In all problems, EX denotes the expectation of a random variable X.

Q1 (25 points) Consider a (s, S) inventory policy in a supermarket: Whenever the storage level drops below s, the policy immediately places an order to bring the inventory level back to S. Suppose the customers arrive according to a Poisson process with rateλ and order a random amount of items, which is distributed according to a random variable X having cumulative distribution function F with density f and a mean of m. Assume the replenish time is negligible. Assume the customer whose request takes the inventory level below s cannot receive any of the new orders.

(a) Let T denotes times between successive orders, give an expression for $E(T)$.

(b) Give an expression for the long-run rate that the supermarket places orders to replace its stock.

(c) Let $X(t)$ be the inventory of goods at time t, find the expression of $\mathbf{P}(X(t) \geq x)$ as $t \to \infty$.

(d) Find the expression of $\mathbf{P}(X(t) \geq x)$ as $t \to \infty$ when cdf F is exponential.

(e) Is the cumulative distribution function of the limiting inventory level in part (c) continuous? Briefly explain your answer.

(f) Assume the storage level is ∞ and the goods suffer a certain amount of diminishing. Suppose the diminishing arises in accordance with Poisson process with rate μ. The ith diminishing causes damages with an amount W_i. If the diminishing has an initial damage W, then a time later its damage is $We^{-\alpha t}$.

   (i) Assume the damage in goods is additives. Find an expression of the total damage in goods at time t.

   (ii) Find the expected value of the total damage in goods at time t.

Q2 Let $\{(X_n, Y_n)\}_{n=0}^{\infty}$ be a two dimensional **symmetric** random walk.

(a) Compute $\mathbf{P}((X_3, Y_3) = (1, 2))$.

(b) Gives an expression of $\mathbf{P}((X_{200}, Y_{200}) = (0, 0))$.

(c) Gives an expression of $\mathbf{P}((X_{300}, Y_{300}) = (1, 2))$.

(d) For $n \geq 1$. Let M_{2n} denote the number of returns to $(0, 0)$ by time n. Compute the expectation of M_{2n}

(e) Show that the state $(0, 0)$ is recurrent

(f) Define $T \equiv \inf\{n \geq 0 : \max(|X_n|, |Y_n|) = 3\}$

i. Find $\mathbf{E}[T]$

ii. Find $\mathbf{P}[X_T = 3, Y_T = 0]$

Q3 Consider an $M/G/1$ queue system with unlimited waiting space. Customers arrive according to a Poisson process with rate $\lambda = 48$ per hour. Assume the service time is i.i.d random variable, distributed as the gamma random variable S with pdf $g(t)$, $\mathbf{E}(S) = 1$ miniute and $\sqrt{Var(S) = 0.5}$. Assume

$$g(t) \equiv g_S(t) \equiv \frac{128t^3e^{-4t}}{3}$$

and Laplace transform

$$\hat{g}(s) \equiv \mathbf{E}[e^{-sS}] \equiv \int_0^\infty e^{-st}g(t)dt = \left(\frac{s}{4+s}\right)^4$$

Upon service complete, each customer receives a reward. The successive reward can be regarded as i.i.d. random variables distributed as W having a gamma distribution with mean 100 and standard deviation 110 and thus Laplace transform

$$\mathbf{E}[e^{-sW}] \equiv \int_0^\infty e^{-st}g(t)dt = \left(\frac{1}{1+121s}\right)^{(1/1.21)}$$

(a) Given 30 customers arrive in an hour, What are the mean and variance of the number of these arrivals that come during the first 20 minutes of that hour?

(b) What is the probability that the first arrival completes service before the second customer arrives? (Assume that the system is initially empty.)

(c) Let $R(t)$ denote the total amount of reward gained by all customers by time t. What distribution does $R(t)$ follows? Finds its mean and variance.

(d) What is the expected conditional total amount reward gain in a given hour, given that the amount reward in the previous hour is exactly two times the mean?

(e) Let $X(t)$ be the number of customers in the system at time t.

  (i) Is $\{X(t) : t \geq 0\}$ an irreducible aperiodic continuous-time Markov chain? Explain.

  (ii) Find random time T_n, $n \geq 0$, such that $\{X(T_n) : n \geq 0\}$is an irreducible aperiodic discrete-time Markov chain with state-space$\{0, 1, 2, ...\}$.

  (iii) Find the transition probability of the DTMC in the previous part.

1. (30 points) Let $\{(X_n, Y_n)\}_{n=0}^{\infty}$ be a two dimensional symmetric random walk, i.e.,

$$
\begin{aligned}
&P((X_{n+1}, Y_{n+1}) = (x+1, y)|(X_n, Y_n) = (x, y)) \\
&= P((X_{n+1}, Y_{n+1}) = (x-1, y)|(X_n, Y_n) = (x, y)) \\
&= P((X_{n+1}, Y_{n+1}) = (x, y+1)|(X_n, Y_n) = (x, y)) \\
&= P((X_{n+1}, Y_{n+1}) = (x, y-1)|(X_n, Y_n) = (x, y)) = \frac{1}{4}.
\end{aligned}
$$

and $(X_0, Y_0) = (0, 0)$.

(a) Argue that $X_{n+1} - X_n$ and $Y_{n+1} - Y_n$ are dependent random variables.

Sol. No. Because when $X_{n+1} - X_n \neq 0$, it must be that $Y_{n+1} - Y_n = 0$. □

(b) Let $Z_n = X_n + Y_n$. Show that $\{Z_n : n \geq 0\}$ has the Markov property and is a simple random walk. Calculate $P_{i,i+1}$ and $P_{i,i-1}$.

Sol.

$$
Z_{n+1} = \begin{cases} Z_n + 1 & \text{if } (X_{n+1}, Y_{n+1}) = (X_n + 1, Y_n) \text{ or } (X_{n+1}, Y_{n+1}) = (X_n, Y_n + 1), \\ Z_n - 1 & \text{if } (X_{n+1}, Y_{n+1}) = (X_n - 1, Y_n) \text{ or } (X_{n+1}, Y_{n+1}) = (X_n, Y_n - 1). \end{cases}
$$

$$
\begin{aligned}
P_{i,i+1} &= P\left(Z_{n+1} = i+1 | Z_n = i\right) \\
&= P(X_{n+1} + Y_{n+1} = X_n + Y_n + 1 | X_n + Y_n = i) \\
&= P((X_{n+1}, Y_{n+1}) = (X_n + 1, Y_n) | X_n + Y_n = i) \\
&\quad + P((X_{n+1}, Y_{n+1}) = (X_n, Y_n + 1) | X_n + Y_n = i) \\
&= \frac{1}{2}.
\end{aligned}
$$

Similarly, $P_{i,i-1} = P\left(Z_{n+1} = i - 1 | Z_n = i\right) = P\left((X_{n+1}, Y_{n+1}) = (X_n - 1, Y_n)\right) +$ $P\left((X_{n+1}, Y_{n+1}) = (X_n, Y_n - 1)\right) = \frac{1}{2}$. □

(c) Calculate $P(Z_n = 0)$.

Sol. $P\left(Z_n = 0\right) = 0$ if n is an odd number. Otherwise, $P\left(Z_n = 0\right) = \binom{n}{\frac{n}{2}} \frac{1}{2^n}$. □

(d) Calculate $P\left((X_{30}, Y_{30}) = (1, 2)\right)$.

Sol. $P\left((X_{30}, Y_{30}) = (1, 2)\right) \leq P(Z_{30} = 3) = 0$. □

2. (30 points) Consider an $M/G/1$ queueing system with unlimited waiting space. Customers arrive according to a Poisson process with rate λ with inter-arrival times $X_1, X_2, \ldots$. Denote by $N(t)$ the number of arrivals by t. Assume that the service times $Y_1, Y_2, \ldots$ are i.i.d random variables with cdf $F(\cdot)$. Assume that the system is initially empty.

   (a) Given $N(3) = 30$, what is the distribution $N(1)$?

   Sol. Given $N(3) = 30$, the arrival times are i.i.d. uniform random variables on $[0, 3]$. $N(1)|N(3) = 30 \sim \mathit{Binomial}\left(30, \frac{1}{3}\right)$. □

   (b) What is the probability that the first arrival completes service before the second customer arrives?

   Sol. The first arrival completes service before the second customer arrives if and only if the service time for the first customer is smaller than the inter-arrival time for the second customer.

$$P(X_2 \geq Y_1) = \int_0^\infty P(X_2 \geq y)dF(y) = \int_0^\infty e^{-\lambda y} dF(y) = E\left[e^{-\lambda Y_1}\right].$$

□

   (c) Let $M(t)$ denote the total number of customers that complete their service by time t. Calculate $E[M(t)]$.

   Sol. Let $S_n = X_1 + \cdots + X_n$.

$$\begin{aligned}
E\left[M(t)\right] &= E\left[E\left[\sum_{i=1}^{N(t)} 1_{\{S_n+Y_n\leq t\}}\middle| N(t)\right]\right] \\
&= E\left[E\left[\sum_{i=1}^{N(t)} 1_{\{U_n+Y_n\leq t\}}\middle| N(t)\right]\right] \\
&= E\left[E\left[\sum_{i=1}^{N(t)} P(Y_n \leq t - U_n)\middle| N(t)\right]\right] \\
&= E\left[E\left[\sum_{i=1}^{N(t)} F(t - U_n)\middle| N(t)\right]\right] \\
&= E\left[E\left[\sum_{i=1}^{N(t)} \frac{1}{t}\int_0^t F(t-u)du\middle| N(t)\right]\right] \\
&= E\left[N(t)\frac{1}{t}\int_0^t F(t-u)du\right] \\
&= \lambda \int_0^t F(u)du.
\end{aligned}$$

□

(d) Suppose that Y_n is an exponential distribution with rate μ and assume that each waiting customer will abandon the system independently if she has waited for an exponential amount of time with mean $\frac{1}{\mu}$ before starting her service. Then the number of customers in the system $\{X(t) : t \geq 0\}$ is a continuous time Markov Chain.

i. Describe the Q matrix of this CTMC.

Sol.

$$q_{i,i+1} = \lambda, \; q_{i,i-1} = i\mu.$$

□

ii. What is the long-run average abandonment rate?

Sol. This is an ergodic birth-death process.

$$\pi_i q_{i,i+1} = \pi_{i+1} q_{i+1,i} \implies \frac{\pi_{i+1}}{\pi_i} = \frac{\lambda}{(i+1)\mu} \implies \pi_i = e^{-\frac{\lambda}{\mu}} \frac{\left(\frac{\lambda}{\mu}\right)^i}{i!}.$$

Total abandon rate is

$$\begin{aligned}\sum_{i=1}^{\infty}(i-1)\mu\pi_i &= -\mu(1-\pi_0) + \left(\sum_{i=1}^{\infty} i\mu\pi_i\right) \\ &= \lambda - \mu e^{-\frac{\lambda}{\mu}}\end{aligned}$$

□

3. (40 points) Consider the following (s, S) inventory policy. Whenever the storage level drops below s, an order is placed which immediately brings the inventory level back to S. Customers arrive according to a Poisson process with rate λ and inter-arrival times $X_1, X_2, \ldots$. The nth arrival requests Y_n amount of items where $Y_1, Y_2, \ldots,$ are i.i.d. exponential random variables with mean $\frac{1}{\mu}$.

(a) Let T denote the time between successive orders. Calculate $E[T]$.

Sol. In the Poisson process generated by the iid exponentials $Y_1, Y_2, \cdots$ with μ, $N(S-s)+1$ is the number of arrivals that will trigger an order. Thus, $T = \sum_{n=1}^{N(S-s)+1} X_n$ and $E[T] = [m(S-s)+1]E[X_1] = \frac{\mu(S-s)+1}{\lambda}$. □

(b) Calculate $\lim_{t\to\infty} P(I(t) \geq x)$ for any given $x \in [s, S]$ where $I(t)$ is the inventory level at time t.

Sol. In the renewal process where a renewal starts when an order is placed, we say the system is on when $I(t) \geq x$ and off otherwise. Then, $N(S-x)+1$ is the number of arrivals to cause inventory to drop below x in a cycle and

$$\lim_{t\to\infty} P\left(I(t) \geq x\right) = \lim_{t\to\infty} P\left(\text{system is on at } t\right) = \frac{m(S-x)+1}{m(S-s)+1} = \frac{\mu(S-x)+1}{\mu(S-s)+1}.$$

□

(c) Let $S_n = Y_1 + \cdots + Y_n$. What is the probability that the total demand in an order cycle exceeds S?

Sol. Let $S_n = Y_1 + \cdots + Y_n$. The cumulative demand between two orders is $S_{N(S-s)+1}$. Since $S_{N(S-s)+1} - (S-s)$ is the residual time of a Poisson process at time $S-s$, it is exponentially distributed with mean $\frac{1}{\mu}$. Thus, $P\left(S_{N(S-s)+1} > S\right) = P\left(S_{N(S-s)+1} - (S-s) > s\right) = e^{-\mu s}$. □

(d) Note that each order cycle starts with S amount of inventory and drops to $S - S_1$, $S - S_2$, ... Suppose that each unit of inventory costs \$1 per unit time. Derive the expected inventory cost in a cycle and using the renewal reward theory to calculate the long-run average inventory cost.

Sol.

$$
\begin{aligned}
\lim_{t\to\infty}\frac{\int_0^t I(x)dx}{t} &= \frac{E\left[\int_0^T I(x)dx\right]}{E[T]} \\
&= \frac{E\left[\sum_{n=1}^{N(S-s)+1} X_n\left(S-S_{n-1}\right)\right]}{E[T]} \\
&= \frac{E\left[E\left[\sum_{n=1}^{N(S-s)+1} X_n(S-S_{n-1})\middle| N(S-s)\right]\right]}{E[T]} \\
&= \frac{\frac{1}{\lambda}E\left[\sum_{n=1}^{N(S-s)+1}(S-S_{n-1})\middle| N(S-s)\right]}{E[T]} \\
&= \frac{\frac{1}{\lambda}E\left[S(N(S-s)+1)-\sum_{n=1}^{N(S-s)} S_n\middle| N(S-s)\right]}{E[T]} \\
&= \frac{\frac{1}{\lambda}E\left[S(N(S-s)+1)-\sum_{n=1}^{N(S-s)} U_n\middle| N(S-s)\right]}{E[T]} \\
&= \frac{\frac{1}{\lambda}E\left[S(N(S-s)+1)-N(S-s)\frac{S-s}{2}\middle| N(S-s)\right]}{E[T]} \\
&= \frac{\frac{1}{\lambda}[S(\mu(S-s)+1)-\mu(S-s)\frac{S-s}{2}]}{\frac{\mu(S-s)+1}{\lambda}} \\
&= \frac{\frac{S+s}{2}\mu(S-s)+S}{\mu(S-s)+1}
\end{aligned}
$$

where $U_n \sim Uniform[0,S-s]$. □

Final Exam of IEDA5250

2 hours and 50 minutes

1. (25 points) Consider a system with two servers. Potential customers arrive at this system in accordance with a Poisson process with rate of two per hour and all the service times are i.i.d. exponential random variables with mean 1/3 hours. Moreover, assume that

   (i) the arrival process is independent of the service times;

   (ii) if a customer finds that there are 4 customers (including 2 customers being served and 2 customers waiting in the queue) in the system upon his/her arrival will not enter the system;

   (iii) each customer waiting in the queue incurs costs at the rate of 10 dollars per hour.

   Let $X(t)$ denote the number of customers in the system at time t. Then $\{X(t) : t \geq 0\}$ is a birth and death process. **(Justify your answers rigorously.)**

   (a) What are the birth rates and death rates for all states?

   sol. $q_{i,i+1} = 2$ for $i = 0, 1, 2, 3$. $q_{1,0} = 3$, $q_{2,1} = q_{3,2} = q_{4,3} = 6$. □

   (b) Find the long-run proportion of time that the system spends in each state.

   sol. Can use detailed balance equation. $3\pi_4 = \pi_3$, $3\pi_3 = \pi_2$, $3\pi_2 = \pi_1$, $3\pi_1 = 2\pi_0$, $\pi = (81, 54, 18, 6, 2)/161$. □

   (c) What is the rate at which the process enters state 2?

   sol. $\pi_1 q_{12} + \pi_3 q_{32} = 144/161$. □

   (d) Is $X(t)$ time reversible in its steady state?

   sol. Yes, check detailed balance equation. □

   (e) What is the long-run average waiting cost per hour?

   sol. Average number of waiting customers 10/161. Cost 100/161. □

2. (25 points) A mouse finds itself in a building with 100 floors. Each floor has nine rooms, numbered one through nine. Each floor has rooms laid out like

1	2	3
4	5	6
7	8	9

   On each floor, there are doors:

between rooms 1 and 2
between rooms 1 and 4
between rooms 2 and 3
between rooms 2 and 5
between rooms 3 and 6
between rooms 4 and 5
between rooms 4 and 7
between rooms 5 and 6
between rooms 5 and 8
between rooms 6 and 9
between rooms 7 and 8
between rooms 8 and 9

The mouse can go through a door in either direction. From Room 5, the mouse can move next to Rooms 2, 4, 6 and 8 on the same floor, or the Mouse can move to Room 5 on the floor above or to Room 5 on the floor below. The mouse could not move up from room 5 on floor 100 and move down from room 5 on floor 1 Suppose that, in each move, The Mouse is equally likely to make each of the available moves. Suppose that the Mouse starts in Room 1 on Floor 1 **(Justify your answers rigorously.)**

(a) Does the limit distribution exist? Briefly explain your answer

Sol. The limiting steady-state distribution does not exist, because of periodicity. □

(b) What is the probability that the Mouse is in Room 1 on Floor 1 after making two moves?

Sol.

$$\mathbb{P}(\textbf{Return}) = \frac{1}{2} \cdot \frac{1}{3} \cdot + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3}$$

□

(c) What is the probability that the Mouse is in Room 1 on Floor 1 after making three moves?

Sol. The probability is 0 because the Markov chain is periodic with period 2 □

(d) What is the expected number of moves made by Markov Mouse before he first returns to the initial room, Room 1 on Floor 1?

Sol. The expected number of moves between successive visits to Room 1 on Floor 1 is the reciprocal of the stationary probability. The stationary probability is the number of doors out of that room divided by the sum of the number of doors out of all the rooms. Hence numerator is 2. The denominator is

$$4 * 100 * 2 + 4 * 100 * 3 + 1 * 98 * 6 + 1 * 2 * 5 = 2598$$

Hence the expected moves should be 1299. □

(e) Consider the probability that Markov Mouse is in Room 1 on Floor 1 after n moves. Give an approximation for this probability for large values of n.

Sol. $P_{1,1}^{2n+1} = 0$ and $P_{1,1}^{2n} = \frac{2}{1299}$ □

3. (25 points) The movement of a particle follows a random walk on a circle with N nodes. Moreover, assume that

(i) the particle at node j, $1 \leq j \leq N$, will move one step, independently of all else, either clockwise with probability p $(0 < p < 1)$ or counterclockwise with probability $q = 1 - p$;

(ii) at time 0 the particle is at node 1.

For $n \geq 1$, let X_n denote the location of the particle after n steps. Define $X_0 = 1$.Then $\{X_n : n = 0, 1, \cdots\}$ is a discrete-time Markov chain.**(Justify your answers rigorously.)**

(a) Write down the stationary distribution $\pi = (\pi_1, \ldots, \pi_N)$, you may skip the intermediate steps for this question.

sol. $\pi = (1/N, \ldots, 1/N)$. □

(b) Let $\tau_1 = \inf\{n > 0 : X_n = 1\}$, find its expectation $E\tau_1$.

sol. $E\tau_1 = \frac{1}{\pi_1} = N$. □

Assume that $p = q = \frac{1}{2}$ for the next three questions.

(c) Let σ_n be the first time that in total n states have been visited, i.e., $\sigma_1 = 0$, $\sigma_2 = 1$, but σ_n is random for $n > 2$. Find $E[\sigma_3 - \sigma_2]$.

sol. This is the same as the exiting time in a gambler's ruin problem with $n = 3, i = 2$. $E[\sigma_3 - \sigma_2] = 2$ □

(d) Find $E[\sigma_N]$.

sol. $E[\sigma_{n+1} - \sigma_n] = n$. $E[\sigma_N] = 1 + 2 + \cdots + N - 1 = N(N-1)/2$. □

(e) Find the distribution of the particle's position at the first time it has visited all states, i.e., find $P(X_{\sigma_N} = j)$ for each $1 \le j \le N$.

sol. $X_{\sigma_n} = j$ means visiting $j-1$ before visiting j and visiting $j+1$ before visiting j. Let τ_j be the first time it visits j. $P(\tau_j > \tau_{j+1}, \tau_j > \tau_{j-1}) = P(\tau_j > \tau_{j+1} > \tau_{j-1}) + P(\tau_j > \tau_{j-1} > \tau_{j+1})$. Using gambler's ruin results, for $j \neq 1$, $P(\tau_j > \tau_{j+1} > \tau_{j-1}) = \frac{j-2}{N-2} \times \frac{1}{N-1}$ and $P(\tau_j > \tau_{j-1} > \tau_{j+1}) = \frac{N-j}{N-2} \times \frac{1}{N-1}$. $P(X_{\sigma_n} = j) = \frac{1}{N-1}$ for $j \neq 1$ and $P(X_{\sigma_n} = 1) = 0$. □

4. (25 points) Consider an $M/M/\infty$ queue in which customers arrive according to a Poisson process having rate λ. Each customer starts to receive service immediately upon arrival from one of an unlimited number of servers. Suppose that the service times are IID exponential random variables with mean $\frac{1}{\mu}$.
Suppose that, on arrival, each customer will choose the lowest numbered server that is free. Thus we can think of all arrivals occurring at server 1. Those customers who find server 1 free begin service there. Those customers finding server 1 busy immediately overflow and become arrivals at server 2. Those customers finding both servers 1 and 2 busy immediately overflow and become arrivals at server 3, and so forth. **(Justify your answers rigorously.)**

(a) Let $N(t)$ be the number of customers that find server 1 is busy and choose server 2 in the time interval $[0, t]$. Is it a Markov process? Briefly explain your answer

Sol. It is not a Markov process, because the future beyond some time t depends upon whether server 1 is busy or not at time t. □

(b) What is the long-run proportion of time that both servers 1 and 2 are busy?

Sol. Consider the steady-state probability in the $M/M/2$ model. In particular

$$\mathbb{P}(\text{busy}) = \frac{\alpha^2/2}{1 + \alpha + \alpha^2/2}$$

where $\alpha = \frac{\lambda}{\mu}$ □

(c) Starting with an empty system, what is the expected time until server 2 first becomes busy?

Sol. Let $T_{i,2}$ be the time until server 2 first becomes busy, starting with the system empty. We want to find $\mathbf{E}T_{0,2}$

$$\begin{aligned} \mathbf{E}T_{0,2} &= \frac{1}{\lambda} + \mathbf{E}T_{1,2} \\ &= \frac{1}{\lambda} + \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}\mathbf{E}T_{0,2} \end{aligned}$$

Solve the equation gives us $\mathbf{E}T_{0,2} = \frac{2\lambda+\mu}{\lambda^2}$ □

(d) What is the expected number of busy servers among the first two servers in steady-state?

Sol.

$$\mathbf{E}N = 1 \cdot P(N = 1) + 2 \cdot P(N = 2) = \frac{2\alpha + 2\alpha^2}{2\alpha + \alpha^2 + 2}$$

where $\alpha = \frac{\lambda}{\mu}$ $\square$

(e) What proportion of time is server 2 busy?

Sol. Let $I_i = 1$ if server i is busy in steady state.

$$\mathbf{E}[I_1] = P(\text{server 1 busy}) = \frac{\alpha}{1 + \alpha}$$

$$E[I_2] = E[I_1 + I_2] - E[I_1] = \frac{2\alpha^2 + \alpha^3}{(2 + 2\alpha + \alpha^2)(1 + \alpha)}$$

$\square$