A. 1. $2A-B$

$$= 2\begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}2\\4\\6\end{bmatrix} - \begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}-2\\-1\\0\end{bmatrix}$$

2. $$\|A\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

$$\theta = \arccos\left(\frac{A \cdot \vec{x}}{\|A\| \cdot |1|}\right) = \arccos\left(\frac{1}{\sqrt{14}}\right) \approx 75^\circ$$

3. $$\hat{A} = \frac{A}{\|A\|} = \left[\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right]^T = \begin{bmatrix}\frac{\sqrt{14}}{14}\\ \frac{\sqrt{14}}{7}\\ \frac{3\sqrt{14}}{14}\end{bmatrix}$$

4. $$\cos\alpha = \frac{A_x}{\|A\|} = \frac{1}{\sqrt{14}} = \frac{\sqrt{14}}{14}$$

$$\cos\beta = \frac{A_y}{\|A\|} = \frac{2}{\sqrt{14}} = \frac{\sqrt{14}}{7}$$

$$\cos\gamma = \frac{A_z}{\|A\|} = \frac{3}{\sqrt{14}} = \frac{3\sqrt{14}}{14}$$

5 $$A \cdot B = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 4 + 10 + 18 = 32 = B \cdot A$$

6. $$A \cdot B = 32 = \|A\| \cdot \|B\| \cdot \cos\theta = 32$$

$$\theta = \arccos\left(\frac{32}{\|A\|\|B\|}\right) = \arccos\left(\frac{32}{\sqrt{14}\sqrt{77}}\right) \approx 13^\circ$$

7. $\vec{n} \perp A$

$\vec{n} \cdot A = 0 \Rightarrow 1x + 2y + 3z = 0$

for example $x = 2$

$y = -1$

$z = 0$

A is perpendicular to $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

8 $A \times B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2\times6 - 3\times5 \\ 3\times4 - 1\times6 \\ 1\times5 - 2\times4 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$

$B \times A = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5\times3 - 2\times6 \\ 6\times1 - 4\times3 \\ 4\times2 - 5\times1 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$

9 $\vec{n} \cdot A = 0$ $\vec{n} \cdot B = 0$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x + 2y + 3z = 0$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 4x + 5y + 6z = 0$

for $z = -1$ $\begin{cases} x + 2y = 3 \\ 4x + 5y = -6 \end{cases}$ $x = -1$, $y = 2$

$\vec{n} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

10. $\text{rank}(A) = \text{rank}(B) = \text{rank}(C) = 1$

form $X = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 5 & 1 \\ 3 & 6 & 3 \end{bmatrix} \underset{r_3 - 3r_1}{\overset{r_2 - 2r_1}{\sim}} \begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & 3 \\ 0 & -6 & 6 \end{bmatrix} \overset{r_3 - 2r_2}{\sim} \begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$\text{rank}(X) = 2 \Rightarrow A, B, C$ linear correlation

11 $A^T = [1 \quad 2 \quad 3]$

$A^T B = [1\,2\,3] \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 4 + 10 + 18 = 32$

$B^T = [4 \quad 5 \quad 6]$

$A B^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [4\,5\,6] = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$

B. 1. 2A−B

$$=2\begin{bmatrix}1&2&3\\4&-2&3\\0&5&-1\end{bmatrix}-\begin{bmatrix}1&2&1\\2&1&-4\\3&-2&1\end{bmatrix}=\begin{bmatrix}1&2&5\\6&-5&10\\-3&12&-3\end{bmatrix}$$

2

$$AB=\begin{bmatrix}1&2&3\\4&-2&3\\0&5&-1\end{bmatrix}\begin{bmatrix}1&2&1\\2&1&-4\\3&-2&1\end{bmatrix}$$

$$=\begin{bmatrix}1+4+9&2+2-6&1-8+3\\4-4+9&8-2-6&4+8+3\\0+10-3&0+5+2&0-20-1\end{bmatrix}$$

$$=\begin{bmatrix}14&-2&-4\\9&0&15\\7&7&-21\end{bmatrix}$$

$$BA=\begin{bmatrix}1&2&1\\2&1&-4\\3&-2&1\end{bmatrix}\begin{bmatrix}1&2&3\\4&-2&3\\0&5&-1\end{bmatrix}$$

$$=\begin{bmatrix}1+8+0&2-4+5&3+6-1\\2+4+0&4-2-20&6+3+4\\3-8+0&6+4+5&9-6-1\end{bmatrix}$$

$$=\begin{bmatrix}9&3&8\\6&-18&13\\-5&15&2\end{bmatrix}$$

3. $(AB)^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$

$\because (AB)^T = B^T A^T$

$\therefore B^T A^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$

4

$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{vmatrix} = 2 + 0 + 60 - 0 + 8 - 15 = 55$

$|C| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{vmatrix} = 15 - 12 + 12 + 15 - 24 - 6 = 0$

5 the orthogonal matrix's columns and rows are orthogonal unit victors. $Q^T = Q^{-1}$ $QQ^T = I$

$AA^T \neq I$ $CC^T \neq I$

$BB^T = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 14 \end{bmatrix}$ B's row vector form a orthogonal set.

6.

$$A^{-1} = \frac{A^*}{|A|} \qquad |A| \neq 0$$

$$M_{11} = -13 \qquad M_{12} = -4 \qquad M_{13} = 20$$

$$M_{21} = -17 \qquad M_{22} = -1 \qquad M_{23} = 5$$

$$M_{31} = 12 \qquad M_{32} = -9 \qquad M_{33} = -10$$

$$A^{-1} = \frac{1}{55}\begin{bmatrix} -13 & 17 & 12 \\ 4 & -1 & 9 \\ 20 & -5 & -10 \end{bmatrix} = \begin{bmatrix} \frac{-13}{55} & \frac{17}{55} & \frac{12}{55} \\ \frac{4}{55} & \frac{-1}{55} & \frac{9}{55} \\ \frac{4}{11} & \frac{-1}{11} & \frac{-2}{11} \end{bmatrix}$$

$$B^{-1} = \frac{B^*}{|B|} \qquad |B| = -42 \neq 0$$

$$M_{11} = -7 \qquad M_{12} = 14 \qquad M_{13} = -7$$

$$M_{21} = 4 \qquad M_{22} = -2 \qquad M_{23} = -8$$

$$M_{31} = -9 \qquad M_{32} = -6 \qquad M_{33} = -3$$

$$B^{-1} = -\frac{1}{42}\begin{bmatrix} -7 & -4 & -9 \\ -14 & -2 & 6 \\ -7 & 8 & -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{2}{21} & \frac{3}{14} \\ \frac{1}{3} & \frac{1}{21} & \frac{-1}{7} \\ \frac{1}{6} & \frac{-4}{21} & \frac{1}{14} \end{bmatrix}$$

C. 1. $(A-\lambda E)x=0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)-6$

$$= \lambda^2 - 3\lambda - 4$$

$$= (\lambda-4)(\lambda+1) = 0$$

$\lambda_1 = 4 \quad \lambda_2 = -1$

for $\lambda_1 = 4$

$A-\lambda_1 I = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \quad \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 = \frac{2}{3}x_2$

$x_2 = x_2 \quad \Rightarrow X = \begin{bmatrix} \frac{2}{3}x_2 \\ x_2 \end{bmatrix}$

$x_2 = 1 \quad V_1 = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$

for $\lambda_2 = -1$

$A-\lambda_2 I = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 = -x_2$

$x_2 = x_2$

$x_2 = 1 \quad V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

2. V is $\begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix}$

where V^{-1} is $\begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{bmatrix}$

$$V^{-1} A V = \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -1 & -1 \end{bmatrix}$$

3. $\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\frac{2}{3} + 1 = \frac{1}{3}$

4. eigenvector of B is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = -1 + 1 = 0.$

5. these two eigenvector are perpendicular to each other and this matrix is a symmetric matrix. $B = B^T$
the eigenvalues are all real number, and the eigen-
-vectors are real vectors.

D. 1. $f(x)' = 2x$

$f''(x) = 2$

2. $\dfrac{\partial g}{\partial x} = 2x$

$\dfrac{\partial g}{\partial y} = 2y$

3. $\nabla g(x, y) = \left(\dfrac{\partial g}{\partial x}, \dfrac{\partial g}{\partial y} \right) = (2x, 2y)$

4. $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

σ is the standard deviation

μ is the mean or expectation of the distribution.