

Personalized Robo-Advising: Enhancing Investment through Client Interaction*

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Abstract

Automated investment managers, or robo-advisors, have emerged as an alternative to traditional financial advisors. The viability of robo-advisors crucially depends on their ability to offer personalized financial advice. We introduce a novel framework, in which a robo-advisor interacts with a client to solve an adaptive mean-variance portfolio optimization problem. The risk-return tradeoff adapts to the client's risk profile, which depends on idiosyncratic characteristics, market returns, and economic conditions. We show that the optimal investment strategy includes both myopic and intertemporal hedging terms which reflect the dynamic risk profile of the client. We characterize the optimal portfolio personalization via a tradeoff faced by the robo-advisor between receiving information from the client in a timely manner and mitigating behavioral biases in the communicated risk profile. We argue that the optimal portfolio's Sharpe ratio and return distribution improve if the robo-advisor counters the client's tendency to reduce market exposure during economic contractions when the market risk-return tradeoff is more favorable.

1 Introduction

Automated investment managers, commonly referred to as robo-advisors, have gained widespread popularity in recent years. The value of assets under management by robo-advisors is the highest in the United States, and exceeded \$650 billion in 2020. Prominent robo-advising firms include Vanguard Personal Advisor Services, with about \$140 billion of assets under management, Schwab Intelligent Portfolios (\$40bn),

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Wealthfront (\$20bn), and Betterment (\$18bn). Robo-advisors are also on the rise in other parts of the world. They currently manage over \$100 billion in Europe, and are exhibiting rapid growth in Asia where China (\$75bn) and Japan (\$30bn) have become the world's second and third largest markets in terms of assets under management (Statista [2020]).

The first robo-advisors were launched in the wake of the 2008 financial crisis and the ensuing loss of trust in established financial services institutions. Examples of pioneering robo-advising firms are Betterment and Wealthfront, which began offering services formerly considered exclusive to the general public, including individuals who did not meet the minimum investment levels of traditional financial advisors. In the years that followed, industry incumbents - such as Vanguard and Charles Schwab - followed suit and began offering their own robo-advising services, taking advantage of their existing customer bases to quickly gain a large market share. With the focus of most robo-advisors being on passive long-term portfolio management, the rise of robo-advising since the financial crisis has also been compounded by the seismic shift towards passive investing and low cost exchange-traded funds.¹

Robo-advising is a term that encompasses various forms of digital financial advice for investment management and trading. A recent survey paper by D'Acunto and Rossi [2020] distinguishes between digital tools and services that aid in short-term active trading, where the client is significantly involved in the strategy implementation, and those that focus on long-term passive investing, where the robo-advisor performs the majority of investment decisions. We develop a framework for automated investment management which belongs to the latter category, where the robo-advisor uses quantitative methods and algorithms to manage the client's portfolio. We explore how the robo-advisor can use features of the client's risk profile to determine a frequency of interaction with the client that ensures a high level of portfolio personalization.² We also analyze the dilemma faced by the robo-advisor when it comes to either catering to the client's wishes, i.e., investing in accordance with the client's risk profile, or going against the client's wishes in order to seek better investment performance.

In our framework, the robo-advisor interacts repeatedly with the client to learn about changes in her risk profile. This risk profile is characterized by a risk aversion process, which captures three fundamental aspects of individual investor behavior. First, the client's risk aversion is sensitive to the passage of time, consistent with empirical research that has identified a positive and potentially nonlinear trend in risk aversion as a function of age (see, e.g., Brooks et al. [2018] and Hallahan et al. [2004]). This type of time-dependence has also been used in the construction of *target-date funds*, which are widely used retirement portfolios that

¹See <https://www.cnbc.com/2018/09/14/the-trillion-dollar-etf-boom-triggered-by-the-financial-crisis.html>

²Robo-advisors are fiduciaries under the Investment Advisers Act of 1940, and as such are subject to the duty of acting in the client's best interest.

reduce market risk as the client gets closer to retirement.³ Second, risk aversion is impacted by idiosyncratic shocks, such as a change in disposable income or an increase in the client's educational level or financial literacy (see, e.g., Hallahan et al. [2004] and Allgood and Walstad [2016]). These shocks in particular capture changes in unobservable factors driving the client's risk aversion, consistent with empirical studies that have identified a substantial variation in idiosyncratic risk preferences that are unexplained by consumer attributes and demographic characteristics (see, e.g., Guiso and Paiella [2008], Sahm [2012], and Van de Venter et al. [2012]). Third, the client's risk aversion depends on realized market returns and prevailing economic conditions. Specifically, the countercyclical nature of risk aversion has been extensively documented in the literature (see Bucciol and Miniaci [2018], Cohn et al. [2015], and Guiso et al. [2018], among others, for recent empirical and experimental evidence).

The robo-advisor adopts a multi-period mean-variance investment criterion with a finite investment horizon. It is worth noting that most robo-advising firms employ asset allocation algorithms based on mean-variance analysis, citing, among other things, its tractability and the intuitive interpretation of the risk-return tradeoff (see, for instance, Beketov et al. [2018]). While these robo-advising algorithms assume the risk-return tradeoff coefficient to be constant throughout the investment period, and statically reoptimize the portfolio in response to changes in the client's risk profile, our framework enables the construction of a portfolio strategy that dynamically adapts to *the robo-advisor's model* of the client's risk aversion process. Such a model is constructed from two sources of information. First, the robo-advisor observes both realized market returns and changes in the state of the economy, and continuously updates its model to reflect this information. Second, at times of interaction with the client, the robo-advisor receives information about the idiosyncratic component of the client's risk aversion.

We show that the stock market allocation resulting from the dynamic mean-variance optimization consists of two components. The first component is akin to the standard single-period Markowitz strategy, but also takes into account the expectation and variance of the optimal strategy's return throughout the investment period. The second component is intertemporal hedging demand, which depends on the relation between the current market return and future portfolio returns. A novel feature of our model is that this relation is driven by the client's dynamic risk profile, in contrast to extant literature (e.g., Liu [2007] and Basak and Chabakauri [2010]) where it depends only on the correlation between market returns and changes in a stochastic state variable.

To understand why intertemporal hedging demand arises in our model, consider the case where positive

³Target-date funds (TDFs) are similar to the "set-and-forget" portfolios offered by the first generation of robo-advisors, where changes in allocation over time were solely based on the client's age. While TDFs have virtually stayed the same since their inception over a quarter century ago, the technology-driven personalized advice of robo-advisors has continued to evolve.

(negative) market returns have a tempering (inflating) effect on the client's risk aversion process. A positive market return thus leads to increased future stock market exposure, whereas a negative return has the opposite effect. This relation amplifies the effect of the current market return on the variance of the client's terminal wealth, making the stock market a less attractive investment option. As a result, for a client whose risk aversion is negatively correlated with market returns, there is a negative hedging demand, i.e., the robo-advisor reduces her market exposure relative to a client whose risk aversion is not sensitive to market returns.

We show the existence of a tradeoff between the rate of information acquisition from the client and the accuracy of the acquired information. On the one hand, if interaction does not occur at all times, the robo-advisor may not always have access to up-to-date information about the client's risk profile. On the other hand, information communicated to the robo-advisor may not be representative of the client's true risk aversion, as the client is subject to behavioral biases, such as trend-chasing.⁴ For example, if recent market returns have exceeded expectations, the client may feel overly exuberant and communicate a risk aversion value that is lower than her actual risk aversion. Vice versa, following a market underperformance, the client may feel overly pessimistic and exaggerate her risk aversion when communicating with the robo-advisor. The suboptimality of high interaction frequencies then hinges on the fact that reducing the frequency of interaction mitigates the effect of the client's biases, which is consistent with the notion of myopic loss aversion, introduced in Benartzi and Thaler [1995].⁵ The rationale for this effect in our model is that over longer time periods, fluctuations in realized market returns average out, dampening their impact on the client's risk aversion.

We introduce a measure of portfolio personalization to analyze the aforementioned tradeoff. This measure is defined in terms of the difference between the client's actual risk aversion and the robo-advisor's model of it. The lower this measure, the closer the investment strategy implemented by the robo-advisor is to the strategy matching the client's risk profile. We show that for a fixed level of behavioral bias, quantified by the client's sensitivity to market return fluctuations, this measure is minimized by a unique interaction frequency. Such a frequency allows the robo-advisor to strike a balance between obtaining information from the client in a timely manner and ensuring that the communicated information is not overly biased by recent market returns. Our result is supported by the practices of current robo-advising firms, which encourage clients to refrain from making frequent changes to their risk profiles, and even limit their ability to do so.⁶ We also show that the optimal interaction frequency is decreasing in the amount of behavioral bias, and

⁴See Kahneman and Riepe [1998] for a discussion of common behavioral biases and strategies to overcome them.

⁵Loss aversion refers to the notion that negative market returns (losses) have a greater impact than positive returns (gains) on the risk-taking behavior of investors.

⁶Wealthfront provides the following guidelines: "Our software limits our clients to one risk-score change per month. We encourage people who attempt more than three risk-score changes over the course of a year to try another investment manager."

increasing in the rate at which the idiosyncratic component of the client's risk aversion changes. Therefore, a higher level of personalization is achieved for clients with limited behavioral biases and stable risk profiles.

We compare the investment performance of strategies that invest the same proportion of wealth in the stock market during periods of economic growth but differ in their market exposure during contractions. We show analytically that the Sharpe ratio of such strategies is generally increasing in the proportion of wealth allocated to the stock market in a state of contraction, when the market risk-return tradeoff is more favorable.⁷ Because individual investors typically have a countercyclical risk aversion process, the client is inclined to reduce market exposure during economic contractions and, as a result, attain a lower Sharpe ratio from her investment. This brings up the question of to what extent, if any, the robo-advisor should deviate from the strategy implied by the client's risk profile in order to improve the investment performance. We show that the Sharpe ratio of the optimal portfolio is concave with respect to the change in the market allocation when the economy moves to a state of contraction. This means that the Sharpe ratio drops more if the allocation is reduced, compared to how much it improves if the allocation is increased by the same amount. Our analysis thus suggests that a good middle ground for the robo-advisor is to rebalance the portfolio and maintain prespecified weights throughout the business cycle. This investment pattern is also in line with the long-term investment principle of riding out business cycles, and further reinforced by the fact that, in practice, it is difficult to detect changes in economic regimes and thus estimate the prevailing market risk-return tradeoff.

The remainder of the paper is organized as follows. In Section 2, we briefly review the related literature. In Section 3, we introduce our robo-advising framework. In Section 4, we present the solution to the associated optimal investment problem. In Section 5, we introduce and study performance metrics for the robo-advisor's investment strategy. Section 6 offers concluding remarks. Appendix A contains theoretical and computational properties of the optimal investment strategy. Appendix B and Appendix C contain proofs and auxiliary results related to Sections 4 and 5, respectively.

2 Literature Review

Our study contributes to the growing literature on robo-advising. D'Acunto and Rossi [2020] describe the main components of robo-advising systems and propose a classification in terms of four main features: (i) portfolio personalization, (ii) client involvement, (iii) client discretion, and (iv) human interaction. We offer a quantitative framework that aligns with this classification and is consistent with that of the most prominent

⁷Since the seminal work of Fama and French [1989], it has been extensively documented that the market risk-return tradeoff is countercyclical, i.e., higher at business cycle troughs than peaks.

stand-alone robo-advising firms. We proceed to briefly describe each of the four features and how they fit into our framework.

(i) *Portfolio personalization* refers to the robo-advisor's ability to offer financial advice tailored to the client's needs. A common criticism of robo-advising is the lack of customization, with risk profiling based on information that is too limited.⁸ In our framework, the robo-advisor achieves portfolio personalization by repeatedly soliciting information from the client. We do not explicitly model the information collecting process, which can be based on both online questionnaires (i.e., self-reported information) and automated gathering of information from the client (e.g., data on savings and spending behavior).

(ii) *Client involvement* refers to the client's participation in the design of the investment strategy. At one end of the spectrum, there are robo-advisors that require the client to approve every single trading decision. In this case, the client is actively involved and the term *robo-advisor* is descriptive, because the client decides in what way to follow the *advice* provided. Our model lies at the other end of the spectrum, where the robo-advisor automatically *manages* the portfolio on the client's behalf and for which the term *robo-manager* is more descriptive.

(iii) *Client discretion* is the client's ability to override the robo-advisor's recommendation. In our framework, the client's behavioral biases can be viewed as the client overriding the robo-advisor's recommendations. This is considered a low level of discretion. It is consistent with the operations of robo-advisors focusing on long-term investing, where the client is allowed to adjust the level of portfolio risk, but has no control over which parts of the investment portfolio are modified.

(iv) *Human interaction* refers to the degree of interaction between the client and a human-advisor. In our framework, there is no human contact which is also the case for automated robo-advisors that strive to minimize operating costs.

The above classification is qualitative in nature. The quantitative components of our framework are constructed to resemble the algorithmic principles of actual robo-advisors. Beketov et al. [2018] provide an industry overview based on the analysis of over 200 robo-advisors globally. Their study shows that a large majority of robo-advisors use an asset allocation framework based on mean-variance analysis, with the asset universe consisting of low cost exchange-traded funds. They also show that risk profiling of clients is primarily done using online questionnaires.

Rossi and Utkus [2019a] conduct an extensive survey to study the "needs and wants" of individuals when

⁸Risk profiling can, in principle, be infinitely customizable. Barron's 2019 annual ranking of robo-advisors (available at <https://webreprints.djreprints.com/4642511400002.pdf>) reports the latest growth in robo-advising to be in cash management, with robo-advisors aiming to become everyday money managers of their clients, in addition to long-term investment planners. Robo-advising firms already offer cash management services such as direct deposits of paychecks, automatic bill payments, and FDIC-insured checking and savings accounts. This implies increased access to data which can in turn be used by robo-advisors to improve investment recommendations.

they hire financial advisors. They show that for traditionally-advised clients, algorithm aversion and the inability to interact with a human are the main obstacles for switching to robo-advising, while robo-advised clients do not have the same need for trust and access to expert opinion. Our model lends theoretical support to the notion that robo-advisors may be less suitable for algorithmic-averse clients. Namely, the robo-advisor can improve the portfolio performance by going against the wishes of the client, i.e., by investing in a way that may seem counterintuitive to the client. Doing so is more challenging for a robo-advisor than for a human-advisor, because market returns are random, and an algorithmic-averse client will be less forgiving to the robo-advisor in the event of adverse market returns.

From a methodological perspective, our work contributes to the literature on *time-inconsistent* stochastic control (Björk and Murgoci [2013]). Other related works include Li and Ng [2010] who solve a multi-period version of the classical Markowitz problem, and Basak and Chabakauri [2010] who solve a continuous-time version of the same problem. Björk et al. [2014] solve the dynamic mean-variance problem in continuous time, with the mean-variance utility function applied to returns as in the classical single-period mean-variance analysis. A recent study of Dai et al. [2019] further develops a dynamic mean-variance framework based on log-returns, with the aim of generating policies conforming with conventional investment wisdom. In all of these works, the risk-return tradeoff is assumed to be constant throughout the investment horizon. By contrast, in our model, the risk-return tradeoff coefficient is stochastic, with explicitly modeled dynamics that can be used to generate investment policies tailored to the client's risk profile.

Our paper is also related to the literature on portfolio optimization models where the security price dynamics are driven by stochastic factors. Unlike Liu [2007] and Basak and Chabakauri [2010], intertemporal hedging demand does not arise because of dependence between the stochastic factor and market returns, which are assumed to be uncorrelated in our model. Rather, intertemporal hedging terms appear because of the correlation between market returns and future changes in the investor's risk aversion. Additionally, the stochastic factor itself impacts the client's risk aversion dynamics, which in turn determine the optimal allocation. It is widely agreed upon that risk aversion varies with the business cycle, and this characteristic is highly relevant for robo-advising where a significant challenge is to keep clients invested in the stock market during economic downturns.⁹

⁹The seminal work of Campbell and Cochrane [1999] provides an asset pricing framework that rationalizes the counter-cyclicity of both risk aversion and the market risk-return tradeoff; see, also, Lettau and Ludvigson [2010] for a survey of the literature and extensive empirical evidence.

3 Modeling Framework

Our robo-advising framework has four main components: (i) a regime switching model of market returns, (ii) a mechanism of interaction between the client and the robo-advisor, (iii) a dynamic model for the client's risk preferences, and (iv) an optimal investment criterion. We describe each of these components in Sections 3.1-3.4.

3.1 Market and Wealth Dynamics

The market consists of a risk-free money market account, $(B_n)_{n \geq 0}$, and a risky asset, $(S_n)_{n \geq 0}$, whose dynamics are described by the equations

$$B_{n+1} = (1 + r(Y_n))B_n, \quad S_{n+1} = (1 + Z_{n+1}(Y_n))S_n,$$

where $B_0 = 1$ and $S_0 > 0$ is a given constant. The risky asset is representative of the overall stock market, and for this reason we will refer to returns of the risky asset as market returns.

The above dynamics are modulated by an observable state process, $(Y_n)_{n \geq 0}$, which captures macroeconomic conditions affecting interest rates and market returns (cf. Hamilton [1989]). We assume the economic state variable $(Y_n)_{n \geq 0}$ to be a time-homogeneous Markov chain with transition matrix P , taking values in a finite set $\mathcal{Y} := \{1, 2, \dots, M\}$, for some $M \geq 1$. Conditioned on $Y_n = y \in \mathcal{Y}$, the risk-free interest rate $r(y)$ is constant, while the risky asset's return, $Z_{n+1}(y)$, admits a probability density function $f_{Z|y}$ which depends only on the current economic state y , and has mean $\mu(y)$ and variance $0 < \sigma^2(y) < \infty$.

For notational simplicity, we omit the dependence on the economic state. Hence, we use Z_{n+1} in place of $Z_{n+1}(Y_n)$, and denote its state-dependent mean and variance by $\mu_{n+1} := \mu(Y_n)$ and $\sigma_{n+1}^2 := \sigma^2(Y_n)$, respectively. Similarly, we use r_{n+1} in place of $r(Y_n)$, and let $R_{n+1} := 1 + r_{n+1}$. We denote by $\tilde{Z}_{n+1} := Z_{n+1} - r_{n+1}$ the excess return of the risky asset over the risk-free rate, which has state-dependent mean and variance given by $\tilde{\mu}_{n+1} := \mu_{n+1} - r_{n+1}$ and σ_{n+1}^2 , respectively.

We denote by X_n the wealth of the client at time n , allocated between the risky asset and the money market account, and use π_n to denote the amount invested in the risky asset. For a given self-financing trading strategy $\pi := (\pi_n)_{n \geq 0}$, the wealth process $(X_n^\pi)_{n \geq 0}$ follows the dynamics

$$X_{n+1}^\pi = R_{n+1}X_n^\pi + \tilde{Z}_{n+1}\pi_n, \quad X_0^\pi = x_0. \quad (3.1)$$

The initial wealth, $x_0 \in \mathbb{R}^+$, and the initial state of the economy, $Y_0 \in \mathcal{Y}$, are assumed to be non-random.

The random variables $(Y_n)_{n \geq 1}$ and $(Z_n)_{n \geq 0}$ are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which additionally supports a sequence $(\epsilon_n)_{n \geq 1}$ of independent real-valued random variables. This source of randomness captures idiosyncratic changes to the client's risk preferences. Moreover, $(\epsilon_n)_{n \geq 1}$ is independent of $(Y_n)_{n \geq 0}$ and $(Z_n)_{n \geq 1}$. We use $(\mathcal{F}_n)_{n \geq 0}$ to denote the filtration generated by the three stochastic processes in our model:

$$\mathcal{F}_n := \sigma(Y_{(n)}, Z_{(n)}, \epsilon_{(n)}), \quad (3.2)$$

where $Y_{(n)} := (Y_0, \dots, Y_n)$, $Z_{(n)} := (Z_1, \dots, Z_n)$, and $\epsilon_{(n)} := (\epsilon_1, \dots, \epsilon_n)$. We will use analogous notation to denote the paths of other stochastic processes throughout the paper.

Remark 3.1. In our discrete-time model, the price of the risky asset is not restricted from becoming negative. However, at any given time the optimal investment strategy presented in Section 4 does not depend on the current price of the risky asset, but only on the first two moments of its returns distribution. Hence, for the optimal investment problem to be well-defined, it is sufficient to assume the existence of a returns process $(Z_n)_{n \geq 1}$ with a finite variance as described above. \square

3.2 Interaction between Client and Robo-Advisor

A key component of the proposed framework is the interaction between the client and the robo-advisor. Following an initial interaction at the beginning of the investment process, the client and the robo-advisor interact repeatedly throughout the investment period. At each interaction time, the client translates the information communicated by the client into a numerical value, herein referred to as a risk aversion parameter. In our quantitative framework, we abstract from the construction of such a mapping, effectively assuming that the client communicates directly a single risk aversion parameter to the robo-advisor.¹⁰ Note that between consecutive interaction times, the robo-advisor receives no input from the client.

The interaction schedule, denoted by $(T_k)_{k \geq 0}$, is an increasing sequence of stopping times with respect to the filtration $(\mathcal{F}_n)_{n \geq 0}$, defined in (3.2). That is, $T_0 = 0$, $T_k < T_{k+1}$, and $\{T_k \leq n\} \in \mathcal{F}_n$, for any $n \geq 0$. Hence, interaction can be triggered by any combination of client-specific events, changes in the state of the economy, and market events such as a cascade of negative market returns.

The rule used to determine the interaction schedule $(T_k)_{k \geq 0}$ is decided at the beginning of the investment

¹⁰The majority of robo-advisors elicit risk preferences by means of online questionnaires (Beketov et al. [2018]). The client is presented with questions regarding, e.g., demographics, investment goals, education and financial literacy, and potential reactions to hypothetical gambles and market events. We refer to Charness et al. [2013] for an outline of the pros and cons of different methods used to assess risk preferences, and Cox and Harrison [2008] for a more comprehensive overview.

process, and used throughout it. For future reference, we also define the process $(\tau_n)_{n \geq 0}$, where

$$\tau_n := \sup\{T_k : T_k \leq n\},$$

is the last interaction time prior to and including time n .

3.3 Client's Risk Aversion Process

We first introduce the *client's actual risk aversion process*, $(\gamma_n^C)_{n \geq 0}$, which is an \mathbb{R}_0^+ -valued stochastic process adapted to the filtration $(\mathcal{F}_n)_{n \geq 0}$ defined in (3.2). This means that at time n , the client's risk aversion may have shifted from its initial value due to changes in economic regimes, $Y_{(n)}$, and realized market returns $Z_{(n)}$, as well as because of idiosyncratic shocks to the client's risk aversion, $\epsilon_{(n)}$.

We also introduce the $(\mathcal{F}_n)_{n \geq 0}$ -adapted process $(\xi_n)_{n \geq 0}$, where $\xi_n \in \mathbb{R}_0^+$ is the risk aversion parameter *communicated by the client* at the most recent interaction time, τ_n . Observe that the process $(\xi_n)_{n \geq 0}$ depends on the interaction schedule $(T_k)_{k \geq 0}$ and, by construction, it remains constant between consecutive interaction times where no new information comes from the client, i.e., $\xi_n = \xi_{\tau_n}$.

The robo-advisor then constructs a *model of the client's risk aversion process*, denoted by $(\gamma_n)_{n \geq 0}$, and uses it to solve the optimal investment problem. While it is generally desirable for the model $(\gamma_n)_{n \geq 0}$ to accurately track the client's actual risk aversion, $(\gamma_n^C)_{n \geq 0}$, the two processes may not coincide due to the following reasons. First, the client and the robo-advisor may not interact at all times, so the robo-advisor does not always have access to up-to-date information about the client's risk preferences. Specifically, while the robo-advisor observes both market returns and changes in the economic regime, it cannot observe in real time the idiosyncratic shocks to the client's risk aversion. Second, even if interaction were to occur at all times, information communicated by the client may not be representative of the client's true risk preferences due to her behavioral biases (see Section 5.1 for further details).

Formally, the risk aversion process $(\gamma_n)_{n \geq 0}$ is an \mathbb{R}_0^+ -valued stochastic process adapted to the *robo-advisor's filtration*, $(\mathcal{F}_n^R)_{n \geq 0}$, which is generated by the random variables $(D_n)_{n \geq 0}$, defined as

$$D_n := (Y_{(n)}, Z_{(n)}, \tau_{(n)}, \xi_{(n)}) \in \mathcal{D}_n, \quad \mathcal{D}_n := \mathcal{Y}^{n+1} \times \mathbb{R}^n \times \mathbb{N}^{n+1} \times (\mathbb{R}_0^+)^{n+1}. \quad (3.3)$$

It then follows that at time n , the risk aversion process may be written as

$$\gamma_n := \gamma_n(D_n) \in \mathcal{F}_n^R, \quad (3.4)$$

for a measurable function $\gamma_n : \mathcal{D}_n \mapsto \mathbb{R}_0^+$.

Remark 3.2. The random variable D_n in (3.3) can be decomposed as $D_n = (M_n, I_n)$, where $M_n := (Y_{(n)}, Z_{(n)})$, and $I_n := (\tau_{(n)}, \xi_{(n)})$. This shows that the robo-advisor's filtration $(\mathcal{F}_n^R)_{n \geq 0}$ has two sources of information. The first component, M_n , reflects the fact that the robo-advisor has the ability to process all available information about the market and the economy. The second component, I_n , is the result of its interaction with the client, and contains both the history of interaction times and the communicated risk aversion values. Under mild conditions, the robo-advisor's filtration grows with the frequency of interaction, and, if interaction occurs at all times, it becomes equal to the filtration $(\mathcal{F}_n)_{n \geq 0}$.¹¹ \square

3.4 Investment Criterion

The robo-advisor's objective is to optimally invest the client's wealth, taking into account the stochastic nature of the client's risk preferences. For this purpose, we develop a dynamic version of the standard Markowitz [1952] mean-variance problem that adapts to the client's changing risk preferences. We proceed to introduce this criterion, which we refer to as an *adaptive mean-variance criterion*.

Let $T \geq 1$ be a fixed investment horizon. For each $n \in \{0, 1, \dots, T-1\}$, $x \in \mathbb{R}_0$, $d \in \mathcal{D}_n$, and a control law $\pi := (\pi_n)_{n \geq 0}$, we consider the mean-variance functional

$$J_n(x, d; \pi) := \mathbb{E}_{n,x,d}[r_{n,T}^\pi] - \frac{\gamma_n(d)}{2} \text{Var}_{n,x,d}[r_{n,T}^\pi], \quad (3.5)$$

where, at time n , the *risk-return tradeoff* γ_n is the robo-advisor's model of the client's risk aversion at this time, and $r_{n,T}^\pi$ is the simple return obtained by following the control law π until the terminal date T ,

$$r_{n,T}^\pi := \frac{X_T^\pi - X_n}{X_n}. \quad (3.6)$$

The initial condition in (3.5) is given by $X_n = x$ and $D_n = d$, and both the expectation and the variance are computed with respect to the probability measure $\mathbb{P}_{n,x,d}(\cdot) := \mathbb{P}(\cdot | X_n = x, D_n = d)$. For future reference, we also introduce the probability measure $\mathbb{P}_{n,d}(\cdot) := \mathbb{P}(\cdot | D_n = d)$.

Observe that the risk-return tradeoff γ_n does not depend on the client's wealth. There are important reasons behind this assumption. First, it is consistent with the Markowitz mean-variance criterion, which is recovered as a special case of our model when there is a single investment period. Second, as we will see in Section 4, the optimal investment strategy turns out to be consistent with that of an investor exhibit-

¹¹For example, assume the interaction times to be given by $T_k = k\phi$, for some $\phi \geq 1$. If the sum of idiosyncratic shocks, $(\sum_{k=1}^n \epsilon_k)_{n \geq 0}$ is measurable with respect to the robo-advisor's filtration, then $(\mathcal{F}_n^R)_{n \geq 0}$ increases to $(\mathcal{F}_n)_{n \geq 0}$, as $\phi \downarrow 1$.

ing constant relative risk aversion (CRRA), i.e., whose proportion of wealth invested in the risky asset is independent of the wealth level.¹²

The objective functional (3.5), together with the risk aversion process (3.4), define a family of *sequentially adaptive* optimization problems, in the sense that at each time a new problem arises, with properties that depend on realized market returns, economic state changes, and client-communicated information, but with the same initially specified terminal date. Moreover, observe that it is the robo-advisor that solves the optimization problem, and the initial condition of each optimization problem fixes the value of the stochastic process that generates the robo-advisor's filtration, as well as the value of the client's wealth process.

The control law π in (3.5), also referred to as a strategy or allocation, is such that for each n , the control π_n is a measurable real-valued function of the state variables X_n and D_n . Additionally, admissible control laws are assumed to be self-financing and to satisfy the square-integrability condition $\mathbb{E}[\sum_{n=0}^{T-1} \pi_n^2] < \infty$.

Remark 3.3. We assume the investment horizon T to be deterministic and fixed. A stochastic horizon can be captured through distributional assumptions on the risk aversion process $(\gamma_n)_{n \geq 0}$. For example, the possibility of client death can be modeled by introducing a time-dependent probability p_n of the risk aversion parameter γ_{n+1} becoming “infinite”. The probability p_n would be increasing in n , and infinity would be an absorbing state reached only in the event that the client dies. In the death-state, the client's portfolio is liquidated and the client's entire wealth is allocated to the risk-free asset. For the optimization problem described in this section, a non-zero value of p_n would have the effect of tilting the risky asset allocation upward, because no more investing is possible after death. In the same vein as when accounting for client death, we may impose conditions on the dynamics of the risk aversion process that halt investment if certain market events or economic conditions occur. \square

Remark 3.4. There is a non-zero probability that the client's wealth becomes negative. This can be either due to short-selling or to an extreme negative return of the risky asset (see Remark 3.1). Although the objective functional (3.5) and the optimal strategy presented in Section 4 are well-defined for negative wealth levels, they are not economically meaningful and, in practice, the robo-advisor would liquidate the client's portfolio as soon as the wealth becomes negative.

In Appendix A.3, we modify the optimal strategy in a way that forces liquidation of the entire risky position if the wealth is negative. We show that the change in expectation and variance of the optimal portfolio's return (given in (3.6)), resulting from imposing this liquidation condition, can be bounded by

¹²Empirical evidence suggests that CRRA is a good description of microeconomic behavior (see, e.g., extensive panel data studies carried out in Brunnermeier and Nagel [2008], Chiappori and Paiella [2011], and Sahm [2012]). However, the evidence is not universal. For instance, Calvet and Sodini [2014] and Guiso and Paiella [2008] reject CRRA in favor of decreasing relative risk aversion.

a term that is extremely small for plausible values of the model parameters. It then follows that forcing liquidation has a small impact on the value of the objective functional (3.5).

The liquidation condition is imposed ad hoc on the optimal strategy. Hence, it is myopic in the sense that the modified strategy coincides with the optimal strategy until the wealth achieves a negative value. In Appendix A.3, we also discuss how the optimal investment problem can be adjusted to handle portfolio constraints which are typically imposed by robo-advising firms, such as no borrowing and no short-selling. In that case, the constraints also affect the optimal strategy at times before they become binding, because of the dynamic nature of the optimization problem. \square

Through the wealth dynamics (3.1), the mean-variance functional J_n depends on the control law π restricted to the time points $\{n, n+1, \dots, T-1\}$ and the robo-advisor chooses the control π_n *given* future control decisions $\pi_{n+1:T} := \{\pi_{n+1}, \pi_{n+2}, \dots, \pi_{T-1}\}$. Therefore, any candidate optimal control law π^* is such that for each $n \in \{0, 1, \dots, T-1\}$, $x \in \mathbb{R}_0$, and $d \in \mathcal{D}_n$,

$$\sup_{\pi \in A_{n+1}^*} J_n(x, d; \pi) = J_n(x, d; \pi^*), \quad (3.7)$$

where $A_{n+1}^* := \{\pi : \pi_{n+1:T} = \pi_{n+1:T}^*\}$ is the set of control laws that coincide with π^* after time n . If a control law π^* satisfying (3.7) exists, we define the corresponding value function at time n as

$$V_n(x, d) := J_n(x, d; \pi^*). \quad (3.8)$$

It is worth noticing that when seeking an optimal control π_n^* at time n , the robo-advisor takes into account the dynamics of $\{\gamma_n, \gamma_{n+1}, \dots, \gamma_{T-1}\}$, i.e., the dynamics of the risk-return coefficient *throughout* the investment horizon, with only the *current* value γ_n being known. At the subsequent time $n+1$, the dynamics of the remaining values $\{\gamma_{n+1}, \dots, \gamma_{T-1}\}$ are updated by the robo-advisor, and γ_{n+1} becomes known. If $n+1$ is a non-interaction time, this update is based only on information about market returns and economic states, while if $n+1$ is a time of interaction it also accounts for information received from the client.

4 Optimal Investment Strategy

We analyze the optimization problem defined by the sequence of objective functionals $(J_n)_{0 \leq n < T}$ in (3.5) and the optimality criterion in (3.7). It is well known that, even if the risk-return tradeoff $(\gamma_n)_{n \geq 0}$ is constant through time, the family of optimization problems defined by (3.5) is time-inconsistent in the sense that the Bellman optimality principle does not hold. This means that if, at time n , the control law π^* maximizes the

objective functional J_n , then, at time $n + 1$, the restriction of π^* to the time points $\{n + 1, n + 2, \dots, T - 1\}$ may not maximize J_{n+1} . We refer to Björk and Murgoci [2013] and references therein for a general framework of time-inconsistent stochastic control in discrete time.

As standard in this literature, we view the optimization problem in (3.7) as a multi-player game, where the player at each time $n \in \{0, 1, \dots, T - 1\}$ is thought of as a future self of the client. Player n then wishes to maximize the objective functional J_n , but decides only the strategy π_n at time n , while $\pi_{n+1}, \dots, \pi_{T-1}$ are determined by her future selves. The resulting optimal control strategy, π^* , is the subgame perfect equilibrium of this game, and can be computed using backward induction. At time $n = T - 1$, the equilibrium control π_{T-1}^* is obtained by maximizing J_{T-1} over π_{T-1} , which is a standard single-period optimization problem. For $n < T - 1$, the equilibrium control π_n^* is then obtained by letting player n choose π_n to maximize J_n , given that player n' will use $\pi_{n'}^*$, for $n' = n + 1, n + 2, \dots, T - 1$.

In Appendix B, we derive an extended Hamilton-Jacobi-Bellman (HJB) system of equations satisfied by the value function of the optimization problem. The following result presents the solution to this system, which is the optimal investment strategy for an interaction schedule $(T_k)_{k \geq 0}$ and a risk aversion process $(\gamma_n)_{n \geq 0}$ of the general form, respectively introduced in Sections 3.2 and 3.3.

Proposition 4.1. *The dynamic optimization problem (3.7) admits a solution π^* of the form*

$$\pi_n^*(x, d) = \tilde{\pi}_n^*(d)x, \quad 0 \leq n < T,$$

for $x \in \mathbb{R}_0$ and $d \in \mathcal{D}_n$. The optimal proportion of wealth allocated to the risky asset is given by

$$\tilde{\pi}_n^*(d) = \frac{1}{\gamma_n} \frac{\mathbb{E}_{n,d}[\tilde{Z}_{n+1} \tilde{r}_{n+1,T}^{\pi^*}]}{\text{Var}_{n,d}[\tilde{Z}_{n+1} \tilde{r}_{n+1,T}^{\pi^*}]} - R_{n+1} \frac{\text{Cov}_{n,d}[\tilde{Z}_{n+1} \tilde{r}_{n+1,T}^{\pi^*}, \tilde{r}_{n+1,T}^{\pi^*}]}{\text{Var}_{n,d}[\tilde{Z}_{n+1} \tilde{r}_{n+1,T}^{\pi^*}]}, \quad (4.1)$$

where we recall that R_{n+1} and \tilde{Z}_{n+1} have been defined in Section 3.1, and

$$\tilde{r}_{n+1,T}^{\pi^*} := 1 + r_{n+1,T}^{\pi^*} = \frac{X_T^{\pi^*}}{X_{n+1}},$$

is the terminal value of one dollar invested in the optimal strategy π^* at time $n + 1$.

The above proposition characterizes the structure of the optimal strategy. It shows that the optimal allocation, i.e., the relative fraction of wealth allocated to each asset, is independent of the wealth level. In Appendix A, we use backward induction to develop an explicit representation of the optimal strategy, and discuss the computational complexity of this procedure. As a byproduct, we obtain formulas for the expected value and variance of the optimal portfolio's return, which in turn can be used to compute the

value function (3.8).

It is evident from (4.1) that the optimal strategy can be decomposed into two terms. The first term resembles the standard single-period Markowitz strategy, $\tilde{\mu}_{n+1}/(\gamma_n \sigma_{n+1}^2)$. However, rather than depending only on statistics of the market return between n and $n+1$, it also accounts for the return $\tilde{r}_{n+1,T}^{\pi^*}$ achieved by the optimal portfolio between times $n+1$ and T . The second term in the decomposition is the *intertemporal hedging demand*, which can be rewritten as

$$Cov_{n,d}[\tilde{Z}_{n+1}\tilde{r}_{n+1,T}^{\pi^*}, \tilde{r}_{n+1,T}^{\pi^*}] = Cov_{n,d}[(\tilde{Z}_{n+1} - \tilde{\mu}_{n+1})\tilde{r}_{n+1,T}^{\pi^*}, \tilde{r}_{n+1,T}^{\pi^*}] + \tilde{\mu}_{n+1}Var_{n,d}(\tilde{r}_{n+1,T}^{\pi^*}),$$

where the first term captures the contribution of dynamic risk aversion. To analyze this term, it is convenient to first rewrite it as

$$Cov_{n,d}[(\tilde{Z}_{n+1} - \tilde{\mu}_{n+1})\tilde{r}_{n+1,T}^{\pi^*}, \tilde{r}_{n+1,T}^{\pi^*}] = Cov_{n,d}[(\tilde{Z}_{n+1} - \tilde{\mu}_{n+1})\mathbb{E}_{n+1,D_{n+1}}[\tilde{r}_{n+1,T}^{\pi^*}], \mathbb{E}_{n+1,D_{n+1}}[\tilde{r}_{n+1,T}^{\pi^*}]],$$

which shows that its sign depends predominantly on the covariance between $\tilde{Z}_{n+1} - \tilde{\mu}_{n+1}$, i.e., the *current* excess market return, and $\mathbb{E}_{n+1,D_{n+1}}[\tilde{r}_{n+1,T}^{\pi^*}]$, i.e., the *future* expected portfolio return. This covariance is generally positive, resulting in a negative hedging demand. To see this, observe that we generally expect an above average market return, $\tilde{Z}_{n+1} - \tilde{\mu}_{n+1} > 0$, to push down the future risk aversion values $(\gamma_k)_{n+1 \leq k < T}$; this, in turn, implies a positive relation between $\tilde{Z}_{n+1} - \tilde{\mu}_{n+1}$ and future risky asset allocations $(\tilde{\pi}_k^*)_{n+1 \leq k < T}$. As a result, the expectation $\mathbb{E}_{n+1,D_{n+1}}[\tilde{r}_{n+1,T}^{\pi^*}]$, achieved through the allocations $(\tilde{\pi}_k^*)_{n+1 \leq k < T}$, is larger following a positive value of $\tilde{Z}_{n+1} - \tilde{\mu}_{n+1}$. Altogether, the current market return and the future expected portfolio return thus move in the same direction, amplifying the effect of the current market return on the variance of the terminal wealth and leading to a negative hedging demand.

This is akin to the intertemporal hedging term in Basak and Chabakauri [2010], which arises because of covariation between market returns and the state variable. In their setting, the hedging demand would vanish if market returns and changes in the state variable were uncorrelated, in which case trading in the risky asset cannot hedge fluctuations in the state variable. In our setting, the hedging demand is driven by the dynamic risk aversion process that links current market return with future portfolio returns. This hedging demand appears even though the market returns in our model are conditionally independent of changes in the state variable.¹³

Next, we consider the special case where intertemporal hedging due to dynamic risk aversion vanishes, i.e., when future risk aversion values are conditionally independent of the current market return. In this

¹³Conditionally on the value of Y_n , the market return \tilde{Z}_{n+1} is independent of $Y_{n+1} - Y_n$.

case, the optimal allocation strategy admits a more explicit expression, given only in terms of the first two moments of the return variables \tilde{Z}_{n+1} and $\tilde{r}_{n+1,T}^{\pi^*}$.

Corollary 4.2. *Given $D_n = d$, assume the risk aversion values $(\gamma_{n+1}, \gamma_{n+2}, \dots, \gamma_{T-1})$ to be independent of the market return \tilde{Z}_{n+1} . The optimal proportion of wealth allocated to the risky asset at time n is given by*

$$\tilde{\pi}_n^*(d) = \frac{1}{\gamma_n \sigma_{n+1}^2 \mathbb{E}_{n,d}[(\tilde{r}_{n+1,T}^{\pi^*})^2] + \tilde{\mu}_{n+1}^2 \text{Var}_{n,d}[\tilde{r}_{n+1,T}^{\pi^*}]} \tilde{\mu}_{n+1} \mathbb{E}_{n,d}[\tilde{r}_{n+1,T}^{\pi^*}] - R_{n+1} \frac{\tilde{\mu}_{n+1} \text{Var}_{n,d}[\tilde{r}_{n+1,T}^{\pi^*}]}{\sigma_{n+1}^2 \mathbb{E}_{n,d}[(\tilde{r}_{n+1,T}^{\pi^*})^2] + \tilde{\mu}_{n+1}^2 \text{Var}_{n,d}[\tilde{r}_{n+1,T}^{\pi^*}]},$$

where $\tilde{\mu}_{n+1} = \mathbb{E}_{n,d}[\tilde{Z}_{n+1}]$ and $\sigma_{n+1}^2 = \text{Var}_{n,d}[\tilde{Z}_{n+1}]$.

The optimal allocation given in the corollary can alternatively be written as¹⁴

$$\tilde{\pi}_n^*(d) = \frac{\tilde{\mu}_{n+1}}{\gamma_n \sigma_{n+1}^2} \frac{\mathbb{E}_{n,d}[\tilde{r}_{n+1,T}^{\pi^*}] - R_{n+1} \gamma_n (\mathbb{E}_{n,d}[(\tilde{r}_{n+1,T}^{\pi^*})^2] - (\mathbb{E}_{n,d}[\tilde{r}_{n+1,T}^{\pi^*}])^2)}{\mathbb{E}_{n,d}[(\tilde{r}_{n+1,T}^{\pi^*})^2] + \left(\frac{\tilde{\mu}_{n+1}}{\sigma_{n+1}}\right)^2 (\mathbb{E}_{n,d}[(\tilde{r}_{n+1,T}^{\pi^*})^2] - (\mathbb{E}_{n,d}[\tilde{r}_{n+1,T}^{\pi^*}])^2)}, \quad (4.2)$$

which allows to pin down the relation with the optimal investment strategy in the classical mean-variance setup. Specifically, it is proportional to the single-period Markowitz strategy, $\tilde{\mu}_{n+1}/(\gamma_n \sigma_{n+1}^2)$, i.e., the optimal allocation of a myopic investor whose objective functional spans a single period. This fraction depends on the *current* economic conditions, and the client's *current* risk aversion. The proportionality factor in (4.2) depends on the *future* return of the investment strategy, $\tilde{r}_{n+1,T}^{\pi^*}$, which in turn depends on both future economic conditions and the client's future risk aversion dynamics. Consistently with intuition, this factor is increasing in the first moment of $\tilde{r}_{n+1,T}^{\pi^*}$ and decreasing in its second moment. In the final time period, the proportionality factor is equal to one and the optimal allocation is given explicitly by the Markowitz strategy in this time period.

5 Performance of the Robo-Advising Framework

We analyze the investment performance of the proposed robo-advising framework. Our analysis will be based on a specific model that fits into the general framework, and which we introduce in Section 5.1. In Section 5.2, we consider the interplay between interaction and portfolio personalization, and show the existence of a tradeoff between the *rate of information acquisition* from the client and the *effect of the client's behavioral biases*. In Section 5.3, we explore how the optimal investment strategy is affected by transitions between economic states and the associated changes in the client's risk aversion. We also discuss whether the robo-advisor should cater to client wishes or go against them to improve the investment performance.

¹⁴An extension of formula (4.2) can be derived for the general model considered in Proposition 4.1 (see Appendix B).

5.1 A Robo-Advising Model

In the following paragraphs, we specify each component of the framework introduced in Sections 3.1-3.4.

Market Dynamics. The risky asset has conditionally Gaussian returns. That is, given the economic state $Y_n = y$ at time n , the return Z_{n+1} has a Gaussian distribution with mean $\mu(y)$ and variance $\sigma^2(y)$.

State Process Dynamics. We assume a two-state economy $(Y_n)_{n \geq 0}$. This choice is supported by the methodology of the National Bureau of Economic Research (NBER), which splits business cycles into periods of economic expansions and contractions.¹⁵ We use $y = 1$ to denote the state of economic growth and $y = 2$ to denote the state of recession. We use P to denote the transition probability matrix, so that P_{ij} is the probability that the economy transits from state i to state j in a single time step.

Interaction Schedule. The sequence of interaction times is deterministic and equally spaced. The interaction schedule $(T_k)_{k \geq 0}$ is then characterized by a fixed integer parameter $\phi \geq 1$, with $T_k = k\phi$. The two extreme cases are $\phi = 1$, which corresponds to communication of risk preferences at all times, and $\phi \geq T$, which corresponds to risk preferences communicated only at the beginning of the investment process.

Client's Risk Aversion. The client's risk aversion process, $(\gamma_n^C)_{n \geq 0}$, is of the form

$$\gamma_n^C = e^{\eta_n} \gamma_n^{id} \gamma_n^Y. \quad (5.1)$$

The first component, $(e^{\eta_n})_{n \geq 0}$, is deterministic and time-varying. For example, the specification $e^{\eta_n} = e^{-\alpha(T-n)}$ captures the empirical observation that risk aversion increases with age, with the parameter $\alpha \geq 0$ specifies the rate of increase.

The second component, $(\gamma_n^{id})_{n \geq 0}$, depends on the client's personal circumstances and has stochastic dynamics, $\gamma_n^{id} = \gamma_{n-1}^{id} e^{\epsilon_n}$, where $(\epsilon_n)_{n \geq 1}$ is an i.i.d. sequence of random variables. Specifically, $\epsilon_n = \sigma_\epsilon W_n - \frac{\sigma_\epsilon^2}{2}$ with probability $p_\epsilon \in [0, 1]$, where $\sigma_\epsilon > 0$ and $(W_n)_{n \geq 1}$ is an i.i.d. sequence of standard Gaussian random variables, and $\epsilon_n = 0$ with probability $1 - p_\epsilon$. This component captures idiosyncratic shocks to the client's risk aversion, which are unrelated to economic state transitions and market dynamics. The multiplicative innovation terms $(e^{\epsilon_n})_{n \geq 1}$ are independent and with unit mean, making $(\gamma_n^{id})_{n \geq 0}$ a martingale, devoid of a predictable component.

¹⁵The work of Chauvet and Hamilton [2006] shows that a Markov switching model successfully identifies NBER's business cycle turning points.

The third component, $(\gamma_n^Y)_{n \geq 0}$, is given by $\gamma_n^Y = \bar{\gamma}_n(Y_n)$, where $\bar{\gamma}_n : \mathcal{Y} \mapsto \mathbb{R}_0^+$ is a possibly time-varying function of the current economic state. This component captures the empirical fact that risk aversion varies with the business cycle.

Client's Behavioral Bias. At time n , the risk aversion parameter communicated by the client at the most recent interaction time, τ_n , is of the form

$$\xi_n = \gamma_{\tau_n}^C \gamma_{\tau_n}^Z := \gamma_{\tau_n}^C e^{-\beta \left(\frac{1}{\phi} \sum_{k=\tau_n-\phi}^{\tau_n-1} (Z_{k+1} - \mu_{k+1}) \right)}, \quad (5.2)$$

where $\gamma_{\tau_n}^Z$ is a factor that inflates or deflates the client's actual risk aversion, $\gamma_{\tau_n}^C$, based on recent market returns. Namely, the sum in the exponent is the cumulative excess return of the risky asset over its expected value, since the previous interaction time, $\tau_n - \phi$. Because previous market returns have no predictive value for future market returns, given the state of the economy, the factor $\gamma_{\tau_n}^Z$ is representative of common behavioral biases, and the coefficient $\beta \geq 0$ determines the magnitude of this effect.

We emphasize two important properties of the component $\gamma_{\tau_n}^Z$. First, the convexity of the exponential function allows us to capture loss aversion: the upward risk aversion bias following a market underperformance is greater than the downward bias when the market exceeds expectations by the same amount (Tversky and Kahneman [1979]). Second, in a model with a single economic state y , we have $\mathbb{E}[\gamma_{\tau_n}^Z] = e^{\frac{\beta \sigma^2(y)}{2\phi}}$, which shows that the average effect of the behavioral bias decreases as the time between interaction increases,¹⁶ consistent with the notion of myopic loss aversion, introduced in Benartzi and Thaler [1995].

The bias $\gamma_{\tau_n}^Z$ may be interpreted as the client overriding the robo-advisor's decisions. That is, at an interaction time n , the robo-advisor proposes a portfolio tailored to the client's characteristics, γ_n^C , but the client makes changes to the proposed allocation based on recent market returns. This results in a portfolio allocation consistent with a risk aversion coefficient $\gamma_n^C \gamma_n^Z$. The effect of this override, which is driven by the client's behavioral biases, prevails until the subsequent time of interaction, $n + \phi$, when the client again adjusts the portfolio allocation proposed by the robo-advisor. This time, however, the adjustment is based on market returns realized in the time interval $[n, n + \phi]$.

Robo-Advisor's Model of Client's Risk Aversion. The risk aversion process $(\gamma_n)_{n \geq 0}$ is of the form

$$\gamma_n = e^{\eta_n - \eta_{\tau_n}} \xi_n \frac{\gamma_n^Y}{\gamma_{\tau_n}^Y}. \quad (5.3)$$

¹⁶In a setup with multiple economic states we have, conditionally on $Y_{\tau_n-\phi} = y \in \mathcal{Y}$, that $\mathbb{E}[\gamma_{\tau_n}^Z] = e^{\frac{\beta \sigma^2(y)}{2\phi}} + O(1 - P_{y,y}^\phi)$.

The component ξ_n is the risk aversion parameter communicated by the client at the most recent interaction time, τ_n , and, in the robo-advisor's model, it stays constant until the subsequent time of interaction, $\tau_n + \phi$. On the other hand, the model is adjusted for the passage of time through the factor $e^{\eta_n - \eta_{\tau_n}}$, and for changes in economic states through the ratio $\gamma_n^X / \gamma_{\tau_n}^X$. The above equation can be written in an alternative form as

$$\gamma_n = \gamma_n(d) = \gamma_{\tau_n}^Z \mathbb{E}_{n,d}[\gamma_n^C],$$

where we recall from (3.3) that $D_n = d$ is the information set of the robo-advisor at time n . This shows that the robo-advisor's view of the client risk aversion is equal to the expectation of client's actual risk aversion times bias factor from the previous interaction time.

From the above expression it can be observed that interaction impacts the robo-advisor's inferred value of the client's risk aversion in two different ways. First, between interaction times, the expectation $\mathbb{E}_{n,d}[\gamma_n^C]$ is not updated in response to changes in the idiosyncratic component of the client's risk aversion, because such changes cannot be inferred from observables like market returns or economic indicators. Second, a higher interaction frequency comes at the expense of increased behavioral bias, tilting the robo-advisor's model further away from the client's actual risk aversion.

Calibration of Risk Aversion Process. The multiplicative model introduced in this section provides a parsimonious description of the client's risk aversion. Under this specification, the risk aversion is guaranteed to be non-negative, and its parameters can be estimated from historical portfolio allocations. We next sketch a calibration procedure for models of this form.

Consider a sequence of historical portfolio allocations for a set of self-directed investors. Such data is available to financial firms offering discount brokerage services, in particular the growing number of low-cost online brokerages.¹⁷ Assume that the portfolio allocations are consistent with a myopic mean-variance criterion, for which there exists a one-to-one relationship between allocations and risk aversion levels. Then, the allocations can equivalently be viewed as historical sequences of implied risk aversion coefficients.

For a given client, the risk aversion model can be calibrated using the subset of investors that have demographic characteristics similar to the client.¹⁸ Specifically, the three components of the risk aversion process (5.1) and the behavioral bias in (5.2) can be estimated using the following sequential procedure. First, the temporal component can be inferred from the initial portfolio composition chosen by investors at different life stages. For a personalized service, this component can also be used to incorporate a client-specific

¹⁷See, among others, the seminal work of Barber and Odean [2000] where such a data set is analyzed and Glaser and Weber [2009] for a more recent study using online brokerage data.

¹⁸In practice, robo-advising firms manage a finite number of portfolios and group clients with similar profiles into risk categories. For instance, Wealthfront constructs a composite Risk Score ranging from 0.5 to 10, in increments of 0.5.

time-dependence of risk aversion. Adjusting for the effect of time, the average investment levels during each economic regime (e.g., periods of expansions and contractions - see Section 5.3) then provide information about the process component which depends on the economic state. Next, the extent of behavioral bias can be estimated from the relationship between market returns and subsequent changes in the portfolio's market exposure.¹⁹ In addition, for a given client, data on financial literacy, investment experience, and cognitive abilities, can be used to adjust the estimated behavioral bias (see, e.g., studies by Oechssler et al. [1997] and Seasholes and Feng [2005]). Finally, after adjusting for the passage of time, economic regimes, and bias, the size and frequency of allocation changes made by investors provide information about the idiosyncratic component of the risk aversion process.

Sensitivity of Optimal Allocation to Idiosyncratic Shock and Behavioral Bias. In Figure 1, we display how the optimal allocation in Proposition 4.1 depends on the variability of the idiosyncratic component of the client's risk aversion and the client's behavioral bias. The left panel shows that if the client's idiosyncratic risk preferences fluctuate more due to a larger value of the parameter p_ϵ , then the optimal allocation deviates more from that of the benchmark case with no idiosyncratic shocks. The right panel shows that the client's behavioral bias following a negative market return ($\gamma_n^Z > 1$) has a stronger influence on the optimal allocation than the bias following a positive market return of the same magnitude ($\gamma_n^Z < 1$). This numerical finding is consistent with the notion of loss aversion, as discussed above.

5.2 Interaction and Optimal Personalization

In our framework, the client and robo-advisor do not interact at all times and, thus, the robo-advisor is not always immediately aware of changes in the client's risk profile. While a higher interaction frequency is the only way to reduce this information asymmetry, it also increases the amount of behavioral bias in the communicated information. The goal of this section is to analyze this tradeoff quantitatively.

We start by defining a measure of portfolio personalization. This measure depends on the relation between the client's risk aversion process, $(\gamma_n^C)_{n \geq 0}$, given by (5.1), and the robo-advisor's model of the client's risk aversion, $(\gamma_n)_{n \geq 0}$, given by (5.3). Recall from Section 4 that at time $n = T - 1$, the optimal risky asset allocation is proportional to the risk tolerance parameter $1/\gamma_n$, and it is approximately proportional to $1/\gamma_n$ for $n < T - 1$ (see (4.2)). We therefore choose to define personalization in terms of the expected relative difference between the risk tolerance parameters $1/\gamma_n$ and $1/\gamma_n^C$, averaged over the investment

¹⁹The literature on individual investor performance focuses to a large extent on biases and investor's mistakes. For instance, Statman et al. [2006] study the role of investor overconfidence by examining the lead-lag relationship between market returns and trading volume.

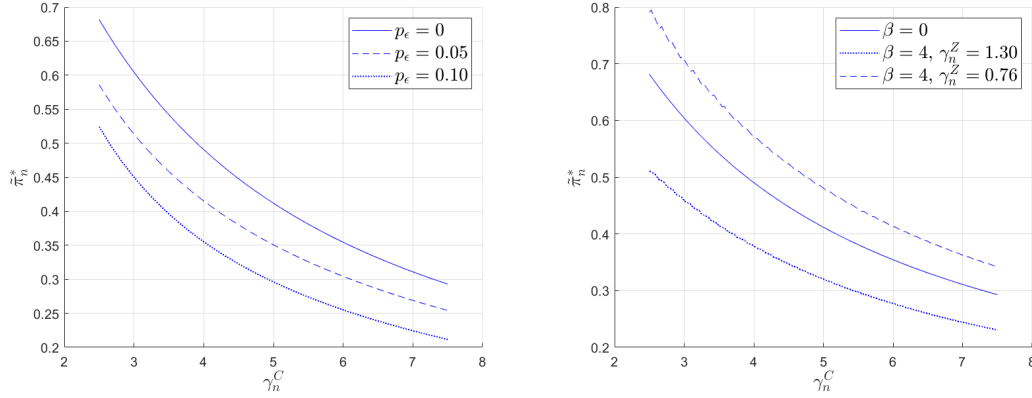


Figure 1: The graphs plot the optimal allocation $\tilde{\pi}_n^*$ as a function of the client's risk aversion γ_n^C , for the model in Section 5.1, where we set $Y_0 = 1$ and P equal to the two-dimensional identity matrix. In both graphs, the solid line refers to the benchmark case of constant risk aversion and no behavioral bias, i.e., $\eta_n = p_\epsilon = \beta = 0$.

Left panel: Impact of idiosyncratic risk aversion shocks, obtained by letting $p_\epsilon \in \{0.05, 0.10\}$ and $\sigma_\epsilon = 0.64$. Right panel: Impact of behavioral bias, obtained by choosing $\beta = 4$ and $\gamma_n^Z \in \{0.76, 1.30\}$. These values of β and γ_n^Z correspond to a cumulative market return between two interaction times which is 1.3 standard deviations above/below its expected value. The annualized market parameters in state Y_0 are set to $r(Y_0) = 0$, $\mu(Y_0) = 0.10$, $\sigma(Y_0) = 0.20$, and one time step in the model corresponds to one month. The investment horizon is $T = 36$ months, and the time between interaction is $\phi = 3$ months. We show the allocations prevailing one year after the start of the investment process, i.e., at time $n = 12$.

period. Specifically, for $\phi \geq 1$ and $\beta \geq 0$, we introduce the *personalization measure*

$$\mathcal{R}(\phi, \beta) := \mathbb{E} \left[\frac{1}{T} \sum_{n=0}^{T-1} \left| \frac{1}{\gamma_n} - \frac{1}{\gamma_n^C} \right| \right]. \quad (5.4)$$

A lower value of $\mathcal{R}(\phi, \beta)$ means a higher level of portfolio personalization. Full personalization is achieved if and only if $\phi = 1$ and $\beta = 0$, which leads to $\mathcal{R}(1, 0) = 0$. This is because the processes $(\gamma_n)_{n \geq 0}$ and $(\gamma_n^C)_{n \geq 0}$ coincide only in this case, when there is no behavioral bias and the client and the robo-advisor interact at all times. Although not explicitly highlighted in the notation, we note that the value of $\mathcal{R}(\phi, \beta)$ also depends on the parameters $p_\epsilon \in [0, 1]$ and $\sigma_\epsilon > 0$, which govern the distribution of idiosyncratic risk aversion shocks.

For a given value of β , our objective is to study the dependence of the measure \mathcal{R} on ϕ . For this, we consider an approximation $\tilde{\mathcal{R}}$ that is analytically more tractable than \mathcal{R} . In the following proposition, we characterize this approximation and show that it is minimized by a unique value of ϕ . In Figure 2, we show graphically that $\tilde{\mathcal{R}}$ and \mathcal{R} are close, for different interaction frequencies and levels of behavioral bias.

Proposition 5.1. *Set $\sigma_0 := \sigma(Y_0)$. The following statements hold:*

(i) *The measure $\mathcal{R}(\phi, \beta)$ satisfies the relation*

$$\frac{T_\phi}{T} \tilde{\mathcal{R}}(\phi, \beta) + \mathcal{E} \leq \mathcal{R}(\phi, \beta) \leq \frac{T_\phi}{T} \tilde{\mathcal{R}}(\phi, \beta) + \mathcal{E}, \quad (5.5)$$

where T_ϕ (resp. T^ϕ) is the largest (resp. smallest) multiple of ϕ that is less than (resp. greater than) or equal to T , the error term \mathcal{E} is of order $O(p_\epsilon^2 \sigma_\epsilon^2 + \beta^2) + O(1 - P_{y_0, y_0}^{T-1})$, and

$$\tilde{\mathcal{R}}(\phi, \beta) := \sqrt{\frac{2}{\pi}} \left(\frac{\beta \sigma_0}{\sqrt{\phi}} \left(1 - \frac{\phi - 1}{2} p_\epsilon \right) + \sqrt{\frac{\beta^2 \sigma_0^2}{\phi} + \sigma_\epsilon^2 \frac{\phi - 1}{2} p_\epsilon} \right).$$

(ii) Let β and p_ϵ be such that $\beta + p_\epsilon > 0$. Then, there exists a unique value of $\phi \geq 1$ that minimizes $\tilde{\mathcal{R}}(\beta, \phi)$.²⁰ Specifically,

$$\arg \min_{\phi \geq 1} \tilde{\mathcal{R}}(\phi, \beta) = \begin{cases} 1, & \beta = 0, \\ \phi_0 \in [1, \infty), & \beta > 0, p_\epsilon > 0, \\ \infty, & p_\epsilon = 0. \end{cases}$$

Furthermore, ϕ_0 satisfies the monotonicity properties

$$\frac{\partial \phi_0}{\partial \beta} > 0, \quad \frac{\partial \phi_0}{\partial \sigma_0} > 0, \quad \frac{\partial \phi_0}{\partial p_\epsilon} < 0, \quad \frac{\partial \phi_0}{\partial \sigma_\epsilon} < 0. \quad (5.6)$$

The error term of order $O(1 - P_{y_0, y_0}^{T-1})$ in the approximation (5.5) results from fixing the economic state throughout the investment period. Typically, transitions between economic states are infrequent (see (5.12) in Section 5.3) and the error term vanishes if the model consists of a single economic state. From the signs of the derivatives given in (5.6), it can be seen that if the economy transits to a state with a higher return volatility, i.e., if σ_0 increases, the optimal time between consecutive interactions also increases. This is because the client's behavioral bias is based on market return fluctuations, which are magnified if the return volatility is higher.

The above proposition confirms two intuitive claims. First, in the absence of behavioral biases, i.e., $\beta = 0$, it is optimal for the robo-advisor to interact with the client at all times. Second, if there are no idiosyncratic risk aversion shocks, i.e., $p_\epsilon = 0$, then it is optimal to never interact. Most interestingly, if both $\beta > 0$ and $p_\epsilon > 0$, then it may be suboptimal to interact at all times. This is because a higher frequency of interaction comes at the expense of increased behavioral bias in the communicated risk aversion parameter. Furthermore, the signs of the derivatives in (5.6) show that a larger value of β increases the optimal time between consecutive interactions, while larger values of p_ϵ and σ_ϵ imply a greater variance of idiosyncratic risk aversion shocks and push down the optimal time between interaction. In the proof of the proposition,

²⁰If $\beta = p_\epsilon = 0$, the values $\mathcal{R}(\phi, \beta)$ and $\tilde{\mathcal{R}}(\phi, \beta)$ are independent of ϕ .

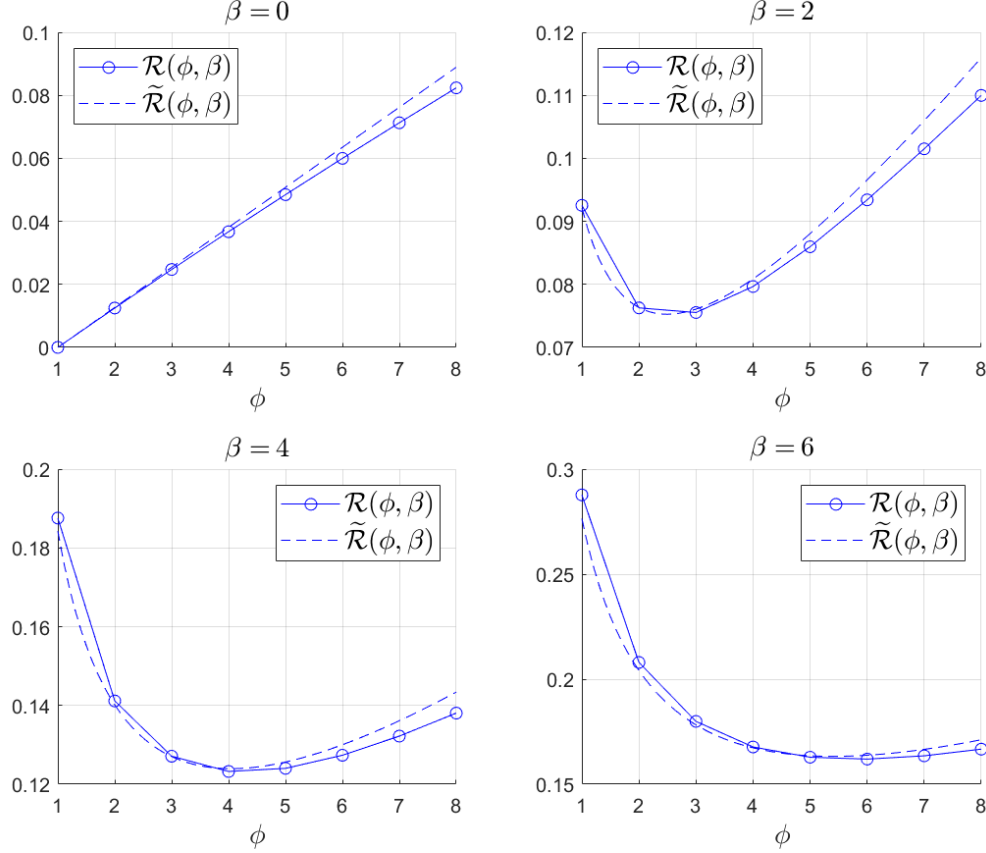


Figure 2: The graphs plot the personalization measure $\mathcal{R}(\phi, \beta)$ and its approximation $\tilde{\mathcal{R}}(\phi, \beta)$ as a function of the time ϕ between consecutive interactions. Each panel corresponds to a different value of the parameter β . We consider the same market parameters and investment horizon as in Figure 1, with $Y_0 = 1$, $P = I_2$, and each time step in the model corresponding to one month. The parameters of the risk aversion model are $\eta_n = 0$, $p_\epsilon = 0.05$, and $\sigma_\epsilon = 0.64$.

we show that

$$\frac{p_\epsilon \sigma_\epsilon}{\beta \sigma_0} < 1 \implies \left. \frac{\partial \tilde{\mathcal{R}}(\phi, \beta)}{\partial \phi} \right|_{\phi=1} < 0,$$

which gives a sufficient condition for when it is optimal to refrain from interacting at all times.²¹ In the above inequality, we compare the rate of change of the idiosyncratic risk-aversion component, as measured by $p_\epsilon \sigma_\epsilon$, with the amount of behavioral bias, quantified by the product $\beta \sigma_0$ of the client's sensitivity to market returns and the volatility of returns. If the latter is greater than the former, then the derivative of $\tilde{\mathcal{R}}$ at $\phi = 1$ is negative, i.e., it is suboptimal to interact at all times.

To ensure a high level of personalization, the robo-advisor must construct a process $(\gamma_n)_{n \geq 0}$ which is as close as possible to the client's actual risk aversion process $(\gamma_n^C)_{n \geq 0}$. A related measure of personalization may thus be directly built on the proximity of the investment strategy corresponding to $(\gamma_n)_{n \geq 0}$ and the

²¹This condition is also close to being necessary. See Eq. (C.2) in Appendix C.

strategy corresponding to $(\gamma_n^C)_{n \geq 0}$. The latter is the strategy that achieves full personalization and can only be attained if interaction occurs at all times and there is no behavioral bias, in which case the two risk aversion processes coincide.

Denote by $(\tilde{\pi}_n^*(\gamma_n))_{0 \leq n < T}$ and $(\tilde{\pi}_n^*(\gamma_n^C))_{0 \leq n < T}$ the optimal allocations corresponding to $(\gamma_n)_{0 \leq n < T}$ and $(\gamma_n^C)_{0 \leq n < T}$, respectively. In line with the definition of $\mathcal{R}(\phi, \beta)$ in (5.4), we introduce the measure

$$\mathcal{S}(\phi, \beta) := \mathbb{E} \left[\frac{1}{T} \sum_{n=0}^{T-1} \left| \frac{\tilde{\pi}_n^*(\gamma_n) - \tilde{\pi}_n^*(\gamma_n^C)}{\tilde{\pi}_n^*(\gamma_n^C)} \right| \right].$$

A smaller value of $\mathcal{S}(\phi, \beta)$ implies a higher level of personalization, and full personalization is achieved if and only if $\mathcal{S}(\phi, \beta) = 0$. We observe that the difference between $\tilde{\pi}_n^*(\gamma_n)$ and $\tilde{\pi}_n^*(\gamma_n^C)$ is not a simple function of the difference between γ_n and γ_n^C . Nevertheless, in Appendix C we establish a direct relation between the two measures \mathcal{R} and \mathcal{S} , namely,

$$\mathcal{S}(\phi, \beta) = \mathcal{R}(\phi, \beta) + O((\phi - 1)p_\epsilon \sigma_\epsilon^2) + O\left(\frac{\beta^2 \sigma^2}{\phi^2}\right), \quad (5.7)$$

which shows that a small difference between $(\gamma_n)_{0 \leq n < T}$ and $(\gamma_n^C)_{0 \leq n < T}$ translates into a small difference between the corresponding investment strategies. In the above equation, the first error term is due to the idiosyncratic component of the client's risk aversion and it vanishes if $\phi = 1$. The second error term originates from the client's behavioral bias and it is maximized at $\phi = 1$.

Figure 3 indicates that the value of ϕ that minimizes $\mathcal{R}(\phi, \beta)$ is a lower bound for the value of ϕ that minimizes $\mathcal{S}(\phi, \beta)$. In other words, minimizing $\mathcal{R}(\phi, \beta)$ provides a conservative estimate for the time between interaction that minimizes the allocation difference $\mathcal{S}(\phi, \beta)$. This is because $\mathcal{R}(\phi, \beta)$ underestimates the impact of the client's behavior bias. Specifically, $\mathcal{R}(\phi, \beta)$ equals the average relative difference between $(\gamma_n^C)_{n \geq 0}$ and $(\gamma_n)_{n \geq 0}$ in any single time interval defined by two consecutive interaction times. Instead, the optimal allocation at any given time depends on all future allocations and, thus, on the future path of the client's risk aversion. As a result, the behavioral bias in future investment decisions feeds into the investment decisions made at earlier times, and this is accounted for by the measure $\mathcal{S}(\phi, \beta)$.

Remark 5.2. Uniformly spaced interaction times yield a conservative estimate for the optimal interaction frequency. If the client were to interact with the robo-advisor at times triggered by market conditions, then we expect interaction to be more likely to occur after a period of extreme returns. The client would either be overly exuberant following a period of positive returns, or overly pessimistic after a period of negative returns. This magnifies the impact of the client's behavioral bias, relative to a uniform interaction schedule, and would result in a larger value for ϕ . \square

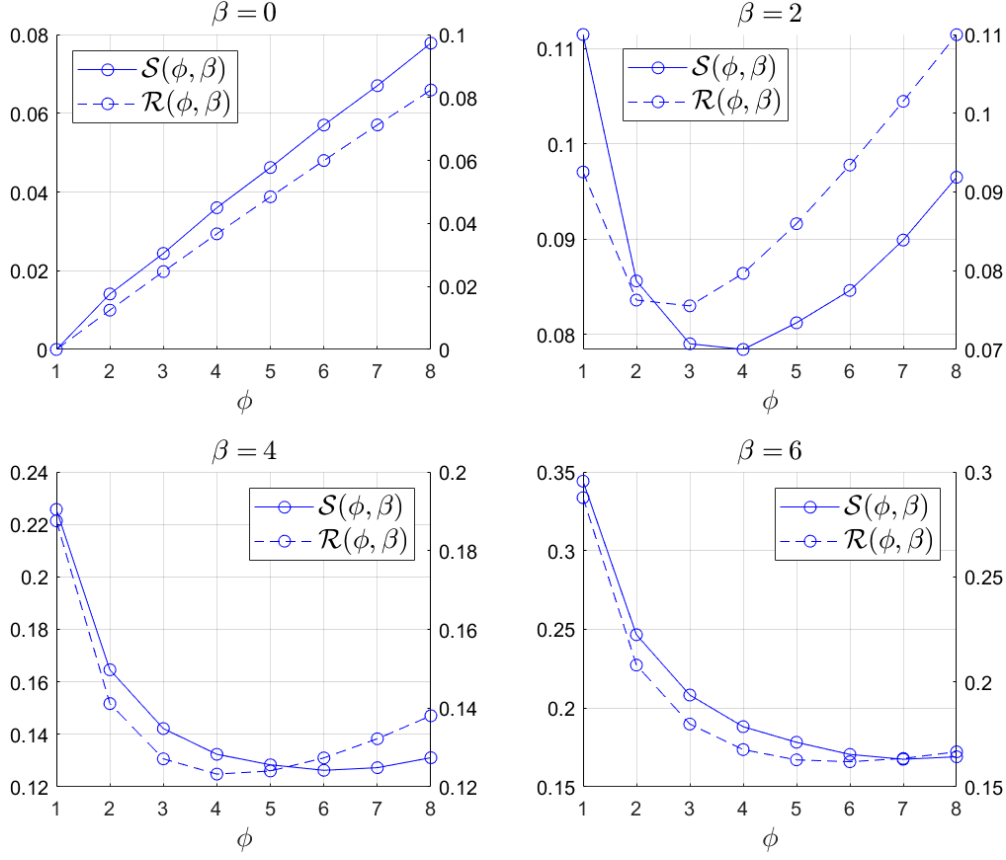


Figure 3: The graphs plot the personalization measures $\mathcal{S}(\phi, \beta)$ (left y -axis) and $\mathcal{R}(\phi, \beta)$ (right y -axis) as a function of the time ϕ between consecutive interactions. Each panel corresponds to a different value of the parameter β . The model parameters are the same as in Figure 2.

5.3 Economic Transitions, Risk Aversion, and Investment Performance

We analyze how the optimal investment strategy is affected by economic state transitions and the corresponding changes in the client's risk aversion. To focus exclusively on the implications of economic transitions on investment decisions, we let the client's risk aversion depend only on the current economic state. In this case, the risk aversion process in (5.3) takes the form $\gamma_n = \bar{\gamma}_n(Y_n)$. We calibrate this process so that the resulting optimal allocation $(\tilde{\pi}_n^*)_{0 \leq n < T}$ is time-homogeneous, given the current economic state. Specifically, in Appendix C (see Proposition C.1) we show that, for a given $\bar{\pi} > 0$ and $\delta > -1$, there exists a unique risk aversion process $(\gamma_n^{(\delta)})_{n \geq 0}$ such that the corresponding optimal allocation is given by

$$\tilde{\pi}_n^* = \begin{cases} \bar{\pi}, & Y_n = 1, \\ \bar{\pi}(1 + \delta), & Y_n = 2. \end{cases} \quad (5.8)$$

That is, in times of growth ($y = 1$), the client's risky asset allocation is equal to $\bar{\pi}$,²² and when the economy transits to the recessionary regime ($y = 2$), the allocation changes to $\bar{\pi}(1 + \delta)$, where the value of δ determines the change in allocation.

It is well known from empirical studies that both the market Sharpe ratio and the risk aversion of retail investors are higher during periods of contractions. Consequently, retail investors may shift wealth away from the risky asset precisely when the benefit of investing, as measured by the expected reward per unit risk, is higher.²³ For these investors, a robo-advisor which *caters to the client's wishes* would construct a risk aversion process $(\gamma_n^{(\delta)})_{n \geq 0}$ such that $\delta < 0$. By contrast, a robo-advisor which *goes against the client's wishes* would construct a risk aversion process $(\gamma_n^{(\delta)})_{n \geq 0}$ such that $\delta \geq 0$, i.e., invest more in the risky asset when the economy is in a state of contraction and the return per unit risk is high.

The remainder of the section proceeds as follows. In Section 5.3.1, we analyze the Sharpe ratio achieved by the optimal investment strategy and its dependence on the parameter δ . In Section 5.3.2, we analyze numerically the dependence of the client's terminal wealth distribution on δ and discuss how the robo-advisor should balance the client's risk preferences with the opportunity to improve the portfolio's investment performance.

5.3.1 Sharpe Ratio of Optimal Investment Strategy

The Sharpe ratio of the strategy (5.8) is defined as

$$s^{\pi^*}(\delta) := \frac{\tilde{\mu}^{\pi^*}}{\sigma^{\pi^*}}, \quad (5.9)$$

where $\tilde{\mu}^{\pi^*}$ and σ^{π^*} are the long-run²⁴ mean and volatility of the excess returns achieved by π^* :

$$\tilde{\mu}^{\pi^*} := \lim_{T \rightarrow \infty} \mathbb{E} \left[\frac{1}{T} \sum_{n=0}^{T-1} (r_{n,n+1}^{\pi^*} - r_{n+1}) \right], \quad (\sigma^{\pi^*})^2 := \lim_{T \rightarrow \infty} \mathbb{E} \left[\frac{1}{T} \sum_{n=0}^{T-1} (r_{n,n+1}^{\pi^*} - r_{n+1} - \tilde{\mu}^{\pi^*})^2 \right].$$

We recall that $r_{n,n+1}^{\pi^*}$ is the return of the strategy π^* , between times n and $n + 1$, and that r_{n+1} is the risk-free rate. In Lemma C.2 we show that the Sharpe ratio (5.9) can be explicitly computed.

Figure 4 indicates that, for a fixed value of δ , the portfolio's Sharpe ratio is increasing in the stationary

²²For example, $\bar{\pi} = 0.60$ corresponds to the classical 60/40 portfolio composition. This strategy was popularized by Jack Bogle, the founder of Vanguard, and is commonly used as a benchmark in portfolio allocation.

²³The economic state variable $(Y_n)_{n \geq 0}$ is assumed to be observable, and it follows from (5.1) that the client's risk aversion reacts instantaneously to a change in economic conditions. In reality, the economic state may be hidden, and its value can only be inferred probabilistically. We expect that extending the model to such a setting would not have a qualitative effect on our results. Namely, the extended model would capture the fact that, throughout the business cycle, the client is on average less willing to invest in the risky asset when the market Sharpe ratio is high.

²⁴We compute the "long-run" Sharpe ratio of the strategy π^* , which is independent of the initial economic state, Y_0 .

probability λ of the recessionary state ($y = 2$). Formally, $\lambda := \lim_{n \rightarrow \infty} \mathbb{P}(Y_n = 2)$, and λ also equals the long-run proportion of time spent in the recessionary state. Additionally, the Sharpe ratio is increasing in $a := \tilde{\mu}(2)/\tilde{\mu}(1)$, which is the relative change in mean excess returns between the two states, and decreasing in $b := \sigma(2)/\sigma(1)$, which is the relative change in the return volatility. These monotonicity properties turn out to hold under mild conditions (see Lemma C.3).

Furthermore, Figure 4 shows that the Sharpe ratio (C.5) is generally increasing in δ , i.e., it is higher if a larger amount of wealth is invested in the risky asset when the economy is in a state of recession and, thus, the market Sharpe ratio is high. However, as the third panel shows, there may be situations when a larger value of δ may result in a smaller Sharpe ratio.

In Lemma C.3, we derive the following characterization for the monotonicity of the Sharpe ratio with respect to δ ,

$$\frac{\partial s^{\pi^*}(\delta)}{\partial \delta} > 0 \iff 1 + \frac{\sigma^2(1)}{\tilde{\mu}^2(1)} > a \left(1 + \frac{b^2 \sigma^2(1)}{a^2 \tilde{\mu}^2(1)} \right) (1 + \delta). \quad (5.10)$$

Whether or not the condition above is satisfied depends on the values of a and b , which determine the difference in market Sharpe ratios between the two economic states. In particular, consider the case where the allocation is the same in both economic states, i.e., $\delta = 0$. For a fixed $b > 1$, it can be observed from (5.10) that there exists a threshold $a^*(b) > b$ that a needs to exceed for the condition to be satisfied. In other words, for the portfolio's Sharpe ratio to increase with additional investment during recessions, the market Sharpe ratio needs to be sufficiently high in such an economic state. The reason for why a marginally higher market Sharpe ratio may be insufficient is that larger market returns in an economic state increase both the numerator and the denominator of the Sharpe ratio, because the “between-state” standard deviation of returns is then higher. We refer to the discussion following Lemma C.3 for additional details.

Condition (5.10) is generally satisfied for empirically plausible parameter values. That is, the portfolio's Sharpe ratio increases if more wealth is allocated to the risky asset when the market Sharpe ratio is high. However, Figure 4 indicates that the gain is modest, and that the Sharpe ratio decreases more when the allocation to the risky asset is decreased, compared to how much it increases when the allocation to the risky asset is increased by the same amount. In Appendix C, we show that $s^{\pi^*}(\delta)$ is concave in a neighborhood of $\delta = 0$, namely,

$$\left. \frac{\partial^2 s^{\pi^*}(\delta)}{\partial \delta^2} \right|_{\delta=0} \propto -\frac{\sigma^2(1)}{\tilde{\mu}^2(1)} \left(a^2 + b^2 \frac{\sigma^2(1)}{\tilde{\mu}^2(1)} \right) \lambda + O(\lambda^2). \quad (5.11)$$

This confirms that in order to maintain a satisfactory Sharpe ratio, it is more important not to reduce the

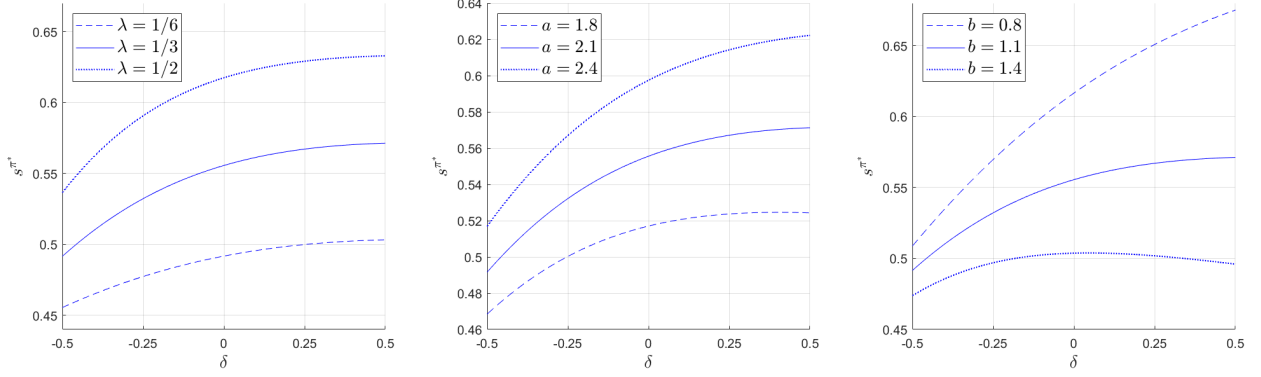


Figure 4: The graphs plot the Sharpe ratio (5.9) as a function of δ . We set the state-dependent parameters r, μ, σ and the transition matrix P to the values specified in (5.12)-(5.13). This specification yields $\lambda = 1/3$, $a = 2.1$, and $b = 1.1$. Left panel: We vary the parameter λ , while keeping a and b fixed. Middle panel: We vary the parameter a , while keeping λ and b fixed. Right panel: We vary the parameter b , while keeping λ and a fixed. Each time step in the model is one month, and we annualize the Sharpe ratio by multiplying (5.9) by $\sqrt{12}$.

risky asset allocation in a state of recession than to increase it.

5.3.2 Wealth Distribution and Cyclicity of Risk Aversion

The portfolio's Sharpe ratio is determined by the first two moments of its returns, which may not be sufficient to characterize the entire return distribution. Therefore, comparing portfolios in terms of their Sharpe ratios therefore leaves out the impact of higher return moments. This is especially relevant in the presence of multiple economic states, given that the unconditional return distribution is then leptokurtic and skewed.

We use numerical simulations to estimate the terminal distribution of the client's wealth, for different values of the parameter δ . We assume monthly portfolio rebalancing (a time step of one month) and set the transition matrix of the state variable $(Y_n)_{n \geq 0}$ to

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 0.95 & 0.05 \\ 0.10 & 0.90 \end{bmatrix}. \quad (5.12)$$

These transition probabilities are based on empirical values reported in Chauvet and Hamilton [2006], who also show that a regime switching model captures well the economic state transitions described by the NBER's business cycle chronology. These values show that, on average, economic expansions last longer than contractions.

We set the state-dependent return parameters in accordance with Tang and Whitelaw [2011], who report historical averages for changes in the mean and volatility of stock market returns, as well as for changes in the market Sharpe ratio, between the peak of the business cycle and the subsequent trough. Specifically, we

set the annual risk-free rate, mean and volatility of market returns to

$$r = (0.015, 0), \quad \mu = (0.081, 0.137), \quad \sigma = (0.155, 0.173). \quad (5.13)$$

Figure 5 shows the simulated distribution of the return of the optimal investment strategy at maturity, and the corresponding annualized rate of return. It is evident from the figure that the skewness and kurtosis are higher when the risky asset allocation is maintained or increased in a state of recession, and that in those cases, the return upside is considerably higher, with a limited additional downside risk. The same pattern is observed from the summary statistics of the simulations, reported in Table 1.

The above findings indicate that both the Sharpe ratio and the terminal wealth achieved by the strategy defined in (5.8) are higher if $\delta \geq 0$, compared to the case $\delta < 0$. In other words, the investment benefits are greater if the risky asset allocation is maintained or increased during recessions ($\delta \geq 0$). The question thus arises of *how far the robo-advisor can reach “against the will” of a client* whose risk preferences are countercyclical and thus consistent with the case $\delta < 0$.

While higher expected returns will be obtained in the long run, the client may suffer from adverse market moves in the short run, and may not have an adequate understanding of the long-term benefits. In a state of recession, with worsening economic outlook and risk aversion rising, the client may be particularly sensitive to what can be perceived as an investment mistake of the robo-advisor. The importance of this dilemma faced by robo-advisors is emphasized by Rossi and Utkus [2019a], who show empirically that algorithm aversion, i.e., the tendency of individuals to prefer a human forecaster over an algorithm, and to more quickly lose confidence in an algorithm than in a human after observing them make the same mistake (Dietvorst et al. [2015]), is one of the main obstacles for the adoption of robo-advising.

Our analysis suggests that a good middle ground for the robo-advisor is to encourage the client to simply maintain a fixed portfolio composition, which is consistent with the long-term investing principle of riding out business cycles. To that end, the robo-advisor may present statistics such as those reported in Table 1 and Figure 5 to the client. Thus, the client may observe that, over time, higher returns can be earned with limited additional risk by maintaining exposure to the risky asset during recessions instead of reducing it.²⁵ This behavior would also be consistent with that of most robo-advising firms. Namely, modifying the portfolio allocation based on economic conditions is a form of *active management* and, in general, robo-advising firms which focus on long-term investing do not engage in such market timing.²⁶ Rather, they urge

²⁵In the context our model, this amounts to “changing” the client’s risk aversion process from $(\gamma_n^{(\delta)})_{n \geq 0}$ such that $\delta < 0$, to $(\gamma_n^{(0)})_{n \geq 0}$. We recall that $(\gamma_n^{(\delta)})$ is the risk aversion process implied by the strategy given in (5.8).

²⁶One of the claimed benefits of robo-advising is that by managing the portfolio on the client’s behalf, the client is helped to resist the temptation of attempting to time the market. The robo-advising firms Betterment and Wealthfront also mention

clients to stay the course through changing market conditions, in order to reap the benefits of long-term investing.

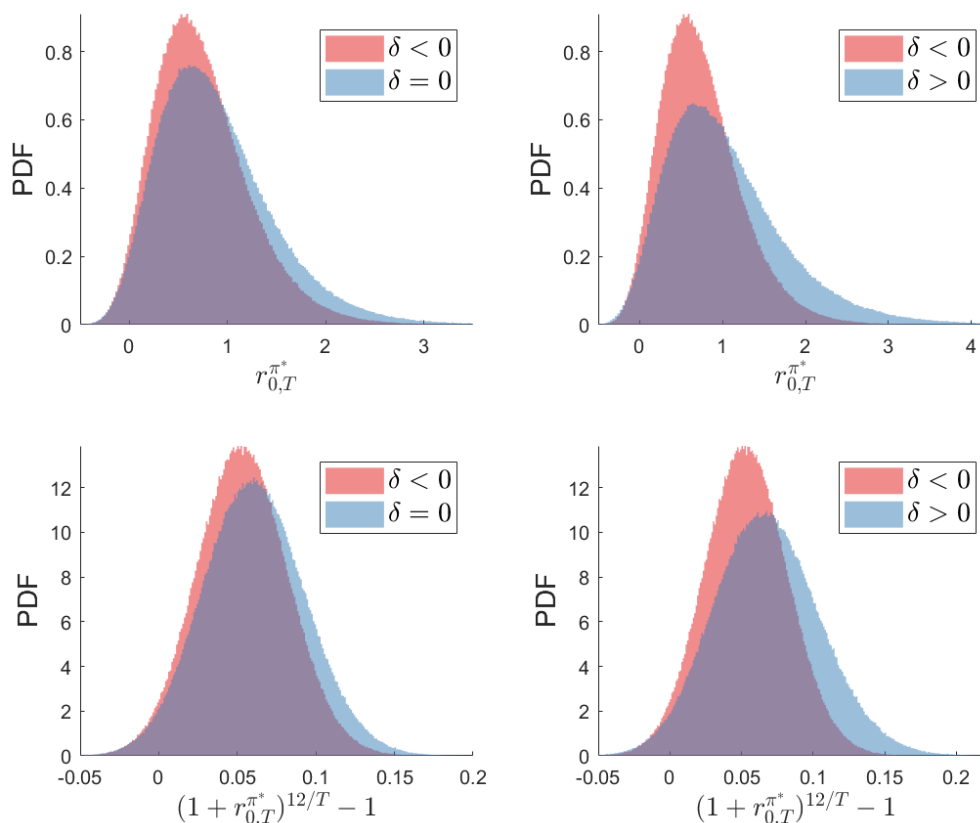


Figure 5: Simulated distributions of the return of the strategy (5.8) between time 0 and time T (top panels), and the corresponding annualized rate of return (bottom panels). The parameters of the strategy are $\bar{\pi} = 0.6$ and $\delta \in \{-0.3, 0, 0.3\}$. One time step in the model corresponds to one month, the market parameters are given in (5.12)-(5.13), the initial economic state is $Y_0 = 1$, and the investment horizon is $T = 120$ months.

	Mean	SD	Skewness	Kurtosis	90% VaR	95% VaR	99% VaR
$\delta < 0$	0.740	0.485	0.838	4.257	-0.179	-0.067	0.117
$\delta = 0$	0.881	0.589	0.970	4.712	-0.213	-0.085	0.118
$\delta > 0$	1.036	0.735	1.231	5.934	-0.233	-0.091	0.134
$\delta < 0$	0.053	0.029	0.066	3.011	-0.017	-0.006	0.012
$\delta = 0$	0.061	0.032	0.092	3.016	-0.019	-0.008	0.012
$\delta > 0$	0.068	0.037	0.162	3.089	-0.021	-0.009	0.014

Table 1: Summary statistics for the simulated return distributions in Figure 5. The top half of the table reports statistics for the return of the strategy throughout the investment period. The bottom half of the table gives the statistics for the annualized rate of return.

empirical work, such as the Dalbar's annual *Quantitative Analysis of Investors* report, which shows that investors who try to time the market tend to perform much worse than a “buy-and-hold” investor, and that the average investor significantly underperforms a broad stock market index.

6 Concluding Remarks

The past decade has witnessed the rapid emergence of robo-advisors, which are online platforms allowing clients to interact directly with automated investment algorithms. Recent literature has provided empirical evidence on the characteristics of those investors who are more likely to switch to robo-advising, and how robo-advising tools affect the portfolio composition of investors.

In this work, we build a novel framework that incorporates important features of modern robo-advising systems. In this framework, a robo-advisor undertakes the task of optimally investing a client's wealth. The client has a risk profile that changes in response to market returns, economic conditions, and idiosyncratic events, and to which the robo-advisor's investment performance criterion dynamically adapts. We derive and analyze optimal investment strategies, and highlight the role of risk aversion process in driving the intertemporal hedging demand.

The frequency of interaction between the client and robo-advisor determines the level of portfolio personalization, which is defined in terms of how accurately the robo-advisor is able to track the client's risk profile. We show that there exists an optimal interaction frequency, which maximizes portfolio personalization by striking a balance between receiving information from the client in a timely manner, and mitigating the effect of the client's behavioral biases.

We quantify how the Sharpe ratio of the optimal investment strategy depends on the allocation of wealth within economic regimes, and study the potential gains achieved by a robo-advisor's strategy during periods of high market Sharpe ratio. The implementation of this strategy may require the robo-advisor to go against the client's wishes, because both the client's risk aversion and the market Sharpe ratio tend to be countercyclical. We show that by simply rebalancing the portfolio to maintain constant weights throughout the business cycle, the portfolio's Sharpe ratio is close to being optimal. We also show that the portfolio return distribution is significantly improved relative to the situation where the investment in the risky asset is reduced during periods of economic contractions.

Our framework can be extended and further aligned to the practices of modern robo-advising systems. Indeed, the market model can be allowed to include multiple tradable assets and portfolio rebalanced only if the portfolio weights have drifted significantly from the target weights, due to price moves of the underlying securities. Robo-advisors generally use such threshold updating rules to rebalance their portfolios (Beketov et al. [2018]), while minimizing expense ratios and maximizing tax efficiency. Furthermore, the regime switching model can be generalized to make the economic state variable hidden, in line with the fact that economic conditions are typically not transparently observed, or observed with a lag. If the state variable is hidden, then the robo-advisor runs the risk of not acting in the client's best interest by making investment

decisions based on misclassified economic conditions. Finally, one may account for noise in the information communicated by the client. As the level of noise is expected to increase with the frequency of interaction, this would act as a further incentive to reduce interaction with the client.

Herein, market returns are assumed to be conditionally independent of economic state transitions. Nevertheless, these transitions are commonly associated with significant market moves, in particular when the economy enters a recession.²⁷ Such market moves would effectively result in the client “buying high” and “selling low”, and thus exacerbate the adverse effects of comonotonicity between risk aversion and the market Sharpe ratio. This would further incentivize the robo-advisor to act against the client’s wishes, e.g., if a market drop is likely to happen *following* a change in economic conditions, the portfolio’s market exposure could be adjusted to account for the probability of a near term market drop.²⁸

A Properties of Optimal Investment Strategy

A.1 Computation of Optimal Investment Strategy

In the following proposition, we provide a characterization of the optimal investment strategy given in Proposition 4.1, which depends on recursively computable expressions.

Proposition A.1. *The optimal allocation in Proposition 4.1 admits the representation*

$$\tilde{\pi}_n^*(d) = \frac{1}{\gamma_n} \frac{\mu_n^{az}(d) - R_{n+1}\gamma_n(\mu_n^{bz}(d) - \mu_n^a(d)\mu_n^{az}(d))}{\mu_n^{bz^2}(d) - (\mu_n^{az}(d))^2}, \quad (\text{A.1})$$

for any $d \in \mathcal{D}_n$, where

$$\begin{aligned} \mu_n^a(d) &:= \mathbb{E}_{n,d}[a_{n+1}(D_{n+1})], & \mu_n^{az}(d) &:= \mathbb{E}_{n,d}[a_{n+1}(D_{n+1})\tilde{Z}_{n+1}], \\ \mu_n^b(d) &:= \mathbb{E}_{n,d}[b_{n+1}(D_{n+1})], & \mu_n^{bz}(d) &:= \mathbb{E}_{n,d}[b_{n+1}(D_{n+1})\tilde{Z}_{n+1}], & \mu_n^{bz^2}(d) &:= \mathbb{E}_{n,d}[b_{n+1}(D_{n+1})\tilde{Z}_{n+1}^2]. \end{aligned} \quad (\text{A.2})$$

²⁷This is also the case in the consumption-based asset pricing frameworks which generate countercyclical variation in risk aversion, such as Campbell and Cochrane [1999]. In their framework, a worsening economic outlook leads to an increase in risk aversion, and investors demand higher reward for carrying risk, which in turn leads to a fall in stock prices.

²⁸Portfolio allocation under regime switching is widely studied. Related to our work, Tang and Whitelaw [2011] show how simple market timing strategies can exploit the fact that Sharpe ratios are countercyclical, and Coudert and Gex [2008] show that risk aversion indices published by financial institutions are good leading indicators of stock market crises. Kritzman et al. [2012] show how a Markov switching model can be used to avoid large portfolio losses related to “event regimes”, during which asset return dynamics start deviating from their past levels and the market risk premium is low.

The sequences $(a_n(d))_{0 \leq n < T}$ and $(b_n(d))_{0 \leq n < T}$ satisfy the recursions

$$\begin{aligned} a_n(d) &:= \mathbb{E}_{n,d}[(R_{n+1} + \tilde{Z}_{n+1}\tilde{\pi}_n^*(d))a_{n+1}(D_{n+1})], \\ b_n(d) &:= \mathbb{E}_{n,d}[(R_{n+1} + \tilde{Z}_{n+1}\tilde{\pi}_n^*(d))^2b_{n+1}(D_{n+1})], \end{aligned} \quad (\text{A.3})$$

with $a_T(d) = b_T(d) = 1$, for all $d \in \mathcal{D}_T$.

The quantities $a_n(d)$ and $b_n(d)$ in Proposition A.1 are the first and second moments of the future value of one dollar invested optimally between time n and the terminal date T , conditionally on $D_n = d$. That is,

$$a_n(d) := \mathbb{E}_{n,d}[1 + r_{n,T}^{\pi^*}], \quad b_n(d) := \mathbb{E}_{n,d}[(1 + r_{n,T}^{\pi^*})^2], \quad (\text{A.4})$$

with the simple return $r_{n,T}^{\pi^*}$ defined in (3.6). Because both $a_n(d)$ and $b_n(d)$ are independent of the wealth x at time n , it follows from (3.8) that the value function at time n is also independent of x , and given by

$$V_n(x, d) = V_n(d) := a_n(d) - 1 - \frac{\gamma_n(d)}{2}(b_n(d) - a_n^2(d)). \quad (\text{A.5})$$

Finally, Proposition A.1 shows that the optimal proportion of wealth allocated to the risky asset, $\tilde{\pi}_n^*$, is also independent of the wealth variable.

The expected values in (A.2) admit integral representations, which in turn can be used to compute the expected values in (A.3). Next, we provide such a representation for $\mu_n^{az}(d)$, both in the general case and in the specific case described in Section 5.1.

Lemma A.2. (a) Assume that given $D_n = d$, the random variable $\epsilon_{(n+1)}$ admits a generalized probability density function, $f_{\epsilon_{(n+1)}|d}$. Then,

$$\mu_n^{az}(d) = \sum_{y' \in \mathcal{Y}} P_{y_n, y'} \int_{z' \in \mathbb{R}} \int_{\epsilon' \in \mathbb{R}^{n+1}} a_{n+1}(D_{n+1}) \tilde{z}' f_{Z|y_n}(z') f_{\epsilon_{(n+1)}|d}(\epsilon') dz' d\epsilon',$$

where $\tilde{z}' := z' - r(y_n)$, and $D_{n+1} = (Y_{(n+1)}, Z_{(n+1)}, \tau_{(n+1)}, \xi_{(n+1)}) \in \mathcal{D}_{n+1}$ is such that $Y_{(n+1)} = (y, y')$, $Z_{(n+1)} = (z, z')$, $\tau_{(n+1)} = (\tau, \tau_{n+1})$, and $\xi_{(n+1)} = (\xi, \xi_{n+1})$. The value of τ_{n+1} is determined by the triplet $(Y_{(n+1)}, Z_{(n+1)}, \epsilon_{(n+1)})$. If $\tau_{n+1} = \tau_n$, then $\xi_{n+1} = \xi_n$, while if $\tau_{n+1} = n+1$, ξ_{n+1} is determined by $(Y_{(n+1)}, Z_{(n+1)}, \epsilon_{(n+1)})$.

(b) For the model in Section 5.1, i.e., where the risk aversion process $(\gamma_n)_{n \geq 0}$ is given by (5.3) and the

interaction times are of the form $T_k = k\phi$, for some $\phi \geq 1$, we have

$$\mu_n^{az}(d) = \begin{cases} \sum_{y' \in \mathcal{Y}} P_{y_n, y'} \int_{z' \in \mathbb{R}} a_{n+1}(D_{n+1}) \tilde{z}' f_{Z|y_n}(z') dz', & \tau_{n+1} < n+1, \\ \sum_{y' \in \mathcal{Y}} P_{y_n, y'} \int_{z' \in \mathbb{R}} \int_{\epsilon' \in \mathbb{R}} a_{n+1}(D_{n+1}) \tilde{z}' f_{Z|y_n}(z') f_{\epsilon}^{(\phi)}(\epsilon') dz' d\epsilon', & \tau_{n+1} = n+1, \end{cases} \quad (\text{A.6})$$

where $f_{Z|y_n}$ is the Gaussian probability density function with mean $\mu(y_n)$ and variance $\sigma^2(y_n)$, $f_{\epsilon}^{(\phi)}$ is the ϕ -fold convolution of the generalized density function of the i.i.d. sequence $(\epsilon_n)_{n \geq 1}$, and²⁹

$$D_{n+1} = \begin{cases} ((y, y'), (z, z'), (\xi, \xi_n)), & \tau_{n+1} < n+1, \\ ((y, y'), (z, z'), (\xi, \xi_{n+1})), & \tau_{n+1} = n+1. \end{cases}$$

If $\tau_{n+1} = n+1$, then ξ_{n+1} is given by an explicit function of $((y, y'), (z, z'), \xi_n, \epsilon')$.

Equation (A.6) shows that two distinct cases need to be considered, depending on whether or not the client's risk preferences are solicited at time n . In both cases, the computation of $\mu_n^{az}(d)$ involves a sum over the distribution of the economic state Y_{n+1} , and an integral with respect to the distribution of the market return Z_{n+1} . Additionally, if $\tau_{n+1} = n+1$, the client's risk preferences are solicited at time $n+1$ and $\mu_n^{az}(d)$ involves an additional integral with respect to the distribution of the cumulative idiosyncratic risk aversion shock $\sum_{k=\tau_n}^n \epsilon_{k+1}$. Hence, the probability distributions $f_{Z|y_n}$, $P_{y_n, \cdot}$, and $f_{\epsilon}^{(\phi)}$, link the optimal allocation at time n to the optimal allocations at time $n+1$. We observe that if $\tau_{n+1} < n+1$, the dynamics are only based on incoming market and economic information, modeled through Z_{n+1} and Y_{n+1} , while, in the case $\tau_{n+1} = n+1$, they also depend on information received from the client, given by $\sum_{k=\tau_n}^n \epsilon_{k+1}$.

Remark A.3. Equation (A.3) shows how the first two moments of $1 + r_{n,T}^{\pi^*}$ can be computed recursively. This can be extended to the m -th finite moment, $\mu_n^{(m)}(d) := \mathbb{E}_{n,d}[(1 + r_{n,T}^{\pi^*})^m]$, which satisfies the recursion

$$\mu_n^{(m)}(d) = \mathbb{E}_{n,d}[(R_{n+1} + \tilde{Z}_{n+1} \tilde{\pi}_n^*(d))^m \mu_{n+1}^{(m)}(D_{n+1})].$$

Under mild conditions, a probability distribution with finite moments of all orders is uniquely determined by its moments (see Billingsley [1995], Sec. 30). In that case, the distribution of the portfolio return $r_{n,T}^{\pi^*}$ can be recovered from a finite set of moments $\mu_n^{(1)}, \dots, \mu_n^{(M)}$, for some $M \geq 1$. \square

²⁹With a deterministic interaction schedule, we omit the interaction times τ_n from the state variable D_n .

A.2 Computational Complexity of Optimal Investment Strategy

The optimal investment strategy in Proposition A.1 is computed using backward induction, which requires discretization of the state variable D_n , for each n . This variable is high-dimensional, as it contains the history of economic states, market returns, interaction times, and communicated risk aversion parameters. However, the optimal investment strategy at time n can in general be written as a function of a lower-dimensional random variable \tilde{D}_n , which is measurable with respect to D_n . For instance, in the model of Section 5.1, the variable \tilde{D}_n consists of four components:³⁰

$$\tilde{D}_n := \left(\xi_n, \sum_{k=\tau_n-\phi}^{\tau_n-1} (Z_{k+1} - \mu_{k+1}), \sum_{k=\tau_n}^{n-1} (Z_{k+1} - \mu_{k+1}), Y_n \right) \in \mathbb{R}_0^+ \times \mathbb{R}^2 \times \mathcal{Y}.$$

That is, \tilde{D}_n consists of the most recently communicated risk aversion parameter, the sum of excess market returns between the two most recent interaction times, the sum of excess market returns since the most recent interaction time, and the current economic state. If a step size $\Delta\gamma$ is used to discretize the risk aversion parameter, a step size Δz is used for each of the return variables, and a general number $|\mathcal{Y}| \geq 1$ of economic states is assumed, the number of points in the discretization grid is of order

$$\frac{T|\mathcal{Y}|}{\Delta\gamma\Delta z} \left(\frac{1}{\Delta z} \mathbf{1}_{\{\phi>1\}} + \mathbf{1}_{\{\phi=1\}} \right),$$

where each point requires numerical integration to compute each of the expected values in (A.2) (see (A.6)).

In the general case, one may use properties of the risk aversion process to reduce the computational complexity. For example, to compute μ_n^{az} in Lemma A.2, the numerical integration can be avoided by evaluating Z_{n+1} at its expected value, μ_{n+1} . This leads to the approximation

$$\mu_n^{az}(d) = \mathbb{E}_{n,d}[a_{n+1}(D_{n+1})\tilde{Z}_{n+1}|Z_{n+1} = \mu_{n+1}] + O\left(\left(\sum_{k=n+1}^{T-1} \mathbb{E}[(\tilde{\pi}_k^*(\gamma_k) - \tilde{\pi}_k^*(\gamma'_k))^2]\right)^{1/2}\right).$$

In the above error term, $(\tilde{\pi}_k^*(\gamma'_k))_{k=n+1}^{T-1}$ are the future optimal allocations corresponding to $(\gamma'_k)_{k=n+1}^{T-1}$, which is the risk aversion process resulting from setting $Z_{n+1} = \mu_{n+1}$. In particular, if the future risk aversion values are independent of Z_{n+1} , then $(\gamma'_k)_{k=n+1}^{T-1} = (\gamma_k)_{k=n+1}^{T-1}$, and the error term vanishes. In general, the effect of a single market return on the future risk aversion path is limited - either temporary, or quickly diluted by the effect of future returns. For instance, in the risk aversion model of Section 5.1, the return Z_{n+1} impacts the risk aversion process only in a time interval bounded by two consecutive interaction times.

³⁰The functional J_n in (3.5) depends on $d \in \mathcal{D}_n$. In this example, for a fixed $\tilde{d} \in \mathbb{R}_0^+ \times \mathbb{R}^2 \times \mathcal{Y}$, the value of J_n is the same for any $d \in \mathcal{D}_n$ such that if $D_n = d$ then $\tilde{D}_n = \tilde{d}$.

The computational complexity can also be reduced by using properties of the economic state transition matrix. In particular, for a single time step, the transition probability out of the current state is generally small (see (5.12)). In the computation of $\mu_n^{az}(d)$, this leads to the approximation

$$\mu_n^{az}(d) = \sum_{y' \in \tilde{\mathcal{Y}}} P_{y_n, y'} \mathbb{E}_{n, d} [a_{n+1}(D_{n+1}) \tilde{Z}_{n+1} | Y_{n+1} = y'] + O\left(1 - \sum_{y' \in \tilde{\mathcal{Y}}} P_{y_n, y'}\right),$$

where $\tilde{\mathcal{Y}} \subseteq \mathcal{Y}$. For instance, if $\tilde{\mathcal{Y}} = \{y_n\}$, we condition on staying in the current economic state, while if $\tilde{\mathcal{Y}} = \{y_n - 1, y_n, y_n + 1\}$, we condition on staying in the current state or transitioning to neighboring states.

A.3 Investment Constraints

The dynamic optimization problem analyzed in Section 3 can be enhanced to handle portfolio constraints typically imposed by robo-advising firms. For example, we may assume that the space of admissible strategies is restricted to control laws $\pi = (\pi_n)_{0 \leq n < T}$ such that the proportion of wealth invested in the risky asset, denoted by $\tilde{\pi} = (\tilde{\pi}_n)_{0 \leq n < T}$, satisfies

$$-\infty < \underline{\pi}_n \leq \tilde{\pi}_n \leq \bar{\pi}_n < \infty, \quad (\text{A.7})$$

where $(\underline{\pi}_n)_{0 \leq n < T}$ and $(\bar{\pi}_n)_{0 \leq n < T}$ are deterministic sequences. In particular, no short-selling and no borrowing constraints correspond to $\underline{\pi} \equiv 0$ and $\bar{\pi} \equiv 1$, respectively.

In this case, the optimal allocation formula is given by a truncated version of the unconstrained optimal allocation formula. Namely, in the unconstrained case, the proof of Proposition 4.1 shows that the optimal allocation $\tilde{\pi}_n^*$ is obtained by maximizing a second order polynomial h_n with a negative leading order coefficient (see (B.9)). Under the above portfolio constraint, the optimal allocation is thus obtained by truncating the outcome of the unconstrained allocation formula (A.1) from below by $\underline{\pi}_n$, and from above by $\bar{\pi}_n$.

We next consider the effect of truncating the optimal investment strategy π^* to zero whenever the wealth level is negative. This results in the investment strategy π^o given by

$$\pi_n^o(x, d) = \pi_n^*(x, d) \mathbf{1}_{\{x \geq 0\}}. \quad (\text{A.8})$$

Hence, π^o coincides with the optimal strategy π^* if the wealth variable x is non-negative, but switches to a risk-free portfolio when the wealth becomes negative. In the following lemma, we study the difference between the strategies π^o and π^* .

Lemma A.4. *Let π^* be the optimal strategy in Proposition 4.1 under the constraint (A.7) with $\underline{\pi} \equiv 0$ and $\bar{\pi} \equiv 1$. For $x > 0$ and $d \in \mathcal{D}_n$, we have the following relations between the expectation and variance of the portfolio return achieved under the strategy π^* and the constrained strategy π^o defined in (A.8):*

$$\begin{aligned}\mathbb{E}_{n,x,d}[1 + r_{n,T}^{\pi^o}] &= \mathbb{E}_{n,d}[1 + r_{n,T}^{\pi^*}] + O(\mathbb{P}_{n,d}(\tilde{Z}_{n+1} < -1)), \\ \text{Var}_{n,x,d}[1 + r_{n,T}^{\pi^o}] &= \text{Var}_{n,d}[1 + r_{n,T}^{\pi^*}] + O(\mathbb{P}_{n,d}(\tilde{Z}_{n+1} < -1)).\end{aligned}$$

Furthermore, for $x > 0$ and $d \in \mathcal{D}_n$, the objective functional (3.5) evaluated at π^o satisfies the relation

$$J_n(x, d, \pi^o) = V_n(d) + O(\mathbb{P}_{n,d}(\tilde{Z}_{n+1} < -1)),$$

where $V_n(d)$ is the value function in (A.5).

Observe that in the previous lemma, the expectation and variance terms under π^o do not depend on the value of x , as long as $x > 0$. The same is true if $x < 0$, in which case they only depend on the risk-free rate throughout the investment horizon, and can be written as

$$\mathbb{E}_{n,x,d}[1 + r_{n,T}^{\pi^o}] = \mathbb{E}_{n,d}\left[\prod_{k=n}^{T-1} R_{k+1}\right], \quad \text{Var}_{n,x,d}[1 + r_{n,T}^{\pi^o}] = \text{Var}_{n,d}\left[\prod_{k=n}^{T-1} R_{k+1}\right].$$

The proof of the lemma shows that the constants in the \mathcal{O} -terms admit explicit upper bounds depending only on n and can thus be uniformly bounded for all $0 \leq n < T$. The order of the \mathcal{O} -terms can also be uniformly bounded by a constant independent of n and d , because

$$\mathbb{P}_{n,d}(\tilde{Z}_{n+1} < -1) \leq \max_{y \in \mathcal{Y}} \mathbb{P}(\tilde{Z}_1 < -1 | Y_0 = y).$$

For reasonable parameter values, the above probability is very small. For example, standard annualized parameter values for the mean and variance of \tilde{Z}_1 are $\tilde{\mu} = 0.1$ and $\sigma = 0.2$. In that case, the event $\{Z_1 < -1\}$ corresponds to a negative return whose magnitude exceeds five standard deviations. As the discrete step size in the model gets smaller (e.g., monthly rather than annual), the probability becomes smaller, and vanishes in the continuous-time limit. This can be seen explicitly for the model considered in Section 5.1, where returns are conditionally Gaussian. In that case, we have the bound

$$\mathbb{P}(\tilde{Z}_{n+1} < -1 | Y_n = y) \leq \sigma(y) e^{-\frac{1}{2\sigma^2(y)}},$$

and the right-hand side converges to zero as $\sigma(y)$ tends to zero.

B Proofs and Auxiliary Results for Section 4

We start by proving a lemma that will be used in the proof of Proposition 4.1.

Lemma B.1. *Let $\pi = (\pi_n)_{n \geq 0}$ be an admissible control law of the form*

$$\pi_n(x, d) = \tilde{\pi}_n(d)x, \quad (\text{B.1})$$

for any $x \in \mathbb{R}$, and $d \in \mathcal{D}_n$. Then, the return moments

$$a_n^\pi(d) := \mathbb{E}_{n,d}[1 + r_{n,T}^\pi], \quad b_n^\pi(d) := \mathbb{E}_{n,d}[(1 + r_{n,T}^\pi)^2],$$

satisfy the recursions in (A.3) with $\pi^ = \pi$.*

Proof of Lemma B.1: For $n = T - 1$, it follows from the wealth dynamics (3.1) that

$$\mathbb{E}_{n,d}[1 + r_{n,T}^\pi] = \mathbb{E}_{n,d}[R_{n+1} + \tilde{Z}_{n+1}\tilde{\pi}_n(d)] = R_{n+1} + \tilde{\mu}_{n+1}\tilde{\pi}_n(d).$$

Next, let $n \in \{0, 1, \dots, T - 2\}$ and assume that the result holds for $n + 1, n + 2, \dots, T - 1$. Then,

$$\begin{aligned} \mathbb{E}_{n,d}[1 + r_{n,T}^\pi] &= \mathbb{E}_{n,x,d} \left[(R_{n+1} + \tilde{Z}_{n+1}\tilde{\pi}_n(d)) \prod_{k=n+1}^{T-1} (R_{k+1} + \tilde{Z}_{k+1}\tilde{\pi}_k(D_k)) \right] \\ &= \mathbb{E}_{n,x,d} \left[(R_{n+1} + \tilde{Z}_{n+1}\tilde{\pi}_n(d)) \mathbb{E}_{n+1, X_{n+1}^\pi, D_{n+1}} \left[\prod_{k=n+1}^{T-1} (R_{k+1} + \tilde{Z}_{k+1}\tilde{\pi}_k(D_k)) \right] \right] \\ &= \mathbb{E}_{n,x,d} [(R_{n+1} + \tilde{Z}_{n+1}\tilde{\pi}_n(d)) a_{n+1}^\pi(D_{n+1})] \\ &= \mathbb{E}_{n,d} [(R_{n+1} + \tilde{Z}_{n+1}\tilde{\pi}_n(d)) a_{n+1}^\pi(D_{n+1})], \end{aligned}$$

The assertion for the b_n -sequence is shown in the same way. □

Next, we derive the extended HJB system of equations necessarily satisfied by an optimal control law for the optimization problem defined by (3.5)-(3.7). To simplify notation, we introduce the random variable

$$\tilde{D}_n := (X_n, D_n) \in \tilde{\mathcal{D}}_{n+1} := \mathbb{R}_0 \times \mathcal{D}_n.$$

Furthermore, we write

$$\tilde{D}_{n+1}^\pi = (X_{n+1}^\pi, D_{n+1}),$$

where X_{n+1}^π is obtained by applying the control law π to the wealth x at time n .

Proposition B.2. *Assume that an optimal control law π^* for the optimization problem (3.5)-(3.7) exists. Then, the value function (3.8) satisfies the recursive equation*

$$V_n(\tilde{d}) = \sup_{\pi} \left\{ \mathbb{E}_{n,\tilde{d}}[V_{n+1}(\tilde{D}_{n+1}^\pi)] - \left(\mathbb{E}_{n,\tilde{d}}[f_{n+1,n+1}(\tilde{D}_{n+1}^\pi; \tilde{D}_{n+1}^\pi)] - \mathbb{E}_{n,\tilde{d}}[f_{n+1,n}(\tilde{D}_{n+1}^\pi; \tilde{d})] \right) \right. \\ \left. - \left(\mathbb{E}_{n,\tilde{d}} \left[\frac{\gamma_{n+1}(D_{n+1})}{2} \left(\frac{g_{n+1}(\tilde{D}_{n+1}^\pi)}{X_{n+1}^\pi} \right)^2 \right] - \frac{\gamma_n(d)}{2} \left(\mathbb{E}_{n,\tilde{d}} \left[\frac{g_{n+1}(\tilde{D}_{n+1}^\pi)}{x} \right] \right)^2 \right) \right\}, \quad (\text{B.2})$$

for $0 \leq n < T$ and $\tilde{d} = (x, d) \in \tilde{\mathcal{D}}_n$, with terminal condition

$$V_T(\tilde{d}) = 0, \quad \tilde{d} \in \tilde{\mathcal{D}}_T.$$

Herein, for any fixed $k = 0, 1, \dots, T$ and $\tilde{d}' = (x', d') \in \tilde{\mathcal{D}}_k$, the function sequence $(f_{n,k}(\cdot; \tilde{d}'))_{0 \leq n \leq T}$, where $f_{n,k}(\cdot; \tilde{d}') : \tilde{\mathcal{D}}_n \mapsto \mathbb{R}$, is determined by the recursion

$$f_{n,k}(\tilde{d}; \tilde{d}') = \mathbb{E}_{n,\tilde{d}}[f_{n+1,k}(\tilde{D}_{n+1}^{\pi^*}; \tilde{d}')], \quad \tilde{d} \in \tilde{\mathcal{D}}_n, \quad 0 \leq n < T, \\ f_{T,k}(\tilde{d}; \tilde{d}') = \frac{x}{x'} - 1 - \frac{\gamma_k(d')}{2} \left(\frac{x}{x'} \right)^2, \quad \tilde{d} \in \tilde{\mathcal{D}}_T,$$

and the function sequence $(g_n)_{0 \leq n \leq T}$, where $g_n : \tilde{\mathcal{D}}_n \mapsto \mathbb{R}$, is determined by the recursion

$$g_n(\tilde{d}) = \mathbb{E}_{n,\tilde{d}}[g_{n+1}(\tilde{D}_{n+1}^{\pi^*})], \quad \tilde{d} \in \tilde{\mathcal{D}}_n, \quad 0 \leq n < T, \\ g_T(\tilde{d}) = x, \quad \tilde{d} \in \tilde{\mathcal{D}}_T.$$

Furthermore, the function sequences admit the probabilistic representations

$$f_{n,k}(\tilde{d}; \tilde{d}') = \mathbb{E}_{n,\tilde{d}} \left[\frac{X_T^{\pi^*}}{x'} - 1 - \frac{\gamma_k(d')}{2} \left(\frac{X_T^{\pi^*}}{x'} \right)^2 \right], \quad g_n(\tilde{d}) = \mathbb{E}_{n,d}[X_T^{\pi^*}].$$

Proof of Proposition B.2: We write the objective functional at time n as

$$J_n(\tilde{d}; \pi) = \mathbb{E}_{n,\tilde{d}}[F_n(\tilde{d}, X_T^\pi)] + G_n(\tilde{d}, \mathbb{E}_{n,\tilde{d}}[X_T^\pi]),$$

where, for $\tilde{d} = (x, d) \in \tilde{\mathcal{D}}_n$ and $y \in \mathbb{R}$,

$$F_n(\tilde{d}, y) := \frac{y}{x} - 1 - \frac{\gamma_n(d)}{2} \left(\frac{y}{x} \right)^2, \quad G_n(\tilde{d}, y) := \frac{\gamma_n(d)}{2} \left(\frac{y}{x} \right)^2. \quad (\text{B.3})$$

The proof now consists of two parts. First, we derive the recursive equation satisfied by the sequence of objective functionals for any admissible control law π . Then, we derive the system of equations that an optimal control law π^* must satisfy.

Step 1: Recursion for $J_n(\tilde{d}; \pi)$. For a given admissible control law π , and fixed $0 \leq k \leq T$ and $\tilde{d}' \in \tilde{\mathcal{D}}_k$, we define the function sequences

$$\begin{aligned} f_{n,k}^\pi(\tilde{d}; \tilde{d}') &:= \mathbb{E}_{n,\tilde{d}}[F_k(\tilde{d}', X_T^\pi)], \quad \tilde{d} \in \tilde{\mathcal{D}}_n, \quad 0 \leq n \leq T, \\ g_n^\pi(\tilde{d}) &:= \mathbb{E}_{n,\tilde{d}}[X_T^\pi], \quad \tilde{d} \in \tilde{\mathcal{D}}_n, \quad 0 \leq n \leq T, \end{aligned} \quad (\text{B.4})$$

and note that, by the law of iterated expectations,

$$f_{n,k}^\pi(\tilde{d}; \tilde{d}') = \mathbb{E}_{n,\tilde{d}}[f_{n+1,k}^\pi(\tilde{D}_{n+1}^\pi; \tilde{d}')], \quad g_n^\pi(\tilde{d}) = \mathbb{E}_{n,\tilde{d}}[g_{n+1}^\pi(\tilde{D}_{n+1}^\pi)]. \quad (\text{B.5})$$

For $\tilde{D}_{n+1} \in \tilde{\mathcal{D}}_{n+1}$, the objective functional at time $n+1$ can then be written as

$$\begin{aligned} J_{n+1}(\tilde{D}_{n+1}; \pi) &= \mathbb{E}_{n+1, \tilde{D}_{n+1}}[F_{n+1}(\tilde{D}_{n+1}, X_T^\pi)] + G_{n+1}(\tilde{D}_{n+1}, \mathbb{E}_{n+1, \tilde{D}_{n+1}}[X_T^\pi]) \\ &= f_{n+1, n+1}^\pi(\tilde{D}_{n+1}; \tilde{D}_{n+1}) + G_{n+1}(\tilde{D}_{n+1}, g_{n+1}^\pi(\tilde{D}_{n+1})). \end{aligned}$$

Taking expectations with respect to $\mathbb{P}_{n,\tilde{d}}$ and applying the control law π at time n yields

$$\mathbb{E}_{n,\tilde{d}}[J_{n+1}(\tilde{D}_{n+1}^\pi; \pi)] = \mathbb{E}_{n,\tilde{d}}[f_{n+1, n+1}^\pi(\tilde{D}_{n+1}^\pi; \tilde{D}_{n+1}^\pi)] + \mathbb{E}_{n,\tilde{d}}[G_{n+1}(\tilde{D}_{n+1}^\pi, g_{n+1}^\pi(\tilde{D}_{n+1}^\pi))].$$

Adding and subtracting $J_n(\tilde{d}; \pi)$ gives

$$\begin{aligned} \mathbb{E}_{n,\tilde{d}}[J_{n+1}(\tilde{D}_{n+1}^\pi; \pi)] &= J_n(\tilde{d}; \pi) + \mathbb{E}_{n,\tilde{d}}[f_{n+1, n+1}^\pi(\tilde{D}_{n+1}^\pi; \tilde{D}_{n+1}^\pi)] - \mathbb{E}_{n,\tilde{d}}[F_n(\tilde{d}, X_T^\pi)] \\ &\quad + \mathbb{E}_{n,\tilde{d}}[G_{n+1}(\tilde{D}_{n+1}^\pi, g_{n+1}^\pi(\tilde{D}_{n+1}^\pi))] - G_n(\tilde{d}, \mathbb{E}_{n,\tilde{d}}[X_T^\pi]). \end{aligned}$$

Using (B.4)-(B.5), we then obtain

$$\begin{aligned} J_n(\tilde{d}; \pi) &= \mathbb{E}_{n,\tilde{d}}[J_{n+1}(\tilde{D}_{n+1}^\pi; \pi)] - \left(\mathbb{E}_{n,\tilde{d}}[f_{n+1, n+1}^\pi(\tilde{D}_{n+1}^\pi; \tilde{D}_{n+1}^\pi)] - \mathbb{E}_{n,\tilde{d}}[f_{n+1, n}^\pi(\tilde{D}_{n+1}^\pi; \tilde{d})] \right) \\ &\quad - \left(\mathbb{E}_{n,\tilde{d}}[G_{n+1}(\tilde{D}_{n+1}^\pi, g_{n+1}^\pi(\tilde{D}_{n+1}^\pi))] - G_n(\tilde{d}, \mathbb{E}_{n,\tilde{d}}[g_{n+1}^\pi(\tilde{D}_{n+1}^\pi)]) \right). \end{aligned} \quad (\text{B.6})$$

Step 2: Recursion for $V_n(\tilde{d})$. Assume the existence of an optimal control law π^* and consider a control law

π that coincides with π^* after time n , i.e., $\pi_k \equiv \pi_k^*$, for $k = n+1, \dots, T-1$. By definition, we then have

$$J_n(\tilde{d}; \pi) \leq V_n(\tilde{d}) = J_n(\tilde{d}; \pi^*), \quad \tilde{d} \in \tilde{\mathcal{D}}_n.$$

Recall the definition of function sequences in (B.4). For the optimal control law π^* , let

$$f_{n,k}(\tilde{d}; \tilde{d}') := f_{n,k}^{\pi^*}(\tilde{d}; \tilde{d}'), \quad g_n(\tilde{d}) := g_n^{\pi^*}(\tilde{d}). \quad (\text{B.7})$$

Since π and π^* coincide after time n , for any $\tilde{D}_{n+1} \in \tilde{\mathcal{D}}_{n+1}$ we have that

$$J_{n+1}(\tilde{D}_{n+1}; \pi) = V_{n+1}(\tilde{D}_{n+1}), \quad f_{n+1,k}(\tilde{D}_{n+1}; \tilde{d}') = f_{n+1,k}^{\pi}(\tilde{D}_{n+1}; \tilde{d}'), \quad g_{n+1}(\tilde{D}_{n+1}) = g_{n+1}^{\pi}(\tilde{D}_{n+1}).$$

In turn, using the recursion (B.6) for $J_n(d; \pi)$, we may write for $\tilde{d} \in \tilde{\mathcal{D}}_n$,

$$\begin{aligned} V_n(\tilde{d}) = \sup_{\pi} \Big\{ & \mathbb{E}_{n,\tilde{d}}[V_{n+1}(\tilde{D}_{n+1}^{\pi})] - \left(\mathbb{E}_{n,\tilde{d}}[f_{n+1,n+1}(\tilde{D}_{n+1}^{\pi}; \tilde{D}_{n+1}^{\pi})] - \mathbb{E}_{n,\tilde{d}}[f_{n+1,n}(\tilde{D}_{n+1}^{\pi}; \tilde{d})] \right) \\ & - \left(\mathbb{E}_{n,\tilde{d}}[G_{n+1}(\tilde{D}_{n+1}^{\pi}, g_{n+1}(\tilde{D}_{n+1}^{\pi}))] - G_n(\tilde{d}, \mathbb{E}_{n,\tilde{d}}[g_{n+1}(\tilde{D}_{n+1}^{\pi})]) \right) \Big\}, \end{aligned}$$

with the terminal condition $V_T(\tilde{d}) = 0$, for $\tilde{d} \in \mathcal{D}_T$. The recursions and probabilistic representations of $(f_{n,k}(\cdot; \tilde{d}'))_{0 \leq n \leq T}$ and $(g_n)_{0 \leq n \leq T}$ then follow from (B.3)-(B.5), and (B.7). \square

Proof of Proposition 4.1: Assuming the existence of an optimal control law π^* , the value function at time $n+1$ satisfies

$$V_{n+1}(\tilde{D}_{n+1}) = f_{n+1,n+1}(\tilde{D}_{n+1}; \tilde{D}_{n+1}) + \frac{\gamma_{n+1}(D_{n+1})}{2} \left(\frac{g_{n+1}(\tilde{D}_{n+1})}{X_{n+1}} \right)^2.$$

Plugging the above expression into the HJB equation (B.2) gives

$$V_n(\tilde{d}) = \sup_{\pi} \left\{ \mathbb{E}_{n,\tilde{d}}[f_{n+1,n}(\tilde{D}_{n+1}^{\pi}; \tilde{d})] + \frac{\gamma_n(d)}{2} \left(\frac{\mathbb{E}_{n,\tilde{d}}[g_{n+1}(\tilde{D}_{n+1}^{\pi})]}{x} \right)^2 \right\}. \quad (\text{B.8})$$

Next, we look for a candidate optimal control law of form (B.1), i.e., $\pi_n(\tilde{d}) = \tilde{\pi}_n(d)x$. It then follows from Lemma B.1, along with (B.3), (B.4), and (B.7) that, for this candidate optimal law,

$$\begin{aligned} f_{n+1,n}(\tilde{d}; \tilde{d}') &= a_{n+1}^{\pi}(d) \frac{x}{x'} - 1 - \frac{\gamma_n(\tilde{d}')}{2} b_{n+1}^{\pi}(d) \left(\frac{x}{x'} \right)^2, \\ g_{n+1}(\tilde{d}) &= a_{n+1}^{\pi}(d)x. \end{aligned}$$

Plugging the above expression into (B.8), using the wealth dynamics (3.1) and, for brevity, writing R for R_{n+1} , $\tilde{\pi}_n$ for $\tilde{\pi}_n(d)$, a_{n+1}^π for $a_{n+1}^\pi(D_{n+1})$, and b_{n+1}^π for $b_{n+1}^\pi(D_{n+1})$, yields

$$\begin{aligned} V_n(\tilde{d}) &= \sup_{\pi} \left\{ \mathbb{E}_{n,\tilde{d}} \left[a_{n+1}^\pi \frac{X_{n+1}^\pi}{x} - 1 - \frac{\gamma_n(d)}{2} b_{n+1}^\pi \left(\frac{X_{n+1}^\pi}{x} \right)^2 \right] + \frac{\gamma_n(d)}{2} \left(\mathbb{E}_{n,\tilde{d}} \left[a_{n+1}^\pi \frac{X_{n+1}^\pi}{x} \right] \right)^2 \right\} \\ &= \sup_{\pi} \left\{ \mathbb{E}_{n,\tilde{d}} \left[a_{n+1}^\pi (R + \tilde{Z}_{n+1} \tilde{\pi}_n) - 1 - \frac{\gamma_n(d)}{2} b_{n+1}^\pi (R + \tilde{Z}_{n+1} \tilde{\pi}_n)^2 \right] + \frac{\gamma_n(d)}{2} \left(\mathbb{E}_{n,\tilde{d}} [a_{n+1}^\pi (R + \tilde{Z}_{n+1} \tilde{\pi}_n)] \right)^2 \right\} \\ &= \sup_{\pi} \left\{ \mathbb{E}_{n,\tilde{d}} \left[a_{n+1}^\pi (R + \tilde{Z}_{n+1} \tilde{\pi}_n) - 1 - \frac{\gamma_n(d)}{2} b_{n+1}^\pi (R^2 + 2R\tilde{\pi}_n \tilde{Z}_{n+1} + \tilde{\pi}_n^2 \tilde{Z}_{n+1}^2) \right] \right. \\ &\quad \left. + \frac{\gamma_n(d)}{2} \left(R^2 (\mathbb{E}_{n,\tilde{d}} [a_{n+1}^\pi])^2 + 2R\tilde{\pi}_n \mathbb{E}_{n,\tilde{d}} [a_{n+1}^\pi] \mathbb{E}_{n,\tilde{d}} [\tilde{Z}_{n+1}] + \tilde{\pi}_n^2 (\mathbb{E}_{n,\tilde{d}} [\tilde{Z}_{n+1}])^2 \right) \right\} =: \sup_{\pi} h_n(\tilde{\pi}_n), \end{aligned} \quad (\text{B.9})$$

where h_n is a second order polynomial with derivative

$$\begin{aligned} h'_n(\tilde{\pi}_n) &= \mathbb{E}_{n,\tilde{d}} [a_{n+1}^\pi \tilde{Z}_{n+1}] - R\gamma_n(d) (\mathbb{E}_{n,\tilde{d}} [b_{n+1}^\pi \tilde{Z}_{n+1}] - \mathbb{E}_{n,\tilde{d}} [a_{n+1}^\pi] \mathbb{E}_{n,\tilde{d}} [\tilde{Z}_{n+1}]) \\ &\quad - \gamma_n(d) (\mathbb{E}_{n,\tilde{d}} [b_{n+1}^\pi \tilde{Z}_{n+1}^2] - (\mathbb{E}_{n,\tilde{d}} [a_{n+1}^\pi \tilde{Z}_{n+1}])^2) \tilde{\pi}_n. \end{aligned}$$

By Jensen's inequality, $\mathbb{E}_{n,\tilde{d}} [b_{n+1}^\pi \tilde{Z}_{n+1}^2] - (\mathbb{E}_{n,\tilde{d}} [a_{n+1}^\pi \tilde{Z}_{n+1}])^2 > 0$, and solving for $h'_n(\tilde{\pi}_n) = 0$ yields the formulation in Proposition 4.1. One can then easily check that this solves the HJB equation (B.2). To obtain the allocation formula (4.1), we first use (A.3) and the law of iterated expectations to rewrite the identities in (A.2),

$$\begin{aligned} \mu_n^a(d) &= \mathbb{E}_{n,d} [\mathbb{E}_{n+1,D_{n+1}} [1 + r_{n+1,T}^{\pi^*}]] = \mathbb{E}_{n,d} [1 + r_{n+1,T}^{\pi^*}], \\ \mu_n^{az}(d) &= \mathbb{E}_{n,d} [\tilde{Z}_{n+1} \mathbb{E}_{n+1,D_{n+1}} [1 + r_{n+1,T}^{\pi^*}]] = \mathbb{E}_{n,d} [\tilde{Z}_{n+1} (1 + r_{n+1,T}^{\pi^*})], \\ \mu_n^{bz}(d) &= \mathbb{E}_{n,d} [\tilde{Z}_{n+1} \mathbb{E}_{n+1,D_{n+1}} [(1 + r_{n+1,T}^{\pi^*})^2]] = \mathbb{E}_{n,d} [\tilde{Z}_{n+1} (1 + r_{n+1,T}^{\pi^*})^2], \\ \mu_n^{bz^2}(d) &= \mathbb{E}_{n,d} [\tilde{Z}_{n+1}^2 \mathbb{E}_{n+1,D_{n+1}} [(1 + r_{n+1,T}^{\pi^*})^2]] = \mathbb{E}_{n,d} [\tilde{Z}_{n+1}^2 (1 + r_{n+1,T}^{\pi^*})^2]. \end{aligned}$$

It then follows from the definition of variance and covariance that

$$\begin{aligned} \mu_n^{bz^2}(d) - (\mu_n^{az}(d))^2 &= \text{Var}_{n,d} [\tilde{Z}_{n+1} (1 + r_{n+1,T}^{\pi^*})], \\ \mu_n^{bz}(d) - \mu_n^a(d) \mu_n^{az}(d) &= \text{Cov}_{n,d} [\tilde{Z}_{n+1} (1 + r_{n+1,T}^{\pi^*}), 1 + r_{n+1,T}^{\pi^*}], \end{aligned}$$

and (4.1) is obtained by plugging the above into formula (A.1). □

Proof of Proposition A.1: It follows directly from the proof of Proposition 4.1. □

Proof of Lemma A.4: Observe that no short-selling and borrowing implies $0 \leq \tilde{\pi}_n^*(d) \leq 1$. Let

$$A_n(d) := \{1 + \tilde{Z}_{n+1}\tilde{\pi}_n^*(d) < 0\},$$

which, for any positive wealth level at time n , is the event of the wealth becoming negative at time $n + 1$. Recall from (A.4) the return moments $a_n(d)$ and $b_n(d)$ for the optimal strategy π^* , and, for $x > 0$, define for π^o the return moments

$$a_n^o(d) := \mathbb{E}_{n,x,d}[1 + r_{n,T}^{\pi^o}], \quad b_n^o(d) := \mathbb{E}_{n,x,d}[(1 + r_{n,T}^{\pi^o})^2],$$

which are independent of the value of x . This is because the probability of the wealth becoming negative, $\mathbb{P}_{n,x,d}(X_k^{\pi^o} < 0 \text{ for some } k \in \{n+1, \dots, T-1\})$, is independent of the value of $x > 0$.

We use induction to show that, for any $d \in \mathcal{D}_n$,

$$|a_n^o(d) - a_n(d)| \leq 2(T - (n+1))K^{T-n}\mathbb{P}_{n,d}(\tilde{Z}_{n+1} < -1) := K_n.$$

where $K := \max_{y \in \mathcal{Y}} (1 + r(y) + \mathbb{E}[\tilde{Z}_1 | Y_0 = y])$ is a constant. First, for $n = T - 1$, we have $a_n^o(d) = a_n(d)$. Then, for a general $n \in \{0, 1, \dots, T - 2\}$, we have

$$\begin{aligned} a_n^o(d) &= \mathbb{E}_{n,d}[(R_{n+1} + \tilde{Z}_{n+1}\tilde{\pi}_n^*(d))a_{n+1}^o(D_{n+1})\mathbf{1}_{A_n^c}] + \mathbb{E}_{n,d}[(R_{n+1} + \tilde{Z}_{n+1}\tilde{\pi}_n^*(d))a_{n+1}^o(D_{n+1})\mathbf{1}_{A_n}] \\ &= \mathbb{E}_{n,d}[(R_{n+1} + \tilde{Z}_{n+1}\tilde{\pi}_n^*(d))a_{n+1}^o(D_{n+1})] - \mathbb{E}_{n,d}[(R_{n+1} + \tilde{Z}_{n+1}\tilde{\pi}_n^*(d))a_{n+1}^o(D_{n+1})\mathbf{1}_{A_n}] \\ &\quad + \mathbb{E}_{n,d}[(R_{n+1} + \tilde{Z}_{n+1}\tilde{\pi}_n^*(d))\left(\prod_{k=n+1}^{T-1} R_{k+1}\right)\mathbf{1}_{A_n}], \end{aligned}$$

so, assuming the conjecture to hold for $n + 1$, and by using $0 \leq \tilde{\pi}_n^* \leq 1$, $R_{n+1} = 1 + r_{n+1}$, and that \tilde{Z}_{n+1} has mean $\mu_{n+1} - r_{n+1}$, we obtain

$$\begin{aligned} |a_n^o(d) - a_n(d)| &\leq K_{n+1}\mathbb{E}_{n,d}[|R_{n+1} + \tilde{Z}_{n+1}\tilde{\pi}_n^*(d)|] + \mathbb{E}_{n,d}[|R_{n+1} + \tilde{Z}_{n+1}\tilde{\pi}_n^*(d)|a_{n+1}^o(D_{n+1})\mathbf{1}_{A_n}] \\ &\quad + \mathbb{E}_{n,d}[|R_{n+1} + \tilde{Z}_{n+1}\tilde{\pi}_n^*(d)|\left(\prod_{k=n+1}^{T-1} R_{k+1}\right)\mathbf{1}_{A_n}] \\ &\leq K_{n+1}K + 2K^{T-n}\mathbb{P}_{n,d}(\tilde{Z}_{n+1} < -1) \\ &\leq 2(T - (n+2))K^{T-n}\mathbb{P}_{n,d}(\tilde{Z}_{n+1} < -1) + 2K^{T-n}\mathbb{P}_{n,d}(\tilde{Z}_{n+1} < -1) \\ &= K_n. \end{aligned}$$

A similar result can be derived for the second moment $b^o(d)$. The results for the variance and objective functional then follow. \square

Proof of Lemma A.2: The formulas for $\mu_n^{az}(d)$ follow directly from (A.2)-(A.3) and the definition of the random variable D_{n+1} . In part (a), the value of τ_{n+1} is determined by $(Y_{(n+1)}, Z_{(n+1)}, \epsilon_{(n+1)})$, because the interaction schedule $(T_k)_{k \geq 0}$ is adapted to the filtration $(\mathcal{F}_n)_{n \geq 0}$ in (3.2). Conditionally on $\tau_{n+1} = n+1$, the value of ξ_{n+1} is determined by $(Y_{(n+1)}, Z_{(n+1)}, \epsilon_{(n+1)})$ for the same reason. In part (b), (5.1) and (5.2) yield that if $\tau_{n+1} = n+1$, then

$$\xi_{n+1} = e^{\eta_{n+1} - \eta_{\tau_n}} \xi_{\tau_n} e^{\sum_{k=\tau_n}^n \epsilon_{k+1}} \frac{\bar{\gamma}_{n+1}(Y_{n+1})}{\bar{\gamma}_{\tau_n}(Y_{\tau_n})} \frac{\gamma_{n+1}^Z}{\gamma_{\tau_n}^Z},$$

where γ_{n+1}^Z and $\gamma_{\tau_n}^Z$ depend on $\{Y_{\tau_n - \phi}, \dots, Y_n\}$ and $\{Z_{\tau_n - \phi + 1}, \dots, Z_{n+1}\}$, and $\tau_n = \phi - 1$ is the interaction time prior to $n+1$. \square

Proof of (4.2): The optimal allocation in Proposition 4.1 can be written as

$$\tilde{\pi}_n^*(d) = \frac{\tilde{\mu}_{n+1}}{\gamma_n \sigma_{n+1}^2} \frac{\mathbb{E}_{n,d} \left[\frac{\tilde{Z}_{n+1}}{\tilde{\mu}_{n+1}} \tilde{r}_{n+1,T}^{\pi^*} \right] - R_{n+1} \gamma_n \left(\mathbb{E}_{n,d} \left[\frac{\tilde{Z}_{n+1}}{\tilde{\mu}_{n+1}} (\tilde{r}_{n+1,T}^{\pi^*})^2 \right] - \mathbb{E}_{n,d} \left[\frac{\tilde{Z}_{n+1}}{\tilde{\mu}_{n+1}} \tilde{r}_{n+1,T}^{\pi^*} \right] \mathbb{E}_{n,d} [\tilde{r}_{n+1,T}^{\pi^*}] \right)}{\mathbb{E}_{n,d} \left[\left(\frac{\tilde{Z}_{n+1}}{\sigma_{n+1}} \tilde{r}_{n+1,T}^{\pi^*} \right)^2 \right] - \left(\mathbb{E}_{n,d} \left[\frac{\tilde{Z}_{n+1}}{\sigma_{n+1}} \tilde{r}_{n+1,T}^{\pi^*} \right] \right)^2}.$$

This simplifies to formula (4.2) under the assumption of Corollary 4.2, in which case the expected values in the previous formula factor into products of expectations. \square

C Proofs and Auxiliary Results for Section 5

Proof of Proposition 5.1: From the definitions of γ_n and γ_n^C , it follows that

$$\mathcal{R}(\phi, \beta) = \frac{1}{T} \sum_{n=0}^{T-1} \mathbb{E} \left[\left| \frac{\gamma_n^{id}}{\gamma_{\tau_n}^{id} \gamma_{\tau_n}^Z} - 1 \right| \right]. \quad (\text{C.1})$$

Let $Y_0 = y_0$ and extend the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to include an independent sequence of random variables $(Z_n)_{n \leq 0}$, such that $Z_n \sim \mathcal{N}(\tilde{\mu}(y_0), \sigma^2(y_0))$. For a given n , define the event $A_n := \{Y_{\tau_n - \phi} = \dots = Y_{\tau_n - 1} = y_0\}$, on which the economic state variable is fixed. We then have

$$\mathbb{E} \left[\left| \frac{\gamma_n^{id}}{\gamma_{\tau_n}^{id} \gamma_{\tau_n}^Z} - 1 \right| \right] = \mathbb{E} \left[\left| e^{\beta \left(\frac{1}{\phi} \sum_{k=\tau_n}^{\tau_n-1} Z_{k+1} - \mu_{k+1} \right) + \sum_{k=\tau_n}^{n-1} \epsilon_{k+1}} - 1 \right| A_n \right] + O(1 - P_{y_0, y_0}^{\tau_n-1}),$$

where $P_{y_0, y_0}^{\tau_n - 1} := (P^{\tau_n - 1})_{y_0, y_0}$ is the probability of staying in state y_0 from time zero to time $\tau_n - 1$. For the first term appearing in the exponent, we have

$$\beta \left(\frac{1}{\phi} \sum_{k=\tau_n - \phi}^{\tau_n - 1} Z_{k+1} - \mu_{k+1} \right) \Big| A_n \sim \mathcal{N} \left(0, \frac{\beta^2 \sigma_0^2}{\phi} \right).$$

For the second term, we define the random variable $J := |\{\tau_n \leq k \leq n - 1 : \epsilon_{k+1} \neq 0\}| \sim \text{Bin}(n - \tau_n, p_\epsilon)$, which is equal to the number of idiosyncratic risk aversion jumps between times τ_n and n . Hence,

$$\sum_{k=\tau_n}^{n-1} \epsilon_{k+1} \Big| \{J = j\} \sim \mathcal{N} \left(-j \frac{\sigma_\epsilon^2}{2}, j \sigma_\epsilon^2 \right).$$

For $Z \sim \mathcal{N}(0, 1)$, $a \in \mathbb{R}$, and $b \geq 0$, we have $\mathbb{E}[e^{a+bZ} - 1] = b\mathbb{E}[|Z|] + O(|a| + b^2)$. Therefore, conditioning on the value of J and using the absolute Gaussian moment formula gives

$$\mathbb{E} \left[\left| \frac{\gamma_n^{id}}{\gamma_{\tau_n}^{id} \gamma_{\tau_n}^Z} - 1 \right| \right] = \sqrt{\frac{2}{\pi}} \left((1 - p_\epsilon)^{n - \tau_n} \frac{\beta \sigma_0}{\sqrt{\phi}} + (n - \tau_n) p_\epsilon (1 - p_\epsilon)^{n - \tau_n - 1} \sqrt{\frac{\beta^2 \sigma_0^2}{\phi} + \sigma_\epsilon^2} \right) + \Theta + O(1 - P_{y_0, y_0}^{\tau_n - 1}),$$

where we have denoted by Θ a term of order $O(p^2 \sigma_\epsilon^2 + \beta^2)$. Next, plugging the above expression into (C.1), and noting that the sum is the same for each interval bounded by consecutive times of interaction, we obtain

$$\begin{aligned} \mathcal{R}(\phi, \beta) &\geq \frac{T_\phi}{T} \sqrt{\frac{2}{\pi}} \frac{1}{\phi} \left(\frac{\beta \sigma_0}{\sqrt{\phi}} \sum_{n=0}^{\phi-1} (1 - p_\epsilon)^n + \sqrt{\frac{\beta^2 \sigma_0^2}{\phi} + \sigma_\epsilon^2} \sum_{n=0}^{\phi-1} n p_\epsilon (1 - p_\epsilon)^{n-1} \right) + \Theta + O(1 - P_{y_0, y_0}^{T - \phi - 1}) \\ &= \frac{T_\phi}{T} \sqrt{\frac{2}{\pi}} \frac{1}{\phi} \left(\frac{\beta \sigma_0}{\sqrt{\phi}} \frac{1 - (1 - p_\epsilon)^\phi}{p_\epsilon} + \sqrt{\frac{\beta^2 \sigma_0^2}{\phi} + \sigma_\epsilon^2} \frac{1 - (1 - p_\epsilon)^\phi - \phi(1 - p_\epsilon)^{\phi-1} p_\epsilon}{p_\epsilon} \right) + \Theta + O(1 - P_{y_0, y_0}^{T - \phi - 1}) \\ &= \frac{T_\phi}{T} \tilde{\mathcal{R}}(\phi, \beta) + \Theta + O(1 - P_{y_0, y_0}^{T - 1}), \end{aligned}$$

where the final equality follows from using a third-order Taylor approximation for $(1 - p_\epsilon)^\phi$ and the fact that $P_{y_0, y_0}^{T - \phi - 1}$ is increasing in $\phi \geq 1$. The upper bound in (5.5) follows from a symmetric argument.

Next, to show that there exists a unique minimizing ϕ for $\tilde{\mathcal{R}}(\phi, \beta)$, we define the function

$$f(\phi) := \sqrt{\frac{2}{\pi}} \tilde{\mathcal{R}}(\phi, \beta) = \frac{\beta \sigma_0}{\sqrt{\phi}} \left(1 - \frac{\phi - 1}{2} p_\epsilon \right) + \sqrt{\frac{\beta^2 \sigma_0^2}{\phi} + \sigma_\epsilon^2} \frac{\phi - 1}{2} p_\epsilon.$$

Observe that $f \geq 0$ and that if $p_\epsilon = 0$ and $\beta = 0$, then $f \equiv 0$. If $p_\epsilon = 0$ and $\beta > 0$, then f is strictly decreasing for $\phi \geq 1$, and converges to zero as $\phi \rightarrow \infty$. If $\beta = 0$ and $p_\epsilon > 0$, then f is strictly increasing for

$\phi \geq 1$, and equal to zero for $\phi = 1$. Furthermore,

$$\begin{aligned} f'(\phi) &= -\frac{1}{2} \frac{\beta \sigma_0}{\phi^{3/2}} \left(1 - \frac{\phi - 1}{2} p_\epsilon\right) - \frac{\beta \sigma_0 p_\epsilon}{\sqrt{\phi} 2} - \frac{1}{2\sqrt{\frac{\beta^2 \sigma_0^2}{\phi} + \sigma_\epsilon^2}} \frac{\beta^2 \sigma_0^2 \phi - 1}{\phi^2} p_\epsilon + \sqrt{\frac{\beta^2 \sigma_0^2}{\phi} + \sigma_\epsilon^2} \frac{p_\epsilon}{2} \\ &= \frac{1}{2} \frac{\beta \sigma_0 p_\epsilon}{\phi^2} \left(-\frac{\sqrt{\phi}}{p_\epsilon} + \frac{1}{2} \left(1 - \frac{1}{\sqrt{1 + \frac{\sigma_\epsilon^2 \phi}{\beta^2 \sigma_0^2}}}\right) \sqrt{\phi} (\phi - 1) + \left(\sqrt{1 + \frac{\sigma_\epsilon^2 \phi}{\beta^2 \sigma_0^2}} - 1\right) \phi^{3/2}\right) =: \frac{1}{2} \frac{\beta \sigma_0 p}{\phi^2} g(\phi), \end{aligned}$$

and it follows that the derivative of f at $\phi = 1$ satisfies

$$f'(1) = \frac{1}{2} \beta \sigma_0 p_\epsilon \left(\sqrt{1 + \frac{\sigma_\epsilon^2}{\beta^2 \sigma_0^2}} - 1 - \frac{1}{p_\epsilon}\right) \geq 0 \iff \frac{p_\epsilon \sigma_\epsilon^2}{\beta^2 \sigma_0^2} \geq 2 + \frac{1}{p_\epsilon}.$$

Observe that, since the value p_ϵ is in general small, we have

$$\frac{p_\epsilon \sigma_\epsilon^2}{\beta^2 \sigma_0^2} \geq 2 + \frac{1}{p_\epsilon} \iff \frac{p_\epsilon \sigma_\epsilon}{\beta \sigma_0} \geq \sqrt{1 + 2p_\epsilon} \approx 1, \quad (\text{C.2})$$

which shows that $p_\epsilon \sigma_\epsilon / (\beta \sigma_0) > 1$ is, at this approximation level, a sufficient and necessary condition for $f'(1) < 0$. Furthermore,

$$\begin{aligned} g'(\phi) &= -\frac{1}{2} \frac{\phi^{-1/2}}{p_\epsilon} + \left[\frac{1}{2} \left(1 - \frac{1}{\sqrt{1 + \frac{\sigma_\epsilon^2 \phi}{\beta^2 \sigma_0^2}}}\right) \sqrt{\phi} (\phi - 1)\right]' + \frac{3}{2} \left(\sqrt{1 + \frac{\sigma_\epsilon^2 \phi}{\beta^2 \sigma_0^2}} - 1\right) \phi^{1/2} + \frac{1}{2\sqrt{1 + \frac{\sigma_\epsilon^2 \phi}{\beta^2 \sigma_0^2}}} \frac{\sigma_\epsilon^2}{\beta^2 \sigma_0^2} \phi^{3/2} \\ &\geq \frac{1}{2} \phi^{-1/2} \left(-\frac{1}{p_\epsilon} + \left(\sqrt{1 + \frac{\sigma_\epsilon^2}{\beta^2 \sigma_0^2}} - 1\right) \phi\right) \geq \frac{1}{2} \phi^{-1/2} g(1). \end{aligned}$$

It then follows that if $g(1) \geq 0$, then $g'(\phi) \geq 0$, for all $\phi \geq 1$. Hence, $f'(\phi) \geq 0$ for all $\phi \geq 1$, and f is minimized for $\phi = 1$. If $g(1) < 0$, and thus $f'(1) < 0$, it can be shown that g is strictly convex and $g(\phi) \rightarrow \infty$, as $\phi \rightarrow \infty$. Hence, there exists a unique point where $g(\phi) = 0$, and where f is minimized. Finally, from the form of the function f , it is easy to see the sign of the derivatives of the optimal value of ϕ with respect to $\beta, \sigma_0, p_\epsilon$, and σ_ϵ . \square

Proof of (5.7): For the two investment strategies we denote the coefficient $\mu_n^a(D_n)$ in (A.2) by $\mu_n^a(\gamma_n)$ and $\mu_n^a(\gamma_n^C)$, and show inductively that

$$\mathbb{E}[|\mu_n^a(\gamma_n) - \mu_n^a(\gamma_n^C)|] = O((\phi - 1)p_\epsilon \sigma_\epsilon^2) + O\left(\frac{\beta^2 \sigma^2}{\phi^2}\right).$$

Analogous identities can be obtained for μ_n^b , μ_n^{az} , $\mu_n^{az^2}$, and μ_n^{bz} . One may, then, inductively show that

$$\tilde{\pi}_n^*(\gamma_n) = \frac{\gamma_n^C}{\gamma_n} \tilde{\pi}_n^*(\gamma_n^C) + O((\phi - 1)p_\epsilon \sigma_\epsilon^2) + O\left(\frac{\beta^2 \sigma^2}{\phi^2}\right),$$

and the result follows. \square

The following proposition shows that there exists a one-to-one relation between risk aversion processes and investment strategies that depend only on time and the economic state. For such investment strategies, the proof shows how the corresponding risk aversion process can be constructed using backward induction.

Proposition C.1. *Fix a sequence $(\tilde{\pi}_n)_{0 \leq n < T}$, with $\tilde{\pi}_n : \mathcal{Y} \mapsto \mathbb{R}_0^+$. Then, there exists a unique risk aversion process $(\gamma_n)_{n \geq 0}$, with $\gamma_n : \mathcal{Y} \mapsto \mathbb{R}_0^+$, such that $(\tilde{\pi}_n)_{0 \leq n < T}$ is the optimal relative risky asset allocation corresponding to $(\gamma_n)_{0 \leq n < T}$.*

Proof of Proposition C.1: We construct the process $(\gamma_n)_{0 \leq n < T}$ using backward induction. First, for $n = T - 1$, we have $\gamma_n(y) = \tilde{\mu}(y)/(\tilde{\pi}_n(y)\sigma^2(y))$. For $n < T - 1$, consider the function

$$h : \left(0, \frac{1}{R(y)} \frac{\mu_n^a}{\mu_n^b - (\mu_n^a)^2}\right) \mapsto (-\tilde{\pi}_n(y), \infty), \quad h(x) = \frac{\tilde{\mu}(y)}{x\sigma^2(y)} \frac{\mu_n^a - R(y)x(\mu_n^b - (\mu_n^a)^2)}{\mu_n^b + \left(\frac{\tilde{\mu}(y)}{\sigma(y)}\right)^2 (\mu_n^b - (\mu_n^a)^2)} - \tilde{\pi}_n(y),$$

where $R(y) := 1 + r(y)$, $\mu_n^a = \mu_n^a(y)$, and $\mu_n^b = \mu_n^b(y)$. The function h is strictly decreasing on its domain, and we set the value of $\gamma_n(y)$ to be the unique root of h . Using Proposition A.1 and (4.2), it is simple to check that $(\tilde{\pi}_n)_{0 \leq n < T}$ is the optimal allocation corresponding to the constructed process $(\gamma_n)_{0 \leq n < T}$. \square

In the following lemma, we compute the Sharpe ratio (5.9) for a strategy π of the form

$$\tilde{\pi}_n = \bar{\pi}(Y_n). \tag{C.3}$$

That is, at any time n , in economic state $y \in \mathcal{Y}$, the proportion of wealth invested in the risky asset is given by $\bar{\pi}(y)$, for some function $\bar{\pi} : \mathcal{Y} \mapsto \mathbb{R}_0^+$. For brevity, we write $\bar{\pi}_y$ for $\bar{\pi}(y)$. Additionally, we denote the mean $\tilde{\mu}(y)$ and standard deviation $\sigma(y)$ of excess market returns by $\tilde{\mu}_y$ and σ_y , respectively, and the stationary distribution of the Markov chain $(Y_n)_{n \geq 0}$ by $(\lambda_y)_{y \in \mathcal{Y}}$.

Lemma C.2.

(a) The Sharpe ratio (5.9) of the strategy π in (C.3) satisfies

$$s^\pi = \frac{\tilde{\mu}^\pi}{\sigma^\pi} = \frac{\sum_{y \in \mathcal{Y}} \lambda_y \tilde{\mu}_y \bar{\pi}_y}{\sqrt{\sum_{y \in \mathcal{Y}} \lambda_y (\sigma_y^2 \bar{\pi}_y^2 + (\tilde{\mu}_y \bar{\pi}_y - \tilde{\mu}^\pi)^2)}}. \quad (\text{C.4})$$

Moreover,

$$(i) \ s^\pi \leq \max_{y \in \mathcal{Y}} \frac{\tilde{\mu}_y}{\sigma_y}, \quad (ii) \ s^\pi \rightarrow \frac{\lambda_{y'}}{\sqrt{\lambda_{y'}(1 - \lambda_{y'})^2 + \lambda_{y'}^2 \sum_{y \neq y'} \lambda_y}}, \quad \text{if } \tilde{\mu}_{y'} \rightarrow \infty,$$

and the maximum in (i) is achieved if and only if $|\mathcal{Y}| = 1$.

(b) If $|\mathcal{Y}| = 2$ and $\bar{\pi}_2 = \bar{\pi}_1(1 + \delta)$, formula (C.4) can be written as

$$s^{\pi^*} = \frac{1}{\sqrt{\frac{\sigma^2(1)}{\tilde{\mu}^2(1)} \frac{1 - \lambda_2 + \lambda_2 b^2 (1 + \delta)^2}{(1 + \lambda_2(a(1 + \delta) - 1))^2} + \frac{1 - \lambda_2 + \lambda_2 a^2 (1 + \delta)^2}{(1 + \lambda_2(a(1 + \delta) - 1))^2} - 1}}, \quad (\text{C.5})$$

where $a := \tilde{\mu}(2)/\tilde{\mu}(1)$ and $b := \tilde{\mu}(2)/\tilde{\mu}(1)$.

The previous lemma shows that the Sharpe ratio of π is bounded by the largest market Sharpe ratio, and is equal to the market Sharpe ratio in the case of a single economic state. From (i) it also follows that if all states have the same market Sharpe ratio $\theta > 0$, then the Sharpe ratio of π is strictly smaller than θ . This can be seen by looking at the contribution of state y to the variance in the Sharpe ratio formula (C.4), which consists of a “within-state” variance term, $\sigma_y^2 \bar{\pi}_y^2$, and a “between-state” variance term, $(\tilde{\mu}_y \bar{\pi}_y - \tilde{\mu}^\pi)^2$, where $\tilde{\mu}^\pi = \sum_{y \in \mathcal{Y}} \lambda_y \tilde{\mu}_y \bar{\pi}_y$ is the mean excess return of the strategy. Without the between-state term, the Sharpe ratio of π would be equal to θ .

The between-state variance term is also the reason why a larger expected return in a given economic state does not always translate into a higher Sharpe ratio of π , as can be seen from the limit in (ii). More concretely, if $|\mathcal{Y}| = 2$, the Sharpe ratio converges to $\sqrt{\lambda_2/\lambda_1}$ if $\tilde{\mu}_2 \rightarrow \infty$. Hence, if λ_2 is small, i.e., if little time is spent in state $y = 2$, then the Sharpe ratio effect of higher expected returns in this state is dominated by the effect of increasing between-state variance. In Lemma C.3, we focus on the case of two economic states, and study the sensitivity of the Sharpe ratio of π with respect to the model parameters.

Proof of Lemma C.2: Using that the portfolio return $r_{n,n+1}^\pi$ is independent of the investment horizon T and that Y_n converges in distribution to a random variable with distribution $(\lambda_y)_{y \in \mathcal{Y}}$, we have

$$\begin{aligned}\tilde{\mu}^\pi &= \lim_{n \rightarrow \infty} \mathbb{E}[r_{n,n+1}^\pi - r_{n+1}] = \mathbb{E}[r_{0,1}^\pi - r_1 | Y_0 \sim \lambda] = \sum_{y \in \mathcal{Y}} \lambda_y \tilde{\mu}_y \bar{\pi}_y, \\ (\sigma^\pi)^2 &= \lim_{n \rightarrow \infty} \mathbb{E}[(r_{n,n+1}^\pi - r_{n+1} - \tilde{\mu}^\pi)^2] = \mathbb{E}[(r_{0,1}^\pi - r_1 - \tilde{\mu}^\pi)^2 | Y_0 \sim \lambda] = \sum_{y \in \mathcal{Y}} \lambda_y (\sigma_y^2 \bar{\pi}_y^2 + (\tilde{\mu}_y \bar{\pi}_y - \tilde{\mu}^\pi)^2).\end{aligned}$$

By Jensen's inequality we deduce

$$s^\pi \leq \frac{\sum_{y \in \mathcal{Y}} \lambda_y \tilde{\mu}_y \bar{\pi}_y}{\sqrt{\sum_{y \in \mathcal{Y}} \lambda_y \sigma_y^2 \bar{\pi}_y^2}} \leq \frac{\sum_{y \in \mathcal{Y}} \lambda_y \tilde{\mu}_y \bar{\pi}_y}{\sum_{y \in \mathcal{Y}} \lambda_y \sigma_y \bar{\pi}_y} = \frac{1}{\sum_{y \in \mathcal{Y}} \frac{1}{\tilde{\mu}_y / \sigma_y} \frac{\lambda_y \tilde{\mu}_y \bar{\pi}_y}{\sum_{y' \in \mathcal{Y}} \lambda_{y'} \tilde{\mu}_{y'} \bar{\pi}_{y'}}} \leq \max_{y \in \mathcal{Y}} \frac{\tilde{\mu}_y}{\sigma_y}.$$

If $\tilde{\mu}_{y'} \rightarrow \infty$ for some $y' \in \mathcal{Y}$, then

$$s^\pi \rightarrow \frac{\lambda_{y'} \bar{\pi}_{y'}}{\sqrt{\lambda_{y'} (\bar{\pi}_{y'} - \lambda_{y'} \bar{\pi}_{y'})^2 + \sum_{y \neq y'} \lambda_y \lambda_{y'}^2 \bar{\pi}_{y'}^2}} = \frac{\lambda_{y'}}{\sqrt{\lambda_{y'} (1 - \lambda_{y'})^2 + \lambda_{y'}^2 \sum_{y \neq y'} \lambda_y}}.$$

The special case in part (b) follows by simple calculations. \square

Lemma C.3. Let $u := (\sigma(1)/\tilde{\mu}(1))^2$. We have the following characterizations for the sensitivities of the Sharpe ratio $s^{\pi^*}(\delta)$ defined in (C.5):

$$\begin{aligned}\frac{\partial s^{\pi^*}(\delta)}{\partial a} > 0 &\iff a < \frac{1+u}{1+\delta} + \frac{\lambda}{1-\lambda} u b^2 (1+\delta), \\ \frac{\partial s^{\pi^*}(\delta)}{\partial b} < 0 &\iff b > 0, \\ \frac{\partial s^{\pi^*}(\delta)}{\partial \lambda} > 0 &\iff \lambda > \frac{(a^2 + u b^2)(1+\delta)^2 - (1+u)(2a(1+\delta) - 1)}{(a(1+\delta) - 1)((a^2 + u b^2)(1+\delta)^2 - (1+u))}, \\ \frac{\partial s^{\pi^*}(\delta)}{\partial \delta} > 0 &\iff 1+u > a \left(1 + \frac{b^2}{a^2} u\right) (1+\delta).\end{aligned}$$

Furthermore,

$$\lim_{a \rightarrow \infty} s^{\pi^*}(\delta) = \sqrt{\frac{\lambda}{1-\lambda}}, \quad \lim_{b \rightarrow \infty} s^{\pi^*}(\delta) = 0, \quad s^{\pi^*}(\delta)|_{\lambda=0} = \frac{\tilde{\mu}(1)}{\sigma(1)}, \quad s^{\pi^*}(\delta)|_{\lambda=1} = \frac{\tilde{\mu}(2)}{\sigma(2)}.$$

The inequality conditions for the parameters in the previous lemma are not restrictive. For instance, let us consider the values of the derivatives in the case $\delta = 0$, which corresponds to equal market allocation in

both states. First, for the a -derivative, we have

$$\left. \frac{\partial s^{\pi^*}(\delta)}{\partial a} \right|_{\delta=0} > 0 \iff a < 1 + u + \frac{\lambda}{1-\lambda} ub^2 < 1 + u,$$

which is trivially satisfied for realistic parameter values.³¹ Next, for the λ -derivative, we consider two cases:

$$\left. \frac{\partial s^{\pi^*}(\delta)}{\partial \lambda} \right|_{\delta=0} > 0 \iff \lambda > \frac{a^2 + ub^2 - (1+u)(2a-1)}{(a-1)(a^2 + ub^2 - (1+u))} = \begin{cases} \frac{1}{a+1}, & b = a, \\ \frac{a-(1+u)}{a(a+u)-(1+u)}, & b = \sqrt{a}. \end{cases}$$

First, in the case $b = a$, the market Sharpe ratio is the same in both states. The Sharpe ratio of π^* is then maximized for $\lambda \in \{0, 1\}$, in which case it is equal to the market Sharpe ratio, while for $0 < \lambda < 1$ the Sharpe ratio is strictly lower (see discussion following Lemma C.2), and it is minimized at $\lambda = 1/(a+1)$. We also consider the case $b = \sqrt{a}$, and assume $a > 1$, leading to a higher market Sharpe ratio in the recessionary state, consistent with empirical evidence. (The case $a < 1$ can be similarly analyzed.) The portfolio's Sharpe ratio is then increasing in λ for $0 < \lambda < 1$, because, as mentioned above, the condition $a < 1 + u$ is in general satisfied. Note that for the parameter values in (5.13), $b \approx a^{0.15} < a^{1/2}$, and the portfolio's Sharpe ratio is increasing for $0 < \lambda < 1$.

Finally, for the δ -derivative, we assume that $\tilde{\mu}(1) < \sigma(1)$, which typically holds for stock market returns. For a fixed value of $b > 0$, one can then show that the right-hand side of the inequality is decreasing in a for $0 < a < a_0(b)$, with $a_0(b) := b\sigma(1)/\tilde{\mu}(1)$, and is increasing in a , for $a > a_0(b)$. The inequality is clearly not satisfied for $a = b$ and is satisfied for $a = a_0(b)$ if and only if

$$1 + u > 2b\sqrt{u}.$$

The above inequality holds for standard parameter values, such as those in (5.13) with monthly portfolio rebalancing. In other words, there exists a threshold $a^*(b) > b$ that a needs to exceed for $s^{\pi^*}(\delta)$ to be increasing at $\delta = 0$. Notably, there also exists an upper threshold $a^{**}(b) > a^*(b)$ such that if $a \geq a^{**}(b)$, then the condition is no longer satisfied.³² These thresholds are due to the between-state variance of portfolio returns, and we again refer to the discussion following Lemma C.2 for more details.

³¹For example, for the parameter values in (5.13) and monthly portfolio rebalancing, $a \approx 2.1$ and $u \approx 66.2$.

³²For plausible values of the parameters, such as those in (5.13), the upper bound is very large and not exceeded.

Proof of Lemma C.3: To show the result for the a -derivative, let

$$\begin{aligned} h(a, \lambda, \delta) &:= \frac{(1-\lambda)(1+u) + \lambda(a^2 + ub^2)(1+\delta)^2}{(1 + \lambda(a(1+\delta) - 1))^2} = \frac{(1-\lambda)(1+u) + \lambda ub^2(1+\delta)^2 + \lambda a^2(1+\delta)^2}{(1 - \lambda + \lambda a(1+\delta))^2} \\ &=: \frac{c_0 + c_1 a^2}{(d_0 + d_1 a)^2}, \end{aligned}$$

and note that $s^{\pi^*}(\delta)$ in (C.5) is increasing in a if and only if h is decreasing in a . We then have

$$\frac{\partial h(a, \lambda, \delta)}{\partial a} = \frac{2c_1 a(d_0 + d_1 a)^2 - 2(d_0 + d_1 a)d_1(c_0 + c_1 a^2)}{(d_0 + d_1 a)^4} = \frac{2c_1 a(d_0 + d_1 a) - 2d_1(c_0 + c_1 a^2)}{(d_0 + d_1 a)^3},$$

and the result is obtained by solving $\frac{\partial h(a, \lambda, \delta)}{\partial a} < 0$ for a . The results for the λ - and δ -derivatives are obtained in the same way by noting that $s^{\pi^*}(\delta)$ is decreasing in λ and δ if and only if h is decreasing in λ and δ . The rest of the lemma follows directly from the formula for $s^{\pi^*}(\delta)$. \square

Proof of (5.11): Let $u := (\sigma(1)/\tilde{\mu}^2(1))^2$. The Sharpe ratio formula (C.5) can be written as

$$s^{\pi^*}(\delta) = \frac{1 + \lambda(a-1) + \lambda a \delta}{\sqrt{(\lambda+u)(1-\lambda) + \lambda(a^2(1-\lambda) + ub^2)(1+\delta)^2 + 2a\lambda(\lambda-1)(1+\delta)}} =: \frac{d_0 + d_1 \delta}{\sqrt{c_0 + c_1(1+\delta) + c_2(1+\delta)^2}}.$$

Let $\kappa(\delta) := \sqrt{c_0 + c_1(1+\delta) + c_2(1+\delta)^2}$. Straightforward computations yield

$$\begin{aligned} \frac{\partial s^{\pi^*}(\delta)}{\partial \delta} &= \frac{d_1 \kappa^2(\delta) - \frac{1}{2}(d_0 + d_1 \delta)(c_1 + 2c_2(1+\delta))}{\kappa^3(\delta)}, \\ \frac{\partial^2 s^{\pi^*}(\delta)}{\partial \delta^2} &= -\frac{(d_0 + d_1 \delta)(c_2 \kappa^2(\delta) - \frac{3}{4}(c_1 + 2c_2(1+\delta))^2) + d_1(c_1 + 2c_2(1+\delta))\kappa^2(\delta)}{\kappa^5(\delta)}. \end{aligned}$$

Evaluating the second derivative at $\delta = 0$, we obtain

$$\begin{aligned} \left. \frac{\partial^2 s^{\pi^*}(\delta)}{\partial \delta^2} \right|_{\delta=0} &= -\frac{d_0(c_0 c_2 - \frac{3}{4}c_1^2 - 2c_1 c_2 - 2c_2^2) + d_1(c_1 + 2c_2)(c_0 + c_1 + c_2)}{\kappa^5(0)} \\ &= \left(\frac{s^{\pi^*}(0)}{d_0} \right)^5 \left(-(a^2 u + b^2 u^2)\lambda + O(\lambda^2) \right), \end{aligned}$$

and the result follows. \square

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