1. Let X be a continuous random variable with pdf $f_X$. Let $Y = g(X)$, where $g$ is a strictly increasing function. Use the pdf of $Y$ derived in class to show that

$$E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

2. $X$ is said to be a *standard normal random variable* if its pdf is

$$f_X(x) = (1/\sqrt{2\pi}) e^{-x^2/2}.$$

Find the pdf of $Y = X^2$, and find its mean and variance.

3. (a) Let $X$ be a discrete random variable whose range is the nonnegative integers. Show that

$$\mathsf{E}X = \sum_{k=0}^{\infty} (1 - F_X(k)),$$

where $F_X(k) = P(X \leq k)$.

(b) Let $X$ be a continuous, nonnegative random variable, i.e., $f(x) = 0$ *for* $x < 0$. Show that

$$\mathsf{E}X = \int_0^{\infty} (1 - F_X(x)) dx,$$

where $F_X(x)$ is the cdf of $X$. Compare this with part (a).

(c) Let $X$ be a continuous random variable. Use part (b) to show that

$$\mathsf{E}X = \int_0^{\infty} (1 - F_X(x)) dx - \int_{-\infty}^{0} F_X(x) dx,$$

where $F_X(x)$ is the cdf of $X$.

(d) Let $X$ and $Y$ be two nonnegative random variables. Show that

$$\mathsf{E}[XY] = \int_0^{\infty} \int_0^{\infty} \mathsf{P}(X > x, Y > y) dx dy.$$

4. Let $(X, Y, Z)$ be a random vector with the following density:

$$f(x, y, z) = \frac{1 - \sin x \sin y \sin z}{8\pi^3}, \quad 0 \leq x, y, z \leq 2\pi.$$

Show that $(X, Y, Z)$ are pairwise independent but not independent.

5. $X$ and $Y$ are independent exponential random variables with rate $\lambda$ and $\mu$. Define

$$Z = \min\{X, Y\}, \quad W = \begin{cases} 1 & Z = X \\ 0 & Z = Y \end{cases}.$$

(a) Find the joint distribution of $Z$ and $W$.

(b) Find the marginal distribution of $Z$ and $W$, respectively.

(c) Show that $Z$ and $W$ are independent.

6. A random varialbe $X$ has a beta distribution if $f_X(x) = C(\alpha,\beta)x^{\alpha-1}(1-x)^{\beta-1}$, for $x \in [0,1]$. $C$ is a constant factor so that $f_X(x)$ integrates to one. We know that $\mathsf{E}[X] = \frac{\alpha}{\alpha+\beta}$ and $\operatorname{Var} X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

   Now consider the following hierarchy model. Let $(X_1, P_1)$,...,$(X_n, P_n)$ be independent random vectors with

$$X_i \big| P_i \sim \text{Bernoulli}(P_i)$$
$$P_i \sim \text{beta}(\alpha, \beta)$$

   This model might be appropriate, for example, if we are measuring the success of a drug on $n$ patients and because the patients are different, their success rates are not constant. A random variable of interest is $Y = \sum_{i=1}^{n} X_i$, the total number of successes.

   (a) Show that $\mathsf{E}[Y] = n\alpha/(\alpha+\beta)$.

   (b) Show that $\operatorname{Var}(Y) = n\alpha\beta/(\alpha+\beta)^2$, and hence $Y$ has the same mean and variance as a binomial$(n, \alpha/(\alpha+\beta))$. What is the distribution of $Y$?

   (c) Suppose now that the model is

$$X_i \big| P_i \sim \text{binomial}(n_i, P_i), \quad i = 1, \ldots, k$$
$$P_i \sim \text{beta}(\alpha, \beta)$$

   For $Y = \sum_{i=1}^{k} X_i$, find $\mathsf{E}[Y]$ and $\operatorname{Var}(Y)$.