Lecture notes for TMA4190 Introduction to Topology

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Disclaimer

These are typed up versions of the notes I wrote down during the lectures thought by Fernando Abellán in the spring of 2025. Take them as they are, they probably contain errors.

1 Introduction.

09.01

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2 Metric spaces, continuous functions

10.01

2.1 Metric spaces

Definition 2.1 (Metric Space)**.** *A metric space is a pair (X, d), where X is a set and d is a map $d : X \times X \to \mathbb{R}$:*

1. $\forall x, y \in X : d(x, y) \geq 0$ *and* $d(x, y) = 0 \iff x = y$
2. $\forall x, y \in X : d(x, y) = d(y, x)$
3. $\forall x, y, z \in X : d(x, z) \leq d(x, y) + d(y, z)$

Definition 2.2 (Continuity)**.** *Let $(X, d_X), (Y, d_Y)$ be metric spaces. A map $f : X \to Y$ is continuous at $x \in X$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t.*

$$d_X(p, q) < \delta \implies d_Y(f(p), f(q)) < \varepsilon$$

Definition 2.3 (Balls)**.** *Let (X, d_X) be metric space and let $p \in X$ and $r > 0$. We define the*

· $B(p, r) = \{x \in X \mid d(p, x) < r\}$

· $\overline{B}(p, r) = \{x \in X \mid d(p, x) \leq r\}$

Definition 2.4 (Open and closed subsets)**.** *Let (X, d) be a metric space. A subset $U \subseteq X$ is open if $\forall p \in U, \exists \varepsilon > 0$ s.t.*

$$B(p, \varepsilon) \subseteq U$$

We say that $Z \subseteq X$ is closed if $Z^{\mathsf{c}} = X \setminus Z$ is open.

Proposition 2.1. *Let (X, d) be a metric space. Then $B(x, r)$ is open and $\overline{B}(x, r)$ is closed $\forall x \in X, \forall r > 0$.*

Proof. We first show that $B(x, r)$ is open. Let $y \in B(x, r)$. Define $\varepsilon = r - d(x, y) > 0$, and consider $z \in B(y, \varepsilon)$. Then

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \varepsilon = d(x, y) + r - d(x, y) = r,$$

so $B(y, \varepsilon) \subseteq B(x, r)$.

Next, we show that $\overline{B}(x, r)$ is closed. Need to show that $\overline{B}(x, r)^{\mathsf{c}}$ is open. Pick $y \in \overline{B}(x, r)^{\mathsf{c}}$, and define $\varepsilon = d(x, y) - r > 0$. Take $z \in B(y, \varepsilon)$. Then

$$d(x, y) \leq d(x, z) + d(z, y) < d(x, z) + \varepsilon = d(x, z) + d(x, z) - r,$$

so $r < d(x, z)$. This shows that $B(y, \varepsilon) \subseteq \overline{B}(x, r)^{\mathsf{c}}$. $\overline{B}(x, r)$ is thus closed. $\square$

Definition 2.5 (Neighbourhood)**.** *Let (X, d) be a metric space. $B \subseteq X$ is a neighbourhood (nbh.) of $p \in X$ if $\exists \varepsilon > 0$ s.t. $B(p, \varepsilon) \subseteq B$*

Theorem 2.1. *Let $f : X \to Y$ be a map between metric spaces. Then f is continuous at $p \in X$ iff. $\forall B$ nbh. of $f(p), \exists$ nbh. A of p such that $f(A) \subseteq B$.*

Proof. We show both directions.

$\Rightarrow$) Assume f is cont. as p. Let B be a nbh. of $f(p)$. By definition of the nbh., there exists an $\varepsilon > 0$ such that there exists a ball $B(f(p), \varepsilon) \subseteq B$. By continuity of f at p, there exists a $\delta > 0$ such that

$$d(p, y) < \delta \implies d(f(p), f(y)) < \varepsilon.$$

That is, $\forall y \in B(p, \delta)$ we have that $f(y) \in B(f(p), \varepsilon)$. Thus

$$f(B(p, \delta)) \subseteq B(f(p), \varepsilon) \subseteq B$$

So we have found a nbh. of p, namely $B(p, \delta)$.

$\Leftarrow$) Assume that for all nbh. B of $f(p)$ there exists a nbh. A of p s.t. $f(A) \subseteq B$. We need to show that f is continuous at p. Given $\varepsilon > 0$, consider the following nbh. of $f(p)$: $B(f(p), \varepsilon)$. By assumption there exists a nbh. of p, A, such that $f(A) \subseteq B(f(p), \varepsilon)$. A is a nbh., so there exists a $\delta > 0$ such that

$$B(p, \delta) \subseteq A.$$

Also

$$f(B(p, \delta)) \subseteq B(f(p), \varepsilon).$$

Let $z \in B(p, \delta)$. That is, $d(p, z) < \delta$. By the previous inclusion we get that $d(f(p), f(z)) < \varepsilon$, so f is continuous at p.

$\square$

Theorem 2.2. *A map of metric spaces $f : X \to Y$ is continuous at every point iff. $V \subseteq Y$ open then $f^{-1}(V) \subseteq X$ is also open.*

Proof. We show both directions.

$\Rightarrow$) Assume that f is continuous at every point in X. Take $V \subseteq Y$ open. Let $x \in f^{-1}(V)$. Now V is a nbh. of $f(x)$, and by theorem 2.1 there exists a nbh. A of x such that $f(A) \subseteq V$. So

$$B(x, \varepsilon) \subseteq A \implies f(B(x, \varepsilon)) \subseteq f(A) \subseteq V \implies B(x, \varepsilon) \subseteq f^{-1}(V)$$

$\Leftarrow$) Let $x \in X$, and let B be a nbh. of $f(x)$. Then there exists a ball $B(f(x), \varepsilon) \subseteq B$. By assumption $f^{-1}(B(f(x), \varepsilon))$ is open. In particular $f^{-1}(B(f(x), \varepsilon))$ is a nbh. of x. In addition

$$f(f^{-1}(B(f(x), \varepsilon))) \subseteq B(f(x), \varepsilon)$$

and so by theorem 2.1 f is continuous at x. Since x was arbitrary, f is continuous everywhere.

$\square$

3 Topological spaces: First definitions and examples

16.01

3.1 Topological spaces

Definition 3.1 (Topological space)**.** *A topological space is a pair (X, τ), where X is a set and $\tau \subseteq \mathcal{P}(X)$ is a collection of subsets of X. We call the elements in τ the open sets in X. τ satisfies*

T1) $\emptyset, X \subseteq \tau$.

T2) Given a collection $\{U_i\}_{i\in I}$ of open sets, then $\bigcup_{i\in I} U_i$ is open.

T3) Given a finite collection $\{V_j\}_{j\in J}$, $|J| < \infty$ of open sets, then $\bigcap_{j\in J} V_j$ is open.

Proposition 3.1. *Let (X, d) be a metric space. Then X is a topological space with $\tau = \{U \subseteq X \mid \forall x \in U, \exists \varepsilon > 0 \text{ s.t. } B(x, \varepsilon) \subseteq U\}$.*

Proof. We show that the three axioms in 3.1 are satisfied.

T1) Trivially true.

T2) Given $\{U_i\}_{i\in I}$ such that $U_i \subseteq \tau$ for all $i \in I$. Take some $x \in \cup_{i\in I} U_i$. Then there exists some i_0 such that $x \in U_{i_0}$. Since U_{i_0} is open there exists $\varepsilon_{i_0} > 0$ such that
$$B(x, \varepsilon_{i_0}) \subseteq U_{i_0} \subseteq \bigcup_{i\in I} U_i.$$

T3) Given a finite index set J and $\{V_j\}_{j\in J}$ such that $V_j \in \tau$ for all $j \in J$. Let $x \in \cap_{j\in J} V_j$. Then $x \in V_j$ for all $j \in J$. Let $\varepsilon_j > 0$ be such that $B(x, \varepsilon_j) \subseteq V_j$. Define $\varepsilon = \min_j \varepsilon_j$. Then
$$B(x, \varepsilon_j) \subseteq \bigcap_{j\in J} V_j.$$

□

Example 3.1. *It is important that J is finite. Take $X = \mathbb{R}$. Let $V_n = (-1/n, 1/n)$. Then*
$$\bigcap_{n\in\mathbb{N}} V_n = \{0\} \notin \tau.$$

Example 3.2. *Let X be a set and let $\tau_{dis} = \mathcal{P}(X)$. τ_{dis} is a topology on X.*

Example 3.3. *Let X be a set and let $\tau_{ind} = \{\emptyset, X\}$. τ_{ind} is a topology on X.*

Example 3.4. *Given (X, τ), then $(X, \tau) = (X, \tau_{dis})$ iff. $\{x\}$ is open $\forall x \in X$.*

Example 3.5. *Let X be a set and recall the discrete metric $\delta_X : X \times X \to \mathbb{R}$. Then X is equal to (X, τ_{dis}) as a topological space.*

Example 3.6. *Let X be a set, and declare $U \subseteq X$ to be open if $X \setminus U$ is finite. We call the collection τ_{cof} the cofinite topology.*

Example 3.7. *Let $\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. $U \subseteq \hat{\mathbb{N}}$ is open if either $\infty \notin U$ or $\infty \in U$ and U^{c} is finite.*

Definition 3.2 (Neighbourhood)**.** *Let (X, τ) be a topological space. Then $U \subseteq X$ is a neighbourhood if $x \in U$ and U is open.*

Theorem 3.1. *Let (X, τ) be a topological space. $U \subseteq X$ is open iff. $\forall x \in U$ there exists a nbh. V_x of x such that $V_x \subseteq U$.*

Proof. We show both directions.

$\Rightarrow$) Assume U open. Then for all $x \in U$ U is a nbh. of x and $U \subseteq U$.

$\Leftarrow$) Assume that $\forall x \in U$ there exists nbhs. V_x such that $V_x \subseteq U$. Then

$$\bigcup V_x = U$$

is open.

$\square$

Definition 3.3 (Continuity)**.** *A map of topological spaces $f : X \to Y$ is continuous if $\forall V \subseteq Y$ open then*

$$f^{-1}(V) \subseteq X$$

is open in X.

Example 3.8. *$id : X \to X$ is cont. under the same topology.*

Example 3.9. *$f : X \to Y, f(x) = y \forall x \in X$. Let $V \subseteq Y$. Then*

$$f^{-1}(V) = \begin{cases} X & y \in V \\ \emptyset & y \notin V \end{cases}$$

and since $\emptyset, X \in \tau$, f is continuous.

Example 3.10. *$f : X \to Y$, where X has the discrete topology. f is continuous since all $f^{-1}(V)$ are open in X.*

Example 3.11. *$f : X \to Y$, where Y has the indiscrete topology. f is continuous since $f^{-1}(Y) = X$ and $f^{-1}(\emptyset) = \emptyset$ are both opens.*

Non-Example 3.1. *$f : (\mathbb{R}, \tau_E) \to (\mathbb{R}, \tau_{dis})$ is not continuous since $f^{-1}(\{x\}) = \{x\}$ is not open wrt. τ_E.*

Definition 3.4 (Coarser, finer)**.** *Let X be a set, and let τ_1, τ_2 be topologies on X. τ_1 is coarser than τ_2 if $\tau_1 \subset \tau_2$. τ_2 is finer than τ_1 if $\tau_1 \subset \tau_2$.*

4 Solutions to exercise sheet 0

17.01

Exercise class. Might add this in the future.

5 Topological spaces: Continuous maps, homeomorphisms, closure, interior

23.01

Proposition 5.1. *$f : X \to Y, g : Y \to Z$ cont.. Then $g \circ f : X \to Z$ cont..*

Proof. Let $V \subseteq Y$ be open and use that $(g \circ f)^{-1}(V) = g^{-1}(f^{-1}(V))$ is open. $\square$

Definition 5.1 (Homeomorphic topological spaces)**.** *A pair X, Y of topological spaces are homeomorphic if $\exists$ cont. maps $f : X \to Y, g : Y \to X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$.*

Example 5.1. *Let $X = \{a, b\}, \tau_1 = \{\emptyset, X, \{a\}\}, \tau_2 = \{\emptyset, X, \{b\}\}$. Then*

$$f : (X, \tau_1) \to (X, \tau_2)$$

is a homeomorphism.

Warning: A homeomorphism is a continuous bijection, but a continuous bijection is not necessarily a homeomorphism.

Non-Example 5.1. *$f : (\mathbb{R}, \tau_{dis}) \to (\mathbb{R}, \tau_E)$ is a continous bijection, but not a homeomorphism.*

Theorem 5.1. *Let X, Y, Z be topological spaces. The relation of being homeomorphic ($\sim$) has the following properties:*

1. $X \sim X$
2. $X \sim Y \implies Y \sim X$
3. $X \sim Y, Y \sim Z \implies X \sim Z$

Proof. Straightforward:

1. $X \xrightarrow{\text{id}} X$ is a homeomorphism.
2. $X \sim Y$. Then there exists continuous functions f, g such that $g \circ f = \text{id}_X, f \circ g = \text{id}_Y$. So $Y \sim X$.
3. $X \sim Y, Y \sim Z$, then by composing arrows, we see that $X \sim Z$.

$$\begin{tikzcd}
X \arrow[r, bend left, "f_1"] & Y \arrow[l, bend left, "g_1"] \arrow[r, bend left, "f_2"] & Z \arrow[l, bend left, "g_2"]
\end{tikzcd}$$

$\square$

Definition 5.2 (Closed space)**.** *X top. space.* $Z \subseteq X$ *is closed if* $Z^{\mathsf{c}} = X \setminus Z$ *is open.*

Proposition 5.2. *X top. space. Then*

1. $\emptyset, X$ *is closed.*
2. $\{Z_i\}_{i \in I}$ *a collection of closed subsets. Then* $\bigcap_{i \in I} Z_i$ *is closed.*
3. $\{Z_j\}_{j \in J}$ *a finite collection of closed subsets. Then* $\bigcup_{j \in J} Z_j$ *is closed.*

Proof. We prove the three axioms in 3.1:

T1) $\emptyset^{\mathsf{c}} = X$ is open and $X^{\mathsf{c}} = \emptyset$ is open.

T2)
$$\left(\bigcap_{i \in I} Z_i\right)^{\mathsf{c}} = \bigcup_{i \in I} Z_i^{\mathsf{c}}$$
is open since Z_i^{c} is open.

T3)
$$\left(\bigcup_{j \in J} Z_j\right)^{\mathsf{c}} = \bigcap_{j \in J} Z_j^{\mathsf{c}}$$
is open since Z_j^{c} is open and the intersection is finite.

□

Example 5.2. *In* (X, τ_{dis})*, everything is open and closed.*

Definition 5.3 (Closure, interior)**.** *Let X be a a top. space and let* $A \subseteq X$*. The closure of A is*
$$\overline{A} = \bigcap_{A \subseteq Z, Z\, closed} Z$$
The interior of A is
$$A^\circ = \bigcup_{U \subseteq A, Z\, open} U$$

Proposition 5.3. *X top. space.* $A \subseteq X$*. A is closed* $\iff \overline{A} = A$*. A is open* $\iff A^\circ = A$*.*

Proof. If $\overline{A} = A$, then A is closed since it is a intersection of closed sets. If A is closed, then
$$\overline{A} = \bigcap_{\substack{A \subseteq Z \\ Z \text{ closed}}} Z = A \cap \bigcap_{\substack{A \subseteq Z \\ Z \text{ closed} \\ A \neq Z}} Z = A$$

□

Example 5.3. *In* $\mathbb{R}$*:* $\overline{(a,b]} = [a,b]$.

Definition 5.4 (Boundary point)**.** *X top. space. $A \subseteq X$. $x \in X$ is a boundary point of A if $\forall U \ni x$ nbh. of X: $U \cap A \neq \emptyset$ and $U \cap A^{\mathsf{c}} \neq \emptyset$.*

Definition 5.5 (Dense)**.** *$A \subseteq X$ is dense if $\overline{A} = X$.*

Example 5.4. $\mathbb{Q}$ *is dense in* $\mathbb{R}$.

6 Basis of a topology, subspace topology

24.01

Proposition 6.1. *Let $f : X \to Y$ be a map of topoligical spaces. Then TFAE:*

1. *f is continuous.*
2. *For all closed subsets $Z \subseteq Y$, $f^{-1}(Z)$ is also closed.*
3. *For all subsets $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.*

Proof. Let us show that proposition 1 is equivalent to proposition 2.

Take $Z \subseteq Y$ closed. Then Z^{c} is open, so $f^{-1}(Z^{\mathsf{c}}) = (f^{-1}(Z))^{\mathsf{c}}$ is open iff. f is continuous. Thus $f^{-1}(Z)$ is closed for all closed subsets $Z \subseteq Y$.

Let us show that 2 is equivalent to 3.

$\Rightarrow$) Note that $\bar{(}f(A))$ is closed, so $\overline{f(A)}^{-1}$ is also closed. We have that

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)})$$

and since $\overline{A}$ is the smallest closed set that contains A we get that

$$\overline{A} \subseteq f^{-1}(\overline{f(A)}).$$

Applying f at both sides yields

$$f(\overline{A}) \subseteq \overline{f(A)}.$$

$\Leftarrow$) Assume that $\forall A \subseteq X$ we have $f(\overline{A}) \subseteq \overline{f(A)}$. Take $Z \subseteq Y$ closed. Consider $f^{-1}(Z) \subseteq X$ and use the assumtion:

$$f(\overline{f^{-1}(Z)}) \subseteq \overline{f(f^{-1}(Z))} = \overline{Z} = Z.$$

So

$$\overline{f^{-1}(Z)} \subseteq f^{-1}(Z)$$

but obviously $\overline{f^{-1}(Z)} \subseteq f^{-1}(Z)$, so $\overline{f^{-1}(Z)} = f^{-1}(Z)$, and hence $f^{-1}(Z)$ is closed.

□

Proposition 6.2. *Let X be a set and let τ_1 and τ_2 be topologies on X. Then $\tau_1 \cap \tau_2$ is a topology on X.*

Proof. Use that every open is in the intersection of τ_1 and τ_2, so in exploit that τ_i is a topology. □

Given a set X, and topologies τ_1, τ_2 on X. Then $\tau_1 \cup \tau_2$ is in general not a topology on X.

Non-Example 6.1. *Let $X = \{a, b, c\}$ and define two topologies on X:*

$$\tau_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$$

and

$$\tau_1 = \{\emptyset, X, \{c\}, \{b, c\}\}.$$

Then

$$\tau_1 \cup \tau_2 = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{b, c\}\}$$

is not a topology on X.

Definition 6.1. *(Basis) Let $\mathscr{B} \subseteq \mathcal{P}(X)$. We say that $\mathscr{B}$ is a basis for a topology on X if the following holds:*

B1) $\forall x \in X, \exists B \in \mathscr{B}$ *s.t.* $x \in B$.

B2) Given $B_1, B_2 \in \mathscr{B}$ *and* $x \in B_1 \cap B_2, \exists B_3 \in \mathscr{B}$ *s.t.* $x \in B_3 \subseteq B_1 \cap B_2$.

Proposition 6.3. *Let $\mathscr{B}$ be a basis for a topolopy on X and let*

$$\tau = \{U \subseteq X \mid \forall x \in U, \exists B \in \mathscr{B} \text{ s.t. } x \in B \subseteq U\}$$

Then τ defines a topology on X.

Proof. We show the three axioms.

T1) $\emptyset \in \tau$ is trivial. $X \in \tau$ follows immediatly from the definition of a basis.

T2) Let $\{U_i\}_{i \in I}$ be a collection of opens. Then for $x \in \bigcup U_i$ there exists a i_0 such that $x \in U_{i_0}$. Since U_{i_0} is open, there exists a basis element $B \in \mathscr{B}$ such that $x \in B \subseteq U_{i_0}$. So immediatly

$$x \in B \subseteq U_{i_0} \subseteq \bigcup_{i \in I} U_i$$

and $\bigcup U_i$ is therefore open.

T3) Let $\{V_j\}_{j \in J}$ be a finite collection of opens. Take some $x \in \bigcap V_j$. Use induction on $|J| = n$.

Base case: $n = 2$.

$$x \in V_1 \cap V_2$$

Then there exists $B_1, B_2 \in \mathscr{B}$ such that $x \in B_i \subseteq V_i$. So

$$x \in B_1 \cap B_2 \subseteq V_1 \cap V_2$$

Then by B2) there exists some $B_3 \in \mathscr{B}$ such that

$$x \in B_3 \subseteq B_1 \cap B_2 \subseteq V_1 \cap V_2$$

Induction step: $n-1 \mapsto n$.

$$x \in \bigcap_{j=1}^{n-1} V_j \cap V_n$$

By the induction hypothesis there exists some $B \in \mathscr{B}$ such that

$$x \in B \subseteq \bigcap_{j=1}^{n-1} V_j$$

Also, there exists $B_2 \in \mathscr{B}$ such that $x \in B_2 \subseteq V_n$. Then, by B2) there exists $B_3 \in \mathscr{B}$ such that

$$x \in B_3 \subseteq B_1 \cap B_2 \bigcap_{n=1}^{n-1} V_j \cap V_n$$

$\square$

Example 6.1. *Given a basis $\mathscr{B}$ for X. Then $U \subseteq X$ is open iff.*

$$U = \bigcup_{i \in I} B_i, B_i \in \mathscr{B} \forall i \in I$$

.

Proposition 6.4. *Let $f : X \to Y$ be a map, and suppose that $\mathscr{B}$ is a basis for a topology on Y. Then f is continuous iff. $f^{-1}(B)$ is open in $X, \forall B \in \mathscr{B}$.*

Proof. We show both directions

$\Rightarrow$) This direction is obvious.

$\Leftarrow$) Take $U \subseteq X$ open. Write $U = \bigcup B_i$. Then

$$f^{-1}(U) = f^{-1}\left(\bigcup B_i\right) = \bigcup f^{-1}(B_i)$$

is open since $f^{-1}(B_i)$ is open.

$\square$

6.1 Constructing topological spaces

6.2 Subspaces

Proposition 6.5. *Let X be a topological space. Let $A \subset X$ be a proper subset. Then the collection*

$$\tau_A = \{U \cap A \mid U \text{ open in } X\}$$

is a topology on A.

Proof. We show the three axioms

T1) Obvious.

T2) Take a collection of opens: $\{U_i \cap A\}$. Then

$$\bigcup_{i \in I} U_i \cap A = A \cap \bigcup_{i \in I} U_i \in \tau_A$$

T3) Take a finite collection of opens: $\{V_j \cap A\}$. Then

$$\bigcap_{j \in J} V_j \cap A = A \cap \bigcap_{j \in J} V_j \in \tau_A$$

$\square$

Proposition 6.6. *Let X be a topological space. Let $A \subseteq X$ be a subset. Then $L \subseteq A$ is closed in the subspace topology on A iff. $\exists K \subseteq X$ which is closed and such that $K \cap A = L$.*

Proof. Observe that $L \subseteq A$ is closed $\iff$ $A \setminus L$ open $\iff$ $\exists U \subseteq X$ open s.t. $U \cap A = A \setminus L$.

($\Rightarrow$) Let $K = U^{\mathsf{c}} = X \setminus U$. Then

$$K \cap A = (X \setminus U) \cap A = A \setminus (A \cap U) = A\ (A \setminus L) = L$$

($\Leftarrow$) Let $K \subseteq X$ be closed and such that $K \cap A = L$. Need to show that $A \setminus L$ is open under the subspace topology. Consider $U = K^{\mathsf{c}} = X \setminus K$. U is open.

$$U \cap A = (X \setminus K) \cap A = (A \setminus (K \cap A)) \cap A = (A \setminus L) \cap A = A \setminus L$$

$\square$

7 Subspace topology, product topology, universal properties

30.01

Definition 7.1 (Subspace topology)**.** *Given $A \subseteq X$, X topological space. We define the subspace topology by*

$$\tau_A = \{U \cap A \mid U \text{ open in } X\}$$

Proposition 7.1. *Let $A \subseteq X$, where X is a topological space with basis $\mathscr{B}$. Then the collection*

$$\mathscr{B}_A = \{B \cap A \mid B \in \mathscr{B}\}$$

is a basis for the subspace topology τ_A.

Proof. We need to show that $\mathscr{B}_A$ actually is a basis and then that the topology it generates coincides with the subspace topology.

We show that we actually have a basis.

B1) Given $x \in A$. In particular $x \in X$, so there exists $B \in \mathscr{B}$ such that $x \in B$. Also, since $x \in A$ we have that $x \in B \cap A \in \mathscr{B}_A$.

B2) Given $x \in (B_1 \cap A) \cap (B_2 \cap A) = (B_1 \cap B_2) \cap A$. Since $x \in B_1$ and $x \in B_2$ we get $B_3 \subseteq \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Then

$$x \in B_3 \cap A \subseteq (B_1 \cap B_2) \cap A = (B_1 \cap A) \cap (B_2 \cap A)$$

Denote the topology generated by $\mathscr{B}_A$ by $\overline{\tau}$. We claim that $\overline{\tau} = \tau_A$. Let us show that $\overline{\tau} \subseteq \tau_A$. Let $U \subseteq A$ be open in $\overline{\tau}$. Then $U = \bigcup (B_i \cap A)$, where B_i open in X. Hence $B_i \cap A$ open in τ_A, so $\overline{\tau} \subseteq \tau_A$.

Next, show that $\tau_A \subseteq \overline{\tau}$. Take $U \cap A \in \tau_A$. Take $x \in U \cap A$. Since U open, there exists $B \in \mathscr{B}$ such that $x \in B \subseteq X$. Thus $x \in B \cap A \subseteq U \cap A$. $\square$

Remark 7.1. *Let $A \subseteq X$, X topological space. Then the canonical inclusion ι is continuous, where A has the subspace topology.*

Theorem 7.1. *Let X, Y be a topological spaces and let $A \subseteq X$ be a subset of X. Let $f : Y \to A$ be a map, and let $\iota : A \to X$ be the inclusion map. Then the supspace topology on A is the unique topology on A which satisfies the following*

$$f \text{ is continuous} \iff \iota \circ f \text{ is continuous.} \tag{1}$$

$$\begin{tikzcd} & X \\ Y \arrow[ur, "\iota\circ f"] \arrow[r, "f"] & A \arrow[u, hook, "\iota"'] \end{tikzcd}$$

Proof. We first show that the subspace topology satisfies (1) and then that it is the unique topology that satisfies (1).

Suppose $f : Y \to A$ is continuous. Since $\iota : A \to X$ is continuous, we get by composition that $\iota \circ f$ is continuous.

Suppose that $\iota \circ f : Y \to X$ is continuous. Let $U \subseteq X$ be a open subset of X. Then

$$f^{-1}(U \cap A) = f^{-1}(\iota^{-1}(U)) = (\iota \circ f)^{-1}(U)$$

is open, so $\iota \circ f$ is continuous.

Suppose there exist another topology $\hat{\tau}$ on A such that (1) holds. The following diagram commutes by (1).

$$\begin{tikzcd} & X \\ (A, \hat{\tau}) \arrow[ur, "\iota \circ \mathrm{id}"] \arrow[r, "\mathrm{id}"] & (A, \hat{\tau}) \arrow[u, hook, "\iota"'] \end{tikzcd}$$

So, by continuity of the identity map, $(\iota \circ \mathrm{id})^{-1}(U) = U \cap A$ is open in $(A, \hat{\tau})$. Hence $\tau_A \subseteq \hat{\tau}$. Now, consider

$$\begin{tikzcd} & X \\ (A, \tau_A) \arrow[ur, "\iota \circ \mathrm{id}"] \arrow[r, "\mathrm{id}"] & (A, \hat{\tau}) \arrow[u, hook, "\iota"'] \end{tikzcd}$$

and note that $\iota \circ \mathrm{id}$ is continuous. So id is continuous by (1), and hence $\hat{\tau} \subseteq \tau_A$.

We have shown that $\hat{\tau} \subseteq \tau_A$ and that $\tau_A \subseteq \hat{\tau}$ so $\tau_A = \hat{\tau}$. $\square$

7.1 Product spaces

Proposition 7.2. *X, Y topological spaces. Then*

$$\mathscr{B}_{X \times Y} = \{U \times V \mid U \subseteq X \textit{ open}, V \subseteq Y \textit{ open}\}$$

is a basis for $X \times Y$.

Proof. We show the the two properties in the definition.

B1) Let $(x, y) \in X \times Y$. Note that $X \times Y$ is a basis element since X is open in X and Y is open in Y.

B2) Let $(U_1 \times V_1), (U_2 \times V_2)$ be basis elements and let

$$(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2).$$

Then, the basis element $(U_1 \cap U_2) \times (V_2 \cap V_2)$ is such that

$$(x, y) \in (U_1 \cap U_2) \times (V_2 \cap V_2) \subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$$

since $(U_1 \cap U_2) \times (V_2 \cap V_2) = (U_1 \times V_1) \cap (U_2 \times V_2)$.

$\square$

Definition 7.2 (Product Topology)**.** *Let X, Y be topological spaces. Then we define the product topology $\tau_{X\times Y}$ to be the topology generated by $\mathscr{B}_{X\times Y}$.*

Proposition 7.3. *Let X, Y be topological spaces. Let $\mathscr{B}_X, \mathscr{B}_Y$ be bases for X and Y respectivly. Then the collection*

$$\mathscr{B}_X \times \mathscr{B}_Y = \{B_X \times B_Y \mid B_X \in \mathscr{B}_X, B_Y \in \mathscr{B}_Y\}$$

is a basis for $X \times Y$ that generates the product topology.

Proof. We need to show that it is in fact a basis and then that is generates the product topology.

B1) Let $(x, y) \in X \times Y$. In particular, $x \in X$ and $y \in Y$, so we get basis elements $B_x \ni x, B_y \ni y$. Hence $(x, y) \in B_x \times B_y \in \mathscr{B}_X \times \mathscr{B}_Y$.

B2) Given basis elements $B_X^1 \times B_Y^1, B_X^2 \times B_Y^2 \in \mathscr{B}_X \times \mathscr{B}_Y$. Let

$$x \in (B_X^1 \times B_Y^1) \cap (B_X^2 \times B_Y^2) = (B_X^1 \cap B_X^2) \times (B_Y^1 \cap B_Y^2)$$

Now, since $\mathscr{B}_X, \mathscr{B}_Y$ are bases for X and Y respectivly we get B_X^3, B_Y^3 such that

$$x \in B_X^3 \subseteq B_X^1 \cap B_X^2$$

and

$$y \in B_Y^3 \subseteq B_Y^1 \cap B_Y^2$$

Hence

$$(x, y) \in B_X^3 \times B_Y^3 \subseteq (B_X^1 \cap B_X^2) \times (B_Y^1 \cap B_Y^2) = (B_X^1 \times B_Y^1) \cap (B_X^2 \times B_Y^2)$$

Next, we show that it generates the product topology $\tau_{X\times Y}$. We show that $\tau_{\mathscr{B}_X\times\mathscr{B}_Y} \subseteq \tau_{X\times Y}$ and that $\tau_{X\times Y} \subseteq \tau_{\mathscr{B}_X\times\mathscr{B}_Y}$.

$\subseteq$) Take $W \in \tau_{X\times Y}$. Then $W = \bigcup_{i\in I} U_{X_i} \times V_{Y_i}$. Let $(x, y) \in W$. Then $\exists j$ such that $(x, y) \in U_{X_j} \times V_{Y_j}$. So there exists basis elements B_{X_j}, B_{Y_j} such that

$$(x, y) \in B_{X_j} \times B_{Y_j} \subseteq U_{X_j} \times U_{Y_j}$$

Hence $\tau_{X\times Y} \subseteq \tau_{\mathscr{B}_X\times\mathscr{B}_Y}$.

$\supseteq$) Take $W \in \tau_{\mathscr{B}_X\times\mathscr{B}_Y}$. Then $W = \bigcup_{i\in I} B_{X_i} \times B_{Y_i}$, and since every B_{X_i}, B_{Y_i} are opens, we get that $\tau_{\mathscr{B}_X\times\mathscr{B}_Y} \subseteq \tau_{X\times Y}$.

$\square$

Remark 7.2. *We have two canonical maps*

$$\begin{aligned} \pi_X : X \times Y &\to X \\ (x, y) &\mapsto x \\ \pi_Y : X \times Y &\to Y \\ (x, y) &\mapsto y \end{aligned}$$

Note, π_X is continuous, since for $U \subseteq X$ open, then $U \times Y$ open in $X \times Y$. $\pi_x(U \times Y) = U$. Same for π_Y.

8 Solutions to exercise sheet 1

31.01
Exercises

9 Universal property of the product topology, quotient topology

06.02

Theorem 9.1. *Let X, Y be topological spaces. The product topology on $X \times Y$ is the unique topology on the product s.t. the following universal property holds*

$$f : Z \to X \times Y \text{ cont.} \iff \pi_X \circ f \text{ and } \pi_Y \circ f \text{ are cont.} \tag{2}$$

$$\begin{tikzcd}
 & Z \arrow[dl] \arrow[d, dashed, "f"'] \arrow[dr] & \\
X & X \times Y \arrow[l, "\pi_x"'] \arrow[r, "\pi_y"] & Y
\end{tikzcd}$$

Proof. We show first that the product topology satisfies (2) and next that it is the unique topology that does so.

Equip $X \times Y$ with the product topology.

$\Rightarrow$) By function composition.

$\Leftarrow$) Let $U \times V \subseteq X \times Y$ be open. Then

$$\begin{aligned}
f^{-1}(U \times V) &= f^{-1}(U \times Y \cap X \times V) \\
&= f^{-1}(U \times Y) \cap f^{-1}(X \times V) \\
&= f^{-1}({\pi_X}^{-1}(U)) \cap f^{-1}({\pi_Y}^{-1}(V)) \\
&= (\pi_X \circ f)^{-1}(U) \cap (\pi_Y \circ f)^{-1}(V)
\end{aligned}$$

And since $\pi_i \circ f$ are continuous, $f^{-1}(U \times V)$ is open. Hence f is continuous.

Now suppose that $\hat{\tau}$ also satisfies (2). We show that $\hat{\tau} \subseteq \tau_{X \times Y}$ and that $\tau_{X \times Y} \subseteq \hat{\tau}$.

Let $X \hat{\times} Y$ denote the product with $\hat{\tau}$ as topology. Consider the following diagram.

$$\begin{tikzcd}
 & X \times Y \arrow[dl, "\pi_X"'] \arrow[d, "\mathrm{id}"] \arrow[dr, "\pi_Y"] & \\
X & X \hat{\times} Y \arrow[l, "\hat{\pi}_X"'] \arrow[r, "\hat{\pi}_Y"] & Y
\end{tikzcd}$$

Since π_X, π_Y are continuous, and $\pi_X = \hat{\pi}_X \circ \mathrm{id}, \pi_Y = \hat{\pi}_Y \circ \mathrm{id}$, by (2)

$$\mathrm{id} : X \times Y \to X \hat{\times} Y$$

is also continuous. This implies that $\hat{\tau} \subseteq \tau_{X \times Y}$.

Now, replace $X \times Y$ with $X \hat{\times} Y$:

$$\begin{tikzcd}
 & X\hat{\times}Y \arrow[dl, "\hat{\pi}_X"'] \arrow[d, "\mathrm{id}"] \arrow[dr, "\hat{\pi}_Y"] & \\
X & X\hat{\times}Y \arrow[l, "\hat{\pi}_X"'] \arrow[r, "\hat{\pi}_Y"] & Y
\end{tikzcd}$$

Since $\mathrm{id} : X\hat{\times}Y \to X\hat{\times}Y$ is continuous (by (2) once again), $\hat{\pi}_X$ and $\hat{\pi}_Y$ are continuous. Take $U \subseteq X, V \subseteq Y$ opens. Then $\hat{\pi}_X^{-1}(U) = U \times Y$ and $\hat{\pi}_Y^{-1}(V) = X \times V$ are both open. Hence $U \times V = (U \times Y) \cap (X \times V)$ is open in $\hat{\tau}$. Since $\tau_{X\times Y}$ is generated by $\{U \times V\}$, we get that $\tau_{X\times Y} \subseteq \hat{\tau}$.

We have shown that $\tau_{X\times Y} \subseteq \hat{\tau}$ and that $\hat{\tau} \subseteq \tau_{X\times Y}$, so $\hat{\tau} = \tau_{X\times Y}$. $\square$

Corollary 9.1.1. *Given A, B topological spaces. Then the universal property says that the map*

$$\begin{aligned}
\hom_{Top}(A, X \times Y) &\longrightarrow \hom_{Top}(A, X) \times \hom_{Top}(A, Y) \\
(\varphi : A \to X \times Y) &\longmapsto (\pi_X \circ \varphi : A \to X, \pi_Y \circ \varphi : A \to Y)
\end{aligned}$$

exists and is an iso.

9.1 Quotient spaces

Lemma 9.2. *Let $\varphi : X \to A$ be a surjection of sets, and let $\sim$ be the equivalence relation given by $x \sim y \iff \varphi(x) = \varphi(y)$. Then the induced map*

$$\begin{aligned}
\overline{\varphi} : X/\sim &\to A \\
[x] &\to \varphi(x)
\end{aligned}$$

is a bijection.

$$\begin{tikzcd}
X \arrow[rr, two heads, "\varphi"] \arrow[dr] & & A \\
 & X/\sim \arrow[ur, "\overline{\varphi}"'] &
\end{tikzcd}$$

Proof. We first show that $\overline{\varphi}$ is well defined. Consider $x, y \in [x]$. Then $x \sim y$, so $\varphi(x) = \varphi(y)$. $\overline{\varphi}$ is surjective since φ is surjective. Now, sps. $\overline{\varphi}([x]) = \overline{\varphi}([y])$. Then $\varphi(x) = \varphi(y) \iff x \sim y \iff [x] = [y]$, so $\overline{\varphi}$ is injective. Hence $\overline{\varphi}$ is both inj. and surj., so it is a bijection. $\square$

Definition 9.1 (Quotient topology)**.** *Let X be a topological space. Let $\pi : X \to A$ be a surjection. The quotient topology on A is formed by declaring $U \subseteq A$ to be open iff. $\pi^{-1}(U)$ open in X.*

Proposition 9.1. *The quotient topology is a topology. Moreover, it is the finest topology that makes π from above continuous.*

Proof. We show the three properties of a topology.

T1) $\emptyset$ and A are both open in A since $\pi^{-1}(\emptyset) = \emptyset$ and $\pi^{-1}(A) = X$ are opens in X.

T2) Let $\{U_i\}_{i\in I}$ be a collection of opens in A. Then

$$\pi^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} \pi^{-1}(U_i)$$

is open.

T3) Let $\{V_j\}_{j\in J}$ be a finite collection of opens in A. Then

$$\pi^{-1}\left(\bigcap_{j\in J} V_j\right) = \bigcup_{j\in J} \pi^{-1}(V_j)$$

is open.

Now suppose $\hat{\tau}$ is a topology on A such that π is continuous. Then, given $U \subseteq A$ open in $\hat{\tau}$, $\pi^{-1}(U)$ is open in X, so U is also open in the quotient topology. Hence, the quotient topology is the finest topology that makes π continuous. □

Definition 9.2. *Given $\pi : X \to A$ surjective, where A has the quotient topology. Then we call π the quotient map.*

Example 9.1. *Let $\pi : \mathbb{R} \to \{a, b, c\} = X$. be defined by*

$$x \mapsto \begin{cases} a & x = 0 \\ b & x < 0 \\ c & x > 0 \end{cases}$$

The inverse images are $\pi^{-1}(a) = \{0\}, \pi^{-1}(b) = (-\infty, 0), \pi^{-1}(c) = (0, \infty)$. Hence, the quotient topology is $\tau_X = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$.

Definition 9.3. *We say that a map $f : X \to Y$ between topological spaces is open if for all $U \subseteq X$ open, then $f(U)$ is open. The map is closed if for all $V \subseteq X$ closed, then $f(V)$ is closed.*

10 Quotients, open maps, universal property of the quotient topology

07.02

Definition 10.1 (Equivalence relation)**.** *Let X be a set. Then $R \subseteq X \times X$ is an equivalence relation if*

1. $(x, x) \in R \, \forall x \in X$
2. $(x, y) \in R \implies (y, x) \in R$
3. $(x, y) \in R, (y, z) \in R \implies (x, z) \in R$

Definition 10.2 (Equivalence class)**.** *The equivalence class of x in a set X equipped with equivalence relation $\sim$ is defined as*

$$[x] = \{y \in X \mid x \sim y\}$$

Lemma 10.1. *Let X be a set equipped with equivalence relation $\sim$. Then*

$$[x] = [y] \iff x \sim y$$

Proof. If $[x] = [y]$, then $y \in [x]$ so $y \sim x$, which proves one direction. Assume now that $x \sim y$. Pick some $z \in [y]$. Then $y \sim z$ by the definition of the equivalence class. Use the assumption and the transitivity of the equivalence relation to arrive at

$$x \sim y, y \sim z \implies x \sim z.$$

So $z \in [x]$, and since z was arbitrary, $[y] \subseteq [x]$. The other inclusion is similar. □

Definition 10.3. *Let X be a set and let $\sim$ be an equivalence relation on X. Denote by $X/\sim$ the set of equivalence classes:*

$$X/\sim = \{[x] \mid x \in X\}$$

Proposition 10.1. *Let X, Y be sets, and let $\sim$ be an equivalence relation on X. Given $f : X \to Y$, then f factors through $X/\sim$ (the diagram commutes) iff. $\forall x, y \in X$ we have that $x \sim y \implies f(x) = f(y)$.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\
X/\sim \arrow[ur, dashed, "\overline{f}"'] &
\end{tikzcd}$$

Proof. Assume that f factors through $X/\sim$. If $x \sim y$, so that $\pi(x) = \pi(y)$ we get

$$f(x) = \overline{f}(\pi(x)) = \overline{f}(\pi(y)) = f(y).$$

For the other direction, define $\overline{f}([x]) = f(x)$. This mapping is well defined since if $[x] = [y]$ then $x \sim y$, and so $f(x) = f(y)$ by assumption. We see that f factors through $X/\sim$ since $f(x) = \overline{f}([x]) = \overline{f}(\pi(x))$. □

Definition 10.4 (quotient topology)**.** *Let X be a toplogical space. Let $\pi : X \to A$ be a surjection. We define a topology on A by declaring $U \subseteq A$ to be open if $\pi^{-1}(U)$ is open in X.*

Definition 10.5 (open and closed maps)**.** *A map $f : X \to Y$ of topological spaces X and Y is said to be open if $\forall U \subseteq X$ open then $f(U)$ open in Y. The map is said to be closed if $\forall V \subseteq X$ closed then $f(V)$ closed in Y.*

Theorem 10.2. *Let $f : X \to Y$ be a cont. bijection of topological spaces. TFAE:*

1. *f is a homeomorphism.*
2. *f is open.*
3. *f is closed.*

Proof. Sps. f is a homeomorphism. Then f is trivially open and closed, since f^{-1} is continuous.

Sps. f is open. Since f is bij., there exists a inverse map $g = f^{-1}$. Take $U \subseteq X$ open. Then $f(U) = g^{-1}(U)$ is open in Y, so g is continuous. Hence f is a homeomorphism.

Similar proof for when f is closed. □

Theorem 10.3. *Let $\pi : X \to A$ be a continuous surjection. Then*

1. *π is open $\implies$ π is a quotient map.*
2. *π is closed $\implies$ π is a quotient map.*

Proof. Suppose π is open. We need to show that $U \subseteq A$ is open iff. $\pi^{-1}(U)$ is open in X.

$\Rightarrow$) Given $U \subseteq A$ is open, then $\pi^{-1}(U)$ is open since π is continuous.

$\Leftarrow$) Given $\pi^{-1}(U) \subseteq X$ open. Then $\pi(\pi^{-1}(U)) = U$ since π surjective. U is open since π is an open map.

Same for π closed. □

Theorem 10.4. *Let X be a topological space. Let $\pi : X \to A$ be a continuous surjection. Then the quotient topology on A is the unique topology on A satisfying the universal property*

$$g : A \longrightarrow Y \text{ cont.} \iff \begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "\pi"'] & Y \\ A \arrow[ur, "g"'] & \end{tikzcd} \quad g \circ \pi \text{ cont.} \tag{3}$$

Proof. We show both directions, and then uniqueness.

$\Rightarrow$) Assume g is cont. Since π is cont. by construction and composition of cont. functions is cont., then so is $g \circ \pi$.

$\Leftarrow$) Assume $g \circ \pi$ is cont.. Take $U \subseteq Y$ open. Then

$$(g \circ \pi)^{-1}(U) = \pi^{-1}(g^{-1}(U))$$

is open since $g \circ \pi$ continuous. Since π is a quotient map, $g^{-1}(U)$ is open, so g is continuous.

Let τ_q denote the quotient topology on A. Assume there exists another topology $\hat{\tau}$ on A such that the universal property holds. Consider

$$\begin{tikzcd}
(X, \tau_X) \arrow[r, "\pi'"'] \arrow[d, "\pi'"] & (A, \hat{\tau}) \\
(A, \hat{\tau}) \arrow[ur, "\mathrm{id}"] &
\end{tikzcd}$$

The identity map is continuous, and since $\hat{\tau}$ satisfies the universal property $\mathrm{id} \circ \pi' = \pi'$ is continuous. So a open subset $U \subseteq A$ gives ${\pi'}^{-1}(U)$ open. Hence $hat\tau \subseteq \tau_q$.

Now, consider the diagram

$$\begin{tikzcd}
(X, \tau_x) \arrow[r, "\pi"'] \arrow[d, "\pi'"] & (A, \tau_q) \\
(A, \hat{\tau}) \arrow[ur, "\mathrm{id}"] &
\end{tikzcd}$$

Both π and π' are continuous by assumption. Since $\hat{\tau}$ is such that the universal property is satisfied id has to be continous. Thus $\tau_q \subseteq \hat{\tau}$. $\square$

10.1 Connected topological spaces

Definition 10.6 (separation)**.** *Let X be a topological space. A pair of opens $U, V \subseteq X$ such that $U, V \neq \emptyset, X$ form a separation of X if*

1. $U \cup V = X$
2. $U \cap V = \emptyset$

Definition 10.7 (connectivity)**.** *A topological space X is connected if no separation exists. X is disconnected if it is not connected.*

11 Connected spaces, path connectedness

13.02

Definition 11.1 (clopen). *Let X be a topological space. $U \subseteq X$ is clopen if it is both closed and open.*

Proposition 11.1. *Let X be a topological space. X is connected iff. all the clopen subsets are $\emptyset$ and X.*

Proof. We prove both directions.

$\Rightarrow$) Assume X is connected. Sps. there exist a clopen subset $U \neq \emptyset$, $U \neq X$. Then $X = U \cup U^{\mathsf{c}}$ is a separation.

$\Leftarrow$) Assume there does not exist clopen subset other than $\emptyset$ and X. Sps. there exists a separation $X = U \cup V$. Then U is clopen since $U^{\mathsf{c}} = V$ is clopen.

□

Proposition 11.2. *Let X be a topological space and let $\{0, 1\}$ be a discrete space of two points. Then X is connected iff. every continuous map $f : X \to \{0, 1\}$ is constant.*

Proof. We show both directions.

$\Rightarrow$) Assume X is connected. Sps. there exists a continuous non-constant map $f : X \to \{0, 1\}$, where $\{0, 1\}$ is a discrete space. Then $f^{-1}(0)$ is clopen. If $f^{-1}(0) = \emptyset$, then $f(X) = 1$, so it is constant. If $f^{-1}(0) = X$, then $f(X) = 0$, so it is constant. Hence, such a map f cannot exist.

$\Leftarrow$) Assume every continuous map f is constant. Suppose X has a separation $X = U \cup V$. Consider

$$f(x) = \begin{cases} 0 & x \in U \\ 1 & x \in V \end{cases}$$

f is well defined since $U \cap V = \emptyset$. f is non-constant since $U \neq \emptyset, V \neq \emptyset$. f is continuous since $f^{-1}(0) = U$ is open and $f^{-1}(1) = V$ is open. Hence X cannot have a separation.

□

Proposition 11.3. *Let $f : X \twoheadrightarrow Y$ be a continuous surjection of topological spaces. If X is connected, then Y is connected.*

Proof. Assume that X is connected. Sps. there exists a separation of Y, $Y = U \cup V$. We show that $f^{-1}(U) \cup f^{-1}(V)$ is a separation of X.

1. f is continuous, so $f^{-1}(U), f^{-1}(V)$ are both opens.
2. $f^{-1}(U) \neq \emptyset, f^{-1}(V) \neq \emptyset$ since f surjective.

3. $f^{-1}(U) \neq X, f^{-1}(V) \neq X$ since f surjective.
4. $f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) = f^{-1}(Y) = X$
5. $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$

So $f^{-1}(U), f^{-1}(V)$ forms a separation of X, which is a contradiction. Hence Y is connected. $\square$

Definition 11.2 (connected subspace)**.** *Let X be a topological space and let $A \subseteq X$. A is a connected subspace of X if A is connected in the subspace topology.*

Lemma 11.1. *Let $A \subseteq X$ be a connected subspace. Assume $X = U \cup V$ is a separation of X. Then $A \subseteq U$ or $A \subseteq V$.*

Proof. Suppose $A \subsetneq U$ and $A \subsetneq V$. Define $A_U = A \cap U, A_V = A \cap V$. Then A_U, A_V forms a separation of A.

1. $A_U, A_V \neq \emptyset$ since $A, U, V \neq \emptyset$ and U, V forms a separation of X.
2. A_U, A_V are opens since A, U, V opens.
3. $A_U \cup A_V = (A \cap U) \cup (A \cap V) = A \cap (U \cup V) = A$.
4. $A_U \cap A_V = (A \cap U) \cap (A \cap V) = A \cap (U \cap V) = \emptyset$.

$\square$

Lemma 11.2. *Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a family of connected subspaces of X, such that*

$$\bigcap_{\lambda \in \Lambda} A_\lambda \neq \emptyset$$

Then, $\bigcup_{\lambda \in \Lambda} A_\lambda$ is a connected subspace of X.

Proof. Sps. $\bigcup_{\lambda \in \Lambda} A_\lambda$ has a separation U, V. Take $p \in \bigcap_{\lambda \in \Lambda} A_\lambda$. Since each A_λ is connected, each A_λ is either contained in U or in V. Without loss of generality we can assume that $p \in U$. Then $A_\lambda \subseteq U$ for all $\lambda \in \Lambda$, so $V = \emptyset$. Contradiction, so U, V is not a separation. Hence $\bigcup_{\lambda \in \Lambda} A_\lambda$ is connected. $\square$

Theorem 11.3. *Given connected topological spaces X, Y. Then $X \times Y$ is connected.*

Proof. Note that $\{x\} \times Y$ and $X \times \{y\}$ are connected for all $x \in X, y \in Y$. Observe that $\{x_0\} \times Y \cup (X \times \{y\})$ is connected since $\{x_0\} \times Y \cap (X \times \{y\}) \neq \emptyset$. Pick some $x_0 \in X$, and define $A_y = \{x_0\} \times Y \cup (X \times \{y\})$ for all $y \in Y$. Now,

1. $\bigcup_{y \in Y} A_y = X \times Y$ since given $(x, y) \in X \times Y$, then $(x, y) \in A_y$.
2. $\bigcap_{y \in Y} A_y \neq \emptyset$ since $\{x_0\} \times Y \in A_y$ for all A_y.

$\square$

Example 11.1. $\mathbb{R}$ *connected* $\implies \mathbb{R}^n$ *connected.*

Theorem 11.4. $\mathbb{R}$ *is connected.*

Proof. Suppose $\mathbb{R} = U \cup V$ is a separation of $\mathbb{R}$. In particular, we have $a \in U, b \in V$ such that $a < b$. Let $A = [a, b] \cap U, B = [a, b] \cap V$. So, $a \in A, b \in B$ and b is an upper bound for A. So $c = \sup A$ is such that

$$a \leq c \leq b$$

Observe that

$$\begin{aligned} A \cup B &= ([a, b] \cap U) \cup ([a, b] \ \cap V) = [a, b] \\ A \cap B &= ([a, b] \cap U) \cap ([a, b] \ \cap V) = \emptyset \end{aligned}$$

So $c \in A \cup B$, which implies $c \in A$ or $c \in B$. Goal: Show that $c \notin A$ and $c \notin B$, which is a contradiction and hence $\mathbb{R}$ is connected.

Suppose $c \in B$. Then $c \neq a$. Since B is open in $[a, b]$ there exists some d such that $(d, c] \subseteq B$. Then $A \cap (d, c] \neq \emptyset$, which is a contradiction.

Suppose $c \in A$. Since A open in $[a, b]$ there exists some d such that $\emptyset \neq [c, d) \subseteq A$. But $d \leq c$ since $c = \sup A$, which is a contradiction.

Hence, $\mathbb{R}$ is connected. □

Theorem 11.5 (Generalized Intermediate Theorem)**.** *Let X be a connected topological space. Let $f : X \to \mathbb{R}$ be continuous. Given $a, b \in \mathbb{R}$ such that $\exists r \in \mathbb{R}$ such that*

$$f(a) < r < f(b)$$

Then $\exists \alpha \in X$ such that $f(\alpha) = r$.

Proof. Suppose no such α exists. Define

$$U_{<r} = f^{-1}\left((-\infty, r)\right) \tag{4}$$

$$U_{>r} = f^{-1}\left((r, +\infty)\right) \tag{5}$$

Claim: $X = U_{<r} \cup U_{>r}$ is a separation of X.

1. $a \in U_{<r}, b \in U_{>r}$ so they are non empty.
2. $b \notin U_{<r}, a \notin U_{>r}$ so they are not X.
3. $U_{<r} \cap U_{>r} = \emptyset$ since if $c \in U_{<r} \cap U_{>r}$ then $c < r$ and $c > r$ which is a contradiction.
4. $U_{<r} \cup U_{>r} = X$ since $f^{-1}(\mathbb{R}) = X$.

Hence, $U_{<r} \cup U_{>r} = X$ is a separation, but X was assumed to be connected which is a contradiction. □

Definition 11.3 (path)**.** *A path in a topological space X is a continous map $\gamma : [0, 1] \to X$.*

Definition 11.4 (path connected)**.** *A topological space X is path connected if $\forall x, y \in X$ there exists a path γ such that $\gamma(0) = x$ and $\gamma(1) = y$.*

Proposition 11.4. *Let X be a topological space. If X is path connected, then X is connected.*

Proof. Assume that X is path connected. Sps. that $X = U \cup V$ is a separation of X. Then there exists points $x \in U, y \in V$. Since X is path connected, there exists a path $\gamma : I \to X$ such that $\gamma(0) = x, \gamma(1) = y$. Now, consider the image of γ. Since I is connected and γ is continuous and surjective onto its image, $\gamma([0, 1])$ is also connected. Since $U \cup V$ is a separation of X, $\gamma(I) \subseteq U$ without loss of generality. So $\gamma(1) = y \in U$, which is a contradiction since $y \in V$ and $U \cap V = \emptyset$. Hence, X has to be connected. $\square$

12 Connected spaces, Hausdorff spaces, compact spaces.

14.02

12.1 Hausdorff spaces

Definition 12.1 (Hausdorff)**.** *A space X is Hausdorff if $\forall x, y \in X, x \neq y$ there exists open subsets $U, V \subseteq X$ with $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.*

Example 12.1. *Every discrete space is Hausdorff.*

Non-Example 12.1. *$X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ is not Hausdorff.*

Non-Example 12.2. *X with the indiscrete topology, $|X| > 1$, is not Hausdorff.*

Proposition 12.1. *Every metric space is Hausdorff.*

Proof. Let X be a metric space. Let $x, y \in X, x \neq y$. Let $\varepsilon = d(x, y)$. Then $d(x, y) > 0$. Construct balls around x, y with radius $\delta = \epsilon/2$. Then you win. □

Theorem 12.1. *Let X be a Hausdorff space. Then the subset $\{x\} \subseteq X$ is closed for all $x \in X$.*

Proof. Pick $x \in X$. Let $y \in \{x\}^{\mathsf{c}}$. Since X Hausdorff, there exists open subsets $U, V \subseteq X$ such that $x \in U, y \in V, U \cap V = \emptyset$. Since $x \notin V$, $V \subseteq \{x\}^{\mathsf{c}}$. Since y was arbitrary, $\{x\}^{\mathsf{c}}$ is open, so $\{x\}$ is closed. Since x was arbitrary this holds for all $x \in X$. □

Theorem 12.2. *Let X, Y be a Hausdorff space. Then $X \times Y$ is Hausdorff.*

Proof. Pick $(x_1, y_1), (x_2, y_2) \subseteq X \times Y$. Since X is Hausdorff, so there exists opens $U \ni x_1, V \ni x_2$, $U \cap V = \emptyset$. Now we have opens $U \times Y \ni (x_1, y_1)$, $V \times Y \ni (x_2, y_2)$ such that

$$(U \times Y) \cap (V \times Y) = (U \cap V) \times Y = \emptyset \times Y = \emptyset$$

□

Theorem 12.3. *Let X be Hausdorff. Let $A \subseteq X$ be a subspace. Then A is Hausdorff.*

Proof. Let $x, y \in A$. Since X Hausdorff, there exists opens $U \ni x, V \ni y$, $U \subseteq X, V \subseteq X$. Now $U \cap A \ni x$ and $V \cap A \ni y$ are opens in A and

$$(U \cap A) \cap (V \cap A) = (U \cap V) \cap A = \emptyset$$

□

Theorem 12.4. *Let X be a topological space. X is Hausdorff iff. $\Delta \subseteq X \times X, \Delta = \{(x, x) \mid x \in X\}$ is closed.*

Proof. We show both directions.

⇒) Assume X is Hausdorff. Claim: Δ^{c} is open. Pick $(x, y) \in \Delta^{\mathsf{c}}$. Since $x \neq y$ and X Hausdorff we can find opens $U \ni x, V \ni y$ such that $U \cap V = \emptyset$. Now, $(x, y) \in U \times V$, but we need to show that $U \times V \subseteq \Delta^{\mathsf{c}}$. Suppose $(a, b) \in U \times V$ and $(a, b) \in \Delta$. Then $a = b$, but $U \cap V = \emptyset$, which is a contradiction. Hence $U \times V \subseteq \Delta^{\mathsf{c}}$. (x, y) was arbitrary, so Δ^{c} is open, hence Δ is closed.

⇐) Assume Δ is closed. Then Δ^{c} is open. That is, for every $x \neq y$, $(x, y) \in \Delta^{\mathsf{c}}$, and we can find a open $U \times V \ni (x, y)$. Suppose $U \cap V \neq \emptyset$. Then there exists a point $z \in U, z \in V$. But then $(z, z) \in U \times V$ and $(z, z) \in \Delta$, which is a contradiction. So $U \cap V = \emptyset$. Hence, X is Hausdorff.

□

Definition 12.2 (cover)**.** *Let X be a topological space. We say that a I-indexed familiy of opens in X, $\{U_i\}_{i \in I}$, is a cover of X if*

$$\bigcup_{i \in I} U_i = X$$

Definition 12.3 (refinement)**.** *Let X be a topological space and let $\{U_i\}_{i \in I}$ be a cover. Let $J \subseteq I$. We say that the J-indexed family $\{U_j\}_{j \in J}$ is a refinement of $\{U_i\}_{i \in I}$ if*

$$\bigcup_{j \in J} U_j = X$$

Definition 12.4 (compact space)**.** *A space X is compact if every cover $\{U_i\}_{i \in I}$ admits a refinement $J \subseteq I$ with $|J| < \infty$.*

Non-Example 12.3. *$\mathbb{R}$ is not compact.*

Proof. $\{(-n, n)\}_{n \in \mathbb{N}}$ is a cover. Sps. $\mathbb{R}$ is compact. Then there exists a finite subset of $\mathbb{N}$, say J, that covers $\mathbb{R}$. Let r be the maximum of J. Then $|x| < r$ for all $x \in \mathbb{R}$, which is absurd. So $\mathbb{R}$ is not compact. □

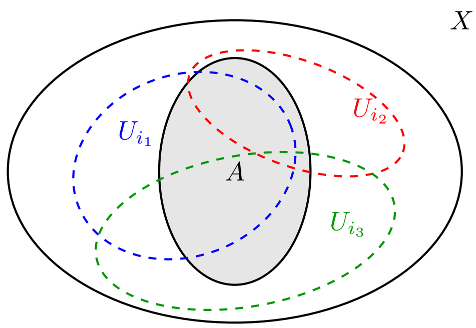
Example 12.2. *X finite topological space. Then X is compact.*

Example 12.3. *X indiscrete topology. Then X compact.*

Definition 12.5 (compact subspace)**.** *Let $A \subseteq X$ be a subspace. Then A is compact in X if it is compact in the subspace topology.*

Lemma 12.5. *Let $A \subseteq X$ be a subspace. Then A is compact as a subspace iff. given a family $\{U_i\}_{i \in I}$, U_i open in X and $A \subseteq \bigcup_{i \in I} U_i$, then there exists a finite refinement $J \subseteq I$ such that*

$$A \subseteq \bigcup_{j \in J} U_j$$



Proof. We show both directions.

⇒) Assume A is compact as a subspace. Let $\{U_i\}_{i\in I}$ be a familiy of opens in X such that $A \subseteq \bigcup_{i\in I} U_i$. Then $\bigcup_{i\in I} U_i \cap A$ covers A since

$$\bigcup_{i\in I} U_i \cap A = A \cap \bigcup_{i\in I} U_i = A$$

Now, since A is compact as a subspace, there exists a finite refinement $J \subseteq I$ such that

$$A = \bigcup_{j\in J} U_j \cap A \subseteq \bigcup_{j\in J} U_j$$

⇐) Assume that every family $\{U_i\}_{i\in I}$ such that $A \subseteq \bigcup_{i\in I} U_i$ admits a finite refinement $J \subseteq I$ such that

$$A \subseteq \bigcup_{j\in J} U_j$$

Then we need to show that A is compact as a subspace. Pick a cover of A:

$$\{V_i\}_{i\in I} = \{U_i \cap A\}_{i\in I}$$

Since $A \subseteq \bigcup_{i\in I} U_i$, then by assumtion we can refine it and get

$$A = \bigcup_{\alpha=0}^{n} V_{i_\alpha}$$

□

Theorem 12.6. *Let X be a compact space. Let $A \subseteq X$ be closed. Then A is compact.*

Proof. Let $\{U_i\}_{i\in I}$ be a family of opens such that $A \subseteq \bigcup_{i\in I} U_i$. Now since A closed, then A^{c} open. We have that

$$\bigcup_{i\in I} U_i \cup A^{\mathsf{c}} = X$$

and since X compact, we have a finite refinement $J \subseteq I \cup *$. Now, J contains finitely many indices such that $A \subseteq \bigcup_{j \in J} U_j$ and by lemma 12.5, A is compact. $\square$

Theorem 12.7. *Let X be Hausdorff. Given $K \subseteq X$, K compact. Then K is closed.*

Proof. Since X is Hausdorff, we can do the following. Let $x \notin K$. For every $y \in K$, pick nbhs. $U_y \ni x, V_y \ni y$ such that $U_y \cap V_y = \emptyset$. Now $\{V_y\}_{y \in Y}$ is a cover of K. Since K is compact, there exists a finite refinement J such that

$$K \subseteq \bigcup_{j \in J} V_{y_j}$$

Now, let $U = \cap_{j \in J} U_{y_j}$. Note that $x \in U$ and U is open. Also

$$U \cap K \subseteq U \cap \left(\bigcup_{j \in J} V_{y_j} \right) = \bigcup_{j \in J} U \cap V_{y_j} = \emptyset$$

Since $x \notin K$ was arbitrary K^{c} is open, hence K is closed. $\square$

Motivation: this theorem makes life easier when we want to prove that a surjective map is a quotient map by combining 12.7 and 10.3. Here is an example:

Example 12.4. *We want to show that*

$$\begin{aligned} f : [0,1] &\longrightarrow \mathbb{S}^1 \\ t &\longmapsto (\cos 2\pi t, \sin 2\pi t) \end{aligned}$$

is a quotient map.

Proof. f is continuous and surjective. Let $K \subseteq [0,1]$ closed. Since $[0,1]$ compact and K is closed, K is compact by theorem 12.6. Now, $f(K) \subseteq \mathbb{S}^1$ and since $\mathbb{S}^1$ is Hausdorff we get that $f(K)$ is closed by theorem 12.7. Hence f is a continuous, surjective and closed map, so f is a quotient map by 10.3. $\square$

13 Compact spaces, product of compact spaces is compact.

Proposition 13.1. *Let $f : X \to Y$ be a continuous surjection. If X is compact, then Y is compact.*

Proof. Let $\{U_i\}_{i \in I}$ be a cover of Y. Then $\{f^{-1}(U_i)\}_{i\ inI}$ is a cover of X since f is continuous. Now, X is compact, so there exists a finite refinement of I, say $J \subseteq I$. Now $\bigcup_{j \in J} U_j = Y$ since

$$Y = f(X) = f\left(\bigcup_{j \in J} f^{-1}(U_j)\right) = \bigcup_{j \in J} f\left(f^{-1}(U_j)\right) = \bigcup_{j \in J} U_j$$

□

13.1 Product of compact spaces

Lemma 13.1 (Tubular neighborhood lemma)**.** *Let X, Y be topological spaces and let Y be compact. Fix $x \in X$ such that $U \subseteq X \times Y$ is a open subset of $X \times Y$ and such that $\{x\} \times Y \subseteq U$. Then there exists an open subset $W_x \subseteq X$ such that $x \in W_x$ and $W_x \times Y \subseteq U$.*

Proof. Since U is open, for all points (x, y), where x is the fiexd point from the statement, we have an open $W_y \times V_y \subseteq U$. Then $\{V_y\}_{y \in Y}$ is a cover of Y. But Y is compact, so there exists a finite number of points $y_0, \ldots, y_n$ such that

$$Y = \bigcup_{i \in I} V_{y_i}$$

Let $W_x = \bigcap_{i \in I} W_{y_i}$. W_x is open since I is finite. Now

$$W_x \times Y \subseteq \bigcup_{i \in I} W_x \times V_{y_i} \subseteq U$$

□

Theorem 13.2. *Let X, Y be compact topological spaces. Then $X \times Y$ is compact.*

Proof. Let $\{U_i\}_{i \in I}$ be a cover of $X \times Y$. Pick $x \in X$. Now $\{x\} \times Y \simeq Y$, and since Y compact and $\{U_i\}_{i \in I}$ covers $\{x\} \times Y$ we have a finite refinement $I_x \subseteq I$ such that $\{x\} \times Y \subseteq \bigcup_{i \in I_x} U_i$. Define $U_x = \bigcup_{i \in I_x} U_i$.

By the Tubular neighborhood lemma there exists an open subset $W_x \subseteq U_x$ such that $x \in W_x$ and $W_x \times Y \subseteq U_x$.

x was arbitrary, so we get a cover of X: $\{W_x\}_{x \in X}$. X is compact so we pick a finite refinement $j = 0, \ldots, n$ such that $\bigcup_{j=0}^{n} W_{x_j} = X$.

Then
$$\bigcup_{j=0}^{n} \bigcup_{i \in I_{x_j}} U_i = \bigcup_{j=0}^{n} U_{x_j} \supseteq \bigcup_{j=0}^{n} W_{x_j} \times Y = X \times Y$$

And the double union on the left hand side of the equation is finite since it is a finite union of a finite union. □

13.2 The closed interval is compact

Theorem 13.3. *Let $[a, b]$ ba a closed interval in $\mathbb{R}$. Then $[a, b]$ is compact.*

Proof. Let $\{U_i\}_{i \in I}$ be a cover of $[a, b]$. Define

$$S := \{x \in [a, b] \mid [a, x] \text{covered by finitely many } i\text{-s in } I\}$$

Goal: Show that $b \in S$.

Observe that S is non-empty since $a \in S$. Furthermore, S is bounded above by b. Let $c = \sup S$. We will show that $c \in S$ and that $c = b$.

1. $c \in S$.

   $a < c \leq b$ since $U_{i_0} = [a, a + \epsilon) \ni a$. We can find $\epsilon > 0$ such that $(c - \epsilon, c\ \epsilon) \subseteq U_{i_\alpha}$. Since $c = \sup S$ there exists

   $$x \in (c - \epsilon, c + \epsilon)$$

   with $x \in S$. So $[a, x]$ is covered by finitely many U_i-s. Also

   $$c \in (c - \epsilon, c + \epsilon)$$

   so $[a, x] \cup [x, c]$ is covered by finitely many U_i-s. Thus $c \in S$.

2. $c = b$.

   Suppose $c < b$. Then there exists $\epsilon > 0$ such that

   $$(c - \epsilon, c + \epsilon) \subseteq U_{i_\alpha}$$

   And $c + \epsilon < b$. Same argument as before shows that $\exists \epsilon' < \epsilon$ such that $c + \epsilon' \in S$. But $c = \sup S$ and $c + \epsilon' > c$, which is a contradiction.

So $c = b \in S$, hence $[a, b]$ compact. □

Definition 13.1 (boundedness)**.** *Let (X, d) be a metric space. Then $A \subseteq X$ is bounded if there exists a constant $L > 0$ such that $d(x, y) < L$ for all $x, y \in A$.*

Theorem 13.4 (Heine-Borel)**.** *Let $A \subseteq \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$. Then A is compact iff. A is closed and bounded.*

Proof. We show both directions.

⇒) Assume $A \subseteq \mathbb{R}$ is compact. Consider the cover $\{B(a,n)\}_{n\in\mathbb{N}}$. Since A compact there exists a finite refinement $N \in \mathbb{N}$ such that

$$A \subseteq B(a,N) = \bigcup_{n=0}^{N} B(a,n)$$

Then A is bounded. A is closed since $\mathbb{R}^n$ is closed.

⇐) Suppose A is closed and bounded. Since A is bounded there exists some L such that $d(a,b) < L$ for all $a,b \in A$. Define

$$P = \prod_{i}^{n} [a_i - L, a_i + L] \subseteq \mathbb{R}^n \tag{6}$$

P is a finite product of closed intervals, and since the closed interval is compact, P is compact. Then $A \subseteq P$ is closed, and hence it is compact in P.

□

Theorem 13.5 (Generalised extreme value theorem)**.** *Let $f : X \to \mathbb{R}$ be a continuous map. If X is compact then there exist $m, M \in X$ such that $f(m) \leq f(x) \leq f(M)$ for all $x \in X$.*

Proof. Note that $f(X) \subseteq \mathbb{R}$ is a compact subspace of $\mathbb{R}$. So by Heine-Borel, $f(X)$ is closed and bounded. There exists α, β such that

$$\alpha \leq f(X) \leq \beta$$

Since $\inf f(X), \sup f(X) \in \overline{f(X)}$ and $f(X)$ is closed we get

$$\inf f(X), \sup f(X) \in f(X)$$

□

Now we can argue that $f : [0,1] \to S^1, t \mapsto (\cos 2\pi t, \sin 2\pi t)$ is a quotient map:

Example 13.1. *Let $Z \subseteq [0,1]$ be a closed subset of $[0,1]$. Since $[0,1]$ is compact, Z is also compact. Now f is a continuous surjection so $f(Z)$ is compact. S^1 is Hausdorff, so $f(Z)$ has to be closed. Thus f is closed and f is thus a quotient map.*

14 Solutions to Exercise sheet 2

21.02

15 Homotopy between maps, homotopy as equivalence relation and path homotopy

Example 15.1. *$\mathbb{S}^1$ is compact, but $\mathbb{R}$ is not. So $\mathbb{S}^1 \not\simeq \mathbb{R}$.*

Example 15.2. *$\mathbb{R} \setminus \{p\}$ is not connected. Assume that there exists a homeomorphism*

$$f : \mathbb{R} \to \mathbb{R}^2$$

f induces a map

$$\hat{f} : \mathbb{R} \setminus \{p\} \to \mathbb{R}^2 \setminus \{f(p)\}$$

but $\mathbb{R}^2 \setminus \{f(p)\}$ is connected. Hence $\mathbb{R} \not\simeq \mathbb{R}^2$.

Denote by I the closed interval $[0, 1] \subseteq \mathbb{R}$ with the usual topology.

15.1 Homotopy theory

15.2 Homotopies

Definition 15.1 (Homotopy)**.** *Let $f, g : X \to Y$ be continuous maps of topological spaces. We say that f is homotopic to g if there exists a continuous map $H : I \times X \to Y$ such that*

$$H(0, x) = f(x) \tag{7}$$

$$H(1, x) = g(x) \tag{8}$$

We denote that two maps f and g are homotopic by writing $f \simeq g$.

Definition 15.2 (Nullhomotopy)**.** *A continuous map $f : X \to Y$ is nullhomotopic if it is homotopic to a constant map.*

Example 15.3. *Consider two maps $f, g : X \to \mathbb{R}^n$. They are homopotic via the homotopy*

$$H(t, x) = (1 - t)f(x) + tg(x) \tag{9}$$

Now, H is continuous, and Fernando says:

> *If this is not continuous, life has no purpose.*

To see that H is continous, consider the following diagrams and argue by composition of known continous maps:

$$I \times X \xrightarrow{\gamma \times \varphi} \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\oplus} \mathbb{R}^n$$

$$(t, x) \longrightarrow ((1 - t)f(x), tg(x)) \longrightarrow (1 - t)f(x) + tg(x)$$

where γ and φ are defined by

$$\begin{array}{llllll} \varphi: & I\times X & \xrightarrow{id\times g} & \mathbb{R}\times\mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & (t,x) & \longrightarrow & (t,g(x)) & \longrightarrow & tg(x) \\ \gamma: & I\times X & \xrightarrow{id\times f} & \mathbb{R}\times\mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & (t,x) & \longrightarrow & (t,f(x)) & \longrightarrow & (1-t)f(x) \end{array}$$

From this we conclude that any two maps with target in $\mathbb{R}^n$ are homotopic. In particular, every map is nullhomotopic.

Definition 15.3. *Let X, Y be topological spaces. Define*

$$\hom_{Top}(X,Y) = \{f : X \to Y \mid f\,\textit{cont.}\} \tag{10}$$

Lemma 15.1 (Pasting lemma)**.** *Let $X = A \cup B$ be a topological space where A, B are closed subsets. Suppose $f : A \to Y$ and $g : B \to Y$ are continuous maps such that*

$$f|_{A\cap B} = g|_{A\cap B}.$$

Then there exists a continuous map $h : X \to Y$ such that

$$h|_A = f, h|_B = g.$$

Proof. Define $h : X \to Y$ by

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

h is well defined since f and g agree on the intersection of A and B. Take a closed subset $Z \subseteq Y$. Then

$$\begin{aligned} h^{-1}(Z) &= (h^{-1}(Z) \cap A) \cup (h^{-1}(Z) \cap B) \\ &= f^{-1}(Z) \cup g^{-1}(Z) \end{aligned}$$

And since f, g are cont. and Z closed we get that $f^{-1}(Z) \subseteq A \subseteq X$ and $g^{-1}(Z) \subseteq B \subseteq X$ are closed. Furthermore, a finite union of closed subsets is closed, so $h^{-1}(Z)$ is closed. $\square$

Theorem 15.2. *Homotopies are an equivalence relation on* $\hom_{Top}(X,Y)$ *.*

Proof. We prove reflexivity, symmetry and transitivity.

1) Define $H : I\times X \to X$ by sending (t,x) to $f(x)$. Then $H(0,x) = H(1,x) = f(x)$.

2) Let H be a homotopy of f and g. Define $\overline{H}$ by

$$\overline{H}: \quad I \times X \longrightarrow I \times X \xrightarrow{H} Y$$

$$(t, x) \longrightarrow (1-t, x)$$

Then it is easy to see that $g \simeq f$.

3) Let H_1 and H_2 be homotopies for $f \simeq g$ and $g \simeq h$ respectivly. Define

$$H_3(t, x) = \begin{cases} H_1(2t, x) & 0 \leq t \leq 1/2 \\ H_2(2t-1, x) & 1/2 \leq t \leq 1 \end{cases} \tag{11}$$

H_3 is continuous by the pasting lemma. It is easy to check that it is a homotopy.

□

Definition 15.4. *Notation:*

$$[X, Y] = \hom_{Top}(X, Y)/\simeq \tag{12}$$

$$[f] = \{g : X \to Y \mid f \simeq g, g \in \hom_{Top}(X, Y)\} \tag{13}$$

Definition 15.5. *Denote by $*$ the singelton set.*

$$[*, Y] = Y/(y_0 \sim y_1 \textit{if a path exists}) = \pi_0(Y) \tag{14}$$

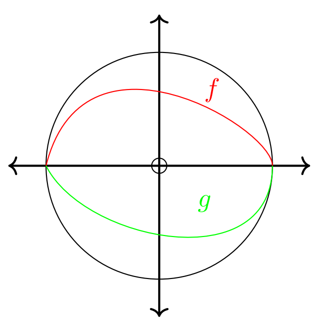
Y path connected $\iff \pi_0(Y) = *$.

15.3 Path homotopies

Definition 15.6 (Path homotopy)**.** *Let $f, g : I \to X$ be paths from x_0 to x_1. We say that f is path homotopic to g and write $f \simeq_p g$ if there exists a continuous function $H : I \times I \to X$ such that*

$$H(0, s) = f(s), H(1, s) = g(s)$$
$$H(t, 0) = x_0, H(t, 1) = x_1$$

Example 15.4. *Let $D^2 = \{(x, y) \in \mathbb{R}^2\} \mid x^2 + y^2 \leq 1$ be the unit disk. Consider $D^2 \setminus \{(0, 0)\}$. Then $f \not\simeq_p g$.*



Definition 15.7. $\hom_{Top}^{x_0,x_1} = \{f : I \to X \mid f\, cont., f(0) = x_0, f(1) = x_1\}$.

Theorem 15.3. *Path homotopies define an equivalence relation on* $\hom_{Top}^{x_0,x_1}$.

Proof. Similar as the proof of homotopy equivalence relation. $\square$

16 Concatenation of paths, associativity , unitality

20.02

Definition 16.1. *Fernando notation:*

$$X(x_0, x_1) = \hom_{Top}^{x_0,x_1}(I, X)/\simeq_p$$

Example 16.1. $\mathbb{R}^n(x_0, x_1) = *$.

Definition 16.2 (loop)**.** *A loop is a path from x to x.*

Definition 16.3 (path concatination)**.** *Let $f, g : I \to X$ be paths from x_0 to x_1 and from x_1 to x_2 respectivly. Define the concatination of f and g as:*

$$(g * f)(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1 \end{cases} \tag{15}$$

Proposition 16.1. *Path concatination is a well defined operation on path homotopy equivalence classes. That is:*

$$\begin{aligned} X(x_0, x_1) \times X(x_1, x_2) &\longrightarrow X(x_0, x_2) \\ ([f], [g]) &\longmapsto [g * f] \end{aligned}$$

is well defined.

Proof. Let $f, f' : I \to X$ be path homotopic maps from x_0 to x_1. Let $g, g' : I \to X$ be path homotopic maps from x_1 to x_2. Hence, $f, f' \in [f]$ and $g, g' \in [g]$. We need to show that

$$g * f \simeq_p g' * f'.$$

Let H_1 be a path homotopy of f and f'. Let H_2 be a path homotopy of g and g'. Define

$$H_3(t, s) = \begin{cases} H_1(t, 2s) & 0 \le s \le 1/2 \\ H_2(t, 2s-1) & 1/2 \le s \le 1 \end{cases}$$

H_3 is well defined since

$$H_3(t, 1/2) = \begin{cases} H_1(t, 1) \\ H_2(t, 0) \end{cases} = \begin{cases} x_1 \\ x_1 \end{cases} = x_1$$

Check that H_3 is a path homotopy of $g * f$ and $g' * f'$.

1.
$$\begin{aligned} H_3(0, s) &= \begin{cases} H_1(0, 2s) & 0 \le s \le 1/2 \\ H_2(0, 2s-1) & 1/2 \le s \le 1 \end{cases} \\ &= \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1 \end{cases} \\ &= (g * f)(s) \end{aligned}$$

2.

$$H_3(1,s) = \begin{cases} H_1(1,2s) & 0 \le s \le 1/2 \\ H_2(1,2s-1) & 1/2 \le s \le 1 \end{cases}$$
$$= \begin{cases} f'(2s) & 0 \le s \le 1/2 \\ g'(2s-1) & 1/2 \le s \le 1 \end{cases}$$
$$= (g' * f')(s)$$

□

Notation: Only f might be used instead of $[f]$ for the equivalence class.

Theorem 16.1. *The operation of concatination enjoys the following properties:*

1) Associativity.
$$(h * g) * f \simeq_p h * (g * f)$$

2) Left/right units.
$$f * c_x \simeq_p f \simeq_p c_y * f$$

3) Left/right inverses.
$$f * \overline{f} \simeq_p c_y, \overline{f} * f \simeq_p c_x$$

Proof. We show the three properties.

1. Associativity. Let $f \in X(x_0, x_1), g \in X(x_1, x_2), h \in X(x_2, x_3)$. We have the following

$$((h * g) * f)(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(4s-2) & 1/2 \le s \le 3/4 \\ h(4s-3) & 3/4 \le s \le 1 \end{cases} \tag{16}$$

$$(h * (g * f))(s) = \begin{cases} f(4s) & 0 \le s \le 1/4 \\ g(4s-1) & 1/4 \le s \le 1/2 \\ h(2s-1) & 1/2 \le s \le 1 \end{cases} \tag{17}$$

A path homotopy for $(h * g) * f$ and $h * (g * f)$ is

$$H(t,s) = \begin{cases} f\left(\frac{4s}{1+t}\right) & 0 \le s \le \frac{1+t}{4} \\ g\left(4s-t-1\right) & \frac{1+t}{4} \le s \le \frac{2+t}{4} \\ h\left(\frac{4s-t-2}{2-t}\right) & \frac{2+t}{4} \le s \le 1 \end{cases} \tag{18}$$

H is continuous by the pasting lemma. And

$$H(0,s) = \begin{cases} f(4s) & 0 \le s \le 1/4 \\ g(4s-1) & 1/4 \le s \le 1/2 \\ h(2s-1) & 1/2 \le s \le 1 \end{cases} = (h * (g * f))(s) \tag{19}$$

$$H(1,s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(4s-2) & 1/2 \le s \le 3/4 \\ h(4s-3) & 3/4 \le s \le 1 \end{cases} = ((h * g) * f)(s) \tag{20}$$

$$H(t,0) = f(0) = x_0 \tag{21}$$

$$H(t,1) = h(1) = x_3 \tag{22}$$

Which shows that H is indeed a path homotopy.

2. Left/right units. Let c_x be the constant path at x and let f be a path from x to y. We show that $f * c_x \simeq_p f$.

$$(f * c_x)(s) = \begin{cases} x & 0 \le s \le 1/2 \\ f(2s-1) & 1/2 \le s \le 1 \end{cases} \tag{23}$$

The following homotopy works.

$$H(t,s) = \begin{cases} x & 0 \le s \le \frac{1-t}{2} \\ f(2s-1) & 1/2 \le s \le 1 \end{cases} \tag{24}$$

It is continuous by the pasting lemma once again. We check:

$$H(0,s) = \begin{cases} x & 0 \le s \le 1/2 \\ f(2s-1) & 1/2 \le s \le 1 \end{cases} = (f * c_x)(s) \tag{25}$$

$$H(1,s) = \begin{cases} x & 0 \le s \le 0 \\ f(s) & 0 \le s \le 1 \end{cases} = f(s) \tag{26}$$

$$H(t,0) = x \tag{27}$$

$$H(t,1) = y \tag{28}$$

Hence, $f * c_x \simeq_p f$. Showing that $c_y \simeq_p f$ follows the same argument.

3. Left/right inverses. Let f be a path from x_0 to x_1. Let $\overline{f}$ be defined by $\overline{f}(t) = f(1-t)$. We show that $\overline{f} * f \simeq_p c_{x_0}$.

$$(\overline{f} * f)(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ f(2-2s) & 1/2 \le s \le 1 \end{cases} \tag{29}$$

Define

$$H(t,s) = \begin{cases} x_0 & 0 \le s \le t/2 \\ f(2s-t) & t/2 \le s \le 1/2 \\ f(2-2s-t) & 1/2 \le s \le 1-t/2 \\ x_0 & 1-t/2 \le s \le 1 \end{cases} \tag{30}$$

Then

$$H(0,s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ f(2-2s) & 1/2 \le s \le 1 \end{cases} \qquad = (\overline{f} * f)(s) \tag{31}$$

$$H(1,s) = \begin{cases} x_0 & 0 \le s \le 1/2 \\ f(2s-1) & 1/2 \le s \le 1/2 \\ f(2-2s-1) & 1/2 \le s \le 1/2 \\ x_0 & 1/2 \le s \le 1 \end{cases} \qquad = c_{x_0}(s) \tag{32}$$

$$H(t,0) = x_0 \tag{33}$$

$$H(t,1) = x_0 \tag{34}$$

Again, showing $c_{x_1} \simeq_p f * \overline{f}$ is similar. □

Fernando defines a group and group homomorphisms. See wikipedia article on groups.

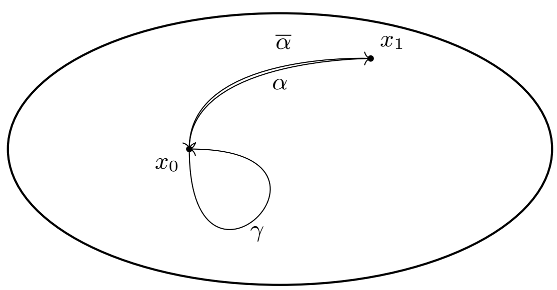
Definition 16.4 (Fundamental group)**.** *Let X be a topological space and let $x_0 \in X$ be a point. Define the fundamental group of X at x_0 as*

$$\begin{aligned} \pi_1(X, x_0) &= X(x_0, x_0) \\ &= \{f : I \to X \mid f(0) = f(1) = x_0\}/\simeq_p \end{aligned}$$

$$\text{Top} \xrightarrow{\simeq} \text{Grp}$$

Theorem 16.2. *Let X be a topological space. Let $\alpha : I \to X$ be a path from x_0 to x_1. Then there exists a group isomorphism*

$$T_\alpha : \pi_1(X, x_0) \to \pi_1(X, x_1)$$



Proof. Define $T_\alpha(\gamma) = \alpha * \gamma * \overline{\alpha}$. We show that T_α is an isomorphism of groups. First,

$$\begin{aligned} T_\alpha(\varphi * \gamma) &= \alpha * \varphi * \gamma * \overline{\alpha} \\ &= \alpha * \varphi * \overline{\alpha} * \alpha * \gamma * \overline{\alpha} \\ &= T_\alpha(\varphi) * T_\alpha(\gamma) \end{aligned}$$

so T_α is a group homomorphism. T_α is bijective, since $T_{\overline{\alpha}}$ is the inverse:

$$(T_{\overline{\alpha}} \circ T_\alpha)(\gamma) = \overline{\alpha} * \alpha * \gamma * \overline{\alpha} * \alpha = \gamma$$
$$(T_\alpha \circ T_{\overline{\alpha}})(\gamma) = \alpha * \overline{\alpha} * \gamma * \alpha * \overline{\alpha} = \gamma$$

□

17 Fundamental group, fundamental group of a product of spaces

06.03

Definition 17.1 (Simply connected)**.** *A topological space X is simply connected if it is path connected and $\pi_1(X, x_0)$ is trivial.*

Example 17.1. $\mathbb{R}^n$ *is simply connected.*

Definition 17.2 (Based space)**.** *A based space is a pair (X, x_0) where X is a topological space and $x_0 \in X$ is a point.*

Definition 17.3 (Based map)**.** *A based map $f : (X, x_0) \to (Y, y_0)$ is a continuous map such that $f(x_0) = f(y_0)$.*

Proposition 17.1. *Let $f : (X, x_0) \to (Y, y_0)$ be a based map. Then there exists a group homomorphism*

$$f_* = \pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

Proof. IDEA: define $f_*(\gamma) = f \circ \gamma$. Show well-definedness and show that it is a grp. hom. We define $f_* = \pi_1(f)$ by

$$f_*(\gamma) = (\pi_1(f))\,(\gamma) = f \circ \gamma$$

This is a well defined map; take $\gamma \simeq_p \gamma'$. Then

$$f \circ H : I \times I \to X \to Y$$

is a homotopy of $f_*(\gamma)$ and $f_*(\gamma')$:

1. $(f \circ H)(0, s) = (f \circ \gamma)(s))$.
2. $(f \circ H)(1, s) = (f \circ \gamma')(s)$.
3. $(f \circ H)(t, 0) = f(x_0) = y_0$.
4. $(f \circ H)(t, 1) = f(x_0) = y_0$.

f_* is a homomorphism of groups. Take $\gamma_1, \gamma_2 \in \pi_1(X, x_0)$. Then

$$f_*(\gamma_2) * (f_*(\gamma_1)) = \begin{cases} (f \circ \gamma_1)(2s) & 0 \leq s \leq 1/2 \\ (f \circ \gamma_2)(2s-1) & 1/2 \leq s \leq 1 \end{cases} = f_*(\gamma_2 * \gamma_1)$$

□

17.1 Category theory

Definition 17.4 (Category). *A category $\mathcal{C}$ is given by the following*

- *a set of objects $ob(\mathcal{C})$. Notation: $X \in \mathcal{C}$ means $X \in ob(\mathcal{C})$.*
- *for every object pair $X, Y \in ob(\mathcal{C})$ a set $\mathcal{C}(X, Y)$ of morphisms from X to Y.*
- *for every $X, Y, Z \in \mathcal{C}$ a map*

$$\begin{aligned} \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) &\to \mathcal{C}(X,Z) \\ (f,g) &\mapsto g \circ f \end{aligned}$$

- *Composition of morphisms is associative.*
- *For all $X \in \mathcal{C}$ there exists a morphism $I_x \in \mathcal{C}(X, X)$ called the identity such that*

$$I_x \circ f = f, g \circ I_x = g$$

for all $f : a \to X$, $g : X \to b$.

Example 17.2. *Let **Set** be the category whose objects are sets and whose morphisms are functions.*

Example 17.3. *Let **Top** be the category whose objects are topological spaces and whose morphisms are continuous functions.*

Example 17.4. *Let **Grp** be the category whose objects are groups and whose morphisms are group homomorphisms.*

Example 17.5. *Let $\boldsymbol{Top}_*$ be the category whose objects are based spaces and whose morphisms are based maps.*

Definition 17.5 (isomorphism). *Let $\mathcal{C}$ be a category. A morphism $f : X \to Y$ is an isomorphism if there exists a morphism $g : Y \to X$ such that*

$$\begin{aligned} f \circ g &= I_Y \\ g \circ f &= I_X \end{aligned}$$

Definition 17.6 (functor). *Let $\mathcal{C}, \mathcal{D}$ be categories. A functor*

$$F : \mathcal{C} \to \mathcal{D}$$

is given by functions

$$\begin{aligned} ob(\mathcal{C}) &\longrightarrow ob(D) \\ c &\longmapsto F(c) \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}(C,Y) &\longrightarrow \mathcal{D}(F(X),F(Y)) \\ f &\longmapsto F(f) \end{aligned}$$

that satisfy

$$\begin{aligned} F(g \circ f) &= F(g) \circ F(f) \\ F(I_X) &= I_{F(X)} \end{aligned}$$

Lemma 17.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F preserves isomorphisms.*

Proof. Let f be iso. Then we have

$$\begin{aligned} F(I_X) &= I_{F(X)} = F(g \circ f) = F(g) \circ F(f) \\ F(I_Y) &= I_{F(Y)} = F(f \circ g) = F(f) \circ F(g) \end{aligned}$$

so $F(f)$ iso. $\square$

Example 17.6. *There is a functor*

$$\mathbf{Grp} \xrightarrow{U} \mathbf{Set}$$

that is forgetful.

Theorem 17.2. *The fundamental group is a functor*

$$\pi_1 : \mathbf{Top}_* \longrightarrow \mathbf{Grp}$$

Proof. Given $(X, x_0), (Y, y_0)$ based spaces and $f, g : (X, x_0) \to (Y, y_0)$ based maps. Proposition 17.1 gives the group homomorphism we need:

$$\begin{aligned} \pi_1(f) = f_* : \pi_1(X, x_0) &\longrightarrow \pi_1(Y, y_0) \\ [\gamma] &\longmapsto [f \circ \gamma] \end{aligned}$$

The identity map is sent to the identity map:

$$\begin{tikzcd} (X, x_0) \arrow[rr, "\mathrm{id}"] \arrow[dr, phantom, "\Downarrow \pi_1"] & & (X, x_0) \\ & & \\ \pi_1(X, x_0) \arrow[rr, "\mathrm{id}"'] & & \pi_1(X, x_0) \\ {[\gamma]} \arrow[rr, mapsto] & & {[\mathrm{id} \circ \gamma]} \end{tikzcd}$$

and preserve compositions. Composing in **Grp**:

$$\begin{array}{ccccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) & \xrightarrow{g_*} & \pi_1(Z, z_0) \\ \\ [\gamma] & \longmapsto & [f \circ \gamma] & \longmapsto & [g \circ f \circ \gamma] \end{array}$$

First composing in $\mathbf{Top}_*$:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(g \circ f)_*} & \pi_1(Z, z_0) \\ \\ [\gamma] & \longrightarrow & [(g \circ f) \circ \gamma] \end{array}$$

which are equal since, $g \circ f \circ \gamma = (g \circ f) \circ \gamma$. □

Corollary 17.2.1. *Let X, Y be topological spaces and suppose $X \simeq Y$. Then $\forall x_0 \in X$ we have that $\pi_1(X, x_0) \simeq \pi_1(Y, f(x_0))$.*

Proof. Since $X \simeq Y$, we have a homeomorphism $f : X \to Y$. This induces the map

$$(X, x_0) \to (Y, f(x_0))$$

By lemma 17.1 we get an isomorphism of groups $\pi_1(X, x_0) \simeq \pi_1(Y, f(x_0))$. □

Theorem 17.3. *Let $(X, x_0), (Y, y_0)$ be based spaces. Then there exists a canonical isomorphism of groups*

$$\pi_1(X \times Y, (x_0, y_0)) \xrightarrow{\sim} \pi_1(X, x_0) \times \pi_1(Y, y_0) \tag{35}$$

Proof. The projection maps

$$\begin{aligned} p_X &: (X \times Y, (x_0, y_0)) \to (X, x_0) \\ p_Y &: (X \times Y, (x_0, y_0)) \to (Y, y_0) \end{aligned}$$

and their images under π_1 induces a map

$$\phi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

defined by

$$\phi([f]) = ([p_X \circ f], [p_Y \circ f]) = ([f_X], [f_Y]) \tag{36}$$

ϕ is a group homomorphism. Let $[f], [f'] \in \pi_1(X \times Y, (x_0, y_0))$.

$$\begin{aligned} \phi([f] * [f']) &= ([p_X \circ (f * f')], [p_Y \circ (f * f')]) \\ &= ([p_X \circ f] * [p_X \circ f'], [p_Y \circ f] * [p_Y \circ f']) \\ &= ([p_X \circ f], [p_Y \circ f]) * ([p_X \circ f'], [p_Y \circ f']) \\ &= \phi([f]) * \phi([f']) \end{aligned}$$

Now we need to show that ϕ is a bijection. First we show injectivity. Assume that $\phi([f]) = ([f_X], [f_Y]) = ([c_{x_0}], [c_{y_0}])$ is the identity. Let H_X, H_Y be the two path homotopies. The universal property of the product gives the map

$$\begin{aligned} H : I \times I &\to X \times Y \\ (t, s) &\mapsto (H_X(t, s), H_Y(t, s)) \end{aligned}$$

which is a path homotopy of $f \simeq_p c_{(x_0, y_0)}$:

$$\begin{aligned} H(t, 0) &= H(t, 1) = (H_X(t, 0), H_Y(t, 0)) = (H_X(t, 1), H_Y(t, 1)) = (x_0, y_0) \\ H(0, s) &= (H_X(0, s), H_Y(0, s)) = (f_X, f_Y) \\ H(1, s) &= (H_X(1, s), H_Y(1, s)) = (c_{x_0}, x_{y_0}) \end{aligned}$$

Now we show surjectivity: Given $([\alpha], [\beta]) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$ we see that $\phi([(\alpha, \beta)]) = ([\alpha], [\beta])$. $\square$

18 Solutions to Exercise sheet 3

14.03

Exercises. Maybe add these in the future.

19 Homotopy equivalences and the fundamental group

20.03

Definition 19.1. *Let $f, g : (X, x_0) \to (Y, y_0)$ be based maps We say that a homotopy*

$$H : I \times X \to Y$$

is a based homotopy if $H(t, x_0) = y_0$ for all t.

Lemma 19.1. *Let $f, g : (X, x_0) \to (Y, y_0)$ be based maps which are homotopic via a based homotopy. Then*

$$\pi_1(f) = \pi_1(g)$$

Proof. Let $\gamma \in \pi_1(X, x_0)$. We need to show that $f_*(\gamma) \simeq_p g_*(\gamma)$. That is, there exists a path homotopy for $f \circ \gamma$ and $g \circ \gamma$. Let $H : I \times X \to X$ be a based homotopy for f and g. Then

$$\hat{H} : I \times I \xrightarrow{\mathrm{id} \times \gamma} I \times X \xrightarrow{H} Y$$

is this path homotopy.

1. $\hat{H}(0, s) = H(0, \gamma(s)) = (f \circ \gamma)(s)$
2. $\hat{H}(1, s) = H(1, \gamma(s)) = (g \circ \gamma)(s)$
3. $\hat{H}(t, 0) = H(t, \gamma(0)) = y_0$
4. $\hat{H}(t, 1) = H(t, \gamma(1)) = y_0$

□

Definition 19.2 (retract, retraction)**.** *Let $A \subseteq X$ be a subspace. We say that A is a retract of X if there exists a map*

$$r : X \to A$$

such that $r \circ \iota = id_A$. We call r a retraction.

Example 19.1. *Consider $\mathbb{S}^1 \subseteq \mathbb{R}^2 \setminus \{(0,0)\}$. Then $r(x) = x/||x||$ is a retraction.*

Lemma 19.2. *Let $x_0 \in A \subseteq X$, and suppose that A is a retract of X. Then the induced group homomorphism*

$$\pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$$

is injective.

Proof. Let $r : X \to A$ be a retraction. Since $x_0 \in A$ then $r(x_0) = x_0$. I get group homomorphisms

$$\pi_1(A, x_0) \xrightarrow{\pi_1(\iota)} \pi_1(X, x_0) \xrightarrow{\pi_1(r)} \pi_1(A, x_0) \tag{37}$$

That compose to the identity: $\pi_1(r) \circ \pi_1(\iota) = id$. Since the composite is injective, $\pi_1(\iota)$ is injective. $\square$

Definition 19.3 (deformation retract)**.** *Let $A \subseteq X$ be a subspace. A homotopy*

$$H : I \times X \to X$$

is a deformation retract if the following holds

1. $H(0, x) = x$
2. $H(1, x) \in A$
3. $H(t, a) = a, \forall a \in A$

Remark 19.1. *A deformation retract is a retract.*

Proof. Let $A \subseteq X$ and $H : I \times X \to X$ be a deformation retract. Then we get the retract $r : X \to A$ by

$$H(1, x) : X \xrightarrow{r} A \xrightarrow{\iota} X$$

$\square$

Theorem 19.3. *Let $x_0 \in A \subseteq X$ and suppose that A is a deformation retract of X. Then we have a group isomorphism*

$$\pi_1(\iota) : \pi_1(A, x_0) \to \pi_1(X, x_0)$$

Proof. My notes here are hardly intelligible, so I provide my own proof. (I think this is what Fernando wrote).

By lemma 19.2 $\pi_1(\iota)$ is an injective group homomorphism. We need to show that it is surjective, i.e. that $\pi_1(\iota) \circ \pi_1(r) = \text{id}$. Let $H : I \times X \to X$ be the deformation retract. Let $H(1, x) = r : X \to A$ denote the retract. Now, H is a based homotopy of id and $r \circ \iota$:

$$\begin{aligned} H(t, x_0) &= x_0 && \text{since } x_0 \in A. \\ H(0, x) &= x \\ H(1, x) &= (\iota \circ r)(x) \end{aligned}$$

By lemma 19.1 $\pi_1(\iota) \circ \pi_1(r) = \text{id}$, so $\pi_1(\iota)$ is surjective.

Hence, $\pi_1(\iota)$ is an isomorphism of groups. $\square$

Exercise 19.1. *Show that $\mathbb{S}^1 \subseteq \mathbb{R}^2 \setminus \{(0, 0)\}$ is a deformation retract.*

Definition 19.4 (homotopy equivalence)**.** *A map $f : X \to Y$ is a homotopy equivalence if there exists a continuous map*

$$g : Y \to X$$

such that $g \circ f \sim id_X$ and $f \circ g \sim id_Y$.

Example 19.2. *Let $A \subseteq X$ be a deformation retract. Then $\iota : A \to X$ is a homotopy equivalence.*

Example 19.3. $0 \in \mathbb{R}^n$. $\mathbb{R}^n \sim *$.

Definition 19.5 (Contractible space)**.** *A space is said to be contractible if it is homotopy equivalent to the point space.*

Definition 19.6 (homotopy type)**.** *We say that X and Y have the same homotopy type if X is homotopy equivalent to Y. Notation:*

$$[X] = \{Y \, top.\ space \mid X \text{ homotopic equivalent to } Y\}$$

Example 19.4. $[\mathbb{R}^n] = [*]$.

Lemma 19.4. *Let $f, g : X \to Y$ be maps. Suppose H is a homotopy*

$$H : I \times X \to Y$$

between f, g. Given $x_0 \in X$, and let

$$\alpha = H(\cdot, x_0) : I \to X$$

Then we have a commutative diagram of groups:

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "\pi_1(f)"] \arrow[dr, "\pi_1(g)"'] & \pi_1(Y, f(x_0)) \arrow[d, "\hat{\alpha}"] \\
& \pi_1(Y, g(x_0))
\end{tikzcd}$$

where $\hat{\alpha}([\gamma]) = [\alpha * \gamma * \overline{\alpha}]$.

Proof. We need to show that $\hat{\alpha} \circ \pi_1(f) = \pi_1(g)$. That is;

$$[\alpha * (f \circ \gamma) * \overline{\alpha}] = [g \circ \gamma]$$

for all $\gamma \in \pi_1(X, x_0)$. Let γ be a loop based at x_0. Define

$$H'(s,t) = \begin{cases} \overline{\alpha}(4s) & 0 \le s \le t/4 \\ H(1-t, \gamma(\frac{4s-t}{4-3t})) & t/4 \le s \le 1 - t/2 \\ \alpha(2s-1) & 1 - t/2 \le s \le 1 \end{cases} \tag{38}$$

Check that H' is a path homotopy of $\alpha * (f \circ \gamma) * \overline{\alpha}$ and $g \circ \gamma$:

$$H'(0,t) = \overline{\alpha}(0) = H(1, x_0) = g(x_0)$$

$$H'(1,t) = \alpha(1) = H(1, x_0) = g(x_0)$$

$$H'(s,0) = \begin{cases} \overline{\alpha}(4s) & 0 \le s \le 0 \\ H(1, \gamma(s)) & 0 \le s \le 1 \\ \alpha(2s-1) & 1 \le s \le 1 \end{cases} = H(1,\gamma(s)) = (g \circ \gamma)(s)$$

$$\begin{aligned} H'(s,1) &= \begin{cases} \overline{\alpha}(4s) & 0 \le s \le 1/4 \\ H(0, \gamma(4s-1)) & 1/4 \le s \le 1/2 \\ \alpha(2s-1) & 1/2 \le s \le 1 \end{cases} \\ &= \begin{cases} \overline{\alpha}(4s) & 0 \le s \le 1/4 \\ f(\gamma(4s-1)) & 1/4 \le s \le 1/2 \\ \alpha(2s-1) & 1/2 \le s \le 1 \end{cases} = (\alpha * (f \circ \gamma) * \overline{\alpha})(s) \end{aligned}$$

□

Theorem 19.5. *Let $f : X \to Y$ be a homotopy equivalence. Then the map*

$$\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

is a group isomorphism.

Proof. Since f is a homotopy equivalence there exists a map $g : Y \to X$ and a homotopy between $g \circ f$ and id_X. By lemma 19.4 we get the following diagram

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "\pi_1(g \circ f)"] \arrow[dr, "\mathrm{id}"'] & \pi_1(X, (g \circ f)(x_0)) \arrow[d, "\hat{\alpha}"] \\ & \pi_1(X, x_0) \end{tikzcd}$$

since $\hat{\alpha} \circ \pi_1(g) \circ \pi_1(f) = \mathrm{id}$, $\pi_1(f)$ is injective. Next, since $f \circ g \sim \mathrm{id}_Y$ we get

$$\begin{tikzcd} \pi_1(Y, f(x_0)) \arrow[r, "\pi_1(f \circ g)"] \arrow[dr, "\mathrm{id}"'] & \pi_1(Y, (f \circ g \circ f)(x_0)) \arrow[d, "\hat{e}"] \\ & \pi_1(Y, x_0) \end{tikzcd}$$

Here my notes are not easy to decode, but I think the proof goes as follows:
So $\hat{e} \circ \pi_1(f) \circ \pi_1(g) = \mathrm{id}$, hence $\hat{e} \circ \pi_1(f)$ is surjective. Since $\hat{e}$ is just conjugation by a path e, $\pi_1(f)$ is surjective.
Hence, $\pi_1(f)$ is a bijection, and thus an isomorphism. □

20 Solutions to exercise sheet 4

21.03

21 Covering spaces

27.03

Definition 21.1 (Covering spaces)**.** *A continuous surjective map $p : E \to B$ is a covering map if for all $b \in B$ there exists a nbh. $U_b \ni b$ with the following properties:*

$$p^{-1}(U_b) = \bigsqcup_{\lambda \in \Lambda} V_\lambda \tag{39}$$

and

$$p_{|V_\lambda} : V_\lambda \to U_b \tag{40}$$

if a homeomorphism. We say that U_b is an evenly covered neighborhood.

Definition 21.2 (Fiber)**.** *For every $b \in B$ we call $p^{-1}(b)$ the fiber of p at b.*

Lemma 21.1. *Let $p : E \to B$ be a covering map. For all $b \in B$, the fiber $p^{-1}(b)$ is a discrete space.*

Proof. Let $U_b \ni b$ be a nbh. that is evenly covered. Then

$$p^{-1}(b) \subseteq p^{-1}(U_b) = \bigsqcup_{\lambda \in \Lambda} V_\lambda$$

Claim: Each $x \in p^{-1}(b)$ live in exactly one of the V_λ-s. Suppose there exists $x, x' \in V_\lambda$. Then

$$p(x) = p(x') = b$$

which is a contradiction since $p_{|V_\lambda}$ is injective (it is a homeomorphism). If $p^{-1}(b) \cap V_\lambda = \emptyset$ then $p_{|V_\lambda}$ cannot be surjective. Hence $p^{-1}(b) \cap V_\lambda = \{x_\lambda\}$. Hence

$$p^{-1}(b) = \bigsqcup_{\lambda \in \Lambda} \{x_\lambda\}$$

□

Example 21.1. *Given a homeomorphism $p : E \to B$. Then p is a covering map.*

Example 21.2. *Let F be a discrete space. Then the projection map*

$$\pi_X : X \times F \to X$$

is a covering map.

Proof. ${\pi_X}^{-1}(U_b) = U_b \times F \simeq \bigsqcup_{f \in F} U_b$. □

Definition 21.3 (Local homeomorphism)**.** *A continuous map $f : X \to Y$ is a local homeomorphism if for all $x \in X$ there exists a nbh. $U_x \ni x$ s.t.*

$$f_{|U_x} : U_x \to f(U_x)$$

is a homeomorphism.

Proposition 21.1. *If $p : E \to B$ is a covering map, then p is a local homeomorphism.*

Proof. Let $U_{p(e)} \ni p(e)$ be an evenly covered neighborhood. Then

$$p^{-1}(U_{p(e)}) \simeq \bigsqcup_{\lambda \in \Lambda} V_\lambda \ni e$$

We know that there exists a unique λ_0 such that $e \in V_{\lambda_0}$. Then

$$p_{|V_{\lambda_0}} : V_{\lambda_0} \longrightarrow U_{p(e)}$$

is a homeomorphism. □

Theorem 21.2. *The map*

$$p : \mathbb{R} \to \mathbb{S}^1 \tag{41}$$
$$t \mapsto (\cos(2\pi t), \sin(2\pi t)) \tag{42}$$

is a covering map.

Proof. Let $U = \mathbb{S}^1 \setminus \{(1,0)\}$ and $V = \mathbb{S}^1 \setminus \{(-1,0)\}$. It suffices to show that U, V are evenly covered.

$$p^{-1}(U) = \bigsqcup_{\lambda \in \mathbb{Z}} (\lambda, \lambda + 1)$$
$$p^{-1}(V) = \bigsqcup_{\lambda \in \mathbb{Z}} (\lambda - \frac{1}{2}, \lambda + \frac{1}{2})$$

NTS: $p_\lambda : (\lambda, \lambda + 1) \to \mathbb{S}^1 \setminus \{(1,0)\}$ is a homeomorphism. □

Theorem 21.3. *Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be covering maps. Then*

$$p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$$

is a covering map.

Proof. Let $(b_1, b_2) \in B_1 \times B_2$. Let $U_{b_1} \ni b_1, V_{b_2} \ni b_2$ be evenly covered neighborhoods. That is,

$$p_1{}^{-1}(U_{b_1}) = \bigsqcup_{\lambda \in \Lambda} V_\lambda$$
$$p_1{}^{-1}(V_{b_2}) = \bigsqcup_{\omega \in \Omega} W_\omega$$

Then

$$(p_1 \times p_2)^{-1}(U_{b_1} \times V_{b_2}) = \bigsqcup_{\lambda \in \Lambda, \omega \in \Omega} V_\lambda \times W_\omega$$

and

$$(p_1 \times p_2)_{|V_\lambda \times W_\omega} : V_\lambda \times W_\omega \to U_{b_1} \times V_{b_2}$$

is a homeomorphism since it is a product of homeomorphisms. □

Definition 21.4 (Lift). *Let $p : E \to B$ be any map and let $f : X \to B$. We say that $\tilde{f} : X \to E$ is a lift of f if $p \circ \tilde{f} = f$.*

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Lemma 21.4 (Lebesgue number lemma). *Let (X, d) be a compact metric space and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover. Then there exists some $\lambda > 0$ s.t. $\forall x \in X$ there exists some U_α in the cover such that $B(x, \lambda) \subseteq U_\alpha$.*

Proof. technical: todo. □

22 Homotopy lifting property covering spaces

28.03

Theorem 22.1. *Let $p : E \to B$ be a covering map and let $e_0 \in E$ such that $p(e_0) = b_0$. Given a path $\gamma : I \to B$ such that $\gamma(0) = b_0$. Then there exists a unique lift*

$$\tilde{\gamma} : I \to E$$

such that $\tilde{\gamma}(0) = e_0$.

Proof. technical. todo. ☐

Theorem 22.2 (Homotopy lifting property)**.** *Let $p : E \to B$ be a covering map and let $H : I \times I \to B$ such that $H(0,0) = b_0$, and let $e_0 \in E$ such that $p(e_0) = b_0$. Then there exists a unique lift $\tilde{H} : I \times I \to E$ such that $\tilde{H}(0,0) = e_0$. Moreover, if H is a path homotopy then so is $\tilde{H}$.*

Proof. technical. todo. ☐

Proposition 22.1. *Let $p : E \to B$ be a covering map. Let $e_0 \in E$ such that $p(e_0) = b_0$. Then there exists an assignment*

$$\pi_1(B, b_0) \longrightarrow p^{-1}(b_0) \tag{43}$$

$$\gamma \longmapsto \tilde{\gamma}(1) \tag{44}$$

Proof. $\tilde{\gamma}(1) \in p^{-1}(b_0)$ since

$$p(\tilde{\gamma}(1)) = \gamma(1) = \gamma(0) = b_0$$

To show that the assignment is well defined we pick two path homotopic maps $\gamma_1 \simeq_p \gamma_2$. We need to show that $\tilde{\gamma_1}(1) = \tilde{\gamma_2}(1)$. Let $H : I \times I \to B$ be a path homotopy of γ_1 and γ_2. Theorem 22.2 gives a unique path homotopy

$$\tilde{H} : I \times I \to E$$

Note that

$$\tilde{H}(0, s) = \tilde{\gamma_1}$$
$$\tilde{H}(1, s) = \tilde{\gamma_2}$$

Since $\tilde{H}$ is a path homotopy we have that $\tilde{H}(t, 1)$ is constant, so

$$\tilde{H}(0, 1) = \tilde{\gamma_1}(1), \tilde{H}(1, 1) = \tilde{\gamma_2}(1)$$

This means that $\tilde{\gamma_1}(1) = \tilde{\gamma_2}(1)$.

☐

Definition 22.1 (Lifting correspondence)**.** *The assignment*

$$\pi_1(B, b_0) \to p^{-1}(b_0)$$

is called the lifting correspondence.

Theorem 22.3. *Let $p : E \to B$ be a covering map. Then the lifting correspondence is*

1. *surjective if E is path connected.*
2. *bijective if E is simply connected.*

Proof. Let $e_0 \in p^{-1}(b_0)$. Since E is path connected I can pick a path

$$\gamma : I \to E$$

from e_0 to e_1. γ is a lift of $p \circ \gamma$. Hence $(\widetilde{p \circ \gamma})(1) = \gamma(1) = e_1$. Hence, the lifting correspondence is surjective if E is path connected.

Let $\gamma_1, \gamma_2 \in \pi_1(B, b_0)$ such that $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1) = e_1$. Consider the loop

$$\tilde{\gamma}_2^{-1} * \tilde{\gamma}_1$$

Since $\tilde{\gamma}_2^{-1} * \tilde{\gamma}_1$ is a loop and E is simply connected it is homotopic to the constant loop at e_1. This implies that

$$p(\tilde{\gamma}_2^{-1} * \tilde{\gamma}_1) = \gamma_2^{-1} * \gamma_1$$

is homotopic to the constant path at b_0. Hence $\gamma_1 \simeq_p \gamma_2$, so the lifting correspondence is also injective, and thus bijective. $\square$

Corollary 22.3.1. *There exists a bijection*

$$\pi_1(\mathbb{S}^1, *) \leftrightarrow \mathbb{Z}$$

Next time we show that this bijection is a group homomorphism, and hence a group isomorphism. Also: Brouwer fixed point theorem and the fundamental theorem of algebra.

23 Fundamental group of the circle and applications

03.04

Theorem 23.1. *For all $x \in \mathbb{S}^1$ we have that*

$$\pi_1(\mathbb{S}^1, x) \simeq \mathbb{Z}$$

is an isomorphism of groups.

Proof. Since $\mathbb{S}^1$ is path connected it suffices to show the claim for some $x_0 \in \mathbb{S}^1$. Let $x_0 = (1, 0)$. For the lifting correspondence, pick $0 \in \mathbb{R}$ and $p : \mathbb{R} \to \mathbb{S}^1$ sending

$$p(t) = (\cos 2\pi t, \sin 2\pi t)$$

By corollary 22.3.1 we have a bijection, so we need to show that it is a group homomorphism.

Let $\gamma_1, \gamma_2 : I \to \mathbb{S}^1$ be paths. Let $\tilde{\gamma}_1, \tilde{\gamma}_2$ be the unique lifts. Let

$$\tilde{\gamma}_1(1) = n, \tilde{\gamma}_2(1) = m$$

We need to show that

$$(\widetilde{\gamma_2 * \gamma_1})(1) = n + m$$

We do this by constructing the lift of $\gamma_2 * \gamma_1$. Define $h : I \to \mathbb{R}$ by

$$h(s) = \tilde{\gamma}_2(s) + \tilde{\gamma}_1(1)$$

Consider

$$(h * \tilde{\gamma}_1)(s) = \begin{cases} \tilde{\gamma}_1(s)(2s) & 0 \le s \le 1/2 \\ h(2s-1) & 1/2 \le s \le 1 \end{cases}$$

$h * \tilde{\gamma}_1$ satisfies the property we want since

$$(h * \tilde{\gamma}_1)(0) = \tilde{\gamma}_1(0) = 0$$

and

$$(h * \tilde{\gamma}_1)(1) = h(1) = \tilde{\gamma}_1(1) + \tilde{\gamma}_2(1) = n + m$$

Let's verify that $h * \tilde{\gamma}_1$ is actually the lift of $\gamma_2 * \gamma_1$ by showing that $p \circ (h * \tilde{\gamma}_1) = \gamma_2 * \gamma_1$.

$$\begin{aligned} (p \circ (h * \tilde{\gamma}_1))(s) &= \begin{cases} (p \circ \tilde{\gamma}_1)(2s) & 0 \le s \le 1/2 \\ (p \circ h)(2s-1) & 1/2 \le s \le 1 \end{cases} \\ &= \begin{cases} \gamma_1(2s) & 0 \le s \le 1/2 \\ p\left(\tilde{\gamma}_1(1) + \tilde{\gamma}_2(2s-1)\right) & 1/2 \le s \le 1 \end{cases} \\ &= \begin{cases} \gamma_1(2s) & 0 \le s \le 1/2 \\ \gamma_2(2s-1) & 1/2 \le s \le 1 \end{cases} \\ &= (\gamma_2 * \gamma_1)(s) \end{aligned}$$

$\square$

23.1 Applications of the fundamental group of the circle

Lemma 23.2. *Let X be a space such that $\pi_1(X, x) = *$. Suppose there exists a map $f : \mathbb{S}^1 \to X$. Then $\mathbb{S}^1$ cannot be a retract of X.*

Proof. Suppose a retract $r : X \to \mathbb{S}^1$ exists. Then we get:

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r, "f"] \arrow[rr, bend right, "\mathrm{id}"'] & X \arrow[r, "r"] & \mathbb{S}^1 \end{tikzcd}$$

$$\Big\Downarrow \pi_1$$

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & 0 \arrow[r] & \mathbb{Z} \end{tikzcd}$$

which is a contradiction since $\mathrm{id} : \mathbb{Z} \to \mathbb{Z}$ does not factor through 0. □

Theorem 23.3 (Brouwer fixed point)**.** *Let $D^2 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Then given a continuous map $f : D^2 \to D^2$ there exists a point in the disk such that $f(x) = x$.*

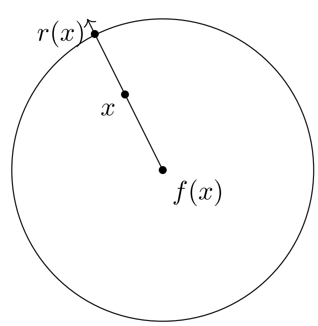
Proof. Suppose there exists a continuous map

$$f : D^2 \to D^2$$

without fixed points. Then there exists a retract

$$r : D^2 \to \mathbb{S}^1$$

defined by constructing the ray at x through $f(x)$ and letting $r(x)$ be the intersection at $\mathbb{S}^1$. This is a contradiction by lemma 23.2 since D^2 is homotopy equivalent to a point.



□

Lemma 23.4. *Let $h : \mathbb{S}^1 \to X$. Then h is nullhomotopic if*

$$\pi_1(h) : \pi_1(\mathbb{S}^1, a) \to \pi_1(X, h(a))$$

is the trivial map.

Proof. My notes are not easy to read. See proof in Hatcher. □

23.2 Study tips

Pass (bare minimum):

1. topological spaces, cont. maps
   interior, closure, metric spaces
2. constructions with topological spaces
   subspaces
   products
   quotients
3. properties of topological spaces
   Hausdorff
   compact
   connected
4. which properties are stable under which constructions?
   find counterexamples, make a table
5. homotopy theory
   homotopy
   path homotopy
   homotopy equivalence
   fundamental group
   $\pi_1(\mathbb{S}^1) = \mathbb{Z}$
   functoriality

Bonus:

1. which properties are stable under homotopy equivalence?

24 Solutions to Exercise sheet 5

04.04