1. A special case of a normal family is one in which the mean and the variance are related, the $N(\theta, a\theta)$ family. If we are interested in testing this relationship, regardless of the value of θ, we are again faced with a nuisance parameter problem.
   (a) Find the LRT of $H_0 : a = 1$ versus $H_1 : a \neq 1$ based on a sample $X_1, \cdots, X_n$ from a $N(\theta, a\theta)$ family, where θ is unknown.
   (b) A similar question can be asked about a related family, the $N(\theta, a\theta^2)$ family. Thus if $X_1, \cdots, X_n$ are iid $N(\theta, a\theta^2)$, where θ is unknown, find the LRT of $H_0 : a = 1$ versus $H_1 : a \neq 1$.

2. Show that for a random sample $X_1, ..., X_n$ from a $N(0, \sigma^2)$ population, the most powerful test of $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$, where $\sigma_0 < \sigma_1$, is given by

$$\phi(\sum X_i^2) = \begin{cases} 1 & \textit{if } \sum X_i^2 > c \\ 0 & \textit{if } \sum X_i^2 \leq c \end{cases}$$

   For a given value of α, the size of the Type I Error, show how the value of c is explicitly determined.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from Uniform$(\theta, \theta + 1)$. To test $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, reject H_0 if $X_{(n)} \geq 1$ or $X_{(1)} > k$ where k is a constant.
   (a) Determin k such that the test have size α.
   (b) Find an expression for the power function of the test in part (a).
   (c) Prove that the test is UMP size α test.

4. Let $\{f_\theta(x)\}$ be a family of density functions with parameter $\theta \in \mathbb{R}$. Assume that $f_\theta(x) > 0$ for all θ and x and that $\frac{\partial^2}{\partial\theta\partial x} \log f_\theta(x)$ exists. Show that this family has MLR in x is equivalent to one of the following conditions:
   (a) $\frac{\partial^2}{\partial\theta\partial x} \log f_\theta(x) \geq 0$ for all x and θ;
   (b) $f_\theta(x)\frac{\partial^2}{\partial\theta\partial x} f_\theta(x) \geq \frac{\partial}{\partial\theta} f_\theta(x) \frac{\partial}{\partial x} f_\theta(x)$ for all x and θ.

5. Consider tests for $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$ based on a single observation X from $N(\theta, 1)$. Consider using $Y = |X|$ in a test.
   (a) Find the probability density $f(y|\theta)$ of Y and show that it depends only on $|\theta|$.
   (b) Show that the probability density $f(y|\theta)$ have MLR.
   (c) Find the UMP size α test based on Y.
   (d) The UMP test in part (c) is not most powerful compared with tests based on X. Find a level α test based on x that have better power at $\theta = -1$. What is the difference of power at $\theta = -1$ for $\alpha = 0.05$?

6. Let $f(x|\theta)$ be the Cauchy scale pdf

$$f(x|\theta) = \frac{\theta}{\pi} \frac{1}{\theta^2 + x^2}, \theta > 0.$$

   (a) Show that this family does not have an MLR.

(b) If X is one observation from $f(x|\theta)$, show that $|X|$ is sufficient for θ, and that the distribution of $|X|$ does have an MLR.

7. Consider the hypothesis test $H_0 : \mu = 0$ versus $H_1 : \mu = \mu_1$ for a normal population $N(\mu, 1)$ with unknown mean μ.

   (a) Given a sample size n, find test such that the Type-I and Type-II error probability are equal $\alpha = 1 - \beta$.

   (b) For the test in part(a), if we want to control $\alpha = 1 - \beta = \gamma$. How much sample do we need?

   (c) Find the Wald'd sequential test such that the Type-I and Type-II error probability are equal $\alpha = 1 - \beta$. Using approximated formula to compute the expected number of sample when the test is terminated.

   (d) Compare the two sample sizes.