Bosons (fully sym. rep.)

$$P|\psi(1,\cdots,N)\rangle = |\psi(1,\cdots,N)\rangle$$

(2) Fermions (fully anti-sym. rep.)

$$P|\psi(1,\cdots,N)\rangle = (-1)^P|\psi(1,\cdots,N)\rangle$$

Hamiltonian for identical particles must be sym. $P\hat{\mathcal{H}} = \hat{\mathcal{H}}$

$\Downarrow$

Eigenstates in a given $\hat{\mathcal{H}}$ with different sym. under permutations

QM the statistic of particles is specified independently to select correct sets of eigenstates (Bosons or Fermions)

For example, N non-interacting particles in a box of box with volume V

$$\hat{\mathcal{H}} = \sum_{\alpha=1}^{N} \hat{\mathcal{H}}_\alpha = \sum_{\alpha=1}^{N}\left(-\frac{\hbar^2}{2m}\nabla_\alpha^2\right) \xrightarrow[\text{diagonalized}]{\text{separately}} \begin{cases} \text{plane wave states } \{|\vec{k}\rangle\} \\ \text{energy } \varepsilon_k = \dfrac{\hbar^2 k^2}{2m} \end{cases}$$

(1) Product Hilbert space (simple multiplication of one-body states)

$$|\vec{k}_1, \cdots, \vec{k}_N\rangle_\otimes \equiv |\vec{k}_1\rangle \cdots |\vec{k}_N\rangle$$

$$\langle \vec{x}_1, \cdots, \vec{x}_N | \vec{k}_1, \cdots, \vec{k}_N\rangle_\otimes \equiv \frac{1}{V^{N/2}} \exp\left(i \sum_{\alpha=1}^{N} \vec{k}_\alpha \cdot \vec{x}_\alpha\right) \qquad \text{(coordinate rep.)}$$

$$\mathcal{H}|\vec{k}_1, \cdots, \vec{k}_N\rangle_\otimes = \left(\sum_{\alpha=1}^{N} \frac{\hbar^2}{2m} k_\alpha^2\right)|\vec{k}_1, \cdots, \vec{k}_N\rangle_\otimes$$

$\Rightarrow$ form appropriate subspaces of sym.

(2) Fermionic subspace

change the configuration

$$|\vec{k}_1, \cdots, \vec{k}_N\rangle_- = \frac{1}{\sqrt{N_-}} \sum_P (-1)^P P |\vec{k}_1, \cdots, \vec{k}_N\rangle_\otimes$$

E.g., two-particle anti-sym.

$$|\vec{k}_1, \vec{k}_2\rangle_- = \frac{(|\vec{k}_1, \vec{k}_2\rangle - |\vec{k}_2, \vec{k}_1\rangle)}{\sqrt{2}}$$

* no duplicated one-particle label $\vec{k}$:

$$P|\vec{k}_1, \cdots, \vec{k}_m, \vec{k}_m, \cdots, \vec{k}_N\rangle = |\vec{k}_1, \cdots, \vec{k}_m, \vec{k}_m, \cdots, \vec{k}_N\rangle \quad (\text{exchange two } \vec{k}_m)$$
$$= (-1)|\vec{k}_1, \cdots, \vec{k}_m, \vec{k}_m, \cdots, \vec{k}_N\rangle$$

$$\Rightarrow |\vec{k}_1, \cdots, \vec{k}_m, \vec{k}_m, \cdots, \vec{k}_N\rangle = 0$$

Therefore $N_- = N!$ for normalization

(3) Bosonic subspace

$$|\vec{k}_1, \cdots, \vec{k}_N\rangle_+ = \frac{1}{\sqrt{N_+}} \sum_P P|\vec{k}_1, \cdots, \vec{k}_N\rangle_\otimes$$

* no restrictions on $\vec{k}$, repeat $n_{\vec{k}}$ times with $\sum_{\vec{k}} n_{\vec{k}} = N$
* Normalization $N_+ = N! \prod_{\vec{k}} n_{\vec{k}}$

E.g., 3-particle bosonic state with 2 1-particle states $|\alpha\rangle$, 1 1-particle state $|\beta\rangle$ ($n_\alpha = 2$, $n_\beta = 1$, $N_+ = 3!2!1!$)

$$|\alpha\alpha\beta\rangle_+ = \frac{|\overset{1}{\alpha}\rangle|\overset{2}{\alpha}\rangle|\beta\rangle + |\overset{1}{\alpha}\rangle|\beta\rangle|\overset{2}{\alpha}\rangle + |\beta\rangle|\overset{1}{\alpha}\rangle|\overset{2}{\alpha}\rangle + |\overset{2}{\alpha}\rangle|\overset{1}{\alpha}\rangle|\beta\rangle + |\beta\rangle|\overset{2}{\alpha}\rangle|\overset{1}{\alpha}\rangle + |\overset{2}{\alpha}\rangle|\beta\rangle|\overset{1}{\alpha}\rangle}{\sqrt{12}}$$

$$= \frac{1}{\sqrt{3}}\left(|\alpha\rangle|\alpha\rangle|\beta\rangle + |\alpha\rangle|\beta\rangle|\alpha\rangle + |\beta\rangle|\alpha\rangle|\alpha\rangle\right)$$

Boson / Fermion Summary

$$|\{\vec{k}\}\rangle_\eta = \frac{1}{\sqrt{N_\eta}} \sum_P \eta^P P |\{\vec{k}\}\rangle, \quad \text{with } \eta = \begin{cases} +1 & \text{for bosons} \\ -1 & \text{for fermions} \end{cases}$$

occupation numbers $\{n_{\vec{k}}\}$, $\sum_{\vec{k}} n_{\vec{k}} = N$

(1) Fermions $|\{\vec{k}\}\rangle_- = 0$ unless $n_{\vec{k}} = 0$ or 1, $N_- = N! \prod_{\vec{k}} n_{\vec{k}}! = N!$

(2) Bosons any $\vec{k}$ may be repeated $n_{\vec{k}}$ times

$${}_+\langle\{\vec{k}\}|\{\vec{k}\}\rangle_+ = \frac{1}{N_+}\sum_{P,P'}\langle P\{\vec{k}\}|P'\{\vec{k}\}\rangle = \frac{N!}{N_+}\sum_P \langle\{\vec{k}\}|P\{\vec{k}\}\rangle = \frac{N! \prod_{\vec{k}} n_{\vec{k}}!}{N_+} = 1$$

* $\langle\{\vec{k}\}|P\{\vec{k}\}\rangle = 0$ unless $\{\vec{k}\} = P\{\vec{k}\}$

$$\Rightarrow N_+ = N! \prod_k n_{\vec{k}}!$$

7.2 Canonical formulation

Canonical density matrix for non-interacting identical particles

$$\langle\{\vec{x}'\}|\rho|\{\vec{x}\}\rangle_\eta = \sum_{\{\vec{k}\}}' \sum_{P,P'} \eta^P \eta^{P'} \langle\{\vec{x}'\}|P'\{\vec{k}\}\rangle \rho(\{\vec{k}\}) \langle P\{\vec{k}\}|\{\vec{x}\}\rangle \frac{1}{N_\eta}$$

where $\rho(\{\vec{k}\}) = \frac{1}{Z_N}\exp\left[-\beta\left(\sum_{\alpha=1}^{N}\frac{\hbar^2 k_\alpha^2}{2m}\right)\right]$

* $\sum'_{\{\vec{k}_1,\cdots,\vec{k}_N\}}$ ensure each identical particle state appears once and only once.

* For fermions, $(-1)^P$ factors cancel out $n_{\vec{k}}$ larger than 1
* For bosons, divide by the resulting over-counting factor of $N!/(\prod_{\vec{k}} n_{\vec{k}}!)$

$$\Rightarrow \sum_{\{\vec{k}\}}' = \sum_{\{\vec{k}\}} \frac{\prod_{\vec{k}} n_{\vec{k}}!}{N!}$$

$$\langle\{\vec{x}'\}|\rho|\{\vec{x}\}\rangle_\eta = \sum_{\{\vec{k}\}} \frac{\prod_{\vec{k}} n_{\vec{k}}!}{N!}\cdot\frac{1}{N!\prod_{\vec{k}} n_{\vec{k}}!}\cdot\sum_{P,P'}\frac{\eta^P\eta^{P'}}{Z_N}\exp\left[-\beta\left(\sum_{\alpha=1}^{N}\frac{\hbar^2k_\alpha^2}{2m}\right)\right]\cdot \langle\{\vec{x}'\}|P'\{\vec{k}\}\rangle\langle P\{\vec{k}\}|\{\vec{x}\}\rangle$$

$dN = \frac{V}{(2\pi)^3}d^3\vec{k} = \rho\, d^3\vec{k}$

$\lim_{V\to\infty}\sum_{\vec{k}} f(\vec{k}) = \int d^3\vec{k}\,\rho f(\vec{k})$

{Section 6.2}

$$\longrightarrow \; \underline{\underline{V\to\infty}} \; \frac{1}{Z_N(N!)^2}\sum_{P,P'}\eta^P\eta^{P'}\int\prod_{\alpha=1}^{N}\frac{V\,d^3\vec{k}_\alpha}{(2\pi)^3}\times\exp\left[-\beta\left(\sum_{\alpha=1}^{N}\frac{\hbar^2k_\alpha^2}{2m}\right)\right] \times \underbrace{\left\{\frac{\exp\left[-i\sum_{\alpha=1}^{N}(\vec{k}_{P\alpha}\cdot\vec{x}_\alpha - \vec{k}_{P'\alpha}\cdot\vec{x}'_\alpha)\right]}{V^N}\right\}}$$

Since $\sum_\alpha f(P\alpha)g(\alpha) = \sum_\beta f(\beta)g(P^{-1}\beta)$

$$= \frac{1}{Z_N(N!)^2}\sum_{P,P'}\eta^P\eta^{P'}\prod_{\alpha=1}^{N}\left[\int\frac{d^3\vec{k}_\alpha}{(2\pi)^3}e^{-i\vec{k}_\alpha\cdot(\vec{x}_{P^{-1}\alpha}-\vec{x}'_{P'^{-1}\alpha})-\beta\hbar^2k_\alpha^2/2m}\right]$$

* Based on Gaussian Integrals in the square brackets

$$\int\frac{dk_{\alpha,x}dk_{\alpha,y}dk_{\alpha,z}}{(2\pi)^3}\exp\left\{\sum_{i=x,y,z}\left[-ik_{\alpha,i}\hat{i}\cdot(\vec{x}_{P^{-1}\alpha}-\vec{x}'_{P'^{-1}\alpha})-\beta\hbar^2k_{\alpha,i}^2/2m\right]\right\}$$

$$=\int\frac{dk_{\alpha,x}dk_{\alpha,y}dk_{\alpha,z}}{(2\pi)^3}\exp\left\{-\frac{\beta\hbar^2}{2m}\sum_{i=x,y,z}\left[k_{\alpha,i}^2+\frac{i2m}{\beta\hbar^2}(\vec{x}_{P^{-1}\alpha,i}-\vec{x}'_{P'^{-1}\alpha,i})k_{\alpha,i}\right]\right\}$$

$$=\underbrace{\int_{-\infty}^{+\infty}\frac{dk_{\alpha,x}dk_{\alpha,y}dk_{\alpha,z}}{(2\pi)^3}\exp\left\{-\frac{\beta\hbar^2}{2m}\sum_{i=x,y,z}\left[k_{\alpha,i}+\frac{im}{\beta\hbar^2}(\vec{x}_{P^{-1}\alpha,i}-\vec{x}'_{P'^{-1}\alpha,i})\right]^2\right\}}$$

$$\times\underbrace{\exp\left\{-\frac{m}{2\beta\hbar^2}\sum_{i=x,y,z}(\vec{x}_{P^{-1}\alpha,i}-\vec{x}'_{P'^{-1}\alpha,i})^2\right\}}$$

$$=\underbrace{\frac{1}{\lambda^3}}\,\underbrace{\exp\left[-\frac{\pi}{\lambda^2}(\vec{x}_{P^{-1}\alpha}-\vec{x}_{P'^{-1}\alpha})^2\right]}$$

Set $\beta = P^{-1}\alpha$

$$\langle\{\vec{x}'\}|\rho|\{\vec{x}\}\rangle_\eta = \frac{1}{Z_N(N!)^2}\sum_{P,P'}\eta^P\eta^{P'}\frac{1}{\lambda^{3N}}\exp\left[-\frac{\pi}{\lambda^2}\sum_{\beta=1}^{N}(\vec{x}_\beta-\vec{x}'_{P'^{-1}P\beta})^2\right]$$

Set $Q = P'^{-1}P$ and

$\eta^P = \eta^{P^{-1}}$; $\eta^Q = \eta^{P'^{-1}P} = \eta^{P'}\cdot\eta^P$

$\sum_P = N!$

$$\langle\{\vec{x}'\}|\rho|\{\vec{x}\}\rangle_\eta = \frac{1}{Z_N\lambda^{3N}N!}\sum_Q\eta^Q\exp\left[-\frac{\pi}{\lambda^2}\sum_{\beta=1}^{N}(\vec{x}_\beta-\vec{x}'_{Q\beta})^2\right]$$

Canonical partition func Z_N (from normalization)

$\Rightarrow \mathrm{tr}(\rho)=1 \quad \Rightarrow \quad \int \prod_{\alpha=1}^{N} d^3\vec{x}_\alpha \langle \{\vec{x}\}|\rho|\{\vec{x}\}\rangle_\eta = 1$

Classical Result

$Z_N = (V/\lambda^3)^N/N!$

(no particle exchange $Q\equiv 1$)

$$Z_N = \frac{1}{N!\lambda^{3N}}\int \prod_{\alpha=1}^{N} d^3\vec{x}_\alpha \sum_Q \eta^Q \exp\left[-\frac{\pi}{\lambda^2}\sum_{\beta=1}^{N}(\vec{x}_\beta - \vec{x}'_{Q\beta})^2\right]$$

* The division by $N!$ justifies the factor applied to CM result for identical particles
* Classical result is only valid at high temperature $T\to\infty$, $\lambda\to 0$, factor $\exp[-\pi(\vec{x}_1-\vec{x}_2)^2/\lambda^2] \to 0$

The lowest order correction: exchange of two particles

(1) exchange of particle 1 and 2 : a factor of $\eta\exp[-2\pi(\vec{x}_1-\vec{x}_2)^2/\lambda^2]$

* $(x_1-x_2)^2$ and $(x_2-x_1)^2$

(2) possible $N(N-1)/2$ pairwise exchanges

$$\Rightarrow Z_N = \frac{1}{N!\lambda^{3N}}\int \prod_{\alpha=1}^{N} d^3\vec{x}_\alpha \left\{1 + \frac{N(N-1)}{2}\eta\exp\left[-\frac{2\pi}{\lambda^2}(\vec{x}_1-\vec{x}_2)^2\right]+\cdots\right\}$$

for $\alpha\geq 3$, $\int d^3\vec{x}_\alpha = V$

$$= \frac{1}{N!\lambda^{3N}}\left\{V^N + V^{N-2}\int d^3\vec{x}_1 d^3\vec{x}_2 \frac{N(N-1)}{2}\eta\exp\left[-\frac{2\pi}{\lambda^2}(\vec{x}_1-\vec{x}_2)^2\right]+\cdots\right\}$$

apply $\begin{cases}\vec{r}_{12} = \vec{x}_2 - \vec{x}_1 \\ \vec{R} = \frac{1}{2}(\vec{x}_2+\vec{x}_1)\end{cases}$

$$= \frac{1}{N!\lambda^{3N}}\left\{V^N + V^{N-1}\int d^3\vec{r}_{12}\frac{N(N-1)}{2}\eta\exp\left[-\frac{2\pi}{\lambda^2}\vec{r}_{12}^{\,2}\right]+\cdots\right\}$$

$$= \frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N\left[1+\frac{N(N-1)}{2V}\eta\left(\frac{\lambda^2}{2}\right)^{3/2}+\cdots\right]$$

$$F = -k_BT\ln Z_N = -k_BT\Big[\underbrace{-N\ln N + N}_{-\ln N!} + N\ln\left(\frac{V}{\lambda^3}\right) + \underbrace{\frac{N^2}{2V}\eta\frac{\lambda^3}{2^{3/2}}}_{\ln(1+t)\sim t \ (t\to 0)} + \cdots\Big]$$

$$= -Nk_BT\ln\left[\frac{e}{\lambda^3}\cdot\frac{V}{N}\right] - \frac{k_BTN^2}{2V}\cdot\frac{\lambda^3}{2^{3/2}}\eta + \cdots$$

$$P = -\frac{\partial F}{\partial V}\Big|_T = \frac{Nk_BT}{V} - \frac{N^2k_BT}{V^2}\cdot\frac{\lambda^3}{2^{5/2}}\eta + \cdots = nk_BT\left[1 - \frac{\eta\lambda^3}{2^{5/2}}n + \cdots\right]$$

First quantum correction $\equiv$ Second virial coefficient

$$B_2 = -\frac{\eta\lambda^3}{2^{5/2}} \begin{cases} + & \text{Fermions} \\ - & \text{Bosons}\end{cases}$$

Classical formulation section 5.1 ; 5.2

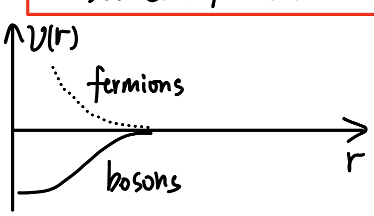
Two-body interaction $\Rightarrow$ a second virial coefficient

$$\ln Z = \ln Z_0 + \frac{N(N-1)}{2V}\int d^3\vec{q}\left[\exp(-\beta\mathcal{V}(\vec{q}))-1\right] + \mathcal{O}\left(\frac{N^3}{V^2}\right)$$

$$\ln Z_N = \ln Z_0 + \ln\left[1+\frac{N(N-1)}{2V}\eta\int d^3\vec{r}_{12}\exp\left(-\frac{2\pi}{\lambda^2}\vec{r}_{12}^{\,2}\right)+\cdots\right]$$

$$\sim \ln Z_0 + \frac{N(N-1)}{2V}\int d^3\vec{r}_{12}\,\eta\exp\left(-\frac{2\pi}{\lambda^2}\vec{r}_{12}^{\,2}\right)$$

$\boxed{\text{Quantum stats. at high } T \equiv \text{Introducing an interaction between particles}}$



$$f(\vec{r}) = e^{-\beta\mathcal{V}(\vec{r})} - 1 = \eta\exp\left(-\frac{2\pi}{\lambda^2}\vec{r}^{\,2}\right)$$

$$\mathcal{V}(\vec{r}) = -k_BT\ln\left[1+\eta\exp(-2\pi\vec{r}^{\,2}/\lambda^2)\right] \xrightarrow{T\to+\infty} -k_BT\eta\exp\left(-\frac{2\pi\vec{r}^{\,2}}{\lambda^2}\right) \begin{cases} + & \text{(repulsive) Fermion} \\ - & \text{(attractive) Boson}\end{cases}$$

7.3 Grand canonical formulation

$$Z_N = \frac{1}{N!\lambda^{3N}} \int \prod_{\alpha=1}^{N} d^3\vec{x}_\alpha \sum_Q \eta^Q \exp\left[-\frac{\pi}{\lambda^2}\sum_{\beta=1}^{N}(\vec{x}_\beta - \vec{x}_{Q\beta})^2\right]$$ $\Rightarrow$ all the sums is a formidable task

Z_N in the energy basis

$$Z_N = tr(e^{-\beta\mathcal{H}}) = \sum_{\{\vec{k}_\alpha\}}' \exp\left[-\beta\sum_{\alpha=1}^{N}\varepsilon(\vec{k}_\alpha)\right] = \sum_{\{n_{\vec{k}}\}}' \exp\left[-\beta\sum_{\vec{k}}\varepsilon(\vec{k})n(\vec{k})\right]$$

Restrictions of symmetry on the allowed values of $\vec{k}$ or $\{n_{\vec{k}}\}$

occupation number $\{n_{\vec{k}}\}$ $\sum_{\vec{k}} n_{\vec{k}} = N$ $\Big\langle$ $n_{\vec{k}} = 0, 1, 2, \cdots$ bosons; $n_{\vec{k}} = 0, 1$ fermions

Remove the First constraint through grand partition funcs

$$Q_\eta(T,\mu) = \sum_{N=0}^{\infty} e^{\beta\mu N} \sum_{\{n_{\vec{k}}\}}' \exp\left[-\beta\sum_{\vec{k}}\varepsilon(\vec{k})n_{\vec{k}}\right]$$

$$= \sum_{N=0}^{\infty}\prod_{\{n_{\vec{k}}\}}' e^{\beta\mu n_{\vec{k}}} \sum_{\{n_{\vec{k}}\}}' \prod_{\vec{k}} \exp\left[-\beta\varepsilon(\vec{k})n_{\vec{k}}\right]$$

$$= \sum_{\{n_{\vec{k}}\}}^{\eta} \prod_{\vec{k}} \exp\left[-\beta(\varepsilon(\vec{k})-\mu)n_{\vec{k}}\right]$$

w.r.t. η

$$\sum_{N=0}^{\infty}\prod_{\{n_{\vec{k}}\}}'\sum_{\{n_{\vec{k}}\}}' = \sum_{\{n_{\vec{k}}\}}$$

$$\sum_{\{n_{\vec{k}}\}}\prod_{\vec{k}} = \prod_{\vec{k}}\sum_{n_{\vec{k}}}$$

* sums over $\{n_{\vec{k}}\}$ can be calculated independently for each $\vec{k}$ subject to restrictions on occupation numbers imposed by particle symmetry.

fermions $n_k = 0$ or 1

$$Q_- = \prod_{\vec{k}}\left\{1+\exp\left[\beta\mu-\beta\varepsilon(\vec{k})\right]\right\}$$

bosons $n_k = 0, 1, 2, \cdots\cdots$

$$Q_+ = \prod_{\vec{k}}\left\{1-\exp\left[\beta\mu-\beta\varepsilon(\vec{k})\right]\right\}^{-1}$$

$$\ln Q_\eta = -\eta\sum_{\vec{k}}\ln\left\{1-\eta\exp\left[\beta\mu-\beta\varepsilon(\vec{k})\right]\right\}$$

(fermions $\eta=-1$; bosons $\eta=+1$)

Different 1-particle states are occupied independently

$$P_\eta(\{n(\vec{k})\}) = \frac{1}{Q_\eta}\prod_{\vec{k}}\exp\left[-\beta(\varepsilon(\vec{k})-\mu)n_{\vec{k}}\right]$$ joint prob. from $Q_\eta(T,\mu)$

Average occupation number of a state of energy $\varepsilon(\vec{k})$

$$\boxed{\langle n_{\vec{k}}\rangle_\eta = -\frac{\partial\ln Q_\eta}{\partial\beta\varepsilon(\vec{k})} = -\frac{+\eta\exp[\beta\mu-\beta\varepsilon(\vec{k})]}{-\eta+\eta^2\exp[\beta\mu-\beta\varepsilon(\vec{k})]} = \frac{1}{z^{-1}\exp(\beta\varepsilon(\vec{k}))-\eta}}$$

where $\boxed{z = \exp(\beta\mu)}$

Average values of the particle number and internal energy

$$N_\eta = \sum_{\vec{k}}\langle n_{\vec{k}}\rangle_\eta = \sum_{\vec{k}}\frac{1}{z^{-1}\exp(\beta\varepsilon(\vec{k}))-\eta}$$

$$E_\eta = \sum_{\vec{k}}\varepsilon(\vec{k})\langle n_{\vec{k}}\rangle_\eta = \sum_{\vec{k}}\frac{\varepsilon(\vec{k})}{z^{-1}\exp(\beta\varepsilon(\vec{k}))-\eta}$$

7.4 Non-relativistic gas

QM are further characterized by spin s. In the absence of the magnetic field, spin degeneracy factor $g = 2s+1$ as multipliers

A non-relativistic gas in 3d w.r.t $\mathcal{E}(\vec{k}) = \frac{\hbar^2 k^2}{2m}$

$$\sum_{\vec{k}} \to V \int d^3\vec{k} \frac{1}{(2\pi)^3}$$

$$\begin{cases} \beta P_\eta = \dfrac{\ln Q_\eta}{V} = -\eta g \displaystyle\int \frac{d^3\vec{k}}{(2\pi)^3} \ln\left[1 - \eta z \exp\left(-\frac{\beta \hbar^2 k^2}{2m}\right)\right] \\ n_\eta \equiv \dfrac{N_\eta}{V} = g \displaystyle\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{z^{-1}\exp(\beta\hbar^2k^2/2m) - \eta} \\ \mathcal{E}_\eta \equiv \dfrac{E_\eta}{V} = g \displaystyle\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} \frac{1}{z^{-1}\exp(\beta\hbar^2k^2/2m) - \eta} \end{cases}$$

Set $x = \beta\hbar^2 k^2/2m$

$$k = \frac{\sqrt{2mk_BT}}{\hbar} x^{1/2} = \frac{2\pi^{1/2}}{\lambda} x^{1/2} \implies dk = \frac{\pi^{1/2}}{\lambda} x^{-1/2} dx$$

$$\beta P_\eta = -\eta g \int \frac{4\pi}{8\pi^3} \left(\frac{2\pi^{1/2}}{\lambda} x^{1/2}\right)^2 \frac{\pi^{1/2}}{\lambda} x^{-1/2} dx \ \ln\left[1 - \eta z e^{-x}\right]$$

$$= -\eta g \frac{2}{\lambda^3\sqrt{\pi}} \int_0^\infty dx\, x^{1/2} \ln\left[1 - \eta z e^{-x}\right]$$

$$= -\eta g \frac{4}{3\lambda^3\sqrt{\pi}} \left[\cancel{x^{3/2}\ln(1-\eta z e^{-x})\Big|_0^\infty}^{\,0} - \int_0^\infty \frac{dx\, x^{3/2} \cdot (-\eta z e^{-x})(-1)}{1 - \eta z e^{-x}} \right]$$

$$= \frac{g}{\lambda^3} \cdot \frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{dx\, x^{3/2} \cdot \eta^2}{z^{-1}e^x - \eta} = \frac{g}{\lambda^3} \cdot \frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{dx\, x^{3/2}}{z^{-1}e^x - \eta}$$

$$n_\eta = \frac{g}{\lambda^3} \cdot \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{dx\, x^{1/2}}{ze^x - \eta}$$

$$\beta\mathcal{E}_\eta = \frac{g}{\lambda^3} \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{dx\, x^{3/2}}{ze^x - \eta}$$

integer non-integer

$$m! = \Gamma(m+1) = \int_0^\infty dx\, x^m e^{-x}$$

e.g., $\Gamma(\frac{3}{2}) = (\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$

$\Gamma(\frac{5}{2}) = (\frac{3}{2})! = (\frac{3}{2})\frac{\sqrt{\pi}}{2}$

Euler's reflection eqn.

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

Legendre duplication formula.

$$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$$

Define funcs $f_m^\eta(z) = \dfrac{1}{(m-1)!} \displaystyle\int_0^\infty \frac{dx\, x^{m-1}}{z^{-1}e^x - \eta}$

$$\Rightarrow \begin{cases} \beta P_\eta = \dfrac{g}{\lambda^3} f_{5/2}^\eta(z) \\ n_\eta = \dfrac{g}{\lambda^3} f_{3/2}^\eta(z) \\ \mathcal{E}_\eta = \dfrac{3}{2} P_\eta \end{cases}$$

thermodynamics of ideal quantum gases as a func of z

Solving for z in terms of density $\Rightarrow$ eqn. of state in $P_\eta(n_\eta, T)$

Behavior of the funcs $f_m^\eta(z)$

① high-Temperature, low-density (non-degenerate) limit, small z

$$f_m^\eta(z) = \frac{1}{(m-1)!}\int_0^\infty \frac{dx\, x^{m-1}}{z^{-1}e^x-\eta} = \frac{1}{(m-1)!}\int_0^\infty dx\, x^{m-1}(ze^{-x})(1-\eta z e^{-x})^{-1}$$

$$= \frac{1}{(m-1)!}\int_0^\infty dx\, x^{m-1}\sum_{\alpha=1}^{\infty}(ze^{-x})^\alpha \eta^{\alpha-1} \qquad (\eta^{\alpha-1}=\eta^{\alpha+1})$$

integer m

$$\int_0^\infty dx\, x^{m-1}e^{-\alpha x} = \frac{1}{\alpha^m}\int_0^\infty d(\alpha x)\,(\alpha x)^{m-1}e^{-\alpha x} = \frac{1}{\alpha^m}\cdot(m-1)!$$

$$= \sum_{\alpha=1}^\infty \eta^{\alpha-1} z^\alpha \frac{1}{(m-1)!}\int_0^\infty dx\, x^{m-1}e^{-\alpha x}$$

$$\overset{\text{integer } m}{=\!=} \sum_{\alpha=1}^\infty \eta^{\alpha-1}\frac{z^\alpha}{\alpha^m} \qquad (\eta=\pm 1)$$

$$= z + \eta\frac{z^2}{2^m} + \frac{z^3}{3^m} + \eta\frac{z^4}{4^m} + \cdots\cdots$$

$f_m^\eta(z)$, $n_\eta(z)$ and $P_\eta(z)$ are small as $z\to 0$

$$\begin{cases} \dfrac{n_\eta\lambda^3}{g} = f_{3/2}^\eta(z) = z + \eta\dfrac{z^2}{2^{3/2}} + \dfrac{z^3}{3^{3/2}} + \eta\dfrac{z^4}{4^{3/2}} + \cdots\cdots \\ \dfrac{\beta P_\eta\lambda^3}{g} = f_{5/2}^\eta(z) = z + \eta\dfrac{z^2}{2^{5/2}} + \dfrac{z^3}{3^{5/2}} + \eta\dfrac{z^4}{4^{5/2}} + \cdots\cdots \end{cases}$$

Solved perturbatively, by the recursive procedure of substituting the solution up to a lower order

$$z = \frac{n_\eta\lambda^3}{g} - \eta\frac{z^2}{2^{3/2}} - \frac{z^3}{3^{3/2}} - \cdots\cdots$$

$$= \left(\frac{n_\eta\lambda^3}{g}\right) - \frac{\eta}{2^{3/2}}\left(\frac{n_\eta\lambda^3}{g}\right)^2 - \cdots\cdots$$

$$= \left(\frac{n_\eta\lambda^3}{g}\right) - \frac{\eta}{2^{3/2}}\left[\left(\frac{n_\eta\lambda^3}{g}\right) - \frac{\eta}{2^{3/2}}\left(\frac{n_\eta\lambda^3}{g}\right)^2\right]^2$$

$$- \frac{1}{3^{3/2}}\left[\left(\frac{n_\eta\lambda^3}{g}\right) - \frac{\eta}{2^{3/2}}\left(\frac{n_\eta\lambda^3}{g}\right)^2\right]^3 - \cdots\cdots$$

$$= \left(\frac{n_\eta\lambda^3}{g}\right) - \frac{\eta}{2^{3/2}}\left(\frac{n_\eta\lambda^3}{g}\right)^2 + 2\frac{\eta^2}{2^3}\left(\frac{n_\eta\lambda^3}{g}\right)^3 \qquad [\text{2nd-order term}]$$

$$- \frac{1}{3^{3/2}}\left(\frac{n_\eta\lambda^3}{g}\right)^3 - \cdots\cdots \qquad [\text{3rd-order term}]$$

$$= \left(\frac{n_\eta\lambda^3}{g}\right) - \frac{\eta}{2^{3/2}}\left(\frac{n_\eta\lambda^3}{g}\right)^2 + \left(\frac{1}{4} - \frac{1}{3^{3/2}}\right)\left(\frac{n_\eta\lambda^3}{g}\right)^3 - \cdots\cdots$$

from $\dfrac{\beta P_\eta \lambda^3}{g} = f^\eta_{5/2}(z) = z + \eta \dfrac{z^2}{2^{5/2}} + \dfrac{z^3}{3^{5/2}} + \eta \dfrac{z^4}{4^{5/2}} + \cdots\cdots$ to $P_\eta(n_\eta, T)$

$$\Rightarrow \frac{\beta P_\eta \lambda^3}{g} = \left(\frac{n_\eta \lambda^3}{g}\right) - \frac{\eta}{2^{3/2}}\left(\frac{n_\eta \lambda^3}{g}\right)^2 + \left(\frac{1}{4} - \frac{1}{3^{3/2}}\right)\left(\frac{n_\eta \lambda^3}{g}\right)^3 \qquad [z]$$

$$+ \frac{\eta}{2^{5/2}}\left(\frac{n_\eta \lambda^3}{g}\right)^2 - \frac{\eta}{2^{5/2}} \cdot 2 \cdot \frac{\eta}{2^{3/2}}\left(\frac{n_\eta \lambda^3}{g}\right)^3 \qquad \left[\eta \frac{z^2}{2^{5/2}}\right]$$

$$+ \frac{1}{3^{5/2}}\left(\frac{n_\eta \lambda^3}{g}\right)^3 + \cdots\cdots \qquad \left[\frac{z^3}{3^{5/2}}\right]$$

$$= \left(\frac{n_\eta \lambda^3}{g}\right) - \frac{\eta}{2^{3/2}}\left(\frac{n_\eta \lambda^3}{g}\right)^2 + \left(\frac{1}{4} - \frac{1}{3^{3/2}}\right)\left(\frac{n_\eta \lambda^3}{g}\right)^3$$

$$+ \frac{\eta}{2^{5/2}}\left(\frac{n_\eta \lambda^3}{g}\right)^2 - \frac{1}{8}\left(\frac{n_\eta \lambda^3}{g}\right)^3 + \frac{1}{3^{5/2}}\left(\frac{n_\eta \lambda^3}{g}\right)^3 + \cdots\cdots$$

similar to virial expansion $\quad P_\eta = n_\eta k_B T\left[1 - \dfrac{\eta}{2^{5/2}}\left(\dfrac{n_\eta \lambda^3}{g}\right) + \left(\dfrac{1}{8} - \dfrac{2}{3^{5/2}}\right)\left(\dfrac{n_\eta \lambda^3}{g}\right)^2 + \cdots\cdots\right]$

Second Virial coefficient $\quad B_2 = -\dfrac{\eta \lambda^3}{2^{5/2} g}$ agree with canonical formulation (Section 7.2) for $g=1$

Natural (dimensionless) expansion parameter $\quad n_\eta \dfrac{\lambda^3}{g}$ (QM effects when $n_\eta \lambda^3 \gtrsim g$)

Quantum degenerate limit

* Fermi and Boson gases are quite different in degenerate limit of low temperature and high densities

7.5 The degenerate fermi gas

fermi occupation number (at zero temperature)

$$\langle n_{\vec{k}}\rangle_- = \begin{cases} \dfrac{1}{e^{\beta(\mathcal{E}(k)-\mu)}+1} & \mathcal{E}(\vec{k})<\mu \\ 0 & \text{otherwise} \end{cases}$$

fermi energy $\mathcal{E}_F$ — limiting value of μ at zero temperature

fermi sea — all one-particle states of energy less than $\mathcal{E}_F$ are occupied

fermi wavenumber k_F

$$N = \sum_{|\vec{k}|\le k_F}(2s+1) = gV\int^{k<k_F}\frac{d^3\vec{k}}{(2\pi)^3} = g\frac{V}{6\pi^2}k_F^3$$

for ideal gas with $\mathcal{E}(\vec{k}) = \frac{\hbar^2k^2}{2m}$

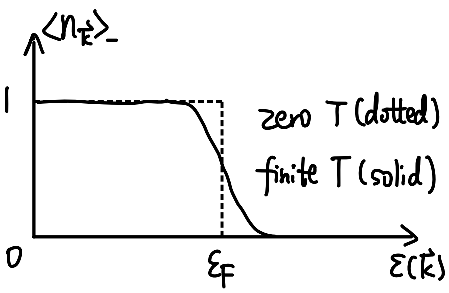
$$\Rightarrow k_F = \left(\frac{6\pi^2 n}{g}\right)^{\frac{1}{3}} \Rightarrow \mathcal{E}_F(n) = \frac{\hbar^2k_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{\frac{2}{3}} \quad \left(n=\frac{N}{V}\right)$$

Classical ideal gas $T=0$ $\Rightarrow$ large density of states (from $\Omega_{classic}\propto V^N/N!$)

QM fermi gas $T=0$ $\Rightarrow$ unique ground state with $\Omega=1$:

one-particle momenta are specified $\Rightarrow$ only one anti-symmetrized state

(all $\vec{k}$ for $|\vec{k}|<k_F$) (Pauli exclusive theorem)



changes abruptly from 1 to 0

derivative is sharply peaked

Fermi sea (at small temperatures): $f_m^-(z)$ for large z

$$f_m^-(z) = \frac{1}{m!}\int_0^\infty dx\, x^m\frac{d}{dx}\left(\frac{-1}{z^{-1}e^x+1}\right) \quad \text{(after integration by parts)}$$

(m is positive non-integer)

Expand around the peak of derivative by $x=\ln z+t$

extending the range of integration to $-\infty<t<+\infty$ $(-\infty<\ln z<+\infty)$

$$f_m^-(z) \simeq \frac{1}{m!}\int_{-\infty}^{+\infty}dt\,(\ln z+t)^m\frac{d}{dt}\left(\frac{-1}{e^t+1}\right)$$

$$= \frac{1}{m!}\int_{-\infty}^{+\infty}dt\sum_{\alpha=0}^{\infty}\binom{m}{\alpha}t^\alpha(\ln z)^{m-\alpha}\frac{d}{dt}\left(\frac{-1}{e^t+1}\right)$$

m is non-integer taylor expansion

$$= \frac{(\ln z)^m}{m!}\sum_{\alpha=0}^{\infty}\frac{m!}{\alpha!(m-\alpha)!}(\ln z)^{-\alpha}\int_{-\infty}^{+\infty}dt\,t^\alpha\frac{d}{dt}\left(\frac{-1}{e^t+1}\right)$$

(Anti-)symmetry of integrand

$$\frac{1}{\alpha!}\int_{-\infty}^{+\infty}dt\,t^\alpha\underbrace{\frac{d}{dt}\left(\frac{-1}{e^t+1}\right)}_{\text{even}} = \begin{cases} 0 & \alpha \text{ odd} \\ \dfrac{2}{(\alpha-1)!}\displaystyle\int_0^\infty dt\,\frac{t^{\alpha-1}}{e^t+1} = 2f_\alpha^-(1) & \alpha \text{ even}\end{cases}$$

undo integral by part.

Dirichlet η (eta)

$$\int_0^\infty\frac{x^n}{e^x+1}dx = n!\cdot\eta(n+1) \Rightarrow f_\alpha^-(1) = \frac{1}{(\alpha-1)!}(\alpha-1)!\cdot\eta(\alpha) = \eta(\alpha)$$

⇒ Sommerfeld expansion

$$\lim_{z\to\infty} f_m^-(z) = \frac{(\ln z)^m}{m!}\sum_{\alpha=0}^{\text{even}} 2\eta(\alpha)\frac{m!}{(m-\alpha)!}(\ln z)^{-\alpha} = \frac{(\ln z)^m}{m!}\,2\left[\underbrace{\frac{1}{2}}_{\text{Abel sum of Grandi's series}} + \frac{\pi^2}{12}\frac{m(m-1)}{(\ln z)^2} + \frac{7\pi^4}{720}\frac{m(m-1)(m-2)(m-3)}{(\ln z)^4} + \cdots\right]$$

$$\boxed{\lim_{z\to\infty} f_m^-(z) = \frac{(\ln z)^m}{m!}\left[1 + \frac{\pi^2}{6}\frac{m(m-1)}{(\ln z)^2} + \frac{7\pi^4}{360}\underbrace{\frac{m(m-1)(m-2)(m-3)}{(\ln z)^4}}_{<0} + \cdots\right]}$$

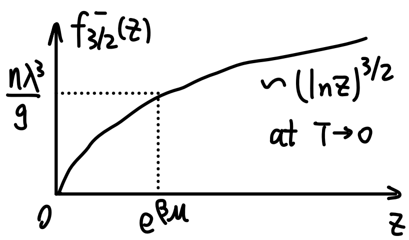
In degenerate limit (high dege.)

$$\boxed{\frac{n\lambda^3}{g} = f_{3/2}^-(z) = \frac{(\ln z)^{3/2}}{(3/2)!}\left[1 + \frac{\pi^2}{6}\frac{3/2\cdot 1/2}{(\ln z)^2} + \cdots\cdots\right] \gg 1}$$

* $\Gamma(\frac{3}{2}) = (\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$
* $\Gamma(\frac{5}{2}) = (\frac{3}{2})! = (\frac{3}{2})\frac{\sqrt{\pi}}{2}$

$\mu = \varepsilon_F(n) \sim n \sim f_{3/2}^-(z)$ as $\underline{\varepsilon_F(n) = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3} \text{ at } T\to 0}$

lowest order results reproduce the expression for ε_F at $T\to 0$



$$\lim_{T\to 0}\ln z = \left[\left(\frac{3}{2}\right)!\, f_{3/2}^-(z)\right]^{2/3} = \left[\frac{3\sqrt{\pi}}{4}\frac{n\lambda^3}{g}\right]^{2/3} = \left[\frac{3\pi^{\frac{1}{2}}}{4}\frac{n\,\hbar^3\cdot(2\pi)^3}{g(2\pi m k_B T)^{3/2}}\right]^{2/3}$$

$$= \frac{\beta\hbar^2}{2m}\left[\frac{3\pi^{\frac{1}{2}}\cdot 8\pi^3 n}{4g\cdot\pi^{3/2}}\right]^{2/3} = \frac{\beta\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3} = \beta\varepsilon_F$$

func $f_{3/2}^-(z)$ determines chemical potential μ

First-order correction on $\ln z$ at zero-temperature limit

$$\lim_{T\to 0}\ln z = \beta\varepsilon_F\left[1 + \frac{\pi^2}{8}(\beta\varepsilon_F)^{-2} + \cdots\right]^{-2/3} = \beta\varepsilon_F\left\{1 + \left(-\frac{2}{3}\right)\left[\frac{\pi^2}{8}(\beta\varepsilon_F)^{-2} + \cdots\right]\right\}$$

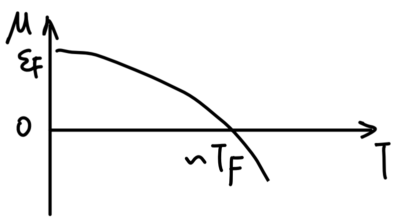
$$= \beta\varepsilon_F\left[1 - \frac{\pi^2}{12}(\beta\varepsilon_F)^{-2} + \cdots\cdots\right]$$

Dimensionless expansion parameter

$$(\beta\varepsilon_F)^{-1} = \frac{k_B T}{\varepsilon_F}$$

Fermion chemical potential

$$\mu = k_B T\ln z \simeq \varepsilon_F\left[1 - \frac{\pi^2}{12}(\beta\varepsilon_F)^{-2} + \cdots\cdots\right]$$



$\triangleq$ derivation of z in section 7.4 non-relativistic gas (High T, low density) ↕ non-degenerate

$\Rightarrow \mu \begin{cases} \mu > 0 & \text{low } T \\ \mu < 0 & \text{high } T \end{cases}$ changes sign at a Temperature of the order of ε_F/k_B

low-temperature expansion for pressure

* $\Gamma(\frac{7}{2}) = (\frac{5}{2})! = \frac{15}{8}\sqrt{\pi}$
* Legendre duplication formula.
* $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$

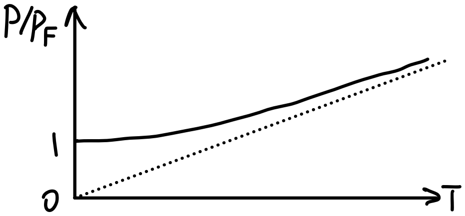
$$\beta P = \frac{g}{\lambda^3} f_{5/2}^-(z) = \frac{g}{\lambda^3}\frac{(\ln z)^{5/2}}{(5/2)!}\left[1 + \frac{\pi^2}{6}\frac{5}{2}\frac{3}{2}(\ln z)^{-2} + \cdots\cdots\right]$$

$$= \frac{g}{\lambda^3}\frac{8(\beta\varepsilon_F)^{5/2}}{15\sqrt{\pi}}\left[1 - \frac{\pi^2}{12}(\beta\varepsilon_F)^{-2} + \cdots\right]^{5/2}\cdot\left[1 + \frac{5\pi^2}{8}(\beta\varepsilon_F)^{-2} + \cdots\cdots\right]$$

$$= n\cdot\left(\lim_{T\to 0}\ln z\right)\frac{(3/2)!}{(5/2)!}\left[1 - \frac{5\pi^2}{24}(\beta\varepsilon_F)^{-2} + \cdots\right]\cdot\left[1 + \frac{5\pi^2}{8}(\beta\varepsilon_F)^{-2} - \cdots\right]$$

$$\boxed{\text{fermi pressure} \quad P_F = \left(\frac{2}{5}\right) n\varepsilon_F}$$

$$\Rightarrow \beta P = \beta P_F\left[1 + \frac{5}{12}\pi^2\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \cdots\cdots\right]$$



Fermi gas at zero T has finite pressure and internal energy

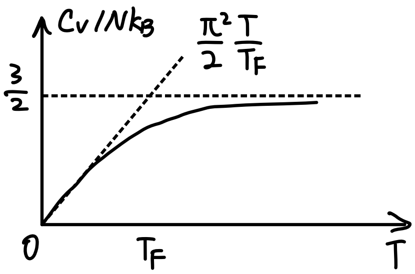
Non-relativistic gas
section 7.4

$$\mathcal{E}_\eta = \frac{E_\eta}{V} = \frac{3}{2}P_\eta$$

low-temperature expansion for internal energy

$$\frac{E}{V} = \frac{3}{2}P = \frac{3}{5}n k_B T_F \left[1 + \frac{5}{12}\pi^2\left(\frac{T}{T_F}\right)^2 + \cdots\right]$$

with fermi temperature $T_F = \varepsilon_F / k_B$



$T \to 0$ slope $= \frac{\pi^2}{2}\frac{T}{T_F}$ for fermi gas in all dimensions

low-temperature heat capacity

$$C_V = \frac{dE}{dT} = \frac{\pi^2}{2} N k_B \left(\frac{T}{T_F}\right) + \mathcal{O}\left(\frac{T}{T_F}\right)^3$$

the prob. of occupying single-particle states $[\exp(\beta\varepsilon - \beta\mu) + 1]^{-1}$ is close to a step func at small temperatures. Only $\varepsilon - \varepsilon_F \approx k_B T$ can be thermally excited.

$\left.\begin{array}{l}\text{Small fraction } T/T_F \text{ of total number of fermions.} \\ \text{Each excited particle gains energies with order of } k_B T\end{array}\right\} \Delta E \sim k_B T \cdot N \frac{T}{T_F}$

$\Rightarrow C_V = dE/dT \sim N k_B T / T_F$ * also valid for interacting fermi gas

Only a small number $N(T/T_F)$ of fermions are excited at small temperature

E.x. magnetic susceptibility
(Promblem set)

CM gas of N non-interacting particles of magnetic moment μ_B

Curie law $\chi \propto \frac{N \mu_B^2}{k_B T}$

QM low-temperature susceptibility saturates to a (Pauli) value

$$\chi \propto \frac{N \mu_B^2}{k_B T_F}$$

7.6 The degenerate bose gas

bose occupation number

$$\langle n_{\vec{k}} \rangle_+ = \frac{1}{\exp[\beta(\varepsilon(\vec{k}) - \mu)] - 1} > 0$$

$\forall \vec{k}$, $\varepsilon(\vec{k}) - \mu > 0 \Rightarrow \mu < \min[\varepsilon(\vec{k})]_{\vec{k}} = 0$ (for $\varepsilon(\vec{k}) = \hbar^2 k^2/2m$)

high temperatures (classical limits)

μ is large and negative and increases towards zero as $k_B T \ln(n\lambda^3/g)$ (first-order of $z = \exp(\beta\mu)$ in section 7.4) as temperature decreases.

QM degenerate limit

$n\lambda^3 \geq g$ (section 7.4), and μ approaches its limiting value of zero
$\Rightarrow$ limiting behavior of $f_m^+(z)$ (section 7.4) as $z = \exp(\beta\mu) \to 1$

$f_m^+(z)$ are monotonically increasing with z in $0 \leq z \leq 1$

Max value attained at $z=1$

$$\zeta_m \equiv f_m^+(1) = \frac{1}{(m-1)!}\int_0^\infty \frac{dx\, x^{m-1}}{e^x - 1} = \frac{1}{(m-1)!}(m-1)! \cdot \zeta(m) = \zeta(m)$$

(Riemann ζ (zeta) : Zeta$[\frac{3}{2}] \approx 2.61238$)

* The integrand has a pole as $x \to 0$, where it behaves as $\int dx\, x^{m-2}$
$\Rightarrow \zeta_m$ is finite for $m > 1$ and infinite for $m \leq 1$

Recursive property ($m > 1$)

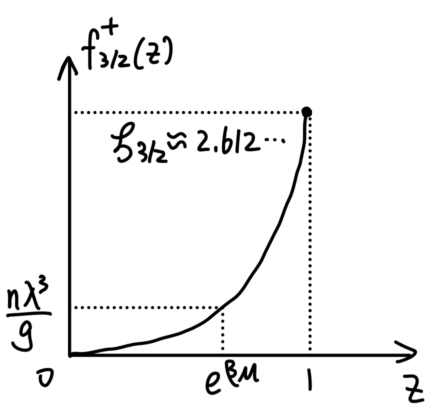
$$\frac{d}{dz} f_m^+(z) = \int_0^\infty dx \frac{x^{m-1}}{(m-1)!} \frac{d}{dz}\left(\frac{1}{z^{-1}e^x - 1}\right)$$

* use $\frac{d}{dz} f(z^{-1}e^x) = -\frac{e^x}{z^2} f' = -\frac{1}{z}\frac{d}{dx} f(z^{-1}e^x)$

$$= -\frac{1}{z}\int_0^\infty dx \frac{x^{m-1}}{(m-1)!}\frac{d}{dx}\left(\frac{1}{z^{-1}e^x - 1}\right) \qquad \{\text{integrate by parts}\}$$

$$= \frac{1}{z}\int_0^\infty dx \frac{x^{m-2}}{(m-2)!}\frac{1}{z^{-1}e^x - 1} = \frac{1}{z} f_{m-1}^+(z)$$

* a sufficiently high derivative of $f_m^+(z)$: $d^n/dz^n (f_m^+(z))|_{z=1} = f_{m-n}^+(z)$
if $m - n \leq 1$, it will be divergent at $z=1$ for all m.



fugacity of a 3d ideal bose gas

The density of excited states for non-relativistic bose gas in 3d is bounded by n^*

$$\left.\begin{aligned} n_x &= \frac{g}{\lambda^3} f^+_{3/2}(z) \le n^* = \frac{g}{\lambda^3}\zeta_{3/2} \\ \frac{n\lambda^3}{g} &= \frac{n}{g}\left(\frac{h}{\sqrt{2\pi m k_B T}}\right)^3 \le \zeta_{3/2} \simeq 2.612\cdots \end{aligned}\right\} \Rightarrow T \ge \frac{h^2}{2\pi m k_B}\left(\frac{n}{g\,\zeta_{3/2}}\right)^{2/3}$$

* At high temperature, the bound is not relevant, and $n_x = n$

The limiting density of excited states at low temperature

$$T_c(n) = \frac{h^2}{2\pi m k_B}\left(\frac{n}{g\,\zeta_{3/2}}\right)^{2/3} \quad \text{and} \quad T \ge T_c(n)$$

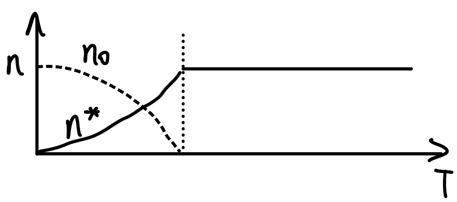
Bose-Einstein condensation

A macroscopic occuption of a single one-particle state

* For $T \le T_c$, $z \equiv 1$ $(\mu = 0)$

$n^* \propto T^{3/2}$ is then less than the total particle density (and independent of it).

Remaining gas particles, with density $n_0 = n - n^*$, occupy the lowest energy state with $\vec{k} = 0$. ($\vec{k} = 0 \Rightarrow \varepsilon(\vec{k}) = 0 \Rightarrow \langle n_{\vec{k}} \rangle_+ \to \infty$)



$T < T_c(n)$, density occupies the ground state.

$n = n_0$ (ground state) + n^* (thermally excited bose particle density)

Bose gas pressure for $T < T_c$ when $z \equiv 1$ $(\mu = 0)$

(section 7.4 Non-relativistic gas)

$$\beta P = \frac{g}{\lambda^3} f^+_{5/2}(1) = \frac{g}{\lambda^3}\zeta_{5/2} = \frac{g}{\lambda^3}\zeta\left(\tfrac{5}{2}\right) \simeq 1.341\frac{g}{\lambda^3} \propto T^{\frac{3}{2}}$$

(Riemann ζ (zeta) : Zeta[$\frac{5}{2}$] $\simeq 1.34149$)

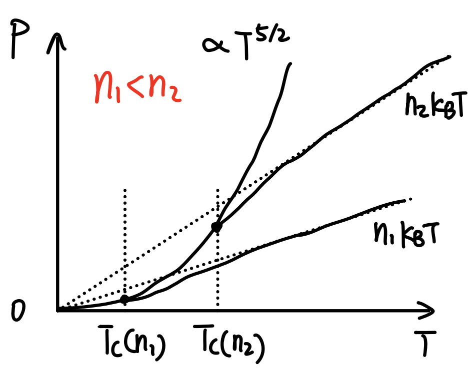
i) $P \propto T^{5/2}$ as $T \to 0$; ii) independent of density

(only the excited fraction n^* has finite momentum and contributes to pressure.)

At fixed temperature, increase density (reducing volume)

$$v^* = \frac{1}{n^*} = \frac{\lambda^3}{g\,\zeta_{3/2}}$$

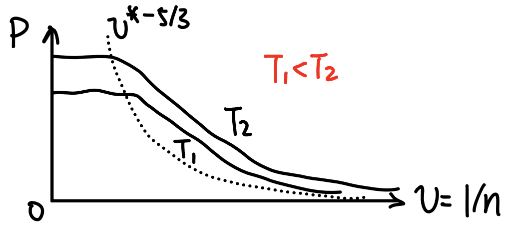
$\left\{ \text{since } \frac{\partial P}{\partial v} = \frac{\partial P}{\partial n} = 0 \text{ for } v < v^*, \text{ the pressure-volume isotherm is flat.} \right\}$



At High Temperature, $P \sim n k_B T$ as classical gas

or the high-T approximation in section 7.4

$$P_\eta = n_\eta k_B T\left[1 - \frac{\eta}{2^{5/2}}\left(\frac{n_\eta \lambda^3}{g}\right) + \left(\frac{1}{8} - \frac{2}{3^{5/2}}\right)\left(\frac{n_\eta \lambda^3}{g}\right)^2 + \cdots\cdots\right]$$



$v^* \propto \lambda^3 \propto T^{-3/2}$; $P \propto T^{5/2} \propto v^{*\,-2/3\cdot 5/2} \propto v^{*\,-5/3}$ (at $v \sim v^*$)

Flat portion : Coexisting liquid and gas phases

Bose condensation as coexistence of gas with $v = v^*$ and liq with $v = 0$
(no interaction potential)

Bose condensation involves features of discontinuous (first-order), and continuous (second-order) transitions. (There is a finite latent heat while the compressibility diverges.)

Latent heat obtained by Clausius-Clapeyron eqn.

As $\beta P \simeq 1.341\, g/\lambda^3$,

$P = C \cdot T^{5/2}$ (C - constant)

$$\left.\frac{dT}{dP}\right|_{\text{Coexistence}} = \frac{\Delta V}{\Delta S} = \frac{(v^* - v_0)}{L/T_c} \qquad L:\ \text{latent heat}$$

$$\left.\frac{dP}{dT}\right|_{\text{Coexistence}} = \frac{5}{2} C \frac{T^{5/2}}{T} = \frac{5}{2}\frac{P}{T}$$

$$\Rightarrow\ L = T_c (v^* - v_0) \left.\frac{dP}{dT}\right|_{\text{Coexistence}} = \frac{5}{2} P v^* = \frac{5}{2}\frac{g}{\lambda^3}\zeta_{5/2}\, k_B T_c \left(\frac{\lambda^3}{g\zeta_{3/2}}\right)$$

$$\Rightarrow\ L = \frac{5}{2}\frac{\zeta_{5/2}}{\zeta_{3/2}} k_B T_c \simeq 1.28\, k_B T_c$$

Compressibility $K_T = \frac{1}{n}\left.\frac{\partial n}{\partial P}\right|_T$

Section 7.4

$$\begin{cases} \beta P_\eta = \dfrac{g}{\lambda^3} f^\eta_{5/2}(z) \\ n_\eta = \dfrac{g}{\lambda^3} f^\eta_{3/2}(z) \end{cases}$$

and $\frac{d}{dz} f^+_m(z) = \frac{1}{z} f^+_{m-1}(z)$

$$\begin{cases} \dfrac{dP}{dz} = \dfrac{g k_B T}{\lambda^3}\dfrac{1}{z} f^+_{3/2}(z) \\ \dfrac{dn}{dz} = \dfrac{g}{\lambda^3}\dfrac{1}{z} f^+_{1/2}(z) \end{cases}$$

$$\Rightarrow\ K_T = \frac{f^+_{1/2}(z)}{k_B T\, f^+_{3/2}(z)} \quad \text{diverges at the transition}$$

as $\lim_{z\to 1} f^+_{1/2}(1) = \zeta_{1/2} = \zeta\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty$

* isotherms approach the flat coexistence portion tangentially.

Energy expression in grand canonical ensemble (Section 7.4)

$$\varepsilon_\eta \equiv \frac{E_\eta}{V} = \frac{3}{2} P_\eta$$

$$E = \frac{3}{2} PV = \frac{3}{2} V \frac{g}{\lambda^3} k_B T\, f^+_{5/2}(z) \propto T^{5/2} f^+_{5/2}(z)$$

$$C_{V,N} = \left.\frac{dE}{dT}\right|_{V,N} = \frac{3}{2} V \frac{g}{\lambda^3} k_B T \left[\frac{5}{2T} f^+_{5/2}(z) + \frac{1}{z} f^+_{3/2}(z) \left.\frac{dz}{dT}\right|_{V,N}\right]$$

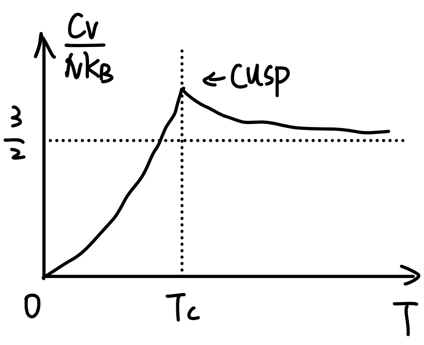
w.r.t $d/dz\,[f^+_m(z)] = f^+_{m-1}(z)/z$

$$n_\eta \equiv \frac{N_\eta}{V} = \frac{g}{\lambda^3} f^\eta_{3/2}(z)$$

with fixed particle number

$$\left.\frac{dN}{dT}\right|_V = 0 = \frac{g}{\lambda^3} V \left[\frac{3}{2T} f^+_{3/2}(z) + \frac{1}{z} f^+_{1/2}(z) \left.\frac{dz}{dT}\right|_{V,N}\right] \Rightarrow \boxed{\frac{T}{z}\left.\frac{dz}{dT}\right|_{V,N} = -\frac{3}{2}\frac{f^+_{3/2}(z)}{f^+_{1/2}(z)}}$$

$$\Rightarrow\quad \frac{C_V}{V k_B} = \frac{3}{2}\frac{g}{\lambda^3}\left[\frac{5}{2} f^+_{5/2}(z) - \frac{3}{2}\frac{f^+_{3/2}(z)^2}{f^+_{1/2}(z)}\right]$$



① Expansion in power of z indicates at high temperature (section 7.4) heat capacity is greater than classical value

$$\frac{C_V}{Nk_B} = \frac{3}{2}\left[1 + \frac{n}{2^{7/2}}\lambda^3 + \cdots\cdots\right]$$

② At low temperature, $z = 1$ ($T < T_c$ and $\mu = 0$)

$$\frac{C_V}{Nk_B} = \frac{3}{2}\frac{g}{n\lambda^3}\left[\frac{5}{2}f^+_{5/2}(1) - \underbrace{\cancel{\frac{3}{2}\frac{f^+_{3/2}(1)^2}{f^+_{1/2}(1)}}}_{\to 0}\right] \to \frac{15}{4}\frac{g}{n\lambda^3}\zeta_{5/2}$$

$$\left\{T_c(n) = \frac{h^2}{2\pi m k_B}\left(\frac{n}{g\,\zeta_{3/2}}\right)^{2/3}\right\} \Rightarrow \boxed{\frac{C_V}{Nk_B} = \frac{15}{4}\frac{\zeta_{5/2}}{\zeta_{3/2}}\left(\frac{T}{T_c}\right)^{3/2}}$$

Origin of $T^{3/2}$ behavior of the heat capacity at low T

At $T=0$ all particles occupy $\vec{k}=0$ state

At small but finite T, the occupation of states of finite momentum up to a value of approximately k_m; $\hbar^2 k_m^2/2m = k_B T$

Each state has energy proportional to $k_B T$

\# of states: $1, 2, 3, \cdots, k_m$

with d dimension: k_m^d

The excitation energy in d dimension is $E_x \propto V k_m^d k_B T$

The heat capacity is $C_V \propto V k_m^d k_B \propto V k_B T^{d/2}$

Example: (1) use to calculate the heat capacities of phonon (or photon) gas

phonon: $\varepsilon(k) \propto k^2 \Rightarrow C_V \propto T^{d/2}$

photon: $\varepsilon(k) \propto k \Rightarrow C_V \propto T^d$

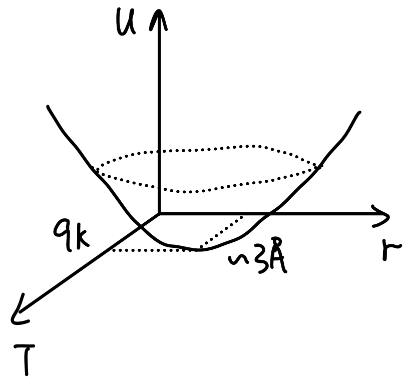
(2) total number of excitation is not conserved, corresponding to $\mu = 0$

lack of conservation only persist up to the transition temperature, where all particles are excited out of reservoir with $\mu = 0$ at $\vec{k} = 0$

(3) C_V is continuous at T_c, reaching maximum approximate $1.92 k_B N$ but has a discontinuous derivative at this point (coexistence to gas)

7.7 Superfluid He^4

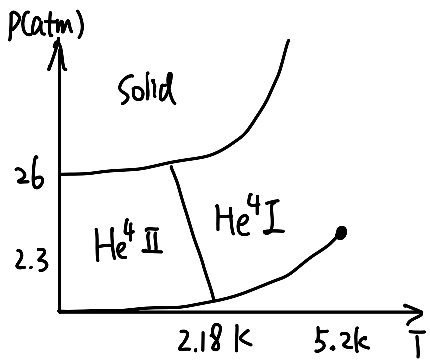
He: $1s^2$ noble gas



(1) The weakness of interaction makes He a universal wetting agent.
(2) He atoms have a stronger interaction for all other molecules, so they easily spread over the surface of any substance.
(3) light mass of He results in large zero-point fluctuation at $T=0$, sufficient to melt a solid phase, thus He^4 remains in a liq. phase at ordinary pressures.
* Quantum liquid at zero temperature has zero entropy

interactions would not significantly change non-interactions picture.

He^3 (3 nucleons) follows fermi statistics (section 7.4)
He^4 (4 nucleons) follows boson statistics (section 7.5)



During evaporation, He^4 I is active and turbulent.
He^4 II is quiescent

* Evaporation process is accompanied by release of latent heat, which cools down liquid

* For ordinary fluid, a finite pressure difference between the containers proportional to viscosity, maintains the flow

superfluid
mechano-caloric effect
fountain effect

He II flows is in the limit of zero pressure difference resulting in zero viscosity
heat → lose He II ; cool → achieve He II
superfluid spontaneously moves up a tube from a heated container

He II normal density vanishes as $T \to 0$, approximately as T^4 by torsional oscillators.

Hypothesis!
Transition to the superfluid state is related to Bose-Einstein condensation

(1) Critical temperature of an ideal bose gas of $v = 46.2 Å^3$

$$T_c = \frac{h^2}{2\pi m k_B}\left(\frac{n}{g\zeta_{3/2}}\right)^{2/3} = \frac{h^2}{2\pi m k_B}\left(\frac{1}{v\zeta_{3/2}}\right)^{2/3} \approx 3.14K \sim 2.18K$$

(2) He^3 (fermion) does not have a similar transition
(At only a few milli-K, He^3 pair becomes boson)

Ordinary fluid: slow process of heat diffusion.

(3) A bose condensate accounts for the thermo-mechanical properties of He II. As P is only the function of temperature and not density (section 7.6), this fits the absence of boiling activity of He II (quiescent). There is increased pressure near hot spot. The fluid flows in response to pressure and removes them very rapidly (at speed of sound in the medium)

Tisza's two-fluid model

(4) postulates the coexistence of two components for $T < T_c$

(a) Normal component of density ρ_n, moving with velocity $\vec{v}_n$ and having finite entropy density s_n

(b) Superfluid component of density ρ_s, no viscosity, no vorticity ($\nabla \times \vec{u}_s = 0$) no entropy density $s_s = 0$

In super-leak experiments, superfluid components pass through, reducing entropy and temperature.

In tensional oscillators, normal components stick to and get dragged by it. (from $\rho_n \to \rho_s$ as T decreases)

Difference between He II and ideal bose condensate.

(1) Interaction in liquid phase. Bose-Einstein condensation has infinite (∞) compressibility, but He II has finite volume (density) and incompressibility.

(2) Even at $T=0$, low-energy spectrum $\mathcal{E}(\vec{k}) = \hbar^2 k^2/2m$ admits many excitations. External body moving through the fluid loses energy and excites modes, leading to finite viscosity (ideal bose condensate is not superfluid)

(3) C_V and ρ_s are quite different. Measured C_V diverges at transition with a characteristic shape similar to λ

	Measured	ideal bose gas
C_V	vanishes at low T, as T^3	$T^{3/2}$
ρ_s	vanishes $(T-T_c)^{2/3}$ at transition	
ρ_n	vanishes T^4 as $T \to 0$	$T^{3/2}$

$\Rightarrow$ A different spectrum of low-energy excitations

Landau suggests

The spectrum of low-energy excitation ~ phonon (interaction between particles)

low energy excitations of a classical liquid : longitudinal sound waves

solid : 2 transverse and 1 longitudinal sound waves

but in section 7.6, that is for photon (?)

For phonon, linear spectrum of excitations ($\varepsilon(k) \propto k$) leads to T^3 vanishing.

Speed of sound waves (in the problem) (T^3) $\Rightarrow$ $v \simeq 240$ m/s

Assumption:

The spectrum of excitation bends down and has a minimum in the vicinity of $k_0 \simeq 2 Å^{-1}$

The excitation in the vicinity of the minimum $\Rightarrow$ rotons

$$\boxed{\varepsilon_{roton}(\vec{k}) = \Delta + \frac{\hbar^2}{2\mu}(k-k_0)^2}$$

$\Delta \simeq 8.6K$; $\mu \simeq 0.16\, m_{He}$

$\Rightarrow$ Proved by neutron scattering measurements.