CS228 Logic for Computer Science 2022

Lecture 4: Formal proofs

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Topic 4.1

Formal proofs

Consequence to derivation

Let us suppose for a (in)finite set of formulas Σ and a formula F, we have $\Sigma \models F$.

Can we syntactically infer $\Sigma \models F$ without writing the truth tables, which may be impossible if the size of Σ is infinite?

We call the syntactic inference "derivation". We derive the following statements.

$$\Sigma \vdash F$$

Example: derivation

Example 4.1

Let us consider the following simple example.

$$\underbrace{\Sigma \cup \{F\}}_{\text{Left hand side(lhs)}} \vdash F$$

If F occurs in lhs, then F is clearly a consequence of the lhs.

Therefore, we should be able to derive the above statement.

Proof rules

A proof rule provides us a means to derive new statements from the old statements.

premises

$$\textsc{RuleName}\ \frac{\text{Stuff-already-there}}{\text{Stuff-to-be-added}}\ \text{Conditions to be met}$$

conclusion

A derivation proceeds by applying the proof rules.

What rules do we need for the propositional logic?

Proof rules - Basic

$$\textsc{Assumption}\frac{}{\Sigma \vdash F}F \in \Sigma$$

$$\textsc{Monotonic}\frac{\Sigma \vdash F}{\Sigma' \vdash F}\Sigma \subseteq \Sigma'$$

Derivation

Definition 4.1

A derivation is a list of statements that are derived from the earlier statements.

Example 4.2

A derivation due to the previous rules

Since assumption does not depend on any other statement, no need to refer.

1. $\{p \lor q, \neg\neg q\} \vdash \neg\neg q$ — Assumption
2. $\{p \lor q, \neg\neg q, r\} \vdash \neg\neg q$ — Monotonic applied to 1

We need to point at an earlier statement.

Proof rules for Negation

$$\textsc{DoubleNeg}\frac{\Sigma \vdash F}{\Sigma \vdash \neg\neg F}$$

Example 4.3

The following is a derivation

1. $\{p \vee q, r\} \vdash r$ *Assumption*
2. $\{p \vee q, \neg\neg q, r\} \vdash r$ *Monotonic applied to 1*
3. $\{p \vee q, \neg\neg q, r\} \vdash \neg\neg r$ *DoubleNeg applied to 2*

Proof rules for $\wedge$

$$\wedge-\text{INTRO}\frac{\Sigma \vdash F \quad \Sigma \vdash G}{\Sigma \vdash F \wedge G} \quad \wedge-\textsc{Elim}\frac{\Sigma \vdash F \wedge G}{\Sigma \vdash F} \quad \wedge-\textsc{Symm}\frac{\Sigma \vdash F \wedge G}{\Sigma \vdash G \wedge F}$$

Example 4.4

The following is a derivation

1. $\{p \wedge q, \neg\neg q, r\} \vdash p \wedge q$ *Assumption*
2. $\{p \wedge q, \neg\neg q, r\} \vdash p$ *$\wedge$-Elim applied to 1*
3. $\{p \wedge q, \neg\neg q, r\} \vdash q \wedge p$ *$\wedge$-Symm applied to 1*

Proof rules for $\vee$

$$\vee-\textsc{intro}\frac{\Sigma \vdash F}{\Sigma \vdash F \vee G} \qquad \vee-\textsc{Symm}\frac{\Sigma \vdash F \vee G}{\Sigma \vdash G \vee F}$$

$$\vee-\textsc{def}\frac{\Sigma \vdash F \vee G}{\Sigma \vdash \neg(\neg F \wedge \neg G)} \qquad \vee-\textsc{def}\frac{\Sigma \vdash \neg(\neg F \wedge \neg G)}{\Sigma \vdash F \vee G}$$

$$\vee-\textsc{Elim}\frac{\Sigma \vdash F \vee G \qquad \Sigma \cup \{F\} \vdash H \qquad \Sigma \cup \{G\} \vdash H}{\Sigma \vdash H}$$

Commentary: We will use the same rule name if a rule can be applied in both the directions. For example, $\vee-\textsc{def}$.

Example : distributivity

Example 4.5

Let us show if we have $\Sigma \vdash (F \wedge G) \vee (F \wedge H)$, we can derive $\Sigma \vdash F \wedge (G \vee H)$.

1. $\Sigma \vdash (F \wedge G) \vee (F \wedge H)$ — Premise
2. $\Sigma \cup \{F \wedge G\} \vdash F \wedge G$ — Assumption
3. $\Sigma \cup \{F \wedge G\} \vdash F$ — $\wedge$-Elim applied to 2
4. $\Sigma \cup \{F \wedge G\} \vdash G \wedge F$ — $\wedge$-Symm applied to 2
5. $\Sigma \cup \{F \wedge G\} \vdash G$ — $\wedge$-Elim applied to 4
6. $\Sigma \cup \{F \wedge G\} \vdash G \vee H$ — $\vee$-Intro applied to 5
7. $\Sigma \cup \{F \wedge G\} \vdash F \wedge (G \vee H)$ — $\wedge$-Intro applied to 3 and 6

Example : distributivity (contd.)

8. $\Sigma \cup \{F \land H\} \vdash F \land H$ — Assumption
9. $\Sigma \cup \{F \land H\} \vdash F$ — $\land$-Elim applied to 8
10. $\Sigma \cup \{F \land H\} \vdash H \land F$ — $\land$-Symm applied to 8
11. $\Sigma \cup \{F \land H\} \vdash H$ — $\land$-Elim applied to 10
12. $\Sigma \cup \{F \land H\} \vdash H \lor G$ — $\lor$-Intro applied to 11
13. $\Sigma \cup \{F \land H\} \vdash G \lor H$ — $\lor$-Symm applied to 12
14. $\Sigma \cup \{F \land H\} \vdash F \land (G \lor H)$ — $\land$-Intro applied to 9 and 13

15. $\Sigma \vdash F \land (G \lor H)$ — $\lor$-elim applied to 1, 7, and 14

Topic 4.2

Rules for implication and others

Proof rules for $\Rightarrow$

$$\Rightarrow\text{-}\textsc{Intro}\ \frac{\Sigma \cup \{F\} \vdash G}{\Sigma \vdash F \Rightarrow G} \qquad \Rightarrow\text{-}\textsc{Elim}\ \frac{\Sigma \vdash F \Rightarrow G \quad \Sigma \vdash F}{\Sigma \vdash G}$$

$$\Rightarrow\text{-}\textsc{Def}\ \frac{\Sigma \vdash F \Rightarrow G}{\Sigma \vdash \neg F \lor G} \qquad \Rightarrow\text{-}\textsc{Def}\ \frac{\Sigma \vdash \neg F \lor G}{\Sigma \vdash F \Rightarrow G}$$

Example: central role of implication

Example 4.6

Let us prove $\{\neg p \vee q, p\} \vdash q$.

1. $\{\neg p \vee q, p\} \vdash p$ — *Assumption*
2. $\{\neg p \vee q, p\} \vdash \neg p \vee q$ — *Assumption*
3. $\{\neg p \vee q, p\} \vdash p \Rightarrow q$ — $\Rightarrow$*-Def applied to 2*
4. $\{\neg p \vee q, p\} \vdash q$ — $\Rightarrow$*-Elim applied to 1 and 3*

All the rules so far

Commentary: Note that we are not writing parentheses on conjunctions etc. They are there but not written for ease as we discussed earlier.

$$\textsc{Assumption}\frac{}{\Sigma \vdash F} F \in \Sigma \qquad \textsc{Monotonic}\frac{\Sigma \vdash F}{\Sigma' \vdash F} \Sigma \subseteq \Sigma' \qquad \textsc{DoubleNeg}\frac{\Sigma \vdash F}{\Sigma \vdash \neg\neg F}$$

$$\wedge-\textsc{intro}\frac{\Sigma \vdash F \quad \Sigma \vdash G}{\Sigma \vdash F \wedge G} \qquad \wedge-\textsc{Elim}\frac{\Sigma \vdash F \wedge G}{\Sigma \vdash F} \qquad \wedge-\textsc{Symm}\frac{\Sigma \vdash F \wedge G}{\Sigma \vdash G \wedge F}$$

$$\vee-\textsc{intro}\frac{\Sigma \vdash F}{\Sigma \vdash F \vee G} \qquad \vee-\textsc{Symm}\frac{\Sigma \vdash F \vee G}{\Sigma \vdash G \vee F} \qquad \vee-\textsc{def}\frac{\Sigma \vdash F \vee G}{\Sigma \vdash \neg(\neg F \wedge \neg G)} *$$

$$\vee-\textsc{Elim}\frac{\Sigma \vdash F \vee G \qquad \Sigma \cup \{F\} \vdash H \qquad \Sigma \cup \{G\} \vdash H}{\Sigma \vdash H}$$

$$\Rightarrow-\textsc{Intro}\frac{\Sigma \cup \{F\} \vdash G}{\Sigma \vdash F \Rightarrow G} \qquad \Rightarrow-\textsc{Elim}\frac{\Sigma \vdash F \Rightarrow G \quad \Sigma \vdash F}{\Sigma \vdash G} \qquad \Rightarrow-\textsc{def}\frac{\Sigma \vdash F \Rightarrow G}{\Sigma \vdash \neg F \vee G} *$$

* Works in the both directions

Example: another proof

Example 4.7

Let us prove $\emptyset \vdash (p \Rightarrow q) \lor p$.

1. $\{\neg p\} \vdash \neg p$ — *Assumption*
2. $\{\neg p\} \vdash \neg p \lor q$ — $\lor$*-Intro applied to 1*
3. $\{\neg p\} \vdash p \Rightarrow q$ — $\Rightarrow$*-Def applied to 2*
4. $\{\neg p\} \vdash (p \Rightarrow q) \lor p$ — $\lor$*-Intro applied to 3*

Case 1

5. $\{p\} \vdash p$ — *Assumption*
6. $\{p\} \vdash p \lor (p \Rightarrow q)$ — $\lor$*-Intro applied to 5*
7. $\{p\} \vdash (p \Rightarrow q) \lor p$ — $\lor$*-Symm applied to 6*

Case 2

8. $\{\} \vdash p \Rightarrow p$ — $\Rightarrow$*-Intro applied to 5*
9. $\{\} \vdash \neg p \lor p$ — $\Rightarrow$*-Def applied to 8*

Only two cases

10. $\{\} \vdash (p \Rightarrow q) \lor p$ — $\lor$*-Elim applied to 4, 7, and 9*

Proof rules for $\Leftrightarrow$

$$\Leftrightarrow -\mathrm{DEF}\frac{\Sigma \vdash F \Leftrightarrow G}{\Sigma \vdash G \Rightarrow F} \qquad \Leftrightarrow -\mathrm{DEF}\frac{\Sigma \vdash F \Leftrightarrow G}{\Sigma \vdash F \Rightarrow G}$$

$$\Leftrightarrow -\mathrm{DEF}\frac{\Sigma \vdash G \Rightarrow F \qquad \Sigma \vdash F \Rightarrow G}{\Sigma \vdash G \Leftrightarrow F}$$

Exercise 4.1

Define similar rules for $\oplus$

Commentary: $\top$ and $\perp$ symbols are not covered in the proof system. They are also macros. $\top$ represents $\neg p \lor p$ and $\perp$ represents $p \land \neg p$ for some variable p. We can introduce Def rules to handle the symbols. However, in practice, they are not very useful.

Topic 4.3

Soundness

Soundness

We need to show that

Theorem 4.1

if

proof rules derive a statement $\Sigma \vdash F$

then

$\Sigma \models F$.

Proof.

We will make an inductive argument. We will assume that the theorem holds for the premises of the rules and show that it is also true for the conclusions. ...

Proving soundness

Proof(contd.)

Consider the following rule

$$\wedge - \textsc{Elim}\frac{\Sigma \vdash F \wedge G}{\Sigma \vdash F}$$

Consider model $m \models \Sigma$. By the induction hypothesis, $m \models F \wedge G$.

Using the the truth table, we can show that if $m \models F \wedge G$ then $m \models F$.

$m(F)$	$m(G)$	$m(F \wedge G)$
0	0	0
0	1	0
1	0	0
1	1	1

Therefore, $\Sigma \models F$. ...

Proof

Proof.
Consider one more rule

$$\Rightarrow -\textsc{Intro}\frac{\Sigma \cup \{F\} \vdash G}{\Sigma \vdash F \Rightarrow G}$$

Consider model $m \models \Sigma$. There are two possibilities.

- **case** $m \models F$:
  Therefore, $m \models \Sigma \cup \{F\}$. By the induction hypothesis, $m \models G$. Therefore, $m \models (F \Rightarrow G)$.
- **case** $m \not\models F$: Therefore, $m \models (F \Rightarrow G)$.

Therefore, $\Sigma \vdash F \Rightarrow G$.

Similarly, we draw truth table or case analysis for each of the rules to check the soundness. $\square$

Topic 4.4

Problems

Exercise: the other direction of distributivity

Exercise 4.2

Show if we have $\Sigma \vdash F \land (G \lor H)$*, we can derive* $\Sigma \vdash (F \land G) \lor (F \land H)$*.*

Hint: Case split on G and $\neg G$.

Exercise: proving a puzzle

Exercise 4.3

a. Convert the following argument into a propositional statement, i.e., $\Sigma \vdash F$.

If the laws are good and their enforcement is strict, then crime will diminish. If strict enforcement of laws will make crime diminish, then our problem is a practical one. The laws are good. Therefore our problem is a practical one. (Hint: needed propositional variables G, S, D, P) (Source : Copi, Introduction of logic)

b. Write a formal proof proving the statement in the previous problem.

Redundant rules

Exercise 4.4

Show that the following rule(s) can be derived from the other rules.

1. $\vee$-*Symm*
2. $\Rightarrow$-*Elim*

Redundancy***

Exercise 4.5

Find a minimal subset of the proof rules which has no redundancy, i.e., none of the rules can be derived from others. Prove that the subset has no redundancy.

End of Lecture 4