1. Slide 7. Sensitivity analysis of Product 2 gives "Allowable Increase = 16.82" and "Allowable Decrease = 1.16."

a. Show how the above two values are calculated.

b. Explain how the optimal solution may (or may not) change when the parameter 25.00 changes within/beyond the range.

Solution

a. $\mathbf{c}_B = (18, 25, 12, 15)$.

$$\mathbf{B}^{-1} = \begin{pmatrix} 0.972 & -0.764 & 0.290 & 0.077 \\ -0.149 & 0.312 & -0.533 & 0.314 \\ -0.814 & -0.348 & 0.560 & 0.511 \\ 0.054 & 1.022 & 0.689 & -1.000 \end{pmatrix}$$

Non-basic variables are x_3, y_1, y_2, y_3, y_4. The corresponding $\mathbf{N}$ is $(A_3, \mathbf{I})$.

When c_2 increases to $c_2 + h$, the reduced cost:

$(c_1, c_2 + h, c_4, c_5)\mathbf{B}^{-1}(A_3, \mathbf{I}) - (c_3, 0, 0, 0, 0) \geq 0$.

We have

$$\begin{aligned} 23.522 - 0.141h &\geq 10 \\ 4.82 - 0.149h &\geq 0 \\ 5.20 + 0.312h &\geq 0 \\ 8.96 - 0.533h &\geq 0 \\ 0.36 + 0.314h &\geq 0 \end{aligned}$$

At $h = -1.15$, the fifth inequality reaches equality, and at $h = 16.81$, the fourth inequality reaches equality. Thus, $-1.15 \leq h \leq 16.81$. The final result can be in a range due to the rounding.

b. Within the range: the optimal solution remains unchanged because the non-basic variable x_3 and the basic variables x_1, x_2, x_4, x_5 remain the same. Therefore, $x_B = \mathbf{B}^{-1}\mathbf{b}$ remains the same. (The optimal value will change due to the change of c_2.)

Beyond the range: the optimal solution will change because as h increases over 16.82, y_3 will not be the non-basic variable; as h decreases below -1.15, y_4 will not be the non-basic variable.

2. Slide 11. Sensitivity analysis of M2 gives "Allowable Increase = 30.36" and "Allowable Decrease = 15.31."

a. Show how the above two values are calculated.

b. Explain how the optimal solution may (or may not) change when the parameter 200.00 changes within/beyond the range.

Solution

a. Method 1:

From the perspective of primal, $x_{\mathbf{B}} = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$

$\mathbf{B}^{-1}$ is the same as above, $\mathbf{b} = (160, 200, 120, 280)^T$.

When b_2 increases to $b_2 + h$, the basic feasible solution satisfies:

$$\begin{aligned} 58.96 - 0.76h &\geq 0 \\ 62.63 + 0.31h &\geq 0 \\ 10.576 - 0.348h &\geq 0 \\ 15.64 + 1.02h &\geq 0 \end{aligned}$$

We have $-15.33 \leq h \leq 30.39$.

Method 2: From the perspective of dual, $\mathbf{c}_B = (160, 200, 120, 280)$.

$$(\mathbf{B}^{-1})^T = \begin{pmatrix} 0.972 & -0.764 & 0.290 & 0.077 \\ -0.149 & 0.312 & -0.533 & 0.314 \\ -0.814 & -0.348 & 0.560 & 0.511 \\ 0.054 & 1.022 & 0.689 & -1.000 \end{pmatrix}$$

When b_2 increases to $b_2 + h$, the reduced cost:

$(b_1, b_2 + h, b_3, b_4)\mathbf{B}^{-1} \geq (0, 0, 0, 0)$.

We have $-15.33 \leq h \leq 30.39$.

b. Within the range: the optimal solution will change due to a modification in the right-hand-side parameter. Although the basic and non-basic variables remain the same, the alteration of the right-hand-side parameter will lead to a corresponding change in $x_{\mathbf{B}} = \mathbf{B}^{-1}\mathbf{b}$. Consequently, the optimal value will also be affected. (The optimal solution of dual remains unchanged because the non-basic and basic variables remains unchanged.)

Beyond the range: the optimal solution will change because as h increases over 30.39, x_4 will not be the basic variable; as h decreases below -15.33, x_5 will not be the basic variable. (When b_2 is negative, the problem is not feasible.)

3. Slide 10. Consider the LP in standard form.

a. Write the dual of the LP.

b. Use this pair of primal/dual to demonstrate the proof of strong duality in slide 27.

Solution

a. Let $\boldsymbol{\pi}$ be the dual variable associated with the constraints in the primal, the dual is written as:

$$
\begin{aligned}
(D)\min \quad & 160\pi_1 + 200\pi_2 + 120\pi_3 + 280\pi_4 \\
s.t. \quad & 1.2\pi_1 + 0.7\pi_2 + 0.9\pi_3 + 1.4\pi_4 \geq 18 \\
& 1.3\pi_1 + 2.2\pi_2 + 0.7\pi_3 + 2.8\pi_4 \geq 25 \\
& 0.7\pi_1 + 1.6\pi_2 + 1.3\pi_3 + 0.5\pi_4 \geq 10 \\
& 0\pi_1 + 0.5\pi_2 + 1.0\pi_3 + 1.2\pi_4 \geq 12 \\
& 0.5\pi_1 + 1.0\pi_2 + 0.8\pi_3 + 0.6\pi_4 \geq 15 \\
& \pi_i \geq 0, i = 1, 2, 3, 4
\end{aligned}
$$

b. For the primal, the optimal solution is
$\mathbf{P}^* = (\mathbf{P}_B^T, \mathbf{P}_N) = (58.96, 62.63, 10.58, 15.64, 0, 0, 0, 0, 0)$.

Let $\boldsymbol{\pi} = \mathbf{c}_B\mathbf{B}^{-1} = (4.82, 5.20, 8.96, 0.36)$, we have $\boldsymbol{\pi}A = \boldsymbol{\pi}(\mathbf{B}, \mathbf{N}) = (\mathbf{c}_B, \mathbf{c}_B\mathbf{B}^{-1}\mathbf{N}) = (18, 25, 12, 15, 23.53, 4.82, 5.20, 8.96, 0.36) \geq (18, 25, 12, 15, 10, 0, 0, 0, 0) = (\mathbf{c}_B, \mathbf{c}_N) = \mathbf{c}$, indicating $\boldsymbol{\pi}$ is feasible to (D).

Since $\boldsymbol{\pi}\mathbf{b} = (4.82, 5.20, 8.963, 0.363) \cdot (160, 200, 120, 280)^T = 2988.4$ and
$\mathbf{c}_B\mathbf{P}_B = (18, 25, 12, 15) \cdot (58.96, 62.63, 10.576, 15.64)^T = 2988.542$, we can obtain $\boldsymbol{\pi}\mathbf{b} \approx \mathbf{c}_B\mathbf{P}_B = \mathbf{c}\mathbf{P}^*$, indicating $\boldsymbol{\pi}$ is optimal to (D). This demonstrates the proof of strong duality.

Remark: The '$\approx$' results from the rounding, theoretically, it is the equal sign.