1. We will make use of the law of total expectation from time to time.

   (a) It follows that

$$\begin{aligned}
\phi_Z(t) = \mathbb{E}[e^{tZ}] = \mathbb{E}\left[\mathbb{E}[e^{tZ}|Y]\right] &= \sum_{y=1}^{\infty} \mathbb{E}\left[e^{t\sum_{i=1}^{y} X_i}\right] \mathbb{P}(Y=y) \\
&\qquad\qquad \text{(law of total expectation)} \\
&= \sum_{y=1}^{\infty} \mathbb{E}\left[\prod_{i=1}^{y} e^{tX_i}\right] \mathbb{P}(Y=y) = \sum_{y=1}^{\infty} \prod_{i=1}^{y} \mathbb{E}\left[e^{tX_i}\right] \mathbb{P}(Y=y) \\
&\qquad\qquad \text{(independence)} \\
&= \sum_{y=1}^{\infty} \left(\mathbb{E}\left[e^{tX_1}\right]\right)^y \mathbb{P}(Y=y) \qquad \text{(identical distribution)} \\
&= \sum_{y=1}^{\infty} [\phi_{X_1}(t)]^y \, \mathbb{P}(Y=y) = \mathbb{E}[[\phi_{X_1}(t)]^Y] \quad \text{if it exists.}
\end{aligned}$$

   The existence of $\phi_Z(t)$ is guaranteed if $t \in \mathbb{R}$ is chosen such that $\phi_{X_1}(t) \leq 1$ (in which $\phi_Z(t)$ is bounded above by $\sum_{y=1}^{\infty} \mathbb{P}(Y=y) = 1$). Note that this may not be a necessary condition, since the actual "radius of convergence" could depend on the (tail) distribution of Y, but we are only given with a very general setting.

   (b) Suppose $\lambda > 0$ and $0 < p < 1$. Note that the mgf of $X_1 \sim \text{Exponential}(\lambda)$ is given by

$$\begin{aligned}
\phi_{X_1}(t) &= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} \, \mathrm{d}x = \lambda \int_0^{\infty} e^{-(\lambda - t)x} \, \mathrm{d}x \\
&= \begin{cases} \frac{\lambda}{\lambda - t}, & t < \lambda \\ \infty, & \text{otherwise} \end{cases}.
\end{aligned}$$

   Then for $t < \lambda$, it follows that

$$\begin{aligned}
\phi_Z(t) &= \mathbb{E}[[\phi_{X_1}(t)]^Y] \\
&= \sum_{y=1}^{\infty} \left(\frac{\lambda}{\lambda - t}\right)^y p(1-p)^{y-1} \\
&= \frac{p}{1-p} \sum_{y=1}^{\infty} \left(\frac{\lambda(1-p)}{\lambda - t}\right)^y \\
&= \begin{cases} \frac{p}{1-p} \frac{\lambda(1-p)}{(\lambda - t) - \lambda(1-p)} = \frac{\lambda p}{\lambda p - t}, & \left|\frac{\lambda(1-p)}{\lambda - t}\right| < 1 \iff t < \lambda p \\ \infty, & \text{otherwise} \end{cases}
\end{aligned}$$

   is essentially the mgf of $\text{Exponential}(\lambda p)$. Hence $Z \sim \text{Exponential}(\lambda p)$.

(c) We have

$$\begin{aligned}
\mathbb{E}[Z] &= \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{Y} X_i \middle| Y\right]\right] = \sum_{y=1}^{\infty} \mathbb{E}\left[\sum_{i=1}^{y} X_i\right] \mathbb{P}(Y=y) \\
&= \sum_{y=1}^{\infty}\left(\sum_{i=1}^{y} \mathbb{E}\left[X_i\right]\right) \mathbb{P}(Y=y) \\
&= \sum_{y=1}^{\infty}\left(y \mathbb{E}\left[X_1\right]\right) \mathbb{P}(Y=y) \\
&= \mathbb{E}\left[X_1\right] \sum_{y=1}^{\infty} y \mathbb{P}(Y=y) = \mathbb{E}[X_1]\mathbb{E}[Y].
\end{aligned}$$

(d) We have

$$\begin{aligned}
\operatorname{Var}(Z) &= \mathbb{E}[Z^2] - \mathbb{E}[Z]^2 = \mathbb{E}\left[\mathbb{E}\left[\left(\sum_{i=1}^{Y} X_i\right)^2 \middle| Y\right]\right] - \mathbb{E}[X_1]^2\mathbb{E}[Y]^2 \\
&= \sum_{y=1}^{\infty} \mathbb{E}\left[\left(\sum_{i=1}^{y} X_i\right)^2\right] \mathbb{P}(Y=y) - \mathbb{E}[X_1]^2\mathbb{E}[Y]^2 \\
&= \sum_{y=1}^{\infty}\left[\operatorname{Var}\left(\sum_{i=1}^{y} X_i\right) + \left(\mathbb{E}\sum_{i=1}^{y} X_i\right)^2\right] \mathbb{P}(Y=y) - \mathbb{E}[X_1]^2\mathbb{E}[Y]^2 \\
&= \sum_{y=1}^{\infty}\left[y\operatorname{Var}\left(X_1\right) + \left(y\mathbb{E}[X_1]\right)^2\right] \mathbb{P}(Y=y) - \mathbb{E}[X_1]^2\mathbb{E}[Y]^2 \\
&= \operatorname{Var}\left(X_1\right) \sum_{y=1}^{\infty} y \mathbb{P}(Y=y) + \mathbb{E}[X_1]^2 \sum_{y=1}^{\infty} y^2 \mathbb{P}(Y=y) - \mathbb{E}[X_1]^2\mathbb{E}[Y]^2 \\
&= \operatorname{Var}\left(X_1\right)\mathbb{E}[Y] + \mathbb{E}[X_1]^2\mathbb{E}[Y^2] - \mathbb{E}[X_1]^2\mathbb{E}[Y]^2 \\
&= \operatorname{Var}\left(X_1\right)\mathbb{E}[Y] + \mathbb{E}[X_1]^2\operatorname{Var}(Y).
\end{aligned}$$

2. By the independence of X_1 and X_2, the joint distribution of (X_1, X_2) is

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2) = \frac{1}{2\pi}\exp\left\{-\frac{x_1^2+x_2^2}{2}\right\}$$

for $x_1, x_2 \in \mathbb{R}$. Now consider the transformation $\mathbf{y}(\mathbf{x}) = (y_1(\mathbf{x}), y_2(\mathbf{x})) = \left(\sqrt{x_1^2+x_2^2}, \frac{x_1}{x_2}\right)$ whose domain $D = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 \neq 0\}$ is open, and whose range is $\mathbb{R}^+ \times \mathbb{R}$. Observe that

$$\begin{aligned}
x_1^2 &= \frac{x_1^2}{y_1^2} y_1^2 = \frac{x_1^2}{x_1^2+x_2^2} y_1^2 = \frac{\frac{x_1^2}{x_2^2}}{\frac{x_1^2}{x_2^2}+1} y_1^2 = \frac{y_1^2 y_2^2}{y_2^2+1}, \\
x_2^2 &= \frac{x_1^2}{y_2^2} = \frac{y_1^2}{y_2^2+1}.
\end{aligned}$$

Hence if we partition $D = D_1 \cup D_2$, where D_1 and D_2 enforce $x_2 > 0$ and $x_2 < 0$ respectively, then $\mathbf{y}$ would be one-to-one on each D_i.

Moreover, $\mathbf{y}$ has continuous partial derivatives

$$\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{bmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ \frac{1}{x_2} & -\frac{x_1}{x_2^2} \end{bmatrix}$$

and $\det\left(\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)}\right) = -\frac{\sqrt{x_1^2+x_2^2}}{x_2^2} \neq 0$ on D. By Inverse Function Theorem, $\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)}$ exists and satisfies

$$\mathbf{J}(y_1, y_2) = \det\left(\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)}\right) = \frac{1}{\det\left(\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)}\right)} = -\frac{x_2^2}{\sqrt{x_1^2+x_2^2}} = -\frac{y_1}{y_2^2+1}$$

regardless of which D_i does (x_1, x_2) belong to. Hence the joint density of (Y_1, Y_2) is

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= \sum_{i=1}^{2} f_{X_1,X_2}(x_1, x_2)|\mathbf{J}(y_1, y_2)| \\ &= \frac{1}{\pi}\exp\left\{-\frac{y_1^2}{2}\right\}\frac{y_1}{y_2^2+1} \\ &= \underbrace{y_1\exp\left\{-\frac{y_1^2}{2}\right\}}_{f_{Y_1}(y_1)}\underbrace{\frac{1}{\pi(y_2^2+1)}}_{f_{Y_2}(y_2)} \end{aligned}$$

for $(y_1, y_2) \in \mathbb{R}^+ \times \mathbb{R}$, and being able to separate into product of marginal densities concludes the independence of Y_1 and Y_2.

3. (a) Note that the mgf of $\mathbf{X}$ is given by

$$\phi_{\mathbf{X}}(\mathbf{t}) = \exp\left\{\frac{1}{2}\mathbf{t}^\top\mathbf{t}\right\}, \qquad \mathbf{t} \in \mathbb{R}^n$$

as $\mathbf{X} \sim \mathbf{N}_n(\mathbf{0}, \mathbf{I}_n)$. Then consider the mgf of $\mathbf{Y}$. Since, for each $\mathbf{t} \in \mathbb{R}^n$,

$$\begin{aligned} \phi_{\mathbf{Y}}(\mathbf{t}) &= \mathbb{E}\left[\exp\{\mathbf{t}^\top\mathbf{Y}\}\right] = \mathbb{E}\left[\exp\{\mathbf{t}^\top\mathbf{A}\mathbf{X}\}\right] \\ &= \mathbb{E}\left[\exp\{(\mathbf{A}^\top\mathbf{t})^\top\mathbf{X}\}\right] = \phi_{\mathbf{X}}(\mathbf{A}^\top\mathbf{t}) \\ &= \exp\left\{\frac{1}{2}(\mathbf{A}^\top\mathbf{t})^\top(\mathbf{A}^\top\mathbf{t})\right\} = \exp\left\{\frac{1}{2}\mathbf{t}^\top\mathbf{A}\mathbf{A}^\top\mathbf{t}\right\} \\ &= \exp\left\{\frac{1}{2}\mathbf{t}^\top\mathbf{t}\right\} && (\text{since } \mathbf{A}\mathbf{A}^\top = \mathbf{I}_n) \end{aligned}$$

coincides with the mgf of $\mathbf{X}$ at $\mathbf{t}$ near $\mathbf{0}$, we conclude that $\mathbf{Y} \sim \mathbf{N}_n(\mathbf{0}, \mathbf{I}_n)$, as desired.

(b) We have

$$\begin{aligned} (n-1)S^2 &= \sum_{i=1}^{n} X_i^2 - n\bar{X}^2 \\ &= \mathbf{X}^\top\mathbf{X} - n\bar{X}^2 \\ &= \mathbf{X}^\top\mathbf{A}^\top\mathbf{A}\mathbf{X} - n\bar{X}^2 && (\text{since } \mathbf{A}^\top\mathbf{A} = \mathbf{I}_n) \\ &= \mathbf{Y}^\top\mathbf{Y} - Y_1^2 \\ &= \sum_{i=1}^{n} Y_i^2 - Y_1^2 = \sum_{i=2}^{n} Y_i^2. \end{aligned}$$

Furthermore, we know from (a) that each $Y_i \overset{\text{iid}}{\sim} N(0, 1)$. Hence R.H.S. follows χ^2_{n-1} by definition.

4. (a) Recall that $F_Y(y) = 1 - e^{-y}$ for $y > 0$. Hence for $\tau \in (0,1)$,

$$\begin{aligned} Q_Y(\tau) &= \inf\{y : 1 - e^{-y} \geq \tau\} \\ &= \inf\{y : y \geq -\log(1-\tau)\} \\ &= -\log(1-\tau). \end{aligned}$$

(b) For $u \geq 0$,

$$\begin{aligned} L(u) &= \mathbb{E}[(Y-u)(\tau - \mathbb{1}\{Y < u\})] \\ &= \mathbb{E}[\tau(Y-u)] + \mathbb{E}[u\mathbb{1}\{Y < u\}] - \mathbb{E}[Y\mathbb{1}\{Y < u\}] \\ &= \tau(1-u) + u\mathbb{P}(Y < u) - \int_0^u ye^{-y}\,\mathrm{d}y \\ &= \tau(1-u) + u(1-e^{-u}) - \left(1 - ue^{-u} - e^{-u}\right) \\ &= (\tau - 1)(1-u) + e^{-u}. \end{aligned}$$

(c) Since we have

$$L'(u) = 1 - \tau - e^{-u} = 0 \iff u = -\log(1-\tau)$$

and $L''(u) = e^{-u} > 0$, we conclude that $u = -\log(1-\tau)$ globally minimizes $L(u)$.

5. Let $Z_i := -\log U_i$ for each i. Notice that $\mathbb{P}(Z_i \leq z) = 1 - \mathbb{P}(U_i < e^{-z}) = 1 - e^{-z}$ if $z > 0$ and 0 otherwise, which matches with the cdf of an exponential random variable with rate 1. Hence $Z_i \overset{\text{iid}}{\sim} \text{Exponential}(1)$, and it follows that $\mathbb{E}[Z_1] = \text{Var}(Z_1) = 1$. By Central Limit Theorem, we have

$$\sqrt{n}\left(\frac{\sum_{i=1}^n Z_i}{n} - 1\right) \Rightarrow N(0,1).$$

Apply the Delta Method with $g(x) = g'(x) = e^x$ gives

$$\sqrt{n}\left(\exp\left\{\frac{\sum_{i=1}^n Z_i}{n}\right\} - e\right) \Rightarrow N(0, e^2),$$

but this is exactly the desired result since

$$Y_n = \left(\prod_{i=1}^n U_i\right)^{-\frac{1}{n}} = \exp\left\{-\frac{1}{n}\log\left(\prod_{i=1}^n U_i\right)\right\} = \exp\left\{-\frac{\sum_{i=1}^n \log U_i}{n}\right\} = \exp\left\{\frac{\sum_{i=1}^n Z_i}{n}\right\}.$$

6. Suppose that $n \geq 2$ and recall that the pdf and cdf of $U[0,1]$ are $f(x) = \mathbb{1}\{0 < x < 1\}$ and $F(x) = x\mathbb{1}\{0 \leq x < 1\} + \mathbb{1}\{x \geq 1\}$ respectively. Then we have the joint density of $(X_{(1)}, X_{(n)})$ being

$$\begin{aligned} f_{X_{(1)},X_{(n)}}(u,v) &= \frac{n!}{(1-1)!(n-1-1)!(n-n)!}u^{1-1}(v-u)^{n-1-1}(1-v)^{n-n} \\ &= \frac{n!}{(n-2)!}(v-u)^{n-2}, \quad 0 < u < v < 1. \end{aligned}$$

Then consider the one-to-one transformation $(R,S) = \mathbf{g}(X_{(1)}, X_{(n)}) = (X_{(n)} - X_{(1)}, X_{(1)})$ on (X_1, X_n) whose corresponding inverse is $\mathbf{h}(R,S) = (h_1(R,S), h_2(R,S)) = (S, R+S)$. This will imply the joint density of (R,S) as follows:

$$\begin{aligned} f_{R,S}(r,s) &= \frac{n!}{(n-2)!}(r+s-s)^{n-2}\left|\det\left(\frac{\partial(h_1,h_2)}{\partial(r,s)}\right)\right| \\ &= \frac{n!}{(n-2)!}r^{n-2}, \qquad r,s > 0, r+s < 1. \end{aligned}$$

Thus the required pdf is essentially the marginal density of R:

$$\begin{aligned} f_R(r) &= \int_0^{1-r} \frac{n!}{(n-2)!} r^{n-2}\,\mathrm{d}s \\ &= n(n-1)(1-r)r^{n-2}, \quad 0 < r < 1. \end{aligned}$$

For each $s > 0$, the cdf of $S_n := 2n(1-R)$ is

$$\begin{aligned} F_{S_n}(s) &= \mathbb{P}(2n(1-R) \le s) = 1 - \mathbb{P}\left(R \le 1 - \frac{s}{2n}\right) \\ &= 1 - n(n-1)\int_0^{1-\frac{s}{2n}} (1-r)r^{n-2}\,\mathrm{d}r \mathbb{1}\{s < 2n\} \\ &= 1 - \left[\left(1 - \frac{s}{2n}\right)^n + \frac{s}{2}\left(1 - \frac{s}{2n}\right)^{n-1}\right]\mathbb{1}\{s < 2n\} \\ &= 1 - \left(1 - \frac{s}{2n}\right)^{n-1}\left(1 - \frac{s}{2n} + \frac{s}{2}\right)\mathbb{1}\{s < 2n\} \\ &\overset{n\to\infty}{\to} 1 - \exp\left\{-\frac{s}{2}\right\}\left(1 + \frac{s}{2}\right) \\ &= -\exp\left\{-\frac{x}{2}\right\}\left(1 + \frac{x}{2}\right)\bigg|_{x=0}^{s} = \int_0^s \underbrace{\frac{x\exp\left\{-\frac{x}{2}\right\}}{4}}_{\text{density of } \chi_4^2}\,\mathrm{d}x, \end{aligned}$$

where R.H.S. is the cdf of χ_4^2 for all $s > 0$. Moreover, it can be observed from $F_{S_n}(s) = 1 - \mathbb{P}\left(R \le 1 - \frac{s}{2n}\right)$ that if $s \le 0$, then we have $F_{S_n}(s) = 0$, which always agree with the cdf of χ_4^2 on the left-half of $\mathbb{R}$. Hence, we conclude that $S_n \Rightarrow \chi_4^2$.