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GROUP PROJECT

Ito's Lemma and it's Application

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Abstract

Ito's lemma, which is one of the most important tools of stochastic analysis in finance, relates the change in the price of the derivative security to the change in the price of the underlying asset. Applications of Ito's lemma to geometric Brownian motion asset price process, Black-Scholes model are discussed in detail. The related concepts, risk-neutral and delta hedging are also included.

Keywords: Ito's lemma, Geometric Brownian motion, Black-Scholes Model

1 Introduction

Ito's lemma is one of the essential formulas in financial mathematics, and it was proposed by mathematician Kiyoshi Ito. It is necessary because it makes the pricing of derivatives possible by pointing out the rules for differentiating functions of stochastic processes. One of its applications is the Black-Scholes Model, the most famous mathematical model used to deal with the pricing of options, and Ito's lemma is the central part of its proof. To understand the whole concept, we need to make some introduction first.

When talking about Brownian motion, it was founded by Robert Brown and further explained in Physics by Albert Einstein. In 1918, the Wiener process was proposed by Norbert Wiener, who gives the rigorous definition and description of Brownian motion in mathematics view. People found that Brownian motion is a continuous stochastic process, and the one-dimensional Brownian motion trend in the time domain is quite similar to the trend of stock price. It was also surprising that Louis Bachelier had already analyzed the stock price and pricing of options in his thesis *Théorie de la speculation* in 1900. Anyway, this observation makes standard Brownian motion becomes vital in describing the stock price.

One of the properties of Brownian motion is that it is continuous, but not differentiable everywhere, which means the classical calculus is inapplicable in Brownian motion. To explain it further, for a Bronian motion, the quadratic variation states that the fluctuation is too frequent so that we cannot derive a time interval leads to the sum of the squares of the displacement difference will approximately equal to 0. Hence, classical calculus fails to deal with Brownian motion problems. However, this problem is precisely what Ito's lemma is dealing with.

The remainder of this paper is organized as follows. We describe the detail and proof of Ito's lemma in Section 2 and Geometric Brownian Motion in Section 3. The applications of Ito's lemma are given in Section 4.

2 Ito's Lemma

2.1 Quadratic variation

Consider a time interval $[0, T]$, divides the time interval into N parts $\{0 = t_0 < t_1 < \cdots < t_N = T\}$. For any continuous function $f(t)$, the quadratic variation is defined as $\sum_{i=0}^{N-1}[f(t_{i+1}) - f(t_i)]^2$. For a continuous and differentiable function $f(t)$, according to mean value theorem from classical calculus,

$$\begin{aligned}\sum_i (f(t_{i+1}) - f(t_i))^2 &\leq \sum_i (t_{i+1} - t_i)^2 f'(\varepsilon)^2 \\ &\leq \max_{S\in[0,T]} f'(\varepsilon)^2 \sum_i (t_{i+1} - t_i)^2 \\ &\leq \max_{S\in[0,T]} f'(s)^2 \max_i\{t_{i+1} - t_i\}T.\end{aligned} \tag{1}$$

When the time interval is divided into infinite sections, which means N is large enough, $\max_i\{t_{i+1} - t_i\}T \to 0$,

$$\lim_{N\to\infty} \sum_i (f(t_{i+1}) - f(t_i))^2 \leq \max_{S\in[0,T]} f'(s)^2 \lim_{S\in[0,T]} \max_i\{t_{i+1} - t_i\}T = 0. \tag{2}$$

Now, change $f(t)$ to standard Brownian Motion $B(t)$, which is not differentiable over the interval $[0, T]$, we have

$$\begin{aligned}&\lim_{N\to\infty} \sum_i (B(t_{i+1}) - B(t_i))^2 \\ =&\sum_i x_i^2 \\ =&\sum_i Y_i \\ =&n(\frac{1}{n}\sum_i Y_i) \\ =&T.\end{aligned} \tag{3}$$

Here, the standard Brownian motion has property $B(t_{i+1}) - B(t_i) \sim N(0, \frac{T}{N})$. Hence, $E(Y_i) = E(X_i^2) = \frac{T}{N}$, and the last equation is obtained by *Strong Law of Large Number*. The result implies that no matter how small the interval $[t_i, t_{i+1}]$ is, the sum square of the displacement won't be zero because of the extremely frequent fluctuation. We rewrite the equation (3) in infinitesimal form

$$(dB_t)^2 = dt, \tag{4}$$

this result is essential for the proof of *Ito's lemma*.

2.2 Proof of Ito's Lemma

2.2.1 Preliminary

Geometric Brownian motion shares with Brownian motion the propery that a future price depends on the present and all past prices only through the present price. It also solves the problem that stock price could be negative while using Wiener process to model the stock price. However, while investing in an asset, it is the return on the investment that we really concern about.

Denote S_t as the stock price, R_t as the return of the stock, we have

$$\begin{aligned} R_t &= \frac{S_{t+\Delta t} - S_t}{S_t} \\ &= \mu \Delta t + \sigma (\Delta t)^{\frac{1}{2}} \Phi \\ &= \mu \Delta t + \sigma (B_{t+\Delta t} - B_t), \end{aligned} \tag{5}$$

where μ is the mean of the return, σ is the standard deviation of the return, and Φ is a standard normal distribution. Let $\Delta t \to 0$, the equation can be wrote in differential form as

$$dS_t = \mu S_t dt + \sigma S_t dB_t. \tag{6}$$

However, B_t is not differentiable. In order to find out the solution of this stochastic differential equation, Ito's lemma will be applied.

2.2.2 Details of the proof

Suppose we need to compute $f(B_t)$ for some smooth function f. According to classical calculus, $df = (\frac{dB_t}{dt})dt$. However, this does not work since B_t is not differentiable. Here, dB_t can be interpreted as how much the Brownian motion change over the small time scale.

Then we tried to use $df = f'(B_t)dB_t$ to estimate f, however it is also wrong because of the quadratic variation $(dB_t)^2 = dt$.

Consider the general form of Taylor expansion,

$$\begin{aligned} f(B_{t+\Delta t} - B_t) &= f(B_t) + f'(B_t)dB_t + \frac{f''(B_t)}{2}(dB_t)^2 + \cdots \\ &\approx f(B_t) + f'(B_t)dB_t + \frac{f''(B_t)}{2}dt, \end{aligned} \tag{7}$$

which is the basic form of *Ito's formula*. Now we consider a smooth bivariate function $f(t, x)$ and expand by the Taylor expansion,

$$\begin{aligned} f(t + \Delta t, x + \Delta x) = f(t, x) &+ \frac{\partial f(t,x)}{\partial t}\Delta t + \frac{\partial f(t,x)}{\partial x}\Delta x \\ &+ \frac{1}{2}(\frac{\partial f^2(t,x)}{\partial t^2}(\Delta t)^2 + \frac{\partial f^2(t,x)}{\partial x^2}(\Delta x^2) \\ &+ 2\frac{\partial^2 f(t,x)}{\partial t \partial x}\Delta x \Delta t) + \cdots \end{aligned} \tag{8}$$

$$\begin{aligned}
df &\approx f(t+\Delta t, x+\Delta x) - f(t,x) \\
&= \frac{\partial f(t,x)}{\partial t}\Delta t + \frac{\partial f(t,x)}{\partial x}\Delta x \\
&+ \frac{1}{2}(\frac{\partial f^2(t,x)}{\partial t^2}(\Delta t)^2 + \frac{\partial f^2(t,x)}{\partial x^2}(\Delta x^2) \\
&+ 2\frac{\partial^2 f(t,x)}{\partial t \partial x}\Delta x \Delta t)
\end{aligned} \tag{9}$$

Now change one of the variable into Brownian motion B_t,

$$\begin{aligned}
f(t+\Delta t, B_t + dB_t) - f(t, B_t) &= \frac{\partial f(t,x)}{\partial t}\Delta t + \frac{\partial f(t,x)}{\partial x} dB_t \\
&+ \frac{1}{2}(\frac{\partial f^2(t,x)}{\partial t^2}(\Delta t)^2 + \frac{\partial f^2(t,x)}{\partial x^2}(dB_t^2) \\
&+ 2\frac{\partial^2 f(t,x)}{\partial t \partial x} dB_t \Delta t) \\
&= (\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2})dt + \frac{\partial f}{\partial x} dB_t
\end{aligned} \tag{10}$$

Recall the definition of $O(h)$, when $f(h) = O(h)$, it means that $lim_{h\to 0}\frac{f(h)}{h} = 0$.

As the limit $dt \to 0$, we have the terms dt^2 and $(dt \cdot dB_t)$ which are approaching to 0 faster than $(dB_t)^2$. From the quadratic variance of the Geometric Brownian motion, the term $(dB_t)^2$, which is O(dt), is equivalent to dt (based on the equation(4)).

Therefore, we get the above equation.

Finally we consider a smooth function $f(t, X_t)$, where X_t is a stochastic process satisfies $dX_t = \mu dt + \sigma dB_t$, then

$$\begin{aligned}
df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2 \\
&= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(\mu dt + \sigma dB_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mu dt + \sigma dBt)^2 \\
&= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(\mu dt + \sigma dB_t) + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}dt \\
&= (\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2})dt + \sigma\frac{\partial f}{\partial x}dB_t
\end{aligned} \tag{11}$$

Hence, for stochastic process with drift $dX_t = \mu dt + \sigma dB_t$, the general form of *Ito's Lemma* is

$$df = (\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2})dt + \sigma\frac{\partial f}{\partial x}dB_t. \tag{12}$$

3 Geometric Brownian Motion

3.1 Geometric Brownian Motion and Stock Price Valuation

The geometric Brownian motion model is a mathematical model used by modern finance to describe stock price evolution over time. For example, the famous Black-Scholes option pricing formula is derived based on the geometric Brownian motion model. Financial scientists and mathematicians connect geometric Brownian motion and stock price because the geometric Brownian motion has some "nice mathematical properties" . During the research and the observation of the actual stock markets, financial mathematicians found that, to a certain degree, the geometric Brownian motion shares similarity with the stock price. For example,

- Intuitively, stock prices look a lot like a random walk.
- The expectation of geometric Brownian motion is independent of the price of the random process (the stock price), which is consistent with our expectation of the actual market.
- Geometric Brownian motion process only consider positive prices, just like stock prices in reality.
- Geometric Brownian motion process takes on the same 'roughness' step as the price trajectory people observe in the stock markets.

3.2 Derivation of Geometric Brownian Motion

Firstly, Ito's lemma could be primarily employed to solve the stochastic differential equation (SDE), which is usually used to model the stock price based on the Geometric Brownian motion.
The basic stochastic differential equation is as follow:

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{13}$$

The model assume that the stock price S_t follows the *Geometric Brownian motion* (i.e. the return of the stock market will follow a normal distribution with the constant expectation μ and variance σ).
To solve this equation, the most important trick is to apply the transformation of S_t:

$$X_t = log(S_t) \tag{14}$$

We can apply Ito's calculus here and make use of Ito's lemma:

$$g(s) = log(s) \tag{15}$$

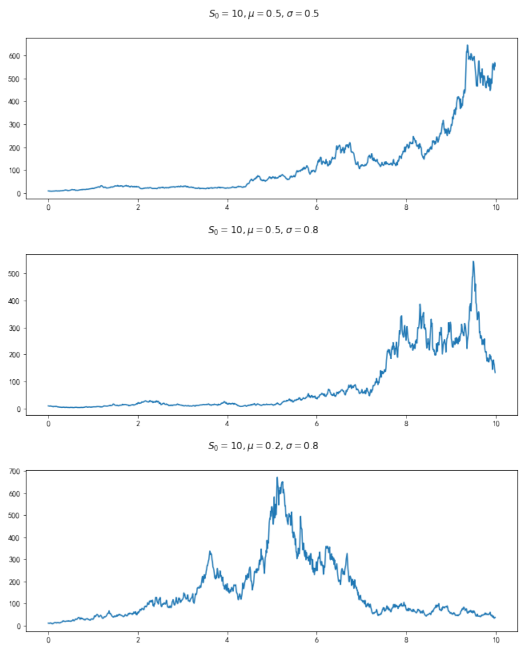

Figure 1: The simulation of Geometric Brownian Motion

$$\frac{\partial}{\partial s} g(s) = \frac{1}{s} \tag{16}$$

$$\frac{\partial^2}{\partial s^2} g(s) = -\frac{1}{s^2} \tag{17}$$

where

$$X_t = g(S_t) = log(S_t) \tag{18}$$

From the Ito's calculus we also have:

$$\begin{aligned} (dS_t)^2 &= (\mu S_t dt + \sigma S_t dB_t)^2 \\ &= \mu^2 {S_t}^2 (d_t)^2 + 2\mu\sigma S_t^2 dt dB_t + \sigma^2 S_t^2 (dB_t)^2 \\ &= \sigma^2 S_t^2 dt \end{aligned} \tag{19}$$

According to the Ito's lemma (1) and the SDE:

$$\begin{aligned} dX_t &= d\log(S_t) \\ &= \frac{\partial}{\partial S_t} g(S_t) dS_t + \frac{1}{2} \frac{\partial^2}{\partial S_t^2} g(S_t)(dS_t)^2 \\ &= \frac{1}{S_t} dS_t - \frac{1}{2} \frac{1}{S_t^2} (dS_t)^2 \\ &= \frac{1}{S_t} (\mu S_t dt + \sigma S_t dB_t) - \frac{1}{2} \frac{1}{S_t^2} \sigma^2 S_t^2 dt \\ &= (\mu - \frac{1}{2}\sigma^2) dt + \sigma dB_t \end{aligned} \tag{20}$$

By integrating the two terms, and use the property of Brownian Motion $W_0 = 0$, we could get the equation of X_T:

$$X_T = X_0 + (\mu - \frac{1}{2}\sigma^2)T + \sigma B_t \tag{21}$$

Therefore, by using $e^{X_0} = S_0$, we finally get the equation of the stock price after the T time period from the initial point:

$$\begin{aligned} S_T &= e^{X_T} \\ &= e^{X_0} e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma B_t} \\ S_T &= S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma B_t} \end{aligned} \tag{22}$$

4 Black-Scholes Equation

4.1 Risk-Neutral Valuation and Delta

To better understand Ito's lemma and Black-Scholes-Merton Equation, we need to introduce two fundamental concepts: risk-neutral valuation and Delta. The risk-neutral valuation is also named as risk-neutral pricing theory. The meaning of risk-neutral refers to a mindset where an investor is insensitive to risk when making an investment decision.

Black Scholes equation is based on risk neutral valuation, and the critical foundation is the geometric Brownian motion. The geometric Brownian motion can view as a continuous approximation of binary trees. Meanwhile, the binary tree always assumes that the stock price will either go up or go down at each step. Since there only exist two possibilities, we can use the two-element linear equations to solve them.

The other important concept is deltas a hedging. Delta hedging is an options trading strategy designed to reduce or directional hedge risk from fluctuations in the underlying asset price. This method uses options to offset the risk of holding a single other option or the entire portfolio.

4.2 Derivation of Black-Scholes Equation

The largest risk would also be considered while solving the Black-Scholes equation. For the sake of clarity, we only focus on the option which targets the stock market and we may give out some assumptions first.

- The asset price follows a lognormal random walk.
- The risk-free interest rate r and the volatility of the underlying asset σ are know functions of time over the life of the option.

- There are no associated transaction costs.
- The underlying asset pays no dividentds during the life of the option.
- There are no arbitrage opportunities.
- Trading of the underlying asset can take place continuously.
- Short selling is allowed.
- Fractional shares of the underlying asset may be traded.

By abiding the assumptions above, we can know that the price of the underlying asset follows a geometric Brownian motion. Hence we may use our prior knowledge to derive the Black-Scholes equation.

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t \tag{23}$$

The only source of uncertainty in this equation is B_t, which is a Brownian motion. Hence this equation states that the infinitesimal rate of return on the assets has an expected value of μ and a variance of $\sigma^2 dt$ The payoff of an option $V(S,T)$ is known. And by Ito's lemma we can have:

$$dV_t = (\mu S_t \frac{\partial V_t}{\partial S_t} + \frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_t}{\partial S_t^2})dt + \sigma S_t \frac{\partial V_t}{\partial S_t} dB_t \tag{24}$$

To analyze the relationship between the option value V_t and the stock price S_t, we then construct a portfolio π, whose variation over a small time period dt is deterministic. We let

$$\Pi = -V + \Delta S \tag{25}$$

the equation shows a risk-free portfolio which combining the option price V with hedging term Δ times the stock price S (not determined yet but constant throughout each time step).

We may observe that

$$\begin{aligned} d\Pi &= -dV_t + \Delta dS_t \\ &= -\sigma S_t \frac{\partial V_t}{\partial S_t} dB_t - (\mu S_t \frac{\partial V_t}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + \frac{\partial V_t}{\partial t})dt + \Delta(\sigma S_t dB_t + \mu S_t dt) \\ &= -\sigma S_t (\frac{\partial V_t}{\partial S_t} - \Delta) dB_t - (\mu S_t (\frac{\partial V_t}{\partial S_t} - \Delta) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + \frac{\partial V_t}{\partial t})dt \end{aligned} \tag{26}$$

To make the portfolio value stable throughout time, we choose $\Delta = (\frac{\partial V_t}{\partial S_t}$ and we have:

$$d\Pi = -(\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + \frac{\partial V_t}{\partial t})dt \tag{27}$$

After deriving the equation, we can see that this equation does not depend on dB_t and therefore must be riskless during time dt. Furthermore, in the basic assumption of Black-Scholes Model, there is one that specify that arbitrage opportunities do not exist, hence Π must earn the same return as other short-term risk-free instruments. It follows that

$$d\Pi = r\Pi dt \tag{28}$$

where r is the risk-free interest rate.

If we equate two formulas for $d\Pi$ we can obtain:

$$(\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + \frac{\partial V_t}{\partial t})dt = r(V_t - \frac{\partial V_t}{\partial S_t} S_t)dt \tag{29}$$

By reorganizing and simplifying the equation, we can get:

$$\frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + rS_t \frac{\partial V_t}{\partial S_t} - rV_t = 0 \tag{30}$$

4.3 Delta Hedging

The delta hedging is one of major ways to implement the risk neutral. The idea of risk neutral is that in a complete market with no arbitrage opportunities, the prices of derivatives are determined by the discounted expected value of the future payoff and the return rate of the portfolio should equal to risk-free rate at the current time.

The delta of the option is how much price of stock (or other underlying assets) we should buy to hedge the risk from each share of options, which has been get from the equation(26).

$$\Delta = \frac{\partial V_t}{\partial S_t} \tag{31}$$

For the price of European Call and Put options, the BS-model can be solved theoretically as:

$$\begin{aligned} C_t &= S_0 N(d_1) - e^{-r(T-t)} K N(d_2) \\ P_t &= e^{-r(T-t)} K N(d_2) - S_0 N(d_1) \end{aligned} \tag{32}$$

where K is the strike price, r is the risk-free interest rate, t is the time to maturity and

$$\begin{aligned} d_1 &= \frac{ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \end{aligned} \tag{33}$$

The Δ of the futures is:

$$\Delta_{F_{t,T}} = e^{(r-q)(T-t)} \tag{34}$$

The Δ of the option with respect to the futures which is the time to maturity is:

$$\begin{aligned}\Delta_{C_t/F_{t,T}} = \frac{\partial C_t}{\partial F_{t,T}} = \frac{\partial C_t/\partial S_t}{\partial F_{t,T}/\partial S_t} = e^{-rT}N(d_1)\\ \Delta_{P_t/F_{t,T}} = \frac{\partial P_t}{\partial F_{t,T}} = \frac{\partial P_t/\partial S_t}{\partial F_{t,T}/\partial S_t} = -e^{-rT}N(-d_1)\end{aligned} \tag{35}$$

By using delta with respect to the futures to hedge and make the portfolio risk-free, it could potentially reduce investment risk and the transaction costs.