Final Exam of IEDA5250

2 hours and 50 minutes

1. (25 points) Consider a system with two servers. Potential customers arrive at this system in accordance with a Poisson process with rate of two per hour and all the service times are i.i.d. exponential random variables with mean 1/3 hours. Moreover, assume that

   (i) the arrival process is independent of the service times;

   (ii) if a customer finds that there are 4 customers (including 2 customers being served and 2 customers waiting in the queue) in the system upon his/her arrival will not enter the system;

   (iii) each customer waiting in the queue incurs costs at the rate of 10 dollars per hour.

   Let $X(t)$ denote the number of customers in the system at time t. Then $\{X(t) : t \geq 0\}$ is a birth and death process. **(Justify your answers rigorously.)**

   (a) What are the birth rates and death rates for all states?

   sol. $q_{i,i+1} = 2$ for $i = 0, 1, 2, 3$. $q_{1,0} = 3$, $q_{2,1} = q_{3,2} = q_{4,3} = 6$. □

   (b) Find the long-run proportion of time that the system spends in each state.

   sol. Can use detailed balance equation. $3\pi_4 = \pi_3$, $3\pi_3 = \pi_2$, $3\pi_2 = \pi_1$, $3\pi_1 = 2\pi_0$, $\pi = (81, 54, 18, 6, 2)/161$. □

   (c) What is the rate at which the process enters state 2?

   sol. $\pi_1 q_{12} + \pi_3 q_{32} = 144/161$. □

   (d) Is $X(t)$ time reversible in its steady state?

   sol. Yes, check detailed balance equation. □

   (e) What is the long-run average waiting cost per hour?

   sol. Average number of waiting customers 10/161. Cost 100/161. □

2. (25 points) A mouse finds itself in a building with 100 floors. Each floor has nine rooms, numbered one through nine. Each floor has rooms laid out like

1	2	3
4	5	6
7	8	9

   On each floor, there are doors:

between rooms 1 and 2
between rooms 1 and 4
between rooms 2 and 3
between rooms 2 and 5
between rooms 3 and 6
between rooms 4 and 5
between rooms 4 and 7
between rooms 5 and 6
between rooms 5 and 8
between rooms 6 and 9
between rooms 7 and 8
between rooms 8 and 9

The mouse can go through a door in either direction. From Room 5, the mouse can move next to Rooms 2, 4, 6 and 8 on the same floor, or the Mouse can move to Room 5 on the floor above or to Room 5 on the floor below. The mouse could not move up from room 5 on floor 100 and move down from room 5 on floor 1 Suppose that, in each move, The Mouse is equally likely to make each of the available moves. Suppose that the Mouse starts in Room 1 on Floor 1 **(Justify your answers rigorously.)**

(a) Does the limit distribution exist? Briefly explain your answer

Sol. The limiting steady-state distribution does not exist, because of periodicity. □

(b) What is the probability that the Mouse is in Room 1 on Floor 1 after making two moves?

Sol.

$$\mathbb{P}(\textbf{Return}) = \frac{1}{2} \cdot \frac{1}{3} \cdot + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3}$$

□

(c) What is the probability that the Mouse is in Room 1 on Floor 1 after making three moves?

Sol. The probability is 0 because the Markov chain is periodic with period 2 □

(d) What is the expected number of moves made by Markov Mouse before he first returns to the initial room, Room 1 on Floor 1?

Sol. The expected number of moves between successive visits to Room 1 on Floor 1 is the reciprocal of the stationary probability. The stationary probability is the number of doors out of that room divided by the sum of the number of doors out of all the rooms. Hence numerator is 2. The denominator is

$$4 * 100 * 2 + 4 * 100 * 3 + 1 * 98 * 6 + 1 * 2 * 5 = 2598$$

Hence the expected moves should be 1299. □

(e) Consider the probability that Markov Mouse is in Room 1 on Floor 1 after n moves. Give an approximation for this probability for large values of n.

Sol. $P_{1,1}^{2n+1} = 0$ and $P_{1,1}^{2n} = \frac{2}{1299}$ □

3. (25 points) The movement of a particle follows a random walk on a circle with N nodes. Moreover, assume that

   (i) the particle at node j, $1 \le j \le N$, will move one step, independently of all else, either clockwise with probability p $(0 < p < 1)$ or counterclockwise with probability $q = 1 - p$;

   (ii) at time 0 the particle is at node 1.

For $n \ge 1$, let X_n denote the location of the particle after n steps. Define $X_0 = 1$.Then $\{X_n : n = 0, 1, \cdots\}$ is a discrete-time Markov chain.**(Justify your answers rigorously.)**

(a) Write down the stationary distribution $\pi = (\pi_1, \ldots, \pi_N)$, you may skip the intermediate steps for this question.

sol. $\pi = (1/N, \ldots, 1/N)$. □

(b) Let $\tau_1 = \inf\{n > 0 : X_n = 1\}$, find its expectation $E\tau_1$.

sol. $E\tau_1 = \frac{1}{\pi_1} = N$. □

Assume that $p = q = \frac{1}{2}$ for the next three questions.

(c) Let σ_n be the first time that in total n states have been visited, i.e., $\sigma_1 = 0$, $\sigma_2 = 1$, but σ_n is random for $n > 2$. Find $E[\sigma_3 - \sigma_2]$.

sol. This is the same as the exiting time in a gambler's ruin problem with $n = 3, i = 2$. $E[\sigma_3 - \sigma_2] = 2$ □

(d) Find $E[\sigma_N]$.

sol. $E[\sigma_{n+1} - \sigma_n] = n$. $E[\sigma_N] = 1 + 2 + \cdots + N - 1 = N(N-1)/2$. □

(e) Find the distribution of the particle's position at the first time it has visited all states, i.e., find $P(X_{\sigma_N} = j)$ for each $1 \le j \le N$.

sol. $X_{\sigma_n} = j$ means visiting $j-1$ before visiting j and visiting $j+1$ before visiting j. Let τ_j be the first time it visits j. $P(\tau_j > \tau_{j+1}, \tau_j > \tau_{j-1}) = P(\tau_j > \tau_{j+1} > \tau_{j-1}) + P(\tau_j > \tau_{j-1} > \tau_{j+1})$. Using gambler's ruin results, for $j \neq 1$, $P(\tau_j > \tau_{j+1} > \tau_{j-1}) = \frac{j-2}{N-2} \times \frac{1}{N-1}$ and $P(\tau_j > \tau_{j-1} > \tau_{j+1}) = \frac{N-j}{N-2} \times \frac{1}{N-1}$. $P(X_{\sigma_n} = j) = \frac{1}{N-1}$ for $j \neq 1$ and $P(X_{\sigma_n} = 1) = 0$. □

4. (25 points) Consider an $M/M/\infty$ queue in which customers arrive according to a Poisson process having rate λ. Each customer starts to receive service immediately upon arrival from one of an unlimited number of servers. Suppose that the service times are IID exponential random variables with mean $\frac{1}{\mu}$.
Suppose that, on arrival, each customer will choose the lowest numbered server that is free. Thus we can think of all arrivals occurring at server 1. Those customers who find server 1 free begin service there. Those customers finding server 1 busy immediately overflow and become arrivals at server 2. Those customers finding both servers 1 and 2 busy immediately overflow and become arrivals at server 3, and so forth. **(Justify your answers rigorously.)**

(a) Let $N(t)$ be the number of customers that find server 1 is busy and choose server 2 in the time interval $[0, t]$. Is it a Markov process? Briefly explain your answer

Sol. It is not a Markov process, because the future beyond some time t depends upon whether server 1 is busy or not at time t. □

(b) What is the long-run proportion of time that both servers 1 and 2 are busy?

Sol. Consider the steady-state probability in the $M/M/2$ model. In particular

$$\mathbb{P}(\text{busy}) = \frac{\alpha^2/2}{1 + \alpha + \alpha^2/2}$$

where $\alpha = \frac{\lambda}{\mu}$ □

(c) Starting with an empty system, what is the expected time until server 2 first becomes busy?

Sol. Let $T_{i,2}$ be the time until server 2 first becomes busy, starting with the system empty. We want to find $\mathbf{E}T_{0,2}$

$$\begin{aligned} \mathbf{E}T_{0,2} &= \frac{1}{\lambda} + \mathbf{E}T_{1,2} \\ &= \frac{1}{\lambda} + \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}\mathbf{E}T_{0,2} \end{aligned}$$

Solve the equation gives us $\mathbf{E}T_{0,2} = \frac{2\lambda+\mu}{\lambda^2}$ □

(d) What is the expected number of busy servers among the first two servers in steady-state?

Sol.

$$\mathbf{E}N = 1 \cdot P(N=1) + 2 \cdot P(N=2) = \frac{2\alpha + 2\alpha^2}{2\alpha + \alpha^2 + 2}$$

where $\alpha = \frac{\lambda}{\mu}$ □

(e) What proportion of time is server 2 busy?

Sol. Let $I_i = 1$ if server i is busy in steady state.

$$\mathbf{E}[I_1] = P(\text{server 1 busy}) = \frac{\alpha}{1+\alpha}$$

$$E[I_2] = E[I_1 + I_2] - E[I_1] = \frac{2\alpha^2 + \alpha^3}{(2 + 2\alpha + \alpha^2)(1+\alpha)}$$

□