1. (a) The pmf of $X$ can be written as

$$\begin{aligned}\mathbb{P}(X=x) &= \frac{\gamma(x)\theta^x}{c(\theta)} \\ &= \gamma(x)\frac{1}{c(\theta)}\exp\{x\log\theta\},\end{aligned}$$

so result follows from definition.

(b) For each $t = 0, 1, \cdots$, we have

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^n X_i = t\right) &= \sum_{\{\mathbf{x}:\sum_{i=1}^n x_i = t\}} \mathbb{P}(\mathbf{X}=\mathbf{x}) \\ &= \sum_{\sum_{i=1}^n x_i=t}\prod_{k=1}^n \frac{\gamma(x_k)\theta^{x_k}}{c(\theta)} \\ &= \frac{1}{(c(\theta))^n}\sum_{\sum_{i=1}^n x_i=t}\theta^{\sum_{i=1}^n x_i}\prod_{k=1}^n\gamma(x_k) \\ &= \frac{\theta^t}{(c(\theta))^n}\sum_{\sum_{i=1}^n x_i=t}\prod_{k=1}^n\gamma(x_k) \\ &= \frac{\theta^t}{(c(\theta))^n}\gamma_n(t),\end{aligned}$$

which is indeed the desired pmf. To elaborate the last equality, we can write

$$(c(\theta))^n = \prod_{i=1}^n\sum_{x_i=0}^\infty \gamma(x_i)\theta^{x_i}.$$

The coefficient of $\theta^t$ in $(c(\theta))^n$, i.e. $\gamma_n(t)$, is determined by summing up all the coefficients of $\theta^t$ after expanding $(\sum_{x_1=0}^\infty \gamma(x_1)\theta^{x_1})(\sum_{x_2=0}^\infty \gamma(x_2)\theta^{x_2})\cdots(\sum_{x_n=0}^\infty \gamma(x_n)\theta^{x_n})$, which are $\gamma(x_1)\gamma(x_2)\cdots\gamma(x_n)$ that satisfy $x_1+\cdots+x_n=t$.

2. (a) Note that the joint density of the random sample is

$$\begin{aligned}f(\mathbf{x}|\theta) &= \prod_{i=1}^n \frac{1}{\theta}\exp\left\{-\frac{x_i-\theta}{\theta}\right\}\mathbb{1}\{x_i>\theta\} \\ &= \frac{1}{\theta^n}\exp\left\{-\frac{\sum_{i=1}^n x_i - n\theta}{\theta}\right\}\mathbb{1}\{x_{(1)}>\theta\}.\end{aligned}$$

Then if we let $\mathbf{y}$ denote another random sample from $f$, we have

$$\begin{aligned}\frac{f(\mathbf{x}|\theta)}{f(\mathbf{y}|\theta)} \text{ is free of } \theta &\iff \exp\left\{-\frac{\sum_{i=1}^n x_i - \sum_{i=1}^n y_i}{\theta}\right\}\frac{\mathbb{1}\{x_{(1)}>\theta\}}{\mathbb{1}\{y_{(1)}>\theta\}} \text{ is free of } \theta \\ &\iff \sum_{i=1}^n x_i = \sum_{i=1}^n y_i \text{ and } x_{(1)} = y_{(1)}.\end{aligned}$$

Hence, from the checking rule, $(\sum_{i=1}^n X_i, X_{(1)})$ is minimal sufficient for $\theta$.

(b) It can be observed that for each $i$, $X_i - \theta$ follows an exponential distribution with rate $\theta^{-1}$ (i.e. $\mathbb{E}[X_i - \theta] = \theta$), so it follows that $\mathbb{E}[\sum_{i=1}^n X_i] = 2n\theta$. On the other hand,

$$\begin{aligned}\mathbb{P}(X_{(1)} - \theta \leq t) &= 1 - \mathbb{P}(X_{(1)} > t + \theta) \\ &= 1 - \mathbb{P}(X_i > t + \theta \text{ for each } i) \\ &= 1 - \mathbb{P}(X_1 - \theta > t)^n \\ &= 1 - \exp\left\{-\frac{n}{\theta}t\right\}\end{aligned}$$

for $t > 0$ and $\mathbb{P}(X_{(1)} - \theta \leq t) = 0$ otherwise, indicating that $X_{(1)} - \theta$ is exponentially distributed with rate $\frac{n}{\theta}$, so it follows that $\mathbb{E}[X_{(1)}] = \mathbb{E}[X_{(1)} - \theta] + \theta = \frac{\theta}{n} + \theta = \frac{n+1}{n}\theta$.

Thus, by denoting $(S, T) := (\sum_{i=1}^n X_i, X_{(1)})$ and letting $g(S, T) := \frac{S}{2n} - \frac{nT}{n+1}$, we have $\mathbb{E}[g(S, T)] = \theta - \theta = 0$ for all $\theta$ but $\mathbb{P}(g(S, T) = 0) \neq 1$. Hence, $(S, T)$ is not complete.

3. (a) Note that the joint density of the random sample is

$$\begin{aligned}f((\mathbf{x}, \mathbf{y})|\theta) &= \prod_{i=1}^n \frac{1}{2\pi\sqrt{1-\theta^2}} \exp\left\{-\frac{x_i^2 - 2\theta x_i y_i + y_i^2}{2(1-\theta^2)}\right\} \\ &= (2\pi\sqrt{1-\theta^2})^{-n} \exp\left\{-\frac{\sum_{i=1}^n x_i^2 + \sum_{i=1}^n y_i^2 - 2\theta\sum_{i=1}^n x_i y_i}{2(1-\theta^2)}\right\}.\end{aligned}$$

Then if we let $(\mathbf{x}', \mathbf{y}')$ denote another random sample from $f$, we have

$$\begin{aligned}&\frac{f(\mathbf{x}|\theta)}{f(\mathbf{y}|\theta)} \text{ is free of } \theta \\ \iff &\exp\left\{-\frac{\sum_{i=1}^n (x_i^2 + y_i^2 - x_i'^2 - y_i'^2) - 2\theta\sum_{i=1}^n (x_i y_i - x_i' y_i')}{2(1-\theta^2)}\right\} \text{ is free of } \theta \\ \iff &\sum_{i=1}^n (x_i^2 + y_i^2) = \sum_{i=1}^n (x_i'^2 + y_i'^2) \text{ and } \sum_{i=1}^n x_i y_i = \sum_{i=1}^n x_i' y_i'.\end{aligned}$$

Hence $(\sum_{i=1}^n (X_i^2 + Y_i^2), \sum_{i=1}^n X_i Y_i)$ is minimal sufficient for $\theta$ from checking rule.

(b) Denote $(S, T) := (\sum_{i=1}^n (X_i^2 + Y_i^2), \sum_{i=1}^n X_i Y_i)$ and consider the function $g(S, T) := S - 2n$. Note that

$$\begin{aligned}\mathbb{E}[g(S, T)] &= \mathbb{E}\left[\sum_{i=1}^n (X_i^2 + Y_i^2)\right] - 2n \\ &= \sum_{i=1}^n (\mathbb{E}[X_i^2] + \mathbb{E}[Y_i^2]) - 2n \\ &= \sum_{i=1}^n (\mathrm{Var}(X_i) + \mathbb{E}[X_i]^2 + \mathrm{Var}(Y_i) + \mathbb{E}[Y_i]^2) - 2n \\ &= \sum_{i=1}^n (1 + 0 + 1 + 0) - 2n = 0\end{aligned}$$

for all $\theta$, but clearly $\mathbb{P}(g(S, T) = 0) \neq 1$. Therefore, $(S, T)$ is not complete.

(c) The marginal distribution of each $X_i$ is standard normal, which is free of $\theta$. Thus, $T_1 \sim \chi_n^2$ is free of $\theta$. Similarly, $T_2 \sim \chi_n^2$ is free of $\theta$ as well, so each $T_i$ is ancillary.

To show that $(T_1, T_2)$ depends on $\theta$, consider

$$\begin{aligned}\mathrm{Cov}(T_1, T_2) &= \mathrm{Cov}\left(\sum_{i=1}^{n} X_i^2, \sum_{j=1}^{n} Y_j^2\right) \\ &= \sum_{i=1}^{n} \mathrm{Cov}(X_i^2, Y_i^2) + \sum_{i\neq j} \mathrm{Cov}(X_i^2, Y_j^2) \\ &= n\mathrm{Cov}(X_1^2, Y_1^2).\end{aligned}$$

To proceed, we define the random vector

$$\begin{pmatrix} X_1 \\ Z_1 \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ \frac{-\theta}{\sqrt{1-\theta^2}} & \frac{1}{\sqrt{1-\theta^2}} \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix},$$

where we introduced $Z_1 = \dfrac{Y_1 - \theta X_1}{\sqrt{1-\theta^2}}$. By matrix calculation, it can obtained that $(X_1, Z_1) \sim \mathbf{N}_2(\mathbf{0}, \mathbf{I}_2)$, i.e. $Z_1$ is standard normal and is independent with $X_1$. Then

$$\begin{aligned}\mathrm{Cov}(T_1, T_2) &= n\mathrm{Cov}(X_1^2, (\theta X_1 + \sqrt{1-\theta^2} Z_1)^2) \\ &= n\theta^2 \mathrm{Var}(X_1^2) + 2n\theta\sqrt{1-\theta^2}\mathrm{Cov}(X_1^2, X_1 Z_1) + n(1-\theta^2)\mathrm{Cov}(X_1^2, Z_1^2) \\ &= n\theta^2(\mathbb{E}[X_1^4] - \mathbb{E}[X_1^2]^2) + 2n\theta\sqrt{1-\theta^2}(\mathbb{E}[X_1^3 Z_1] - \mathbb{E}[X_1^2]\mathbb{E}[X_1 Z_1]) + 0 \\ &= n\theta^2(3-1) + 2n\theta\sqrt{1-\theta^2}(0-0) = 2n\theta^2.\end{aligned}$$

Here we further used the fact that $\mathbb{E}[X_1^4] = 3$, which can be verified by differentiating the mgf of $X_1$. Since a function of the distribution of $(T_1, T_2)$ depends on $\theta$, the joint distribution itself cannot be free of $\theta$.

4. Note that the joint density of the random sample is

$$\begin{aligned}f(\mathbf{x}|\theta) &= \prod_{i=1}^{n} \frac{1}{\theta}\mathbb{1}\{\theta < x_i < 2\theta\} \\ &= \frac{1}{\theta^n}\mathbb{1}\{x_{(1)} > \theta\}\mathbb{1}\{x_{(n)} < 2\theta\}.\end{aligned}$$

Then if we let $\mathbf{y}$ denote another random sample from $f$, we have

$$\begin{aligned}\frac{f(\mathbf{x}|\theta)}{f(\mathbf{y}|\theta)} \text{ is free of } \theta &\iff \frac{\mathbb{1}\{x_{(1)} > \theta\}\mathbb{1}\{x_{(n)} < 2\theta\}}{\mathbb{1}\{y_{(1)} > \theta\}\mathbb{1}\{y_{(n)} < 2\theta\}} \text{ is free of } \theta \\ &\iff x_{(1)} = y_{(1)} \text{ and } x_{(n)} = y_{(n)}.\end{aligned}$$

Hence, from the checking rule, $(X_{(1)}, X_{(n)})$ is minimal sufficient for $\theta$.

To check completeness, define $U_i = \frac{X_i}{\theta} - 1$ for each $i$, and it follows that each $U_i \overset{\text{iid}}{\sim} \mathrm{U}[0,1]$. Since $\theta > 0$, the order in $X_1, \cdots, X_n$ is preserved in $U_1, \cdots, U_n$, i.e. $U_{(1)} = \frac{X_{(1)}}{\theta} - 1$ and $U_{(n)} = \frac{X_{(n)}}{\theta} - 1$. Moreover, we know from Lecture 2 that $U_{(1)} \sim \mathrm{Beta}(1, n)$ and $U_{(n)} \sim \mathrm{Beta}(n, 1)$, whose expectations are, respectively, $\frac{1}{n+1}$ and $\frac{n}{n+1}$. Therefore we have

$$\begin{aligned}\mathbb{E}[X_{(1)}] &= \theta(1 + \mathbb{E}[U_{(1)}]) = \frac{n+2}{n+1}\theta, \\ \mathbb{E}[X_{(n)}] &= \theta(1 + \mathbb{E}[U_{(n)}]) = \frac{2n+1}{n+1}\theta.\end{aligned}$$

Then the non-trivial function $g(X_{(1)}, X_{(n)}) := \frac{n+1}{2n+1} X_{(n)} - \frac{n+1}{n+2} X_{(1)}$ will satisfy $\mathbb{E}[g(X_{(1)}, X_{(n)})] = \theta - \theta = 0$, indicating that $(X_{(1)}, X_{(n)})$ is not complete.

*Remark.* A more elegant way to disprove completeness is to state that $X_{(n)}/X_{(1)}$ is ancillary for $\theta$, so $\mathbb{E}[X_{(n)}/X_{(1)}] =: c$ is free of $\theta$, and hence $g(X_{(1)}, X_{(n)}) := X_{(n)}/X_{(1)} - c$ satisfies $\mathbb{E}[g(X_{(1)}, X_{(n)})] = 0$.

5. We have $X_i \overset{\text{iid}}{\sim} \mathrm{N}(\theta, \sigma_\theta^2)$, where $\sigma_\theta^2 := 1 + \mathbb{1}\{\theta = 0\}$ is the variance of $X_1$ which depends on $\theta$. Then it follows that $\bar{X} \sim \mathrm{N}\left(\theta, \frac{\sigma_\theta^2}{n}\right)$. Now condition on $\bar{X} = t$, consider the conditional density $f_{\mathbf{X}|\bar{X}}(\mathbf{x}|t)$ for $\mathbf{x} \in \mathbb{R}^n$.

   - Clearly, $f_{\mathbf{X}|\bar{X}}(\mathbf{x}|t) = 0$ on $\{\mathbf{x} : \sum_{i=1}^n x_i \neq nt\}$.
   - For $\sum_{i=1}^n x_i = nt$, we have $\{\mathbf{X} = \mathbf{x}, \bar{X} = t\} = \{\mathbf{X} = \mathbf{x}\}$ and hence

$$\begin{aligned} f_{\mathbf{X}|\bar{X}}(\mathbf{x}|t) &= \frac{f_{\mathbf{X}}(\mathbf{x})}{f_{\bar{X}}(t)} \\ &= \frac{\sqrt{2\pi\sigma_\theta^2/n}}{(\sqrt{2\pi\sigma_\theta^2})^n} \frac{\exp\left\{-\frac{\sum_i (x_i-\theta)^2}{2\sigma_\theta^2}\right\}}{\exp\left\{-\frac{n(t-\theta)^2}{2\sigma_\theta^2}\right\}} \\ &= \frac{1}{(\sqrt{2\pi\sigma_\theta^2})^{n-1}\sqrt{n}} \exp\left\{-\frac{\sum_i (x_i - t)^2}{2\sigma_\theta^2}\right\} \end{aligned}$$

     which is not free of $\theta$ due to the existence of $\sigma_\theta^2$, e.g. plugging in $\theta = 0$ and $\theta = 1$ will produce non-identical conditional densities.

   Hence, by definition, $\bar{X}$ is not sufficient for $\theta$.

   To show completeness, let $g(x)$ be a function satisfying $\mathbb{E}_\theta[g(\bar{X})] = 0$ for all $\theta \in \mathbb{R}$. Then by splitting $g(x) =: g^+(x) - g^-(x)$, where $g^+(x) = \max\{0, g(x)\} \geq 0$ and $g^-(x) = \max\{0, -g(x)\} \geq 0$, we have

$$\int_{-\infty}^{\infty} g^+(x) \exp\left\{-\frac{n(x-\theta)^2}{2\sigma_\theta^2}\right\} \mathrm{d}x = \int_{-\infty}^{\infty} g^-(x) \exp\left\{-\frac{n(x-\theta)^2}{2\sigma_\theta^2}\right\} \mathrm{d}x \tag{*}$$

   for all $\theta$ near 1. In particular, when $\theta = 1$, we denote

$$\int_{-\infty}^{\infty} g^+(x) \exp\left\{-\frac{n(x-1)^2}{2}\right\} \mathrm{d}x = \int_{-\infty}^{\infty} g^-(x) \exp\left\{-\frac{n(x-1)^2}{2}\right\} \mathrm{d}x =: C.$$

   By the non-negativity of both integrands, $C$ must be non-negative. If $C = 0$, then the integrands should be zero a.e., that is,

$$g^+(x) \exp\left\{-\frac{n(x-1)^2}{2}\right\} = g^-(x) \exp\left\{-\frac{n(x-1)^2}{2}\right\} = 0$$

   a.e. or simply $g(\bar{X}) = 0$ a.s. If $C > 0$, then we can treat $\frac{1}{C} g^+(x) \exp\left\{-\frac{n(x-1)^2}{2}\right\}$ and $\frac{1}{C} g^-(x) \exp\left\{-\frac{n(x-1)^2}{2}\right\}$ as densities of some real-valued random variables, say $W_1, W_2$ respectively (since the densities are non-negative and integrate to 1). Now notice that the corresponding moment generating functions of $W_1$ and $W_2$ evaluated at $t = n(\theta - 1)$ satisfy

$$\begin{aligned}
\phi_{W_1}(t) &= \int_{-\infty}^{\infty} \frac{1}{C} g^+(x) \exp\left\{-\frac{n(x-1)^2}{2} + nx(\theta-1)\right\} \mathrm{d}x \\
&= \frac{1}{C} \exp\left\{\frac{n(\theta^2-1)}{2}\right\} \int_{-\infty}^{\infty} g^+(x) \exp\left\{-\frac{n(x-\theta)^2}{2}\right\} \mathrm{d}x \\
&\overset{(*)}{=} \frac{1}{C} \exp\left\{\frac{n(\theta^2-1)}{2}\right\} \int_{-\infty}^{\infty} g^-(x) \exp\left\{-\frac{n(x-\theta)^2}{2}\right\} \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{C} g^-(x) \exp\left\{-\frac{n(x-1)^2}{2} + nx(\theta-1)\right\} \mathrm{d}x = \phi_{W_2}(t)
\end{aligned}$$

for all $\theta$ near 1, i.e. $t = n(\theta - 1)$ near 0. Hence, from the one-to-one correspondence of moment generating function, we conclude that $W_1$ and $W_2$ has the same density a.e., which leads to $g^+(x) = g^-(x)$ or $g(\bar{X}) = 0$ a.s. for any $\theta$, concluding the proof of completeness.

*Remark.* A (probably) less technical approach to show completeness is outlined as follows: Denote $P_\theta := \mathrm{N}(\theta, \sigma_\theta^2/n)$ and $P'_\theta := \mathrm{N}(\theta, 1/n)$ for $\theta \in \mathbb{R}$, so that they only differ at $\theta = 0$. Then sequentially show that

- $\underline{\mathbb{E}_{P_\theta}[g(\bar{X})] = 0 \text{ for all } \theta \implies \mathbb{E}_{P'_\theta}[g(\bar{X})] = 0 \text{ for all } \theta}$ (letting $\theta \to 0$ and use Dominated Convergence),
- $\underline{\mathbb{E}_{P'_\theta}[g(\bar{X})] = 0 \text{ for all } \theta \implies \mathbb{P}_{P'_\theta}(g(\bar{X}) = 0) = 1 \text{ for all } \theta}$ (full-rank exponential family),
- $\underline{\mathbb{P}_{P'_\theta}(g(\bar{X}) = 0) = 1 \text{ for all } \theta \implies \mathbb{P}_{P_\theta}(g(\bar{X}) = 0) = 1 \text{ for all } \theta}$ (preservation of almost sure event).

6. The joint density of the random sample is

$$\begin{aligned}
f(\mathbf{x}|\sigma^2) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\} \\
&= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2\right\} \\
&=: h(\mathbf{x}) c(\sigma^2) \exp\left\{w(\sigma^2) t(\mathbf{x})\right\},
\end{aligned}$$

where $w(\sigma^2) = -\frac{1}{2\sigma^2}$, $t(\mathbf{x}) = \sum_{i=1}^{n} x_i^2$, $c(\sigma^2) = (\sigma^2)^{-\frac{n}{2}}$ and $h(\mathbf{x}) = (2\pi)^{-\frac{n}{2}}$, implying by definition that the joint distribution of $(X_1, \cdots, X_n)$ belongs to a 1-parameter exponential family. Then, the corresponding canonical form can be written as

$$f(\mathbf{x}|\eta) = h(\mathbf{x}) c^*(\eta) \exp\{\eta\, t(\mathbf{x})\}$$

for $\eta = w(\sigma^2) = -\frac{1}{2\sigma^2}$ and some $c^*(\eta)$. Since the natural parameter space

$$\mathcal{H} = \{w(\sigma^2) : \sigma^2 \in \Theta\} = \left\{-\frac{1}{2\sigma^2} : \sigma^2 \in \mathbb{R}^+\right\} = \mathbb{R}^-$$

contains an open interval, such family is of full rank. Therefore, $t(\mathbf{X}) = \sum_{i=1}^{n} X_i^2$ is complete and sufficient for $\sigma^2$.

On the other hand, we define $Z_i := \frac{X_i}{\sigma} \sim \mathrm{N}(0, 1)$ for each $i$, whose distribution is free of $\sigma^2$. This implies

$$S = \frac{\left(\sum_{i=1}^{n} X_i\right)^2}{\sum_{i=1}^{n} X_i^2} = \frac{\left(\sum_{i=1}^{n} Z_i\right)^2}{\sum_{i=1}^{n} Z_i^2}$$

being free of $\sigma^2$, or $S$ is ancillary for $\sigma^2$. By Basu's Theorem, $S$ and $\sum_{i=1}^n X_i^2$ are independent, and hence we have

$$\begin{aligned} n\sigma^2 = \operatorname{Var}\left(\sum_{i=1}^n X_i\right) &= \mathbb{E}\left[\left(\sum_{i=1}^n X_i\right)^2\right] \\ &= \mathbb{E}\left[S\sum_{i=1}^n X_i^2\right] = \mathbb{E}[S]\,\mathbb{E}\left[\sum_{i=1}^n X_i^2\right] \\ &= \mathbb{E}[S]\sum_{i=1}^n \mathbb{E}[X_i^2] = \mathbb{E}[S]n\sigma^2 \end{aligned}$$

which gives $\mathbb{E}[S] = 1$.