IEDA5270 HW2

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1. Let $X_1, X_2, \ldots$ be i.i.d. random variables and Y be a discrete random variable taking positive integer values. Assume that Y and X_i's are independent. Let $Z = \sum_{i=1}^{Y} X_i$.
   (a) Obtain the moment generating function of Z. What is the condition that it exits?
   (b) Use part (a) to derive the distribution of Z when X is Exponential(λ) and Y is Geometric(p).
   (c) Show that $\mathsf{E}[Z] = \mathsf{E}[Y]\mathsf{E}[X_1]$.
   (d) Show that $\text{Var}(Z) = \mathsf{E}[Y]\text{Var}(X_1) + \text{Var}(Y)(\mathsf{E}[X_1])^2$

(a)
$$
\begin{aligned}
\phi_Z(t) &= \mathbb{E}(e^{Zt}) \\
&= \mathbb{E}(e^{t\sum_{i=1}^{Y} X_i}) \\
&= \mathbb{E}\left(\prod_{i=1}^{Y} e^{tX_i}\right) \\
&= \mathbb{E}\left[\mathbb{E}\left(\prod_{i=1}^{Y} e^{tX_i} \mid Y\right)\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{Y} \mathbb{E}(e^{tX_i} \mid Y)\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{Y} \mathbb{E}(e^{tX_i})\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{Y} \phi_{X_i}(t)\right] \\
&= \mathbb{E}\left[\phi_X^Y(t)\right]
\end{aligned}
$$

$\phi_Z(t)$ exists if $\phi_X(t)$ exist

(b) $X \sim \text{Exp}(\lambda)$, $\quad f(x,\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

$$Pr(Y=k) = (1-p)^{k-1} p$$

$$Z = \sum_{i=1}^{Y} X_i = \sum_{i=1}^{n} 1_{\{Y=n\}} X_i$$

Using the law of total expectation

$$\mathbb{E}[e^{itZ}] = \sum_{n=1}^{\infty} \mathbb{E}[e^{itZ} \mid Y=n]\, \mathbb{P}(Y=n)$$

$$\mathbb{E}[e^{itZ} \mid Y=n] = \mathbb{E}[\exp(it(X_1 + \cdots + \lambda_n)) - \left(\frac{\lambda}{\lambda - it}\right)^n$$

$\phi \quad \prod^{n} \quad \prod^{n} \lambda$

$$= E[\exp(it(X_1+\cdots+X_n))] = \left(\frac{\lambda}{\lambda - it}\right)$$

$$\phi_{X_1+X_2+\cdots X_n}(t) = \prod_{i=1}^{n} \phi_{X_i}(t) = \prod_{i=1}^{n} \frac{\lambda}{\lambda - it}$$

since X_i ar independent of Y, so

$$\sum_{n=1}^{\infty} E[e^{itZ} \mid Y=n] P(Y=n) = \sum_{n=1}^{\infty} \left(\frac{\lambda}{\lambda - it}\right)^n (1-p)^{n-1} p$$

$$= \frac{p\lambda}{\lambda - it} \sum_{m=0}^{\infty} \left[\frac{(1-p)\lambda}{\lambda - it}\right]^m = \frac{p\lambda}{\lambda - it} \frac{1}{1 - \frac{(1-p)\lambda}{\lambda - it}} = \frac{p\lambda}{\lambda - it - \lambda + p\lambda} = \frac{p\lambda}{p\lambda - it}$$

$$\phi_Z(t) = \frac{p\lambda}{p\lambda - it}, \quad Z \sim \exp(p\lambda)$$

(c) $$\mathbb{E}(Z) = \mathbb{E}\left(\sum_{i=1}^{Y} X_i\right)$$

$$= \mathbb{E}\left[\mathbb{E}\left(\sum_{i=1}^{Y} X_i \mid Y\right)\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{Y} \mathbb{E}(X_i)\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{Y} \mathbb{E}(X_1)\right]$$

$$= \mathbb{E}(X_1)\, \mathbb{E}\left[\sum_{i=1}^{Y} (1)\right]$$

$$= \mathbb{E}(Y)\, \mathbb{E}(X_1)$$

(d) $$Var(Z) = Var\left(\sum_{i=1}^{Y} X_i\right)$$

$$= \mathbb{E}\left[Var\left(\sum_{i=1}^{Y} X_i \mid Y\right)\right] + Var\left[\mathbb{E}\left(\sum_{i=1}^{Y} X_i \mid Y\right)\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{Y} Var(X_i \mid Y)\right] + Var\left[\sum_{i=1}^{Y} \mathbb{E}(X_i \mid Y)\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{Y} Var(X_i)\right] + Var\left[\sum_{i=1}^{Y} \mathbb{E}(X_i)\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{Y} Var(X_1)\right] + Var\left[\sum_{i=1}^{Y} \mathbb{E}(X_1)\right]$$

$$= Var(X_1)\, \mathbb{E}\left[\sum_{i=1}^{Y} (1)\right] + \mathbb{E}^2(X_1)\, Var\left[\sum_{i=1}^{Y} (1)\right]$$

$$= \mathbb{E}(Y)\, Var(X_1) + Var(Y)(\mathbb{E}(X_1))^2$$

2. Let X_1 and X_2 be independent random variables having the standard normal distribution. Obtain the joint pdf of (Y_1, Y_2), where $Y_1 = \sqrt{X_1^2 + X_2^2}$ and $Y_2 = X_1/X_2$. Are Y_i's independent?

$$Y_1^2 = X_1^2 + X_2^2 \quad\quad X_1^2$$

tain the joint pdf of (Y_1, Y_2), where $Y_1 = \sqrt{X_1^2 + X_2^2}$ and $Y_2 = X_1/X_2$. Are Y_i's independent?

$$X_1^2 = Y_1^2 - X_2^2 = Y_2^2 X_2^2 \qquad X_2^2 = Y_1^2 - X_1^2 = \frac{X_1^2}{Y_2^2}$$

$$X_1^2 = \frac{Y_1^2 Y_2^2}{1+Y_2^2} \qquad X_2^2 = \frac{Y_1^2}{1+Y_2^2}$$

$$X_1 = \frac{Y_1 Y_2}{\sqrt{1+Y_2^2}}, \quad X_2 = \frac{Y_1}{\sqrt{1+Y_2^2}},$$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{y_2}{\sqrt{1+y_2^2}} & \frac{y_1}{(1+y_2^2)^{\frac{3}{2}}} \\ \frac{1}{\sqrt{1+y_2^2}} & -\frac{y_1 y_2}{(1+y_2^2)^{\frac{3}{2}}} \end{vmatrix} = \left| \frac{-y_1 y_2^2 - y_1}{(1+y_2^2)^2} \right| = \frac{y_1}{1+y_2^2}$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}\left(\frac{y_1 y_2}{\sqrt{1+y_2^2}}, \frac{y_1}{\sqrt{1+y_2^2}}\right)|J|$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}\cdot\frac{y_1^2 y_2^2}{1+y_2^2}} \cdot e^{-\frac{1}{2}\frac{y_1^2}{1+y_2^2}} \cdot \left(\frac{y_1}{1+y_2^2}\right)$$

$$= \frac{y_1 e^{-\frac{1}{2}y_1^2}}{2\pi(1+y_2^2)}$$

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{y_1 e^{-\frac{1}{2}y_1^2}}{2\pi(1+y_2^2)}, & \text{if } 0 \le y_1 < \infty, -\infty < y_2 < \infty \\ 0, & \text{otherwise} \end{cases}$$

$$f_{Y_1}(Y_1 = y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_2$$

$$= \frac{y_1 e^{-\frac{1}{2}y_1^2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+y_2^2} dy_2$$

$$= \frac{y_1 e^{-\frac{1}{2}y_1^2}}{2\pi} \cdot \arctan(y_2)\Big|_{-\infty}^{\infty}$$

$$= \frac{1}{2} y_1 e^{-\frac{1}{2}y_1^2}$$

$$f_{Y_2}(Y_2 = y_2) = \int_0^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_1$$

$$= \frac{1}{2\pi(1+y_2^2)} \int_0^{\infty} y_1 e^{-\frac{1}{2}y_1^2} dy_1$$

$$= \frac{1}{2\pi(1+y_2^2)} \cdot \left(-e^{-\frac{x^2}{2}}\right) \Big|_0^{\infty}$$

$$= \frac{}{2\pi(1+y_2^2)} \cdot (-e^{-}) \Big|_0$$

$$= \frac{1}{\pi(1+y_2^2)}$$

$$f_{Y_1,Y_2}(y_1,y_2) = f_{Y_1}(y_1) \cdot f_{Y_2}(y_2)$$

so Y_i is independent.

3. Let's verify $(n-1)S_n^2 \sim \chi_{n-1}^2$ directly. Consider the standard normal random vector $X = (X_1, \ldots, X_n)$. Its covariance matrix is the identity matrix $\Sigma = I_n$. This means that X_i and X_j are independent and $\mathrm{Var}(X_i) = 1$.

   (a) Show that for a matrix $A \in \mathbb{R}^n$, if A is orthonormal (i.e., $AA^T = I_n$), then $Y = AX$ (it is a linear transformed random vector) is also a standard normal vector.

   (b) Let A be an orthonormal matrix and its first row be $(n^{-1/2}, \ldots, n^{-1/2})$.[1] So $Y_1 = \sqrt{n}\bar{X}, Y_2, \ldots, Y_n$ is a standard normal random vector. Then by the orthonormality of A, show that $\sum_{i=2}^n Y_i^2 = \sum_{i=1}^n X_i^2 - n\bar{X}^2$. Therefore, $(n-1)S^2$ is χ_{n-1}^2. (Hint: use the fact that $\sum_{i=1}^n Y_i^2 = (AX)^T AX = X^T A^T AX = \sum_{i=1}^n X_i^2$.)

(a) $X_i \sim N(0,1)$, so $\mathbb{E}[X]=0$, $\mathbb{E}[XX^T]=I_n$, $Cov(X_i, X_j)=0$

$$\mathbb{E}[Y] = \mathbb{E}[AX] = A\cdot\mathbb{E}[X] = A\cdot 0 = 0$$

$$Cov(Y) = \mathbb{E}[(Y-\mathbb{E}(Y))(Y-\mathbb{E}[Y])^T]$$
$$= \mathbb{E}[YY^T]$$
$$= \mathbb{E}[AX(AX)^T]$$
$$= \mathbb{E}[AXX^TA^T]$$
$$= A\cdot\mathbb{E}[XX^T]\cdot A^T$$
$$= AI_nA^T$$
$$= AA^T$$
$$= I_n$$

So Y is a standard normal vector

(b) $\sum_{i=1}^n Y_i^2 = (AX)^T AX$

(b) $\sum_{i=1}^{n} Y_i^2 = (AX)^T AX$

$= (AX)^T AX$

$= X^T A^T A X$

$= \sum_{i=1}^{n} X_i^2$

$Y_1 = \sqrt{n}\,\bar{X}$, so $Y_1^2 = n\bar{X}^2$

$\sum_{i=2}^{n} Y_i^2 = \sum_{i=1}^{n} Y_i^2 - Y_1^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2$

Now, the chi-square distribution with $n-1$ degrees of freedom is given by: $(n-1)S^2 = \sum_{i=2}^{n} Y_i^2 = \sum_{i=1}^{n} X_i^2 - (X_1)^2$

Therefore, $(n-1)S^2$ is χ^2_{n-1}

4. Let Y be a Exponential(1) random variable with PDF $f_Y(y) = e^{-y}$. The τth quantile of Y is defined as

$$Q_Y(\tau) = F_Y^{-1}(\tau) := \inf\{y : F_Y(y) \geq \tau\}, \tau \in (0,1).$$

(a) Find the the τth quantile $Q_Y(\tau)$ of Y for $\tau \in (0,1)$.

(b) Define the loss function as

$$\rho_\tau(y) := y(\tau - \mathbb{I}_{\{y<0\}}) = \begin{cases} (\tau-1)y, & y < 0, \\ \tau y, & y \geq 0. \end{cases}$$

Calculate the expected loss $L(u) := \mathsf{E}[\rho_\tau(Y-u)]$ as a function of $u \geq 0$.

(c) Show that the τth quantile minimizes $L(u)$.

(a) $\int_0^{Q_Y(\tau)} e^{-y}\,dy = \tau$

$1 - e^{-Q_Y(\tau)} = \tau$

$e^{-Q_Y(\tau)} = 1-\tau$

$Q_Y(\tau) = -\ln(1-\tau)$

(b) $L(u) := \mathbb{E}[\rho_\tau(Y-u)]$

5. Let $U_1, U_2, \dots$ be independent random variables having the uniform distribution on $[0,1]$ and $Y_n = (\prod_{i=1}^n U_i)^{-1/n}$. Show that

$$\sqrt{n}(Y_n - e) \Rightarrow N(0, e^2).$$

[Hint: Use delta method.]

$\sqrt{n}\,(g(\bar{y}) - g(1)) \Rightarrow N(0, (g'(1))^2 1)$

let $g(x) = e^x$

$g(1) = e$

$g'(x) = e^x$

$g'(1) = e$

$(g'(1))^2 = e^2$

so $\sqrt{n}\,(e^{\bar{y}} - e) \Rightarrow N\,(0, e^2)$

note: $e^{\bar{y}} = e^{\frac{1}{n}\sum_{i=1}^{n} Y_i}$

$= e^{\frac{1}{n}\sum_{i=1}^{n} - \log(U_i)}$

$= e^{\sum_{i=1}^{n} \log U_i^{-\frac{1}{n}}}$

$= e^{\log(\prod_{i=1}^{n} U_i^{-\frac{1}{n}})}$

$= \left(\prod_{i=1}^{n} U_i\right)^{-\frac{1}{n}}$

so $\sqrt{n}\,(Y_n - e) \Rightarrow N(0, e^2)$

6. Let $(X_1, \ldots, X_n)$ be a random sample from the uniform distribution on the interval $[0,1]$ and let $R = X_{(n)} - X_{(1)}$, where $X_{(i)}$ is the ith order statistic. Derive the density of R and find the limiting distribution of $2n(1-R)$ as $n \to \infty$.

Joint distribution of order statistic n, s, $(n < s)$

$$f_{X_{(n)}, X_{(s)}}(x, y) = \frac{n!}{(n-1)!(s-n-1)!(n-s)!} (F(x))^{n-1} (F(y) - F(x))^{s-n-1} (1 - F(y))^{n-s} f(x) f(y)$$

Thus, $X_i \sim U(0,1) \Rightarrow F_X(x) = x$ and $f_X(x) = 1$

$$f_{X_{(n)}, X_{(1)}}(x_{(1)}, x_{(n)}) = \frac{n!}{(1-1)!(n-1-1)!(n-n)!} x_{(1)}^0 (x_{(n)} - x_{(1)})^{n-2} (1 - x_{(n)})^0 \cdot 1$$

$$= n(n-1)(x_{(n)} - x_{(1)})^{n-2}, \quad 0 < x_{(1)} < x_{(n)} < 1$$

$\therefore R = X_{(n)} - X_{(1)} \Rightarrow X_{(n)} = X_{(1)} + R \Rightarrow X_{(1)} + R < 1 \Rightarrow X_{(1)} < 1 - R$

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so $|J| = \left| \frac{dX_{(n)}}{dR} \right| = 1$

Joint distribution of $(X_{(1)}, R)$: $f_{X_{(1)},R}(x_{(1)}, r) = f_{X_{(1)},X_{(n)}}(x_{(1)}, r)|J|$

$$= n(n-1)r^{n-2}, \quad x_{(1)} < 1 - r$$

pdf of R: $f_R(r) = \int_0^{1-r} f_{X_{(1)},R}(x_{(1)}, r)\, dx_{(1)}$

$$= \int_0^{1-r} n(n-1)r^{n-2}\, dx_{(1)}$$

$$= n(n-1)r^{n-2}(1-r)$$

Let $Y = 2n(1-R) \Rightarrow R = 1 - \frac{Y}{2n}$

$0 < R < 1 \Rightarrow 0 < 1-R < 1 \Rightarrow 0 < 2n(1-R) < 2n$

so $|J| = \left| \frac{dR}{dY} \right| = \frac{1}{2n}$

pdf of Y: $f_Y(y) = f_R(y)|J|$

$$= n(n-1)\left(1-\frac{y}{2n}\right)^{n-2} \frac{y}{2n} \cdot \frac{1}{2n}$$

$$= \frac{1}{4}\left(1-\frac{1}{n}\right)\left(1-\frac{y}{2n}\right)^{n}\left(1-\frac{y}{2n}\right)^{-2} \cdot y, \quad 0 < y < \infty$$

$$= \frac{1}{4} e^{-\frac{y}{2}} y \quad (\text{as } n \to \infty)$$