Homework 1 (Stats 620, Winter 2017)

Due Thursday Jan 19, in class

1. **(a)** Let N denote a nonnegative integer-valued random variable. Show that

$$\mathbb{E}[N] = \sum_{k=1}^{\infty} \mathbb{P}\{N \geq k\} = \sum_{k=0}^{\infty} \mathbb{P}\{N > k\}.$$

(b) In general show that if X is nonnegative with distribution F, then

$$\begin{aligned} \mathbb{E}[X] &= \int_0^\infty \overline{F}(x)\, dx, \\ \mathbb{E}[X^n] &= \int_0^\infty n x^{n-1} \overline{F}(x)\, dx. \end{aligned}$$

Comment: These identities will be useful later in the course.

Solution:

(a) Let

$$I_k = \begin{cases} 1 & \text{if } N \geq k \\ 0 & \text{else} \end{cases} \tag{1}$$

and

$$J_k = \begin{cases} 1 & \text{if } N > k \\ 0 & \text{else} \end{cases}$$

Then, $N = \sum_{k=1}^{\infty} I_k = \sum_{k=0}^{\infty} J_k$. Taking expectations gives

$$\mathbb{E}(N) = \sum_{k=1}^{\infty} \mathbb{E}(I_k) = \sum_{k=0}^{\infty} \mathbb{E}(J_k)$$

Since $E[I_k] = P[N \geq k]$ and $E[J_k] = P[N > k]$, the result is shown.

(b)

$$\mathbb{E}(X) = \int_0^\infty x\, dF(x) = -\int_0^\infty x\, d\overline{F}(x)$$

Integration by parts gives

$$\mathbb{E}(X) = -x\overline{F}(x)\Big|_0^\infty + \int_0^\infty \overline{F}(x)\, dx$$

Since $\mathbb{E}(X) < \infty$, $\lim_{x\to\infty} x\overline{F}(x) = 0$, giving

$$\mathbb{E}(X) = \int_0^\infty \overline{F}(x)\, dx$$

The second part of (b) is a very similar integration by parts.

2. Let X_n denote a binomial random variable, $X_n \sim \text{Binomial}\,(n, p_n)$ for $n \geq 1$. If $np_n \to \lambda$ as $n \to \infty$, show that

$$\mathbb{P}\{X_n = i\} \to e^{-\lambda}\lambda^i/i! \qquad \text{as } n \to \infty.$$

Hint: Write out the required binomial probability, expanding the binomial coefficient into a ratio of products. Taking logarithms may be helpful to show that $\lim_{n\to\infty} c_n = c$ implies $\lim_{n\to\infty}(1 - c_n/n)^n = e^{-c}$.

Solution:

$$\begin{aligned}\mathbb{P}(X_n = i) &= \binom{n}{i} {p_n}^i (1-p_n)^{n-i} \\ &= \frac{n \times n-1 \times \cdots \times n-i+1}{i!} \times \frac{(np_n)^i}{n^i} \times \frac{(1-p_n)^n}{(1-p_n)^i}\end{aligned}$$

As $n \to \infty$, $(n-k)/n \to 1$ for $k = 0, 1, \cdots, i-1$ and so $\lim_{n\to\infty} n!/(n-i)!n^i = 1$. Define $\lambda_n = np_n$. Now, $np_n \to \lambda$ implies $(np_n)^i \to \lambda^i$ and $(1-p_n)^i \to 1$ for all fixed i. Now,

$$\begin{aligned}\log \lim_{n\to\infty}(1-\lambda_n/n)^n &= \lim_{n\to\infty} n \log(1-\lambda_n/n) \\ &= \lim_{n\to\infty} n(-\lambda_n/n + o(1/n)) \qquad (2) \\ &= -\lim_{n\to\infty} \lambda_n = -\lambda\end{aligned}$$

So, $\lim_{n\to\infty}(1-\lambda_n/n)^n = \exp(-\lambda)$,where (2) is justified by a Taylor expansion of $\log(1-x)$ and $a_n = o(b_n)$ means $a_n/b_n \to 0$.

3. Let F be a continuous distribution function and let U be a uniformly distributed random variable, $U \sim \text{Uniform}\,(0,1)$.

(a) If $X = F^{-1}(U)$, show that X has distribution function F.

(b) Show that $-\log(U)$ is an exponential random variable with mean 1.

Comment: Part (b) gives a way to simulate exponential random variables using a computer with a random number generator producing $U[0,1]$ random variables.

Solution:

(a)

$$\begin{aligned}\mathbb{P}(X \leq x) &= \mathbb{P}(F^{-1}(U) \leq x) \\ &= \mathbb{P}(U \leq F(x)) \qquad (3) \\ &= \int_0^{F(x)} du = F(x)\,.\end{aligned}$$

Note that in (3), we used the fact that F is non-decreasing. Otherwise the equality does not necessarily hold. For instance let $F(t) = \exp\{-t\}$ and note that $F(0) = 1$, $\lim_{t\to\infty} F(t) = 0$ and F is decreasing. In this case, $\mathbb{P}(F^{-1}(U) \leq x) = \mathbb{P}(U \geq F(x))$.

(b) Note that U has the same distribution as $1-U$. If $F(x) = 1-e^{-x}$, then $F^{-1}(x) = -\log(1-x)$ so (a) shows that $-\log(U)$ has c.d.f. e^{-x}, i.e. X is exponentially distributed with parameter $\lambda = 1$. Thus, $\mathbb{E}(X) = 1/\lambda = 1$.

4. Let $f(x)$ and $g(x)$ be probability density functions, and suppose that for some constant c, $f(x) \le cg(x)$ for all x. Suppose we can generate random variables having density function g, and consider the following algorithm.

   Step 1. Generate Y, a random variable having density function g.

   Step 2. Generate $U \sim$ Uniform $(0, 1)$.

   Step 3. If $U \le \frac{f(Y)}{cg(Y)}$ set $X = Y$. Otherwise, go back to Step 1.

   Assuming that successively generated random variables are independent, show that:

   (a) X has density function f.

   (b) the number of iterations of the algorithm needed to generate X is a geometric random variable with mean c.

   Comment: The procedure investigated in this problem is a standard computational tool for simulating a random variable with a given target density $f(x)$.

   Solution: **(b)** Define infinite sequences $\{Y_n, n \ge 1\}$ and $\{U_n, n \ge 1\}$ where Y_n are i.i.d. with density g and U_n are iid uniform $(0, 1)$. Define $E_n = \{U_n \le f(Y_n)/cg(Y_n)\}$. E_n is a sequence of independent trials and the algorithm stops at the first "success." Thus the number of iterations is geometric with parameter

$$\begin{aligned} p &= \mathbb{P}(E_n) = \mathbb{E}(\mathbb{P}(E_n|Y_n)) \\ &= \int_{-\infty}^{\infty} \frac{f(y)}{cg(y)} g(y)\, dy = \frac{1}{c}. \end{aligned}$$

   The mean number of iterations is $1/p = c$.

   (a)

$$\begin{aligned} \mathbb{P}(Y_n \le x|E_n) &= \mathbb{P}(Y_n \le x, E_n)/\mathbb{P}(E_n) \\ &= c\,\mathbb{E}(\mathbb{P}(Y_n \le x, E_n|Y_n)) \\ &= c\int_{-\infty}^{x} \frac{f(y)}{cg(y)} g(y)\, dy = F(x), \end{aligned}$$

   where $F(x) = \int_{-\infty}^{x} f(u)\, du$. On the nth iteration, conditional on the algorithm terminating at that iteration, Y_n has conditional c.d.f. F, so X has density f.

5. If $X_1, X_2, \ldots, X_n$ are independent and identically distributed exponential random variables with parameter λ, show that $S = \sum_{i=1}^{n} X_n$ has a gamma distribution with parameters (n, λ). That is, show that the density function of S is given by

$$f(t) = \lambda e^{-\lambda t}(\lambda t)^{n-1}/(n-1)!, \qquad t \ge 0.$$

   Instruction: Use moment generating functions for this question.

   Solution: Let $Y = \sum_{i=1}^{n} X_i$. Since $X_i \sim$ Exponential (λ), the MGF of X_i is

$$\mathbb{E}(e^{tX_i}) = \lambda/(\lambda - t)\,, 0 \le t < \lambda\,.$$

Then

$$
\begin{aligned}
\mathbb{E}(e^{tY}) &= \mathbb{E}(\exp\{t\sum_{i=1}^{n} X_i\}) \\
&= \mathbb{E}(\prod_{i=1}^{n} e^{tX_i}) \\
&= \prod_{i=1}^{n} \mathbb{E}(e^{tX_i}) && \text{using independence} \\
&= \lambda^n/(\lambda - t)^n && \text{for } 0 \le t < \lambda.
\end{aligned}
$$

This can be recognized as the MGF of a Gamma (n, λ) random variable. Since the MGF, when exists, determines a distribution uniquely, the result is proved.

Recommended reading:
These homework problems derive from Chapter 1 of Ross "Stochastic Processes," all of which is relevant material. There are too many examples to study them all! Some suggested examples are 1.3(A), 1.3(C), 1.5(A), 1.5(D), 1.9(A).

Supplementary exercises: 1.5, 1.22.
These are optional, but recommended. Do not turn in solutions—they are in the back of the book. To prove Booles's inequality (B5 in the notes for the first class), one can write $\bigcup_{i=1}^{\infty} E_i$ as a disjoint union via defining $F_i = E_i \cap E_{i-1}^c \cap \cdots \cap E_1^c$.

Homework 2 (Stats 620, Winter 2017)

Due Tuesday January 31, in class

Questions are derived from problems in *Stochastic Processes* by S. Ross.

1. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Calculate $\mathbb{E}\big[N(t)N(t+s)\big]$.

   Comment: Please state carefully where you make use of basic properties of Poisson processes, such as stationary, independent increments.

   Solution: Note that $N(t)N(t+s) = N(t)[N(t) + N(t+s) - N(t)]$. Thus

   $$\mathbb{E}[N(t)N(t+s)] = \mathbb{E}[(N(t))^2] + \mathbb{E}[N(t)(N(t+s) - N(t))] = \mathbb{E}[(N(t))^2] + \mathbb{E}[N(t)]\mathbb{E}[N(s)]$$

   where we used the property of stationary and independent increments. Since for any $t > 0$, $N(t) \sim$Poisson(λt), whence $\mathbb{E}[N(t)] = \lambda t$, $\mathrm{Var}[N(t)] = \lambda t$. It follows that

   $$\mathbb{E}[N(t)N(t+s)] = \lambda t + (\lambda t)^2 + \lambda s \lambda t\,.$$

2. Suppose that $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson processes with rates λ_1 and λ_2. Show that $\{N_1(t)+N_2(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1+\lambda_2$. Also, show that the probability that the first event of the combined process comes from $\{N_1(t), t \geq 0\}$ is $\lambda_1/(\lambda_1+\lambda_2)$, independently of the time of the event.

   Solution: We check that $N(t) = N_1(t) + N_2(t)$ satisfies Definition 1.
   (i) $N(t) = 0$.
   (ii) Note that $N_1(t)$ and N_2 have independent increments. Moreover, $N_1(t)$ and $N_2(t)$ are independent.
   (iii) Indeed, for any $t, s > 0$,

$$\begin{aligned}
\mathbb{P}(N(t+s) - N(t) = n) &= \sum_{k=0}^{n} \mathbb{P}(N_1(t+s) - N_1(t) = n-k | N_2(t+s) - N_2(t) = k) \\
&\quad \times \mathbb{P}(N_2(t+s) - N_2(t) = k) \\
&= \sum_{k=0}^{n} \frac{(\lambda_1 s)^{n-k}}{(n-k)!} \exp\{-\lambda_1 s\} \frac{(\lambda_2 s)^k}{k!} \exp\{-\lambda_2 s\} \\
&= \exp\{-(\lambda_1+\lambda_2)s\} \sum_{k=0}^{n} \frac{(\lambda_1 s)^{n-k}(\lambda_2 s)^k}{(n-k)!k!} \\
&= \frac{((\lambda_1+\lambda_2)s)^n}{n!} \exp\{-(\lambda_1+\lambda_2)t\}\,.
\end{aligned}$$

Now to show that the probability of the first arrival is from $N_1(t)$. Let X be the first arrival time for $N(t)$, and X_1, X_2 the corresponding times for $N_1(t)$ and $N_2(t)$.

One way to do is, observing that, $X \sim$ Exponential$(\lambda_1 + \lambda_2)$,

$$\begin{aligned}
\mathbb{P}(\text{first event from } N_1(t)|X = x) &= \lim_{\delta_x \to 0} \mathbb{P}(X_1 < X_2 | X \in [x, x + \delta_x]) \\
&= \lim_{\delta_x \to 0} \frac{\mathbb{P}(X_1 \in [x, x + \delta_x])\mathbb{P}(X_2 > x) + o(\delta_x)}{\mathbb{P}(X \in [x, x + \delta_x])} \\
&= \frac{e^{-\lambda_1 x}(\lambda_1 \delta_x + o(\delta_x))e^{-\lambda_2 x} + o(\delta_x)}{e^{-(\lambda_1+\lambda_2)x}((\lambda_1 + \lambda_2)\delta_x + o(\delta_x))} = \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\end{aligned}$$

As required, this probability does not depend on the first event time for $N(t)$.

3. Buses arrive at a certain stop according to a Poisson process with rate λ. If you take the bus from that stop then it takes a time R, measured from the time at which you enter the bus, to arrive home. If you walk from the bus stop the it takes a time W to arrive home. Suppose that your policy when arriving at the bus stop is to wait up to a time s, and if a bus has not yet arrived by that time then you walk home.
(a) Compute the expected time from when you arrive at the bus stop until you reach home.
(b) Show that if $W < 1/\lambda + R$ then the expected time of part (a) is minimized by letting $s = 0$; if $W > 1/\lambda + R$ then it is minimizes by letting $s = \infty$ (that is, you continue to wait for the bus); and when $W = 1/\lambda + R$ all values of s give the same expected time.
(c) Give an intuitive explanation of why we need only consider the cases $s = 0$ and $s = \infty$ when minimizing the expected time.
<u>Solution</u>: **(a)** Let $E_s = \mathbb{E}$(journey time for strategy s). The journey time is the function of the first arrival time of the rate λ Poisson process of bus arrivals. This has Exponential(λ) distribution (prop 2.2.1). So

$$E_s = \int_0^\infty \lambda e^{-\lambda t}[(t + R)\mathbf{1}(t \leq s) + (s + W)\mathbf{1}(t > s)]\, dt$$

where $\mathbf{1}$ is the indicator function. Thus

$$\begin{aligned}
E_s &= \int_0^s \lambda t e^{-\lambda t}\, dt + R\int_0^s \lambda e^{-\lambda t}\, dt + (s + W)\int_s^\infty \lambda e^{-\lambda t}\, dt \\
&= \frac{1 - e^{-\lambda s}}{\lambda} + R(1 - e^{-\lambda s}) + W e^{-\lambda s}
\end{aligned}$$

(b) Writing $E_s = (W - R - \frac{1}{\lambda})e^{-\lambda s} + \frac{1}{\lambda} + R$. We see that E_s is a decreasing function of s for $(W - R - 1/\lambda) > 0$, and increasing function for $(W - R - 1/\lambda) < 0$ and constant if $(W - R - 1/\lambda) = 0$.
(c) From the memoryless property of the exponential distribution , if it was worth waiting some time $s_0 > 0$ for a bus, and the bus has not arrived at s_0, then resetting time suggests that it must be worth waiting another s_0 time units. Thus, if the optimal s is positive, it must be infinite.

4. Cars pass a certain street location according to a Poisson process with rate λ. A person wanting to cross the street at that location waits until she can see that no cars will come by

in the next T time units. Find the expected time that the person waits before starting to cross. (Note, for instance, that if no cars will be passing in the first T time units then the waiting time is 0.)

Comment: An elegant approach is to condition on the first arrival time.
Solution: Let W be the waiting time, and let X be the first arrival time.

$$\begin{aligned}
\mathbb{E}(W) &= \mathbb{E}\big[\mathbb{E}(W|X)\big] \\
&= \int_0^\infty \mathbb{E}(W|X=x)\lambda e^{-\lambda x}\,dx \\
&= \int_0^\infty \big[(\mathbb{E}(W)+x)\mathbf{1}(x<T)+0\times\mathbf{1}(x\geq T)\big]\lambda e^{-\lambda x}\,dx\,,
\end{aligned}$$

where the last equality follows by the fact that when $X<T$, we can use the memoryless property to reset the clock. Thus,

$$\mathbb{E}(W)=\mathbb{E}(W)\int_0^T \lambda e^{-\lambda x}\,dx+\int_0^T \lambda x e^{-\lambda x}\,dx\,,$$

which gives

$$\mathbb{E}(W)=\frac{1}{\lambda}\big[e^{\lambda T}-(1+\lambda T)\big]\,.$$

5. Individuals enter a system in accordance with a Poisson process having rate λ. Each arrival independently makes its way through the states of the system. Let $\alpha_i(s)$ denote the probability that an individual is in state i a time s after it arrived. Let $N_i(t)$ denote the number of individuals in state i at time t. Show that the $N_i(t), i\geq 1$, are independent and $N_i(t)$ is Poisson with mean equal to

$$\lambda\,\mathbb{E}\big[\text{amount of time an individual is in state } i \text{ during its first } t \text{ units in the system}\big].$$

Comment: You will probably want to make use of Theorem 2.3.1 of Ross. This question is similar to a multivariate version of Proposition 2.3.2, and you may need the multinomial distribution. If n independent experiments each give rise to outcomes $1,\dots,r$ with respective probabilities $p_1,\dots,p_r$, and X_i counts the number of outcomes of type i, then $X_1,\dots,X_r$ are *multinomial*. For $\sum_{i=1}^r n_i=n$,

$$\mathbb{P}(X_1=n_1,\dots,X_r=n_r)=\frac{n!}{n_1!\dots n_r!}\prod_{i=1}^r p_i^{n_i}.$$

Solution: Although not explicit in the question, we suppose there are countably infinite states. Let $N(t)$ be the arrival process, so $N(t)=\sum_{i=1}^\infty N_i(t)$.

$$\begin{aligned}
\mathbb{P}(N_i(t)=n_i\,\forall\, i\in\mathbb{N}) &= \mathbb{E}(\mathbb{P}(N_i(t)=n_i\,\forall\, i\in\mathbb{N}|N(t))) \\
&= \mathbb{P}(N_i(t)=n_i\,\forall\, i\in\mathbb{N}|N(t)=n)\mathbb{P}(N(t)=n)
\end{aligned}$$

where $n = \sum_i n_i$. Now let $U_1, \ldots, U_n$ be n i.i.d. random variables uniformly distributed on $[0, t]$. Theorem 2.3.1 of Ross asserts that conditional on $N(t) = n$, the arrival times in $[0, t]$ $S_1, \ldots, S_n$ have the same distribution as the ordered random variables: $U_{(1)}, \cdots, U_{(n)}$. Let U denote one uniform random variable on $[0, t]$. We define

$$\beta_i \triangleq \mathbb{P}(\text{arrival at time } U, \text{in state } i \text{ at time } t) = \frac{1}{t}\int_0^t \alpha_i(t-s)\, ds = \frac{1}{t}\int_0^t \alpha_i(s) ds \,.$$

Thus,

$$\begin{aligned}
\mathbb{P}(N_i(t) = n_i \,\forall\, i) &= \mathbb{P}\left(\sum_{k=1}^n \mathbf{1}_{\{k\text{–th arrival is at state } i \text{ at time } t\}} = n_i \,\forall\, i\right) \\
&= \mathbb{P}\left(\sum_{k=1}^n \mathbf{1}_{\{\text{the arrival at time } U_{(k)} \text{ is at state } i \text{ at time } t\}} = n_i \,\forall\, i\right) \\
&= \mathbb{P}\left(\sum_{k=1}^n \mathbf{1}_{\{\text{the arrival at time } U_k \text{ is at state } i \text{ at time } t\}} = n_i \,\forall\, i\right) \\
&= \frac{n!}{\prod_{i\in\mathcal{I}} n_i!} \prod_i \beta_i^{n_i} \times \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= \prod_{i\in\mathcal{I}} \frac{(\lambda t \beta_i)^{n_i}}{(n_i)!} e^{-\lambda t \beta_i} \\
&= \prod_{i\in\mathcal{I}} \mathbb{P}(N_i(t) = n_i)\,,
\end{aligned}$$

where $\mathbb{P}(N_i(t) = n_i)$ is calculated in a similar way. We have thus obtained that $\{N_i(t), i \geq 1\}$ are independent Poisson$(\lambda t \beta_i)$ random variables.

To complete the proof, define random variable $\mathbf{1}_i(s)$ be 1 if an individual is in state i after s time units and 0 otherwise. Then

$$\begin{aligned}
\mathbb{E}(\text{time in } i \text{ during individual's first } t \text{ units in the system}) &= \mathbb{E}\big[\int_0^t \mathbf{1}_i(s)\, ds\big] \\
&= \int_0^t \big[\mathbb{E}\mathbf{1}_i(s)\big]\, ds = \int_0^t \alpha_i(s)\, ds = t\beta_i\,.
\end{aligned}$$

Recommended reading:
Sections 2.1 through 2.4, excluding 2.3.1.

Supplementary exercises: 2.14, 2.22.
These are optional, but recommended. Do not turn in solutions—they are in the back of the book.

Homework 3 (Stats 620, Winter 2017)

Due Tuesday February 7, in class
Questions are derived from problems in *Stochastic Processes* by S. Ross.

1. Prove the renewal equation

$$m(t) = F(t) + \int_0^t m(t-x)\, dF(x)$$

Hint: One approach is to use the identity $\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X|Y]\big]$ for appropriate choices of X and Y.

Solution:

$$\begin{aligned}
m(t) &= \mathbb{E}(N(t)) \\
&= \mathbb{E}(\mathbb{E}(N(t)|X_1)) \\
&= \int_0^t \mathbb{E}(N(t)|X_1 = x)\, dF(x) \text{ since } X_1 > t \Rightarrow N(t) = 0 \\
&= \int_0^t \mathbb{E}(1 + N(t-x))\, dF(x) \text{ since renewals are i.i.d.} \\
&= \int_0^t \big[1 + m(t-x)\big]\, dF(x) \\
&= F(t) + \int_0^t m(t-x)\, dF(x)\,. \qquad (1)
\end{aligned}$$

2. Prove that the renewal function $m(t), 0 \le t < \infty$ uniquely determines the interarrival distribution F.
Hint: Laplace transforms may be useful.

Solution: Note that there are two definitions of Laplace transform. Under the definition of Ross,

$$\widetilde{F}(s) = \int_0^\infty e^{-st}\, dF(t)\,,$$

and we also have the Laplace transform of the convolution $F * G(t) = \int_0^\infty F(t-s)dG(s)$:

$$\begin{aligned}
\widetilde{F * G}(s) &= \int_0^\infty \exp\{-st\} d\left(\int_0^\infty F(t-x)dG(x)\right) = \int_0^\infty \exp\{-st\} \int_0^\infty dF(t-x)dG(x) \\
&= \int_0^\infty \int_x^\infty \exp\{-st\} dF(t-x)dG(x) = \int_0^\infty \int_0^\infty \exp\{-s(t+x)\} dF(t)dG(x) \\
&= \int_0^\infty \exp\{-sx\} \int_0^\infty \exp\{-st\} dF(t)dG(x) \\
&= \int_0^\infty \exp\{-sx\} \widetilde{F}(s) dG(x) = \widetilde{F}(s)\widetilde{G}(s)\,.
\end{aligned}$$

Thus the Laplace transform of Equation (1) becomes

$$\widetilde{m}(s) = \widetilde{F}(s) + \widetilde{m}(s)\widetilde{F}(s)$$

so

$$\widetilde{F}(s) = \frac{\widetilde{m}(s)}{1 + \widetilde{m}(s)} . \tag{2}$$

By the uniqueness of Laplace transforms, $\widetilde{m}(s)$ uniquely determines F. Another way to obtain Relation (2) is to calculate the Laplace transform of the identity

$$m(t) = \sum_{n=1}^{\infty} F_n(t) ,$$

where F_n is the n-th convolution of F and $\widetilde{F}_n(s) = \widetilde{F}^n(s)$. If we use another definition of Laplace transform:

$$\widetilde{F}(s) = \int_0^{\infty} e^{-st} F(t) dt ,$$

the calculation becomes slightly different. In particular, the Laplace transform of $\int_0^t m(t - s)dF(s)$ becomes $\widetilde{m}(s)s\widetilde{F}(s)$. In this case, the Relation (2) becomes

$$\widetilde{F}(s) = \frac{\widetilde{m}(s)}{1 + s\widetilde{m}(s)} .$$

3. Let $\{N(t), t \geq 0\}$ be a renewal process and suppose that for all n and t, conditional on the event that $N(t) = n$, the event times $S_1, \ldots, S_n$ are distributed as the order statistics of a set of independent uniform $(0, t)$ random variables. Show that $\{N(t), \geq 0\}$ is a Poisson process. **Hint**: Consider $\mathbb{E}[N(s) \mid N(t)]$ and then use the result of Problem 2.
Solution: Following the hints

$$\mathbb{E}[N(s)|N(t) = n] = \mathbb{E}[\sum_{i=1}^{n} \mathbf{1}(U_{(i)} \leq s)]$$

where $U_{(1)}, \cdots, U_{(n)}$ are the order statistics of n i.i.d. Unif$[0, t]$ random variables $U_1, \cdots, U_n$. Thus

$$\begin{aligned} \mathbb{E}[N(s)|N(t) = n] &= \mathbb{E}[\sum_{i=1}^{n} \mathbf{1}(U_{(i)} \leq s)] \\ &= \mathbb{E}[\sum_{i=1}^{n} \mathbf{1}(U_i \leq s)] \text{ since ordering does not affect the sum} \\ &= \sum_{i=1}^{n} \mathbb{P}[U_i \leq s] = ns/t . \end{aligned}$$

Thus

$$m(s) = \mathbb{E}[\mathbb{E}[N(s)|N(t)]] = \frac{s}{t}\mathbb{E}[N(t)] = \frac{s}{t}m(t)\,.$$

The only solution to this is $m(s) = as$ for some constant a. This is exactly the renewal function for a rate a Poisson process. Using the result from question 2 completes the argument.

4. The random variables $X_1, \ldots, X_n$ are said to be exchangeable if $X_{i_1}, \ldots, X_{i_n}$ has the same joint distribution as $X_1, \ldots, X_n$ whenever $i_1, i_2, \ldots, i_n$ is a permutation of $1, 2, \ldots, n$. That is, they are exchangeable if the joint distribution function $\mathbb{P}\{X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n\}$ is a symmetric function of $(x_1, x_2, \ldots, x_n)$. Let $X_1, X_2, \ldots$ denote the interarrival times of a renewal process.

   (a) Argue that conditional on $N(t) = n$, $X_1, \ldots, X_n$ are exchangeable. Would $X_1, \ldots, X_n, X_{n+1}$ be exchangeable (conditional on $N(t) = n$)?

   (b) Use (a) to prove that for $n > 0$

$$\mathbb{E}\left[\frac{X_1 + \cdots + X_{N(t)}}{N(t)}\big|N(t) = n\right] = \mathbb{E}[X_1|N(t) = n].$$

   (c) Prove that

$$\mathbb{E}\left[\frac{X_1 + \cdots + X_{N(t)}}{N(t)}\big|N(t) > 0\right] = \mathbb{E}[X_1|X_1 < t].$$

   Hint: One approach to (a) involves computing

$$\mathbb{E}\big\{\mathbb{P}[X_1 \leq x_1, \ldots, X_n \leq x_n, N(t) = n\big|X_1, \ldots, X_n]\big\}.$$

   Solution: **(a)** Employing the hint, we write

$$\begin{aligned}
&\mathbb{P}[X_1 \leq x_1, \ldots, X_n \leq x_n, N(t) = n] \\
&= \int_{y_1 \leq x_1} \cdots \int_{y_n \leq x_n} \mathbb{P}\Big[X_{n+1} > t - \sum_{i=1}^{n} y_i\Big]\, dF(y_1) \ldots dF(y_n) \\
&= \int_0^1 \cdots \int_0^1 I_{\{y_1 \leq x_1, \ldots, y_n \leq x_n\}} \bar{F}\Big(t - \sum_{i=1}^{n} y_i\Big)\, dF(y_1) \ldots dF(y_n).
\end{aligned}$$

   Changing the order of integration, by Fubini's theorem, we see that the integral is unchanged by permutations of $x_1, \ldots, x_n$.

   (b) First note that

$$\mathbb{E}\left[\frac{X_1 + \cdots + X_{N(t)}}{N(t)}\big|N(t) = n\right] = \sum_{i=1}^{n} \frac{1}{n}\mathbb{E}[X_i|N(t) = n]\,.$$

   By the exchangeability established in part (a), $\mathbb{E}[X_i|N(t) = n] = \mathbb{E}[X_1|N(t) = n]$, $i = 1, \cdots, n$. So the required result follows.

(c)

$$\begin{aligned}
\mathbb{E}\left[\frac{X_1+\cdots+X_{N(t)}}{N(t)}|N(t)>0\right] &= \sum_{n=1}^{\infty}\mathbb{E}[\frac{X_1+\cdots+X_{N(t)}}{N(t)}|N(t)=n]\mathbb{P}[N(t)=n|N(t)>0] \\
&= \sum_{n=1}^{\infty}\mathbb{E}[X_1|N(t)=n]\mathbb{P}[N(t)=n|N(t)>0] \\
&= \mathbb{E}[X_1|N(t)>0]=\mathbb{E}[X_1|X_1<t]\,.
\end{aligned}$$

5. Consider a miner trapped in a room that contains three doors. Door 1 leads her to freedom after two-days' travel; door 2 returns her to her room after four-days' journey; and door 3 returns her to her room after eight-days' journey. Suppose at all times she is equally to choose any of the three doors, and let T denote the time it takes the miner to become free.
(a) Define a sequence of independent and identically distributed random variables $X_1, X_2, \ldots$ and a stopping time N such that

$$T=\sum_{i=1}^{N} X_i.$$

Note: You may have to imagine that the miner continues to randomly choose doors even after she reaches safety.
(b) Use Wald's equation to find $\mathbb{E}[T]$.
(c) Compute $\mathbb{E}\big[\sum_{i=1}^{N} X_i|N=n\big]$ and note that it is not equal to $\mathbb{E}\big[\sum_{i=1}^{n} X_i\big]$.
(d) Use part (c) for a second derivation of $\mathbb{E}[T]$.

<u>Solution</u>: **(a)** Define

$$X=\begin{cases} 2 & \text{Door 1 (probability 1/3)} \\ 4 & \text{Door 2 (probability 1/3)} \\ 8 & \text{Door 3 (probability 1/3)} \end{cases}$$

and $N=\min\{n: X_n=2\}$. Clearly N is a stopping time as the event $N=n$ is determined by the first n observations of X.
(b) Using Wald's theorem, $\mathbb{E}[T]=\mathbb{E}[N]\mathbb{E}[X]$. Further $\mathbb{E}[N]=3$ since N follows a geometric distribution with parameter $p=1/3$. Also $\mathbb{E}[X]=14/3$. Thus $\mathbb{E}[T]=14$.
(c)

$$\begin{aligned}
\mathbb{E}[\sum_{i=1}^{N} X_i|N=n] &= \mathbb{E}[\sum_{i=1}^{N} X_i|X_1\neq 2,\cdots,X_{n-1}\neq 2, X_n=2]\,, \\
&= 2+(n-1)\mathbb{E}[X_i|X_i\neq 2]=2+(n-1)6=6n-4 \\
\mathbb{E}[\sum_{i=1}^{n} X_i] &= n\mathbb{E}[X_i]=14n/3\,.
\end{aligned}$$

(d)$\mathbb{E}[T]=\mathbb{E}[\mathbb{E}[\sum_{i=1}^{N} X_i|N]]=\mathbb{E}[6N-4]=6\times 3-4=14.$

Recommended reading:
Sections 3.1 through 3.3.

Supplementary exercise: 3.7.
Optional, but recommended. Do not turn in a solution—it is in the back of the book.

Homework 4 (Stats 620, Winter 2017)

Due Tuesday Feb 14, in class
Questions are derived from problems in *Stochastic Processes* by S. Ross.

1. Let $A(t)$ and $Y(t)$ denote respectively the age and excess at t. Find:
(a) $\mathbb{P}\{Y(t) > x|A(t) = s\}$.
(b) $\mathbb{P}\{Y(t) > x|A(t + x/2) = s\}$.
(c) $\mathbb{P}\{Y(t) > x|A(t + x) > s\}$ for a Poisson process.
(d) $\mathbb{P}\{Y(t) > x, A(t) > y\}$.
(e) If $\mu < \infty$, show that, with probability 1, $A(t)/t \to 0$ as $t \to \infty$.

Hint: For (d), use a regenerative process argument (E.g. Ross, section 3.7) to find $\lim_{t\to\infty} \mathbb{P}(Y(t) > x, A(t) > y)$. For (e), you may use without proof the following results on convergence with probability 1: **(L1)** $\lim_{n\to\infty} S_n/n = \mu$; **(L2)** $\lim_{t\to\infty} N(t) = \infty$; **(L3)** $\lim_{t\to\infty} N(t)/t = 1/\mu$.

Solution: **(a)**

$$\begin{aligned}\mathbb{P}[Y(t) > x|A(t) = s] &= \mathbb{P}[X_{N(t)+1} > s + x|S_{N(t)} = t - s] \\ &= \mathbb{P}[X_1 > s + x|X_1 > s] = \overline{F}(s + x)/\overline{F}(s)\,.\end{aligned}$$

Here is a more formal solution:

$$\begin{aligned}&\mathbb{P}[Y(t) > x|A(t) = s] \\ &= \mathbb{P}[S_{N(t)+1} > t + x|S_{N(t)} = t - s] = \mathbb{P}[X_{N(t)+1} > s + x|S_{N(t)} = t - s] \\ &= \sum_{n=0}^{\infty} \mathbb{P}[X_{n+1} > s + x|S_n = t - s, N(t) = n]\mathbb{P}[N(t) = n|S_{N(t)} = t - s] \\ &= \sum_{n=0}^{\infty} \mathbb{P}[X_{n+1} > s + x|S_n = t - s, X_{n+1} > s]\mathbb{P}[N(t) = n|S_{N(t)} = t - s] \\ &= \sum_{n=0}^{\infty} \mathbb{P}[X_{n+1} > s + x|X_{n+1} > s]\mathbb{P}[N(t) = n|S_{N(t)} = t - s]\text{(by independence)} \\ &= \mathbb{P}[X_1 > s + x|X_1 > s] = \overline{F}(s + x)/\overline{F}(s)\end{aligned}$$

(b)

$$p := \mathbb{P}[Y(t) > x|A(t + x/2) = s]$$

For $s \geq x/2$, an argument similar to (a) allows us to write

$$\begin{aligned}p &= \mathbb{P}[\text{no event in } (t + x/2 - s, t + x]|\text{ event at } t + x/2 - s, \text{ no events in } (t + x/2 - s, t + x/2]] \\ &= \mathbb{P}[X_{N(t)+1} > s + x/2|S_{N(t)} = t + x/2 - s, X_{N(t)+1} > s] \\ &= \mathbb{P}[X_1 > s + x/2|X_1 > s] = \overline{F}(s + x/2)/\overline{F}(s)\end{aligned}$$

For $s < x/2$, $\{A(t + x/2) = s\} \Rightarrow \{Y(t) \leq s - x/2\}$. It follows that $p = 0$.

(c)

$$\begin{aligned} q &\equiv \mathbb{P}[Y(t) > x | A(t+x) > s] \\ &= \mathbb{P}\big[\text{no event in } [t, t+x] | \text{ no events in } [t+x-s, t+x]\big] \end{aligned}$$

for $0 \le s \le x$, since the process is Poisson with independent increments,

$$q = \mathbb{P}\big[\text{no event in } [t, t+x-s]\big] = \exp^{-\lambda(x-s)},$$

where λ is the rate of the Poisson process. For $s > x$, $q = 1$.

(d)

$$\begin{aligned} &\mathbb{P}(Y(t) > x, A(t) > y) = \mathbb{P}(X_{N(t)+1} > t - S_{N(t)} + x, t - S_{N(t)} > y) \\ &\quad = \mathbb{P}(X_1 > t + x, t > y | S_{N(t)} = 0)\mathbb{P}(S_{N(t)} = 0) \\ &\quad\quad + \int_0^t \mathbb{P}(X_{N(t)+1} > t - S_{N(t)} + x, t - S_{N(t)} > y | S_{N(t)} = s, X_{N(t)+1} > t - S_{N(t)}) dF_{S_{N(t)}}(s) \\ &\quad = \mathbf{1}_{\{t>y\}}\mathbb{P}(X_1 > t + x | X_1 > t)\mathbb{P}(S_{N(t)} = 0) \\ &\quad\quad + \int_0^t \mathbf{1}_{\{t-s>y\}}\mathbb{P}(X > t + x - s | X > t - s) dF_{S_{N(t)}}(s) \\ &\quad = \mathbf{1}_{\{t>y\}}\overline{F}(t+x) + \int_0^{t-y} \overline{F}(t + x - s) dm(s). \end{aligned}$$

Let $P_t = \mathbb{P}[Y(t) > x, A(t) > y]$. Define a regenerative process to be "on" at t if $S_{N(t)} < t - y$ and $S_{N(t)+1} > t + x$. Thus, P_t is the probability that the process is "on" at time t. By the regenerative process limit theorem

$$\begin{aligned} \lim_{t\to\infty} P_t &= \frac{\mathbb{E}[\text{time "on" during a cycle}]}{\mathbb{E}[\text{time of the cycle}]} \\ &= \frac{\mathbb{E}[\max(X_1 - (x+y), 0)]}{\mu} \\ &= \frac{1}{\mu}\int_{x+y}^{\infty} (z - x - y)\, dF(z) \end{aligned}$$

(e)

$$\begin{aligned} \lim_{t\to\infty} \frac{A(t)}{t} &= \lim_{t\to\infty} \frac{t - S_{N(t)}}{t} \\ &= 1 - \lim_{t\to\infty} \frac{S_{N(t)}}{N(t)} \lim_{t\to\infty} \frac{N(t)}{t} \\ &= 1 - \mu/\mu \quad \text{(by (L1), (L2) and (L3))} \\ &= 0 \end{aligned}$$

2. Consider a single-server bank in which potential customers arrive in accordance with a renewal process having interarrival distribution F. However, an arrival only enters the bank if the

server is free when he or she arrives; otherwise, the individual goes elsewhere without being served. Would the number of events by time t constitute a (possibly delayed) renewal process if an event corresponds to a customer:
(a) entering the bank?
(b) leaving the bank after being served?
What if F were exponential?

Solution: Let X_i denote the length of the i-th service and let Y_i denote the time from the end of i-th service until the start of the $i+1$-th service. Let Y_0 denote the time when first arrival enters the bank (and gets service). Note that X_i and Y_i may be dependent when the arrival is not a Poisson process.
(a) In this case, each cycle consists of $Z_i = X_i + Y_i, i = 1, 2, \dots$ and $Z_0 = Y_0$. Since X_i and Y_i are independent of X_j and Y_j with $j = 1, \dots, i-1$, $\{Z_i\}_{i\in\mathbb{N}}$ are i.i.d copies. We thus have a delayed renewal process.
(b) In this case, $Z_i = Y_{i-1} + X_i$. When X_i and Y_i are dependent, $\{Z_i\}_{i\in\mathbb{N}}$ are not i.i.d. copies. We do not have a (delayed) renewal process. One counter example can be constructed as in the sequel. Suppose the service distribution is given by

$$Y_1 = \begin{cases} 1 & \text{w.p. } 0.5 \\ 10 & \text{w.p. } 0.5 \end{cases}$$

and the interarrival times of the customers to the bank $Z_n \sim F$ are given by, $Z_1 = 6$ w.p. 1. Then, given a previous interval between departures $S_n - S_{n-1} = 3$, we know that the next arrival will enter the bank at time $S_n + 4$.

If F is exponential **(a)** still gives a delayed renewal process. **(b)** now results in a (non-delayed) renewal process, since the memoryless property implies that Y_i is independent of X_i, $i \in \mathbb{N}$. Hence, $\{Z_i\}_{i\in\mathbb{N}}$ are i.i.d. copies.

3. On each bet a gambler, independently of the past, either wins or loses 1 unit with respective probability p and $1-p$. Suppose the gambler's strategy is to quit playing the first time she wins k consecutive bets. At the moment she quits

   (a) find her expected winnings.
   (b) find the expected number of bets that she has won.

   Hint: It may help you to look at Example 3.5(A) in Ross.

   Solution: Let

$$Y_n = \begin{cases} 1 & \text{if } n\text{th game is a win} \\ 0 & \text{else} \end{cases}$$

   and

$$X_n = \begin{cases} 1 & \text{if } n\text{th game is a win} \\ -1 & \text{else} \end{cases}$$

   and let $N = \inf\{n \geq k : \sum_{m=n-k+1}^{n} X_m = k\} = \inf\{n \geq k : \sum_{m=n-k+1}^{n} Y_m = k\}$, the first time k consecutive games are won. Let $W = \sum_{i=1}^{N} X_i$, the gamblers total winnings. Also let $N_W = \sum_{i=1}^{N} Y_i$, the number of games won.

(a) From Ross, Example 3.5A, $\mathbb{E}[N] = \sum_{i=1}^{k}(1/p)^i$. Also N is a stopping time w.r.t $X_i, i = 1, 2, \dots$. By Wald's equation, we have

$$\mathbb{E}[W] = \mathbb{E}[N]\,\mathbb{E}[X_1] = \left(\sum_{i=1}^{k}\frac{1}{p^i}\right)(2p-1)$$

(b) N is also a stopping time w.r.t. $Y_i, i = 1, \cdots$, so Wald's equation gives

$$\mathbb{E}[N_W] = \mathbb{E}[N]\mathbb{E}[Y_1] = \left(\sum_{i=1}^{k}\frac{1}{p^i}\right)p = \sum_{i=0}^{k-1}\frac{1}{p^i}$$

4. Prove Blackwell's theorem for renewal reward processes. That is, assuming that the cycle distribution is not lattice, show that, as $t \to \infty$,

$$\mathbb{E}\big[\text{reward in}(t, t+a)\big] \to a\,\frac{\mathbb{E}[\text{reward incycle}]}{\mathbb{E}[\text{time of cycle}]}.$$

Assume that any relevant function is directly Riemann integrable.

Hint: You may adopt an informal approach by assuming that one can write

$$\mathbb{E}\left[\int_t^{t+a} dR(s)\right] = \int_t^{t+a}\mathbb{E}\big[dR(s)\big],$$

and then developing the identity

$$\mathbb{E}[dR(t)] = \mathbb{E}[R_1|X_1 = t]dF(t) + \int_0^t \{\mathbb{E}[R_1|X_1 = t-x]dF(t-x)\}\,dm(x).$$

If you can find a more elegant or more rigorous solution, that would also be good!

Solution: Let $R(t)$ be the reward accumulated by time t. Then,

$$\begin{aligned}\mathbb{E}[\text{reward in } (t, t+a)] &= \mathbb{E}[R(t+a) - R(t)] \\ &= \mathbb{E}[\int_t^{t+a} dR(s)] \\ &= \int_t^{t+a}\mathbb{E}[\,dR(s)]\end{aligned}$$

assuming that the interchange is allowed, e.g. if $R(t)$ is increasing. Now,

$$\begin{aligned}\mathbb{E}[\,dR(t)] &= \mathbb{E}[\mathbb{E}[\,dR(t)|S_{N(t)}]] \\ &= \mathbb{E}[\,dR(t)|S_{N(t)} = 0]\mathbb{P}[S_{N(t)} = 0] + \int_0^\infty \mathbb{E}[\,dR(t)|S_{N(t)} = y]\,dF_{S_{N(t)}}(y) \\ &= \mathbb{E}[\,dR(t)|S_{N(t)} = 0]\overline{F}(t) + \int_0^\infty \mathbb{E}[\,dR(t)|S_{N(t)} = y]\overline{F}(t-y)\,dm(y)\,.\end{aligned}$$

Now, since $R(t)$ only increases when an event occurs,

$$\begin{aligned}\mathbb{E}[\,dR(t)|S_{N(t)}=y] &= \mathbb{E}[R_{N(t)+1}|X_{N(t)+1}=t-y]\mathrm{d}F_{X_{N(t)+1}|S_{N(t)}=y}(t-y)\\ &= \mathbb{E}[R_1|X_1=t-y]\frac{dF(t-y)}{\overline{F}(t-y)}\,.\end{aligned}$$

This established the identity

$$\mathbb{E}[\,dR(t)] = \mathbb{E}[R_1|X_1=t]\,dF(t) + \int_0^t (\mathbb{E}[R_1|X_1=t-y]\,dF(t-y))\,dm(x)\,.$$

Now the key renewal theorem gives

$$\lim_{t\to\infty}\mathbb{E}[\,dR(t)] = \int_0^t \mathbb{E}[R_1|X_1=t]\,dF(t)\,dt = \mathbb{E}[R_1]\,dt\,.$$

Thus

$$\lim_{t\to\infty}\int_t^{t+a}\mathbb{E}[\,dR(t)] = \int_t^{t+a}\mathbb{E}[R_1]\,ds = a\frac{\mathbb{E}[R_1]}{\mu}\,.$$

Another approach: Note that

$$\begin{aligned}\mathbb{E}[\text{reward in }(t,t+a)] &= \mathbb{E}[\sum_{n=1}^{N(t+a)} R_n - \sum_{n=1}^{N(t)} R_n]\\ &= \mathbb{E}[\sum_{n=1}^{N(t+a)+1} R_n] - \mathbb{E}[\sum_{n=1}^{N(t)+1} R_n] + \mathbb{E}[R_{N(t)+1}] - \mathbb{E}[R_{N(t+a)+1}]\end{aligned}$$

Now $N(t+a)+1$ and $N(t)+1$ are stopping times for the sequence $(X_i,R_i), i=1,\cdots$. Thus from (generalized) Wald's equation

$$\mathbb{E}[\sum_{n=1}^{N(t)+1} R_n] = \mathbb{E}[N(t)+1]\mathbb{E}[R]$$

and

$$\mathbb{E}[\sum_{n=1}^{N(t+a)+1} R_n] = \mathbb{E}[N(t+a)+1]\mathbb{E}[R]\,,$$

where $\mathbb{E}[R]$ is the expected reward in a cycle. Thus

$$\begin{aligned}\mathbb{E}[\text{reward in }(t,t+a)] &= \mathbb{E}[N(t+a)+1]\mathbb{E}[R] - \mathbb{E}[N(t)+1]\mathbb{E}[R] + \mathbb{E}[R_{N(t)+1}] - \mathbb{E}[R_{N(t+a)+1}]\\ &= (m(t+a)-m(t))\mathbb{E}[R] + \mathbb{E}[R_{N(t)+1}] - \mathbb{E}[R_{N(t+a)+1}]\,.\end{aligned}$$

Now

$$\lim_{t\to\infty}(m(t+a)-m(t))\mathbb{E}[R] = a\mathbb{E}[R]/\mathbb{E}[X]\,.$$

from Blackwell's theorem. Now, it suffices to show that $\lim_{t\to\infty} \mathbb{E}[R_{N(t)+1}]$ exists and is finite. Indeed,

$$\begin{aligned}
\mathbb{E}[R_{N(t)+1}] &= \mathbb{E}[R_{N(t)+1}|S_{N(t)}=0]\overline{F}(t) + \int_0^t \mathbb{E}[R_{N(t)+1}|S_{N(t)}=s]\overline{F}(t-s)\mathrm{d}m(s) \\
&= \mathbb{E}[R|X>t]\overline{F}(t) + \int_0^t \mathbb{E}[R|X>t-s]\overline{F}(t-s)\,dm(s) \\
&= h(t) + \int_0^t h(t-s)\,dm(s)\,,
\end{aligned}$$

where $h(t)=\mathbb{E}[R|X>t]\overline{F}(t)$. Then by the Key Renewal theorem, we have

$$\lim_{t\to\infty} \mathbb{E}[R_{N(t)+1}] = \frac{1}{\mathbb{E}[X]}\int_0^\infty h(s)\,ds\,.$$

Here we assumed that $h(t)$ is directly Riemann integrable.

5. The life of a car is a random variable with distribution F. An individual has a policy of trading in his car either when it fails or reaches the age of A. Let $R(A)$ denote the resale value of an A-year-old car. There is no resale value of a failed car. Let C_1 denote the cost of a new car and suppose that an additional cost C_2 is incurred whenever the car fails.
(a) Say that a cycle begins each time a new car is purchased. Compute the long-run average cost per unit time.
(b) Say that a cycle begins each time a car in use fails. Compute the long-run average cost per unit time.
Note: In both (a) and (b) you are expected to compute the ratio of the expected cost incurred in a cycle to the expected time of a cycle. The answer should, of course, be the same in both parts.

Solution: **(a)** Clearly,

$$\mathbb{E}[\text{cost per cycle}] = C_1 - \overline{F}(A)R(A) + F(A)C_2$$

and

$$\mathbb{E}[\text{time of cycle}] = \int_0^A x\mathrm{d}F(x) + A(1-F(A))\,.$$

So, treating the cost as the reward, the renewal reward theorem gives

$$\lim_{t\to\infty}\frac{\mathbb{E}[\text{accumulated cost by t}]}{t} = \frac{\mathbb{E}[\text{cost per cycle}]}{\mathbb{E}[\text{time of cycle}]} = \frac{C_1-\overline{F}(A)R(A)+F(A)C_2}{\int_0^A x\,dF(x)+A\overline{F}(A)}$$

(b) The chance that a car fails is $F(A)$, so the number, N, of cars bought between failures has the geometric distribution with parameter $p=F(A)$. We have,

$$\mathbb{E}[\text{cost per cycle}] = \mathbb{E}[NC_1-(N-1)R(A)+C_2] = C_1/F(A) + (1-1/F(A))R(A) + C_2$$

and

$$\mathbb{E}[\text{time of cycle}] = \mathbb{E}[(N-1)A] + \mathbb{E}[\text{car life}|\text{car life} < A] = \overline{F}(A)A/F(A) + \int_0^A x\,dF(x)/F(A)\,.$$

Thus,

$$\lim_{t\to\infty} \frac{\mathbb{E}[\text{accumulated cost by t}]}{t} = \frac{C_1/F(A) + (1 - 1/F(A))R(A) + C_2}{\overline{F}(A)A/F(A) + \int_0^A x\,dF(x)/F(A)}\,.$$

Multiplying numerator and denominator by $F(A)$ gives the same expression as in (a).

Recommended reading:
Sections 3.4 through 3.7, excluding subsections 3.4.3, 3.6.1, 3.7.1. We will not cover the material in Section 3.8, though you may like to look through it.

Supplementary exercises: 3.24, 3.27, 3.35.
These are optional, but recommended. Do not turn in solutions—they are in the back of the book.

Homework 5 (Stats 620, Winter 2017)

Due Tuesday Feb 21, in class
Questions are derived from problems in *Stochastic Processes* by S. Ross.

1. Prove that if the number of state is n, and if state j is accessible from state i, then it is accessible in n or fewer steps.

   Solution: j is accessible from i if, for some $k \geq 0$, $P_{ij}^k > 0$. Now

   $$P_{ij}^k = \sum \prod_{m=1}^{k} P_{i_m i_{m+1}}$$

   where the sum is taken over all sequences $(i_0, i_1, \cdots, i_k) \in \{1, \cdots, n\}^{k+1}$ of states with $i_0 = i$ and $i_k = j$. Now, $P_{ij}^k > 0$ implies that at least one term is positive, say $\prod_{m=1}^k P_{i_m i_{m+1}} > 0$. If a state s occurs twice, say $i_a = i_b = s$ for $a < b$, and $(a, b) \neq (0, k)$, then the sequence of states $(i_0, \cdots, i_{a-1}, i_b, \cdots, i_k)$ also has positive probability, without this repetition. Thus, the sequence $i_0, \cdots, i_k$ can be reduced to another sequence, say $j_0, \cdots, j_r$, in which no state is repeated. This gives $r \leq n - 1$, so $i \neq j$ is accessible in at most $n - 1$ steps. If $i = j$, we cannot remove this repetition! This gives the possibility of $r = n$, when $i = j$, but there are no other repetitions.

2. For states i, j, k with $k \neq j$, let

   $$P_{ij/k}^n = P\{X_n = j, X_\ell \neq k, \ell = 1, ..., n-1 | X_0 = i\}.$$

   (a) Explain in words what $P_{ij/k}^n$ presents.
   (b) Prove that, for $i \neq j$, $P_{ij}^n = \sum_{k=0}^n P_{ii}^k P_{ij/i}^{n-k}$

   Solution:

   (a) $P_{ij/k}^n$ is the probability of being in j at time n, starting in i at time 0, while avoiding k.
   (b) Let N be the (random) time at which $\{X_k\}$ is last in i before time n. Then since $0 \leq N \leq n$,

$$\begin{aligned}
P_{ij}^n &= P[X_n = j | X_0 = i] \\
&= \sum_{k=0}^n P[X_n = j, N = k | X_0 = i] \\
&= \sum_{k=0}^n P[X_n = j, X_k = i, X_l \neq i : k+1 \leq l \leq n | X_0 = i] \\
&= \sum_{k=0}^n P[X_n = j, X_l \neq i : k+1 \leq l \leq n | X_0 = i, X_k = i] P[X_k = i | X_0 = i]
\end{aligned}$$

Now using the Markov property

$$
\begin{aligned}
P_{ij}^n &= \sum_{k=0}^{n} P[X_n = j, X_l \neq i : k+1 \leq l \leq n | X_k = i] P[X_k = i | X_0 = i] \\
&= \sum_{k=0}^{n} P_{ij/i}^{n-k} P_{ii}^k
\end{aligned}
$$

Note that one can also calculate $P_{ij}^n = E[P[X_n = j | X_0 = i, N]]$, but this works out slightly less easily.

3. Show that the symmetric random walk is recurrent in two dimensions and transient in three dimensions.

   Comments: This asks you to extend the argument of Ross Example 4.2(A) to two and three dimensions. You may use either of the definitions of the simple symmetric random walk in d dimensions from the notes.

   Solution: Define the symmetric random walk in d dimensions, $X_n^{(d)} = (X_{(n,1)}, \cdots, X_{(n,d)})$, by

$$
X_{n+1,j} = \begin{cases} X_{n,j} + 1 & \text{with probability } 0.5 \\ X_{n,j} - 1 & \text{with probability } 0.5 \end{cases}
$$

   i.e. each component of $X_n^{(d)}$ carries out an independent symmetric random walk in one dimension. Let $P_n^{(d)} = P[X_n^{(d)} = X_0^{(d)}]$. From example 4.2(A), $P_{2n}^{(1)} \sim 1/\sqrt{\pi n}$, and $P_{2n+1}^{(1)} = 0$. By independence,

$$
P_{2n}^{(d)} \sim (\frac{1}{\sqrt{\pi n}})^d
$$

   Therefore,

$$
\sum_{n=1}^{\infty} P_n^{(2)} \sim \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} = \infty
$$

   and

$$
\sum_{n=1}^{\infty} P_n^{(3)} \sim \frac{1}{\pi^{3/2}} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty
$$

   Thus by Proposition 4.2.3 we get the recurrence of the two dimensional symmetric random walk, and the transience in three (or more) dimensions. Note that it would take a bit more work to fully justify our implicit "interchange" of $\sim$ with infinite summation.

4. A transition probability matrix P is said to be doubly stochastic if

$$
\sum_i P_{ij} = 1 \quad \text{for all } j.
$$

   That is, the column sums all equal 1. If a doubly stochastic chain has n states and is ergodic, calculate its limiting probabilities.

Hint: guess the answer, and then show that your guess satisfies the required equations. Then, by arguing for the uniqueness of the limiting distribution, you will have solved the problem.

Solution: Let $\mu = (1/n, \cdots, 1/n)$. Then since $\sum_j P_{ji} = 1$, $[\mu P]_i = \sum_{j=0}^{n-1} \mu_j P_{ji} = \sum_j P_{ji}/n = 1/n$. Thus, $\mu P = \mu$. The uniqueness of the limiting distribution for an ergodic Markov chain implies that any solution to $\mu P = \mu$ with $\mu_i > 0$ and $\sum_i \mu_i = 1$ is the required limiting distribution.

5. Jobs arrive at a processing center in accordance with a Poisson process with rate λ. However, the center has waiting space for only N jobs and so an arriving job finding N others waiting goes away. At most 1 job per day can be processed, and the processing of this job must start at the beginning of the day. Thus, if there are any jobs waiting for processing at the beginning of a day, then one of them is processed that day, and if no jobs are waiting at the beginning of a day then no jobs are processed that day. Let X_n denote the number of jobs at the center at the beginning of day n.
(a) Find the transition probabilities of the Markov chain $\{X_n, n \geq 0\}$.
(b) Is this chain ergodic? Explain.
(c) Write the equations for the stationary probabilities.

Instructions:
(a). Suppose that the arrival rate λ has units day^{-1}.
(b). You may assert the property that a finite state, irreducible, aperiodic Markov chain is ergodic (see Theorem 4.3.3, the discussion following this theorem, and Problem 4.14).
(c). There is no particularly elegant way to write these equations, and you are not expected to solve them.

Solution: The state space is $0, 1, \cdots, N$. Let $p(j) = \lambda^j e^{-\lambda}/j!$
(a)

$$P_{0k} = \begin{cases} P[k \text{ arrivals}] = p(k) & \text{for } k = 0, \cdots, N-1 \\ P[\geq N \text{ arrivals}] = \sum_{l=N}^{\infty} p(l) & \text{for } k = N \end{cases}$$

$$P_{jk} = \begin{cases} P[k-j+1 \text{ arrivals}] = p(k-j+1) & \text{for } k = j-1, \cdots, N-1 \\ P[\geq N-j+1 \text{ arrivals}] = \sum_{l=N-j+1}^{\infty} p(l) & \text{for } k = N \end{cases}$$

(b) The Markov chain is irreducible, since $P_{0k} > 0$ and $P_{k0}^k = (p(0))^k > 0$. It is aperiodic, since $P_{00} > 0$. A finite state Markov chain which is irreducible and aperiodic is ergodic (since it is not possible for all states to be transient or for any states to be null recurrent).
(c) For $j < N$, the identity $\pi_j = \sum_k \pi_k P_{kj}$ becomes $\pi_j = \sum_{k=0}^{j+1} \pi_k P_{kj}$. This can be rewritten as a recursion

$$\pi_{j+1} = \frac{\pi_j(1 - P_{jj}) - \sum_{k=0}^{j-1} \pi_k P_{kj}}{P_{j+1,j}}$$

An alternative expression is given as follows. Since the long run rate of entering $\{0, \cdots, j\}$ must equal the rate of leaving $\{0, \cdots, j\}$

$$\pi_{j+1} p(0) = \pi_0 \overline{F}(j) + \sum_{k=1}^{j} \pi_k \overline{F}(j-k+1)$$

where $\overline{F}(j) = \sum_{k=j+1}^{\infty} p(k)$.

Recommended reading:
Sections 4.1 through 4.3, excluding examples 4.3(A,B,C).

Supplementary exercises: 4.13, 4.14
These are optional, but recommended. Do not turn in solutions—they are in the back of the book.

Homework 6 (Stats 620, Winter 2017)

Due Thursday March 16, in class

1. In a branching process the number of offspring per individual has a Binomial $(2, p)$ distribution. Starting with a single individual, calculate:
(a) the extinction probability;
(b) the probability that the population becomes extinct for the first time in the third generation.
(c) Suppose that, instead of starting with a single individual, the initial population size Z_0 is a random variable that is Poisson distributed with mean λ. Show that, in this case, the extinction probability is given, for $p > 1/2$, by

$$\exp\{\lambda(1-2p)/p^2\}.$$

Instructions: (b) If X_n is the size of the nth generation, this question is asking you to find $P(X_3 = 0, X_2 > 0, X_1 > 0 | X_0 = 1)$. This can be done by brute force calculation, or by using probability generating functions.

Solution:

(a) Say $p_j = \mathbb{P}[X_1 = j | X_0 = 1] = \binom{2}{j} p^j (1-p)^{2-j}$ for $j = 0, 1, 2$ and 0 for $j > 2$. Now $\pi_0 = \mathbb{P}[\text{Population dies out}] = \sum_{j=0}^{\infty} \pi_0^j p_j$. Thus

$$\pi_0 = (1-p)^2 + 2\pi_0(1-p)p + \pi_0^2 p^2 .$$

Solving and choosing the smaller root

$$\pi_0 = \begin{cases} 1 & p \leq .5 \\ (\frac{1-p}{p})^2 & p > .5 \end{cases}$$

(b) Let $\phi_n(s) = \mathbb{E}[s^{X_n}]$. It was shown in class that $\phi_n(s) = \phi_1(\phi_{n-1}(s))$. Also its easy to see that $\phi_1(s) = (sp + 1 - p)^2$. Also $\mathbb{P}[X_n = 0] = \phi_n(0)$. Thus the probability of extinction in third generation is

$$\begin{aligned} \phi_3(0) - \phi_2(0) &= \phi_1(\phi_1(\phi_1(0))) - \phi_1(\phi_1(0)) \\ &= 4p^2(1-p)^4 + 6p^3(1-p)^5 + 6p^4(1-p)^6 + 4p^5(1-p)^7 + p^6(1-p)^8 \end{aligned}$$

(c) Probability of extinction when $Z_0 = 1$ and $p > .5$ is $(\frac{1-p}{p})^2$. All families behave independently of each other. Thus when Z_0 has a Poisson distribution with parameter (λ), the probability of extinction equals

$$\sum_{n=0}^{\infty} \left[\frac{(1-p)^2}{p^2}\right]^n P[Z_0 = n] ,$$

which is exactly the probability generating function of Poisson r.v. evaluated at $\frac{(1-p)^2}{p^2}$. Thus probability of extinction equals $e^{\lambda[\frac{(1-p)^2}{p^2} - 1]}$.

2. Consider a time-reversible Markov chain with transition probabilities P_{ij} and limiting probabilities π_i; and now consider the same chain truncated to the states $0, 1, , M$. That is, for the truncated chain its transition probabilities $\overline{P}_{ij}$ are

$$\overline{P}_{ij} = \begin{cases} P_{ij} + \sum_{k>M} P_{ik}, & 0 \le i \le M, j = i \\ P_{ij}, & 0 \le i \ne j \le M \\ 0, & \text{otherwise.} \end{cases}$$

Show that the truncated chain is also time reversible and has limiting probabilities given by

$$\overline{\pi}_i = \frac{\pi_i}{\sum_{i=0}^{M} \pi_i}$$

Solution: Assume that the truncated chain is also irreducible. Simply verify that $\{\overline{\pi}_i\}_{0\le i\le M}$ defined by

$$\overline{\pi}_i = \frac{\pi_i}{\sum_{i=0}^{M} \pi_i},$$

satisfy

$$\overline{P}_{ij}\overline{\pi}_i = \overline{P}_{ji}\overline{\pi}_j\,, \forall 0 \le i, j \le M \quad \text{and} \quad \sum_{i=0}^{M} \overline{\pi}_i = 1\,..$$

Since the original Markov chain is time–reversible, we have

$$P_{ij}\pi_i = P_{ji}\pi_j\,, \forall i, j \ge 0\,.$$

It follows that for any $0 \le i, j \le M$, we have

$$\overline{P}_{ij}\overline{\pi}_i = \frac{\pi_i}{\sum_{i=0}^{M} \pi_i}\left(P_{ij} + \mathbf{1}_{\{i=j\}} \sum_{k>M}^{M} P_{ik}\right) = \frac{\pi_j}{\sum_{i=0}^{M} \pi_i}\left(P_{ji} + \mathbf{1}_{\{i=j\}} \sum_{k>M}^{M} P_{jk}\right) = \overline{P}_{ji}\overline{\pi}_j\,.$$

3. For an ergodic semi-Markov process:
(a) Compute the rate at which the process makes a transition from i into j.
(b) Show that $\sum_i P_{ij}/\mu_{ii} = 1/\mu_{jj}$.
(c) Show that the proportion of time that the process is in state i and headed for state j is $P_{ij}\eta_{ij}/\mu_{ii}$ where $\eta_{ij} = \int_0^\infty \overline{F}_{ij}(t)\,dt$.
(d) Show that the proportion of time that the state is i and will next be j within a time x is

$$\frac{P_{ij}\eta_{ij}}{\mu_{ii}} F_{ij}^e(x),$$

where F_{ij}^e is the equilibrium distribution of F_{ij}

Hint: all parts of this question can be done by defining appropriate renewal-reward processes. For (d), we use the definition $F_{ij}^e(x) = \int_0^x \bar{F}_{ij}(y)\,dy / \int_0^\infty \bar{F}_{ij}(y)$ (see Ross, p131). This is the delay required to make a delayed renewal process with renewal distribution F_{ij} stationary. It arises here since it is also the limiting distribution of the residual life process for a non-lattice renewal process.

Solution: **(a)** Define a (delayed) renewal reward process: a renewal occurs when state i is entered from other states and the reward of each n–th cycle R_n equals 1 if in the n–th cycle, the state after i is j and 0 otherwise. Let $R_{ij}(t)$ be the total number of transitions from i to j by time t. We have

$$\sum_{n=0}^{N(t)} R_n \le R_{ij}(t) \le \sum_{n=0}^{N(t)+1} R_n \le \sum_{n=0}^{N(t)} R_n + 1 \,.$$

Thus the rate at which the process makes a transition from i to j equals

$$\lim_{t\to\infty} \frac{R_{ij}(t)}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]} = \frac{P_{ij}}{\mu_{ii}} \,.$$

(b) Let $R_j(t)$ be the number of visits to state j by time t. Thus

$$\begin{aligned}
\sum_i R_{ij}(t) &= R_j(t) \\
\sum_i \lim_{t\to\infty} \frac{R_{ij}(t)}{t} &= \lim_{t\to\infty} \frac{R_j(t)}{t} \\
\sum_i \frac{P_{ij}}{t} &= \frac{1}{\mu_{jj}}
\end{aligned}$$

(c) Define cycle as in part (a) and the reward in a cycle to be 0 if the transition from i is not into j and T_{ij} the time taken for transition if the transition from i is into j. Thus

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]} = \frac{P_{ij}\mathbb{E}[T_{ij}]}{\mu_{ii}} = \frac{P_{ij}\eta_{ij}}{\mu_{ii}} \,.$$

(d) Define cycle as in last part and the reward in a cycle as 0 if the transition from i is not into j and $\min(x, T_{ij})$ if the transition from i is into j. Thus

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]} = \frac{P_{ij}\mathbb{E}[\min(x, T_{ij})]}{\mu_{ii}} = \frac{P_{ij}\eta_{ij}}{\mu_{ii}} F^e_{ij}(x) \,.$$

4. A taxi alternates between three locations. When it reaches location 1 it is equally likely to go next to either 2 or 3. When it reaches 2 it will next go to 1 with probability 1/3 and to 3 with probability 2/3. From 3 it always goes to 1. The mean times between location i and j are $t_{12} = 20, t_{13} = 30$ and $t_{23} = 30$ (with $t_{ij} = t_{ji}$).
(a) What is the (limiting) probability that the taxi's most recent stop was at location $i, i = 1, 2, 3$?
(b) What is the (limiting) probability that the taxi is heading for location 2?
(c) What fraction of time is the taxi traveling from 2 to 3? Note: Upon arrival at a location the taxi immediately departs.

Solution: First we write the transition matrix:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ 1 & 0 & 0 \end{bmatrix} \,.$$

Now the stationary probabilities can be found by solving $\pi = \pi P$. We have

$$\pi_1 = \frac{6}{14}\,, \pi_2 = \frac{3}{14}\,,\ \text{and}\ \pi_3 = \frac{5}{14}\,.$$

Since $\mu_i = \sum_i P_{ij}\mu_{ij}$, we have $\mu_1 = 25$, $\mu_2 = 80/3$ and $\mu_3 = 30$.

(a) By formula

$$P_i = \frac{\pi_i \mu_i}{\sum_j \pi_j \mu_j}\,,$$

we have

$$P_1 = \frac{15}{38}\,, P_2 = \frac{8}{38}\,,\ \text{and}\ P_3 = \frac{15}{38}\,.$$

They are the correspondingly required limiting probabilities.

(b) Use part (c) of 4.48. Since the taxi can only go to location 2 from location 1, the limiting probability that taxi is headed for location 2 equals

$$P_{12}\eta_{12}(\frac{\mu_1}{P_1})^{-1} = \frac{3}{19}\,.$$

(c) Same argument as in part (b) implies that the proportion of the time that the taxi is traveling from location 2 to location 3 equals

$$P_{23}\eta_{23}(\frac{\mu_2}{P_2})^{-1} = \frac{3}{19}\,.$$

Recommended reading:
Sections 4.5, 4.7, 4.8, 5.1, 5.2. You may skip Section 4.6, which will not be covered in this course.

Supplementary exercise: 4.40
These are optional, but recommended. Do not turn in solutions—they are in the back of the book.

Homework 7 (Stats 620, Winter 2017)

Due Thursday March 23, in class

1. Show that a continuous-time Markov chain is regular, given (a) that $\nu_i < M < \infty$ for all i or (b) that the corresponding embedded discrete-time Markov chain with transition probabilities P_{ij} is irreducible and recurrent.

   Hint: For (a), you may follow the method suggested in the book solution (p. 491).

   Solution:

   (a) Let X_n denote the duration of the nth transition and $N(t)$ the number of transitions up to time t. Define, $\widetilde{X}_n \equiv \nu_i X_n / M$. Thus $\widetilde{X}_n \leq X_n$ and $\widetilde{X}_n \sim \exp(M)$. Let

$$\widetilde{N}(t) = \max\{n : \sum_{i=0}^{n} \widetilde{X}_i \leq t\}.$$

   Then $\widetilde{N}(t)$ is a Poisson random variable with mean Mt and $\widetilde{N}(t) \geq N(t)$ for all t. Thus

$$\begin{aligned} \mathbb{P}[N(t) = \infty] &\leq \mathbb{P}[\widetilde{N}(t) = \infty] \\ &= \lim_{n\to\infty} \sum_{k=n}^{\infty} e^{-M} M^k / k! \\ &= 0 \end{aligned}$$

   (b) Let $N_i(t)$ count visits to i by time t for some i. Recurrence and irreducibility ensure that w.p. 1 there are a finite number of transitions between visits to i. These conditions also imply that $N_i(t)$ is a renewal process, due to which we know that $\mathbb{P}[N_i(t) = \infty] = 0$ for any $t < \infty$. Thus, the number of transitions of the Markov chain, $N(t)$, is also finite.

2. Let $\{X(t), t \geq 0\}$ be a continuous-time Markov chain on the non-negative integers, having transition rates q_{ij}. Let $P(t) = P_{00}(t)$.

   (a) Find $\lim_{t\to 0} \frac{1-P(t)}{t}$.

   (b) Show that $P(t)P(s) \leq P(t+s) \leq 1 - P(s) + P(s)P(t)$.

   (c) Show $|P(t) - P(s)| \leq 1 - P(t-s)$, $s < t$ and conclude that P is continuous.

   Hint: For (a) you should justify your answer but need not prove the necessary limit theorem, so your answer could be quite short! One way to obtain (b) is through two applications of the Chapman-Kolmogorov identity. One way to solve (c) is by algebraic manipulation of (b).

   Solution:

   (a) By Lemma 5.4.1, which is proved as the solution to exercise 5.8 in Ross.

(b) Note that

$$\begin{aligned}
P(t)P(s) &= P_{00}(t)P_{00}(s) \le \sum_{k=0}^{\infty} P_{0k}(t)P_{k0}(s) = P_{00}(t+s) = P(t+s) \\
&= \sum_{k=1}^{\infty} P_{0k}(t)P_{k0}(s) + P_{00}(t)P_{00}(s) \le \sum_{k=1}^{\infty} P_{0k}(s) + P_{00}(t)P_{00}(s) \\
&= (1 - P_{00}(s)) + P(t)P(s)\,.
\end{aligned}$$

(c) By part (b), we have,

$$P(s)P(t-s) \le P(t) \le 1 - P(t-s) + P(t-s)P(s)\,.$$

Thus, substracting $P(s)$ on the inequality above, we obtain

$$P(s)(P(t-s)-1) \le P(t) - P(s) \le 1 - P(t-s) + (P(t-s)-1)P(s)\,. \tag{1}$$

Note that

$$P(s)(P(t-s)-1) \ge P(t-s) - 1$$

and

$$(P(t-s)-1)P(s) \le 0\,,$$

it follows from (**??**) that

$$|P(t) - P(s)| \le 1 - P(t-s)\,.$$

Finally,

$$\begin{aligned}
\lim_{t\to s} |P(t) - P(s)| &\le \lim_{t\to s} 1 - P(t-s) \\
&= 1 - P(0) \\
&= 0
\end{aligned}$$

Thus $P(t)$ is continuous.

3. Suppose that the "state" of a system can be modeled as a two-state continuous-time Markov chain with transition rates $\nu_0 = \lambda, \nu_1 = \mu$. When the state of the system is i, "events" occur in accordance with a Poisson process with rate α_i for $i = 0, 1$. Let $N(t)$ denote the number of events in $(0, t)$.

   (a) Find $\lim_{t\to\infty} N(t)/t$.

   (b) If the initial state is state 0, find $\mathbb{E}[N(t)]$.

   Hint For (a), one approach is to let return times into state 0 form a renewal process, and consider a reward to be the number of "events" in the renewal period. For (b), you are asked to find the exact result for finite t, rather than a limiting result as $t \to \infty$.

   <u>Solution</u>:

   (a) Define a renewal reward process as follows. A renewal occurs when the process enters state 0 and reward in a cycle equals the number of events in that cycle. Let the length of

nth cycle X_n is the sum of time spent in states 0 and 1, say X_{0n} and X_{1n} respectively. Thus $\mathbb{E}[X_n] = \mathbb{E}[X_{0n}] + \mathbb{E}[X_{1n}] = \lambda^{-1} + \mu^{-1}$. Further if R_n is the reward in the nth cycle, with R_{0n} and R_{1n} earned in state 0 and 1 respectively

$$\begin{aligned}\mathbb{E}[R_n] &= \mathbb{E}[R_{0n}] + \mathbb{E}[R_{1n}] = \mathbb{E}[\mathbb{E}[R_{0n}|X_{0n}]] + \mathbb{E}[\mathbb{E}[R_{0n}|X_{0n}]] \\ &= \mathbb{E}[\alpha_0 X_{0n}] + \mathbb{E}[\alpha_1 X_{1n}] \\ &= \alpha_0/\lambda + \alpha_1/\mu\,.\end{aligned} \tag{2}$$

Thus

$$\lim_{t\to\infty} \frac{N(t)}{t} = \lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]} = \frac{\alpha_0\mu + \alpha_1\lambda}{\lambda+\mu}\,.$$

(b) By similar argument as in Equation (**??**), it is clear that $\mathbb{E}[N(t)] = \alpha_0\mathbb{E}[T_0(t)] + \alpha_1\mathbb{E}[T_1(t)]$, where $T_i(t)$ is the time spent in state i up to time t. Thus

$$\begin{aligned}\mathbb{E}[N(t)] &= \alpha_0\mathbb{E}[T_0(t)] + \alpha_1(t - \mathbb{E}[T_0(t)]) \\ &= (\alpha_0 - \alpha_1)\mathbb{E}[T_0(t)] + \alpha_1 t \\ &= \alpha_1 t + (\alpha_0 - \alpha_1)\int_0^t P_{00}(s)\,ds\,.\end{aligned}$$

By the forward equation,

$$P_{00}'(t) = -(\lambda+\mu)P_{00}(t) + \mu\,.$$

With the boundary condition $P_{00}(0) = 1$, we have

$$P_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$$

and

$$\mathbb{E}[T_0(t)] = \frac{\mu t}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\int_0^t e^{-(\lambda+\mu)s}\,ds\,.$$

Finally one can show that

$$\mathbb{E}[N(t)] = \frac{\alpha_0\mu + \alpha_1\lambda}{\lambda+\mu}t + \frac{\alpha_1 - \alpha_0}{(\lambda+\mu)^2}\lambda(e^{-(\lambda+\mu)t} - 1)\,.$$

4. Consider a population in which each individual independently gives birth at an exponential rate λ and dies at an exponential rate μ. In addition, new members enter the population in accordance with a Poisson process with rate θ. Let $X(t)$ denote the population size at time t.

   (a) Explain why $\{X(t), t \geq 0\}$ is a birth/death process. What are its parameters?

   (b) Set up and solve a differential equation to find $\mathbb{E}\big[X(T)|X(0) = i]$.

   <u>Solution</u>:

(a) The Markov property comes from the memorylessness of the exponential distribution for event times. This is a linear birth/death process with immigration, having parameters $\mu_n = n\mu$ and $\lambda_n = n\lambda + \theta$.

(b) Note that

$$\mathbb{E}[X(t+h)|X(0)] = \mathbb{E}[X(t)|X(0)] + (\lambda - \mu)\mathbb{E}[X(t)|X(0)]h + \theta h + o(h)$$

Thus defining $M(t) \equiv \mathbb{E}[X(t)|X(0)]$ we get the differential equation

$$M'(t) = (\lambda - \mu)M(t) + \theta\,.$$

With the initial condition $M(0) = i$, we solve

$$M(t) = \begin{cases} \theta t + i & \text{if } \lambda = \mu \\ (i + \frac{\theta}{\lambda-\mu})e^{(\lambda-\mu)t} - \frac{\theta}{\lambda-\mu} & \text{otherwise} \end{cases}$$

Recommended reading:
Sections 5.3, 5.4, 5.5.

Supplementary exercise: 5.14
Optional, but recommended. Do not turn in a solution—it is in the back of the book.

Homework 8 (Stats 620, Winter 2017)

Due Thursday March 30, in class
Questions are derived from problems in *Stochastic Processes* by S. Ross.

1. Let $\{N(t)\}$ be a Poisson process with rate λ. Since $\{N(t)\}$ is also a continuous time Markov chain, it can also be defined in terms of its transition rates q_{ij} and rates ν_i of leaving i (together with its initial distribution $P_i(0) = \mathbb{P}\{N(0) = i\}$). For some fixed time T, let $N^*(t) = N(T-t)$ for $0 \le t \le T$. $\{N^*(t)\}$ is an inhomogeneous continuous time Markov chain, so it is specified by time-dependent parameters $q^*_{ij}(t)$ and $\nu^*_i(t)$ (together with its initial distribution $P^*_i(0) = \mathbb{P}\{N^*(0) = i\}$).

   (a) Write down expressions for q_{ij}, ν_i and $P_i(0)$.

   (b) Obtain expressions for $q^*_{ij}(t)$, $\nu^*_i(t)$ and $P^*_i(0)$. Note that q^*_{ij} does not depend on λ. Note also that duration in state i viewed in reverse time is different from forward in time (as mentioned in the notes, they are the same if the Markov chain is stationary).

   Hint: One approach is to employ Theorem 2.3.1 of Ross, which you may use without proof.

   Solution: (a) $\{N(t)\}$ has states 0, 1, 2,$\cdots$. The renewal process starts at 0, so $P_0(0) = 1$ and $P_i(0) = 0$ for $i > 0$. From the construction of Poisson processes, we have

$$P[N(t+h) = j | N(t) = i] = \begin{cases} \lambda h + o(h) & \text{for} \quad j = i+1 \\ o(h) & \text{for} \quad j > i+1, \\ 0 & \text{for} \quad j < i. \end{cases}$$

   Hence,

$$q_{ij} = \begin{cases} \lambda & \text{for } j = i+1, \\ 0 & \text{for } j > i+1 \text{ or } j < i. \end{cases}$$

   Since the only nonzero transition rate is $q_{i,i+1} = \lambda$, we have $\nu_i = \lambda$ (and in the Q matrix notation, $q_{i,i} = -\lambda$) for all $i = 0, 1, 2, \cdots$.

   (b) We apply Bayes' rule to get backward transition rates.

$$\begin{aligned} &P[N^*(t+h) = i-1 | N^*(t) = i] = P[N(T-t-h) = i-1 | N(T-t) = i] \\ &= \frac{P[N(T-t-h) = i-1] \cdot P[N(T-t) = i | N(T-t-h) = i-1]}{P[N(T-t) = i]}. \end{aligned}$$

   The above probability can be computed as

$$\frac{\frac{e^{-\lambda(T-t-h)}(\lambda(T-t-h))^{i-1}}{(i-1)!} \cdot (\lambda h + o(h))}{\frac{e^{-\lambda(T-t)}(\lambda(T-t))^{i}}{i!}} = \frac{i}{T-t}(h + o(h)).$$

   Thus we have $q^*_{i,i-1} = \frac{i}{T-t}$. The infinitesimal probabilities

$$P[N(T-t) = i | N(T-t-h) = j] = \begin{cases} o(h) & \text{for} \quad j < i-1 \\ 0 & \text{for} \quad j > i \end{cases}$$

similarly gives that $q^*_{i,j} = 0$ for $j > i$ or $j < i-1$. Therefore, $\lambda^*_i = -\frac{i}{T-t}$. The initial condition of the reverse chain is simply given by the Poisson distribution with rate λT.

$$P^*_i(0) = P_i(T) = \frac{e^{-\lambda T}(\lambda T)^i}{i!}.$$

2. The following problems arises in molecular biology. The surface of a bacterium consists of several sites at which foreign molecules—some acceptable and some not—become attached. We consider a particular site and assume that molecules arrive at the site according to a Poisson process with parameter λ. Among these molecules a proportion α are acceptable. Unacceptable molecules stay at the site for a length of time which is exponentially distributed with parameter μ_1, whereas an acceptable molecule remains at the site for an exponential time with departure rate μ_2. An arriving molecule will become attached only if the site is free of other molecules.

(i) What percentage of the time is the site occupied with an acceptable molecule?

(ii) What fraction of arriving acceptable molecules become attached?

Solution:

Consider a continuous–time Markov chain with 3 states. Define states 0, 1 and 2 as the site being free, attached to an unacceptable molecule and attached to an acceptable molecule respectively. Thus, the transition rate matrix Q is

$$Q = \begin{pmatrix} -\lambda & \lambda\alpha & \lambda(1-\alpha) \\ \mu_2 & -\mu_2 & 0 \\ \mu_1 & 0 & -\mu_1 \end{pmatrix}.$$

(i) We have the balance equations

$$\begin{aligned} \mu_2 P_2 &= \alpha\lambda P_0 & (1) \\ \mu_1 P_1 &= (1-\alpha)\lambda P_0 & (2) \\ P_0 + P_1 + P_2 &= 1 & (3) \end{aligned}$$

The long run fraction of time that an acceptable molecule is attached equals

$$P_1 = \frac{\alpha\mu_2^{-1}}{\lambda^{-1} + (1-\alpha)\mu_1^{-1} + \alpha\mu_2^{-1}}.$$

(ii) Using the PASTA property (Poisson Arrivals See Time Averages), the fraction of arriving acceptable molecules that become attached equals the proportion that the site is free, which equals

$$P_0 = \frac{\lambda^{-1}}{\lambda^{-1} + (1-\alpha)\mu_1^{-1} + \alpha\mu_2^{-1}}.$$

3. An undirected graph has n vertices and edges between all $n(n-1)/2$ vertex pairs. A particle moves along the graph as follows: Events occur along the edge (i,j) according to independent Poisson processes with rates λ_{ij}. An event on edge (i,j) causes the edge to become "excited". If the particle is at vertex i the moment that the edge (i,j) becomes excited then the particle instantaneously moves from to vertex j. Let P_j denote the (limiting) proportion of the time that the particle is at vertex j. Explain why the position of the particle follows a continuous time Markov chain, and hence show that $P_j = 1/n$.

   Hint: use time reversibility.

   Solution:

   Let $X(t)$ denote the position of the particle at time t. Using Definition 2.1.2 of a Poisson process for $j \neq i$

$$\begin{aligned} & \mathbb{P}(X(t+h) = j | X(t) = i, X(s); s < t) \\ = \ & \mathbb{P}(\text{an event only on edge } (i,j) \text{ during } (t, t+h]) + o(h) \\ = \ & h\lambda_{ij} + o(h) \end{aligned}$$

   Thus $\{X(t), t \geq 0\}$ is a continuous–time Markov chain.
   Now $q_{ij} = \lambda_{ij} = \lambda_{ji} = q_{ji}$, since the graph is undirected. It is easy to verify that $P_j = 1/n$ satisfies $\sum_{j=1}^{n} P_j = 1$ and $P_i q_{ij} = P_j q_{ji}$ for $j \neq i$. Thus $\{X(t)\}$ is reversible with stationary distribution $\{P_j\}$. Also P_j is the proportion of the time that the particle is at vertex j.

4. Verify that $X_n/m^n, n \geq 1$, is a martingale when X_n is the size of the n^{th} generation of a branching process whose mean umber of offspring per individual is m.

   Solution:

   To check a sequence of random variables $Z_1, Z_2, \ldots$ is a martingale, we need to check

   (1) $\mathbb{E}|Z_n| < \infty \,, \forall n \in \mathbb{N}$;
   (2) $\mathbb{E}(Z_{n+1}|Z_1, \ldots, Z_n) = Z_n \,, \forall n \in \mathbb{N}$.

   In this case, we have

$$\mathbb{E}[Z_{n+1}|Z_1, \cdots, Z_n] = \mathbb{E}[Z_{n+1}|X_1, \cdots, X_n] = \mathbb{E}[\frac{X_{n+1}}{m^{n+1}}|X_1, \cdots, X_n] = \frac{1}{m^{n+1}} m X_n = Z_n \,,$$

   and

$$\mathbb{E}|Z_n| = \mathbb{E}Z_n = \mathbb{E}Z_1 = 1 \,.$$

   We have thus shown that Z_n is a martingale.

5. Consider the Markov chain which at each transition either goes up 1 with probability p or down 1 with probability $q = 1 - p$. Argue that $(q/p)^{S_n}$, $n \geq 1$, is a martingale.

   Solution:

   Let $S_n = \sum_{i=1}^{n} X_i$. We have

$$\mathbb{E}|Z_n| = \mathbb{E}Z_n \leq \sum_{k=-n}^{n} (q/p)^k < \infty \,, \forall n \in \mathbb{N}$$

and

$$\begin{aligned}
\mathbb{E}[Z_{n+1}|Z_1,\cdots,Z_n] &= \mathbb{E}[Z_{n+1}|S_1,\cdots,S_n] = \mathbb{E}[(q/p)^{X_{n+1}+S_n}||S_1,\cdots,S_n] \\
&= (q/p)^{S_n}\,\mathbb{E}[(q/p)^{X_{n+1}}|S_1,\cdots,S_n] = (q/p)^{S_n}\,\mathbb{E}[(q/p)^{X_{n+1}}] \\
&= Z_n\left((q/p)\,p + (q/p)^{-1}\,q\right) = Z_n\,.
\end{aligned}$$

We have thus shown that Z_n is a martingale.

6. Consider a Markov chain $\{X_n, n \geq 0\}$ with $P_{NN} = 1$. Let $P(i)$ denote the probability that this chain eventually enters state N given that it starts in state i. Show that $\{P(X_n), n \geq 0\}$ is a martingale.

   Hint: One approach involves defining $A = \{X_\infty = N\}$ and showing that $\mathbb{E}\big[\mathbb{P}(A \mid X_n) \mid X_{n-1}] = \mathbb{E}\big[\mathbb{P}(A \mid X_n, X_{n-1}) \mid X_{n-1}\big] = \mathbb{P}(A \mid X_{n-1})$.

   Solution:

   First note that if $Z = g(Y)$ and $\mathbb{E}[X|Y] = h(g(Y))$ then

$$\mathbb{E}[X|Z] = \mathbb{E}[\mathbb{E}[X|Y,Z]|Z] = \mathbb{E}[\mathbb{E}[X|Y]|Z] = \mathbb{E}[h(g(Y))|Z] = \mathbb{E}[h(Z)|Z] = h(Z)\,. \tag{4}$$

   Now let $A = \{X_\infty = N\}$.

$$\begin{aligned}
\mathbb{E}[\mathbb{P}(A|X_{n+1})|X_n,\cdots,X_1] &= \mathbb{E}[\mathbb{P}(A|X_{n+1})|X_n] \text{ (by Markov Property)} \\
&= \mathbb{E}[\mathbb{P}(A|X_{n+1},X_n)|X_n] \text{ (by Markov Property)} \\
&= \mathbb{P}(A|X_n) \text{ (by equation 6.1.2 Ross)}\,.
\end{aligned}$$

   Thus, by (4),

$$\mathbb{E}[\mathbb{P}(A|X_{n+1})|\mathbb{P}(A|X_n),\cdots,\mathbb{P}(A|X_1)] = \mathbb{P}(A|X_n)\,.$$

   It is also clear that $\mathbb{E}(|\mathbb{P}(A|X_n)|) \leq 1$. Hence $\mathbb{P}(A|X_n)$ is a martingale.

Recommended reading:
Sections 5.6 (up to Prop. 5.6.3, i.e. pp 257–261), 6.1, 6.2.

Supplementary exercise: 6.2, 6.7
Optional, but recommended. Do not turn in solutions.