Qualifying Exam of SOR 2021

Instructions:

1. Duration: 3 hours;
2. Only giving the final result without providing the ideas and methods may get no points (unless the question explicitly waives);
3. Open book; open notes.

In all problems, EX denotes the expectation of a random variable X.

Q1 (25 points) Consider a (s, S) inventory policy in a supermarket: Whenever the storage level drops below s, the policy immediately places an order to bring the inventory level back to S. Suppose the customers arrive according to a Poisson process with rateλ and order a random amount of items, which is distributed according to a random variable X having cumulative distribution function F with density f and a mean of m. Assume the replenish time is negligible. Assume the customer whose request takes the inventory level below s cannot receive any of the new orders.

(a) Let T denotes times between successive orders, give an expression for $E(T)$.

(b) Give an expression for the long-run rate that the supermarket places orders to replace its stock.

(c) Let $X(t)$ be the inventory of goods at time t, find the expression of $\mathbf{P}(X(t) \geq x)$ as $t \to \infty$.

(d) Find the expression of $\mathbf{P}(X(t) \geq x)$ as $t \to \infty$ when cdf F is exponential.

(e) Is the cumulative distribution function of the limiting inventory level in part (c) continuous? Briefly explain your answer.

(f) Assume the storage level is ∞ and the goods suffer a certain amount of diminishing. Suppose the diminishing arises in accordance with Poisson process with rate μ. The ith diminishing causes damages with an amount W_i. If the diminishing has an initial damage W, then a time later its damage is $We^{-\alpha t}$.

   (i) Assume the damage in goods is additives. Find an expression of the total damage in goods at time t.

   (ii) Find the expected value of the total damage in goods at time t.

Q2 Let $\{(X_n, Y_n)\}_{n=0}^{\infty}$ be a two dimensional **symmetric** random walk.

(a) Compute $\mathbf{P}\left((X_3, Y_3) = (1, 2)\right)$.

(b) Gives an expression of $\mathbf{P}\left((X_{200}, Y_{200}) = (0, 0)\right)$.

(c) Gives an expression of $\mathbf{P}\left((X_{300}, Y_{300}) = (1, 2)\right)$.

(d) For $n \geq 1$. Let M_{2n} denote the number of returns to $(0, 0)$ by time n. Compute the expectation of M_{2n}

(e) Show that the state $(0, 0)$ is recurrent

(f) Define $T \equiv \inf\{n \geq 0 : \max\left(|X_n|, |Y_n|\right) = 3\}$

i. Find $\mathbf{E}[T]$

ii. Find $\mathbf{P}[X_T = 3, Y_T = 0]$

Q3 Consider an $M/G/1$ queue system with unlimited waiting space. Customers arrive according to a Poisson process with rate $\lambda = 48$ per hour. Assume the service time is i.i.d random variable, distributed as the gamma random variable S with pdf $g(t)$, $\mathbf{E}(S) = 1$ miniute and $\sqrt{Var(S) = 0.5}$. Assume

$$g(t) \equiv g_S(t) \equiv \frac{128t^3e^{-4t}}{3}$$

and Laplace transform

$$\hat{g}(s) \equiv \mathbf{E}[e^{-sS}] \equiv \int_0^\infty e^{-st}g(t)dt = \left(\frac{s}{4+s}\right)^4$$

Upon service complete, each customer receives a reward. The successive reward can be regarded as i.i.d. random variables distributed as W having a gamma distribution with mean 100 and standard deviation 110 and thus Laplace transform

$$\mathbf{E}[e^{-sW}] \equiv \int_0^\infty e^{-st}g(t)dt = \left(\frac{1}{1+121s}\right)^{(1/1.21)}$$

(a) Given 30 customers arrive in an hour, What are the mean and variance of the number of these arrivals that come during the first 20 minutes of that hour?

(b) What is the probability that the first arrival completes service before the second customer arrives? (Assume that the system is initially empty.)

(c) Let $R(t)$ denote the total amount of reward gained by all customers by time t. What distribution does $R(t)$ follows? Finds its mean and variance.

(d) What is the expected conditional total amount reward gain in a given hour, given that the amount reward in the previous hour is exactly two times the mean?

(e) Let $X(t)$ be the number of customers in the system at time t.

  (i) Is $\{X(t) : t \geq 0\}$ an irreducible aperiodic continuous-time Markov chain? Explain.

  (ii) Find random time T_n, $n \geq 0$, such that $\{X(T_n) : n \geq 0\}$is an irreducible aperiodic discrete-time Markov chain with state-space$\{0, 1, 2, ...\}$.

  (iii) Find the transition probability of the DTMC in the previous part.