Midterm Exam IEDA 5270

Name ____________________ Student ID ____________________

Question 1 (10 points)
Let $X \sim \text{Binomial}(n, \theta)$ and $Y \sim \text{Binomial}(n, \theta^2)$ be independent with $\theta \in (0, 1)$ being an unknown parameter.

(a) (5 points) Find a minimal sufficient statistic.

(b) (5 points) Is the minimal sufficient statistic complete?

Question 2 (20 points)
Suppose that $c_1, c_2, \ldots, c_n$ are known positive constants and that X_i follows a gamma distribution with shape parameter 2 and scale parameter θc_i with $\theta > 0$, i.e., with a density of

$$f(x) = (\theta c_i)^{-2} x e^{-x/(\theta c_i)}, \ x \geq 0.$$

Suppose $\{X_i\}$ are mutually independent.

(a) (5 points) Compute the Cramér-Rao lower bound (CRLB) for the variance of all unbiased estimators of θ.

(b) (5 points) Find the maximum likelihood estimator of θ, $\widehat{\theta}_{\text{MLE}}$.

(c) (5 points) Is $\widehat{\theta}_{\text{MLE}}$ unbiased? Does it achieve the CRLB?

(d) (5 points) Consider the class of all estimators for θ of the forms $\widehat{\theta} = \sum_{i=1}^{n} d_i X_i$. Find $d_1, d_2, \ldots, d_n$ so that $\widehat{\theta}$ minimizes the mean-squared error $\mathsf{E}[(\widehat{\theta} - \theta)^2]$.

Question 3 (10 points)
Suppose $X_1, \ldots, X_n$ are independent normal variables, each with unit variance and with $\mathsf{E}[X_i] = \alpha t_i + \beta t_i^2, i = 1, 2, \ldots, n$. Here α and β are unknown parameters and $t_1, \ldots, t_n$ are known constants. Find the UMVUE for α and β.

Question 4 (10 points)
Suppose $X_1, \ldots, X_n$ is a random sample from the discrete uniform distribution on points $1, 2, \ldots, \theta$, where θ is an integer.

(a) Consider $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$, where $\theta_0 > 0$ is a known integer. Show that

$$T_1(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} > \theta_0 \\ \alpha & X_{(n)} \leq \theta_0 \end{cases}$$

is a UMP test of size α.

(b) (5 points) Consider $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, where $\theta_0 > 0$ is a known integer. Show that

$$T_2(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} > \theta_0 \text{ or } X_{(n)} \leq \theta_0 \alpha^{1/n} \\ 0 & \text{otherwise} \end{cases}$$

is a UMP test of size α.

Question 5 (25 points)

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be i.i.d. with $X_i \sim N(0,1)$ and $Y_i | X_i = x \sim N(x\theta, 1)$. Here, given $X_1 = x_1, \ldots, X_n = x_n$, the variables Y_i are conditionally independent.

(a) (5 points) Find the MLE $\widehat{\theta}$ for θ.

(b) (5 points) Find the Fisher Information $\mathcal{I}(\theta)$ for a single observation (X_1, Y_1).

(c) (5 points) Determine the limiting distribution of $\sqrt{n}(\widehat{\theta} - \theta)$ and use it to give a $1 - \alpha$ asymptotic confidence interval for θ based on $\mathcal{I}(\widehat{\theta})$.

(d) (5 points) Compare the interval in part (c) with a $1 - \alpha$ asymptotic confidence interval based on observed Fisher information.

(e) (5 points) Determine the exact distribution of $\sqrt{\sum_i X_i^2}(\widehat{\theta} - \theta)$ and use it to find the true coverage probability of the interval in part (d).

Question 6 (15 points)

Let $X_1, \ldots, X_n$ be a random sample from a continuous CDF F_θ on $\mathbb{R}$, parametrized by a real-valued θ. A level $1 - \alpha$ <u>confidence band</u> for the CDF F_θ is a collection of confidence intervals $\{C_t(\boldsymbol{X}) : t \in \mathbb{R}\}$ such that

$$\inf_\theta \mathbb{P}_\theta(F_\theta(t) \in C_t(\boldsymbol{X}) \text{ for all } t \in \mathbb{R}) \geq 1 - \alpha.$$

(a) (5 points) Suppose F_θ is nonincreasing in θ for every t, find a level $1 - \alpha$ confidence band for the CDF F_θ.

(b) (5 points) Suppose $X_1, \ldots, X_n$ is a normal random sample with unknown μ and known σ^2. Find a level $1 - \alpha$ confidence band for the CDF F_θ.

(c) (5 points) Suppose $X_1, \ldots, X_n$ is a normal random sample with known μ and known σ^2. Find a level $1 - \alpha$ confidence band for the CDF F_θ.