Computational Physics Final Project Proposal

Studying Hydrodynamics Using Spectral Methods

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Spectral methods may be viewed as an extreme development of the class of discretization schemes for differential equations. The key elements of the it are the trial functions (expansion or approximating functions) and the test functions (weight functions). One commonly used spectral method is the Fourier Galerkin method, where the trial functions and the test functions are trigonometric polynomials. Here I'd like to use the Fourier Galerkin method to solve the Navier-Stokes equations, and then study some or one of the complex phenomenas, like the turbulence, jet, and instability.

The basic Navier-Stokes equations for incompressible fluid is:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} \tag{1}$$

$$\nabla \mathbf{u} = 0 \tag{2}$$

with initial condition $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ and periodic boundary condition (assume the periodicity lengths in all three directions are 2π): $\mathbf{u}(\mathbf{x} + 2\pi\mathbf{e}_j, t) = \mathbf{u}(\mathbf{x}, t), j = 1, 2, 3$

Then we take Fourier transforms to $\mathbf{u}$ and p:

$$\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{\mathbf{u}}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{3}$$

$$p(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{p}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{4}$$

In Fourier space, the Navier-Stokes equations become:

$$\hat{p}_{\mathbf{k}} = -\frac{1}{|\mathbf{k}|^2} i\mathbf{k} \cdot \hat{\mathbf{f}}_{\mathbf{k}} \tag{5}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu|\mathbf{k}|^2\right) \hat{\mathbf{u}}_{\mathbf{k}} = \hat{\mathbf{f}}_{\mathbf{k}} - \mathbf{k}\frac{(\mathbf{k} \cdot \hat{\mathbf{f}}_{\mathbf{k}})}{|\mathbf{k}|^2} \tag{6}$$

where $\hat{\mathbf{f}}_{\mathbf{k}} = -(\widehat{\mathbf{u} \cdot \nabla \mathbf{u}})_{\mathbf{k}}$ is the nonlinear terms.

In Fourier space, we don't have any spatial derivatives, so it just looks like ordinary differential equations for the Fourier coefficients. Therefore, we can calculate the Fourier coefficients using FFT, then use RK4 or Verlet method to solve the ordinary differential equation, then take the inverse Fourier transforms, and we can get the time evolution of velocity and pressure in real space.

For the whole project, I will firstly use the Fourier Galerkin method to solve the most simple and basic Burger's equation as a test. Then, extend the method to solve the one-dimension and two-dimension Navier-Stokes equations with some simple initial conditions and periodic boundary conditions. Last, try to study some nonlinear or chaotic phenomenon, like the Kelvin-Helmholtz instabilities, and turbulence and jets.

There are probably some predictable difficulties in this project. First, how to calculate the Fourier transform of the nonlinear terms. Second, how to determine the initial condition for different situations which will automatically satisfy the NS equations. Third, the two dimensional discrete Fourier transforms.