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Dedication

Abstract

Útdráttur á ensku sem er að hámarki 250 orð.

Útdráttur

Hér kemur útdráttur á íslensku sem er að hámarki 250 orð. Reynið að koma útdráttum á eina blaðsíðu en ef tvær blaðsíður eru nauðsynlegar á seinni blaðsíða útdráttar að hefjast á oddatölusíðu (hægri síðu).

Preface

Formála má sleppa og skal þá fjarlægja þessa blaðsíðu. Formáli skal hefjast á oddatölu blaðsíðu og nota skal Section Break (Odd Page).

Ekki birtist blaðsíðutal á þessum fyrstu síðum ritgerðarinnar en blaðsíðurnar teljast með og hafa áhrif á blaðsíðutal sem birtist með rómverskum tölum fyrst á efnisyfirliti.

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Abbreviations

Í þessum kafla mega koma fram listar yfir skammstafanir og/eða breytuheiti. Gefið kaflanum nafn við hæfi, t.d. Skammstafanir eða Breytuheiti. Þessum kafla má sleppa ef hans er ekki þörf.
The section could be titled: Glossary, Variable Names, etc.

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1. Introduction

$\mathcal{O}(U)$ for open $U \subset \mathbb{C}$ is the family of functions holomorphic on U. $C(X)$ for open topological space X is the family of continuous functions from X to $\mathbb{C}$. We will use $\mathbb{D}$ to refer to the open unit disk $\{z \in \mathbb{C};\ |z| < 1\}$. The closed unit disk will get no special notation, but will be referred to by $\overline{\mathbb{D}}$. We will use $\mathbb{T}$ to refer to the open unit circle $\{z \in \mathbb{C};\ |z| = 1\}$.

2. Preliminaries

2.1. Measure theory

During introductory courses in measure theory the focus is often solely on positive measures [Tao, 2014], which are then simply referred to as 'measures'. This restriction is immediately felt when studying functional analysis (see for example 2.3.7). We, therefore, need the following definitions:

Definition 2.1.1. Let $\mathcal{F}$ be a σ-algebra and $\mu : \mathcal{F} \to Y$, where Y is a subset of $\mathbb{C}$ or $\overline{\mathbb{R}}$. We say that μ is *countably additive* if

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu(E_n)$$

for all disjoint collections $(E_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$. We also say that

(i) μ is a *positive measure* if it is countably additive and $Y = [0, \infty]$.

(ii) μ is a *real valued measure* if it is countably additive and $Y = [-\infty, \infty[$ or $Y =]-\infty, \infty]$.

(iii) μ is a *complex measure* if it is countably additive and $Y = \mathbb{C}$.

Allowing $Y = \overline{\mathbb{R}}$ in (ii) would lead to trouble, for example if $\mu(\{x\}) = \infty$ and $\mu(\{y\}) = -\infty$ what is $\mu(\{x, y\})$?

For those not familiar with complex measures, some care must be taken. A prime example is the notion of null sets. They still play a great role, but their definition is different. We can motivate this difference by letting $X = \{x, y\}$, $\mathcal{F}$ be the powerset of X, and define μ by $\mu(\{x\}) = -1$ and $\mu(\{y\}) = 1$. For μ to be a measure we need $\mu(X) = 0$ to hold. Blindly applying our definitions from positive measures would label X as a μ-null set. This would of course technically be a valid definition, but leads to problems further down the road, for example

$$\int_X f \, d\mu = 0$$

does not hold generally (once we define integration by complex measure, of course). To avoid this specific pitfall we define a measurable set E to be *μ-null* if $\mu(F) = 0$ holds for all measurable F such that $F \subset E$. The above example also shows us another pitfall, $E \subset F \implies \mu(E) \leqslant \mu(F)$ doesn't hold generally.

If we have a complex measure $\mu : \mathcal{F} \to \mathbb{C}$ we may want a positive measure λ that dominates it, in the sense that $|\mu(E)| \leqslant \lambda(E)$. We would also want λ to be 'small' in some sense. We will refer to a collection $(E_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ as a *partition of* E if they are pairwise disjoint and their union is E. The measure λ mentioned earlier will then fulfill

$$\lambda(E) = \sum_{n\in\mathbb{N}} \lambda(E) \geqslant |\mu(E)|.$$

It so happens that defining λ by

$$\lambda(E) = \sup \left\{ \sum_{n\in\mathbb{N}} |\mu(E_n)|; \text{ where } (E_n)_{n\in\mathbb{N}} \text{ is any partition of } E \right\}$$

yields a measure. This measure is referred to as *the total variation of* μ and denoted by $|\mu|$ (for example, $|\mu|(E)$). A proof that $|\mu|$ is a measure and more detail on complex measures can be found in chapter 6 of [Rudin, 1987].

2.2. Complex analysis in one variable

2.2.1. The disk algebra

Definition 2.2.1. A function is said to be in the *disk algebra* if it is holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. We will refer to this family of function $\mathcal{A}$.

It can be shown using Morera's theorem that a sequence of holomorphic function that converges uniformly has a holomorphic limit [Axelsson, 2014]. The same holds for continuous functions. So, naturally, we conclude that a sequence of functions in $\mathcal{A}$ that converges uniformly has a limit in $\mathcal{A}$.

2.2.2. Carathéodory

Definition 2.2.2. A continuous function $\gamma : [0, 1] \to \mathbb{C}$ is said to be a *Jordan curve* if $\gamma(0) = \gamma(1)$ and

$$\gamma(s) = \gamma(t) \implies s = t \qquad \text{for all } s, t \in]0, 1[.$$

The definition above can be restated as: A Jordan curve is a closed simple curve. The term 'simple' here means that the curve is not self-intersecting. The name stems from a famous result by Camille Jordan stating that $\mathbb{C}\backslash\gamma([0,1])$ has two connected component, one of which is simply connected. The simply connected component will be called *the domain bounded by* γ. The proof of this result is rather technical and outside the scope of this thesis [Greene and Krantz, 2006, Whyburn, 1958]. The result is however necessary to make statement such as 'let U be the domain bounded by γ'. An example of this is the following theorem:

Theorem 2.2.3 (Carathéodory)**.** *Let Ω_1 and Ω_2 be domains in $\mathbb{C}$ each bounded by a Jordan curve and $\Phi : \Omega_1 \to \Omega_2$ be a conformal mapping. There exists a continuous injection $\hat{\Phi} : \overline{\Omega_1} \to \overline{\Omega_2}$ that extends Φ.*

A proof of this can be found in section 13.2 of [Greene and Krantz, 2006].

2.2.3. The Riemann Mapping Theorem

We will start of with a definition.

Definition 2.2.4. A map $f : U \to V$, with open $U, V \subset \mathbb{C}$ is said to be *conformal* if it is holomorphic, bijective, and its inverse is holomorphic.

The fact that the inverse is homomorphic is actually redundant. It can be shown that a holomorphic bijection has a holomorphic inverse. [Greene and Krantz, 2006]

Theorem 2.2.5 (Riemann mapping theorem [Greene and Krantz, 2006])**.** *If $U \subset \mathbb{C}$, $U \neq \mathbb{C}$ is homeomorphic to $\mathbb{D}$ then there exists a conformal mapping from $\mathbb{D}$ to U.*

Corollary 2.2.6. *If U and V are both homeomorphic to $\mathbb{D}$ then there exists a conformal mapping from U to V.*

Proof. The Riemann mapping theorem gives us Φ_1, a conformal mapping from $\mathbb{D}$ to U, and Φ_2, a conformal mapping from $\mathbb{D}$ to V. The desired conformal mapping from U to V is then $\Phi_2 \circ \Phi_1^{-1}$. □

In this thesis the disk algebra $\mathcal{A}$ is of special interest so a version of the Riemann mapping theorem that considers continuity at the boundary is desirable. We can combine the Riemann mapping theorem and Carathéodory's theorem to achieve the desired theorem. The only thing to prove is that a homeomorphism f maps the boundary of a bounded set U to the boundary of $f(U)$ and that a Jordan curve under a homeomorphism is still a Jordan curve.

Corollary 2.2.7. *If K is homeomorphic to $\overline{\mathbb{D}}$ then there exists a continuous, injective $\Phi : \overline{\mathbb{D}} \to K$ such that its restriction to $\mathbb{D}$ is a conformal mapping.*

Proof. First, let U be a bounded, open set in $\mathbb{C}$, $f : \overline{U} \to \mathbb{C}$ be an injective continuous map, and let's show that $\partial f(U) = f(\partial U)$. Let $p \in \partial U$ and $B_r(z) = \{z \in \mathbb{C};\ |r - z| < r\}$. Let's also assume that $f(p)$ is in the interior of $f(U)$ and show that it leads to a contradiction. There exists an $r > 0$ such that $B_r(p) \subset f(U)$, because the interior of a set is always open. This gives us as an open neighbourhood $f^{-1}(B_r(p)) \subset U$ of p, but $p \in \partial U$ implies that no such neighbourhood exists. So $f(p) \in \partial f(U)$, implying $\partial f(U) \supset f(\partial U)$. We can use the same argument to show that $\partial f(U) \subset f(\partial U)$, since f is bijective if we consider it as a map from $\overline{U}$ into $f(\overline{U})$. So $f(\partial U) = \partial f(U)$.

Second, let U and V be open sets in $\mathbb{C}$, $\gamma : [0, 1] \to \partial\mathbb{D}$ a Jordan curve, $f : U \to V$ be homeomorphism, Let $\lambda = f \circ \gamma$ and $s, t \in]0, 1[$ such that $\lambda(s) = \lambda(t)$. It suffices to show that $s = t$, since $\lambda(0) = \lambda(1)$ obviously holds. We have that $f(z) = f(w)$ implies $z = w$, since f is bijective, and therefore injective. So $\lambda(s) = (f \circ \gamma)(s) = (f \circ \gamma)(t) = \lambda(t)$ implies that $\gamma(s) = \gamma(t)$. But γ is a Jordan curve, so $\gamma(s) = \gamma(t)$ implies $s = t$. So λ is also a Jordan curve.

Finally, we can prove the corollary. Let f be a homeomorphism from $\overline{\mathbb{D}}$ to K. The Riemann mapping theorem also gives us a conformal map $\Phi : \mathbb{D} \to f(\mathbb{D})$. We know that $\overline{\mathbb{D}}$ is compact, so $K = f(\overline{\mathbb{D}})$ is also compact, since the image of a compact set under a continuous mapping is also compact. Moreover, K is bounded. So $f(\partial\mathbb{D}) = \partial f(\mathbb{D}) = \partial K$ according to our first step and ∂K is a Jordan curve according to the second step, since $\partial\mathbb{D}$ is a Jordan curve. So Φ is a conformal map between two domains, each bounded by a Jordan curve. This allows us to use Carathéodory's theorem to extend Φ continuously and injectively to $\overline{\mathbb{D}}$, concluding the proof. $\square$

The Riemann mapping theorem is a strong tool when analyzing holomorphic functions on simply connected domains. We can often solve things for the unit disk (or unit square as in 3.1.1) and then map that solution to a general simply connected domain.

We will briefly touch on multivariate complex analysis so we will need a definition of a holomorphic function in $\mathbb{C}^n$. If $U \subset \mathbb{C}^n$ is open and $f : U \to \mathbb{C}$ such that f is holomorphic in each variable separately we say that f is *holomorphic on U*.

2.3. Functional analysis

Definition 2.3.1. Let X be a topological space. We say that X is *locally compact* if for each $x \in X$ there exists a open set U_x such that $x \in U_x$ and $\overline{U_x}$ is compact.

Definition 2.3.2. Let X be a locally compact space. We say a complex function f *vanishes at infinity* if for all $\varepsilon > 0$ there exists a compact set K such that $|f(x)| < \varepsilon$ for all $x \in X \backslash K$. The family of all such functions is referred as $C_0(X)$.

Let X be compact. Then if $f \in C(X)$ and $\varepsilon > 0$ we set $K = X$ and see that $|f(x)| < \varepsilon$ vacuously holds for all $x \in X \backslash K = \varnothing$. So $C(X)$ and $C_0(X)$ are identical in this case.

Definition 2.3.3. Let α be a linear map from a normed vector space X into a normed vector space Y. We define a its *norm* by

$$\|\alpha\| = \sup\{\|\alpha(x)\|;\ \|x\| < 1\}$$

and say α is *bounded* if its norm is finite.

Theorem 2.3.4. *If α is a linear map from a normed vector space X into a normed vector space Y then the following properties are equivalent:*

(i) *α is bounded.*

(ii) *α is continuous.*

(iii) *α is continuous at $x \in X$.*

This theorem allows us to use the terms 'bounded' and 'continuous' interchangeably when talking about linear mappings between normed vector spaces. A proof of this theorem can be found on page 96 in [Rudin, 1987].

Let's recall what the L^p spaces are. Let X be a space with σ-algebra $\mathcal{F}$, and μ be a measure on that σ-algebra. We say a function is in $\mathcal{L}^1(X, \mu)$ if

$$\int |f| \, d\mu < \infty$$

and we say it is $\mathcal{L}^p(X, \mu)$ if $|f|^p$ is in $\mathcal{L}^1$, for $0 < p < \infty$. We then define an equivalence relation $\sim$ such that $f \sim g$ if and only if $\mu(\{x \in X;\ f(x) \neq g(x)\}) = 0$.

We define $L^p(X,\mu)$ to be the quotient space $\mathcal{L}(X,\mu)/\sim$. The space $L^p(X,\mu)$ has one major benefit over $\mathcal{L}^p(X,\mu)$,

$$\|f\|_p = \left(\int |f|^p \, d\mu\right)^{1/p}$$

is a norm on $L^p(X,\mu)$. We sometimes write $L^p(\mu)$ if the space in question is obvious from context and $L^p(X)$ if the measure is obvious from context.

2.3.1. Hahn-Banach

There are many related theorems going by the name 'Hahn-Banach Theorem'. These are sometimes split in two groups, 'separation theorems' and 'extension theorems'. When proving 3.3.1 we need the following Hahn-Banach separation theorem:

Theorem 2.3.5. *Let X be a normed vector space, $A \neq \emptyset$ be a closed convex subset of X, and $p \in X \backslash A$. Then there exists a continuous linear functional f such that* $\sup\{f(x);\ x \in A\} < 1$ *and* $f(p) = 1$.

Corollary 8.15 in [Pryce, 1973] is very similar. It states that, for same X, A, and p, there exists a continuous linear functional f such that $f(p) < \inf\{f(x);\ x \in A\}$. With such a functional given, ... TODOTODOTODO

2.3.2. The Riesz representation theorem

Let X be a space with σ-algebra and measure μ, choose p and q such that $1 < p, q < \infty$ and $1/p + 1/q = 1$, and $g \in L^q(\mu)$. We can then define a linear transform

$$\alpha(f) = \int fg \, d\mu.$$

The Hölder inequality tells us it is bounded.

Theorem 2.3.6 (The Riesz representation theorem for bounded linear functionals on $L(X,\mu)$ with σ-finite X [Rudin, 1987])**.** *Let $1 < p < \infty$, q be such that $1/p+1/q = 1$, μ be a σ-finite positive measure on X, and α be a bounded linear functional on $L^p(\mu)$. There then exists a unique $g \in L^q(\mu)$ such that*

$$\alpha(f) = \int fg \, d\mu$$

for all $f \in L^p(\mu)$ *and*

$$\|\alpha\| = \|g\|_q.$$

Note that the theorem above also hold for $p = 1$ and $q = \infty$, but we will not need that result.

Theorem 2.3.7 (The Riesz representation theorem for bounded linear functionals on $C_0(X)$ with locally compact X [Rudin, 1987])**.** *Let X be a locally compact Hausdorff space and α be a bounded linear functional on $C_0(X)$. There then exists a measure μ such that*

(i)

$$\alpha(f) = \int_X f \, d\mu$$

for all $f \in C_0(X)$ *and*

(ii)

$$\|\alpha\| = |\mu|(X).$$

3. Rudin-Carleson theorem

3.1. Rudin-Carleson theorem

Theorem 3.1.1 (Rudin-Carleson theroem). *Let E be a closed subset of $\mathbb{T}$ of Lebesgue-measure 0, let f be a continuous function on E, and let T be a subset of $\mathbb{C}$ homeomorphic to $\overline{\mathbb{D}}$ such that $f(\overline{\mathbb{D}}) \subset T$. Then there exists an $F \in \mathcal{A}$, such that $F = f$ on E and $F(\overline{\mathbb{D}}) \subset T$.*

We will break the proof into several lemmas.

Lemma 3.1.2. Let H be a closed set of Lebesgue-measure 0. Then there exists an integrable function $\mu > 1$ such that μ is continuous on $\mathbb{T}\backslash H$, $\mu = +\infty$ on H, if $w \in H$ then $\mu(z) \xrightarrow{z \to w} +\infty$, and μ has a bounded derivative on any closed subarc of $\mathbb{T}\backslash H$.

Recall from set theory that if (X, d) is a metric space, A is a subset of X and x is a point of X then the distance between x and A is

$$d(x, A) = \inf_{a \in A} d(x, a).$$

Furthermore, if A is closed then the infimum is obtained at some of A, namely, there exists a point $y \in A$ such $d(x, y) = d(x, A)$. Let $X = [0, 1]$ and H be some closed set of Lebesgue-measure zero. We will, for reasons that will be made clear later, assume that $0, 1 \subset H$. We can now define function $V_H : [0, 1] \to [0, 1]$ and $H_H : [0, 1] \to [0, 1]$ such that

$$d(x, H \cap [0, x]) = d(x, V_H(x)) \quad \text{and} \quad d(x, H \cap [x, 1]) = d(x, H_H(x)).$$

Intuitively, $H_H(x)$ is the point in H that's to the right of x and is closest to x and $V_H(x)$ is the point in H that's to the left of x and is closest to x. So all points x in $[0, 1]\backslash H$ are on the open interval $]V_H(x), H_H(x)[$. We will define function $f : [0, 1] \to [0, \infty]$ by

$$f_H(x) = 2 \log \left(\frac{H_H(x) - V_H(x)}{2} \right) - \log((x - V_H(x))(H_H(x) - x))$$

and show that this is the desired function.

Let's first define a family of functions indexed with $a, b \in \mathbb{R}$ by

$$f_{a,b} : [a, b] \to [0, \infty], x \mapsto 2\log\left(\frac{b-a}{2}\right) - \log((x-a)(b-x))$$

and show they fulfill the following properties:

1. $f_{a,b}(a + (b-a)2^{-n}) \leqslant n$,
2. $f_{a,b}(x) = f_{a,b}(b + a - x)$.

The first point gives us a handy estimate and the second tells us $f_{a,b}$ is symmetric around $(a+b)/2$. Note that for the first point it suffices to consider when $a = 0$, since a denotes a simple translation. We then have that

$$\begin{aligned} f_{0,b}(b2^{-n}) &= 2\log b - 2\log 2 - \log(b2^{-n}(b - b2^{-n})) \\ &= 2\log b - 2\log 2 - \log b + n\log 2 - \log b - \log(1 - 2^{-n}) \\ &= n - (2 + \log(1 - 2^{-n})) \\ &\leqslant n. \end{aligned}$$

To show the second property we consider the case where $a = -b$, that is we translate so that the midpoint between them is 0. Then

$$\begin{aligned} f_{-b,b}(-x) &= 2 - \log((-x + b)(b + x)) \\ &= 2 - \log((b - x)(x - (-b))) \\ &= f_{-b,b}(x). \end{aligned}$$

These two points together tell us that we can bound $f_{a,b}$ above by

$$g_{a,b}(x) = \begin{cases} n, & \text{if } 2^{-n} \leqslant \frac{x-a}{b} < 2^{-n+1} \\ g(a + b - x), & \text{if } \frac{x-a}{b} > \frac{1}{2} \end{cases},$$

namely a function taking integer values. It is more convenient to look at $g_{a,b}$ instead of $f_{a,b}$ because the former can trivially be shown to be the limit of a monotone sequence of simple functions. Let's look at the case where $a = 0$ and $b = 1$. Let $g = g_{0,1}$ and

$$g_n(x) = \begin{cases} g(x), & \text{if } g(x) \leqslant n \\ 0, & \text{else.} \end{cases}$$

We can now integrate g by

$$\begin{aligned}\int g \; d\sigma &= \int \lim_{n\to\infty} g_n \; d\sigma \\ &= \lim_{n\to\infty} \int g_n \; d\sigma \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} k2^{-k} \\ &= \sum_{k\in\mathbb{N}} k2^{-k} \\ &< \infty\end{aligned}$$

where the second equality is by monotone convergence and the last is by some test. This rationale also holds for general a and b, since a does not effect the value of the integral and b scales it.

Let's now go back to f_H. We can define a function

$$g_H(x) = g_{V_H(x), H_H(x)}(x)$$

and a sequence of functions

$$g_n(x) = \begin{cases} g_H(x), & \text{if } g(x) \leqslant n \\ 0, & \text{else.} \end{cases}$$

We have that $(g_n)_{n\in\mathbb{N}}$ is monotone with limit g, similar to before, and $f \leqslant g$. Note that, independent of H, we have

$$\sigma(f^{-1}(k)) = 2^{-k}$$

so

$$\begin{aligned}\int f_H \; d\sigma &\leqslant \int g_H \; d\sigma \\ &= \int \lim_{n\to\infty} g_n \; d\sigma \\ &= \lim_{n\to\infty} g_n \; d\sigma \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} k2^{-k} \\ &= \sum_{k\in\mathbb{N}} k2^{-k} \\ &< \infty\end{aligned}$$

by the same reasoning as before.

Note that we left some fundementals unmentioned. Specifically the fact that f_H and g_H are measureable. To show that g is measureable it suffices to show that $(g_n)_{n\in\mathbb{N}}$ are measureable. We, therefore, can show that the preimage under g_n of each point is measureble, since g_n is simple, for each n. But the preimage is a (possibly uncountable) union of half-open intervals so it is the union of a closed set and an open set. So g is measureable.

Proof of 3.1.2. □

Lemma 3.1.3. If f is a simple continuous function on E such that $\operatorname{Re} f \geqslant 0$, then there exists an $F \in \mathcal{A}$ such that $F = f$ on E and $\operatorname{Re} F \geqslant 0$ on $\overline{\mathbb{D}}$.

Proof. It suffices to show that this holds if f takes only two values on E, since simple functions are finite linear combinations of characteristic functions. Let's assume these values are 0 and $\alpha \neq 0$, with $\operatorname{Re} \alpha \geqslant 0$, $E_0 = f^{-1}(0)$, and $E_1 = f^{-1}(\alpha)$. Our assumption that f only takes two values then implies that $E_0 \cup E_1 = E$.

Let $u_H(z)$ be the Poisson integral of the function from the above lemma. This function is continuous on $\mathbb{T}\backslash H$, $u_H|_H = \infty$, and $\lim_{z\to w} u_H(z) = \infty$ for $w \in H$. ... We now define

$$g_H(z) = \begin{cases} u_H(z) + iv_H(z), & z \in \mathbb{D}\backslash H \\ \infty, & \text{otherwise} \end{cases}.$$

By our construction of u_H we see the $\operatorname{Re} g > 1$, so it has a well defined square root. Let's call it h_H and define

$$q = \frac{h_{E_1}}{h_{E_0} + h_{E_1}}.$$

Note that $|\arg h_H(z)| < \pi/4$ since if a $w \in \mathbb{C}$ had an argument outside of this range then its square would have and argument outside of the range $[-\pi/2, \pi/2]$ meaning $\operatorname{Re} w^2 < 0$. Also, $q(z) = 0$ if and only if $h_{E_0} = \infty$, so q is zero only on E_0, and $q(z) = 1$ if and only if $h_{E_1} = \infty$, so q is one only on E_1. We now want to show that $0 \leqslant \operatorname{Re} q \leqslant 1$. We will let $z, w \in \mathbb{C}$, with $|\arg z|, |\arg w| < \pi/4$ and $\operatorname{Re} z, \operatorname{Re} w > 1$, and show that $0 < \operatorname{Re} z/(w+z) < 1$.

Note first that

$$\frac{z}{w+z} = \frac{1}{w/z+1}$$

so

$$\arg \frac{z}{w+z} = -\arg\left(\frac{w}{z}+1\right)$$

and

$$|\arg w/z| = |\arg w - \arg z| \leqslant |\arg w| + |\arg z| < \pi/4 + \pi/4 = \pi/2.$$

So w/z is in the right halfplane and, therefore $w/z+1$ is as well. So $0 < \text{Re } z/(w+z)$. Note that $0 < \text{Re } z/(w+z) \implies 0 > \text{Re } -w/(w+z)$ due to z and w being constrained in the same manner. So

$$\begin{aligned} 0 &> \text{Re } \frac{-w}{w+z} \\ &= \text{Re } \frac{z-(z+w)}{w+z} \\ &= \text{Re } \left(\frac{z}{w+z} - 1\right) \\ &= \text{Re } \frac{z}{w+z} - 1 \\ \implies 1 &> \text{Re } \frac{z}{z+w}. \end{aligned}$$

So we have shown that

$$0 < \text{Re } \frac{z}{z+w} < 1.$$

We have now constructed a function q that maps $\overline{\mathbb{D}}$ to the ribbon $\{z;\ 0 \leqslant \text{Re } z \leqslant 1\}$. We then let Φ be the conformal mapping from the ribbon $\{z;\ 0 \leqslant \text{Re } z \leqslant 1\}$ to $\{z;\ 0 \leqslant \text{Re } z \leqslant \text{Re } \alpha\}$. We will also choose Φ such that $\Phi(0) = 0$ and $\Phi(1) = \alpha$. We can then let $F = \Phi \circ q$ and conclude the proof.

□

Lemma 3.1.4. If f is a simple continuous function on E that maps E into $T \subset \mathbb{C}$ homeomorphic to $\overline{\mathbb{D}}$, then there exists a $F \in \mathcal{A}$, such that $F = f$ on E and F maps $\overline{\mathbb{D}}$ into T.

Proof. Let $z_0 \in T \backslash f(E)$ and Φ be a conformal mapping from the right halfplane to the interior of T such that $\Phi(\infty) = z_0$. There exists a $g \in \mathcal{A}$ that extends $\Phi^{-1} \circ f$, according to Lemma 3.1.3. The desired function in then obtained with $F = \Phi \circ g$. □

Lemma 3.1.5. If f is a continue function on E which maps E into

$$S = \{z;\ |\max(\text{Re } z, \text{Im } z)| \leqslant 1\},$$

then there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of simple continuous function on E such that

$$f(x) = \sum_{n \in \mathbb{N}} f_n(z) \text{ and } f_n(E) \subset 2^{-n} S.$$

Proof. We will set $f_0 = 0$ and construct f_n iteratively. Assuming $f_0, f_1, ..., f_{n-1}$ have been constructed such that

$$\lambda_{n-1}(E) \subset 2^{1-n}S$$

with $\lambda_{n-1} = f - \sum_{k=0}^{n-1} f_k$. According to **??** we have can write E as the union of disjoint closed sets $E_1, E_2, ..., E_p$ such that the oscillation of λ_{n-1} is less than 2^{-n} on each E_k. So we can define $Q_k \subset 2^{1-n}S$ for $k = 1, 2, ..., p$ such that $Q_k = 2^{-n}S + a_k$ for some $a_k \in S$. We can choose $c_k \in Q_k \cap 2^{-n}S$ since Q_k has side length 2^{-n} and is a subset of 2^{-n+1} and both are closed. We can now define

$$f_n(z) = c_k, \qquad z \in E_k,\ k = 1, 2, ..., p.$$

If we then look at $\lambda_n = f - \sum_{k=0}^{n} f_k = \lambda_{n-1} - f_n$ we see that $\lambda_n(E) \subset 2^{-n}S$ due to the way we decomposed E into $E_1, E_2, ..., E_p$ using the oscillations of λ_{n-1}. This means we can continue the process. □

Proof of 3.1.1. Let's first show the result for $T = S$.

Let f_n be the functions from Lemma 3.1.5. According to Lemma 3.1.4 we have functions $g_n \in \mathcal{A}$ which extend f_n and map $\overline{\mathbb{D}}$ into $2^{-n}S$. We then define

$$F = \sum_{n \in \mathbb{N}} g_n$$

on $\overline{\mathbb{D}}$. To show that F is in $\mathcal{A}$ it suffices to show that series converges uniformly. Let $M_n = 2^{-n+1}$ and note that $|g_n(z)| \leqslant \sqrt{2} \cdot 2^{-n} < 2^{-n+1} = M_n$ and

$$\sum_{n \in \mathbb{N}} M_n = \sqrt{2} \sum_{n \in \mathbb{N}} 2^{-n} = \sqrt{2} < \infty$$

so the Weierstrass M-test tells us that F converges uniformly. We also have that

$$\text{Re } F = \sum_{n \in \mathbb{N}} \text{Re } g_n \leqslant \sum_{n \in \mathbb{N}} 2^{-n} = 1.$$

It can be shown in the same manner that Im $F \leqslant 1$, so F maps into S. Lastly, for $z \in E$ we have that

$$F(z) = \sum_{n \in \mathbb{N}} g_n(z) = \sum_{n \in \mathbb{N}} f_n(z) = f(z)$$

so F is an extension of f.

To prove the result for a general T we first let $\Phi : T \to S$ be the map provided to us by 2.2.7. We will also let $g = f \circ \Phi$. Note that it maps E into S, so we can use what we showed above to find $G \in \mathcal{A}$ that extends g and maps into S. We finally set $F = G \circ \Phi^{-1}$. On E we have that

$$F = G \circ \Phi^{-1} = g \circ \Phi^{-1} = f \circ \Phi \circ \Phi^{-1} = f,$$

so F extends f. It is also a composition of functions in $\mathcal{A}$ so it is also in $\mathcal{A}$. □

3.2. F. and M. Riesz theorem

The main result of this section is that the annihilating measures of

$$\mathcal{A}|_{\mathbb{T}} = \{f|_{\mathbb{T}};\ f \in \mathcal{A}\}$$

are absolutely continuous with respect to the Lebesgue measure. We will show this to be a corollary of the F. and M. Riesz theorem, which we will prove in the manner of [Rudin, 1987]. To attain the main result of this section we need some lemmas and definitions. To prove one of the lemmas we will also use the following two famous theorems:

Definition 3.2.1. Let $\mathcal{F}$ be a family of complex functions on a metric space (X, d).

We say that the family is *pointwise bounded* if for all $x \in X$ there exists a constant $M < \infty$ such that

$$|f(x)| < M, \text{ for all } f \in \mathcal{F}.$$

Note that M may depend on x.

We say that the family is *equicontinuous* if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \text{ for all } f \in \mathcal{F} \text{ and } x, y \in X \text{ such that } d(x, y) < \delta.$$

Note here that δ is globally defined and only dependent on ε.

Theorem 3.2.2 (Bolzano-Weierstrass)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of numbers in $\mathbb{R}^n$, such that $|a_n| < M < \infty$, for all $k \in \mathbb{N}$. There than exists and infinite $S \subset \mathbb{N}$ such that $(a_n)_{n\in S}$ is convergent.*

Proof. Let's first assume that the sequence is in $\mathbb{R}$, that no element in it is repeated infinitely often (there is nothing to prove in that case), and that $a_n \in]0, 1[$ for all $n \in \mathbb{N}$. The last assumption can be done with out loss of generality by studying the sequence $((a_n + M)/(2M))_{n\in\mathbb{N}}$ instead. We will obtain the subsequence by a diagonal process. Let $S_0 = \mathbb{N}$, $S_0^- = \{n \in S_0;\ a_n < 1/2\}$, and $S_0^+ = \{n \in S_0;\ a_n > 1/2\}$. We then set $S_1 = S_0^-$ if it is infinite, but $S_1 = S_0^+$ otherwise. This gives us a subsequence $(a_n)_{n\in S_1}$ such that

$$\sup_{n\in S_1} a_n - \inf_{n\in S_1} a_n < 1/2.$$

We can then repeat this to get a sequence of sets $(S_n)_{n\in\mathbb{N}}$ such that $S_0 \supset S_1 \supset S_2 \supset ...$ and

$$\sup_{n\in S_k} a_n - \inf_{n\in S_k} a_n < 2^{1-k},$$

for all $k \in \mathbb{N}$. Specifically, if we have S_k we set

$$U = m2^{-k},\ L = (m+1)2^{-k}$$

$S_k^- = \{n \in S_k;\ a_n < (U+L)/2\}$, and $S_k^+ = \{n \in S_k;\ a_n > (U+L)/2\}$. We now set $S_{k+1} = S_k^-$ if it has infinitely many elements, otherwise we set $S_{k+1} = S_k^+$. We conclude our construction by setting

$$S = \bigcup_{n \in \mathbb{N}} r_n,$$

where r_n is the n-th smallest element of S_n. This gives us the convergent sequence $(a_n)_{n \in S}$ with limit

$$\sum_{k=0}^{\infty} \delta_k 2^{-k}$$

where

$$\delta_k = \begin{cases} 0, & \text{if we chose } S_k^- \\ 1, & \text{if we chose } S_k^+ \end{cases}.$$

To show the result for $\mathbb{R}^n$ we can start by finding a subsequence such that the first coordinate is convergent. We can then choose a subsequence thereof such that the second coordinate is also convergent. Now the first two coordinates are convergent. If we do this $n-2$ more times we get a desired subsequence. □

Remark 3.2.3. The theorem above clearly holds for sequences in $\mathbb{C}$ as well.

Theorem 3.2.4 (Ascoli-Arzela)**.** *Let $\mathcal{F}$ be a pointwise bounded equicontinuous collection of complex functions on a metric space (X, d), and X contains a countable dense subset. Then every sequence in $\mathcal{F}$ contains a subsequence that converges uniformly on every compact subsets of X.*

Proof. Let E be a countable dense subset of X, $(f_n)_{n \in \mathbb{N}}$ be a series in $\mathcal{F}$, and $x_1, x_2, ...$ be an enumeration of E. We will prove the theorem in two steps. The first step is finding a subsequence of $(f_n)_{n \in \mathbb{N}}$ that's pointwise convergent on E using the pointwise boundedness along with Bolzano-Weierstrass. The second step is using the equicontinuity to show that this gives us uniform continuity on compact subsets.

Let's first set $S_0 = \mathbb{N}$. Pointwise boundedness gives us that the sequence $(f_n(x_1))_{n \in S_0}$ has a convergent subsequence. Let S_1 index that subsequence. We can use this process to generate sets $S_0 \supset S_1 \supset ...$ such that $(f_n(x_k))_{n \in S_k}$ is convergent. We then set

$$S = \bigcup_{k \in \mathbb{N}} r_k$$

where r_n is the k-th smallest element of S_k. We now have concluded the first step of the proof.

We will now assume the $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent on E, let K be a compact subset of X, and $\varepsilon > 0$. Equicontinuity gives us a $\delta > 0$ such that $d(x, y) < \delta$ implies that $|f_n(x) - f_n(y)| < \varepsilon/3$, for all n. Let's now cover K with m balls of radius $\delta/2$ and call the k-th ball B_k. We can now set p_k as a point in $B_k \cap E$. This point exists because E is dense in X. Pointwise convergence on E let's us chose an N such that $|f_{n_1}(p_k) - f_{n_2}(p_k)| < \varepsilon/3$ for $k = 1, 2, ..., m$ and all $n_1, n_2 > N$. Let's conclude by setting $x \in K$. Then there is a k such that $x \in B_k$ and thus $d(x, p_k) < \delta$. The choice of δ and N then gives us that

$$\begin{aligned}
|f_{n_1}(x) - f_{n_2}(x)| &\leqslant |f_{n_1}(x) - f_{n_1}(p_k)| + |f_{n_1}(p_k) - f_{n_2}(p_k)| + |f_{n_2}(p_k) - f_{n_2}(x)| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
&= \varepsilon.
\end{aligned}$$

□

Definition 3.2.5. The function

$$P_r(t) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{int}$$

for $0 \leqslant r < 1$, $t \in \mathbb{R}$, is referred to as *the Poisson kernel on* $\mathbb{D}$, or sometimes simply *the Poisson kernel.* We also rewrite $P_r(\theta - t)$ by setting $z = re^{it}$ as

$$P(z, e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2}.$$

This stems from the fact that

$$P_r(\theta - t) = \operatorname{Re}\left(\frac{e^{it} + z}{e^{it} - z}\right) = \frac{1 - r^2}{1 - 2r\cos(\theta - r) + r^2}.$$

For $f \in L^1(\mathbb{T})$ we define *the Poisson integral of* f by

$$P[f](z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})P(z, e^{it})\ dt$$

For complex measure μ we define *the Poisson integral of* μ by

$$P[d\mu](z) = \int_T P(z, e^{it})\ d\mu(e^{it})$$

Let's take a look at some of the properties of the Poisson kernel and Poisson integral. Note that both $P[f]$ and $P[d\mu]$ are defined on $\mathbb{D}$ because of the way we defined the Poisson kernel.

We see that, since

$$P_r(t) = \frac{1 - r^2}{1 - 2r\cos(t) + r^2},$$

both $P_r(t) > 0$ and $P_r(t) = P_r(-t)$. It will also come of use to know that for $n \neq 0$

$$in \int_{-\pi}^{\pi} e^{int}\ dt = (e^{in\pi} - e^{-in\pi}) = (e^{in\pi} - e^{-i(2n\pi - n\pi)}) = (e^{in\pi} - e^{in\pi}) = 0,$$

so

$$\begin{aligned}
\int_{-\pi}^{\pi} P_r(t)\ dt &= \int_{-\pi}^{\pi} \sum_{n\in\mathbb{Z}} r^{|n|} e^{int}\ dt \\
&= \sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} r^{|n|} e^{int}\ dt \\
&= \int_{-\pi}^{\pi} dt \\
&= 2\pi.
\end{aligned}$$

This naturally leads us to the following lemma:

Lemma 3.2.6. Let μ be a complex Borel measure, and $u = P[d\mu]$. Then

$$\|u_r\|_1 \leqslant \|\mu\|.$$

Proof. We have that

$$\begin{aligned}
\|u\|_1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})|\ d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \int_{\mathbb{T}} P(re^{i\theta}, e^{it})\ d\mu(e^{it}) \right| d\theta \\
&\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{\mathbb{T}} P(re^{i\theta}, e^{it})\ d|\mu(e^{it})| d\theta \\
&= \int_{\mathbb{T}} \frac{1}{2\pi} \int_{-\pi}^{\pi} P(re^{i\theta}, e^{it})\ d\theta d|\mu(e^{it})| \\
&= \int_{\mathbb{T}} d|\mu(e^{it})| \\
&= |\mu|(\mathbb{T}) \\
&= \|\mu\|.
\end{aligned}$$

□

The main reason we are developing these tools is to prove the F. and M. Riesz theorem. The theorem gives us a sufficient condition for when a measure μ (on $\mathbb{T}$) is absolutely continuous with regards to the Lebesgue-measure (on $\mathbb{T}$). The proof is rather simple, we let f be the Poisson integral of μ and h be a function such that its Poisson integral is f. We then show that $d\mu = h\ dm$. The following lemmas show that all this is in fact possible.

Lemma 3.2.7. Let u be harmonic in $\mathbb{D}$ and

$$\sup_{0<r<1} \|u_r\|_1 = M < \infty.$$

Then there exists a unique complex Borel measure μ on $\mathbb{T}$ such that $u = P[d\mu]$.

Lemma 3.2.8. Let u be harmonic in $\mathbb{D}$ and

$$\sup_{0<r<1} \|u_r\|_2 = M < \infty.$$

Then there exists a unique function f in $L^2(\mathbb{T})$ such that $u = P[f]$.

We will need the following lemma in the proofs of 3.2.7 and 3.2.8:

Lemma 3.2.9. Let X be a separable Banach space, $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of linear functionals on X, and $\sup_n \|\Gamma_n\| = M < \infty$. Then there exists a subsequence $(\Gamma_{n_i})_{i\in\mathbb{N}}$ such that the limit

$$\Gamma x = \lim_{k\to\infty} \Gamma_{n_k}\ x$$

exists for every $x \in X$. Furthermore, Γ is linear and $\|\Gamma\| \leqslant M$.

Proof. We have that $|\Gamma_n\ x| \leqslant M\|x\|$ and

$$\begin{aligned} |\Gamma_n\ x - \Gamma_n\ y| &= |\Gamma_n(x-y)| \\ &\leqslant M\|x-y\|. \end{aligned}$$

The first inequality gives us pointwise boundedness and the second gives us equicontinuity. Now, since singletons are compact, Ascoli-Arzela gives us a subsequence, let's index it by S, such that $(\Gamma_n x)_{n\in S}$ is convergent for all $x \in X$. Let's now define Γ by

$$\Gamma(x) = \lim_{k\in S} \Gamma_k\ x,$$

see the

$$\begin{aligned}\Gamma(x)+\Gamma(y) &= \lim_{k\in S}\Gamma_k\ x + \lim_{k\in S}\Gamma_k\ y\\ &= \lim_{k\in S}\left(\Gamma_k\ x+\Gamma_k\ y\right)\\ &= \lim_{k\in S}\Gamma_k(x+y)\\ &= \Gamma(x+y),\end{aligned}$$

where the second equality holds because addition is continuous, and $a\Gamma(x)=\Gamma(ax)$ obviously holds. So Γ is linear. Lastly

$$\begin{aligned}\|\Gamma\| &= \sup\{|\Gamma x|;\ \|x\|\leqslant 1\}\\ &= \sup\left\{\left|\lim_{n\in S}\Gamma_n x\right|;\ \|x\|\leqslant 1\right\}\\ &\leqslant \sup\{M;\ \|x\|\leqslant 1\}\\ &= M.\end{aligned}$$

□

Proof of 3.2.7. Let Γ_r, for $r\in[0,1[$, be linear functionals on $C(\mathbb{T})$ defined by

$$\Gamma_r g = \int_{\mathbb{T}} g u_r\ d\sigma.$$

If $\|g\|\leqslant 1$ is assumed we get that

$$\Gamma_r\ g = \int_{\mathbb{T}} g u_r\ d\sigma \leqslant \int_{\mathbb{T}} u_r\ d\sigma = \|u_r\|_1 \leqslant M.$$

so

$$\|\Gamma_r\|\leqslant M.$$

By Lemma 3.2.9 and the Riezs representation theorem we get a measure μ on $\mathbb{T}$ with $\|\mu\|\leqslant M$, and a sequence $(r_n)_{n\in\mathbb{N}}$ on $[0,1[$ with limit 1, such that

$$\lim_{n\to\infty}\int_{\mathbb{T}} g u_{r_n}\ d\sigma = \int_{\mathbb{T}} g\ d\mu \tag{3.1}$$

for all $g\in C(\mathbb{T})$. Let's now define functions h_k on $\overline{\mathbb{D}}$ by $h_k(z)=u(r_k z)$. We get that, since u is harmonic on $r\mathbb{D}$ for $r\in]0,1[$, the functions h_k are harmonic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. So each of them can be represented by the Poisson integral of their restriction to $\mathbb{T}$. Note that $h_k(e^{it}) = u_{r_k}(e^{it})$, so

$$\begin{aligned} u(z) &= \lim_{n\to\infty} u(r_n z) \\ &= \lim_{n\to\infty} h_n(z) \\ &= \lim_{n\to\infty} \int_{\mathbb{T}} P(z, e^{it}) h_n(e^{it})\ d\sigma(e^{it}) \\ &= \lim_{n\to\infty} \int_{\mathbb{T}} P(z, e^{it}) u_{r_n}(e^{it})\ d\sigma(e^{it}) \\ &= \int_{\mathbb{T}} P(z, e^{it})\ d\mu(e^{it}) \\ &= P[d\mu](z), \end{aligned}$$

where the fifth equality is achieved by putting $g = P(z, e^{it})$ into 3.1. This concludes the proof of existence.

Let's assume that $P[d\mu] = 0$, and let $f \in C(\mathbb{T})$, $u = P[f]$ and $v = P[d\mu]$. We firstly have the symmetry

$$P(re^{i\theta}, e^{it}) = P(re^{it}, e^{i\theta}).$$

This symmetry is due to

$$|e^{it} - re^{i\theta}| = |1 - re^{i(\theta - t)}| = |1 - re^{i(t-\theta)}| = |e^{i\theta} - re^{it}|,$$

which is geometrically intuitive. The first and last equalities hold because the distance between two points doesn't change under rotation and the second equality holds because the distance from z to a real number a is the same distance from $\overline{z}$ to a. We now obtain

$$\begin{aligned} \int_{\mathbb{T}} u_r\ d\mu &= \int_{\mathbb{T}} \int_{-\pi}^{\pi} P(re^{i\theta}, e^{it}) f(e^{i\theta})\ d\theta d\mu(e^{it}) \\ &= \int_{-\pi}^{\pi} f(e^{i\theta}) \int_{\mathbb{T}} P(re^{it}, e^{i\theta})\ d\mu(e^{it}) d\theta \\ &= \int_{-\pi}^{\pi} f(e^{i\theta}) v_r\ d\theta \\ &= \int_{\mathbb{T}} f v_r\ d\sigma. \end{aligned}$$

If we let $r \to 1$ we get

$$\int_{\mathbb{T}} f\ d\mu = 0.$$

This holds for all $f \in C(\mathbb{T})$, so the measure μ represents zero in the dual of $C(\mathbb{T})$. The Riesz representation theorem then tells us that $|\mu|(\mathbb{T}) = 0$, so $\mu = 0$.

Now let λ and ν be measures on $\mathbb{T}$ such that $P[d\lambda] = P[d\nu]$. We have that $P[d(\lambda - \nu)] = 0$, so, as shown above $\lambda - \nu = 0$. Moreover $\lambda = \nu$, which concludes the proof of uniqueness. $\square$

Proof of 3.2.8. The proof of existence is almost identical to that proof of existence in 3.2.7 and the uniqueness is shown in the same manner. The differences in the existence proofs are in the beginning when we are choosing what function spaces we use and which theorems to references. Now we define Γ_r in the same way, except we define it on $L^2(\mathbb{T})$. We again use 3.2.9 and the Riesz representation theorem to show that there exists a function f in $L^2(\mathbb{T})$ with $\|f\|_2 \leqslant M$ and

$$\lim_{n\to\infty} \int_{\mathbb{T}} g u_{r_n} \, d\sigma = \int_{\mathbb{T}} gf \, d\sigma$$

for all g in $L^2(\mathbb{T})$. The remaining calculations remain the same. □

We see by this proof that 3.2.8 could be generalized trivially to find a function f in $L^p(\mathbb{T})$ such that $P[f] = u$ for harmonic u with

$$\sup_{0<r<1} \|u_r\|_p = M < \infty$$

and $p > 1$. We can't use this method fir $p = 1$ since we will need to use the Riesz representation theorem for the exponent conjugate of p, and the theorem only holds if it is in $[1, \infty[$.

We have not yet shown any condition for a function f that implies there exists a $h \in L^1(\mathbb{T})$ such that $P[h] = f$. For that we need the following definition:

Definition 3.2.10. For a function f holomorphic on $\mathbb{D}$ we define

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta}|^p \, d\theta \right)^{1/p}$$

and we say that f is in H^p if $\|f\|_{H^p} < \infty$. These spaces are referred to as the *Hardy spaces.*

A useful result from the theory of Hardy spaces is if $f \in H^1$ then there exist $g, h \in H^2$ such that $f = g \cdot h$ [Rudin, 1987, Theorem 17.10].

Lemma 3.2.11. Let $f \in H^1$. Then there exists a $h \in L^1(\mathbb{T})$ such that $f = P[h]$.

Proof. TODO □

Theorem 3.2.12 (F. and M. Riesz theorem)**.** *If μ is a complex Borel measure on $\mathbb{T}$ and*

$$\int e^{-int} \, d\mu = 0$$

for $n = -1, -2, ...,$ then $\mu \ll m$.

Proof. Let $f = P[d\mu]$. If we set $z = re^{i\theta}$ we get that

$$P(z, e^{it}) = P_r(\theta - t) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{in(\theta-t)} = \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} e^{-int}.$$

We can use the assumption of the theorem to write f as a power series by

$$\begin{aligned} f(z) &= \int_{\mathbb{T}} P(z, e^{it})\ d\mu(e^{it}) \\ &= \int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} e^{-int}\ d\mu(e^{it}) \\ &= \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} \int_{\mathbb{T}} e^{-int}\ d\mu(e^{it}) \\ &= \sum_{n=0}^{\infty} r^{n} e^{in\theta} \int_{\mathbb{T}} e^{-int}\ d\mu(e^{it}) \\ &= \sum_{n=0}^{\infty} \hat{\mu}_n z^n, \end{aligned}$$

where $\hat{\mu}_n$ is the n-th Fourier coefficient of μ. This along with 3.2.6 gives us that $f \in H^1$. We can now define a $h \in L^1(\mathbb{T})$, by 3.2.11, such that $f = P[h]$. It follows from 3.2.7 that $d\mu = hd\sigma$. TODO ☐

Corollary 3.2.13. *Every annihilating measures of $\mathcal{A}|_{\mathbb{T}}$ is absolutely continuous with regards to the Lebesgue-measure on $\mathbb{T}$.*

Proof. Let μ be an annihilating measure of $\mathcal{A}|_{\mathbb{T}}$. By definition we have that

$$\int f\ d\mu = 0$$

for all $f \in \mathcal{A}|_{\mathbb{T}}$. Now since $t \mapsto e^{-int}$ is entire for $n = -1, -2, ...$ we have that their restriction to $\mathbb{T}$ are in $\mathcal{A}|_{\mathbb{T}}$. Thus,

$$\int e^{-int}\ d\mu = 0$$

for all $n = -1, -2, ...$ and $\mu \ll m$. ☐

3.3. A generalization of the Rudin-Carleson theorem

This borrows from [Bishop, 1962].

Theorem 3.3.1 (General Rudin-Carleson theorem)**.** *Let X be a compact Hausdorff space, $V = (C(X), \|\cdot\|_\infty)$, B be a closed subspace of $C(X)$, $B^\perp$ be the annihilating measures of B, S be a closed subset of X that is μ-null for all $\mu \in B^\perp$, f be a continuous function on S, and $\Xi > 0$ be a continuous function on X such that $|f| < \Xi$ on S. Then there exists a $F \in B$ such that $F = f$ on S and $|F| < \Xi$ on X.*

Let's start with the following lemma:

Lemma 3.3.2. Assume $|f| < r < 1$ on S. Then there exists a $F \in B$ such that $F = f$ on S and $\|F\| < 1$.

Proof. Let U_r be the subset of B defined by $U_r = \{g;\ \|g\| < r\}$ and ϕ be the mapping from B to $C(S)$ that sends a member of B to its restriction on S. It suffices to show that $f \in \phi(U_r)$. Let's first show that $f \in \overline{\phi(U_r)} =: V_r$, by assuming otherwise, and showing it leads to a contradiction. Note that if $f, g \in U_r$ and $t \in [0,1]$ then

$$\|tf + (1-t)g\| \leqslant t\|f\| + (1-t)\|g\| \leqslant tr + (1-t)r = r$$

so $tf + (1-t)g$ is also in U_r, showing it is convex. Its closure, V_r, is then convex as well.

By Hahn-Banach we can define a bounded linear functional α, such that $\alpha(f) > 1$ and $|\alpha| < 1$, on V_r. We can then define a measure μ_1 by the Riesz representation theorem that fulfills

$$\alpha(g) = \int g\ d\mu_1$$

for all $g \in C(S)$. We will refer to the associated functional on B by $\beta(g) = \alpha(\phi(g))$. Since $\phi(g) \in V_r$ for all $g \in U_r$ we have that

$$\beta(g) = \alpha(\phi(g)) < 1,$$

for all $g \in U_r$, due to the construction of α. From this we get

$$\begin{aligned}\|\beta\| &= \sup\{|\beta(g)|;\ \|g\| < 1\} \\ &= \sup\{(1/r)|\beta(g)|;\ \|g\| < r\} \\ &\leqslant 1/r.\end{aligned}$$

Let's denote the Riesz representation of β by μ_2, set $\mu = \mu_1 - \mu_2$, and note that $\mu \in B^\perp$. But

$$0 = \left|\int_S f\ d\mu\right| \geqslant \int_S f\ d\mu_1 - r\|\mu_2\| \geqslant \int_S f\ d\mu_1 - r\frac{1}{r} > 1 - r\frac{1}{r} = 0.$$

This is the contradiction that gives that $f \in V_r$. We can now take a F_1 in U_r, and therefore also in B such that $|f - F_1| < \lambda/2$ on S, with $\lambda := 1 - r$. Remember that $F_1 \in U_r$ implies that $\|F_1\| < r$. Now let $f_1 = f - F_1$ and use the same method as above to obtain an F_2 such that $\|F_2\| < \lambda/2$ and $|f - F_2| < \lambda/4$ on S. Iterating this process yields a sequence $(F_n)_{n \in \mathbb{N}}$ from B that fulfill $\|F_n\| < 2^{1-n}\lambda$ for $n > 1$ and

$$\left| f - \sum_{k=1}^{n} F_k \right| < 2^{-n}\lambda$$

on S for $n > 1$. We finally let

$$F = \sum_{k=1}^{\infty} F_k.$$

Now $F \in B$,

$$\|F\| \leqslant \|F_1\| + \|F - F_1\| \leqslant r + \sum_{k=2}^{\infty} 2^{1-n}\lambda = r + \lambda = 1,$$

and $F = f$ on S. □

Proof of 3.3.1. Let B_0 be the closed subspace of $C(X)$ consisting of function g such that $\Xi \cdot g \in B$. We have that $B_0^\perp = B^\perp$, since $\Xi > 0$. So we can use Lemma 3.3.2 for B_0 instead of B and f/Ξ instead of f. This gives us a $F_0 \in B_0$ such that $\Xi \cdot F_0 = f$ on S and $\|F_0\| < 1$. We set $F = \Xi \cdot F_0$ which is in B by the construction of B_0. Also note that $|F| < \Xi$ on X and

$$F = \Xi \cdot F_0 = \Xi \cdot f/\Xi = f$$

on S. □

Alternative proof of 3.1.1. Let $X = \mathbb{T}$, $B = \mathcal{A}|_{\mathbb{T}}$, and S be a closed set of Lebesgue-measure zero. Then, according to 3.2.13, S is also a $B^\perp$-null. So all requirements of 3.3.1 are met. □

A. Further applications of the general Rudin-Carleson theorem

Definition A.0.1. Let K be a closed subset of $S = \{z \in \mathbb{C}^n;\ |z| = 1\}$. We then say K is a

1. *zero set* if there exists a function in $f \in \mathcal{A}^n$ such that $K = f^{-1}(0)$.
2. *peak set* if there exists a function in $f \in \mathcal{A}^n$ such that $K = f^{-1}(1)$ and $|f(z)| < 1$ for $z \in \overline{\mathbb{B}}^n \backslash K$.
3. *interpolation set* if every continuous function on K extends via $\mathcal{A}^n$.
4. *peak-interpolation set* if every non-zero continuous function f on K extends to $F \in \mathcal{A}^n$ such that $|f(z)| < \|F\|$ for $z \in \overline{\mathbb{B}}^n \backslash K$.
5. *null set* if K has measure zero with regards to all annihilating measures of $\mathcal{A}^n$.
6. *totally null* if K has measure zero with regards to all measure μ such that

   $$f(0) = \int_{\mathbb{B}} f \ d\mu$$

   for all $f \in \mathcal{A}^n$.

Theorem A.0.2. *The six classes of sets described in A.0.1 are equivalent.*

The first few sections of chapter 10 in [Rudin, 1980] are dedicated to prove this large theorem. One implication, namely that all null sets are peak-interpolation sets, follows directly from 3.3.1.

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