Question 1: Refer to Lecture 4, slides 23 to 24

Slide 23 shows the optimal solutions to 5 instances of transportation problem with 3 source nodes and 4 destination nodes.

1) In all these optimal solutions, there are at most 6 non-zero flows. Is this a coincidence or is it always true?

2) For the problem of the grand coalition, the shadow price for one constraint is 0. Is this a coincidence or is it always true?

**Solution**

1) It is always true. In this transportation problem, let there be $m$ suppliers and $n$ consumers, then we have $m + n = 7$ constraints. Since the total supply should be equal to the total demand, there is one redundant constraint. The coefficience matrix contains at most $m + n - 1 = 6$ linearly independent row vectors, so we can only obtain at most 6 basic variables. Therefore, the optimal solution only has at most 6 non-zero elements.

2) It is always true. Since one constraint is redundant, the basis matrix $B$ of the primal problem contains at most 6 linearly independent row vectors. Let $c_B$ be the vector of costs of the basic variables of the primal. Let $p = c_B B^{-1}$, $p$ is the basic variables for the dual problem and contains at most 6 elements, i.e., the number of non-zero dual variables is at most 6. Thus, there is always a shadow price with the value of 0.

Question 2: Refer to Lecture 5, slides 13 to 16

It is found that the company does not have enough buses to serve all orders. The company considers two options, rejecting some orders and renting some outside buses. Assume that each order j has a profit $p_j$, and the cost of renting one outside bus is b.

1) Develop a modified network flow model to determine which orders should be rejected.

2) Develop a modified network flow model to determine the number of outside buses to rent.

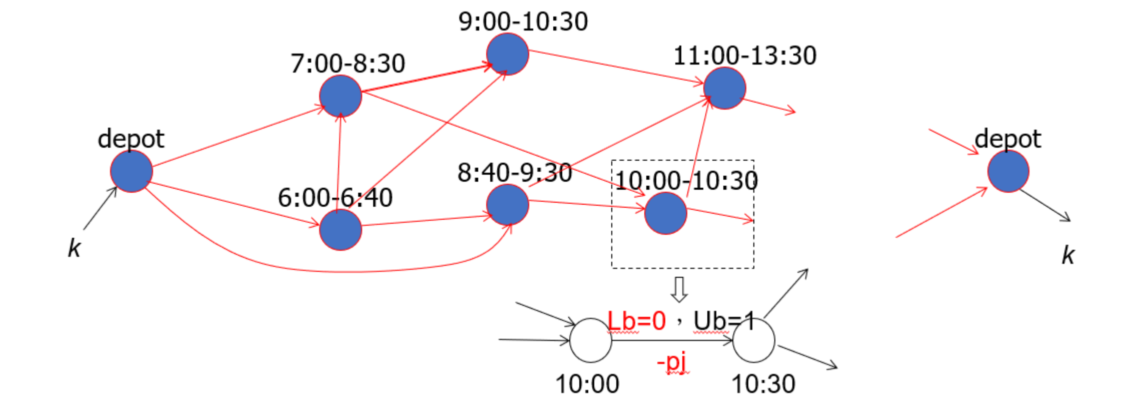
3) Is it possible to build a network flow model which can find an optimal decision that considers both options together?

**Solution**

1) Determine which orders should be rejected. We need to do the following modifications. The lower bound of the arc of the order $j$ should be set as 0. When a flow goes through the arc of the order $j$, the cost is $-p_j$.

Remark: profit = revenue - cost.

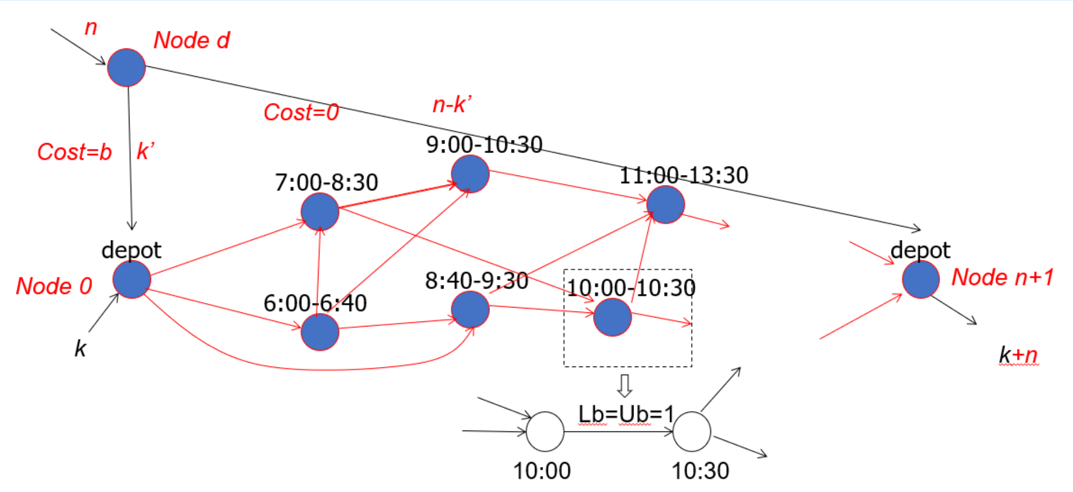
The modified network flow is:




2) Determine the number of outside buses to rent.

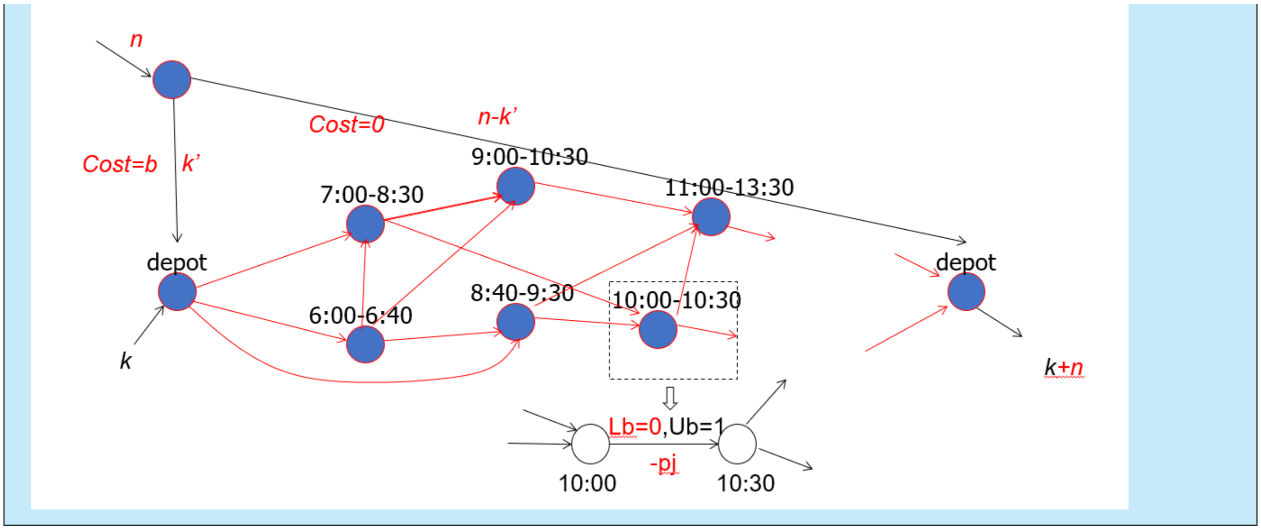
Notice that the cost of renting one outside bus is a constant value, the bus renter should cover the travelling expenses, thus we assume the rented bus should start from the depot.

Add one dummy node, $d$, connecting depot nodes $0, n+1$. We can easily find that the number of outside buses to rent is no larger than $n$. (Or you set it as a large integer $M$.) Let the amount of flow into node $d$ is $n$. The number of rented buses is the flow from node d to node 0, the lower bound is 0, the upper bound is n, the corresponding cost is b; the number of rest buses is the flow from node d to node n+1, the lower bound is 0, the upper bound is n, the corresponding cost is 0.




Remark: If you assume that there are $k'$ buses to rent, when changing $k$ to $k+k'$, this problem will not be a network flow problem. Because the flow into the depot remains constant in the network flow model, the use of an integer variable for $k'$ in this context is not consistent.

3) Yes. We can combine all the changes in 1) and 2) together to obtain the modified model.




If you write the model to represent, I have the following for you to refer to.

1) Suppose there are $n$ nodes representing the orders. The set of nodes can be expressed as $N = \{1, 2, \ldots, n\}$. For each order $j$, we split the node $j$ to two nodes $j$ and $j'$. Besides, add two nodes, node 0 and $n+1$, which represent the initial and end depots, respectively. Let $A$ be the set of all possible arcs. We denote by $\delta^-(i)$ the set of flows into the node $i$, $\delta^+(i)$ the set of flows out of the node $i$. We use $x_{ij}$ to denote the amount of flow through arc $(i, j)$. Let the upper bound of flow $x_{ij}$ be $n$. Numbers $c_{ij}$ represent the cost per unit of flow along arc $(i, j)$. The objective function is to minimize the total flow costs.

The constraint $1 \le x_{jj'} \le 1$ should be changed to $0 \le x_{jj'} \le 1$. And the term $\sum_j -x_{jj'}p_j$ should be added to the objective function.

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} - \sum_j x_{jj'}p_j \\
s.t. \quad & \sum_{i\in\delta^+(0)} x_{0i} = k \text{ (For node 0)} \\
& \sum_{i\in\delta^-(n+1)} x_{i,n+1} = k \text{ (For node n+1)} \\
& \sum_{j\in\delta^-(i)} x_{ji} = \sum_{j\in\delta^+(i)} x_{ij}, \ \forall i \in N \text{ (For node 1 to n)} \\
& 0 \le x_{jj'} \le 1, \ \forall j \\
& 0 \le x_{ij} \le n, \ \forall (i,j) \in A
\end{aligned}
$$

Remark: In this case, the upper bound of flow $x_{ij}$ could be 1. Please model it as a network flow problem.

For the optimal solution, if $x_{jj'} = 0$, then order $j$ is rejected.

2) After these modifications, let $A'$ be the new set of all arcs. (We just add one dummy node and two arcs.)

The flow from node $d$ to node 0 is exactly the number of outside buses to rent, denoted by $x_{d,0}$. The cost of this flow, $c_{d,0}$, is $b$. The flow from node $d$ to node $n+1$ is $n - x_{d,0}$ and the corresponding cost is 0.

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in A'} c_{ij}x_{ij} \\
s.t. \quad & \sum_{i\in\delta^+(0)} x_{0i} = k + x_{d,0} \\
& \sum_{i\in\delta^-(n+1)} x_{i,n+1} = k + n \\
& \sum_{j\in\delta^-(i)} x_{ji} = \sum_{j\in\delta^+(i)} x_{ij}, \ \forall i \in N \\
& x_{d,0} + x_{d,n+1} = n \text{ (For node d)} \\
& 1 \le x_{jj'} \le 1, \ \forall j \\
& 0 \le x_{ij} \le n, \ \forall (i,j) \in A'
\end{aligned}
$$

For the optimal solution, $x_{d,0}$ is the number of outside buses to rent.

3)

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in A'} c_{ij}x_{ij} - \sum_{j} x_{jj'}p_j \\
s.t. \quad & \sum_{i\in\delta^+(0)} x_{0i} = k \\
& \sum_{i\in\delta^-(n+1)} x_{i,n+1} = k \\
& x_{d,0} + x_{d,n+1} = n \\
& \sum_{j\in\delta^-(i)} x_{ji} = \sum_{j\in\delta^+(i)} x_{ij}, \ \forall i \in N \\
& 0 \le x_{jj'} \le 1, \ \forall j \\
& 0 \le x_{ij} \le n, \ \forall (i,j) \in A'
\end{aligned}
$$