1. It can be observed that each $Y_i$ is a Bernoulli random variable with probability of success $p = \mathbb{P}(X_i < 0) = \mathbb{P}(X_i - \mu < -\mu) = \Phi(-\mu)$, i.e. $\mu = -\Phi^{-1}(p)$. Then due to the invariance property of MLE, it remains to find an MLE for $p$. Consider the log-likelihood

$$\begin{aligned} l(p|\mathbf{y}) &= \log \prod_{i=1}^{n} p^{y_i}(1-p)^{1-y_i} \\ &= \sum_{i=1}^{n} y_i \log(p) + \left(n - \sum_{i=1}^{n} y_i\right) \log(1-p) \end{aligned}$$

whose partial derivatives w.r.t. $p$ are

$$l'(p) = \frac{\sum_{i=1}^{n} y_i}{p} - \frac{n - \sum_{i=1}^{n} y_i}{1-p} \quad \text{and} \quad l''(p) = -\frac{\sum_{i=1}^{n} y_i}{p^2} - \frac{n - \sum_{i=1}^{n} y_i}{(1-p)^2}.$$

Since $l''(p) < 0$, we can solve the first-order condition and conclude that an MLE for $p$ is $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, and hence an MLE for $\mu$ is $\hat{\mu} = -\Phi^{-1}(\hat{p}) = -\Phi^{-1}(\frac{1}{n}\sum_{i=1}^{n} Y_i)$.

2. (a) Let $h(X_1)$ be an unbiased estimator of $\mu$, i.e. we have $\mathbb{E}[h(X_1) - X_1] = 0$ for all symmetric distributions.

   In particular, it implies $\mathbb{E}[h(X_1) - X_1] = 0$ for all distributions from the family $\{\mathrm{N}(\mu, 1) : \mu \in \mathbb{R}\}$. Since such family is a 1-parameter exponential family of full rank, we have $X_1$ itself being complete and sufficient for $\mu$.

   From completeness, we have $\mathbb{P}(h(X_1) - X_1 = 0) = 1$ for all distributions in $\{\mathrm{N}(\mu, 1) : \mu \in \mathbb{R}\}$. Since $X_1$ is absolutely continuous, we must have $h(x) = x$ a.e., but this indicates that for all symmetric distributions, the only unbiased estimator of $\mu$ will be $h(X_1) = X_1$ itself (up to a null set).

   Since another unbiased estimator for $\mu$ (that has a smaller variance) does not exist, we conclude that $X_1$ is the UMVUE of $\mu$.

   (b) From Factorization theorem, it can be easily obtained that $(X_{(1)}, X_{(n)})$ is sufficient for $(\theta_1, \theta_2)$. To check completeness, note that the joint density of $(X_{(1)}, X_{(n)})$ is

$$\begin{aligned} f_{X_{(1)},X_{(n)}}(u, v) &= \frac{n!}{(n-2)!} \frac{1}{(2\theta_2)^2} \left(\frac{u - \theta_1 + \theta_2}{2\theta_2}\right)^{1-1} \left(\frac{v-u}{2\theta_2}\right)^{n-1-1} \left(1 - \frac{v - \theta_1 + \theta_2}{2\theta_2}\right)^{n-n} \\ &= \frac{n!}{(n-2)!(2\theta_2)^n}(v-u)^{n-2}, \quad \theta_1 - \theta_2 < u < v < \theta_1 + \theta_2. \end{aligned}$$

   Let $g(X_{(1)}, X_{(n)})$ be a function with $\mathbb{E}[g(X_{(1)}, X_{(n)})] = 0$ for all $\theta_1 \in \mathbb{R}, \theta_2 \in \mathbb{R}^+$, so for all $a < b$ up to a null set,

$$\begin{aligned} 0 &= \int_a^b \int_u^b g(u, v)(v-u)^{n-2}\,\mathrm{d}v\,\mathrm{d}u \\ &= -\int_a^b g(a, v)(v-a)^{n-2}\,\mathrm{d}v && \text{(differentiate w.r.t. } a) \\ &= -g(a, b)(b-a)^{n-2} && \text{(differentiate w.r.t. } b) \\ &= g(a, b). \end{aligned}$$

   Thus $g(u, v)$ must be zero a.e. on the support of $(X_{(1)}, X_{(n)})$, i.e. $(X_{(1)}, X_{(n)})$ is complete and sufficient for $(\theta_1, \theta_2)$. Moreover, from Lecture 2 we also know that

$$\frac{X_{(1)} - \theta_1 + \theta_2}{2\theta_2} \sim \mathrm{Beta}(1, n) \quad \text{and} \quad \frac{X_{(n)} - \theta_1 + \theta_2}{2\theta_2} \sim \mathrm{Beta}(n, 1),$$

whose expectations are $\frac{1}{n+1}$ and $\frac{n}{n+1}$ respectively. Hence,

$$\mathbb{E}\left[\frac{1}{2}(X_{(1)} + X_{(n)})\right] = \frac{1}{2}\left(\frac{2\theta_2}{n+1} + \theta_1 - \theta_2 + \frac{2n\theta_2}{n+1} + \theta_1 - \theta_2\right) = \theta_1,$$

By Lehmann-Scheffe's Theorem, the UMVUE for $\theta_1$ is $\frac{1}{2}(X_{(1)} + X_{(n)})$.

(c) Assume the contrary, i.e. $T = T(\mathbf{X})$ is the unique UMVUE of $\mu$ for all symmetric distributions. Then the following should hold at once:

1. $T$ is the UMVUE of $\mu$ for all distributions in $\{\mathrm{N}(\mu, 1) : \mu \in \mathbb{R}\}$. Clearly in this case $T = \bar{X}$.

2. $T$ is the UMVUE of $\mu$ for all distributions in $\{\mathrm{U}[\mu - \theta, \mu + \theta] : \theta > 0\}$. From (b) we know that we should have $T = \frac{1}{2}(X_{(1)} + X_{(n)})$.

By the uniqueness of $T$, the statements above would imply $\bar{X} = \frac{1}{2}(X_{(1)} + X_{(n)})$, but apparently this is false in general !!!

Hence, such $T$ does not exist.

3. (a) By chain rule, we have

$$\begin{aligned}\mathcal{I}_\eta(\eta) &= \mathrm{Var}\left(\frac{\partial l(\eta|X_1)}{\partial \eta}\right) = \mathrm{Var}\left(\frac{\partial l(\theta|X_1)}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right) \\ &= \left(\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right)^2 \mathrm{Var}\left(\frac{\partial l(\theta|X_1)}{\partial \theta}\right) = \frac{\mathcal{I}_\theta(\theta)}{(h'(\theta))^2}.\end{aligned}$$

(b) Unbiasedness of $\hat{\theta}$ is shown by

$$\mathbb{E}[\hat{\theta}] = \frac{1}{n}\sum_{i=1}^n \frac{\mathbb{E}|X_i|}{\int_{-\infty}^\infty |x| f_0(x)\,\mathrm{d}x} = \frac{1}{n}\sum_{i=1}^n \frac{\theta \int_{-\infty}^\infty \left|\frac{x}{\theta}\right| f_0(\frac{x}{\theta})\,\mathrm{d}\frac{x}{\theta}}{\int_{-\infty}^\infty |x| f_0(x)\,\mathrm{d}x} = \frac{1}{n}\sum_{i=1}^n \theta = \theta.$$

To obtain the asymptotic distribution for $\hat{\theta}$, let $Z_i = \frac{X_i}{\theta}$ for each $i$. Apparently each $Z_i$ are iid with pdf $f_0$. Then $\mu = \mathbb{E}|Z_i|$ and $\sigma^2 = \mathrm{Var}|Z_i|$ are known values. As such,

$$\mathrm{Var}(\hat{\theta}) = \frac{1}{n^2\mu^2}\sum_{i=1}^n \mathrm{Var}|X_i| = \frac{\theta^2\sigma^2}{n\mu^2} \to 0 \text{ as } n \to \infty,$$

i.e. the MSE of $\hat{\theta}$ converges to 0, indicating that $\hat{\theta}$ is consistent. Moreover, by Central Limit Theorem, $\frac{\hat{\theta} - \theta}{\sqrt{\mathrm{Var}(\hat{\theta})}} \Rightarrow \mathrm{N}(0, 1)$, or equivalently $\hat{\theta} \Rightarrow \mathrm{N}(\theta, \frac{\theta^2\sigma^2}{n\mu^2})$.

4. (a) Note that $\bar{X} \sim \mathrm{N}(\mu, \frac{\sigma^2}{n})$ is a complete and sufficient statistics for $\mu$, and the mgf of $\bar{X}$ is given by

$$\mathbb{E}\left[e^{t\bar{X}}\right] = \exp\left\{\mu t + \frac{\sigma^2}{2n}t^2\right\}.$$

Hence, $\exp\left\{t\bar{X} - \frac{\sigma^2 t^2}{2n}\right\}$ is an unbiased estimator for $e^{\mu t}$ using the complete and sufficient statistics. By Lehmann-Scheffe's Theorem, it must be the UMVUE for $e^{\mu t}$.

Moreover, the variance of the UMVUE is found to be

$$\begin{aligned}
\mathrm{Var}\left(\exp\left\{t\bar{X}-\frac{\sigma^2t^2}{2n}\right\}\right) &= \mathbb{E}\left[\exp\left\{t\bar{X}-\frac{\sigma^2t^2}{2n}\right\}^2\right]-e^{2\mu t}\\
&= \exp\left\{-\frac{\sigma^2t^2}{n}\right\}\underbrace{\mathbb{E}\left[e^{2t\bar{X}}\right]}_{\text{mgf of }2\bar{X}\sim\mathrm{N}(2\mu,\frac{4\sigma^2}{n})}-e^{2\mu t}\\
&= \exp\left\{-\frac{\sigma^2t^2}{n}+2\mu t+\frac{2\sigma^2t^2}{n}\right\}-e^{2\mu t}\\
&= e^{2\mu t}\left(\exp\left\{\frac{\sigma^2t^2}{n}\right\}-1\right),
\end{aligned}$$

and the CRLB is

$$\frac{1}{\mathcal{I}_n(\mu)}\left(\frac{\mathrm{d}e^{\mu t}}{\mathrm{d}\mu}\right)^2=\frac{t^2e^{2\mu t}}{n/\sigma^2}=e^{2\mu t}\left(\frac{\sigma^2t^2}{n}\right),$$

so desired results follow since $e^x-1>x$ for all $x>0$, and $\lim_{x\to 0}\frac{e^x-1}{x}=1$.

(b) Let $p=\mathbb{P}(X_1<t)$ and apparently an unbiased estimator for $p$ is $\hat{p}=\mathbb{1}\{X_1<t\}$. Moreover, note that $X_1-\bar{X}\sim\mathrm{N}(0,\frac{n-1}{n}\sigma^2)$ is ancillary for $\mu$, hence is independent with $\bar{X}$ by Basu's Theorem. Then we have

$$\begin{aligned}
\mathbb{E}[\hat{p}|\bar{X}=s] &= \mathbb{P}(X_1<t|\bar{X}=s)\\
&= \mathbb{P}(X_1-\bar{X}<t-s|\bar{X}=s)\\
&= \mathbb{P}(X_1-\bar{X}<t-s)\\
&= \Phi\left(\frac{t-s}{\sigma}\sqrt{\frac{n}{n-1}}\right),
\end{aligned}$$

where $\Phi(\cdot)$ is the cdf of $\mathrm{N}(0,1)$. By Lehmann-Scheffe's Theorem, $\Phi\left(\frac{t-\bar{X}}{\sigma}\sqrt{\frac{n}{n-1}}\right)$ is the UMVUE for $p$.

5. (a) Consider the log-likelihood

$$l(\mu_1,\mu_2,\sigma^2|\mathbf{x},\mathbf{y})=-\frac{m+n}{2}\log(2\pi\sigma^2)-\frac{\sum_{i=1}^n(x_i-\mu_1)^2+\sum_{j=1}^m(y_j-\mu_2)^2}{2\sigma^2}$$

whose partial derivatives w.r.t. unknown parameters are, respectively,

$$\begin{aligned}
\frac{\partial l(\mu_1,\mu_2,\sigma^2)}{\partial\mu_1} &= \frac{\sum_{i=1}^n(x_i-\mu_1)}{\sigma^2}, \quad \frac{\partial l(\mu_1,\mu_2,\sigma^2)}{\partial\mu_2}=\frac{\sum_{j=1}^m(y_j-\mu_2)}{\sigma^2},\\
\frac{\partial l(\mu_1,\mu_2,\sigma^2)}{\partial\sigma^2} &= -\frac{m+n}{2\sigma^2}+\frac{\sum_{i=1}^n(x_i-\mu_1)^2+\sum_{j=1}^m(y_j-\mu_2)^2}{2(\sigma^2)^2}.
\end{aligned}$$

Solving $\frac{\partial l}{\partial\mu_1}=\frac{\partial l}{\partial\mu_2}=\frac{\partial l}{\partial\sigma^2}=0$ at once gives the unique solution

$$(\hat{\mu}_1,\hat{\mu}_2,\hat{\sigma}^2)=\left(\frac{\sum_{i=1}^n X_i}{n},\frac{\sum_{j=1}^m Y_j}{m},\frac{\sum_{i=1}^n(X_i-\hat{\mu}_1)^2+\sum_{j=1}^m(Y_j-\hat{\mu}_2)^2}{m+n}\right).$$

By observing the concavity of $l(\mu_1,\mu_2,\sigma^2)$, we conclude that $(\hat{\mu}_1,\hat{\mu}_2,\hat{\sigma}^2)$ is the desired MLE.

(b) Unbiasedness of $s^2_{\text{pooled}}$ can be shown via

$$\begin{aligned}\mathbb{E}[s^2_{\text{pooled}}] &= \frac{1}{m+n-2}\left[\mathbb{E}\sum_{i=1}^{n}(X_i-\bar{X})^2 + \mathbb{E}\sum_{j=1}^{m}(Y_j-\bar{Y})^2\right] \\ &= \frac{1}{m+n-2}\left[(n-1)\sigma^2 + (m-1)\sigma^2\right] = \sigma^2.\end{aligned}$$

With Lehmann-Scheffe's Theorem, if we want to show that $s^2_{\text{pooled}}$ is UMVUE, then it remains to show that $s^2_{\text{pooled}}$ is a function of complete and sufficient statistics. Consider the joint density of $(X_1,\cdots,X_n,Y_1,\cdots,Y_m)$:

$$\begin{aligned}f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}|\mu_1,\mu_2,\sigma^2) &= \prod_{i=1}^{n}(2\pi\sigma^2)^{-\frac{1}{2}}\exp\left\{-\frac{(x_i-\mu_1)^2}{2\sigma^2}\right\}\prod_{j=1}^{m}(2\pi\sigma^2)^{-\frac{1}{2}}\exp\left\{-\frac{(y_j-\mu_2)^2}{2\sigma^2}\right\} \\ &= C(\mu_1,\mu_2,\sigma^2)\exp\left\{\frac{\mu_1\sum_{i=1}^{n}x_i}{2\sigma^2}+\frac{\mu_2\sum_{j=1}^{m}y_j}{2\sigma^2}-\frac{\sum_{i=1}^{n}x_i^2+\sum_{j=1}^{m}y_j^2}{2\sigma^2}\right\},\end{aligned}$$

where $C$ is some function of the unknown parameters. It can be easily checked that the joint density above belongs to a 3-parameter minimal exponential family of full rank. Hence the statistics

$$\left(\sum_{i=1}^{n}X_i,\sum_{j=1}^{m}Y_j,\sum_{i=1}^{n}X_i^2+\sum_{j=1}^{m}Y_j^2\right)$$

are complete and sufficient. Moreover,

$$\begin{aligned}s^2_{\text{pooled}} &= \frac{1}{m+n-2}\left[\sum_{i=1}^{n}(X_i-\bar{X})^2+\sum_{j=1}^{m}(Y_j-\bar{Y})^2\right] \\ &= \frac{1}{m+n-2}\left[\sum_{i=1}^{n}X_i^2+\sum_{j=1}^{m}Y_j^2-\frac{1}{n}\left(\sum_{i=1}^{n}X_i\right)^2-\frac{1}{m}\left(\sum_{j=1}^{m}Y_j\right)^2\right]\end{aligned}$$

can be constructed through the complete and sufficient statistics, so the desired result follows from Lehmann-Scheffe's Theorem.