1. (a) We will quote some results from Question 1 of HW5. Firstly, the maximum likelihood over the entire parameter space is

$$\sup_{(a,\theta)\in\Theta} L(a,\theta|\mathbf{x}) = \left(\frac{1}{\sqrt{2\pi\hat{a}\hat{\theta}}}\right)^n \exp\left\{-\frac{\sum_{i=1}^n (x_i-\hat{\theta})^2}{2\hat{a}\hat{\theta}}\right\} = \left(\frac{e^{-\frac{1}{2}}}{\sqrt{2\pi s_n^2}}\right)^n.$$

Under $H_0 : a = a_0$, the MLE for θ satisfies

$$-\frac{n}{2\theta} + \frac{\sum_{i=1}^n x_i^2}{2a_0\theta^2} - \frac{n}{2a_0} = 0 \implies \hat{\theta}_0 = \frac{-a_0 + \sqrt{a_0 + 4n^{-1}\sum_{i=1}^n x_i^2}}{2},$$

indicating that

$$\sup_{(a,\theta)\in\Theta_0} L(a,\theta|\mathbf{x}) = \left(\frac{1}{\sqrt{2\pi a_0\hat{\theta}_0}}\right)^n \exp\left\{-\frac{\sum_{i=1}^n (x_i-\hat{\theta}_0)^2}{2a_0\hat{\theta}_0}\right\}.$$

Since the LRT rejects H_0 at level α when

$$\begin{aligned}\lambda(\mathbf{x}) &= \frac{\sup_{(a,\theta)\in\Theta_0} L(a,\theta|\mathbf{x})}{\sup_{(a,\theta)\in\Theta} L(a,\theta|\mathbf{x})} \\ &= \left(\frac{s_n^2}{a_0\hat{\theta}_0}\right)^{\frac{n}{2}} \exp\left\{\frac{n}{2} - \frac{\sum_{i=1}^n (x_i-\hat{\theta}_0)^2}{2a_0\hat{\theta}_0}\right\} < c,\end{aligned}$$

where the threshold c depends on α, by inverting the rejection region, we conclude that a $1-\alpha$ confidence set for a is the set $\{a_0 : \lambda(\mathbf{x}) \geq c\}$.

(b) We have

$$\frac{\bar{X}-\theta}{\sqrt{\theta/n}} \sim \mathrm{N}(0,1),$$

whose distribution does not depend on θ. Therefore, it is a pivotal quantity satisfying

$$\begin{aligned}1-\alpha &= \mathbb{P}\left(-z(1-\alpha/2) \leq \frac{\bar{X}-\theta}{\sqrt{\theta/n}} \leq z(1-\alpha/2)\right) \\ &= \mathbb{P}\left(\frac{(\bar{X}-\theta)^2}{\theta/n} \leq z(1-\alpha/2)^2\right) \\ &= \mathbb{P}\left(\theta^2 - \left(2\bar{X} + \frac{z(1-\alpha/2)^2}{n}\right)\theta + \bar{X}^2 \leq 0\right)\end{aligned}$$

Hence, a $1-\alpha$ confidence set for θ is $\left\{\theta : \theta^2 - \left(2\bar{X} + \frac{z(1-\alpha/2)^2}{n}\right)\theta + \bar{X}^2 \leq 0\right\}$.

(c) We have

$$\frac{\bar{X}-\theta}{\sqrt{S^2/n}} = \frac{\frac{\bar{X}-\theta}{\sqrt{\theta/n}}}{\sqrt{\frac{(n-1)S^2}{\theta}/(n-1)}} \sim t_{n-1}$$

since it is in the form of $Z/\sqrt{V/(n-1)}$, where $Z \sim \mathrm{N}(0,1)$ and $V \sim \chi^2_{n-1}$ are independent. Therefore, it is a pivotal quantity satisfying

$$\begin{aligned}1-\alpha &= \mathbb{P}\left(-t_{n-1}(1-\alpha/2) \leq \frac{\bar{X}-\theta}{\sqrt{S^2/n}} \leq t_{n-1}(1-\alpha/2)\right) \\ &= \mathbb{P}\left(\bar{X} - t_{n-1}(1-\alpha/2)\sqrt{\frac{S^2}{n}} \leq \theta \leq \bar{X} + t_{n-1}(1-\alpha/2)\sqrt{\frac{S^2}{n}}\right),\end{aligned}$$

so a $1-\alpha$ C.I. for θ is $\bar{X} \mp t_{n-1}\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{S^2}{n}}$.

(d) We have

$$\frac{(n-1)S^2}{\theta} \sim \chi^2_{n-1}.$$

Therefore, it is a pivotal quantity satisfying

$$\begin{aligned}
1-\alpha &= \mathbb{P}\left(\chi^2_{n-1}(\alpha/2) \le \frac{(n-1)S^2}{\theta} \le \chi^2_{n-1}(1-\alpha/2)\right) \\
&= \mathbb{P}\left(\frac{(n-1)S^2}{\chi^2_{n-1}(1-\alpha/2)} \le \theta \le \frac{(n-1)S^2}{\chi^2_{n-1}(\alpha/2)}\right),
\end{aligned}$$

so a $1-\alpha$ C.I. for θ is $\left[\frac{(n-1)S^2}{\chi^2_{n-1}(1-\alpha/2)}, \frac{(n-1)S^2}{\chi^2_{n-1}(\alpha/2)}\right]$.

2. It is clear that $\theta > 0$ and the cdf of X is $\mathbb{P}(X \le x) = x^\theta \mathbb{1}\{0 \le x \le 1\} + \mathbb{1}\{x > 1\}$.

(a) From formulation Y is non-negative. For $y > 0$, we have

$$\mathbb{P}(Y \le y) = \mathbb{P}\left(-\frac{1}{\log(X)} \le y\right) = \mathbb{P}(X \le e^{-\frac{1}{y}}) = e^{-\frac{\theta}{y}}.$$

Hence,

$$\begin{aligned}
\mathbb{P}\left(\frac{Y}{2} \le \theta \le Y\right) &= \mathbb{P}\left(\theta \le Y \le 2\theta\right) \\
&= \mathbb{P}(Y \le 2\theta) - \mathbb{P}(Y \le \theta) = e^{-\frac{1}{2}} - e^{-1}
\end{aligned}$$

for all $\theta > 0$, so it is the desired confidence coefficient.

(b) Let $U := X^\theta$. Then

$$\mathbb{P}(U \le u) = \mathbb{P}\left(X^\theta \le u\right) = u\mathbb{1}\{0 \le u \le 1\} + \mathbb{1}\{u > 1\},$$

indicating that $U \sim \text{Unif}(0,1)$ is a pivotal quantity, and we can use it to construct a C.I. For instance, since we have

$$\begin{aligned}
e^{-\frac{1}{2}} - e^{-1} &= \mathbb{P}(1 - e^{-\frac{1}{2}} + e^{-1} \le U \le 1) \\
&= \mathbb{P}(1 - e^{-\frac{1}{2}} + e^{-1} \le X^\theta \le 1) \\
&= \mathbb{P}\left(\frac{\log(1 - e^{-\frac{1}{2}} + e^{-1})}{\log(X)} \ge \theta \ge 0\right),
\end{aligned}$$

a required C.I. for θ would be given by $\left[0, \dfrac{\log(1 - e^{-\frac{1}{2}} + e^{-1})}{\log(X)}\right]$.

(c) Note that $\theta \in [Y/2, Y] \iff \theta \in \left[-\frac{1}{2\log(X)}, -\frac{1}{\log(X)}\right]$, and hence the length of C.I. in part (a) is

$$-\frac{1}{\log(X)} + \frac{1}{2\log(X)} = -\frac{1}{2\log(X)}.$$

Since

$$\begin{aligned}
(1 - e^{-\frac{1}{2}})^2 > 0 &\iff 1 - e^{-\frac{1}{2}} + e^{-1} > e^{-\frac{1}{2}} \\
&\iff \log(1 - e^{-\frac{1}{2}} + e^{-1}) > -\frac{1}{2} \\
&\iff \frac{\log(1 - e^{-\frac{1}{2}} + e^{-1})}{\log(X)} < -\frac{1}{2\log(X)},
\end{aligned}$$

we conclude that the length of C.I. that we proposed in (b) is shorter than that in (a).

3. (a) We know that $\dfrac{\bar{X}-\mu}{\sqrt{\sigma^2/n}} \sim \mathrm{N}(0,1)$ so a $1-\alpha$ C.I. for μ is

$$\left[\bar{X} - z\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{\sigma^2}{n}}, \bar{X} + z\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{\sigma^2}{n}}\right],$$

whose length is $2z\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{\sigma^2}{n}}$. Hence, $2z\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{\sigma^2}{n}} \leq \delta \iff n \geq \frac{4z\left(1-\frac{\alpha}{2}\right)^2\sigma^2}{\delta^2}$ so we pick $n = \left\lceil \frac{4z\left(1-\frac{\alpha}{2}\right)^2\sigma^2}{\delta^2} \right\rceil$.

(b) We know that $\dfrac{\bar{X}-\mu}{\sqrt{S^2/n}} \sim t_{n-1}$ so a $1-\alpha$ C.I. for μ is

$$\left[\bar{X} - t_{n-1}\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{S^2}{n}}, \bar{X} + t_{n-1}\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{S^2}{n}}\right],$$

whose expected length is

$$\begin{aligned} 2t_{n-1}\left(1-\frac{\alpha}{2}\right)\mathbb{E}\left[\sqrt{\frac{S^2}{n}}\right] &= 2t_{n-1}\left(1-\frac{\alpha}{2}\right)\frac{\sigma}{\sqrt{n(n-1)}}\mathbb{E}\left[\sqrt{\frac{(n-1)S^2}{\sigma^2}}\right] \\ &= 2t_{n-1}\left(1-\frac{\alpha}{2}\right)\frac{\sqrt{2}\Gamma(n/2)}{\Gamma((n-1)/2)\sqrt{n-1}}\sqrt{\frac{\sigma^2}{n}}, \end{aligned}$$

where we used the moment properties for the χ-distribution with $n-1$ DoF. Hence, we choose n as the least integer satisfying

$$2t_{n-1}\left(1-\frac{\alpha}{2}\right)\frac{\sqrt{2}\Gamma(n/2)}{\Gamma((n-1)/2)\sqrt{n-1}}\sqrt{\frac{\sigma^2}{n}} \leq \delta.$$

The close form is not easy to obtain, though.

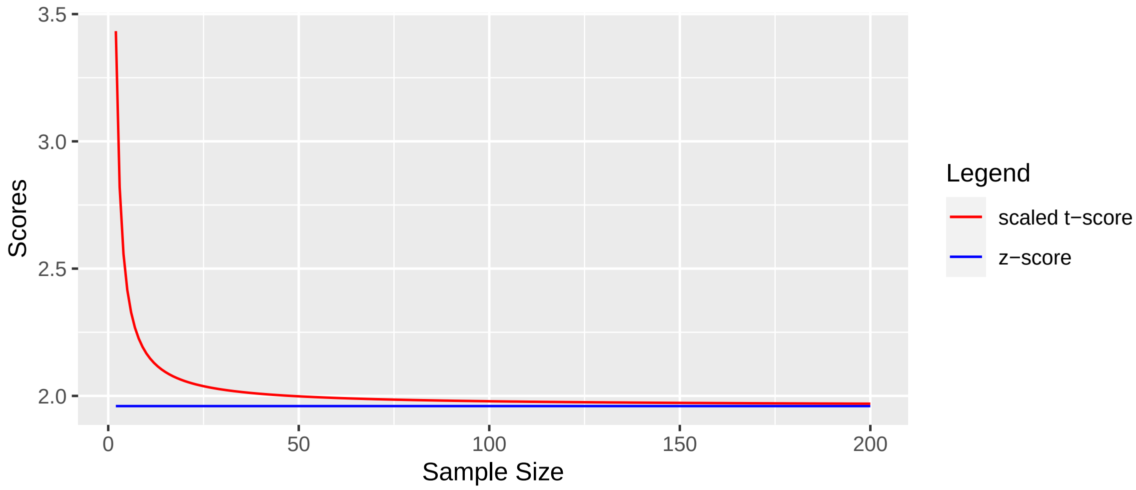
(c) We compare their expected lengths, and it remains to compare the z-score $z\left(1-\frac{\alpha}{2}\right)$ and the "adjusted" t-score $t_{n-1}\left(1-\frac{\alpha}{2}\right)\frac{\sqrt{2}\Gamma(n/2)}{\Gamma((n-1)/2)\sqrt{n-1}}$. It is observed that the t-score is always larger for a small α and $n \geq 2$, indicating that the expected length in (b) is larger than that in (a). Moreover, their difference gets closer to 0 when n becomes larger (which can be proved by $t_{n-1}(1-\alpha/2) \to z(1-\alpha/2)$ and $\frac{\sqrt{2}\Gamma(n/2)}{\Gamma((n-1)/2)\sqrt{n-1}} \to 1$ when $n \to \infty$). For demonstration, we plot the case for $\alpha = 0.05$ as follows.

```r
library(ggplot2)

alpha <- 0.05
n <- 2:200
z <- rep(qnorm(1-alpha/2), length(n))
t <- qt(1-alpha/2, df=n) * sqrt(2/(n-1)) * gamma(n/2) / gamma((n-1)/2)
colors <- c("z-score"="blue", "scaled t-score"="red")

data <- data.frame(n, z, t)
ggplot(data, aes(x=n)) +
geom_line(aes(y=z, color="z-score")) +
geom_line(aes(y=t, color="scaled t-score")) +
```

```
labs(x="Sample Size",
     y="Scores",
     color="Legend") +
scale_colour_manual(values=colors)
```



4. Assume $\theta > 0$ as we need the Fisher Information to be continuous. Our first direction of finding asymptotic C.I. is to consider the MLE of θ, $\hat{\theta}_n$. As we have derived $l'(\theta)$ in Question 1 of HW5, we have

$$l''(\theta|\mathbf{X}) = \frac{n}{\theta^2} - \frac{3\sum_{i=1}^n X_i^2}{\theta^4} + \frac{2\sum_{i=1}^n X_i}{\theta^3}$$

and hence the expected and observed Fisher Information are, respectively,

$$nI(\theta) = -\frac{n}{\theta^2} + \frac{3n\mathbb{E}[X_1^2]}{\theta^4} + \frac{2n\mathbb{E}[X_1]}{\theta^3} = \frac{3n}{\theta^2},$$
$$-l''(\hat{\theta}_n|\mathbf{x}) = -\frac{n}{\hat{\theta}_n^2} + \frac{3\sum_{i=1}^n X_i^2}{\hat{\theta}_n^4} - \frac{2\sum_{i=1}^n X_i}{\hat{\theta}_n^3},$$

where the explicit form of $\hat{\theta}_n$ is also provided in HW5.

Now, the asymptotic normality of MLE under regular families indicates $\sqrt{nI(\theta)}(\hat{\theta}_n - \theta) \Rightarrow \mathrm{N}(0,1)$, where $nI(\theta)$ could be replaced by $nI(\hat{\theta}_n)$ or $-l''(\hat{\theta}_n|\mathbf{x})$ by Continuous Mapping Theorem and Slutsky's Theorem. Hence, we have a $1-\alpha$ asymptotic C.I. for θ being

$$\left[\hat{\theta}_n - \frac{z\left(1-\frac{\alpha}{2}\right)}{\sqrt{nI(\hat{\theta}_n)}}, \hat{\theta}_n + \frac{z\left(1-\frac{\alpha}{2}\right)}{\sqrt{nI(\hat{\theta}_n)}}\right],$$

another one being

$$\left[\hat{\theta}_n - \frac{z\left(1-\frac{\alpha}{2}\right)}{\sqrt{-l''(\hat{\theta}_n|\mathbf{x})}}, \hat{\theta}_n + \frac{z\left(1-\frac{\alpha}{2}\right)}{\sqrt{-l''(\hat{\theta}_n|\mathbf{x})}}\right].$$

Alternatively, we may find the asymptotic C.I. based on $\bar{X}$ as the estimator for θ. Then we have $\dfrac{\bar{X}-\theta}{\sqrt{S^2/n}} \Rightarrow \mathrm{N}(0,1)$ (since $t_{n-1} \Rightarrow \mathrm{N}(0,1)$). Hence a $1-\alpha$ asymptotic C.I. for θ is

$$\left[\bar{X} - z\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{S^2}{n}}, \bar{X} + z\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{S^2}{n}}\right].$$

By Slutsky's Theorem, we can maintain its asymptotic normality when we replace S^2 with any estimator that converges to θ^2 in probability, say $\bar{X}^2$ (or $\bar{X}^4/S^2$ or whatever). Then another $1-\alpha$ asymptotic C.I. for θ is

$$\left[\bar{X} - z\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{\bar{X}^2}{n}}, \bar{X} + z\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{\bar{X}^2}{n}}\right].$$

Remark. The answers are not unique. One can construct other asymptotic C.I.s (for instance, profile confidence intervals, apply Delta Method to $\sqrt{n}(\bar{X}-\theta)$, etc.) provided that they can establish the correct asymptotic distributions and can show that the answers are indeed intervals.

5. (a) From Central Limit Theorem, we have

$$\sqrt{n}(\bar{X}_n - \theta) \Rightarrow \mathrm{N}(0, \theta).$$

Furthermore, we know from Delta Method that

$$\sqrt{n}(g(\bar{X}_n) - g(\theta)) \Rightarrow \mathrm{N}(0, \theta[g'(\theta)]^2)$$

if g is differentiable with $g'(\theta) \neq 0$. Therefore we want $\theta[g'(\theta)]^2 = 1$, which can be achieved if $g(\theta) = 2\sqrt{\theta}$.

(b) From (a) we have

$$\begin{aligned}
1-\alpha &\leftarrow \mathbb{P}\left(-z\left(1-\frac{\alpha}{2}\right) \leq \sqrt{n}(2\sqrt{\bar{X}_n} - 2\sqrt{\theta}) \leq z\left(1-\frac{\alpha}{2}\right)\right) \\
&= \mathbb{P}\left(\sqrt{\bar{X}_n} - \frac{z\left(1-\frac{\alpha}{2}\right)}{2\sqrt{n}} \leq \sqrt{\theta} \leq \sqrt{\bar{X}_n} + \frac{z\left(1-\frac{\alpha}{2}\right)}{2\sqrt{n}}\right) \\
&= \mathbb{P}\left(\left(\max\left\{0, \sqrt{\bar{X}_n} - \frac{z\left(1-\frac{\alpha}{2}\right)}{2\sqrt{n}}\right\}\right)^2 \leq \theta \leq \left(\sqrt{\bar{X}_n} + \frac{z\left(1-\frac{\alpha}{2}\right)}{2\sqrt{n}}\right)^2\right).
\end{aligned}$$

Hence, a $1-\alpha$ asymptotic C.I. for θ is

$$\left[\left(\max\left\{0, \sqrt{\bar{X}_n} - \frac{z\left(1-\frac{\alpha}{2}\right)}{2\sqrt{n}}\right\}\right)^2, \left(\sqrt{\bar{X}_n} + \frac{z\left(1-\frac{\alpha}{2}\right)}{2\sqrt{n}}\right)^2\right].$$