1 Linear Programming

1.1 Standard Form of LP

Decision variables $\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$
Objective coefficients $\mathbf{c} = (c_1, c_2, \dots, c_n)^\top$
Right-hand-side constraints $\mathbf{b} = (b_1, b_2, \dots, b_m)^\top$
Structural coefficients $A \in \mathbb{R}^{m\times n}$
Maximize $z = \sum_{j=1}^n c_j x_j$
subject to $\sum_{j=1}^n a_{ij} x_j = b_i, i = 1, ..., m$
. $x_j \geq 0, j = 1, ..., n$
where $b_i \geq 0, n > m$

Transforming an LP to Standard Form

- Nonzero lower bound
  suppose $x_j \geq l_j, l_j \neq 0$, replace $x_j = x'_j + l_j, x'_j \geq 0$
- Non-positive upper bound
  suppose $x_j \geq u_j, u_j \leq 0$, replace $x_j = u_j - x'_j$
- Unrestricted (or Free) Variables
  - $x_1 + x_2 = 8$, replace $x_2 = 8 - x_1$ if x_2 is free
  - define $x_j^+, x_j^- \geq 0$, use $x_j^+ - x_j^-$ substitute x_j
- Inequality Constraints
  define slack variable $s_1 \geq 0$, $ax \geq b \leftrightarrow ax - s_1 = b$

1.2 Solving LP

Basic Feasible Solution

- **Definition**. For an LP in the standard form, a basic solution is called a BFS if it satisfies the non-negativity constraints.
- **Theorem**
  - $\exists$ a feasible solution $\leftrightarrow \exists$ a BFS.
  - $\exists$ an optimal FS $\leftrightarrow \exists$ an optimal BFS.
- **Additional remark**. Each BFS corresponds to a corner point in the graphic representation of LP.

Given $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$ with $\mathbf{x}_N = 0$. Write $\mathbf{A} = (\mathbf{B}, \mathbf{N})$ (columns of $\mathbf{A}$ corresponding to variables in $\mathbf{x_B}, \mathbf{x_N}$)
$\mathbf{Ax} = \mathbf{Bx_B} + \mathbf{Nx_N} = \mathbf{Bx_B} = \mathbf{b}, \mathbf{x_B} = \mathbf{B^{-1}b}$
Proof. Assume $\mathbf{x}$ is a feasible solution, but not BFS. Can find a $\mathbf{y}$ ($\mathbf{Ay} = 0$) making $\mathbf{x} + k\mathbf{y}$ to be BFS.

Simplex method A simplex form $\leftrightarrow$ a BFS:

- Each basic variable corresponds to a row, and value of basic variable is right-hand side of row
- The value of the objective function is equal to the right-hand side of row 0
- Every basic variable appears in one and only one equation, but not row 0
- Each basic variable has the coefficient 1 in the equation it appears
- Each equation has only one basic variable
- Variable z only appears in row 0 with coefficient 1

The BFS is optimal iff row 0 has no negative numbers

Optimality Test Suppose $\mathbf{x} = (\mathbf{x_B}, \mathbf{x_N})$ is a BFS
$\mathbf{x_B} = \mathbf{B^{-1}(b - Nx_N)} = \mathbf{B^{-1}b}$
$\mathbf{z} = \mathbf{c_Bx_B} + \mathbf{c_Nx_N} = \mathbf{c_BB^{-1}b}$
if non-basic variable becomes non-zero:
$\mathbf{z} = \mathbf{c_Bx_B} + \mathbf{c_Nx_N} = \mathbf{c_BB^{-1}(b - Nx_N)} + \mathbf{c_Nx_N}$
$= \mathbf{c_BB^{-1}b} - (\mathbf{c_BB^{-1}N} - \mathbf{c_N})\mathbf{x_N}$
consider a non-basic variable x_k increased
$\mathbf{z} = \mathbf{c_BB^{-1}1b} - (\mathbf{c_BB^{-1}A_k} - \mathbf{c_k})\mathbf{x_k}$
$\bar{c_k} = \mathbf{c_BB^{-1}A_k} - c_k$ referred as reduced cost
The current basic solution is optimal if and only if the reduced cost is nonnegative for all non-basic variables.

Ratio Test non-basic variable x_k increased
$\mathbf{x_B} = \mathbf{B^{-1}(b - Nx_N)} = \mathbf{B^{-1}b} - \mathbf{B^{-1}A_kx_k}$
To keep non-negativity, $(\mathbf{x_B})_\mathbf{i} \geq 0$
ratio test $\underset{i}{\operatorname{argmin}} (\mathbf{B^{-1}b})_\mathbf{i} / (\mathbf{B^{-1}A_k})_\mathbf{i}$

1.3 Sensitivity Analysis

Shadow Price Optimal of dual variable: $\lambda = \mathbf{c_BB}^{-1}$
In the optimal solution, if a constraint is not tight ($<$ or $>$), then its shadow price must be 0, if tight ($=$), may or may not be 0

Constraint Analysis

- Non-basic variable $\mathbf{c}_j + \Delta c_j$
  $\mathbf{c_BB^{-1}A_j} - (c_j + \Delta c_j) \geq 0$
- Basic variable $\mathbf{c}_j + \Delta c_j$
  $(\mathbf{c_B} + \Delta c_j)\mathbf{B}^{-1}(\mathbf{A_N}, \mathbf{I}) - (\mathbf{c_N}, \mathbf{0}, ..) \geq 0$
  optimal solution unchange within range, vice versa
- $\mathbf{b}_r$: $\mathbf{x'_B} = \mathbf{B^{-1}b'} \geq 0$, $\Delta S = \Delta \mathbf{b}_r \cdot \lambda_r$
  optimal solution always change. Basic variable change beyond range, vice versa.

1.4 Duality

Primal: max $z = \sum_{i=1}^n c_i x_i$
s.t. $\sum_{i=1}^n a_{ji} x_i \leq b_j, j = 1, ..., m$
$x_i \geq 0, i = 1, ..., n$
Dual: max $w = \sum_{j=1}^m b_j y_j$
s.t. $\sum_{j=1}^m a_{ji} y_j \geq c_i, i = 1, ..., n$
$y_j \geq 0, j = 1, ..., m$

Primal model (MAX)	Dual model (MIN)
Constraint j is $\leq$	Variable $y_j \geq 0$
Constraint j is $=$	Variable y_j is free
Constraint j is $\geq$	Variable $y_j \leq 0$
Variable $x_i \geq 0$	Constraint i is $\geq$
Variable x_i is free	Constraint i is $=$
Variable $x_i \leq 0$	Constraint i is $\leq$

Weak Duality $Z(x) \leq W(y)$
proof: $\mathbf{cx} \leq \mathbf{yAx} \leq \mathbf{yb}$

Strong Duality If either of Primal or Dual has an optimal feasible bounded solution, then 1. the other problem also has *ofbs*, 2. $z = w$
proof: suppose $\mathbf{x}^* = (\mathbf{x_B}, \mathbf{x_N})$ is optimal solution,
Reduced cost for non-basic variables $\mathbf{c_BB^{-1}N} - \mathbf{c_N} \geq 0$,
Let $\mathbf{y} = \mathbf{c_BB^{-1}}$, we have:

- $\mathbf{yA} = \mathbf{y(B\ N)} = (\mathbf{c_B}, \mathbf{c_BB^{-1}N}) \geq (\mathbf{c_B}, \mathbf{c_B}) = \mathbf{c}$
- $\mathbf{y}$ is feasible to the dual
- $\mathbf{yb} = \mathbf{c_BB^{-1}b} = \mathbf{c_Bx_B} = \mathbf{cx}^*$
- $\mathbf{y}$ is optimal to the dual

Complementary Slackness Any feasible solution $\mathbf{x}, \mathbf{y}$, they both are optimal iff for any i:
(1) $\mathbf{x_i} > 0 \rightarrow (\pi\mathbf{A})_\mathbf{i} = \mathbf{c_i}$, (2) $\mathbf{x_i} = 0 \leftarrow (\pi\mathbf{A})_\mathbf{i} > \mathbf{c_i}$
proof: If 1,2 true, $(\mathbf{yA} - \mathbf{c})\mathbf{x} = \mathbf{0} \rightarrow \mathbf{yAx} = \mathbf{cx}$ or $\mathbf{yb} = \mathbf{cx}$, from strong duality, $\mathbf{x}, \mathbf{y}$ optimal.
If $\mathbf{x}, \mathbf{y}$ optimal, $\mathbf{yb} = \mathbf{cx}$ and $(\mathbf{yA} - \mathbf{c})\mathbf{x} = \mathbf{0}$, implies 1,2.

1.5 Transportation Problem

source with supplying capacity s_i, destination with demand d_i, cost c_{ij}, transportation plan x_{ij}
min $\sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$
$\sum_{j=1}^n x_{ij} = s_i, i = 1, ..., m$
$\sum_{i=1}^m x_{ij} = d_i, i = 1, ..., n$
$x_{ij} \geq 0, i = 1, ..., m, j = 1, ..., n$.
If total supply = demand, coefficience matrix contains $k \leq m + n - 1$ linearly independent row vectors, k basic variables, k non-zero elements in optimal solution.
$\mathbf{p} = \mathbf{c_BB^{-1}}$ is basic variables for dual problem, contains k elements. Thus always a shadow price $= 0$.

1.6 Cooperative Game

N: set of players
$V(S)$: function gives cost of a coalition S(subset of N)
An allocation is a distribution of Ground Coalition V(N), s.t. $\sum_{i\in N} x_i = V(N)$
develop core allocation rules $\sum_{i\in S} x_i \leq V(S), \forall S$

1.7 Max Flow Problem

$G = (N, A)$, flow on arc x_{ij}, capacity of flow in arc U_{ij}, source node s, destination node t, Max flow value $v(x)$. Original network G, feasible flow x, residual network $G(x)$, residual capacity r_{ij}, argumenting path P.

Min cut The capacity of a cut (S,T) is sum of capacities of all forward arcs, CAP(S,T)=$\sum_{i\in S}\sum_{j\in T} u_{ij}$.
Dual LP (min cut): $\pi_i \in \{0, 1\}$ two sets, w_{ij} cut edge
min $\sum u_{ij} w_{ij}$ *s.t.* $\pi_1 - \pi_i + w_{1i} \geq 0, \pi_i - \pi_n + w_{in} \geq 0, -\pi_1 + \pi_n \geq 1, w_{ij} \geq 0, \pi$ free

Weak Duality Theorem for Max Flow Problem
define **flow across the cut**: flow on forward arcs - backward $F_x(S, T) = \sum_{i\in S}\sum_{j\in T} x_{ij} - \sum_{i\in S}\sum_{j\in T} x_{ji}$
Claim: 1. $F_x(S, T) = v$ =flow into t. 2. $F_x(S, T) \leq$ capacity of a cut. **Weak**: $v(x) \leq CAP(S, T)$

Strong Duality: Max Flow Min Cut Theorem
The following are equivalent. $1 \Rightarrow 2, 3 \Rightarrow 1, 2 \Rightarrow 3$

1 A flow x is maximum
2 There is no augmenting path in $G(x)$.
3 There is an s-t cut (S, T) whose capacity is the flow value of x.
* **Corollary**. The maximum flow value is the minimum value of a cut

1.8 Min Cost Network Flow

- **Problem input**
  - Network G=(N,A)
  - Flow cost c_{ij} for each arc (i, j) in A
  - Lower and upper bounds l_{ij}, u_{ij} for each arc
  - Supply or demand $b(j)$ for each node j
- **Decisions** Flow x_{ij} for each arc
- **Objective** min total flow cost $\sum_{(i,j)} x_{ij} c_{ij}$
- **Constraints**
  - Lower and upper bounds
  - Flow conservation
  - A necessary condition for the problem to be feasible is that total supply is equal to total demand

LP Formulation min $\mathbf{cx}$ *s.t.* $\mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq 0$
$A_{ij} \in \{-1, 0, 1\}$, each column has exactly one 1 & -1
LP with Consecutive 1' s in Columns (Each row k is multiplied by -1 and added to row k+1) $\rightarrow$ IP