Cryptography and Security

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Version 3

Lecture 15: Secret Sharing Schemes

Main Topics of this Lecture

1. The need for secret sharing.
2. Several secret sharing schemes.

Scenarios Requiring Secret Sharing

Scenario I: A bank has a vault which must be opened every day. The bank employs four tellers, but they do not trust the combination of any two individual tellers. The bank wants to design a system whereby any three tellers can gain access to the vault, but no two of them can do so. This problem can be solved by means of **secret sharing schemes**.

Scenario II: According to *Time Magazine* (p. 13, May 4, 1992), the control of nuclear weapons in Russia involves a similar "2-out-of-3" access mechanism. The three parties involved are the President, the Defense Minister and the Defense Ministry.

The Idea of Secret Sharing

The problem: A **dealer** has a secret $s \in \mathcal{S}$ to be shared by **a group of** n **participants** in such a way that some subgroups of them can recover the secret, while other subgroups cannot.

Solution: The idea of secret sharing
The dealer designs n functions $f_1, \cdots, f_n$. The dealer then computes

$$s_i = f_i(s),$$

and distributes it to one participant as his/her **share**.

When some subgroup of participants meet together with their shares, they are able to compute s from their shares with a **secret recovering function**.

The Building Blocks of a Secret Sharing Scheme

A secret s: We assume that the secret s is equally likely to be any element of a **Secret Space** $\mathcal{S}$.

A dealer: A special participant who chooses the secret s from $\mathcal{S}$ and computes a share for each participant.

A group of participants P_i**:** they are going to share the secret designed by the dealer.

A share computing procedure: The dealer will follow this procedure to compute a share from s for each participant.

A secret recovering procedure: When "some" subgroup of participants meet together with their shares, they can recover the secret s by following this procedure.

(t, n)-Threshold Schemes

Definition: Let t, n be positive integers, $t \leq n$. A (t, n)**-threshold scheme** is a method of sharing a secret s among a set of n participants in such a way that any t participants can determine s, but no group of $t-1$ or fewer participants can get any information about s.

An (n, n)-Threshold Scheme

Secret: a binary string s of w bits.

Participants: $P_1, P_2, \cdots, P_n$.

Computing the shares: A dealer first chooses $n-1$ random binary strings $s_1, s_2, \cdots, s_{n-1}$ of w bits, and then computes

$$s_n = s \oplus s_1 \oplus s_2 \oplus \cdots \oplus s_{n-1}.$$

The he/she distributes s_i to P_i as the share.

Recovering the secret: When all the participants come together, the secret is computed as

$$s = s_1 \oplus s_2 \oplus \cdots \oplus s_n.$$

The Shamir (t, n)-Threshold Scheme

Secret: An element $s \in \mathbf{Z}_p$, where p is a prime.

Participants: $P_1, P_2, \cdots, P_n$, where $2 \leq t \leq n$.

System parameters: A dealer first chooses n distinct nonzero elements of $\mathbf{Z}_p$, denoted x_i, $1 \leq i \leq n$. The condition is that $n + 1 \leq p$. The dealer then gives x_i to P_i. The values x_i are public.

The Shamir (t, n)-Threshold Scheme

Computing and distributing the shares: The dealer chooses (independently at random) $t-1$ elements of $\mathbf{Z}_p$, a_1, a_2, $\cdots$, a_{t-1}. For each $1 \leq i \leq n$, the dealer computes $y_i = a(x_i)$, where

$$a(x) = (s + a_1 x + a_2 x^2 + \cdots + a_{t-1} x^{t-1}) \bmod p.$$

The dealer then gives y_i to P_i.

Remark: Only the dealer knows s and $a_1, \cdots, a_{t-1}$.

Question: Why the constants x_i cannot be zero?

The Shamir (t, n)-Threshold Scheme

Lemma: Let $i_1, i_2, \cdots, i_t$ be t distinct integers in the set $\{1, 2, \cdots, n\}$. The the following **Vandermonde matrix**

$$A = \begin{bmatrix} 1 & x_{i_1} & x_{i_1}^2 & \cdots & x_{i_1}^{t-1} \\ 1 & x_{i_2} & x_{i_2}^2 & \cdots & x_{i_2}^{t-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{i_t} & x_{i_t}^2 & \cdots & x_{i_t}^{t-1} \end{bmatrix}$$

is invertible over the finite field $\mathbf{Z}_p$.

Remark: It is known that

$$\det(A) = \prod_{1 \le j < k \le t} (x_{i_k} - x_{i_j}) \bmod p.$$

The Shamir (t, n)-Threshold Scheme

Recovering the secret: Suppose that participants $P_{i_1}, P_{i_2}, \cdots, P_{i_t}$ want to determine s. They know that

$$y_{i_j} = a(x_{i_j}), \quad j = 1, 2, \cdots, t$$

and $a(x)$ is the (secret) polynomial chosen by the dealer. So they have

$$\begin{bmatrix} 1 & x_{i_1} & x_{i_1}^2 & \cdots & x_{i_1}^{t-1} \\ 1 & x_{i_2} & x_{i_2}^2 & \cdots & x_{i_2}^{t-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{i_t} & x_{i_t}^2 & \cdots & x_{i_t}^{t-1} \end{bmatrix} \begin{bmatrix} s \\ a_1 \\ \vdots \\ a_{t-1} \end{bmatrix} = \begin{bmatrix} y_{i_1} \\ y_{i_2} \\ \vdots \\ y_{i_t} \end{bmatrix}.$$

Since A is invertible, solving this equation gives s.

The Shamir (t, n)-Threshold Scheme

Theorem: Suppose the participants $P_{i_1}, P_{i_2}, \cdots, P_{i_{t-1}}$ want to determine s. They know that

$$y_{i_j} = a(x_{i_j}), \quad j = 1, 2, \cdots, t-1$$

and $a(x)$ is the (secret) polynomial chosen by the dealer. But the $t-1$ shares give no information on s.

Proof: The $t-1$ participants have

$$\begin{bmatrix} 1 & x_{i_1} & x_{i_1}^2 & \cdots & x_{i_1}^{t-1} \\ 1 & x_{i_2} & x_{i_2}^2 & \cdots & x_{i_2}^{t-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{i_{t-1}} & x_{i_{t-1}}^2 & \cdots & x_{i_{t-1}}^{t-1} \end{bmatrix} \begin{bmatrix} s \\ a_1 \\ \vdots \\ a_{t-1} \end{bmatrix} = \begin{bmatrix} y_{i_1} \\ y_{i_2} \\ \vdots \\ y_{i_{t-1}} \end{bmatrix}.$$

Proof of the Theorem - ctd.

It follows that

$$\begin{aligned} s + a_1 x_{i_1} + a_2 x_{i_1}^2 + \cdots + a_{t-1} x_{i_1}^{t-1} &= y_{i_1} \\ s + a_1 x_{i_2} + a_2 x_{i_2}^2 + \cdots + a_{t-1} x_{i_2}^{t-1} &= y_{i_2} \\ \vdots &= \vdots \\ s + a_1 x_{i_{t-1}} + a_2 x_{i_{t-1}}^2 + \cdots + a_{t-1} x_{i_{t-1}}^{t-1} &= y_{i_{t-1}} \end{aligned}$$

Proof of the Theorem - ctd.

Thus

$$\begin{aligned} a_1 x_{i_1} + a_2 x_{i_1}^2 + \cdots + a_{t-1} x_{i_1}^{t-1} &= y_{i_1} - s \\ a_1 x_{i_2} + a_2 x_{i_2}^2 + \cdots + a_{t-1} x_{i_2}^{t-1} &= y_{i_2} - s \\ \vdots &= \vdots \\ a_1 x_{i_{t-1}} + a_2 x_{i_{t-1}}^2 + \cdots + a_{t-1} x_{i_{t-1}}^{t-1} &= y_{i_{t-1}} - s \end{aligned}$$

Proof of the Theorem - ctd.

Define

$$B = \begin{bmatrix} x_{i_1} & x_{i_1}^2 & \cdots & x_{i_1}^{t-1} \\ x_{i_2} & x_{i_2}^2 & \cdots & x_{i_2}^{t-1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{i_{t-1}} & x_{i_{t-1}}^2 & \cdots & x_{i_{t-1}}^{t-1} \end{bmatrix}.$$

Then the system of $t-1$ equations in the previous slide can be expressed as

$$B \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{t-1} \end{bmatrix} = \begin{bmatrix} y_{i_1} - s \\ y_{i_2} - s \\ \vdots \\ y_{i_{t-1}} - s \end{bmatrix}.$$

Proof of the Theorem - ctd.

Clearly,

$$\begin{aligned}
\det(B) &= x_{i_1} \cdots x_{i_{t-1}} \det \left(\begin{bmatrix} 1 & x_{i_1}^1 & \cdots & x_{i_1}^{t-2} \\ 1 & x_{i_2}^1 & \cdots & x_{i_2}^{t-2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{i_{t-1}}^1 & \cdots & x_{i_{t-1}}^{t-2} \end{bmatrix} \right) \\
&= x_{i_1} \cdots x_{i_{t-1}} \prod_{1 \le j < k \le t-1} (x_{i_k} - x_{i_j}) \bmod p \\
&\neq 0.
\end{aligned}$$

This means that B is invertible.

Proof of the Theorem - ctd.

Note that

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{t-1} \end{bmatrix} = B^{-1} \begin{bmatrix} y_{i_1} - s \\ y_{i_2} - s \\ \vdots \\ y_{i_{t-1}} - s \end{bmatrix}.$$

For any value of s in $\mathbf{Z}_p$, this set of equations has a unique solution $(a_1, \cdots, a_{t-1})$. Hence the $t-1$ shares $y_{i_1}, \ldots, y_{i_{t-1}}$ give no information on s.

The Shamir (t, n)-Threshold Scheme: Example

Secret: An element $s \in \mathbf{Z}_p$, where $p = 17$.

Participants: P_1, P_2, P_3, P_4, P_5.

System parameters: A dealer first chooses 5 distinct nonzero elements of $\mathbf{Z}_p$, $x_i = i$, $1 \leq i \leq 5$. The dealer then gives x_i to P_i. The values x_i are public.

The Shamir (t, n)-Threshold Scheme: Example

Computing and distributing the shares: Set $t = 3$ and let the secret be $s = 13$. The dealer chooses (independently at random) 2 elements of $\mathbf{Z}_p$, $a_1 = 10$ and $a_2 = 2$. The dealer forms

$$a(x) = s + a_1 x + a_2 x^2.$$

Then the shares are

$$\begin{aligned} y_1 &= a(1) = 8, \quad & y_2 &= a(2) = 7, \\ y_3 &= a(3) = 10, \quad & y_4 &= a(4) = 0, \\ y_5 &= a(5) = 11. \end{aligned}$$

The dealer then gives y_i to P_i.

The Shamir (t, n)-Threshold Scheme: Example

Recovering the secret: Suppose that P_1, P_3 and P_5 want to recover the secret s. They solve the following equations

$$\begin{array}{rcrcrcrcl} a(1) & = & s & + & a_1 & + & a_2 & = & 8 & = & y(1) \\ a(3) & = & s & + & 3a_1 & + & 9a_2 & = & 10 & = & y(3) \\ a(5) & = & s & + & 5a_1 & + & 8a_2 & = & 11 & = & y(5) \end{array}$$

The unique solution is $(13, 10, 2)$. So $s = 13$.

Original Chinese Remainder Problem

History: Documented in the Chinese book Sun Zi Suanjing by Sun Zi in about 100 A.D.

Problem 26, Vol. 3 of Sun Zi Suanjing**:**

> "We have a number of things, but do not know exactly how many. If we count them by threes we have two left over. If we count them by fives we have three left over. If we count them by sevens we have two left over. How many things are there?"

Chinese Remainder Problem

Sun's problem in modern terminology: Find an x such that

$$x \bmod 3 = 2,\ x \bmod 5 = 3,\ x \bmod 7 = 2.$$

Chinese Remainder Problem:
Let $m_1, m_2, \cdots, m_n$ be n positive integers that are pairwise relatively prime. Find an integer x such that

$$x \bmod m_i = r_i, \quad i = 1, 2, \cdots, n, \tag{1}$$

where $r_1, r_2, \cdots, r_n$ are any set of integers with $0 \leq r_i < m_i$.

Chinese Remainder Theorem

Let $m_1, \cdots, m_n$ be n positive integers that are pairwise relatively prime. For any set of integers $r_1, \cdots, r_n$ with $0 \leq r_i < m_i$, there is an unique integer $0 \leq x < M$ such that

$$x \bmod m_i = r_i, \quad i = 1, 2, \cdots, n. \tag{2}$$

Furthermore,

$$x = \left(\sum_{i=1}^{n} r_i u_i M_i \right) \bmod M, \quad M = \prod_{i=1}^{n} m_i, \quad M_i = \frac{M}{m_i}$$

and u_i is the multiplicative inverse of $M_i \bmod m_i$, i.e., $u_i M_i = 1 \pmod{m_i}$.

Proof: It is easily checked that x is a solution.

Question: How to prove the uniqueness of x.

Secret Sharing with Chinese Remainder Theorem

Secret: an integer s in $\mathbf{Z}_{5005}$.

Parties involved: P_1, P_2, P_3 and P_4.

Computing shares: A dealer sets $m_1 = 5$, $m_2 = 7$, $m_3 = 11$ and $m_4 = 13$, so that $5005 = m_1 m_2 m_3 m_4$. Then the dealer computes

$$s_i = s \bmod m_i$$

and gives it together with m_i to P_i for each i.

Recovering s: When all four participants come together with their shares. The Chinese Remainder Algorithm is used to recover s. Any group of three participants cannot determine s.

Secret Sharing with Chinese Remainder Theorem

Question: Is the secret sharing scheme using the CRT a $(4, 4)$-threshold scheme?

Answer: No.

Why? Each share gives information on the secret.

Suppose that $s_1 = 0$. Then $s = 5j$ for some j with $0 \leq j \leq 1000$. Thus the number of possible values for s is reduced from 5005 to 1001.

Secret Sharing with Chinese Remainder Theorem

Question: Is possible to use the CRT to construct a (t, n)-threshold scheme?

Answer: Yes. In fact, the Shamir (t, n)-threshold scheme is a special case of a (t, n)-threshold scheme based on "**the Chinese Remainder Theorem for polynomial**".

Reference: C. Ding, D. Pei, A. Salomaa, Chinese Remainder Theorem: Applications in Computing, Codes, Cryptography. Singapore: World Scientific, 1996.

References and other Information

History: The idea of secret sharing goes back to ancient times. The threshold schemes were studied independently by Blakely and Shamir in 1979.

Comments: Secret sharing is mathematically well defined, and some of their properties can be rigorously proved. Many secret sharing schemes have been proposed. Interested readers may consult the following references.

A. Beimel, Secret-sharing schemes: a survey, In: IWCC 2011: Coding and Cryptology pp. 11–46.

`https://link.springer.com/chapter/10.1007/978-3-642-20901-7_2`

D. G. Stinson, Cryptography: Theory and Practice, CRC Press, 1995.

Online Demo of Shamir's Secret Sharing Scheme

In the following URL, you find a demo of the Shamir secret sharing scheme:

`https://simon-frey.com/s4/`