Problem 1:

(a). For $x < 0$, $\varphi(x) = 0$. so $p_n(x) = 0$

For $0 \le x \le a$, $\varphi(x) = e^{-x}$,

$$p_n(x) = \int_0^x \frac{1}{h_n} \varphi\left(\frac{x-v}{h_n}\right) p(v)\, dv$$

$$= \frac{1}{a} \cdot \frac{1}{h_n} \int_0^x e^{-\frac{(x-v)}{h_n}} dv$$

$$= \frac{1}{a} \cdot \frac{1}{h_n} \cdot e^{-\frac{x}{h_n}} \cdot \left(e^{\frac{v}{h_n}} \cdot h_n\right)\Big|_0^x$$

$$= \frac{1}{a}\left(1 - e^{-\frac{x}{h_n}}\right)$$

For $x > a$, $\varphi(x) = e^{-x}$

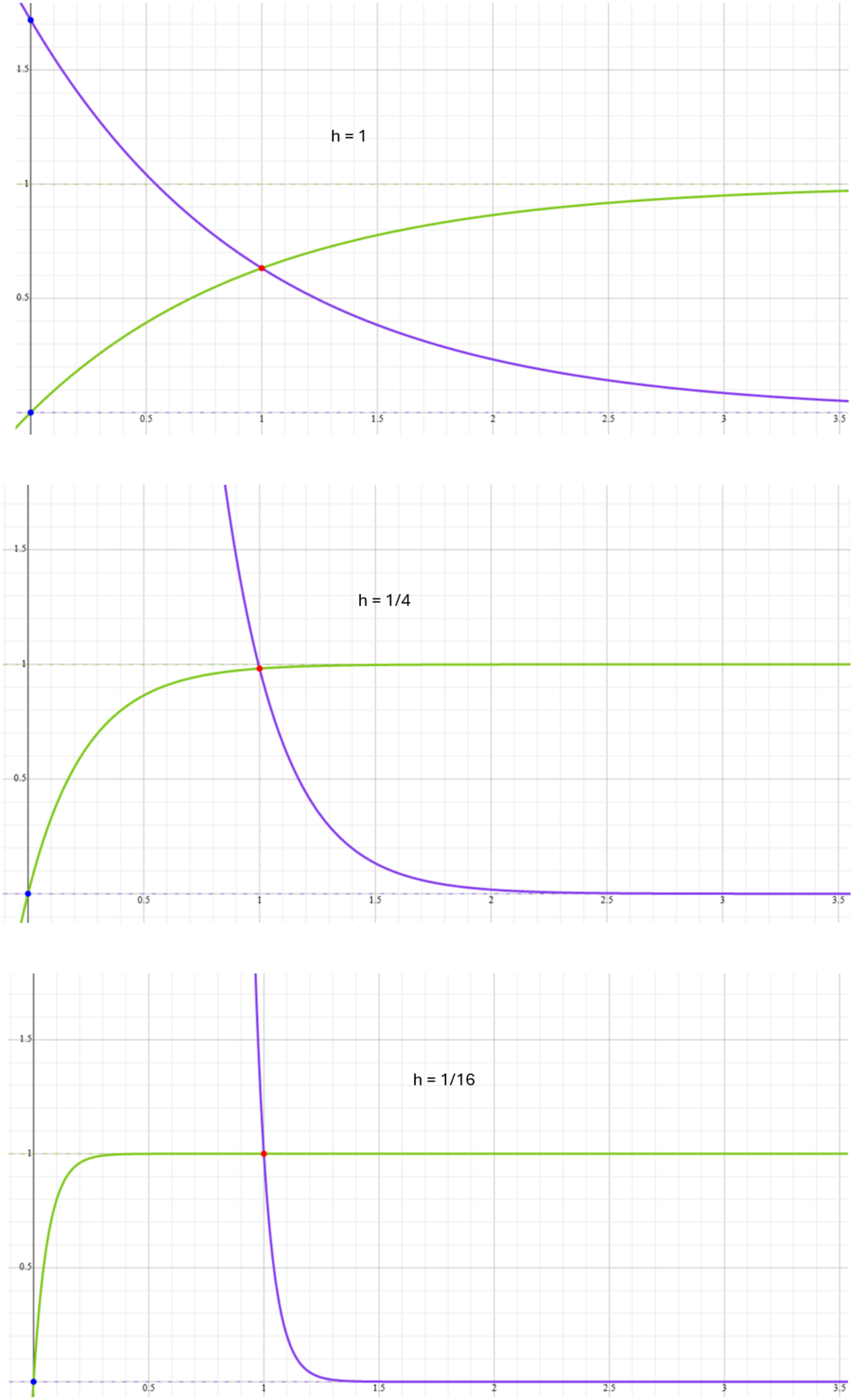
$$p_n(x) = \int_0^a \frac{1}{h_n} \varphi\left(\frac{x-v}{h_n}\right) p(v)\, dv$$

$$= \frac{1}{a} \cdot \frac{1}{h_n} \cdot \int_0^a e^{-\frac{(x-v)}{h_n}} dv$$

$$= \frac{1}{a} \cdot \frac{1}{h_n} \cdot e^{-\frac{x}{h_n}} \left(h_n e^{\frac{a}{h_n}} - h_n\right)$$

$$= \frac{1}{a}\left(e^{a/h_n} - 1\right) e^{-x/h_n}$$

(b)



(c). For $0 < x < a$, $p_n(x) = \frac{1}{a}(1 - e^{-x/h_n})$

$$\text{Bias} = \frac{p(x) - \bar{p}_n(x)}{p(x)} = \frac{\frac{1}{a} - \frac{1}{a}(1 - e^{-x/h_n})}{\frac{1}{a}} = e^{-x/h_n}$$

For Bias ≤ 0.01: $e^{-x/h_n} \le 0.01$

And in 99% range, that means For all $x \ge 0.01a$, Bias should smaller than 0.01.

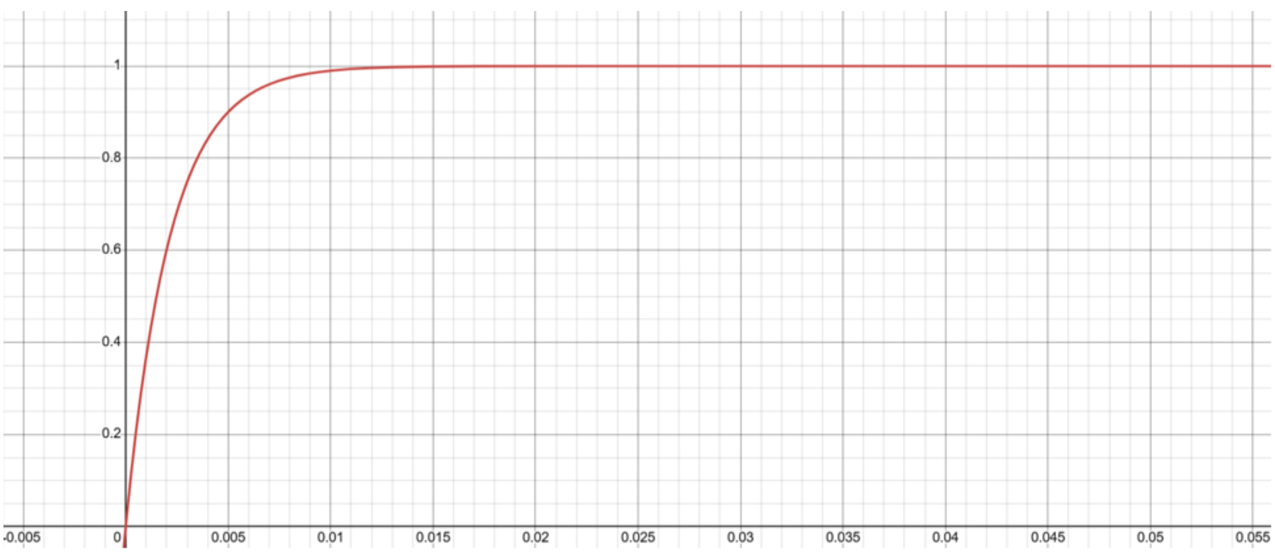
So: $-\frac{0.01a}{h_n} \le \ln 0.01$

$$\frac{0.01a}{h_n} \ge -\ln 0.01$$

$$h_n \le \frac{0.01a}{4.605}$$

So h_n should smaller than $\frac{0.01a}{4.605}$

(d). if $a = 1$, $h_n \le 0.00217$



Problem 2:

(a). To decide w_1 means $g_1(x) > g_2(x)$.

$g_1(x) = p(w_1|x)\,p(w_1) = 2x \cdot \frac{1}{2}$

$g_2(x) = p(w_2|x)\,p(w_2) = 2(1-x) \cdot \frac{1}{2}$

$2x > 2(1-x)$ So $[0, \frac{1}{2}] = R_2$,

$x > \frac{1}{2}$ $[\frac{1}{2}, 1] = R_1$

Error: $p(error) = \int_{R_1} p(w_2|x)\,p(x)\,dx + \int_{R_2} p(w_1|x)\,p(x)\,dx$

$= \cancel{0.75}\ 0.25$

(b). Using NN rule means for all x near x_1 will be classified as w_1, and all x near x_2 will be w_2. And the decision boundary is $\frac{x_1+x_2}{2}$

Since $x_1 < x_2$, That means:

$[0, \frac{x_1+x_2}{2}] = R_1$

$(\frac{x_1+x_2}{2}, 1] = R_2$

And the error will be:

$$p(error) = \int_0^{\frac{x_1+x_2}{2}} p(x|w_2)p(w_2)\,dx + \int_{\frac{x_1+x_2}{2}}^{1} p(x|w_1)p(w_1)\,dx$$

$$= \frac{1}{2}\left(\int_0^{\frac{x_1+x_2}{2}} 2(1-x)\,dx + \int_{\frac{x_1+x_2}{2}}^{1} 2x\,dx \right)$$

(C). If we randomly select x_1 and x_2 from w_1 and w_2, there will have two condition.

① $x_1 < x_2$, same with part (b).

$$p(error) = \frac{1}{2}\left(\int_0^{\frac{x_1+x_2}{2}} 2(1-x)\,dx + \int_{\frac{x_1+x_2}{2}}^{1} 2x\,dx \right.$$

② $x_2 > x_1$

$$p(error) = \frac{1}{2}\left(\int_0^{\frac{x_1+x_2}{2}} 2x\,dx + \int_{\frac{x_1+x_2}{2}}^{1} 2(1-x)\,dx \right.$$

Problem 3:

assume x_i is the center of Voronoi cell,

x_j is the center for other cells.

$\|x_1 - x_i\| \le \|x_1 - x_j\|$

$\|x_2 - x_i\| \le \|x_2 - x_j\|$

x is any point in the line $x_1 x_2$.

so $x = \lambda x_1 + (1-\lambda)(x_2)$, $0 \le \lambda \le 1$

$\because \|x_1 - x_i\| \le \|x_1 - x_j\|$

$\therefore \|x_1 - x_i\|^2 \le \|x_1 - x_j\|^2$

$\|x_1\|^2 + \|x_i\|^2 - 2x_i x_1 \le \|x_1\|^2 + \|x_j\|^2 - 2x_j x_1$

$\|x_i\|^2 - 2x_i x_1 \le \|x_j\|^2 - 2x_j x_1$

also for x_2: $\|x_i\|^2 - 2x_i x_2 \le \|x_j\|^2 - 2x_j x_2$

$\because \|x - x_i\|^2 = \|x\|^2 + \|x_i\|^2 - 2xx_i$, and $x = \lambda x_1 + (1-\lambda)x_2$

$\therefore \|x - x_i\|^2 = \|x\|^2 + \|x_i\|^2 - 2\lambda x_1 x_i - 2x_2 x_i + 2\lambda x_2 x_i$

$\|x - x_j\|^2 = \|x\|^2 + \|x_j\|^2 - 2\lambda x_1 x_j - 2x_2 x_j + 2\lambda x_2 x_j$

$\|x - x_i\|^2 = \|x\|^2 + \|x_i\|^2 - 2\lambda x_1 x_i - 2x_2 x_i + 2\lambda x_2 x_i - \lambda\|x_i\|^2 + \lambda\|x_i\|$

$= \|x\|^2 + \lambda(\|x_i\|^2 - 2x_1 x_i) + (1-\lambda)(\|x_i\|^2 - 2x_i x_2)$

$\|x - x_j\|^2 = \|x\|^2 + \lambda(\|x_j\|^2 - 2x_1 x_j) + (1-\lambda)(\|x_j\|^2 - 2x_j x_2)$

$\because \quad \|x_j\|^2 - 2x \cdot x_j > \|x_i\|^2 - 2x \cdot x_i$

$\& \quad \|x_j\|^2 - 2x_2 x_j > \|x_i\|^2 - 2x_i x_2$

$\& \quad \lambda > 0$

$\therefore \quad \|x - x_i\|^2 < \|x - x_j\|^2$

$\|x - x_i\| < \|x - x_j\|$