XXTitle

Bergur Snorrason

Faculty of XX
University of Iceland
2020

XXTITLE

Bergur Snorrason

XX ECTS thesis submitted in partial fulfillment of a
Magister Scientiarum degree in XX

Advisor
XXNN1
XXNN2

Faculty Representative
XXNN3

M.Sc. committee
XXNN4
XXNN5

Faculty of XX
School of Engineering and Natural Sciences
University of Iceland
Reykjavik, XXmonth 2020

XXTitle
XXShort title (50 characters including spaces)
XX ECTS thesis submitted in partial fulfillment of a M.Sc. degree in XX

Faculty of XX
School of Engineering and Natural Sciences
University of Iceland
XXFaculty street address
XXFaculty postal code, Reykjavik, Reykjavik
Iceland

Telephone: 525 4000

Bibliographic information:
Bergur Snorrason, 2020, XXTitle, M.Sc. thesis, Faculty of XX, University of Iceland.

ISBN XX

Printing: Háskólaprent, Fálkagata 2, 107 Reykjavík
Reykjavik, Iceland, XXmonth 2020

Dedication

Abstract

Útdráttur á ensku sem er að hámarki 250 orð.

Útdráttur

Hér kemur útdráttur á íslensku sem er að hámarki 250 orð. Reynið að koma útdráttum á eina blaðsíðu en ef tvær blaðsíður eru nauðsynlegar á seinni blaðsíða útdráttar að hefjast á oddatölusíðu (hægri síðu).

Preface

Formála má sleppa og skal þá fjarlægja þessa blaðsíðu. Formáli skal hefjast á oddatölu blaðsíðu og nota skal Section Break (Odd Page).

Ekki birtist blaðsíðutal á þessum fyrstu síðum ritgerðarinnar en blaðsíðurnar teljast með og hafa áhrif á blaðsíðutal sem birtist með rómverskum tölum fyrst á efnisyfirliti.

Contents

List of Figures

List of Tables

Abbreviations

Í þessum kafla mega koma fram listar yfir skammstafanir og/eða breytuheiti. Gefið kaflanum nafn við hæfi, t.d. Skammstafanir eða Breytuheiti. Þessum kafla má sleppa ef hans er ekki þörf.
The section could be titled: Glossary, Variable Names, etc.

Acknowledgments

Í þessum kafla koma fram þakkir til þeirra sem hafa styrkt rannsóknina með fjárframlögum, aðstöðu eða vinnu. T.d. styrktarsjóðir, fyrirtæki, leiðbeinendur, og aðrir aðilar sem hafa á einhvern hátt aðstoðað við gerð verkefnisins, þ.m.t. vinir og fjölskylda ef við á. Þakkir byrja á oddatölusíðu (hægri síðu).

1. Introduction

2. Preliminaries

2.1. Measure theory

2.2. Functional analysis

2.2.1. Hahn-Banach

2.2.2. Riesz representation theorem

3. Rudin-Carleson theorem

Theorem 1 (Rudin-Carleson theroem)**.** *Let E be a closed subset of $\mathbb{T}$ of Lebesgue-measure 0, let f be a continuous function on E and let T be a simply connected subset of $\mathbb{C}$ such that $f(\overline{\mathbb{D}}) \subset T$. Then there exists an $F \in \mathcal{A}$, such that $F = f$ on E and $F(\overline{\mathbb{D}}) \subset T$.*

We will break the proof into several lemmas.

Lemma 1. *Let H be a closed set of Lebesbue-measure 0. Then there exists an integrable function $\mu > 1$ such that μ is continuous on $\mathbb{T}\backslash H$, $\mu = +\infty$ on H, if $w \in H$ then $\mu(z) \xrightarrow{z \to w} +\infty$, and μ has a bounded derivative on any closed subarc of $\mathbb{T}\backslash H$.*

Proof. TODO □

Lemma 2. *If f is a simple continuous function on E such that Re $f \geqslant 0$, then there exists an $F \in \mathcal{A}$ such that $F = f$ on E and Re $F \geqslant 0$ on $\overline{\mathbb{D}}$.*

Proof. It suffices to show that this holds if f takes only two values on E, since simple functions are finite linear combinations of characteristic functions. Let's assume these values are 0 and $\alpha \neq 0$, with Re $\alpha \geqslant 0$, $E_0 = f^{-1}(0)$, and $E_1 = f^{-1}(\alpha)$. Our assumption that f only takes two values then implies that $E_0 \cap E_1 = E$.

Let $u_H(z)$ be the Poisson integral of the function from the above lemma with H as E. This function is continuous on $\mathbb{T}\backslash H$, $u_H|_H = \infty$, and $\lim_{z\to w} u_H(z) = \infty$ for $w \in H$. ... We now define

$$g_H(z) = \begin{cases} u_H(z) + iv_H(z), & z \in \mathbb{D}\backslash H \\ \infty, & \text{otherwise} \end{cases}.$$

By our construction of u_H we see the Re $g > 1$, so it has a well defined square root. Let's call it h_H and define

$$q = \frac{h_{E_1}}{h_{E_0} + h_{E_1}}.$$

Note that $|\arg h_H(z)| < \pi/4$ since if a $w \in \mathbb{C}$ had an argument outside of this range then its square would have and argument outside of the range $[-\pi/2, \pi/2]$ meaning $\mathrm{Re}\ w^2 < 0$. Also, $q(z) = 0$ if and only if $h_{E_0} = \infty$, so q is zero only on E_0, and $q(z) = 1$ if and only if $h_{E_1} = \infty$, so q is one only on E_1. We now want to show that $|\mathrm{Re}\ q| \leqslant 1$. We will let $z, w \in \mathbb{C}$, with $|\arg z|, |\arg w| < \pi/4$ and $\mathrm{Re}\ z, \mathrm{Re}\ w > 1$, and show that $0 < \mathrm{Re}\ z/(w+z) < 1$.

Firstly note that

$$\frac{z}{w+z} = \frac{1}{w/z+1}$$

so

$$\arg \frac{z}{w+z} = -\arg\left(\frac{w}{z}+1\right)$$

and

$$|\arg w/z| = |\arg w - \arg z| < |\arg w| + |\arg z| < \pi/4 + \pi/4 = \pi/2.$$

So w/z is in the right halfplane and, since $\arctan(y/x)$ is decreasing in x for positive y, we get

$$\arg \frac{z}{w+z} = -\arg\left(\frac{w}{z}+1\right) < -\arg \frac{w}{z}$$

and thus

$$\left|\arg \frac{z}{w+z}\right| < \left|\arg \frac{w}{z}\right| < \pi/2.$$

We know $\arctan(y/x)$ is decreasing in x for positive y because

$$\frac{d}{dx} \arctan\left(\frac{y}{x}\right) = -\frac{y}{x^2+y^2} < 0.$$

So $0 < \mathrm{Re}\ z/(w+z)$. We can also see that

$$\begin{aligned}
\left|\mathrm{Re}\ \frac{z}{z+w}\right| &= \frac{1}{2}\left|\frac{z}{z+w} + \frac{\overline{z}}{\overline{z}+\overline{w}}\right| \\
&\leqslant \frac{1}{2}\left(\frac{|z|}{|z+w|} + \frac{|\overline{z}|}{|\overline{z}+\overline{w}|}\right) \\
&= \frac{|z|}{|z+w|} \\
&= \frac{1}{1+|w|/|z|} \\
&< 1.
\end{aligned}$$

So we have constructed a function q that maps $\overline{D}$ to the ribbon $\{z;\ 0 \leqslant \mathrm{Re}\ z \leqslant 1\}$. We then let Φ be the conformal mapping from the ribbon $\{z;\ 0 \leqslant \mathrm{Re}\ z \leqslant 1\}$ to $\{z;\ 0 \leqslant \mathrm{Re}\ z \leqslant \mathrm{Re}\ \alpha\}$. We will also choose Φ such that $\Phi(0) = 0$ and $\Phi(1) = \alpha$. We can then let $f = \Phi \circ q$ and conclude the proof.

□

Lemma 3. *If f is a simple continuous function on E that maps E into a simply connected T, then there exists a $F \in \mathcal{A}$, such that $F = f$ on E and F maps $\overline{\mathbb{D}}$ into T.*

Proof. TODO □

Lemma 4. *If f is a continue function on E which maps E into $S = \{z;\ |\max(Re\ z, Im\ z)| \leqslant 1\}$, then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple continuous function on E such that*

$$f(x) = \sum_{n\in\mathbb{N}} f_n(z),$$

$$f_n(E) \subset 2^{-n}S.$$

Proof. TODO □

Proof. TODO □

Corollary 1 (Fatou)**.** *Let E be a closed subset of $\mathbb{T}$ of Lebesgue-measure 0. There exists a function $f \in \mathcal{A}$ that vanishes on E and nowhere else.*

Proof. It's clear from the theorem that there exists a function $f \in \mathcal{A}$ that vanishes on E. TODO □

4. F. and M. Riesz theorem

In this section we will endeavour to show that the annihilating measures of $\mathcal{A}|_{\mathbb{T}}$ are absolutely continuous with respect to the Lebesgue measure. We will show this to be a corollary of the F. and M. Riesz theorem, which we will prove in the manner of Rudin. To attain the main result of this section we need some lemmas and definitions. To prove one of the lemmas we will also use the following two famous theorems:

Definition 1. *Let $\mathcal{F}$ be a family of complex functions on a metric space (X, d).*

We say that the family is pointwise bounded *if for all $x \in X$ there exists a constant $M < \infty$ such that*

$$|f(x)| < M, \text{ for all } f \in \mathcal{F}.$$

Note that M is dependent on x.

We say that the family is equicontinuous *if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|f(x) - f(y)| < \varepsilon, \text{ for all } f \in \mathcal{F} \text{ and } x, y \in X \text{ such that } d(x, y) < \delta.$$

Note here that δ is globally defined and only dependent on ε.

Theorem 2 (Bolzano-Weierstrass)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of numbers in $\mathbb{R}^n$, such that $|a_n| < M < \infty$, for all $k \in \mathbb{N}$. There than exists and infinite $S \subset \mathbb{N}$ such that $(a_n)_{n\in S}$ is convergent.*

Proof. Let's first assume that the sequence is in $\mathbb{R}$, that no element in it is repeated infinitely often (there is nothing to prove in that case), and that $a_n \in]0, 1[$ for all $n \in \mathbb{N}$. The last assumption can be done with out loss of generality by studying the sequence $((a_n + M)/(2M))_{n\in\mathbb{N}}$ instead. We will obtain the subsequence by a diagonal process. Let $S_0 = \mathbb{N}$, $S_0^- = \{n \in S_0;\ a_n < 1/2\}$, and $S_0^+ = \{n \in S_0;\ a_n > 1/2\}$. We then set $S_1 = S_0^-$ if it is infinite, but $S_1 = S_0^+$ otherwise. This gives us a subsequence $(a_n)_{n\in S_1}$ such that

$$\sup_{n\in S_1} a_n - \inf_{n\in S_1} a_n < 1/2.$$

We can then repeat this to get a sequence of sets $(S_n)_{n\in\mathbb{N}}$ such that $S_0 \supset S_1 \supset S_2 \supset ...$ and

$$\sup_{n\in S_k} a_n - \inf_{n\in S_k} a_n < 2^{1-k},$$

for all $k \in \mathbb{N}$. Specifically, if we have S_k we set

$$U = m2^{-k},\ L = (m+1)2^{-k}$$

$S_k^- = \{n \in S_k;\ a_n < (U+L)/2\}$, and $S_k^+ = \{n \in S_k;\ a_n > (U+L)/2\}$. We now set $S_{k+1} = S_k^-$ if it has infinitely many elements, otherwise we set $S_{k+1} = S_k^+$. We conclude our construction by setting

$$S = \bigcup_{n\in\mathbb{N}} r_n,$$

where r_n is the n-th smallest element of S_n. This gives us the convergent sequence $(a_n)_{n\in S}$ with limit

$$\sum_{k=0}^{\infty} \delta_k 2^{-k}$$

where

$$\delta_k = \begin{cases} 0, & \text{if we chose } S_k^- \\ 1, & \text{if we chose } S_k^+ \end{cases}.$$

To show the result for $\mathbb{R}^n$ we can start by finding a subsequence such that the first coordinate is convergent. We can then chose a subsequence thereof such that the second coordinate is also convergent. Now the first two coordinates are convergent. If we do this $n-2$ more times we get a desired subsequence. □

Remark 1. *The theorem above clearly holds for sequences in $\mathbb{C}$ as well.*

Theorem 3 (Ascoli-Arzela)**.** *Let $\mathcal{F}$ be a pointwise bounded equicontinuous collection of complex functions on a metric space (X,d), and X contains a countable dense subset. Then every sequence in $\mathcal{F}$ contains a subsequence that converges uniformly on every compact subsets of X.*

Proof. Let E be a countable dense subset of X, $(f_n)_{n\in\mathbb{N}}$ be a series in $\mathcal{F}$, and $x_1, x_2, ...$ be an enumeration of E. We will prove the theorem in two steps. The first step is finding a subsequence of $(f_n)_{n\in\mathbb{N}}$ that's pointwise convergent on E using the point wise boundedness along with Bolzano-Weierstrass. The second step is using the equicontinuity to show that this gives us uniform continuity on compact subsets.

Let's first set $S_0 = \mathbb{N}$. Pointwise boundedness gives us that the sequence $(f_n(x_1))_{n\in S_0}$ has a convergent subsequence. Let S_1 index that subsequence. We can use this process to generate sets $S_0 \supset S_1 \supset \ldots$ such that $(f_n(x_k))_{n \in S_k}$ is convergent. We then set

$$S = \bigcup_{k \in \mathbb{N}} r_k$$

where r_n is the k-th smallest element of S_k. We now have concluded the first step of the proof.

We will now assume the $(f_n)_{n \in \mathbb{N}}$ is pointwise convergent on E, let K be a compact subset of X, and $\varepsilon > 0$. Equicontinuity gives us a $\delta > 0$ such that $d(x, y) < \delta$ implies that $|f_n(x) - f_n(y)| < \varepsilon/3$, for all n. Let's now cover K with m balls of radius $\delta/2$ and call the k-th ball B_k. We can now set p_k as a point in B_k. This point exists because E is dense in X. Pointwise convergence on E let's us chose an N such that $|f_{n_1}(p_k) - f_{n_2}(p_k)| < \varepsilon/3$ for $k = 1, 2, \ldots, m$ and all $n_1, n_2 > N$. Let's conclude by setting $x \in K$. Then there is a k such that $x \in B_k$ and thus $d(x, p_k) < \delta$. The choice of δ and N then gives us that

$$\begin{aligned}
|f_{n_1}(x) - f_{n_2}(x)| &\leqslant |f_{n_1}(x) - f_{n_1}(p_k)| + |f_{n_1}(p_k) - f_{n_2}(p_k)| + |f_{n_2}(p_k) - f_{n_2}(x)| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
&= \varepsilon.
\end{aligned}$$

□

Definition 2. *Poisson kernel, Poisson integral, Poisson integral of a measure.*

Lemma 5. *Let μ be a complex Borel measure, and $u = P[d\mu]$. Then*

$$\|u_r\|_1 \leqslant \|\mu\|.$$

Proof. First, we need to see that, if $n \neq 0$

$$in \int_{-\pi}^{\pi} e^{int}\, dt = (e^{in\pi} - e^{-in\pi}) = (e^{in\pi} - e^{-i(2n\pi - n\pi)}) = (e^{in\pi} - e^{in\pi}) = 0,$$

so

$$\begin{aligned}
\int_{-\pi}^{\pi} P_r(t)\, dt &= \int_{-\pi}^{\pi} \sum_{n \in \mathbb{Z}} r^{|n|} e^{int}\, dt \\
&= \sum_{n \in \mathbb{Z}} \int_{-\pi}^{\pi} r^{|n|} e^{int}\, dt \\
&= \int_{-\pi}^{\pi} dt \\
&= 2\pi.
\end{aligned}$$

Fubini let's us

$$\begin{aligned}
\|u\|_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})| d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \int_{\mathbb{T}} P(re^{i\theta}, e^{it}) \; d\mu(e^{it}) \right| d\theta \\
&\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \int_{\mathbb{T}} P(re^{i\theta}, e^{it}) \; d|\mu(e^{it})| d\theta \\
&= \int_{\mathbb{T}} \frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i\theta}, e^{it}) \; d\theta d|\mu(e^{it})| \\
&= \int_{\mathbb{T}} \; d|\mu(e^{it})| \\
&= |\mu|(\mathbb{T}) \\
&= \|\mu\|.
\end{aligned}$$

□

Lemma 6. *Let $f \in H^1$. Then there exists a $g \in L^1(\mathbb{T})$ such that $f = P[g]$.*

Proof. □

Lemma 7. *Let u be harmonic in $\mathbb{D}$ and*

$$\sup_{0<r<1} \|u_r\|_1 = M < \infty.$$

Then there exists a unique complex Borel measure μ on $\mathbb{T}$ such that $u = P[d\mu]$.

We will need the following lemma in the proof of 7:

Lemma 8. *Let X be a separable Banach space, $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of linear functionals on X, and $\sup_n \|\Gamma_n\| = M < \infty$. Then there exists a subsequence $\{\Gamma_{n_i}\}$ such that the limit*

$$\Gamma x = \lim_{k\to\infty} \Gamma_{n_k} \, x$$

exists for every $x \in X$. We also have that Γ is linear and $\|\Gamma\| \leqslant M$.

Proof. We have that $|\Gamma_n \, x| \leqslant M\|x\|$ and

$$\begin{aligned}
|\Gamma_n \, x - \Gamma_n \, y| &= |\Gamma_n(x-y)| \\
&\leqslant M\|x-y\|.
\end{aligned}$$

The first inequality gives us pointwise boundedness and the second gives us equicontinuity. Now, since singletons are compact, Ascoli-Arzela gives us a subsequence, let's index it by S, such that $(\Gamma_n x)_{n\in S}$ is convergent for all $x \in X$. Let's now define Γ by

$$\Gamma(x) = \lim_{k\in S} \Gamma_k\ x,$$

see the

$$\begin{aligned}\Gamma(x) + \Gamma(y) &= \lim_{k\in S} \Gamma_k\ x + \lim_{k\in S} \Gamma_k\ y \\ &= \lim_{k\in S} (\Gamma_k\ x + \Gamma_k\ y) \\ &= \lim_{k\in S} \Gamma_k(x+y) \\ &= \Gamma(x+y),\end{aligned}$$

where the third equality holds because addition is continuous, and $a\Gamma(x) = \Gamma(ax)$ obviously holds. So Γ is linear. Lastly

$$\begin{aligned}\|\Gamma\| &= \sup\{|\Gamma x|;\ \|x\| \leqslant 1\} \\ &= \sup\left\{\left|\lim_{n\in S} \Gamma_n x\right|;\ \|x\| \leqslant 1\right\} \\ &\leqslant \sup\{M;\ \|x\| \leqslant 1\} \\ &= M.\end{aligned}$$

□

Proof of 7. Let Γ_r, for $r \in [0,1[$, be linear functionals on $C(\mathbb{T})$ defined by

$$\Gamma_r g = \int_{\mathbb{T}} g u_r d\sigma.$$

If $\|g\| \leqslant 1$ is assummed we get that

$$\Gamma_r\ g = \int_{\mathbb{T}} g u_r\ d\sigma \leqslant \int_{\mathbb{T}} u_r\ d\sigma = \|u_r\|_1 \leqslant M.$$

so

$$\|\Gamma_r\| \leqslant M.$$

By the above lemma and the Riezs representation theorem we get a measure μ on $\mathbb{T}$ with $\|\mu\| \leqslant M$, and a sequence $(r_n)_{n\in\mathbb{N}}$ on $[0,1[$ with limit 1, such that

$$\lim_{n\to\infty} \int_{\mathbb{T}} g u_{r_n}\ d\sigma = \int_{\mathbb{T}} g\ d\mu \tag{4.1}$$

for all $g \in C(\mathbb{T})$. Let's now define functions h_k on $\overline{\mathbb{B}}$ by $h_k(z) = u(r_k z)$. We get that, since u is harmonic on $r\mathbb{B}$ for $r \in]0,1[$, the functions h_k are harmonic on $\mathbb{B}$

and continuous on $\overline{\mathbb{B}}$. So each of them can be represented by the Poisson integral of their restriction to $\mathbb{T}$, according to Ramsford. Note that $h_k(e^{it}) = u_{r_k}(e^{it})$, so

$$\begin{aligned} u(z) &= \lim_{n\to\infty} u(r_n z) \\ &= \lim_{n\to\infty} h_n(z) \\ &= \lim_{n\to\infty} \int_{\mathbb{T}} P(z,e^{it})h_n(e^{it})\ d\sigma(e^{it}) \\ &= \lim_{n\to\infty} \int_{\mathbb{T}} P(z,e^{it})u_{r_n}(e^{it})\ d\sigma(e^{it}) \\ &= \int_{\mathbb{T}} P(z,e^{it})\ d\mu(e^{it}) \\ &= P[d\mu](z), \end{aligned}$$

where the fifth equlity is achived by putting $g = P(z, e^{it})$ into 4.1. This concludes the proof of excistence.

Let's assume that $P[d\mu] = 0$, and let $f \in C(\mathbb{T})$, $u = P[f]$ and $v = P[d\mu]$. We firstly have the symmetry

$$P(re^{i\theta}, e^{it}) = P(re^{it}, e^{i\theta}).$$

This symmetry is due to

$$|e^{it} - re^{i\theta}| = |1 - re^{i(\theta-t)}| = |1 - re^{i(t-\theta)}| = |e^{i\theta} - re^{it}|,$$

which is geometrically intuitive. The first and last equalities hold because the euclidian metric is rotationally invariant, and the second euality holds becuase the distance from z to a real number a is the same distance from $\overline{z}$ to a. We now obtain

$$\begin{aligned} \int_{\mathbb{T}} u_r d\mu &= \int_{\mathbb{T}} \int_{-\pi}^{\pi} P(re^{i\theta}, e^{it}) f(e^{i\theta})\ d\theta d\mu(e^{it}) \\ &= \int_{-\pi}^{\pi} f(e^{i\theta}) \int_{\mathbb{T}} P(re^{it}, e^{i\theta})\ d\mu(e^{it}) d\theta \\ &= \int_{-\pi}^{\pi} f(e^{i\theta}) v_r\ d\theta \\ &= \int_{\mathbb{T}} f v_r\ d\sigma. \end{aligned}$$

If we let $r \to 1$ we get

$$\int_{\mathbb{T}} f\ d\mu = 0.$$

This holds for all $f \in C(\mathbb{T})$, so the measure μ represents zero in the dual of $C(\mathbb{T})$. The Riesz representation theorem then tells us that $|\mu|(\mathbb{T}) = 0$, so $\mu = 0$.

Now let λ and ν be measures on $\mathbb{T}$ such that $P[d\lambda] = P[d\nu]$. We have that $P[d(\lambda - \nu)] = 0$, so, as shown above $\lambda - \nu = 0$. Moreover $\lambda = \nu$, which concludes the proof of uniquness. □

Theorem 4 (F. and M. Riesz theorem)**.** *If μ is a complex Borel measure on $\mathbb{T}$ and*

$$\int e^{-int} d\mu = 0$$

for $n = -1, -2, ...,$ then $\mu \ll m$.

Proof. Let $f = P[d\mu]$. If we set $z = re^{i\theta}$ we get that

$$P(z, e^{it}) = P_r(\theta - t) = \sum_{n \in \mathbb{Z}} r^{|n|} e^{in(\theta - t)} = \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta} e^{-int}.$$

We can use the assumption of the theorem to write f as a power series by

$$\begin{aligned} f(z) &= \int_{\mathbb{T}} P(z, e^{it}) d\mu(e^{it}) \\ &= \int_{\mathbb{T}} \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta} e^{-int} d\mu(e^{it}) \\ &= \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta} \int_{\mathbb{T}} e^{-int} d\mu(e^{it}) \\ &= \sum_{n=0}^{\infty} r^{n} e^{in\theta} \int_{\mathbb{T}} e^{-int} d\mu(e^{it}) \\ &= \sum_{n=0}^{\infty} \hat{\mu}_n z^n, \end{aligned}$$

where $\hat{\mu}_n$ is the n-th Fourier coefficient of μ. This along with 5 gives us that $f \in H^1$. We can now define a $g \in H^1$, by 6, such that $f = P[g]$. It follows from 7 that $d\mu = f d\sigma$. TODO □

Corollary 2. *Let A be the closed subspace of $C(\mathbb{T})$ that consists of all functions that are restriction from $\mathcal{A}$. All measures in $A^{\perp}$ are absolutely continuous with regards to the Lebesgue-measure on $\mathbb{T}$.*

Proof. Let $\mu \in A^{\perp}$. By definition we have that

$$\int f d\mu = 0.$$

Now since $t \mapsto e^{-int}$ is entire for $n = -1, -2, ...$ we have that their restriction to $\mathbb{T}$ are in A. Thus,

$$\int e^{-int} d\mu = 0$$

for all $n = -1, -2, ...$ and $\mu \ll m$. □

5. A generalization of the Rudin-Carleson theorem

This borrows from Bishop.

Theorem 5 (General Rudin-Carleson theorem)**.** *Let X be a compact Hausdorff space, $V = (C(X), \| \cdot \|_\infty)$, B be a closed subspace of $C(X)$, $B^\perp$ be the annihilating measures for B, S be a closed subset of X, and f be a continues function on S. If $\int_S f d\mu = 0$ holds for all $\mu \in B^\perp$ then there exists a function $F \in B$ such that $F = f$ on S.*

Proof. Since f is continuous and S is a closed subset of a compact set, and therefore also compact, f is bounded. So we can, with out loss of generality, assume that $|f| < r < 1$ on S. Let U_r be the subset of B defined by $U_r = \{g;\ \|g\| < r\}$ and ϕ be the mapping from B to $C(S)$ that sends a member of B to its restriction on S. It suffices to show that $f \in \phi(U_r)$. Let's first show that $f \in \overline{\phi(U_r)} =: V_r$, by assuming otherwise, and showing it leads to a contradiction.

We now assume $f \notin V_r$. By Hahn-Banach we can define a bounded linear functional α, such that $\alpha(f) > 1$ and $|\alpha(h)| < 1$, for $h \in V_r$. We can then define a measure μ_1 by the Riesz-representation theorem that fulfills

$$\alpha(g) = \int g d\mu_1$$

for all $g \in C(S)$. We will refer to the associated functional on B by $\beta(g) = \phi(\alpha(g))$. Since $\phi(g) \in V_r$ for all $g \in U_r$ we have that

$$\beta(g) = \alpha(\phi(g)) < 1,$$

for all $g \in U_r$, due to the construction of α. From this we get

$$\begin{aligned} \|\beta\| &= \sup\{|\beta(g)|;\ |g| < 1\} \\ &= \sup\{(1/r)|\beta(g)|;\ |g| < r\} \\ &\leqslant \sup\{(1/r);\ |g| < r\} \\ &= 1/r. \end{aligned}$$

Let's denote the Riesz representation of β by μ_2, set $\mu = \mu_1 - \mu_2$ and see that $\mu \in B^{\perp}$. But

$$0 = \left| \int_S f d\mu \right| \geqslant \int_S f d\mu_1 - r\|\mu_2\| \geqslant \int_S f d\mu_1 - r\frac{1}{r} > 1 - r\frac{1}{r} = 0,$$

where the first equality is the assumption in the theorem. This is the contradiction that gives that $f \in V_r$. We can now take a F_1 in U_r, and therefore also in B such that $|f - F_1| < \lambda/2$ on S, with $\lambda := 1 - r$. Remember that $F_1 \; inU_r$ implies that $\|F_1\| < r$. Now let $f_1 = f - F_1$ and use the same method as above to obtain an F_2 such that $\|F_2\| < \lambda/2$ and $|f - F_2| < \lambda/4$ on S. Iterating this process yields a series $(F_n)_{n\in\mathbb{N}}$ from B that fulfill $\|F_n\| < 2^{1-n}\lambda$ for $n > 1$ and

$$\left| f - \sum_{k=1}^{n} F_k \right| < 2^{-n}\lambda$$

on S for $n > 1$. We finally let

$$F = \sum_{k=1}^{\infty} F_k.$$

Now $F \in B$,

$$\|F\| \leqslant \|F_1\| + \|F - F_1\| = r + \sum_{k=2}^{\infty} 2^{1-n}\lambda = r + \lambda = 1,$$

and $F = f$ on S. □

Corollary 3. *Let X be a compact Hausdorff space, $V = (C(X), \|\cdot\|_\infty)$, B be a closed subspace of $C(X)$, $B^{\perp}$ be the annihilating measures for B, S be a closed subset of X, and f be a continues function on S. If S is $B^{\perp}$-null then there exists a function $F \in B$ such that $F = f$ on S.*

Proof. If S is $B^{\perp}$-null we have that $\int_S f d\mu = 0$ for all $\mu \in B^{\perp}$. □

Remark 2. *The corollary is the version of the theorem from Bishop. Note also that if we set $X = \mathbb{T}$ and $B = \mathcal{A}$ we can use F. and M. Riesz to prove the classical Rudin-Carleson theorem.*

It is of course worth noting an applications of where the corollary fails.

Example 1. *Let $X = \mathbb{T}$, $B = \mathcal{A}$, E be a closed m-null subset of $\partial\mathbb{T}$ that is not dense in E, $F = \{e^{i\theta};\ a \leqslant \theta \leqslant b\}$, and choose a and such that E and F are disjoint and*

$a \neq b$. The last assumption restricts us to E that are not dense in the $\mathbb{E}$. Since $a \neq b$ we obtain that $S := E \cup F$ does not fulfill the requirements of the Rudin-Carelson theorem nor the above corollary. Let's choose f such that $f = 0$ on F, and f is continues on S. We now have for all $\mu \in \mathcal{A}^{\perp}$

$$\begin{aligned} \left|\int_S f d\mu\right| &= \left|\int_E f d\mu + \int_F f d\mu\right| \\ &\leqslant \left|\int_E f d\mu\right| + \left|\int_F f d\mu\right| \\ &= 0 + \left|\int_F f d\mu\right| \\ &= 0. \end{aligned}$$

The F. and M. Riesz theorem tells us the since E is m-null it is also μ-null, which gives the third step. The final step stems from the fact that f vanishes on F. We now see the X, B, and f are all as in theorem so there exists a $F \in B$, such that $F = f$ on S.

A. Annað