The Pricing of Options and Other Derivatives

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CONTENTS

1.Introduction

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Derivatives:

An agreement between two parties which has a value determined by the price of something else.

Four main types

- Future contracts
- Forward agreement
- Swaps agreement
- **Option contracts**

Options:

- Derivatives that based on the value of its underlying asset.
- The right to buy or sell certain asset by certain date for a certain price.

Types of contracts:

- Holders' perspective
  - Call option: buy
  - Put option: sell
- Trading style perspective
  - American options:
    Exercised at any date
  - European options:
    Only exercised at the expiration/maturity date

Trading details

1. Signing option contract:

- The call option’s holder have the right to choose whether to make the transaction.
- Place: International Securities exchange, Chicago Board Options Exchange. (electronic/face-to-face)
- a fund in a margin account is deposited by the buyers.

Trading details

2. The operation of margins:

- Initial margin: deposit in a margin account at the time the contract is entered.
- Daily settlement/ marking to margin: at the end of each day, the margin account is adjusted to reflect the investor’s gain or loss.
- Maintenance margin: ensure the balance in the balance account never become negative.

Clearing margins:

- Used to guarantee the performance of the parties to each transaction.
- Posted in the clearing house.
- Adjusted every trading day.
- Only has initial margin, no maintenance margin.

2.ONE-STEP BINOMIAL MODEL

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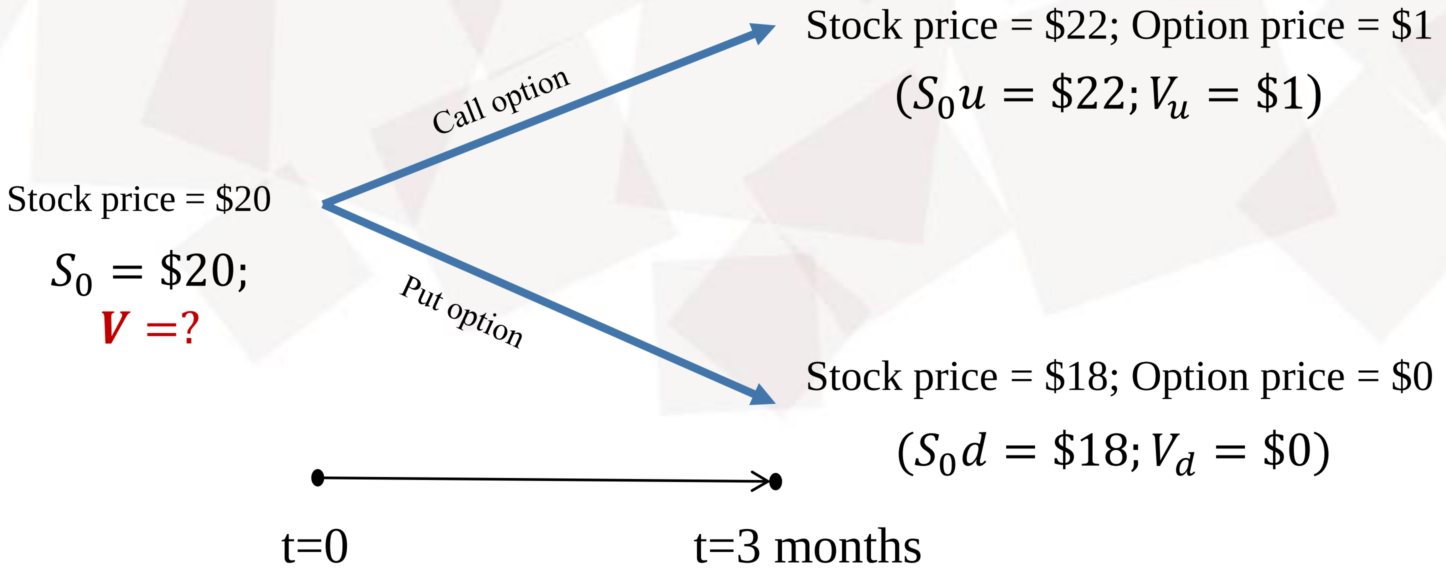
Example 1

A stock price is currently \$20, and it is known that at the end of 3 months it will be either \$22 or \$18. We are interested in valuing a European call option to buy the stock for \$21 in 3 months. This option will have one of two values at the end of the 3 months. If the stock price turns out to be \$22, the value of the option will be \$1; if the stock price turns out to be \$18, the value of the option will be 0.



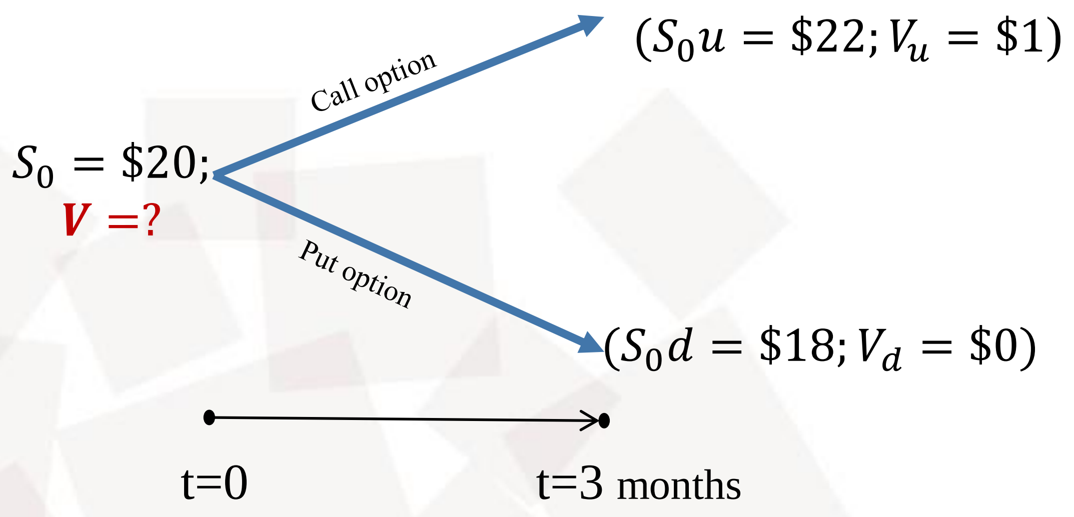
Example 1 (cont.)

$S_0 u * \Delta - V_u = S_0 d * \Delta - V_d$

$22\Delta - 1 = 18\Delta$

$\Delta = 0.25$

Where Δ is the shares of the stock in a portfolio



A riskless portfolio is long 0.25 shares of stock and short 1 option.

- If the stock price moves up to \$22, the value of the portfolio is $22 \times 0.25 - 1 = 4.5$
- If the stock price moves down to \$18, the value of the portfolio is $18 \times 0.25 = 4.5$

Aim: Make sure there is no arbitrage opportunities in this Riskless Portfolios

Example 1 (cont.)

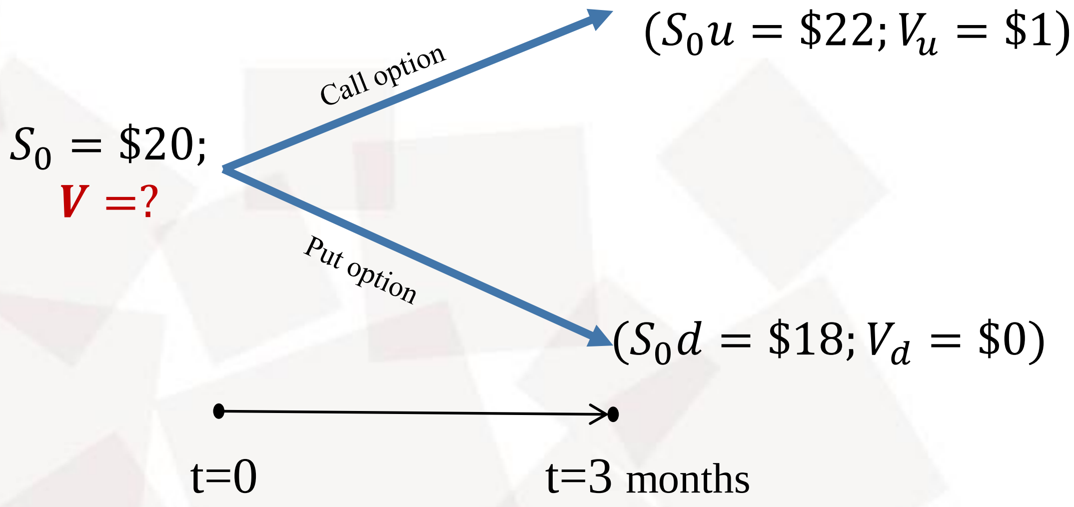
Continuous compounding

$$P = S * e^{-rT}$$

r is the risk-free rate
T is time
S is the value of Portfolio at time T
P is the Present value at time 0

$The\ Present\ value\ of\ Portfolio$

$The\ value\ of\ Portfolio\ at\ time\ t:\ \$4.5$



Suppose the risk-free rate r is 12% per annum and T is 3/12 year.

The present value of Portfolio P $= 4.5e^{-rT} = 4.5e^{-0.12\times3/12} = 4.367$

The price of call option C should be $20 \times 0.25 - C = 5 - C = 4.357; \rightarrow \boldsymbol{V = C = 0.633}$
V is the option price we expected to get.

3. Binomial Options Pricing Model

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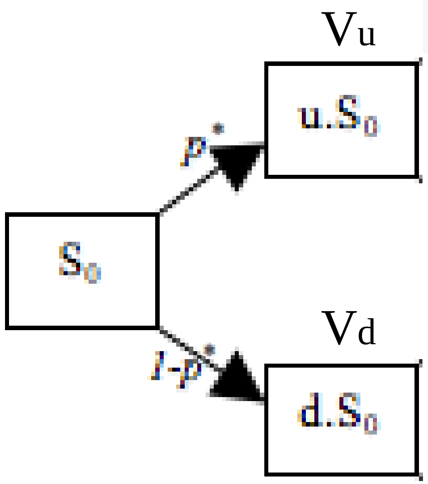
A **risk-neutral** world has two features that simplify the pricing of derivatives:

1. The expected return on a stock (or any other investment) **is the risk-free rate**.
2. The discount rate used for the expected payoff on an option (or any other instrument) **is the risk-free rate.**

p^* : the probability of an up movement in a risk-neutral world

1-p^* : the probability of a down movement in this world

$$p^* = \frac{e^{rT} - d}{u - d}$$



We can generalize the no-arbitrage argument just presented by considering a non-dividend paying stock whose price is $S(0)$ and an European (call or put) option on the stock (or any derivative dependent on the stock) whose current price is V. We suppose that the option lasts for time T and that during the life of the call option the stock price can either move up from $S(0)$ to a new level, $S(0)u$, where $u > 1$, or down from $S(0)$ to a new level, $S(0)d$, where $d < 1$. If the stock price moves up to $S(0)u$, payoff from the option is denoted by V_u; if the stock price moves down to $S(0)d$, the payoff from the option is denoted by V_d.

Present value of the option:

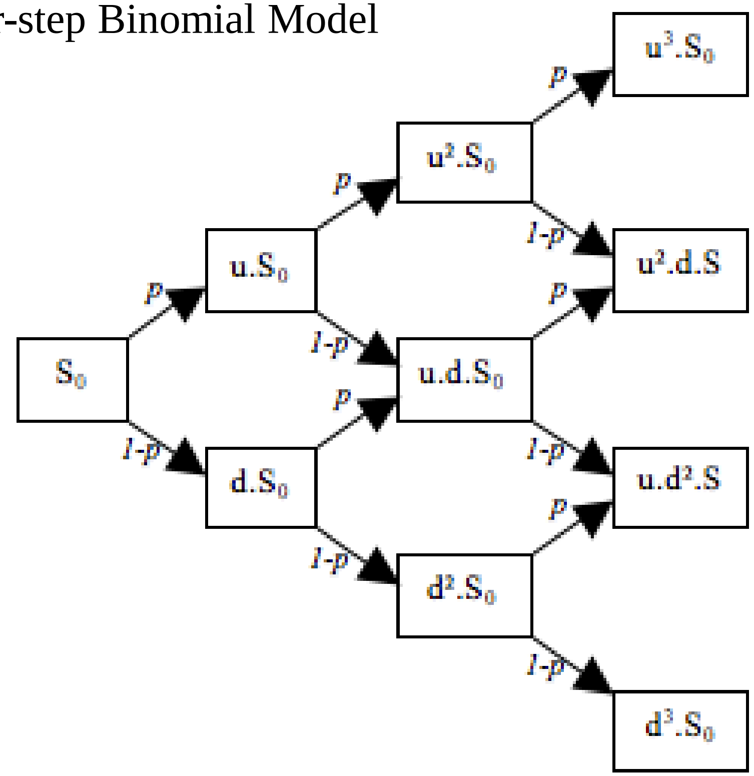
$$S(0)\Delta - V = (S(0)u\Delta - V_u)e^{-rT}.$$

$$\begin{aligned}
V &= S(0)\Delta\left(1-ue^{-rT}\right)+V_u e^{-rT} \\
&= S(0)\left(\frac{V_u - V_d}{S(0)u - S(0)d}\right)\left(1-ue^{-rT}\right)+V_u e^{-rT} \\
&= \frac{V_u\left(1-ue^{-rT}\right)-V_d\left(1-ue^{-rT}\right)+V_u e^{-rT}(u-d)}{u-d} \\
&= \frac{V_u\left(1-de^{-rT}\right)-V_d\left(1-ue^{-rT}\right)}{u-d} \\
&= e^{-rT}\left(p^* V_u + (1-p^*)V_d\right)
\end{aligned}$$

(the value of the option today equals to the expected future payoff in a risk-neutral world discounted at the risk-free rate.)

Cox-Ross-Rubinstein Tree

In four-step Binomial Model



max{S(0)u^3-k,0}

max{S(0)u^{2}d-k,0}

max{S(0)ud^2-k,0}

max{S(0)d^3-k,0}

$$u = e^{\sigma \sqrt{t/n}}$$

$$d = e^{-\sigma \sqrt{t/n}}$$

For binomial tree on futures based on Cox-Ross-Rubinstein tree, the risk neutral probability is:

$$p^* = \frac{1-d}{u-d} = \frac{1-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} = \frac{1-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}(1-e^{-2\sigma\sqrt{\Delta t}})} = \frac{1}{1+e^{\sigma\sqrt{\Delta t}}},$$

$$1-p^* = 1-\frac{1}{1+e^{\sigma\sqrt{\Delta t}}} = \frac{e^{\sigma\sqrt{\Delta t}}}{1+e^{\sigma\sqrt{\Delta t}}} = \frac{1}{1+e^{-\sigma\sqrt{\Delta t}}}.$$

In an N-step binomial model, the price of a European call and put option on a stock S with strike price K to be exercised after N time steps is given by:

$$C_{Eur}(S(0), K, N\Delta t) = e^{-rN\Delta t} E\left[\max\left(S(0)u^i d^{N-i} - K, 0\right)\right]$$

$$= e^{-rN\Delta t} \sum_{i=m}^{N} \binom{N}{i} (p^*)^i (1-p^*)^{N-i} \left(S(0)u^i d^{N-i} - K\right)$$

$$P_{Eur}(S(0), K, N\Delta t) = e^{-rN\Delta t} E\left[\max\left(K - S(0)u^i d^{N-i}, 0\right)\right]$$

$$= e^{-rN\Delta t} \sum_{i=0}^{m-1} \binom{N}{i} (p^*)^i (1-p^*)^{N-i} \left(K - S(0)u^i d^{N-i}\right)$$

where r is the risk-free interest rate, p^* is risk neutral probability and m is the least integer

such that $S(0)u^m d^{N-m} > K$. Then $m = \max\left\{0, \text{the smallest integer } > \dfrac{\ln(K/S(0)) - N\ln d}{\ln(u/d)}\right\}$.

4.The Black-Scholes Model

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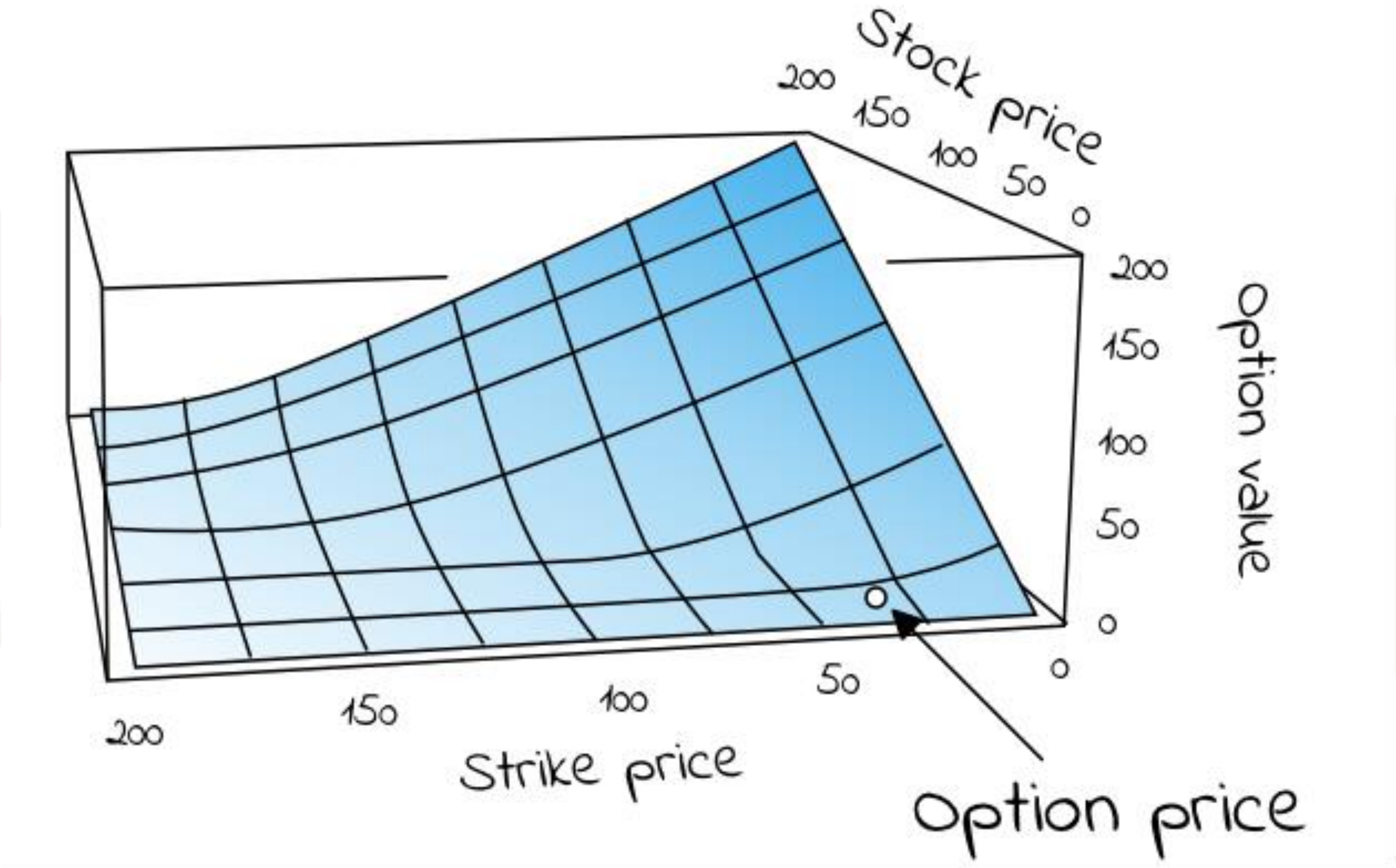

Visual representation of European call option price/value with respect to strike price and stock price

The Black-Scholes equation is the partial differential equation (PDE) that governs the price evolution of European stock options in financial markets operating according to the dynamics of the Black-Scholes model. The equation is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

S: the stock price
t: time
V: the price of the option (as a function of two variables: the stock price *S* and time *t*)
r: is the risk-free interest rate (think interest rate akin to that which you would receive from a money-market fund, government debt or similar “safe” debt securities)
σ: the volatility of the log returns of the underlying security .

Rewrite: $$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}$$

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = rV - rS\Delta$$

Idea from 【risk neutral】

The Black-Scholes *formula* is a solution to the Black-Scholes PDE, given the boundary conditions.

$$C_{E,T} = \max(0, S_T - K)$$

For European call option

$$P_{E,T} = \max(0, K - S_T)$$

For European put option

It calculates the price of European put and call options. That is, it calculates the price of contracts for the right (but not obligation) to buy or sell some underlaying asset at a pre-determined price on a pre-determined date in the future.

Another concept: (from **Cox-Ross-Rubinstein Tree***)*

Denote $\Phi(m, N, p)$ the cumulative binomial distribution with N trials and probability p of success in each trial, that is

$$\Phi(m, N, p) = \sum_{i=0}^{m} \binom{N}{i} p^i (1-p)^{N-i}.$$

Theorem 3.10 (Cox-Ross-Rubinstein Formula)

$$C_{Eur}(S(0), K, N\Delta t) = S(0)\left[1 - \Phi\left(m-1, N, p^* u e^{-r\Delta t}\right)\right] - Ke^{-rN\Delta t}\left[1 - \Phi\left(m-1, N, p^*\right)\right]$$

$$P_{Eur}(S(0), K, N\Delta t) = Ke^{-rN\Delta t}\Phi\left(m-1, N, p^*\right) - S(0)\Phi\left(m-1, N, p^* u e^{-r\Delta t}\right)$$

Find $\lim_{\Delta t \to 0} C_E(S(0), K, N\Delta t)$, according to geometric Brownian motion,the risky asset's price is assumed to behave as an infinitesimal random walk. Then we could get the idea of Black-Scholes model.

What Is the Black Scholes Model?

- The Black Scholes model, also known as the Black-Scholes-Merton (BSM) model, is a mathematical model for pricing an options contract. In particular, the model estimates the variation over time of financial instruments.
- The formula, developed by three economists—Fischer Black, Myron Scholes and Robert Merton—is perhaps the world's most well-known options pricing model. The initial equation was introduced in Black and Scholes' 1973 paper, "The Pricing of Options and Corporate Liabilities," published in the *Journal of Political Economy*.
- In 1997, Scholes and Merton were awarded the Nobel Prize in economics for their work in finding a new method to determine the value of derivatives.

The Black-Scholes model makes certain assumptions:

1. The option is European and can only be exercised at expiration.
2. No dividends are paid out during the life of the option.
3. Markets are efficient (i.e., market movements cannot be predicted).
4. There are no transaction costs in buying the option.
5. The risk-free rate and volatility of the underlying are known and constant.
6. The returns on the underlying asset are normally distributed.

The Black Scholes Formula

$$C = S(0)N(d_1) - Ke^{-rT}N(d_2)$$

$$P = Ke^{-rT}N(-d_2) - S(0)N(-d_1)$$

Where

$$d_1 = \frac{ln\frac{S(0)}{K} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{ln\frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

Notation

$C = Call\ option\ price$

$P = Put\ option\ price$

$K = Strike\ price$

$S = Current\ stock\ (or\ other\ underlying)\ price$

$r = Risk - free\ interest\ rate$

$\sigma = A\ volatility\ parameter\ of$

$\quad a\ non - dividend - paying\ stock$

$t = Time\ to\ maturity$

$N = A\ normal\ distribution$

In general

$$C = F_{t,T}^{P}(S)N(d_1) - F_{t,T}^{P}(K)N(d_2)$$

$$P = F_{t,T}^{P}(K)N(-d_2) - F_{t,T}^{P}(K)N(-d_1)$$

Where

$$d_1 = \frac{\ln\left(F_{t,T}^{P}(S)/F_{t,T}^{P}(K)\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_2 - \sigma\sqrt{T-t} = \frac{\ln\left(F_{t,T}^{P}(S)/F_{t,T}^{P}(K)\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

5.Other Derivatives

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Other derivatives

Futures

Forwards

Swaps

Similarity

Agreements

Value depends on something else

- Deposit
- Bond
- Stock …

Other derivatives

Standardized

Futures

Buy the contract

Pay the futures price

Now

Future

TIME

Forwards

Manufactured (OTC)

Protect enterprises from exchange rate fluctuations

Fix futures rate

Swaps

The borrower can repay the loan in another currency

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Thanks