1. Let $X_1, X_2, \ldots, X_n$ be i.i.d. normal distribution $N(\mu, 1)$ random variables with unknown μ. Instead of recording all the observations, we record only whether the observations is less than 0. In other words, we record $Y_i = \mathbb{I}_{\{X_i<0\}}$, for $i = 1, 2, \ldots, n$. Find a MLE of μ. You may use $\Phi(x)$ to denote the CDF of a standard normal random variable $N(0, 1)$.

2. Let $(X_1, X_2, \ldots, X_n)$ be a random sample from $P \in \mathcal{P}$ containing all *symmetric distributions* with finite means and with proper probability density functions. Here, symmetric means that the pdf is a symetric function, i.e., there exist a constant b such that $f(b-x) = f(b+x)$ for all $x > 0$.
   (a) When $n = 1$, show that X_1 is the UMVUE of $\mu = \mathbb{E}[X]$.
   (b) (For this part only.) Let $(X_1, \ldots, X_n), n \geq 2$, be a random sample from the uniform distribution on the interval $[\theta_1 - \theta_2, \theta_1 + \theta_2]$ where $\theta_2 > 0$. Find the UMVUE of θ_1.
   (c) (Back to the setting of the main question.) When $n > 1$, show that there is no UMVUE of $\mu = \mathbb{E}[X]$. [Hint: Consider the family in Part (b). Use the uniqueness of UMVUE.]

3. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with probability density function

$$f(x;\theta) = \frac{1}{\theta} f_0\left(\frac{x}{\theta}\right),$$

   where $\theta > 0$ is an unknown parameter, $\mathsf{E}[|X_1|] < \infty$ and $f_0(\cdot)$ is known, does not depend on θ and is differentiable.
   (a) Suppose $\eta = h(\theta)$, where $h(\cdot)$ is an invertible differentiable one-to-one function. Find an expression for the Fisher information $I^\eta(\eta)$ based on X_1 in terms of $I^\theta(\theta)$, the Fisher information for θ based on X_1.
   (b) Let

$$\widehat{\theta} = \frac{1}{n} \sum_{i=1}^{n} \frac{|X_i|}{\int_{-\infty}^{\infty} |x| f_0(x) dx}.$$

   Is $\widehat{\theta}$ unbiased? If $\mathsf{E}[X_i^2] < \infty$, is it consistent? What is the asymptotic distribution of $\widehat{\theta}$ as $n \to \infty$?

4. Let $X_1, \cdots, X_n$ be iid Normal(μ, σ^2) random samples, with known $\sigma^2 > 0$.
   (a) Find the UMVUE of $e^{\mu t}$ for a fixed t. Show that the variance of the UMVUE is larger than the Cramér-Rao lower bound but the ratio of the variance and the lower bound converges to 1 as $n \to \infty$.
   (b) Find the UMVUE of $\mathsf{P}(X_1 < t)$ with fixed $t \in \mathbb{R}$.

5. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two independent i.i.d. RV from $\mathcal{N}(\mu_1, \sigma^2)$ and $\mathcal{N}(\mu_2, \sigma^2)$.
   (a) Show that the MLE of (μ_1, μ_2, σ^2) is $\bar{X}$, $\bar{Y}$, and

$$\hat{\sigma}^2 = \frac{1}{m+n}\left(\sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{j=1}^{m}(Y_i - \bar{Y})^2\right).$$

   (b) Let $s^2_{pooled} = \frac{m+n}{m+n-2}\hat{\sigma}^2$. Show that s^2_{pooled} is unbiased, and then show it is UMVUE.