Qualifying Exam of SOR 2022

Instructions:

1. Duration: 3 hours;
2. Only giving the final result without providing the ideas and methods may get no points (unless the question explicitly waives);
3. Open book; open notes.

1. (30 points) Let $\{(X_n, Y_n)\}_{n=0}^{\infty}$ be a two dimensional symmetric random walk, i.e.,

$$
\begin{aligned}
& P((X_{n+1}, Y_{n+1}) = (x+1, y)|(X_n, Y_n) = (x, y)) \\
&= P((X_{n+1}, Y_{n+1}) = (x-1, y)|(X_n, Y_n) = (x, y)) \\
&= P((X_{n+1}, Y_{n+1}) = (x, y+1)|(X_n, Y_n) = (x, y)) \\
&= P((X_{n+1}, Y_{n+1}) = (x, y-1)|(X_n, Y_n) = (x, y)) = \frac{1}{4}.
\end{aligned}
$$

and $(X_0, Y_0) = (0, 0)$.

(a) Argue that $X_{n+1} - X_n$ and $Y_{n+1} - Y_n$ are dependent random variables.

Sol. No. Because when $X_{n+1} - X_n \neq 0$, it must be that $Y_{n+1} - Y_n = 0$. $\square$

(b) Let $Z_n = X_n + Y_n$. Show that $\{Z_n : n \geq 0\}$ has the Markov property and is a simple random walk. Calculate $P_{i,i+1}$ and $P_{i,i-1}$.

Sol.

$$
Z_{n+1} = \begin{cases} Z_n + 1 & \text{if } (X_{n+1}, Y_{n+1}) = (X_n + 1, Y_n) \text{ or } (X_{n+1}, Y_{n+1}) = (X_n, Y_n + 1), \\ Z_n - 1 & \text{if } (X_{n+1}, Y_{n+1}) = (X_n - 1, Y_n) \text{ or } (X_{n+1}, Y_{n+1}) = (X_n, Y_n - 1). \end{cases}
$$

$$
\begin{aligned}
P_{i,i+1} &= P\left(Z_{n+1} = i+1 | Z_n = i\right) \\
&= P(X_{n+1} + Y_{n+1} = X_n + Y_n + 1 | X_n + Y_n = i) \\
&= P((X_{n+1}, Y_{n+1}) = (X_n + 1, Y_n) | X_n + Y_n = i) \\
&\quad + P((X_{n+1}, Y_{n+1}) = (X_n, Y_n + 1) | X_n + Y_n = i) \\
&= \frac{1}{2}.
\end{aligned}
$$

Similarly, $P_{i,i-1} = P\left(Z_{n+1} = i-1 | Z_n = i\right) = P\left((X_{n+1}, Y_{n+1}) = (X_n - 1, Y_n)\right) + P\left((X_{n+1}, Y_{n+1}) = (X_n, Y_n - 1)\right) = \frac{1}{2}$. $\square$

(c) Calculate $P(Z_n = 0)$.

Sol. $P\left(Z_n = 0\right) = 0$ if n is an odd number. Otherwise, $P\left(Z_n = 0\right) = \binom{n}{\frac{n}{2}} \frac{1}{2^n}$. $\square$

(d) Calculate $P\left((X_{30}, Y_{30}) = (1, 2)\right)$.

Sol. $P\left((X_{30}, Y_{30}) = (1, 2)\right) \leq P(Z_{30} = 3) = 0$. $\square$

2. (30 points) Consider an $M/G/1$ queueing system with unlimited waiting space. Customers arrive according to a Poisson process with rate λ with inter-arrival times $X_1, X_2, \ldots$. Denote by $N(t)$ the number of arrivals by t. Assume that the service times $Y_1, Y_2, \ldots$ are i.i.d random variables with cdf $F(\cdot)$. Assume that the system is initially empty.

   (a) Given $N(3) = 30$, what is the distribution $N(1)$?

   Sol. Given $N(3) = 30$, the arrival times are i.i.d. uniform random variables on $[0,3]$. $N(1)|N(3) = 30 \sim \mathit{Binomial}\left(30, \frac{1}{3}\right)$. □

   (b) What is the probability that the first arrival completes service before the second customer arrives?

   Sol. The first arrival completes service before the second customer arrives if and only if the service time for the first customer is smaller than the inter-arrival time for the second customer.

$$P(X_2 \geq Y_1) = \int_0^\infty P(X_2 \geq y)dF(y) = \int_0^\infty e^{-\lambda y} dF(y) = E\left[e^{-\lambda Y_1}\right].$$

□

   (c) Let $M(t)$ denote the total number of customers that complete their service by time t. Calculate $E[M(t)]$.

   Sol. Let $S_n = X_1 + \cdots + X_n$.

$$\begin{aligned}
E\left[M(t)\right] &= E\left[E\left[\sum_{i=1}^{N(t)} 1_{\{S_n+Y_n \leq t\}} \middle| N(t)\right]\right] \\
&= E\left[E\left[\sum_{i=1}^{N(t)} 1_{\{U_n+Y_n \leq t\}} \middle| N(t)\right]\right] \\
&= E\left[E\left[\sum_{i=1}^{N(t)} P(Y_n \leq t - U_n) \middle| N(t)\right]\right] \\
&= E\left[E\left[\sum_{i=1}^{N(t)} F(t - U_n) \middle| N(t)\right]\right] \\
&= E\left[E\left[\sum_{i=1}^{N(t)} \frac{1}{t}\int_0^t F(t-u)du \middle| N(t)\right]\right] \\
&= E\left[N(t)\frac{1}{t}\int_0^t F(t-u)du\right] \\
&= \lambda \int_0^t F(u)du.
\end{aligned}$$

□

(d) Suppose that Y_n is an exponential distribution with rate μ and assume that each waiting customer will abandon the system independently if she has waited for an exponential amount of time with mean $\frac{1}{\mu}$ before starting her service. Then the number of customers in the system $\{X(t) : t \geq 0\}$ is a continuous time Markov Chain.

i. Describe the Q matrix of this CTMC.

Sol.

$$q_{i,i+1} = \lambda, \; q_{i,i-1} = i\mu.$$

□

ii. What is the long-run average abandonment rate?

Sol. This is an ergodic birth-death process.

$$\pi_i q_{i,i+1} = \pi_{i+1} q_{i+1,i} \implies \frac{\pi_{i+1}}{\pi_i} = \frac{\lambda}{(i+1)\mu} \implies \pi_i = e^{-\frac{\lambda}{\mu}} \frac{\left(\frac{\lambda}{\mu}\right)^i}{i!}.$$

Total abandon rate is

$$\begin{aligned} \sum_{i=1}^{\infty} (i-1)\mu\pi_i &= -\mu(1-\pi_0) + \left(\sum_{i=1}^{\infty} i\mu\pi_i\right) \\ &= \lambda - \mu e^{-\frac{\lambda}{\mu}} \end{aligned}$$

□

3. (40 points) Consider the following (s, S) inventory policy. Whenever the storage level drops below s, an order is placed which immediately brings the inventory level back to S. Customers arrive according to a Poisson process with rate λ and inter-arrival times $X_1, X_2, \ldots$. The nth arrival requests Y_n amount of items where $Y_1, Y_2, \ldots,$ are i.i.d. exponential random variables with mean $\frac{1}{\mu}$.

(a) Let T denote the time between successive orders. Calculate $E[T]$.

Sol. In the Poisson process generated by the iid exponentials $Y_1, Y_2, \cdots$ with μ, $N(S-s)+1$ is the number of arrivals that will trigger an order. Thus, $T = \sum_{n=1}^{N(S-s)+1} X_n$ and $E[T] = [m(S-s)+1]E[X_1] = \frac{\mu(S-s)+1}{\lambda}$. □

(b) Calculate $\lim_{t\to\infty} P(I(t) \geq x)$ for any given $x \in [s, S]$ where $I(t)$ is the inventory level at time t.

Sol. In the renewal process where a renewal starts when an order is placed, we say the system is on when $I(t) \geq x$ and off otherwise. Then, $N(S-x)+1$ is the number of arrivals to cause inventory to drop below x in a cycle and

$$\lim_{t\to\infty} P\left(I(t) \geq x\right) = \lim_{t\to\infty} P\left(\text{system is on at } t\right) = \frac{m(S-x)+1}{m(S-s)+1} = \frac{\mu(S-x)+1}{\mu(S-s)+1}.$$

□

(c) Let $S_n = Y_1 + \cdots + Y_n$. What is the probability that the total demand in an order cycle exceeds S?

Sol. Let $S_n = Y_1 + \cdots + Y_n$. The cumulative demand between two orders is $S_{N(S-s)+1}$. Since $S_{N(S-s)+1} - (S-s)$ is the residual time of a Poisson process at time $S-s$, it is exponentially distributed with mean $\frac{1}{\mu}$. Thus, $P\left(S_{N(S-s)+1} > S\right) = P\left(S_{N(S-s)+1} - (S-s) > s\right) = e^{-\mu s}$. □

(d) Note that each order cycle starts with S amount of inventory and drops to $S - S_1$, $S - S_2$, ... Suppose that each unit of inventory costs \$1 per unit time. Derive the expected inventory cost in a cycle and using the renewal reward theory to calculate the long-run average inventory cost.

Sol.

$$\begin{aligned}
\lim_{t\to\infty}\frac{\int_0^t I(x)dx}{t} &= \frac{E\left[\int_0^T I(x)dx\right]}{E[T]} \\
&= \frac{E\left[\sum_{n=1}^{N(S-s)+1} X_n\left(S-S_{n-1}\right)\right]}{E[T]} \\
&= \frac{E\left[E\left[\sum_{n=1}^{N(S-s)+1} X_n(S-S_{n-1})\middle| N(S-s)\right]\right]}{E[T]} \\
&= \frac{\frac{1}{\lambda}E\left[\sum_{n=1}^{N(S-s)+1}(S-S_{n-1})\middle| N(S-s)\right]}{E[T]} \\
&= \frac{\frac{1}{\lambda}E\left[S(N(S-s)+1)-\sum_{n=1}^{N(S-s)} S_n\middle| N(S-s)\right]}{E[T]} \\
&= \frac{\frac{1}{\lambda}E\left[S(N(S-s)+1)-\sum_{n=1}^{N(S-s)} U_n\middle| N(S-s)\right]}{E[T]} \\
&= \frac{\frac{1}{\lambda}E\left[S(N(S-s)+1)-N(S-s)\frac{S-s}{2}\middle| N(S-s)\right]}{E[T]} \\
&= \frac{\frac{1}{\lambda}[S(\mu(S-s)+1)-\mu(S-s)\frac{S-s}{2}]}{\frac{\mu(S-s)+1}{\lambda}} \\
&= \frac{\frac{S+s}{2}\mu(S-s)+S}{\mu(S-s)+1}
\end{aligned}$$

where $U_n \sim Uniform[0,S-s]$. $\square$