IEDA5270 HW1

1

1. Let X be a continuous random variable with pdf f_X. Let $Y = g(X)$, where g is a strictly increasing function. Use the pdf of Y derived in class to show that

$$E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

For any $A \subset \mathbb{R}$,

$\mathbb{P}(Y \in A) = \mathbb{P}(g(X) \in A)$.

Define the inverse

$g^{-1}(A) = \{x \in \mathbb{R} : g(x) \in A\}$,

$\mathbb{P}(Y \in \ A) = \mathbb{P}(X \in \ g^{-1}(A)) = \int_{g^{-1}(A)} f_X(x) dx.$

Because g is a strictly increasing function, so

$$\mathbb{P}(Y \leq \ y) = \int_{-\infty}^{g^{-1}(y)} f_X(x) dx.$$

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) \\ &= \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))} \end{aligned}$$

Because $x = g^{-1}(y)$, so $dx = \dfrac{1}{g'(g^{-1}(y))} dy$

$$\begin{aligned} E[Y] &= \int_{-\infty}^{\infty} y f_Y(y) dy \\ &= \int_{-\infty}^{\infty} g(x) \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))} dy \\ &= \int_{-\infty}^{\infty} g(x) f_X(x) dx \end{aligned}$$

2

2. X is said to be a *standard normal random variable* if its pdf is

$$f_X(x) = (1/\sqrt{2\pi})e^{-x^2/2}.$$

Find the pdf of $Y = X^2$, and find its mean and variance.

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&= 2\int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)dx = \sqrt{\frac{2}{\pi}}\int_0^{\sqrt{y}} e^{-x^2/2}dx
\end{aligned}$$

$$\begin{aligned}
f_Y(y) &= \frac{d}{dy}F_Y(y) \\
&= \frac{F_X(y)}{dx}\frac{d\sqrt{y}}{dy} \\
&= \sqrt{\frac{2}{\pi}}e^{-y/2}\frac{1}{2\sqrt{y}} \\
&= \frac{e^{-y/2}}{\sqrt{2\pi y}}
\end{aligned}$$

$$\begin{aligned}
E[Y] &= \int_0^\infty y f_Y(y)dy \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty y^{-1/2}e^{-y/2}dy \\
&= 1
\end{aligned}$$

$$\begin{aligned}
Var(Y) &= E[(Y - E[Y])^2] = E[Y^2] - E[Y]^2 \\
&= \int_0^\infty y^2 f_Y(y)dy - 1 \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty ye^{-y/2}dy - 1 \\
&= \frac{4}{\sqrt{2\pi}}(-y/2 - 1)e^{-y/2}\Big|_0^\infty - 1 \\
&= 2\sqrt{\frac{2}{\pi}} - 1
\end{aligned}$$

3

(a)

Let X be a discrete random variable whose range is the nonnegative integers. Show that

$$\mathsf{E}X = \sum_{k=0}^{\infty}(1 - F_X(k)),$$

where $F_X(k) = P(X \leq k)$.

$$1 - F_X(k) = P(X > k) = \sum_{i=k+1}^{\infty} p(i)$$

$$\sum_{k=0}^{\infty}(1 - F_X(k)) = \sum_{k=0}^{\infty}\sum_{i=k+1}^{\infty} p(i) = \sum_{i=0}^{\infty} ip(i) = E[X]$$

(b)

Let X be a continuous, nonnegative random variable, i.e., $f(x) = 0$ *for* $x < 0$. Show that

$$\mathsf{E}X = \int_0^{\infty}(1 - F_X(x))dx,$$

where $F_X(x)$ is the cdf of X. Compare this with part (a).

$$1 - F_X(x) = P(X > x) = \int_x^{\infty} f_X(y)dy$$

$$\begin{aligned}
\int_0^{\infty}(1 - F_X(x))dx &= \int_0^{\infty}\int_x^{\infty} f_X(y)dydx \\
&= \int_0^{\infty}\int_0^{y} f_X(y)dxdy \\
&= \int_0^{\infty} xf_X(y)\Big|_0^y dy \\
&= \int_0^{\infty} yf_X(y)dy \\
&= E[X]
\end{aligned}$$

(c)

Let X be a continuous random variable. Use part (b) to show that

$$\mathsf{E}X = \int_0^{\infty}(1 - F_X(x))dx - \int_{-\infty}^{0} F_X(x)dx,$$

where $F_X(x)$ is the cdf of X.

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(y)dy$$

$$
\begin{aligned}
\int_{-\infty}^{0} F_X(x)dx &= \int_{-\infty}^{0}\int_{-\infty}^{x} f_X(y)dydx \\
&= \int_{-\infty}^{0}\int_{y}^{0} f_X(y)dxdy \\
&= \int_{0}^{\infty} xf_X(y)\Big|_{y}^{0} dy \\
&= -\int_{0}^{\infty} yf_X(y)dy
\end{aligned}
$$

According to (b), we have $\int_0^\infty (1 - F_X(x))dx = \int_0^\infty yf_X(y)dy$, so

$$
\int_0^\infty (1 - F_X(x))dx - \int_{-\infty}^{0} F_X(x)dx = \int_{-\infty}^{\infty} yf_X(y)dy = E[X]
$$

(d)

Let X and Y be two nonnegative random variables. Show that

$$
\mathsf{E}[XY] = \int_0^\infty \int_0^\infty \mathsf{P}(X > x, Y > y)dxdy.
$$

$$
\begin{aligned}
E[XY] &= \int_0^\infty \int_0^\infty xyf(x,y)dxdy \\
&= \int_0^\infty \int_0^\infty xyf_X(x)f_Y(y)dxdy \\
&= \int_0^\infty xf_X(x)dx \int_0^\infty yf_Y(y)dy \\
&= \int_0^\infty (1 - F_X(x))dx \int_0^\infty (1 - F_Y(y))dy \\
&= \int_0^\infty P(X > x)dx \int_0^\infty P(Y > y)dy \\
&= \int_0^\infty \int_0^\infty P(X > x, Y > y)dxdy
\end{aligned}
$$

4

Let (X, Y, Z) be a random vector with the following density:

$$
f(x, y, z) = \frac{1 - \sin x \sin y \sin z}{8\pi^3}, \quad 0 \le x, y, z \le 2\pi.
$$

Show that (X, Y, Z) are pairwise independent but not independent.

$$
\begin{aligned}
P(x=i,y=j) &= \int_0^{2\pi} f(i,j,z)dz \\
&= \int_0^{2\pi} \frac{1-\sin i \sin j \sin z}{8\pi^3} dz \\
&= \frac{1}{8\pi^3}(z+\sin i \sin j \cos z)\Big|_0^{2\pi} \\
&= \frac{1}{4\pi^2}
\end{aligned}
$$

Similarly, we can derive: $P(x=i,z=k)=P(y=j,z=k)=\frac{1}{4\pi^2}$

$$
\begin{aligned}
P(x=i) &= \int_0^{2\pi}\int_0^{2\pi} f(i,y,z)dydz \\
&= \int_0^{2\pi}\int_0^{2\pi} \frac{1-\sin i \sin y \sin z}{8\pi^3} dydz \\
&= \frac{1}{8\pi^3}\int_0^{2\pi} (y+\sin i \sin z \cos y)\Big|_0^{2\pi} dz \\
&= \frac{1}{8\pi^3}\int_0^{2\pi} 2\pi dz \\
&= \frac{1}{2\pi}
\end{aligned}
$$

Similarly, we can derive: $P(y=j)=P(z=k)=\frac{1}{2\pi}$

$P(x=i,y=j)=P(x=i)\cdot P(y=j)$

$P(x=i,z=k)=P(x=i)\cdot P(z=k)$

$P(y=j,z=k)=P(y=j)\cdot P(z=k)$

$P(x=i,y=j,z=k)=f(x,y,z)\neq P(x=i)\cdot P(y=j)\cdot P(z=k)$

So (X,Y,Z) are pairwise independent but not independent.

5

X and Y are independent exponential random variables with rate λ and μ. Define

$$
Z=\min\{X,Y\},\quad W=\begin{cases}1 & Z=X\\ 0 & Z=Y\end{cases}.
$$

(a) Find the joint distribution of Z and W.

(b) Find the marginal distribution of Z and W, respectively.

(c) Show that Z and W are independent.

Pdf of exponential distribution $f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0, \\ 0 & x < 0. \end{cases}$

$X \sim \ f(x;\lambda),\ Y \sim \ f(x;\mu)$

(a)

$$\begin{aligned} P(Z > z, W = 1) &= P(Y > X > z) \\ &= \int_z^\infty \mu e^{-\mu y} \int_z^y \lambda e^{-\lambda x} dx dy \\ &= \int_z^\infty \mu e^{-\mu y} (e^{-\lambda\, z} - e^{-\lambda\, y}) dy \\ &= \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)z} \end{aligned}$$

$$P(Z > z, W = 0) = P(X > Y > z) = \frac{\mu}{\lambda + \mu} e^{-(\lambda+\mu)z}$$

(b)

$$P(Z > z) = P(Z > z, W = 0) + P(Z > z, W = 1) = e^{-(\lambda+\mu)z}$$

$$P(W = 0) = P(Z > 0, W = 0) = \frac{\mu}{\lambda + \mu}$$

$$P(W = 1) = P(Z > 0, W = 1) = \frac{\lambda}{\lambda + \mu}$$

(c)

$$P(Z > z, W = 0) = P(Z > z) \cdot P(W = 0)$$

$$P(Z > z, W = 1) = P(Z > z) \cdot P(W = 1)$$

Z and W are independent

6

A random varialbe X has a beta distribution if $f_X(x) = C(\alpha,\beta)x^{\alpha-1}(1-x)^{\beta-1}$, for $x \in [0,1]$. C is a constant factor so that $f_X(x)$ integrates to one. We know that $\mathsf{E}[X] = \frac{\alpha}{\alpha+\beta}$ and $\operatorname{Var} X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

Now consider the following hierarchy model. Let $(X_1,P_1),\ldots,(X_n,P_n)$ be independent random vectors with

$$X_i \big| P_i \sim \text{Bernoulli}(P_i)$$
$$P_i \sim \text{beta}(\alpha,\beta)$$

This model might be appropriate, for example, if we are measuring the success of a drug on n patients and because the patients are different, their success rates are not constant. A random variable of interest is $Y = \sum_{i=1}^n X_i$, the total number of successes.

(a) Show that $\mathsf{E}[Y] = n\alpha/(\alpha+\beta)$.

(b) Show that $\operatorname{Var}(Y) = n\alpha\beta/(\alpha+\beta)^2$, and hence Y has the same mean and variance as a binomial$(n, \alpha/(\alpha+\beta))$. What is the distribution of Y?

(c) Suppose now that the model is

$$X_i \big| P_i \sim \text{binomial}(n_i, P_i), \quad i = 1,\ldots,k$$
$$P_i \sim \text{beta}(\alpha,\beta)$$

For $Y = \sum_{i=1}^k X_i$, find $\mathsf{E}[Y]$ and $\operatorname{Var}(Y)$.

(a)

$$E[X_i] = E(E(X_i|P_i)) = E(P_i) = \alpha/(\alpha+\beta)$$

$$E[Y] = E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i] = n\alpha/(\alpha+\beta)$$

(b)

$$\begin{aligned}
Var(X_i) &= E(Var(X_i|P_i)) + Var(E(X_i|P_i)) \\
&= E(P_i - P_i^2) + Var(P_i) \\
&= E(P_i - P_i^2) + E(P_i^2) - (E(P_i))^2 \\
&= E(P_i)(1 - E(P_i)) \\
&= \alpha\beta/(\alpha+\beta)^2
\end{aligned}$$

$$Var(Y) = Var(\sum_{i=1}^n X_i) = \sum_{i=1}^n Var(X_i) = n\alpha\beta/(\alpha+\beta)^2$$

$$P(Y=k) = \binom{n}{k}\frac{\alpha^{[k]}\beta^{[n-k]}}{(\alpha+\beta)^{[n]}}$$

(c)

$$E(X_i|P_i) = n_i P_i,\ Var(X_i|P_i) = n_i P_i(1-P_i)$$

$$E[X_i] = E(E(X_i|P_i)) = E(n_iP_i) = n_i\alpha/(\alpha+\beta)$$

$$E[Y] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{k} n_i\alpha/(\alpha+\beta)$$

$$\begin{aligned}
Var(X_i) &= E(Var(X_i|P_i)) + Var(E(X_i|P_i)) \\
&= E(n_iP_i(1-P_i)) + Var(n_iP_i) \\
&= n_i(E(P_i) - E(P_i^2)) + n_i^2 Var(P_i) \\
&= n_i(E(P_i) - E(P_i)^2 - Var(P_i) + n_i Var(P_i)) \\
&= n_i \frac{\alpha\beta(\alpha+\beta+n_i)}{(\alpha+\beta)^2(\alpha+\beta+1)}
\end{aligned}$$

$$Var(Y) = \sum_{i=1}^{k} Var(X_i) = \sum_{i=1}^{k} n_i \frac{\alpha\beta(\alpha+\beta+n_i)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$