Beijing Normal University-Hong Kong Baptist University United International College

MATH Introduction to Mathematical Finance

Group Presentation Report

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The Pricing of Options and other Derivatives

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Abstract:

Nowadays, the position of derivatives in finance is becoming more and more important, and the financial market share of derivatives like options and futures is becoming larger and larger, which leads to an extensive discussion in the pricing of derivatives. Since more and more models for pricing the derivatives are introduced, this paper tries to discuss and summarize two of the most basic and classical pricing models. In this paper, the basic concept for the trading of options and other derivatives is firstly introduced, then the models: The Binomial Option Pricing model and the Black-Scholes Model are discussed in detail, and finally give comparation of two models and draw the conclusion.

Content

1. Basic concept about options and other derivatives

This part gives the introduction about the concepts of derivatives and options, and the operation of margin in the option contract.

1.1 The concept of derivatives

Derivatives can be defined as a financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables (Hull. J. C., 2014). It exists in the form of an agreement between two parties. There are four main types of derivatives in the derivatives market: future contracts, option contracts, forward agreement and swaps agreement. This paper mainly focuses on the options and uses it for the explanation of the models.

1.1.1 Option

Option is a kind of derivatives that give its holder the right to sell or buy certain asset in certain time for a certain price according to the contract made by the two parties. From the holders' perspective, the option contract is divided into two types: call option and put option. The holder of the call option is allowed to buy, while the holder of the put option is allowed to sell certain asset according to the contract they made. From the perspective of the trading style, there are two kinds of options in the option market: American option and European option. American options can be exercised at any date from the contract made to the expiration date, the only consideration is the will of the call option's holder. European option is restricted to exercised at the expiration date.

Two kinds of contracts are eventually exercised or not according to the decision of the call option's holder.

1.1.2 Other derivatives

a. Future contracts & forward agreements

Similarity

Both parties to the transaction agree to buy or sell certain assets of a certain quantity and quality at certain prices at certain times in the future.

Difference

The futures contract is a standardized contract formulated by the futures exchange, which has unified regulations on the expiry date of the contract and the type, quantity, and quality of the assets it trades.

A forward contract is a contract set up by the buyer and the seller based on their special needs.

b. Swaps agreement

It is a contract signed by the two parties within the transaction to exchange certain assets at a certain period in the future. The exchange currency specified in the swap contract is the same currency, it is an interest rate swap; if it is a different currency, it is a currency swap.

1.1.3 The main role of derivatives

Derivatives play an important role in facilitating price discovery and fostering greater market transparency. They contribute to the establishment of a market price and thereby enable better assessments of risk.

1.2 The operation of margins

Since the call option's holders have the right to decide whether to carry out contract or not without any loss, in order to reduce the breaking of contract, the margin was introduced.

The investor is required to deposit certain amount of money into a margin account at the time the contract is entered. During the contract period, the margin amount is adjusted each day according to the investor's gain or loss. When the investor loses due to the change of the market, the investor can get a certain rate of money back from the margin account. A maintenance margin is set with amount under the initial margin that guarantees the money in the account is never less than certain amount.

Clearing margin is another kind of margin that managed by the clearing house. The clearing margin is also adjusted every day. The difference between it and the general margin mentioned above is that the clearing margin only has the initial margin, but not have a maintenance margin.

2. Binomial Options Pricing Model

An investor knows the current stock price at any known period. They will try to guess the stock price movements in the future. Under the Binomial Options model, we split the time to expiration of the option into equal periods (weeks, months, quarters, annuals). Then the model follows an iterative method to evaluate the value of each period, considering either an up or down movement ratio and the respective risk-neutral probabilities. Actually, the model creates a binomial distribution of potential stock prices.

It's commonly used for European-style options, which investors can exercise the right at maturity. The model also assumes there's no arbitrage, which means there's no buying while selling at a higher price. Having no-arbitrage can ensures the value of the asset remains unchanged, as a requirement for the Binomial Option Pricing model to work.

Assumptions:

When setting a binomial option pricing model, we need to be aware of the underlying assumptions, to understand the limitations of this approach better.

- At every point in time, the price can go to only two possible new prices, one up and one down (this is in the name, binomial);
- The underlying asset pays no dividends;
- The interest rate (discount factor) is a constant throughout the period;

- There are no transaction costs and no taxes in the market;
- Investors are risk-neutral, indifferent to risk;
- The risk-free rate remains constant.

2.1 One step binomial model

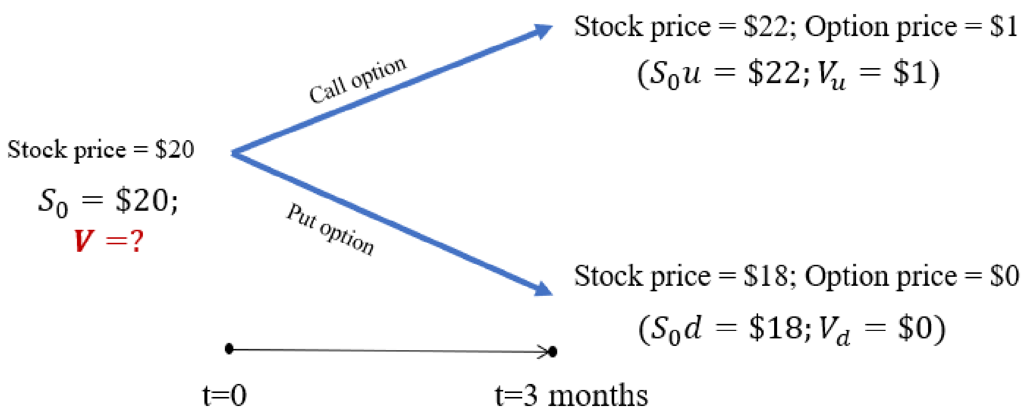
One step binomial model is a simplified model, and it generate the original idea of the pricing methods of options.

2.1.1 Hedging

A stock price is currently \$20, and it is known that at the end of 3 months it will be either \$22 or \$18. We are interested in valuing a European call option to buy the stock for \$21 in 3 months.

This option will have one of two values at the end of the 3 months. If the stock price turns out to be \$22, the value of the option will be \$1; if the stock price turns out to be \$18, the value of the option will be 0.

A binomial tree are draw based on the question:



From the model, the stock price at time 0 is equal to $20, and our goal is to pricing the options at this moment. There are two possible outcomes of this option, the first one we called call option, and the other we can put option. The corresponding stock price and option price for the call option is $22 and $1, for the put option is $18 and $0.

$$S_0u * \Delta - V_u = S_0d * \Delta - V_d$$

$$22\Delta - 1 = 18\Delta$$

$$\Delta = 0.25$$

Where Δ is the shares of the stock in a portfolio

Then this riskless portfolio is long 0.25 shares of stock and short 1 option.

- If the stock price moves up to $22, the value of the portfolio is $22 \times 0.25 - 1 = 4.5$
- If the stock price moves down to $18, the value of the portfolio is $18 \times 0.25 = 4.5$

Hence, the value of portfolio at time t equal to $4.5, and we want to calculate the present value of portfolio. According to the formula $P = S * e^{-rT}$, and the condition provided in the example, the rate-free rate r is 12% per annul, T is 3/12 Year. Then the present value of Portfolio P $= 4.5e^{-rT} = 4.5e^{-0.12\times3/12} = 4.367$. Finally, the price of call option C should be $20 \times 0.25 - C = 5 - C = 4.367$;

$$\boldsymbol{Where\ V = C = 0.633\ is\ the\ option\ price\ we\ expected\ to\ get}$$

2.1.2 Replicating

Considering the no-arbitrage argument just presented by considering a non-dividend paying stock whose price is S(0) and an European (call or put) option on the stock (or any derivative dependent on the stock) whose current price is V. We suppose that the option lasts for time T and that during the life of the call option the stock price can either move up from S(0) to a new level, $S(0)\cdot u$, where u > 1, or down from S(0) to a new level, $S(0)\cdot d$, where d < 1. If the stock price moves up to $S(0)\cdot u$, payoff from the option is denoted by V_u, if the stock price moves down to $S(0)\cdot d$, the payoff from the option is denoted by V_d.

We can also reproduce the risk by replicating an option of premium V with a portfolio of a Δ shares of stock and B in a risk-free bond with interest rate r, that is

$$V = S(0) * \Delta + B$$

Aim at calculating Δ *and* B, we can make the equation

$$(S(0)u\Delta - V_u)e^{-rT} = (S(0)d\Delta - V_d)e^{-rT}$$

Where $V_u = S(0)u\Delta + Be^{rT}$ *and* $V_d = S(0)d\Delta + Be^{rT}$

Then we can get $\Delta = \frac{V_u - V_d}{S(0)u - S(0)d}$ *and* $B = e^{-rT} \frac{uV_d - dV_u}{u-d}$

2.1.3 Risk-neutral

A risk-neutral world has two features that simplify the pricing of derivatives:

a. The expected return on a stock (or any other investment) is the risk-free rate.

b. The discount rate used for the expected payoff on an option (or any other instrument) is the risk-free rate.

The parameter $P^* = \frac{e^{rT} - d}{u - d}$ should be indicated as the risk-neutral probability, and also $1 - P^*$ is the risk-neutral probability of a down movement.

$$V = e^{-rT}(p^* V_u + (1 - p^*)V_d)$$

represents that the value of the option today is its expected future payoff in a risk-neutral world discounted at the risk-free rate.

2.2 Cox-Ross-Rubinstein Binomial Model

$u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$

For binomial tree on futures based on Cox-Ross-Rubinstein tree,

$$p^* = \frac{1-d}{u-d} = \frac{1-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} = \frac{1-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}(1-e^{-2\sigma\sqrt{\Delta t}})} = \frac{1}{1+e^{\sigma\sqrt{\Delta t}}},$$

$$1-p^* = 1-\frac{1}{1+e^{\sigma\sqrt{\Delta t}}} = \frac{e^{\sigma\sqrt{\Delta t}}}{1+e^{\sigma\sqrt{\Delta t}}} = \frac{1}{1+e^{-\sigma\sqrt{\Delta t}}}.$$

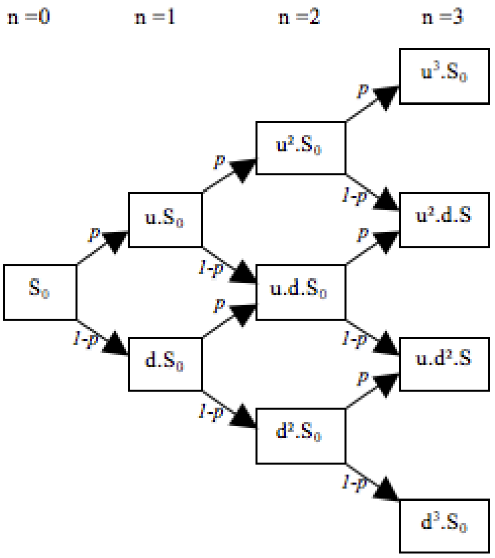
As for 4-step binomial model, At n=3,

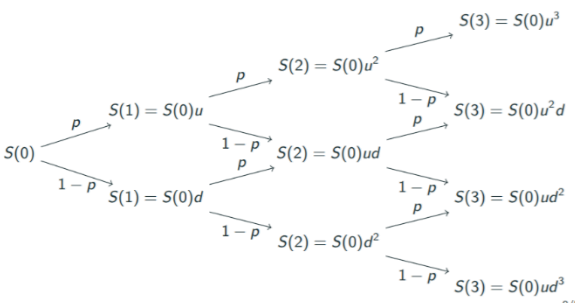
The range of three call option $= \max\{S(0)\cdot u^3 - K, 0\}$

Two call option with one put option $= \max\{S(0)\cdot u^2 d - K, 0\}$

One call option with two put option$= \max\{S(0)\cdot d^2 - K, 0\}$

Four put option $= \max\{S(0)\cdot d^3 - K, 0\}$





For N-step binomial model, we Denote $\phi(m,N,p)=\sum_{i=0}^{m}\binom{N}{i}p^i(1-p)^{N-i}$ as the cumulative binomial distribution with N trials and probability p of success in each trial, Then:

$$
\begin{aligned}
C_{Eur}(S(0),K,N\Delta t) &= e^{-rN\Delta t}\sum_{i=m}^{N}\binom{N}{i}(p^*)^i(1-p^*)^{N-i}\left(S(0)u^id^{N-i}-K\right) \\
&= S(0)\sum_{i=m}^{N}\binom{N}{i}\left(p^*ue^{-r\Delta t}\right)^i\left((1-p^*)de^{-r\Delta t}\right)^{N-i}-Ke^{-rN\Delta t}\sum_{i=m}^{N}\binom{N}{i}(p^*)^i(1-p^*)^{N-i} \\
&= S(0)\left[1-\Phi\left(m-1,N,p^*ue^{-r\Delta t}\right)\right]-Ke^{-rN\Delta t}\left[1-\Phi\left(m-1,N,p^*\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
P_{Eur}(S(0),K,N\Delta t) &= e^{-rN\Delta t}\sum_{i=0}^{m-1}\binom{N}{i}(p^*)^i(1-p^*)^{N-i}\left(K-S(0)u^jd^{N-j}\right) \\
&= Ke^{-rN\Delta t}\sum_{i=0}^{m-1}\binom{N}{i}(p^*)^i(1-p^*)^{N-i}-S(0)\sum_{i=0}^{m-1}\binom{N}{i}\left(p^*ue^{-r\Delta t}\right)^i\left((1-p^*)de^{-r\Delta t}\right)^{N-i} \\
&= Ke^{-rN\Delta t}\Phi\left(m-1,N,p^*\right)-S(0)\Phi\left(m-1,N,p^*ue^{-r\Delta t}\right)
\end{aligned}
$$

where r is the risk-free interest rate, P^* is risk neutral probability and m is the least integer such that $S(0)\cdot u^m d^{N-m}>K$

Then, m=max{0, the smallest integer> $\dfrac{\ln(\frac{K}{S(0)})-N\ln d}{\ln(\frac{u}{d})}$ }

3. Black-Scholes Option Pricing Model

Since the concepts about options was introduced, how to measure its prices be a big problem.

The key part of the problem was solved by Economists Fischer Black and Myron Sholes. They found that when pricing options, the main factors affecting option price, such as stock price, volatility, strike price and contract term, can be quantified, while the risk of the option itself is difficult to quantify. This is the crux of the problem, but also the key breakthrough point. The creative idea is that, instead of attacking head-on, they could go the other way. Since it is hard to quantify the risk of an option, they could find a way to make it irrelevant in pricing. Their method, "Delta hedging", became one of the most important inventions in modern economics. They found that if the stock and a certain number of options together,

$$\frac{stock}{options} = \Delta$$

When the stock price fluctuation disappeared, through the combination can only achieve risk-free return after the hedge, then the value of options can build stock and risk-free capital portfolio replication, the option price is only with the stock price, volatility, executive price, contract term and risk-free interest rate, successfully eliminate the risk of option itself this variable.

This clever idea is very groundbreaking, but there is a barrier to applying it to the market, which is that they assume that the market is always in equilibrium, which is obviously unrealistic. The market is always in a dynamic state of change, and the supply and demand relationship will not always be in an equilibrium state. They need continuous dynamic hedging to keep the fluctuations in the value of stocks and options in the portfolio offset each other.

This problem was solved by another gifted economist, Robert Merton.

He adopted the differential theory of Kiyosi Ito, a Japanese mathematician, and used the concept of continuous time and random variables to simulate dynamic hedging more precisely.

The process of simulation is:

During building the model, the price of the asset (e.g. stock) follows the rule of Geometric Brownian motion and could get the following equation:

$$dS = \mu S dt + \sigma S dW$$

S : the current stock price

t : the time

W : a stochastic variable for Brownian motion

$return\ on\ the\ asset \sim N(\mu, \sigma^2), where\ \mu\ and\ \sigma\ are\ both\ constant.$

As well, in the market, the price of option V is a function about stock price S and time t. Then, by an important lemma —— Itô's lemma for these two variable, the equation will become as:

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW$$

Then, after eliminating and substituting, an important partial differential equation is shown, which is called Black–Scholes equation.

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

V(S,t) : the price of the option (the function of the current stock price S and time t)

r : the risk-free interest rate (e.g. the interest rate that received from a government debt)

σ : the volatility of the log returns of the asset.

The Black–Scholes model is based on this equation, which gives a theoretical estimation to the price of European stock options.

Then the equation could be rewritten to the following form, which is also known as the equation of the "risk neutral".

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS \frac{\partial V}{\partial S}$$

In the left hand side, the first term represents the change in the price of the option price V due to time t increasing which is typically negative (because the value of the option decreases as time moves closer to expiration). The next term reflecting the gains which the portfolio receives from holding the option. It is typically positive. In sum, in the left-hand side, the losses from the first part and the gains from the second part offset one another, resulting in returns at a risk-free rate.

The right-hand side represents the risk-free return from buying the option and the return from selling shares of the stock.

Then the BS formula is easily to got which is the analytic solution to the Black-Scholes equation with following boundary conditions.

$$C_{E,T} = \max(0, S_T - K)$$

For European call option

$$P_{E,T} = \max\,(0,\ K - S_T)$$

For European put option

The Black–Scholes formula is,

$$C = S(0)N(d_1) - Ke^{-rT}N(d_2)$$

$$P = Ke^{-rT}N(-d_2) - S(0)N(-d_1)$$

Where

$$d_1 = \frac{ln\frac{S(0)}{K} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{ln\frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

The meaning of notions in it are,

$C = Call\ option\ price$

$P = Put\ option\ price$

$K = Strike\ price$

$S = Current\ stock\ (or\ other\ underlying)\ price$

$r = Risk - free\ interest\ rate$

$\sigma = A\ volatility\ parameter\ of$

$a\ non - dividend - paying\ stock$

$t = Time\ to\ maturity$

$N = A\ normal\ distribution$

In 1997, this Black-Scholes Option Pricing Model won the 29th Nobel Prize in Economics.

4. The evaluation of the Binomial Options Pricing Model and B-S model

a. Binomial option pricing model:

Advantage:

1. Its derivation is simple
2. A given time period can be subdivided into smaller time units, which can handle more complex budgets (American options and European options)

Disadvantages:

1. The calculation is slower than B-S model

b. The Black-Scholes model:

Advantage:

1. Lay the foundation for the reasonable pricing of various derivative financial instruments priced at market price changes in the derivative financial market including stocks and bonds.
2. The model evaluates At the Money very well.

Disadvantages:

1. The mathematical derivation process is very complicated
2. Cannot be implemented before the option expires

When the limit, the pricing of European options by the Cox-Ross-Rubinstein Tree model has evolved into the B-S model.

5. Summary

This paper tries to describe the methods which including the binomial options pricing model and the Black-Scholes model to estimate the price of options after the introduction and understanding of derivatives.

Firstly, this article elaborates on a simplified model called the one-step binomial method to give the basic concept of pricing options. One step binomial starts with the change in the price of the subject matter at a point in time and covers three situations: hedging, replicating, and risk-neutral to point out the object of option pricing, get the portfolio, and indicate the risk-neutral probability respectively.

Secondly, based on the risk-neutral probability and multiple time points combinations, the binomial options pricing model is built considering either an up or down movement ratio probabilities in every time point to evaluate the value of each period.

The last one is the Black-Scholes model that strictly requires certain assumptions of the price of the asset to estimate the price of options by considering to hedge the options in an investment portfolio and eliminate the risk. This article describes the mathematical process of the Black-Scholes equation, "Delta hedging" which can make quantifying the risk of an option irrelevant in pricing, and the application on the market.

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