IEDA 5230 (Fall 2023) Homework Assignment 3

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A Knapsack problem is defined as follows: Given A, a set of n elements, and a constant c, where each element j has a weight w_j and a value v_j, find a subset of the elements, denoted by B, such that $\sum_{j \in B} w_j \leq c$ and $\sum_{j \in B} v_j$ is maximized.
Assume all w_j, v_j are positive integers, and $w_1 \leq w_2 \leq \ldots \leq w_n$. Let $\sum_{j \in A} w_j = W$ and $\sum_{j \in A} v_j = V$.

1) Develop a dynamic programming of which the time complexity is $O(nW)$ to solve the knapsack problem.
2) Develop a dynamic programming of which the time complexity is $O(nV)$ to solve the knapsack problem.
3) There is a requirement of B such that for any three consecutive elements, $k, k+1$, and $k+2$, at most one of them is in B.
   a. Formulate this problem by integer linear programming.
   b. Develop a dynamic programming to solve this problem.
4) There is a requirement of B such that if element k is in B, at least one of $k-1, k+1$ is in B.
   a. Formulate this problem by integer linear programming.
   b. Develop a dynamic programming to solve this problem.

1 1

Define $F_k(a)$ as the maximum value of the subset of the first k elements, subject to the weight equal to a. Define $B_k(a)$ is the subset.
$F_0(0) = 0, B_0(0) = \emptyset$.
For $k \in \{1, ..., n\}, a \in \{0, 1, ..., W\}$, we have:

$$F_k(a) = \max\{F_{k-1}(a), F_{k-1}(a - w_i) + v_i\},\ B_k(a) = \begin{cases} B_{k-1}(a), & F_{k-1}(a - w_i) + v_i \leq F_{k-1}(a) \\ B_{k-1}(a - w_i) \cup i, & F_{k-1}(a - w_i) + v_i > F_{k-1}(a) \end{cases}$$

The answer is $B_n(c)$

2 2

Define $F_k(a)$ as the minimum weight of the subset of the first k elements, subject to the value equal to a. Define $B_k(a)$ is the subset.
$F_0(0) = 0, B_0(0) = \emptyset$.
For $k \in \{1, ..., n\}, a \in \{0, 1, ..., V\}$, we have:

$$F_k(a) = \min\{F_{k-1}(a), F_{k-1}(a - v_i) + w_i\},\ B_k(a) = \begin{cases} B_{k-1}(a), & F_{k-1}(a - v_i) + w_i \geq F_{k-1}(a) \\ B_{k-1}(a - v_i) \cup i, & F_{k-1}(a - v_i) + w_i < F_{k-1}(a) \end{cases}$$

The answer is $B_n(b)$, where b is the largest element that satisfies $F_n(b) \leq c$.

3 3

a .

Maximize $\sum_{i=1}^{n} x_i v_i$

subject to $\sum_{i=1}^{n} x_i w_i \leq c$

$x_i + x_{i+1} + x_{i+2} \leq 1, i = 1, ..., n-2$

$x_i \in \{0, 1\}, i = 1, ..., n$

b .

Define $F_k(a, x_{k-1}, x_k)$ as the maximum value of the subset of the first k elements, subject to the weight equal to a, when the selection of $(k-1)$th, kth element is x_{k-1}, x_k. Define $B_k(a, x_{k-1}, x_k)$ is the subset.

$F_0(0, 0, 0) = 0, B_0(0, 0, 0) = \emptyset$.

For $k \in \{1, ..., n\}, a \in \{0, 1, ..., W\}$, we have:

$F_k(a, 1, 0) = F_{k-1}(a, 0, 1)$, $B_k(a, 1, 0) = B_{k-1}(a, 0, 1)$,

$F_k(a, 0, 1) = F_{k-1}(a - w_i, 0, 0) + v_i$, $B_k(a, 0, 1) = B_{k-1}(a - w_i, 0, 0) \cup i$,

$F_k(a, 0, 0) = \max\{F_{k-1}(a, 0, 0), F_{k-1}(a, 1, 0)\}$, $B_k(a, 0, 0) = \begin{cases} B_{k-1}(a, 1, 0), & F_{k-1}(a, 0, 0) \leq F_{k-1}(a, 1, 0) \\ B_{k-1}(a, 0, 0), & F_{k-1}(a, 0, 0) > F_{k-1}(a, 1, 0) \end{cases}$

The answer is $B_n(c, d, e)$, where $d, e = \arg\max\limits_{i,j} F_n(a, i, j)$ s.t. $i + j \leq 1$

4 4

a .

Maximize $\sum_{i=1}^{n} x_i v_i$

subject to $\sum_{i=1}^{n} x_i w_i \leq c$

$x_{i-1} - x_i + x_{i+1} \geq 0, i = 2, ..., n-1$

$x_i \in \{0, 1\}, i = 1, ..., n$

b .

Define $F_k(a, x_{k-1}, x_k), B_k(a, x_{k-1}, x_k)$ same as in 3).

$F_1(a, x_0, 1) = \begin{cases} 0, & a < w_1 \\ v_1, & a \geq w_1 \end{cases}$, $B_1(a, x_0, 1) = \begin{cases} \emptyset, & a < w_1 \\ \{1\}, & a \geq w_1 \end{cases}$, $x_0 \in \{0, 1\}$.

For $k \in \{1, ..., n\}, a \in \{0, 1, ..., W\}$, we have:

$F_k(a, 1, 0) = F_{k-1}(a, 1, 1)$, $B_k(a, 1, 0) = B_{k-1}(a, 1, 1)$,

$F_k(a, 0, 0) = \max\{F_{k-1}(a, 0, 0), F_{k-1}(a, 1, 0)\}$, $B_k(a, 0, 0) = \begin{cases} B_{k-1}(a, 1, 0), & F_{k-1}(a, 0, 0) \leq F_{k-1}(a, 1, 0) \\ B_{k-1}(a, 0, 0), & F_{k-1}(a, 0, 0) > F_{k-1}(a, 1, 0) \end{cases}$

$F_k(a, x_{k-1}, 1) = \max\{F_{k-1}(a - w_i, 0, 1) + v_i, F_{k-1}(a - w_i, 1, 1) + v_i\}$,

$B_k(a, x_{k-1}, 1) = \begin{cases} B_{k-1}(a, 0, 1) \cup i, & F_{k-1}(a - w_i, 0, 1) \geq F_{k-1}(a - w_i, 1, 1) \\ B_{k-1}(a, 1, 1) \cup i, & F_{k-1}(a - w_i, 0, 1) < F_{k-1}(a - w_i, 1, 1) \end{cases}$

The answer is $B_n(c, d, e)$, where $d, e = \arg\max\limits_{i,j} F_n(a, i, j)$