1. Let $X_1, X_2, \ldots$ be i.i.d. random variables and Y be a discrete random variable taking positive integer values. Assume that Y and X_i's are independent. Let $Z = \sum_{i=1}^{Y} X_i$.
   (a) Obtain the moment generating function of Z. What is the condition that it exits?
   (b) Use part (a) to derive the distribution of Z when X is Exponential(λ) and Y is Geometric(p).
   (c) Show that $\mathsf{E}[Z] = \mathsf{E}[Y]\mathsf{E}[X_1]$.
   (d) Show that $\mathrm{Var}(Z) = \mathsf{E}[Y]\mathrm{Var}(X_1) + \mathrm{Var}(Y)(\mathsf{E}[X_1])^2$

2. Let X_1 and X_2 be independent random variables having the standard normal distribution. Obtain the joint pdf of (Y_1, Y_2), where $Y_1 = \sqrt{X_1^2 + X_2^2}$ and $Y_2 = X_1/X_2$. Are Y_i's independent?

3. Let's verify $(n-1)S_n^2 \sim \chi^2_{n-1}$ directly. Consider the standard normal random vector $X = (X_1, \ldots, X_n)$. Its covariance matrix is the identity matrix $\Sigma = I_n$. This means that X_i and X_j are independent and $\mathrm{Var}(X_i) = 1$.
   (a) Show that for a matrix $A \in \mathbb{R}^n$, if A is orthonormal (i.e., $AA^T = I_n$), then $Y = AX$ (it is a linear transformed random vector) is also a standard normal vector.
   (b) Let A be an orthonormal matrix and its first row be $(n^{-1/2}, \ldots, n^{-1/2})$.[1] So $Y_1 = \sqrt{n}\bar{X}, Y_2, \ldots, Y_n$ is a standard normal random vector. Then by the orthonormality of A, show that $\sum_{i=2}^n Y_i^2 = \sum_{i=1}^n X_i^2 - n\bar{X}^2$. Therefore, $(n-1)S^2$ is χ^2_{n-1}. (Hint: use the fact that $\sum_{i=1}^n Y_i^2 = (AX)^T AX = X^T A^T AX = \sum_{i=1}^n X_i^2$.)

4. Let Y be a Exponential(1) random variable with PDF $f_Y(y) = e^{-y}$. The τth quantile of Y is defined as
$$Q_Y(\tau) = F_Y^{-1}(\tau) := \inf\{y : F_Y(y) \geq \tau\}, \tau \in (0,1).$$
   (a) Find the the τth quantile $Q_Y(\tau)$ of Y for $\tau \in (0,1)$.
   (b) Define the loss function as
$$\rho_\tau(y) := y(\tau - \mathbb{I}_{\{y<0\}}) = \begin{cases} (\tau-1)y, & y < 0, \\ \tau y, & y \geq 0. \end{cases}$$
   Calculate the expected loss $L(u) := \mathsf{E}[\rho_\tau(Y - u)]$ as a function of $u \geq 0$.
   (c) Show that the τth quantile minimizes $L(u)$.

5. Let $U_1, U_2, \ldots$ be independent random variables having the uniform distribution on $[0,1]$ and $Y_n = (\prod_{i=1}^n U_i)^{-1/n}$. Show that
$$\sqrt{n}(Y_n - e) \Rightarrow N(0, e^2).$$
   [Hint: Use delta method.]

6. Let $(X_1, \ldots, X_n)$ be a random sample from the uniform distribution on the interval $[0,1]$ and let $R = X_{(n)} - X_{(1)}$, where $X_{(i)}$ is the ith order statistic. Derive the density of R and find the limiting distribution of $2n(1-R)$ as $n \to \infty$.

[1] Why this is always possible? Think of each row of A as an orthornormal basis in $\mathbb{R}^n$. We fix one of them, we can always find the remaining $n-1$ bases in the subspace. Mathematically, we can use orthonormal decomposition to find rows of A sequentially.