Midterm Exam IEDA 5270

Name ______________________ Student ID ______________________

Question 1 (10 points)

Let $X \sim \text{Binomial}(n, \theta)$ and $Y \sim \text{Binomial}(n, \theta^2)$ be independent with $\theta \in (0, 1)$ being an unknown parameter.

(a) (5 points) Find a minimal sufficient statistic.

(b) (5 points) Is the minimal sufficient statistic complete?

Solution:

(a)

$$\frac{f(x_1, y_1|\theta)}{f(x_2, y_2|\theta)} = \frac{\binom{n}{x_1}\binom{n}{y_1}\theta^{x_1+2y_1}(1-\theta)^{n-x_1}(1-\theta^2)^{n-y_1}}{\binom{n}{x_2}\binom{n}{y_2}\theta^{x_2+2y_2}(1-\theta)^{n-x_2}(1-\theta^2)^{n-y_2}} \propto \left(\frac{\theta}{1-\theta}\right)^{x_1-x_2}\left(\frac{\theta^2}{1-\theta^2}\right)^{y_1-y_2}$$

does not depend on θ iff $(x_1, y_1) = (x_2, y_2)$.

Then (X, Y) is a minimal sufficient statistic for θ.

(b) Let $g(X, Y) = X^2 - X + Y - nY$. Then

$$\mathsf{E}[g(X, Y)] = n\theta - n\theta^2 + n^2\theta^2 - n\theta + n\theta^2 - n^2\theta^2 = 0$$

holds for all θ. Thus (X, Y) is not complete.

Question 2 (20 points)

Suppose that $c_1, c_2, \ldots, c_n$ are known positive constants and that X_i follows a gamma distribution with shape parameter 2 and scale parameter θc_i with $\theta > 0$, i.e., with a density of

$$f(x) = (\theta c_i)^{-2} x e^{-x/(\theta c_i)},\ x \geq 0.$$

Suppose $\{X_i\}$ are mutually independent.

(a) (5 points) Compute the Cramér-Rao lower bound (CRLB) for the variance of all unbiased estimators of θ.

(b) (5 points) Find the maximum likelihood estimator of θ, $\widehat{\theta}_{\text{MLE}}$.

(c) (5 points) Is $\widehat{\theta}_{\text{MLE}}$ unbiased? Does it achieve the CRLB?

(d) (5 points) Consider the class of all estimators for θ of the forms $\widehat{\theta} = \sum_{i=1}^n d_i X_i$. Find $d_1, d_2, \ldots, d_n$ so that $\widehat{\theta}$ minimizes the mean-squared error $\mathsf{E}[(\widehat{\theta} - \theta)^2]$.

Solution:

$$l \propto -2n\log\theta - \frac{1}{\theta}\sum_{i=1}^{n}\frac{X_i}{c_i}.$$

(a) We have Cramé-Rao inequality:

$$\mathrm{Var}_\theta(\hat{\theta}) \geq \frac{1}{\mathcal{I}(\theta)},$$

where $\mathcal{I}(\theta)$ is computed by

$$\mathcal{I}(\theta) = -\mathsf{E}\left[\frac{\partial l^2}{\partial\theta^2}\right] = -\mathsf{E}\left[\frac{2n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n}\frac{X_i}{c_i}\right] = \frac{2n}{\theta^2}.$$

(b)

$$\hat{\theta}_{\mathrm{MLE}} = \frac{1}{2n}\sum_{i=1}^{n}\frac{X_i}{c_i}.$$

(c)

$$\mathsf{E}[\hat{\theta}_{\mathrm{MLE}}] = \sum_{i=1}^{n}\frac{1}{2nc_i}2c_i\theta = \theta, \quad \mathrm{Var}(\hat{\theta}_{\mathrm{MLE}}) = \frac{\theta^2}{2n}.$$

$\hat{\theta}_{\mathrm{MLE}}$ is unbiased and achieved CRLB.

(d)

$$\mathsf{E}[(\hat{\theta}-\theta)^2] = \mathrm{Var}(\hat{\theta}) + \mathsf{E}[\hat{\theta}-\theta]^2 = \sum_{i=1}^{n}2d_i^2c_i^2\theta^2 + (\sum_{i=1}^{n}2c_id_i\theta - \theta)^2$$

Taking partial derivatives gives $d_i = \frac{1}{2n+1}c_i$ for all i.

Question 3 (10 points)
Suppose $X_1,\ldots,X_n$ are independent normal variables, each with unit variance and with $\mathsf{E}[X_i] = \alpha t_i + \beta t_i^2, i = 1,2,\ldots,n$. Here α and β are unknown parameters and $t_1,\ldots,t_n$ are known constants. Find the UMVUE for α and β.

Solution: $(\sum_{i=1}^n t_iX_i, \sum_{i=1}^n t_i^2X_i)$ is complete and sufficient for (α,β). Hence, it suffices to find a function of $(\sum_{i=1}^n t_iX_i, \sum_{i=1}^n t_i^2X_i)$ that is unbiased for α and β. Note that

$$\mathsf{E}[\sum_{i=1}^{n}t_iX_i] = \alpha\sum_{i=1}^{n}t_i^2 + \beta\sum_{i=1}^{n}t_i^3$$
$$\mathsf{E}[\sum_{i=1}^{n}t_i^2X_i] = \alpha\sum_{i=1}^{n}t_i^3 + \beta\sum_{i=1}^{n}t_i^4$$

Let $\phi_2 = \sum_{i=1}^n t_i^2, \phi_3 = \sum_{i=1}^n t_i^3, \phi_4 = \sum_{i=1}^n t_i^4, T_1 = \sum_{i=1}^n t_i X_i$ and $T_2 = \sum_{i=1}^n t_i^2 X_i$. Then

$$\hat{\alpha} = \frac{T_1\phi_4 - T_2\phi_3}{\phi_2\phi_4 - \phi_3^2}, \quad \hat{\beta} = \frac{T_2\phi_2 - T_1\phi_3}{\phi_2\phi_4 - \phi_3^2}$$

are the UMVUE of α and β, respectively.

Question 4 (10 points)

Suppose $X_1, \ldots, X_n$ is a random sample from the discrete uniform distribution on points $1, 2, \ldots, \theta$, where θ is an integer.

(a) Consider $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$, where $\theta_0 > 0$ is a known integer. Show that

$$T_1(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} > \theta_0 \\ \alpha & X_{(n)} \le \theta_0 \end{cases}$$

is a UMP test of size α.

(b) (5 points) Consider $H_0 : \theta = \theta_0$ versus $H_1 : \theta \ne \theta_0$, where $\theta_0 > 0$ is a known integer. Show that

$$T_2(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} > \theta_0 \text{ or } X_{(n)} \le \theta_0\alpha^{1/n} \\ 0 & \text{otherwise} \end{cases}$$

is a UMP test of size α.

Solution:

(a) The family of pmf for $X_{(n)}$ has monotone likelihood ratio. Then a UMP test of size α is

$$T'(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} > c \\ \gamma & X_{(n)} = c \\ 0 & X_{(n)} < c \end{cases}$$

where γ satisfies

$$\mathsf{E}_{\theta_0}[T'] = \alpha.$$

For any $\theta > \theta_0$, the power of T' is

$$\mathsf{E}_\theta[T'] = 1 - (1-\alpha)\frac{\theta_0^n}{\theta^n}$$

which is equal to $\mathsf{E}_\theta[T_1]$. Combining with $\sup_{\theta \le \theta_0} \mathsf{E}_\theta[T_1] \le \alpha$, we conclude that T_1 is a UMP test of size α.

(b) For $\theta > \theta_0$, we have

$$\mathsf{E}_\theta[T_2] = 1 - (1-\alpha)\frac{\theta_0^n}{\theta^n}$$

Consider hypotheses $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$. The UMP test of size α can be constructed by

$$T''(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} < d \\ \eta & X_{(n)} = d \\ 0 & X_{(n)} > d \end{cases}$$

For $\theta_0 \alpha^{1/n} \le \theta \le \theta_0$, the power of T_2 is equal to $\mathsf{E}_\theta[T''] = \alpha \frac{\theta_0^n}{\theta^n}$. For $\theta \le \theta_0 \alpha^{1/n}$, the power of T_2 is 1. Thus we conclude T_2 has size α and its power is the same as the power of T' when $\theta > \theta_0$ and is no smaller than the power of T'' when $\theta < \theta_0$. Thus T_2 is UMP test of size α.

Question 5 (25 points)

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be i.i.d. with $X_i \sim N(0,1)$ and $Y_i | X_i = x \sim N(x\theta, 1)$. Here, given $X_1 = x_1, \ldots, X_n = x_n$, the variables Y_i are conditionally independent.

(a) (5 points) Find the MLE $\widehat{\theta}$ for θ.

(b) (5 points) Find the Fisher Information $\mathcal{I}(\theta)$ for a single observation (X_1, Y_1).

(c) (5 points) Determine the limiting distribution of $\sqrt{n}(\widehat{\theta} - \theta)$ and use it to give a $1-\alpha$ asymptotic confidence interval for θ based on $\mathcal{I}(\widehat{\theta})$.

(d) (5 points) Compare the interval in part (c) with a $1-\alpha$ asymptotic confidence interval based on observed Fisher information.

(e) (5 points) Determine the exact distribution of $\sqrt{\sum_i X_i^2}(\widehat{\theta} - \theta)$ and use it to find the true coverage probability of the interval in part (d).

Solution:

(a)

$$l \propto -\frac{1}{2} \sum_{i=1}^n (Y_i - X_i \theta)^2$$

$$\hat{\theta} = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n X_i^2}$$

(b)

$$\mathcal{I}(\theta) = -\mathsf{E}_\theta \left[\frac{\partial^2 l}{\partial \theta^2} \right] = \mathsf{E}_\theta \left[X_1^2 \right] = 1$$

(c) $N(0,1)$. A $1-\alpha$ asymptotic confidence interval is defined as

$$\{\theta \in \Theta : \sqrt{n} |\widehat{\theta}_n - \theta| \le z_{\alpha/2}\} = (\theta - z_{\alpha/2}/\sqrt{n}, \theta + z_{\alpha/2}\sqrt{n}).$$

(d) The $1-\alpha$ asymptotic confidence interval is $(\theta - z_{\alpha/2}/\sqrt{\sum_{i=1}^n X_i^2}, \theta + z_{\alpha/2}\sqrt{\sum_{i=1}^n X_i^2})$.

(e) The condition distribution of $\sqrt{\sum_i X_i^2}(\widehat{\theta} - \theta)$ given $X_1 = x_1, \ldots, X_n = x_n$ is $N(0,1)$. So $\sqrt{\sum_i X_i^2}(\widehat{\theta} - \theta) \sim N(0,1)$. The coverage probability in part (d) is exactly $1-\alpha$.

Question 6 (15 points)
Let $X_1, \ldots, X_n$ be a random sample from a continuous CDF F_θ on $\mathbb{R}$, parametrized by a real-valued θ. A level $1-\alpha$ confidence band for the CDF F_θ is a collection of confidence intervals $\{C_t(\boldsymbol{X}) : t \in \mathbb{R}\}$ such that

$$\inf_\theta \mathbb{P}_\theta(F_\theta(t) \in C_t(\boldsymbol{X}) \text{ for all } t \in \mathbb{R}) \geq 1-\alpha.$$

(a) (5 points) Suppose $F_\theta(t)$ is nonincreasing in θ for every t, find a level $1-\alpha$ confidence band for the CDF F_θ.

(b) (5 points) Suppose $X_1, \ldots, X_n$ is a normal random sample with unknown μ and known σ^2. Find a level $1-\alpha$ confidence band for the CDF F_μ.

(c) (5 points) Suppose $X_1, \ldots, X_n$ is a normal random sample with known μ and unknown σ^2. Find a level $1-\alpha$ confidence band for the CDF F_σ.

Solution:

(a) Note that $F_{\theta_1}(t) \geq F_{\theta_2}(t)$ for every t iff $\theta_1 \leq \theta_2$. Let $\Theta = [L(X), U(X)]$ be a $1-\alpha$ confidence interval for θ. Then $\{\{F_\theta(t) : \theta \in \Theta\} : t \in \mathbb{R}\} = \{[F_{U(X)}(t), F_{L(X)}(t)] : t \in \mathbb{R}\}$ is a $1-\alpha$ confidence band for F_θ.

(b) It is obvious that $F_\mu(t)$ is nonincreasing in μ for every t. The confidence interval for μ is $[\bar{X} - \frac{z_{\alpha/2}\sigma}{\sqrt{n}}, \bar{X} + \frac{z_{\alpha/2}\sigma}{\sqrt{n}}]$. Then a $1-\alpha$ confidence band for CDF is given by $\{[F_{\bar{X}+\frac{z_{\alpha/2}\sigma}{\sqrt{n}}}(t), F_{\bar{X}-\frac{z_{\alpha/2}\sigma}{\sqrt{n}}}(t)] : t \in \mathbb{R}\}$.

(c) $F_\sigma(t)$ is nondecreasing for $t < \mu$ and nonincreasing for $t \geq \mu$. The confidence interval for σ is $\left[\sqrt{\frac{(n-1)S^2}{\chi^2_{\alpha/2}}}, \sqrt{\frac{(n-1)S^2}{\chi^2_{1-\alpha/2}}}\right]$. Then a $1-\alpha$ confidence band for CDF is given by

$$\left\{[F_{\sqrt{\frac{(n-1)S^2}{\chi^2_{\alpha/2}}}}(t), F_{\sqrt{\frac{(n-1)S^2}{\chi^2_{1-\alpha/2}}}}(t)] : t < \mu\right\} \cup \left\{[F_{\sqrt{\frac{(n-1)S^2}{\chi^2_{1-\alpha/2}}}}(t), F_{\sqrt{\frac{(n-1)S^2}{\chi^2_{\alpha/2}}}}(t),] : t \geq \mu\right\}$$