IEDA5270 HW2

2023年9月17日 13:38

1. Let $X_1, X_2, \ldots$ be i.i.d. random variables and Y be a discrete random variable taking positive integer values. Assume that Y and X_i's are independent. Let $Z = \sum_{i=1}^{Y} X_i$.
   (a) Obtain the moment generating function of Z. What is the condition that it exits?
   (b) Use part (a) to derive the distribution of Z when X is Exponential(λ) and Y is Geometric(p).
   (c) Show that $\mathsf{E}[Z] = \mathsf{E}[Y]\mathsf{E}[X_1]$.
   (d) Show that $\mathrm{Var}(Z) = \mathsf{E}[Y]\mathrm{Var}(X_1) + \mathrm{Var}(Y)(\mathsf{E}[X_1])^2$

(a) $\phi_Z(t) = \mathbb{E}(e^{Zt})$

$$= \mathbb{E}(e^{t\sum_{i=1}^{Y} X_i})$$

$$= \mathbb{E}(\prod_{i=1}^{Y} e^{tX_i})$$

$$= \mathbb{E}[\mathbb{E}(\prod_{i=1}^{Y} e^{tX_i} \mid Y)]$$

$$= \mathbb{E}[\prod_{i=1}^{Y} \mathbb{E}(e^{tX_i} \mid Y)]$$

$$= \mathbb{E}[\prod_{i=1}^{Y} \mathbb{E}(e^{tX_i})]$$

$$= \mathbb{E}[\prod_{i=1}^{Y} \phi_{X_i}(t)]$$

$$= \mathbb{E}[\phi_X^Y(t)]$$

$\phi_Z(t)$ exists if $\phi_X(t)$ exist

(b) $X \sim \mathrm{Exp}(\lambda)$, $f(x,\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

$$\Pr(Y=k) = (1-p)^{k-1} p$$

$$Z = \sum_{i=1}^{Y} X_i = \sum_{i=1}^{n} 1_{\{Y=n\}} X_i$$

Using the law of total expectation

$$\mathbb{E}[e^{itZ}] = \sum_{n=1}^{\infty} \mathbb{E}[e^{itZ} \mid Y=n]\, \mathbb{P}(Y=n)$$

$$\mathbb{E}[e^{itZ} \mid Y=n] = \mathbb{E}[\exp(it(X_1 + \cdots + \lambda_n)) - (\frac{\lambda}{\lambda - it})^n$$

ϕ ... $\prod^{n}$ ... $\prod^{n}$ λ

$$\cdots (\lambda - it)$$

$$\phi_{X_1+X_2+\cdots X_n}(t) = \prod_{i=1}^{n} \phi_{X_i}(t) = \prod_{i=1}^{n} \frac{\lambda}{\lambda - it}$$

since X_i ar independent of Y, so

$$\sum_{n=1}^{\infty} \mathbb{E}[e^{itZ} \mid Y=n] \mathbb{P}(Y=n) = \sum_{n=1}^{\infty} \left(\frac{\lambda}{\lambda-it}\right)^n (1-p)^{n-1} p$$

$$= \frac{p\lambda}{\lambda-it} \sum_{m=0}^{\infty} \left[\frac{(1-p)\lambda}{\lambda-it}\right]^m = \frac{p\lambda}{\lambda-it} \frac{1}{1-\frac{(1-p)\lambda}{\lambda-it}} = \frac{p\lambda}{\lambda-it-\lambda+p\lambda} = \frac{p\lambda}{p\lambda-it}$$

$$\phi_Z(t) = \frac{p\lambda}{p\lambda-it}, \quad Z \sim \exp(p\lambda)$$

(c) $\mathbb{E}(Z) = \mathbb{E}\left(\sum_{i=1}^{Y} X_i\right)$

$$= \mathbb{E}\left[\mathbb{E}\left(\sum_{i=1}^{Y} X_i \mid Y\right)\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{Y} \mathbb{E}(X_i)\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{Y} \mathbb{E}(X_1)\right]$$

$$= \mathbb{E}(X_1)\,\mathbb{E}\left[\sum_{i=1}^{Y}(1)\right]$$

$$= \mathbb{E}(Y)\,\mathbb{E}(X_1)$$

(d) $Var(Z) = Var\left(\sum_{i=1}^{Y} X_i\right)$

$$= \mathbb{E}\left[Var\left(\sum_{i=1}^{Y} X_i \mid Y\right)\right] + Var\left[\mathbb{E}\left(\sum_{i=1}^{Y} X_i \mid Y\right)\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{Y} Var(X_i \mid Y)\right] + Var\left[\sum_{i=1}^{Y} \mathbb{E}(X_i \mid Y)\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{Y} Var(X_i)\right] + Var\left[\sum_{i=1}^{Y} \mathbb{E}(X_i)\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{Y} Var(X_1)\right] + Var\left[\sum_{i=1}^{Y} \mathbb{E}(X_1)\right]$$

$$= Var(X_1)\,\mathbb{E}\left[\sum_{i=1}^{Y}(1)\right] + \mathbb{E}^2(X_1)\,Var\left[\sum_{i=1}^{Y}(1)\right]$$

$$= \mathbb{E}(Y)\,Var(X_1) + Var(Y)(\mathbb{E}(X_1))^2$$

2. Let X_1 and X_2 be independent random variables having the standard normal distribution. Obtain the joint pdf of (Y_1, Y_2), where $Y_1 = \sqrt{X_1^2 + X_2^2}$ and $Y_2 = X_1/X_2$. Are Y_i's independent?

$$Y_1^2 = X_1^2 + X_2^2 \cdots X_1^2$$

tain the joint pdf of (Y_1, Y_2), where $Y_1 = \sqrt{X_1^2 + X_2^2}$ and $Y_2 = X_1/X_2$. Are Y_i's independent?

$$X_1^2 = Y_1^2 - X_2^2 = Y_2^2 X_2^2 \qquad X_2^2 = Y_1^2 - X_1^2 = \frac{X_1^2}{Y_2^2}$$

$$X_1^2 = \frac{Y_1^2 Y_2^2}{1+Y_2^2} \qquad X_2^2 = \frac{Y_1^2}{1+Y_2^2}$$

$$X_1 = \frac{Y_1 Y_2}{\sqrt{1+Y_2^2}}, \quad X_2 = \frac{Y_1}{\sqrt{1+Y_2^2}},$$

$$|J| = \left|\begin{matrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{matrix}\right| = \left|\begin{matrix} \frac{y_2}{\sqrt{1+y_2^2}} & \frac{y_1}{(1+y_2^2)^{\frac{3}{2}}} \\ \frac{1}{\sqrt{1+y_2^2}} & -\frac{y_1 y_2}{(1+y_2^2)^{\frac{3}{2}}} \end{matrix}\right| = \left|\frac{-y_1 y_2^2 - y_1}{(1+y_2^2)^2}\right| = \frac{y_1}{1+y_2^2}$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}\left(\frac{y_1 y_2}{\sqrt{1+y_2^2}}, \frac{y_1}{\sqrt{1+y_2^2}}\right)|J|$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}\cdot\frac{y_1^2 y_2^2}{1+y_2^2}} \cdot e^{-\frac{1}{2}\frac{y_1^2}{1+y_2^2}} \cdot \left(\frac{y_1}{1+y_2^2}\right)$$

$$= \frac{y_1 e^{-\frac{1}{2}y_1^2}}{2\pi(1+y_2^2)}$$

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{y_1 e^{-\frac{1}{2}y_1^2}}{2\pi(1+y_2^2)}, & \text{if } 0 \le y_1 < \infty, -\infty < y_2 < \infty \\ 0, & \text{otherwise} \end{cases}$$

$$f_{Y_1}(Y_1 = y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_2$$

$$= \frac{y_1 e^{-\frac{1}{2}y_1^2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+y_2^2}\, dy_2$$

$$= \frac{y_1 e^{-\frac{1}{2}y_1^2}}{2\pi} \cdot \arctan(y_2)\Big|_{-\infty}^{\infty}$$

$$= \frac{1}{2} y_1 e^{-\frac{1}{2}y_1^2}$$

$$f_{Y_2}(Y_2 = y_2) = \int_0^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_1$$

$$= \frac{1}{2\pi(1+y_2^2)} \int_0^{\infty} {}_1 e^{-\frac{1}{2}y_1^2}\, dy_1$$

$$= \frac{1}{2\pi(1+y_2^2)} \cdot \left(-e^{-\frac{x^2}{2}}\right) \Big|_0^{\infty}$$

$$= \frac{}{2\pi(1+y_2^2)} \cdot (-e^{-}) \Big|_0$$

$$= \frac{1}{\pi(1+y_2^2)}$$

$$f_{Y_1,Y_2}(y_1,y_2) = f_{Y_1}(y_1) \cdot f_{Y_2}(y_2)$$

so Y_i is independent.

3. Let's verify $(n-1)S_n^2 \sim \chi^2_{n-1}$ directly. Consider the standard normal random vector $X = (X_1, \ldots, X_n)$. Its covariance matrix is the identity matrix $\Sigma = I_n$. This means that X_i and X_j are independent and $\mathrm{Var}(X_i) = 1$.
   (a) Show that for a matrix $A \in \mathbb{R}^n$, if A is orthonormal (i.e., $AA^T = I_n$), then $Y = AX$ (it is a linear transformed random vector) is also a standard normal vector.
   (b) Let A be an orthonormal matrix and its first row be $(n^{-1/2}, \ldots, n^{-1/2})$.[1] So $Y_1 = \sqrt{n}\bar{X}, Y_2, \ldots, Y_n$ is a standard normal random vector. Then by the orthonormality of A, show that $\sum_{i=2}^n Y_i^2 = \sum_{i=1}^n X_i^2 - n\bar{X}^2$. Therefore, $(n-1)S^2$ is χ^2_{n-1}. (Hint: use the fact that $\sum_{i=1}^n Y_i^2 = (AX)^T AX = X^T A^T AX = \sum_{i=1}^n X_i^2$.)

(a) $X_i \sim N(0,1)$, so $\mathbb{E}[X] = 0$, $\mathbb{E}[XX^T] = I_n$, $\mathrm{Cov}(X_i, X_j) = 0$

$$\mathbb{E}[Y] = \mathbb{E}[AX] = A \cdot \mathbb{E}[X] = A \cdot 0 = 0$$

$$\begin{aligned}
\mathrm{Cov}(Y) &= \mathbb{E}[(Y-\mathbb{E}(Y))(Y-\mathbb{E}[Y])^T] \\
&= \mathbb{E}[YY^T] \\
&= \mathbb{E}[AX(AX)^T] \\
&= \mathbb{E}[AXX^TA^T] \\
&= A \cdot \mathbb{E}[XX^T] \cdot A^T \\
&= AI_nA^T \\
&= AA^T \\
&= I_n
\end{aligned}$$

So Y is a standard normal vector

(b) $\sum_{i=1}^n Y_i^2 = (AX)^T AX$

(b) $\sum_{i=1}^{n} Y_i^2 = (AX)^T AX$

$= (AX)^T AX$

$= X^T A^T A X$

$= \sum_{i=1}^{n} X_i^2$

$Y_1 = \sqrt{n}\,\bar{X}$, so $Y_1^2 = n\bar{X}^2$

$\sum_{i=2}^{n} Y_i^2 = \sum_{i=1}^{n} Y_i^2 - Y_1^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2$

Now, the chi-square distribution with $n-1$ degrees of freedom is given by : $(n-1)S^2 = \sum_{i=2}^{n} Y_i^2 = \sum_{i=1}^{n} X_i^2 - (X_1)^2$

Therefore, $(n-1)S^2$ is χ^2_{n-1}

4. Let Y be a Exponential(1) random variable with PDF $f_Y(y) = e^{-y}$. The τth quantile of Y is defined as

$$Q_Y(\tau) = F_Y^{-1}(\tau) := \inf\{y : F_Y(y) \geq \tau\}, \tau \in (0,1).$$

(a) Find the the τth quantile $Q_Y(\tau)$ of Y for $\tau \in (0,1)$.

(b) Define the loss function as

$$\rho_\tau(y) := y(\tau - \mathbb{I}_{\{y<0\}}) = \begin{cases} (\tau-1)y, & y < 0, \\ \tau y, & y \geq 0. \end{cases}$$

Calculate the expected loss $L(u) := \mathsf{E}[\rho_\tau(Y-u)]$ as a function of $u \geq 0$.

(c) Show that the τth quantile minimizes $L(u)$.

(a) $\int_0^{Q_Y(\tau)} e^{-y}\,dy = \tau$

$1 - e^{-Q_Y(\tau)} = \tau$

$e^{-Q_Y(\tau)} = 1-\tau$

$Q_Y(\tau) = -\ln(1-\tau)$

(b) $\rho_\tau(Y-u) = \begin{cases} (\tau-1)(y-u), & y < u \\ \tau(y-u), & y \geq u \end{cases}$

$$L(u) = \mathbb{E}[\rho_\tau(Y-u)]$$

$$= \int_{-\infty}^{\infty} \rho_\tau(y-\theta) f_Y(y)\,dy$$

$$= (\tau-1)\int_{-\infty}^{u}(y-u)f_Y(y)\,dy + \tau\int_{u}^{\infty}(y-u)f_Y(y)\,dy$$

$$= (\tau-1)\left[\int_{-\infty}^{u} y f_Y(y)\,dy - u\int_{-\infty}^{u} f_Y(y)\,dy\right] + \tau\left[\int_{u}^{\infty} y f_Y(y)\,dy - u\int_{u}^{\infty} f_Y(y)\,dy\right]$$

$$= \tau\,\mathbb{E}[Y] - \int_{-\infty}^{u} y f_Y(y)\,dy - \tau + u\int_{-\infty}^{u} f_Y(y)\,dy$$

$$= u\int_{-\infty}^{u} e^{-y}\,dy - \int_{-\infty}^{u} y e^{-y}\,dy$$

(c) $$\mathbb{E}[\rho_\tau(Y - Q_Y(\tau))] = (\tau-1)\int_{-\infty}^{Q_Y(\tau)}(y - Q_Y(\tau))f_Y(y)\,dy + \tau\int_{Q_Y(\tau)}^{\infty}(y - Q_Y(\tau))f_Y(y)\,dy$$

$$= (\tau-1)\left[\int_{-\infty}^{Q_Y(\tau)} y f_Y(y)\,dy - Q_Y(\tau)\int_{-\infty}^{Q_Y(\tau)} f_Y(y)\,dy\right]$$

$$+ \tau\left[\int_{Q_Y(\tau)}^{\infty} y f_Y(y)\,dy - Q_Y(\tau)\int_{Q_Y(\tau)}^{\infty} f_Y(y)\,dy\right]$$

$$= (\tau-1)\left(\mathbb{E}[Y] - Q_Y(\tau)\tau\right) + \tau\left(\mathbb{E}[Y] - Q_Y(\tau)(1-\tau)\right)$$

$$= \mathbb{E}[Y] - Q_Y(\tau)$$

5. Let $U_1, U_2, \ldots$ be independent random variables having the uniform distribution on $[0,1]$ and $Y_n = (\prod_{i=1}^n U_i)^{-1/n}$. Show that

$$\sqrt{n}(Y_n - e) \Rightarrow N(0, e^2).$$

[Hint: Use delta method.]

$$\sqrt{n}\,(g(\bar{y}) - g(1)) \Rightarrow N(0, (g'(1))^2 1)$$

let $g(x) = e^x$

$g(1) = e$

$g'(x) = e^x$

$g'(1) = e$

$(g'(1))^2 = e^2$

so $\sqrt{n}\,(e^{\bar{y}} - e) \Rightarrow N(0, e^2)$

note: $e^{\bar{y}} = e^{\frac{1}{n}\sum_{i=1}^n Y_i}$

$= e^{\frac{1}{n}\sum_{i=1}^n -\log(U_i)}$

$= e^{\sum_{i=1}^n \log U_i^{-\frac{1}{n}}}$

$= e^{\log(\prod_{i=1}^n U_i^{-\frac{1}{n}})}$

$= (\prod_{i=1}^n U_i)^{-\frac{1}{n}}$

so $\sqrt{n}\,(Y_n - e) \Rightarrow N(0, e^2)$

6. Let $(X_1, \ldots, X_n)$ be a random sample from the uniform distribution on the interval $[0,1]$ and let $R = X_{(n)} - X_{(1)}$, where $X_{(i)}$ is the ith order statistic. Derive the density of R and find the limiting distribution of $2n(1-R)$ as $n \to \infty$.

Joint distribution of order statistic n, s, $(n < s)$

$$f_{X_{(n)}, X_{(s)}}(x, y) = \frac{n!}{(n-1)!(s-n-1)!(n-s)!} (F(x))^{n-1} (F(y) - F(x))^{s-n-1} (1-F(y))^{n-s} f(x) f(y)$$

Thus, $X_i \sim U(0,1) \Rightarrow F_X(x) = x$ and $f_X(x) = 1$

$$f_{X_{(n)}, X_{(1)}}(x_{(1)}, x_{(n)}) = \frac{n!}{(1-1)!(n-1-1)!(n-n)!} x_{(1)}^0 (x_{(n)} - x_{(1)})^{n-2} (1 - x_{(n)})^0 \cdot 1$$

$$f_{X_{(n)},X_{(1)}}(x_{(1)},x_{(n)}) = \frac{}{(1-1)!(n-1-1)!(n-n)!} x_{(1)} (x_{(n)}-x_{(1)}) (1-x_{(n)}) \cdot 1$$

$$= n(n-1)(x_{(n)}-x_{(1)})^{n-2}, \quad 0 < x_{(1)} < x_{(n)} < 1$$

$\therefore R = X_{(n)} - X_{(1)} \Rightarrow X_{(n)} = X_{(1)} + R \Rightarrow X_{(1)} + R < 1 \Rightarrow X_{(1)} < 1-R$

So $|J| = \left|\frac{dX_{(n)}}{dR}\right| = 1$

Joint distribution of $(X_{(1)}, R)$: $f_{X_{(1)},R}(x_{(1)}, r) = f_{X_{(1)},X_{(n)}}(x_{(1)}, r)|J|$

$$= n(n-1) r^{n-2}, \quad x_{(1)} < 1-r$$

Pdf of R: $f_R(r) = \int_0^{1-r} f_{X_{(1)},R}(x_{(1)}, r)\, dx_{(1)}$

$$= \int_0^{1-r} n(n-1) r^{n-2} dx_{(1)}$$

$$= n(n-1) r^{n-2}(1-r)$$

Let $Y = 2n(1-R) \Rightarrow R = 1 - \frac{Y}{2n}$

$0 < R < 1 \Rightarrow 0 < 1-R < 1 \Rightarrow 0 < 2n(1-R) < 2n$

So $|J| = \left|\frac{dR}{dY}\right| = \frac{1}{2n}$

pdf of Y: $f_Y(y) = f_R(y)|J|$

$$= n(n-1)\left(1-\frac{y}{2n}\right)^{n-2} \frac{y}{2n} \cdot \frac{1}{2n}$$

$$= \frac{1}{4}\left(1-\frac{1}{n}\right)\left(1-\frac{y}{2n}\right)^{n}\left(1-\frac{y}{2n}\right)^{-2} \cdot y, \quad 0<y<\infty$$

$$= \frac{1}{4} e^{-\frac{y}{2}} y \quad (\text{as } n \to \infty)$$