TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 317, Number 1, January 1990

TOPOLOGICAL SPACES WHOSE BAIRE MEASURE ADMITS A REGULAR BOREL EXTENSION

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Dedicated to Professor Yukihiro Kodama on his 60th birthday

ABSTRACT. A completely regular, Hausdorff space X is called a Mařík space if every Baire measure on X admits an extension to a regular Borel measure. We answer the questions about Mařík spaces asked by Wheeler [29] and study their topological properties. In particular, we give examples of the following spaces: A locally compact, measure compact space which is not weakly Baire-dominated; i.e., it has a sequence $F_n \downarrow \varnothing$ of regular closed sets such that $\bigcap_{n\in\omega} B_n \neq \varnothing$ whenever B_n 's are Baire sets with $F_n \subset B_n$; a countably para-compact, non-Mařík space; a locally compact, non-Mařík space X such that the absolute $E(X)$ is a Mařík space; and a locally compact, Mařík space X for which $E(X)$ is not. It is also proved that Michael's product space is not weakly Baire-dominated.

1. INTRODUCTION

All spaces are assumed to be completely regular, Hausdorff spaces. Unless otherwise specified, measures are finite, nonnegative, σ-additive measures. A *Baire* (*Borel*) *measure* on a space X is a measure defined on the σ-algebra $\mathrm{Ba}(X)$ of Baire sets ($\mathrm{Bo}(X)$ of Borel sets) of X. A (finitely additive) measure μ defined on an algebra $\mathscr{A}$ containing all closed sets is called *regular* if for each $A \in \mathscr{A}$, $\mu(A) = \sup\{\mu(F)\colon F \subset A,\ F \text{ is closed}\}$. In [29, §9], Wheeler fully reviewed a number of interesting topics relating to the problem of when a Baire measure can be extended to a regular Borel measure. In particular, he defined a space X to be a *Mařík space* if every Baire measure on X admits an extension to a regular Borel measure, and asked several questions thereupon.

This paper falls into two parts. In the first part, §§2 and 3, we answer some of his questions. In the second part, §4, we study how Mařík spaces are preserved under various topological operations. Before stating his questions, we recall some definitions and show which spaces are Mařík spaces. A *countably paracompact space* is a space each of whose countable, open covers has a locally finite, open refinement. A space X is said to be (*weakly*) *cozero-dominated*

Received by the editors June 27, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54C50, 28C15; Secondary 54G20.
Key words and phrases. Mařík space, countably paracompact, locally compact, absolute, Michael line, Baire measure, Borel measure, regular Borel extension, measure compact.

if for each decreasing sequence $\{F_n\}_{n\in\omega}$ of (regular) closed sets in X with $\bigcap_{n\in\omega} F_n = \varnothing$ (we write this situation symbolically as $F_n \downarrow \varnothing$), there exists a sequence $\{U_n\}_{n\in\omega}$ of cozero-sets in X such that $F_n \subset U_n$ for each $n \in \omega$ and $U_n \downarrow \varnothing$. Here, a *regular closed set* is a set that is the closure of its interior. If cozero-sets are relaxed to Baire sets, then X is said to be (*weakly*) *Baire-dominated.* A Baire measure μ on a space is called τ*-additive* if, whenever a net $\{Z_\alpha\}_{\alpha\in A}$ of zero-sets decreases to a zero-set Z, $\mu(Z) = \inf\{\mu(Z_\alpha) \colon \alpha \in A\}$. A space X is called *measure compact*, abbreviated as MC, if every Baire measure on X is τ-additive (cf. [29, §8]). The relationship of these spaces to a Mařík space is summarized as the following:

$$\begin{tikzcd}[column sep=small]
\text{(1) normal and countably paracompact} \arrow[dr] & & \text{(2) countably compact} \arrow[dl] & \\
& \text{(3) cozero-dominated} \arrow[d] & & \\
& \text{(4) Baire-dominated} \arrow[dl] \arrow[dr] & & \text{(5) MC} \arrow[dl] \\
\text{(6) weakly Baire-dominated} & & \text{(7) Mařík} &
\end{tikzcd}$$

The implication $(1) \to (3)$ follows from [6, 5.2.2]. $(2) \to (3) \to (4) \to (6)$ are obvious. $(1) \to (7)$ is a classical result of Mařík [18] and is the origin of the name of a Mařík space. $(3) \to (7)$ was proved by Bachman-Sultan [4], and $(4) \to (7)$ is a recent result of Adamski [2]. $(5) \to (7)$ is due to Knowles [15].

Wheeler's questions which we now answer are the following; in his papers [28] and [29], the symbol $(*)$, $((**))$ was used to denote the property of being (weakly) cozero-dominated.

A [29, Problem 8.12]. Is there a locally compact, MC space which is not paracompact?

B [28, Q6]; [29, Problem 9.10]. Is there an MC space which is not (weakly) cozero-dominated?

C [28, p. 95]. Is Michael's product space (see §2 below) cozero-dominated or weakly cozero-dominated?

D [28, Q5]; [29, Problem 9.15]. Is every countably paracompact space a Mařík space?

A *perfect map* is a closed, continuous map such that the inverse image of each point is compact. A perfect map is called *irreducible* if it carries a proper closed subset to a proper subset. A space is called *extremally disconnected* if the closure of every open set is open. Each space X is known to be the image of a unique extremally disconnected space $E(X)$, called the *absolute* of X, under a perfect irreducible map. For details, see [31].

E [28, Q7]. Is it true that X is a Mařík space if and only if $E(X)$ is a Mařík space?

The answers to A and B are positive, and the answers to C, D, and E are negative.

From now on, $|A|$ denotes the cardinality of a set A. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a topological space, a set of ordinals has the order topology. Let ω (ω_1) denote the first infinite (uncountable) ordinal, and let $\mathfrak{c} = 2^{\omega}$. If α is a cardinal, then the inequality $\alpha < m_r$ means that α is not real-valued measurable. For a space X, $Z(X)$ $(\mathrm{Coz}(X))$ is the set of all zero-(cozero-)sets of X, $C(X)$ is the set of all real-valued, continuous functions on X, and $C^*(X)$ is the set of all bounded functions in $C(X)$. A zero-set of the form $f^{-1}(0)$, where $f \in C(X)$, is denoted by $Z(f)$. The letter N is used for the set of natural numbers.

Our terminology and notation follow [6] and [13]. For recent surveys of Baire measures and of Borel measures, the reader is referred to [29] and [10], respectively.

2. MC spaces which are not Baire-dominated

In this section, we give three examples of MC spaces which are not Baire-dominated. They have different features; the first one is locally compact and needs no set theoretic axioms beyond ZFC unlike others; the second one is Michael's product space, which is first countable and submetrizable; and the last one is weakly cozero-dominated. Before proceeding to examples, we show that a nonnormal space yields a space which is not weakly Baire-dominated. Recall that a subspace S of a space X is *C-embedded* in X if every $f \in C(S)$ can be extended continuously over X.

Theorem 2.1. *For each nonnormal space X, there exists a space Y which is not weakly Baire-dominated and which is the countable union of closed, C-embedded copies of X. Moreover, if X is locally compact, then so is Y.*

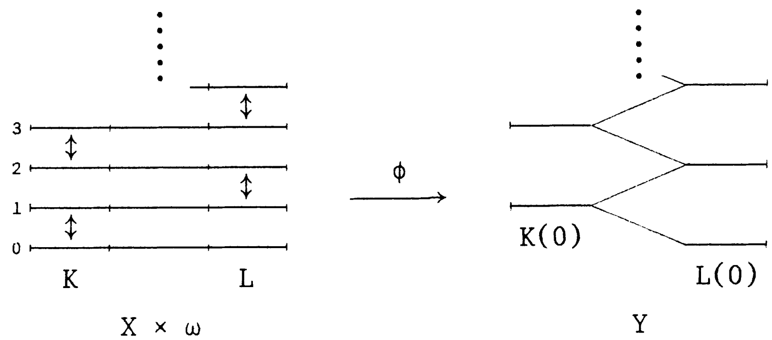

FIGURE 1

Proof. Let X be a nonnormal space. Then there exists a pair K, L of disjoint closed sets for which every zero-set containing K meets L. In fact, take disjoint closed sets K' and L' which can not be separated by disjoint open sets; then, if there is $Z \in Z(X)$ such that $K' \subset Z$ and $Z \cap L' = \varnothing$, then the pair L', Z

is a desired one. Define Y to be the quotient space obtained from the product $X \times \omega$ by identifying points $(k, 2n)$ with $(k, 2n+1)$ for each $k \in K$ and $n \in \omega$, and points $(l, 2n+1)$ with $(l, 2n+2)$ for each $l \in L$ and $n \in \omega$. Let $\phi : X \times \omega \to Y$ be the quotient map. For each $n \in \omega$ and each $A \subset X$, let $A(n) = \phi(A \times \{n\})$ (see Figure 1). Then $Y = \bigcup_{n\in\omega} X(n)$, and each $X(n)$ is closed, C-embedded in Y, and homeomorphic to X. To prove that Y is not weakly Baire-dominated, let $\mathscr{F} = \{Z \cap L : K \subset Z \in Z(X)\}$. Then $\mathscr{F}$ is a filter base closed under countable intersections. For each $n \in \omega$ and each $B \subset Y$, let $B\langle n\rangle = p_n^{-1}(B \cap L(n))$, where p_n is an embedding from X onto $X(n)$. As usual, $S \triangle T = (S - T) \cup (T - S)$.

Claim 1. For each $B \in \mathrm{Ba}(Y)$ and each $0 < n < \omega$, there exists $F \in \mathscr{F}$ such that $(B\langle 0\rangle \triangle B\langle n\rangle) \cap F = \varnothing$.

Proof. Since $B\langle 0\rangle \triangle B\langle n\rangle \subset \bigcup_{i<n}(B\langle i\rangle \triangle B\langle i+1\rangle)$, it suffices to prove that for each $i < n$, there exists $F \in \mathscr{F}$ such that $(B\langle i\rangle \triangle B\langle i+1\rangle) \cap F = \varnothing$. In case i is odd, $B\langle i\rangle \triangle B\langle i+1\rangle = \varnothing$, so we prove only the case $i = 0$; other even cases are similar. Since

$$\mathscr{S} = \{S \subset Y : (S\langle 0\rangle \triangle S\langle 1\rangle) \cap F = \varnothing \text{ for some } F \in \mathscr{F}\}$$

is a σ-algebra, it is sufficient to show that $Z(Y) \subset \mathscr{S}$. To do this, let $Z = Z(f) \in Z(Y)$. Define $f_i = f \circ p_i$ for $i = 0, 1$. Since $f_0|K = f_1|K$,

$$K \subset Z(f_0 - f_1).$$

Thus, if we set $F = Z(f_0 - f_1) \cap L$, then $F \in \mathscr{F}$. Since $(Z(f_0) \triangle Z(f_1)) \cap Z(f_0 - f_1) = \varnothing$ and $Z\langle i\rangle = Z(f_i) \cap L$, $(Z\langle 0\rangle \triangle Z\langle 1\rangle) \cap F = \varnothing$, and hence $Z \in \mathscr{S}$. □

Claim 2. The space Y is not weakly Baire-dominated.

Proof. For each $n \in \omega$, let $D_n = \bigcup_{i>n} X(i)$. Then D_n is regular closed in Y and $D_n \downarrow \varnothing$. Assume that there exist $B_n \in \mathrm{Ba}(Y)$ such that $D_n \subset B_n$ and $B_n \downarrow \varnothing$. For each $n \in \omega$, by Claim 1, there exists $F_n \in \mathscr{F}$ such that

$$(B_n\langle 0\rangle \triangle B_n\langle n+1\rangle) \cap F_n = \varnothing.$$

Since $L(n+1) \subset D_n \subset B_n$, $B_n\langle n+1\rangle = L$, so $F_n \subset B_n\langle 0\rangle$, and hence $F_n(0) \subset B_n \cap L(0)$. Consequently, $\bigcap_{n\in\omega} B_n \supset \bigcap_{n\in\omega} F_n(0) \neq \varnothing$, which is a contradiction. □

Finally, assume that X is locally compact. Since ϕ is a perfect map, it follows from [6, 3.7.21] that Y is then locally compact. Hence the proof is complete. □

The following theorem due to Okada-Okazaki [22] shows that, in Theorem 2.1, if X is MC, then so is Y. A subspace S of a space X is said to be *Baire-embedded* in X if for each $B \in \mathrm{Ba}(S)$, there exists $A \in \mathrm{Ba}(X)$ with $B = A \cap S$. Every C-embedded subspace is Baire-embedded (cf. [5, 8.7]).

Okada-Okazaki's theorem. *If* $X = \bigcup_{n\in\omega} X_n$, *and if each* X_n *is MC and is Baire-embedded in* X, *then* X *is MC.*

Remark 2.2. If there exists a nonparacompact MC space X, then there exists a nonnormal MC space. Consider the product of X with its Stone-Čech compactification βX. By [20, Theorem 5.3], $X \times \beta X$ is MC, while it follows from [6, 5.1.38] that it is not normal.

The preceding theorems and remark show that a positive answer to the question A answers B positively. Thus the following example provides answers to both of the questions.

Example 2.3. There exists a locally compact, nonnormal, MC space X.

Proof. Let D be a discrete space of cardinality ω_1. Consider the product $\beta D \times (\omega + 1)$ and its subspace

$$X = (\beta D \times (\omega + 1)) - ((\beta D - D) \times \{\omega\}).$$

Clearly X is locally compact. If we set $K = (\beta D - D) \times \omega$ and $L = D \times \{\omega\}$, then K and L are disjoint closed in X but cannot be separated by disjoint open sets, and hence X is not normal. Define $T = D \times (\omega + 1)$; then $X = T \cup K$. It is easily checked that T is MC and is Baire-embedded in X. On the other hand, being σ-compact, K is also MC and is Baire-embedded in X by [5, 9.11]. Hence it follows from Okada-Okazaki's theorem that X is MC. □

Remarks 2.4. The space X defined above itself is cozero-dominated. To see this, let $\{F_n\}_{n\in\omega}$ be a sequence of closed sets in X such that $F_n \downarrow \varnothing$. Since each $\beta D \times \{m\}$, $m \in \omega$, is compact, we may assume that $F_n \cap (\beta D \times \{m\}) = \varnothing$ if $m \leq n$. For each $n \in \omega$, define

$$G_n = F_n \cup \left(\bigcup_{n<m<\omega} (\beta D \times \{m\}) \right).$$

Then $G_n \in \mathrm{Coz}(X)$, $F_n \subset G_n$, and $G_n \downarrow \varnothing$. The construction of X was inspired by the argument used by Kato in the proof of his theorem [14, Theorem I].

The second example is Michael's product space $M \times P$. The letters R, Q, and P are used to denote the real numbers, rational numbers, and irrational numbers, respectively, and, unless otherwise stated, are assumed to have the usual topologies inherited from R. The *Michael line* M is the set R topologized by isolating the points of P and leaving the points of Q with their usual neighborhoods. As was proved by Michael (cf. [6, 5.1.32]), $M \times P$ is not normal. On the other hand, Moran proved in [20] that $M \times P$ is MC when $\mathfrak{c} < m_r$. Therefore, by Theorem 2.1, we can make from $M \times P$ an MC space which is not weakly Baire-dominated. Here, in response to the question C, we prove the following theorem:

Theorem 2.5. *Michael's product space* $M \times P$ *is not weakly Baire-dominated.*

The proof is rather difficult and requires the following two theorems, which may be of some interest in their own right. For a space X, let $\mathscr{C}_1(X)$ $(\mathscr{C}_2(X))$

denote the family of all subsets of first (second) category in X, and define $\mathscr{C}_0(X)$ to be the family of all subsets A of X satisfying that every nonempty open set U in X contains a nonempty open subset V such that $V \cap A$ or $V - A$ is in $\mathscr{C}_1(X)$. When no confusion can arise, we shall write $\mathscr{C}_i$ instead of $\mathscr{C}_i(X)$, $i = 0, 1, 2$.

Theorem 2.6. *If X is a hereditarily Lindelöf space, then $\mathscr{C}_0(X)$ is a σ-algebra.*

Proof. Since $V \cap (X - A) = V - A$ and $V - (X - A) = V \cap A$, $A \in \mathscr{C}_0$ implies $X - A \in \mathscr{C}_0$. To complete the proof, suppose that $\{A_n\}_{n\in\omega} \subset \mathscr{C}_0$ is given, and let $A = \bigcup_{n\in\omega} A_n$. To show that $A \in \mathscr{C}_0$, fix any nonempty open set U in X. We have to prove that there exists a nonempty open set $V \subset U$ such that $V \cap A \in \mathscr{C}_1$ or $V - A \in \mathscr{C}_1$. If there exists a nonempty open set $V \subset U$ such that $V - A_n \in \mathscr{C}_1$ for some $n \in \omega$, then $V - A \in \mathscr{C}_1$ since $V - A \subset V - A_n$. So suppose that for each $n \in \omega$ and each nonempty open set $V \subset U$, $V - A_n \in \mathscr{C}_2$. Then, since $A_n \in \mathscr{C}_0$, each nonempty open set $V \subset U$ has a nonempty open subset W such that $W \cap A_n \in \mathscr{C}_1$. For each $n \in \omega$, define $\mathscr{W}_n$ to be the family of all nonempty open sets $W \subset U$ such that $W \cap A_n \in \mathscr{C}_1$, and let $W_n = \bigcup\{W : W \in \mathscr{W}_n\}$. Then W_n is dense in U, and hence $U - W_n \in \mathscr{C}_1$. On the other hand, X being hereditarily Lindelöf, there exists a countable subfamily $\{W_{ni}\}_{i\in\omega}$ of $\mathscr{W}_n$ with $W_n = \bigcup_{i\in\omega} W_{ni}$. Since each $W_{ni} \cap A_n$ is in $\mathscr{C}_1$, $W_n \cap A_n = \bigcup_{i\in\omega}(W_{ni} \cap A_n) \in \mathscr{C}_1$. Thus $U \cap A_n$ is contained in the union of two sets $U - W_n$ and $W_n \cap A_n$ in $\mathscr{C}_1$, so $U \cap A_n \in \mathscr{C}_1$. Consequently, $U \cap A = \bigcup_{n\in\omega}(U \cap A_n) \in \mathscr{C}_1$, which completes the proof. □

For each $A \subset M \times P$, define $A_\triangle = \{x \in P : (x, x) \in A\}$.

Theorem 2.7. *If $B \in \mathrm{Ba}(M \times P)$, then $B_\triangle \in \mathscr{C}_0(P)$.*

Proof. By the preceding theorem $\mathscr{C}_0(P)$ is a σ-algebra. Since $\{B_\triangle : B \in \mathrm{Ba}(M \times P)\}$ is a σ-algebra generated by a family $\mathscr{G} = \{G_\triangle : G \in \mathrm{Coz}(M \times P)\}$, it suffices to prove that $\mathscr{G} \subset \mathscr{C}_0(P)$. Suppose not, and let $G_\triangle \in \mathscr{G} - \mathscr{C}_0(P)$. Then there exists a nonempty open set U of P such that for each nonempty open set $V \subset U$, $V \cap G_\triangle \in \mathscr{C}_2$ and $V - G_\triangle \in \mathscr{C}_2$. Since G is a cozero-set, there exists an increasing sequence $\{G_i\}_{i\in\omega} \subset \mathrm{Coz}(M \times P)$ such that $\mathrm{cl}_{M\times P} G_i \subset G_{i+1}$, $i \in \omega$, and $G = \bigcup_{i\in\omega} G_i$. For each $i, j \in \omega$, define

$$A_{ij} = \{x \in U \cap G_\triangle : \{x\} \times B_j(x) \subset G_i\},$$

where $B_j(x) = \{y \in P : |x - y| < 1/2^j\}$. If $x \in U \cap G_\triangle$, then, since $(x, x) \in G$, $\{x\} \times B_j(x) \subset G_i$ for some $i, j \in \omega$, so $x \in A_{ij}$. Hence $U \cap G_\triangle = \bigcup_{i,j\in\omega} A_{ij}$. By the hypothesis of U, $U \cap G_\triangle \in \mathscr{C}_2$, so there exist $k, l \in \omega$ such that $A_{kl} \in \mathscr{C}_2$. Therefore, there exists a nonempty open set $V \subset U$ such that

$$(1) \qquad V \cap A_{kl} \text{ is dense in } V.$$

Fix such k, l, V, and define

$$B_m = \{x \in V - G_\triangle : (\{x\} \times B_m(x)) \cap G_{k+1} = \varnothing\}$$

for each $m \in \omega$. If $x \in V - G_\triangle$, then, since $(x, x) \notin G$, $(\{x\} \times B_m(X)) \cap G_{k+1} = \varnothing$ for some $m \in \omega$, so $x \in B_m$. Hence $V - G_\triangle = \bigcup_{m\in\omega} B_m$. By the hypothesis of U again, $V - G_\triangle \in \mathscr{C}_2$, so there exist $n \in \omega$ and a nonempty open set $W \subset V$ such that

$$(2) \qquad W \cap B_n \quad \text{is dense in } W.$$

Define $s = \max\{l, n\}$, and pick a point $q \in (\mathrm{cl}_R W) \cap Q$. Then, by (1) and (2), $q \in (\mathrm{cl}_R A_{kl}) \cap (\mathrm{cl}_R B_n)$. Since $q \in \mathrm{cl}_R A_{kl}$,

$$\{q\} \times B_s(q) \subset \mathrm{cl}_{M\times P}\left(\bigcup\{\{x\} \times B_s(x) : x \in A_{kl}\}\right) \subset \mathrm{cl}_{M\times P} G_k .$$

On the other hand, since $q \in \mathrm{cl}_R B_n$,

$$\{q\} \times B_s(q) \subset \mathrm{cl}_{M\times P}\left(\bigcup\{\{x\} \times B_s(x) : x \in B_n\}\right) \subset (M \times P) - G_{k+1} .$$

This contradicts the fact that $\mathrm{cl}_{M\times P} G_k \subset G_{k+1}$. Hence $\mathscr{G} \subset \mathscr{C}_0(P)$, which completes the proof. □

Corollary 2.8. *If $A \in \mathrm{Ba}(M)$, then $A \cap P \in \mathscr{C}_0(P)$.*

Proof. Since $A \in \mathrm{Ba}(M)$, $B = A \times P \in \mathrm{Ba}(M \times P)$. Hence it follows from Theorem 2.7 that $B_\triangle = A \cap P \in \mathscr{C}_0(P)$. □

Proof of Theorem 2.5.

Claim 1. There exists a sequence $\{X_n\}_{n\in\omega}$ of subsets of P such that $X_n \downarrow \varnothing$ and for each $n \in \omega$ and each nonempty open set U of P, $U \cap X_n \in \mathscr{C}_2(P)$.

Proof. By Bernstein's theorem (cf. [15, §40]), there exists a partition $\{Y_n\}_{n\in\omega}$ of P such that $|K \cap Y_n| = \mathfrak{c}$ for each $n \in \omega$ and each uncountable closed set K of P. Define $X_n = \bigcup_{k\geq n} Y_k$, $n \in \omega$. Then, obviously $X_n \downarrow \varnothing$. For our end, let $n \in \omega$ and let U be a nonempty open set of P. If $U \cap X_n \in \mathscr{C}_1$, then $U \cap Y_n \in \mathscr{C}_1$ since $Y_n \subset X_n$, so there exists a sequence $\{D_i\}_{i\in\omega}$ of nowhere dense closed subsets of P with $U \cap Y_n \subset \bigcup_{i\in\omega} D_i$. Let $E = U - \bigcup_{i\in\omega} D_i$. Since U is nonempty open, $U \in \mathscr{C}_2$, and hence so is E. Therefore E is an uncountable Borel set of P. Hence it follows from [15, §37, Theorem 3] that E contains a copy of the Cantor set C. By the property of Y_n, $|C \cap Y_n| = \mathfrak{c}$, while

$$C \cap Y_n \subset (U \cap Y_n) - \bigcup_{i\in\omega} D_i \subset \left(\bigcup_{i\in\omega} D_i\right) - \left(\bigcup_{i\in\omega} D_i\right) = \varnothing,$$

a contradiction. □

For each $n \in \omega$, define

$$F_n = \mathrm{cl}_{M\times P}\left(\bigcup\{\{x\} \times B_n(x) : x \in X_n\}\right),$$

where $B_n(x)$ is the same as in the proof of Theorem 2.7. Then each F_n is regular closed in $M \times P$, and it is easily checked that $F_n \downarrow \varnothing$. To show that $M \times P$ is not weakly Baire-dominated, let $\{B_n\}_{n\in\omega}$ be a sequence in $\mathrm{Ba}(M \times P)$ such that $F_n \subset B_n$ for each $n \in \omega$. We have to prove that $\bigcap_{n\in\omega} B_n \neq \varnothing$.

Claim 2. Suppose that X is a subset of P such that for each nonempty open set U of P, $U \cap X \in \mathscr{C}_2(P)$, and B is a Baire set of $M \times P$ containing $\{(x, x) : x \in X\}$. Then $P - B_\triangle \in \mathscr{C}_1(P)$.

Proof. By Theorem 2.7, $B_\triangle \in \mathscr{C}_0(P)$. Since $X \subset B_\triangle$, $U \cap B_\triangle \in \mathscr{C}_2$ for each nonempty open set U of P. Hence, by the definition of $\mathscr{C}_0(P)$, for each nonempty open set U of P, there exists a nonempty open set $V \subset U$ such that $V - B_\triangle \in \mathscr{C}_1$. Define $\mathscr{V}$ to be the family of all nonempty open sets V of P such that $V - B_\triangle \in \mathscr{C}_1$, and let $W = \bigcup\{V : V \in \mathscr{V}\}$. Then W is open and dense in P, so $P - W$ is nowhere dense in P. On the other hand, P being hereditarily Lindelöf, there exists a countable subfamily $\{V_i\}_{i\in\omega}$ of $\mathscr{V}$ with $W = \bigcup_{i\in\omega} V_i$. Since $V_i - B_\triangle \in \mathscr{C}_1$, $W - B_\triangle = \bigcup_{i\in\omega}(V_i - B_\triangle) \in \mathscr{C}_1$. Since $P - B_\triangle \subset (P - W) \cup (W - B_\triangle)$, $P - B_\triangle \in \mathscr{C}_1$, thus proving the claim. □

Since $\{(x, x) : x \in X_n\} \subset F_n \subset B_n$, it follows from Claim 2 that $P - (B_n)_\triangle \in \mathscr{C}_1$. Thus $(\bigcap_{n\in\omega} B_n)_\triangle = \bigcap_{n\in\omega}(B_n)_\triangle \neq \varnothing$, and hence $\bigcap_{n\in\omega} B_n \neq \varnothing$. The proof of Theorem 2.5 is now completed. □

The following corollary (to the proof of Theorem 2.5) will be used in §4.

Corollary 2.9. *Suppose that X is a subset of P such that for each nonempty open set U of P, $U \cap X \in \mathscr{C}_2(P)$, and $X \subset A \in \mathrm{Ba}(M)$. Then $P - A \in \mathscr{C}_1(P)$.*

Proof. Apply Claim 2 in the proof of Theorem 2.5 by putting $B = A \times P$. Then $P - B_\triangle = P - (A \cap P) = P - A \in \mathscr{C}_1(P)$. □

The third example needs Martin's axiom plus the negation of the continuum hypothesis, abbreviated as $\mathrm{MA} + \neg\mathrm{CH}$, from which $\mathfrak{c} < m_r$ is deduced. Under this assumption, there exist many examples of nonparacompact MC spaces. For example, every nonmetrizable, normal, Moore space of cardinality $\leq \mathfrak{c}$ is this case (cf. [25, §IV]); however, such a space is countably paracompact by itself. Perhaps the most interesting one is the nonnormal space N^{ω_1}, the product of ω_1 many copies of a countable discrete space N. It was proved by Fremlin in [7] that N^{ω_1} is MC under $\mathrm{MA} + \neg\mathrm{CH}$. The space N^{ω_1} has the following properties:

Theorem 2.10. *For each $\lambda > \omega$, N^λ is weakly cozero-dominated but not Baire-dominated.*

Proof. If B is a regular closed set or a Baire set of N^λ, then, by [23, Theorem 3] and [24, Theorem 2.3], it is a set of the form $\pi_\Lambda^{-1}(\pi_\Lambda(B))$ for some countable subset Λ of λ, where π_Λ is the projection from N^λ to N^Λ. Hence the first assertion is easily proved. To prove that N^λ is not Baire-dominated, let

$$F_n = \{f \in N^\lambda : \text{for each } i \leq n, |\{\alpha \in \lambda : f(\alpha) = i\}| \leq 1\}$$

for each $n \in N$. Then F_n is closed in N^λ and $F_n \downarrow \varnothing$. Assume that there exists a sequence $\{B_n\}_{n\in N} \subset \mathrm{Ba}(N^\lambda)$ such that $F_n \subset B_n$ and $B_n \downarrow \varnothing$. Then there exists a countable subset M of λ such that for each $n \in N$, $B_n =$

$\pi_M^{-1}(\pi_M(B_n))$. If we choose a bijection $g : M \to N$, then $g \in \bigcap_{n\in N} \pi_M(F_n) \subset \bigcap_{n\in N} \pi_M(B_n)$, and hence $\bigcap_{n\in N} B_n \neq \varnothing$, which is a contradiction. □

We have been unable to decide if N^{ω_1}, or more generally N^{λ}, is a Mařík space without MA+¬CH. If regularity of the Borel extension is negligible, then we have a much stronger result. To show this, we call a space X a *quasi-Mařík space* if each Baire measure on X admits an extension to a (not necessarily regular) Borel measure on X. For a space X, υX denotes the Hewitt realcompactification of X.

Theorem 2.11. *Assume that X is a quasi-Mařík space and $X \subset Y \subset \upsilon X$. Then Y is a quasi-Mařík space.*

Proof. Let μ be a Baire measure on Y. As is well known, every $B \in \mathrm{Ba}(X)$ extends to a unique $B^{\upsilon} \in \mathrm{Ba}(\upsilon X)$ in such a manner that if $\{B_n\}_{n\in\omega}$ is disjoint, then so is $\{B_n^{\upsilon}\}_{n\in\omega}$. Therefore, if we define $\mu_X(B) = \mu(B^{\upsilon} \cap Y)$ for $B \in \mathrm{Ba}(X)$, then μ_X is a Baire measure on X. By the assumption, μ_X extends to a Borel measure ν_X on X. For each $A \in \mathrm{Bo}(Y)$, define $\nu(A) = \nu_X(A \cap X)$. Then ν is a Borel extension of μ. □

Corollary 2.12. *For any family $\{X_\alpha\}_{\alpha\in\lambda}$ of metric spaces, the product $X = \Pi_{\alpha\in\lambda} X_\alpha$ is a quasi-Mařík space. In particular, N^{λ} is a quasi-Mařík space.*

Proof. Consider a Σ-product Σ of X; i.e., define

$$\Sigma = \{f \in X : |\{\alpha \in \lambda : f(\alpha) \neq g(\alpha)\}| \leq \omega\} \subset X,$$

where g is a fixed point of X. By [26, Theorem 2.2] Σ is C-embedded in X, so $X \subset \upsilon\Sigma$. By [12, Theorem 1] Σ is normal, and by [6, 5.2.9] Σ is countably paracompact. Consequently, Σ is a Mařík space, and hence it follows from Theorem 2.11 that X is a quasi-Mařík space. □

The following questions remain open.

Question 2.13. Is N^{λ} a Mařík space for each cardinal λ?

Question 2.14. Is υX a Mařík space, whenever X is?

Question 2.15. Is every quasi-Mařík space a Mařík space?

By our results, a positive answer to 2.15 answers 2.14 positively, and a positive answer to 2.14 answers 2.13 positively.

3. Non-Mařík spaces

In this section, we answer both of the questions D and E negatively. Let us begin by making criteria to check that a space is not a quasi-Mařík space. Some terms and symbols are needed. The continuous extension of $f \in C^*(X)$ over βX is denoted by f^{β}. Clearly, $Z(f) = Z(f^{\beta}) \cap X$. A measure is called *locally zero* if each point has a neighborhood of measure zero. Let μ be a Baire measure on X; then μ^{β} denotes the Baire measure on βX defined by

$\mu^\beta(B) = \mu(B \cap X)$ for $B \in \mathrm{Ba}(\beta X)$. Define $S(\mu^\beta) = \bigcap\{Z \in Z(\beta X) : \mu^\beta(Z) = \mu^\beta(\beta X)\}$, which is called the *support* of μ^β. A space X is called a *D-space* if for each discrete subspace $S \subset X$, $|S| < \mathfrak{m}_r$ (cf. [29]).

Theorem 3.1. *Let μ be a locally zero, Baire measure on a space X with $\mu(X) > 0$. Assume that there exist $Z_0 = Z(f) \in Z(X)$ such that $S(\mu^\beta) \subset Z(f^\beta)$, a continuous map ψ from Z_0 to a paracompact D-space Y, and an open cover $\mathscr{U}$ of Y satisfying the following condition*: (1) *For each $U \in \mathscr{U}$, there exists $B \in \mathrm{Ba}(X)$ such that $\psi^{-1}(U) \subset B$ and $\mu(B) = 0$. Then μ cannot be extended to any Borel measure on X.*

Proof. Let $\mathscr{S} = \{Z \in Z(\beta X) : \mu^\beta(Z) = \mu^\beta(\beta X)\}$. Then $Z_1, Z_2 \in \mathscr{S}$ implies that $Z_1 \cap Z_2 \in \mathscr{S}$. Thus, $\{Z(f^\beta) \cup Z : Z \in \mathscr{S}\}$ is a net. Since $S(\mu^\beta) \subset Z(f^\beta)$,

$$Z(f^\beta) = \bigcap\{Z(f^\beta) \cup Z : Z \in \mathscr{S}\}. \tag{2}$$

Since βX is MC, μ^β is τ-additive. This can be combined with (2) to yield that

$$\mu^\beta(Z(f^\beta)) = \inf\{\mu^\beta(Z(f^\beta) \cup Z)\} : Z \in \mathscr{S}\} = \mu^\beta(\beta X).$$

Hence, $\mu(Z_0) = \mu^\beta(Z(f^\beta)) = \mu^\beta(\beta X) = \mu(X) > 0$. Suppose that there exists a Borel extension ν of μ. For each $A \in \mathrm{Bo}(Y)$, define $\nu_Y(A) = \nu(\psi^{-1}(A))$. By condition (1), ν_Y is a locally zero, Borel measure on Y. It is known [10, 7.6 and 10.2] that every locally zero, Borel measure on a paracompact D-space is identically zero. Hence $\nu_Y(Y) = 0$, while $\nu_Y(Y) = \nu(Z_0) = \mu(Z_0) > 0$, which is a contradiction. □

Recall from [11, 6.5 and 8.7] that for each $p \in \nu X$, $A^p = \{Z \in Z(X): p \in \mathrm{cl}_{\beta X} Z\}$ is an ultrafilter in $Z(X)$ with the countable intersection property. Define a map $\mu_p : \mathrm{Ba}(X) \to \{0, 1\}$ by $\mu_p(B) = 1$ if B includes some element of A^p, and $\mu_p(B) = 0$ otherwise. Then μ_p is a Baire measure on X such that $S(\mu_p^\beta) = \{p\}$ (cf. also [10, 8.11]).

Corollary 3.2. *Assume that there exist $p \in \upsilon X - X$, $Z_0 \in Z(X)$ with $p \in \mathrm{cl}_{\beta X} Z_0$, and a closed, continuous map ψ from Z_0 to a paracompact D-space Y such that $p \notin \mathrm{cl}_{\beta X} \psi^{-1}(y)$ for each $y \in Y$. Then X is not a quasi-Mařík space.*

Proof. Let μ_p be the $\{0, 1\}$-valued Baire measure on X defined as above. Then $S(\mu_p^\beta) = \{p\} \subset Z(f^\beta)$ by our assumption. For each $y \in Y$, choose $V_y \in \mathrm{Coz}(\beta X)$ such that $\mathrm{cl}_{\beta X} \psi^{-1}(y) \subset V_y$ and $p \notin \mathrm{cl}_{\beta X} V_y$. Then, since $p \in \mathrm{cl}_{\beta X}(X - V_y)$, $\mu_p(X - V_y) = 1$, and hence $\mu_p(X \cap V_y) = 0$. Since ψ is closed, there exists an open neighborhood U_y of y in Y such that $\psi^{-1}(U_y) \subset V_y$. Put $\mathscr{U} = \{U_y : y \in Y\}$; then $\mathscr{U}$ satisfies the condition (1) in the preceding theorem. Hence, μ_p cannot be extended to any Borel measure on X. □

Corollary 3.3. *Assume that there exists $Z_0 \in Z(X)$, which is a paracompact D-space as a subspace, such that $\mathrm{cl}_{\beta X} Z_0 \cap (\upsilon X - X) \neq \varnothing$. Then X is not a quasi-Mařík space.*

Proof. This follows from Corollary 3.2 if we consider the identity of Z_0 as ψ. □

Remarks 3.4. (1) In 3.1, 3.2, and 3.3, "paracompact D-space" can be weakened to a space on which each locally zero, Borel measure is identically zero. Such a space was investigated by Gardner [9] and Adamski [1] and is now called a *weakly Borel measure complete space.* It is known that every weakly θ-refinable D-space is weakly Borel measure complete. For details, see [10].

(2) Two typical examples of non-Mařík spaces were exhibited by Wheeler in [28]; the square S^2 of the Sorgenfrey lines under $\mathfrak{c} < \mathfrak{m}_r$ and the Dieudonné Plank D. We can reconfirm that those spaces are not quasi-Mařík spaces by use of Theorem 3.1 and Corollary 3.3, respectively.

The following example, which is a cubic deformation of the Dieudonné Plank, provides a negative answer to the question D.

Example 3.5. There exists a countably paracompact space X which is not a quasi-Mařík space.

Proof. Step I. Let ω_2 be the second, uncountable, initial ordinal. Note that $\omega_2 \neq \omega^2$; ω^2 denotes the square of ω. Define

$$A = ((\omega_2 + 1) \times (\omega_2 + 1)) - \{(\omega_2, \omega_2)\} \quad \text{and}$$
$$G_A(\alpha) = \{(\alpha_1, \alpha_2) \in A : \alpha < \alpha_i \leq \omega_2,\ i = 1, 2\}$$

for $\alpha \in \omega_2$. Then A is ω_1-compact, i.e., every open cover of cardinality $\leq \omega_1$ has a finite subcover, and it is easily checked that each $f \in C(A)$ is constant on some $G_A(\alpha)$. Let us set $A_\square = A \times \omega^2$. For each $\beta \in \omega^2$, there exist uniquely $n, m \in \omega$ and $i \in \{0, 1\}$ such that $\beta = \omega n + 2m + i$. Let B be the quotient space obtained from $A_\square$ by identifying points $(\omega_2, \alpha, \omega n + 2m)$ with $(\omega_2, \alpha, \omega n + 2m + 1)$ for each $\alpha \in \omega_2$ and $n, m \in \omega$, and points $(\alpha, \omega_2, \omega n + 2m + 1)$ with $(\alpha, \omega_2, \omega n + 2m + 2)$ for each $\alpha \in \omega_2$ and $n, m \in \omega$. Then the quotient map $\phi : A_\square \to B$ is perfect. For each $\alpha \in \omega_2$ and each $n \in \omega$, let

$$G(\alpha) = \phi(G_A(\alpha) \times \omega^2) \quad \text{and}$$
$$B(n) = \phi(A \times \{\beta \in \omega^2 : \omega n \leq \beta < \omega^2\}).$$

In what follows, a subspace S of a space T is said to be *normally placed* in T if for every open set U with $S \subset U$, there exists an open set V such that $S \subset V \subset \mathrm{cl}_T V \subset U$.

Claim 1. For each $n \in \omega$, $B(n)$ is closed and normally placed in B, $B(n+1) \subset \mathrm{int}_B B(n)$, and $\bigcap_{n\in\omega} B(n) = \varnothing$.

Proof. Let U be an open set with $B(n) \subset U$, and let

$$\beta_* = \sup(\pi[A_\square - \phi^{-1}(U)]),$$

where π is the projection from $A_\square$ to ω^2. Since π is closed by [6, 3.7.10], $\beta_* < \omega n$. Define

$$V = \phi[A \times \{\beta \in \omega^2 : \beta_* + 2 \leq \beta < \omega^2\}].$$

Then V is closed in B and $B(n) \subset \mathrm{int}_B V \subset V \subset U$, and hence $B(n)$ is normally placed in B. Other assertions are obvious. □

Claim 2. Each $f \in C(B)$ is constant on some $G(\alpha)$.

Proof. For each $\beta \in \omega^2$, there exists $\alpha_\beta \in \omega_2$ such that $f \circ \phi$ is constant on $G_A(\alpha_\beta) \times \{\beta\}$. Let $\alpha = \sup\{\alpha_\beta : \beta \in \omega^2\}$. Then $\alpha < \omega_2$ and f is constant on $G(\alpha)$. □

Step II. Define $C = B \times (\omega + 1)$ and $C_\nabla = \bigcup_{n\in\omega}(B(n) \times \{n\}) \subset C$.

Claim 3. The space C is ω_1-paracompact; i.e., every open cover of cardinality $\leq \omega_1$ has a locally finite, open refinement.

Proof. In [17] Mack proved that the product of an ω_1-compact space with a metric space is ω_1-paracompact. Hence $A_\square \times (\omega+1) = A \times (\omega^2 \times (\omega+1))$ is ω_1-paracompact. If id is the identity of $\omega + 1$, then $\phi \times \mathrm{id}$ is a perfect map from $A_\square \times (\omega + 1)$ onto C. By [17, Theorem 16] again, C is ω_1-paracompact. □

Claim 4. The set C_∇ is closed and normally placed in C.

Proof. Since $\bigcap_{n\in\omega} B(n) = \varnothing$, C_∇ is closed in C. Let U be an open set with $C_\nabla \subset U$, and let $\pi_B : C \to B$ denote the projection. For each $n \in \omega$, since $B(n)$ is normally placed in B by Claim 1, there exists an open set V_n in B such that

$$B(n) \subset V_n \subset \mathrm{cl}_B V_n \subset \mathrm{int}_B B(n-1) \cap \pi_B[U \cap (B \times \{n\})],$$

where $B(-1) = B$. Define $V = \bigcup_{n\in\omega}(V_n \times \{n\})$. Then V is open in C and $C_\nabla \subset V$. Since $\bigcap_{n\in\omega} \mathrm{cl}_B V_n \subset \bigcap_{n\in\omega} B(n-1) = \varnothing$, $\mathrm{cl}_C V \subset U$. □

Step III. Let D be the set $\omega_1 + 1$ with the topology obtained from the order topology by making all points of ω_1 isolated. Define

$$X = (C \times D) - ((C - C_\nabla) \times \{\omega_1\}),$$

topologized as a subspace of $C \times D$. We show that X is the desired example. Let $D_0 = D - \{\omega_1\}$.

Claim 5. The space X is countably paracompact.

Proof. Let $\{F_n\}_{n\in\omega}$ be a sequence of closed sets in X with $F_n \downarrow \varnothing$. By [6, 5.2.1] it suffices to find open sets W_n in X such that $F_n \subset W_n$ and $\mathrm{cl}_X W_n \downarrow \varnothing$. In case $\bigcap_{n\in\omega} \mathrm{cl}_{C\times D} F_n \neq \varnothing$, there exists a closed set $F \subset C$ with $\bigcap_{n\in\omega} \mathrm{cl}_{C\times D} F_n = F \times \{\omega_1\}$. Since $F \cap C_\nabla = \varnothing$, by Claim 4 there exists an open set $H \subset C$ such that $F \subset H \subset \mathrm{cl}_C H \subset C - C_\nabla$. Let $E = \mathrm{cl}_C H \times D_0$. Since $\mathrm{cl}_C H$ is countably paracompact by Claim 3 and D_0 is discrete, E is countably paracompact. Thus there exist open sets U_n' in E such that $F_n \cap E \subset U_n'$ and

$\mathrm{cl}_E U_n' \downarrow \varnothing$. For each $n \in \omega$, define $U_n = U_n' \cap (H \times D_0)$. Then U_n is open in X and

$$(1) \qquad F_n \cap (H \times D_0) \subset U_n \quad \text{and} \quad \mathrm{cl}_X U_n \downarrow \varnothing.$$

In case $\bigcap_{n\in\omega} \mathrm{cl}_{C\times D} F_n = \varnothing$, define $U_n = H = \varnothing$, $n \in \omega$. Observe that U_n and H then also satisfy (1). Next, for each $n \in \omega$, let $F_n' = \mathrm{cl}_{C\times D} F_n - (H \times D)$. Then F_n' is closed in $C \times D$ and $F_n' \downarrow \varnothing$. We show that there exists a sequence $\{V_n\}_{n\in\omega}$ of open sets in $C \times D$ satisfying that

$$(2) \qquad F_n' \subset V_n \quad \text{and} \quad \mathrm{cl}_{C\times D} V_n \downarrow \varnothing.$$

Since C is countably paracompact, there exist open sets $I_n \subset C$ such that

$$F_n' \cap (C \times \{\omega_1\}) \subset I_n \times \{\omega_1\} \quad \text{and} \quad \mathrm{cl}_C I_n \downarrow \varnothing.$$

Define $F' = \bigcup_{n\in\omega}(F_n' - (I_n \times D))$. Then F' is closed in $C \times D$ and $F' \cap (C \times \{\omega_1\}) = \varnothing$. For each $\xi \in D_0$, let J_ξ be the union of all open sets G in C such that $(G \times (\xi, \omega_1]) \cap F' = \varnothing$, where $(\xi, \omega_1] = \{\eta \in D : \xi < \eta \le \omega_1\}$. Then $\mathscr{J} = \{J_\xi : \xi \in D_0\}$ is an open cover of C such that $J_\xi \subset J_{\xi'}$ if $\xi < \xi'$. By Claim 3 and [17, Theorem 5], there exists a locally finite open refinement $\{K_\xi : \xi \in D_0\}$ of $\mathscr{J}$ such that $\mathrm{cl}_C K_\xi \subset J_\xi$ for each $\xi \in D_0$. Let $L = \bigcup_{\xi\in D_0}(K_\xi \times (\xi, \omega_1])$. Then L is an open set in $C \times D$ such that

$$C \times \{\omega_1\} \subset L \subset \mathrm{cl}_{C\times D} L \subset (C \times D) - F'.$$

Since $C \times D_0$ is countably paracompact, there exist open sets $O_n \subset C \times D_0$ such that

$$F_n' \cap (C \times D_0) \subset O_n \quad \text{and} \quad \mathrm{cl}_{C\times D_0} O_n \downarrow \varnothing.$$

For each $n \in \omega$, let $V_n = (I_n \times D) \cup (O_n - \mathrm{cl}_{C\times D} L)$. Then V_n's are open in $C \times D$ and satisfy (2). Finally, define $W_n = U_n \cup (V_n \cap X)$ for each $n \in \omega$. Then W_n is open in X, and it follows from (1) and (2) that $F_n \subset W_n$ and $\mathrm{cl}_X W_n \downarrow \varnothing$, as required. □

Recall that $X \subset B \times (\omega + 1) \times D$. In the rest, we denote a point of X by a triplet, such as (b, m, η), of points of B, $\omega + 1$, and D. For each $\alpha \in \omega_2$, $n \in \omega$, and $\xi \in D_0$, let

$$G(\alpha, n, \xi) = \{(b, m, \eta) \in X : b \in G(\alpha), n < m \le \omega,\ \xi < \eta \le \omega_1\}.$$

We add an ideal point x_∞ to X, and define a neighborhood base of x_∞ by $\{\{x_\infty\} \cup G(\alpha, n, \xi) : \alpha \in \omega_2, n \in \omega, \xi \in D_0\}$.

Claim 6. The point x_∞ is in υX.

Proof. By [5, 1.16], it suffices to prove that X is C-embedded in $X \cup \{x_\infty\}$. To do this, let $f \in C(X)$. For each $n \le \omega$ and each $\xi \in D_0$, by Claim 2 there exists $\alpha_{n\xi} \in \omega_2$ such that f takes on the constant value $r_{n\xi}$ on $\{(b, n, \xi) : b \in G(\alpha_{n\xi})\}$. Let $\alpha_* = \sup\{\alpha_{n\xi} : n \le \omega, \xi \in D_0\}$; then $\alpha_* < \omega_2$. For each $n \in \omega$, pick $b_n \in G(\alpha_*) \cap B(n)$. Since each G_δ in D is open, there exists $\xi_n \in D_0$ such that f is constant on $\{(b_n, n, \eta): \xi_n < \eta \leq \omega_1\}$. Let $\xi_* = \sup\{\xi_n : n \in \omega\}$. Then, for each $n \in \omega$, $\xi_* < \xi < \xi' < \omega_1$ imply that $r_{n\xi} = r_{n\xi'}$. We denote this constant value by r_n. Pick $b' \in G(\alpha_*)$. Then, for each ξ with $\xi_* < \xi < \omega_1$,

$$\begin{aligned} r_{\omega\xi} &= f((b', \omega, \xi)) = f\left(\lim_{n\to\infty}(b', n, \xi)\right) \\ &= \lim_{n\to\infty} f((b', n, \xi)) = \lim_{n\to\infty} r_n, \end{aligned}$$

so define $r_\omega = \lim_{n\to\infty} r_n$. Then $f((b, n, \eta)) = r_n$ whenever $b \in G(\alpha_*)$, $n \leq \omega$, and $\xi_* < \eta \leq \omega_1$. Extend f over $X \cup \{x_\infty\}$ by setting $f(x_\infty) = r_\omega$. To check that f is continuous at x_∞, let $\varepsilon > 0$. Then there exists $n_0 \in \omega$ such that if $n > n_0$, $|r_n - r_\omega| < \varepsilon$. This implies that if $x \in G(\alpha_*, n_0, \xi_*)$, then $|f(x) - f(x_\infty)| < \varepsilon$. Hence f can be extended continuously to x_∞. □

Claim 7. The space X is not a quasi-Mařík space.

Proof. Let $Z = \pi^{-1}(\omega)$, where π is the projection from X to $\omega + 1$. Then $Z \in Z(X)$ and $x_\infty \in \mathrm{cl}_{\beta X} Z$. Define a map $\psi : Z \to D_0$ by $\psi((b, \omega, \xi)) = \xi$. Then ψ is a closed, continuous map, and for each $\xi \in D_0$, $x_\infty \notin \mathrm{cl}_{\beta X} f^{-1}(\xi)$, since $G(0, 0, \xi) \cap f^{-1}(\xi) = \varnothing$. Clearly D_0 is a paracompact D-space. Hence it follows from Claim 6 and Corollary 3.2 that X is not a quasi-Mařík space. This completes the whole proof. □

We turn to answer question E. By [28, Proposition 4.4], the absolute $E(X)$ of a countably paracompact space X is cozero-dominated and hence a Mařík space. Therefore, Example 3.5 also provides a negative answer to question E. Another counterexample is the Dieudonné Plank D. In his earlier paper [27], Wheeler proved that $E(D)$ is MC, while D is not a Mařík space. However, neither example is locally compact. Here we show that the Dieudonné Plank can easily be modified to a locally compact space.

Example 3.6. There exists a locally compact space Y which is not a quasi-Mařík space, such that $E(Y)$ is cozero-dominated and MC.

Proof. Let $\alpha D = D \cup \{\infty\}$ be the one-point compactification of a discrete space D of cardinality ω_1. Define

$$Y = (\alpha D \times (\omega + 1)) - \{(\infty, \omega)\}.$$

Clearly Y is locally compact. Let $Z = D \times \{\omega\}$. For each $B \in \mathrm{Ba}(Y)$, it can easily be checked that either $|B \cap Z| \leq \omega$ or $|Z - B| \leq \omega$. Define a Baire measure μ on Y by $\mu(B) = 0$ in the former case and $\mu(B) = 1$ in the latter case. Then there exists $y \in \upsilon Y - Y$ such that $S(\mu^\beta) = \{y\}$. Since Z is a discrete zero-set with $y \in \mathrm{cl}_{\beta X} Z$, it follows from Corollary 3.3 that Y is not a quasi-Mařík space. To see that $E(Y)$ is cozero-dominated and MC, let X be the space defined in the proof of Example 2.4. Then the natural map $\phi : X \to Y$ collapsing the set $(\beta D - D) \times \{n\}$ to the point (∞, n) for each $n \in \omega$ is perfect irreducible. Thus, by the uniqueness of the absolute, $E(X) = E(Y)$. As we have proved in 2.4 and 2.5, X is cozero-dominated and MC, and hence so is $E(Y)$ by [27, Theorem 2] and [28, Remark 4.3]. □

The space $\Psi = N \cup \mathscr{R}$, described in [11, 5I, p. 79], is a locally compact, pseudocompact space in which N is dense, and $\mathscr{R}$ is a discrete zero-set with $|\mathscr{R}| \leq \mathfrak{c}$. Since all subsets of Ψ are Borel sets, Ψ is weakly Borel measure complete if $\mathfrak{c} < m_r$. In [19], Mrówka proved that $\mathscr{R}$ can be chosen so that $|\beta\Psi - \Psi| = 1$. We use this Ψ to show that $E(Y)$ need not be a Mařík space even if Y is.

Example 3.7. Assume $\mathfrak{c} < m_r$. Then there exists a pseudocompact, locally compact, Mařík space Y for which $E(Y)$ is not a quasi-Mařík space.

Proof. Let Ψ be the space due to Mrówka stated above, and let $\beta\Psi - \Psi = \{p\}$. Define

$$S = (\beta\Psi \times (\omega + 1)) - \{(p, \omega)\},$$
$$T = ((\omega_1 + 1) \times (\omega + 1)) - \{(\omega_1, \omega)\},$$

and $X = S \oplus T$, where $\oplus$ means the topological sum. The desired space Y is the quotient space obtained from X by identifying points (p, n) with (ω_1, n) for each $n \in \omega$. Let $\phi : X \to Y$ be the quotient map. Since Ψ is pseudocompact and locally compact, so is Y. First, we show that Y is a Mařík space. Although the proof is essentially the same as the proof, due to Wheeler [28, p. 101], that his space $T\#D$ is a Mařík space, we do this in detail for the convenience of the reader. Observe that $|\beta Y - Y| = 1$, and let $\beta Y - Y = \{y\}$. Since Y is pseudocompact, $\beta Y = \upsilon Y$. Let μ be a Baire measure on Y. We have to prove that μ extends to a regular Borel measure. Let

$$r = \inf\{\mu(U) : y \in U^\beta, U \in \mathrm{Coz}(Y)\},$$

where U^β is the unique Baire set of βY $(= \upsilon Y)$ with $U = U^\beta \cap Y$. For each $B \in \mathrm{Ba}(Y)$, define $\mu_1(B) = r$ if $y \in B^\beta$, and $\mu_1(B) = 0$ otherwise. Then μ_1 is a Baire measure on Y. Define $\mu_2 = \mu - \mu_1$. Then, since $\inf\{\mu_2(U) : y \in U^\beta, U \in \mathrm{Coz}(Y)\} = 0$, μ_2 is τ-additive by [15, Theorem 2.4], and hence μ_2 has a regular Borel extension ν_2 (see [15, p. 144]). On the other hand, let

$$\mathscr{F} = \{\phi(E \times \{\omega\}) : E \text{ is a closed unbounded set of } \omega_1\}.$$

For each $A \in \mathrm{Bo}(Y)$, either A or $Y - A$ contains a set $F \in \mathscr{F}$. Define $\nu_1(A) = r$ if A contains a set $F \in \mathscr{F}$, and $\nu_1(A) = 0$ otherwise. Then ν_1 is a regular Borel extension of μ_1. Consequently, μ extends to a regular Borel measure $\nu = \nu_1 + \nu_2$ on Y. Hence Y is proved to be a Mařík space. Next, we show that $E(Y)$ is not a quasi-Mařík space. Since ϕ is perfect irreducible, $E(X) = E(Y)$, and $E(X) = E(S) \oplus E(T)$. It is known [22, Theorem 3.3 and 3, Corollary 4.13] that weakly Borel measure complete spaces are preserved by countable unions and perfect preimages. Using these results, we can check that $E(S)$ is weakly Borel measure complete. Since X is pseudocompact, so is $E(X)$ by [30, Proposition 2.5], and hence $\beta E(X) = \upsilon E(X)$. Consequently,

$$\mathrm{cl}_{\beta E(X)} E(S) \cap (\upsilon E(X) - E(X)) \neq \varnothing.$$

Hence it follows from Corollary 3.3 and Remark 3.4 that $E(X)$, and hence $E(Y)$, is not a quasi-Mařík space. □

Remark 3.8. That the space Ψ is not a quasi-Mařík space under $\mathfrak{c} < m_r$ was first observed by Adamski in [1]. This provides a negative answer to another Wheeler's question [29, Problem 9.16], whether every pseudocompact space is a Mařík space. The following question is yet unanswered.

Question 3.9. Is there a pseudocompact space which is not a (quasi-) Mařík space without assuming $\mathfrak{c} < m_r$?

4. Topological properties of Mařík spaces

Generally speaking, Mařík spaces are badly behaved under topological operations. We can, however, prove a few positive results. We begin by considering how Mařík spaces are preserved in subspaces. A subset $S \subset X$ is called a *generalized Baire set* if for each open set G with $S \subset G$, there exists $B \in \mathrm{Ba}(X)$ such that $S \subset B \subset G$.

Theorem 4.1. *Let X be a Mařík space and Y a Baire-embedded, generalized Baire set of X. Then Y is a Mařík space.*

Proof. Let μ be a Baire measure on Y. For each $A \in \mathrm{Ba}(X)$, define $\mu_X(A) = \mu(A \cap Y)$. Then μ_X is a Baire measure on X, and hence μ_X extends to a regular Borel measure ν_X on X. For each $B \in \mathrm{Bo}(Y)$, define

$$\nu(B) = \inf\{\nu_X(G) : B \subset G, G \text{ is open in } X\}.$$

Then, by [10, Proposition 3.6], ν is a regular Borel measure on Y. Since Y is a Baire-embedded, generalized Baire set, it follows from the next lemma that ν is an extension of μ. □

Lemma 4.2. *Let μ and ν be the same as above, and let Y be the generalized Baire set. Then, for each $A \in \mathrm{Ba}(X)$, $\mu(A \cap Y) = \nu(A \cap Y)$.*

Proof. Let $A \in \mathrm{Ba}(X)$. Then

$$\begin{aligned}\mu(A \cap Y) = \mu_X(A) &= \inf\{\mu_X(U) : A \subset U \in \mathrm{Coz}(X)\} \\ &= \inf\{\nu_X(U) : A \subset U \in \mathrm{Coz}(X)\} \geq \nu(A \cap Y).\end{aligned}$$

To prove the converse, let $\varepsilon > 0$. Then there exists $Z \in Z(X)$ such that $Z \subset A$ and $\mu_X(A) - \varepsilon < \mu_X(Z)$. For each open set G in X with $A \cap Y \subset G$, by the condition of Y there exists $J \in \mathrm{Ba}(X)$ such that $Y \subset J$ and $J \cap (Z - G) = \emptyset$. Define $Z_0 = Z \cap J$; then $Z_0 \in \mathrm{Ba}(X)$ and $Z_0 \subset G$. Since $Z_0 \cap Y = Z \cap Y$,

$$\mu_X(Z) = \mu_X(Z_0) = \nu_X(Z_0) \leq \nu_X(G),$$

so $\mu(A \cap Y) - \varepsilon = \mu_X(A) - \varepsilon < \nu_X(G)$. Since G and ε are arbitrary, it follows that $\mu(A \cap Y) \leq \nu(A \cap Y)$. □

Corollary 4.3. *Let X be a Mařík space and Y a cozero-set of X. Then Y is a Mařík space.*

Remarks 4.4. (1) As Wheeler mentioned in [29], the space $T\#D$ in [28, p. 101] shows that a C-embedded, regular closed subspace of a Mařík space need not be a Mařík space (see also Example 3.7).

(2) We do not know whether the assumption in Theorem 4.1 that Y is Baire-embedded can be removed.

Recall from [2] that a space X is *Baire-separated* if for each pair F_1, F_2 of disjoint closed sets, there exists $B \in \mathrm{Ba}(X)$ such that $F_1 \subset B$ and $B \cap F_2 = \varnothing$. All normal spaces are Baire-separated, but the converse is not true. For example, the space X defined in the proof of Example 2.3 is certainly the case. The proof of the following lemma is left to the reader since it is routine.

Lemma 4.5. *For a space X, the following conditions are equivalent*:

(a) *X is Baire-separated.*

(b) *For each pair F_1, F_2 of disjoint F_σ-sets of X, there exists $B \in \mathrm{Ba}(X)$ such that $F_1 \subset B$ and $B \cap F_2 = \varnothing$.*

(c) *Every F_σ-set of X is a generalized Baire set.*

(d) *Every F_σ-set of X is Baire-embedded in X.*

Theorem 4.6. *Let X be a Baire-separated, Mařík space and Y a generalized Baire set of X. Then Y is a Mařík space.*

Proof. Let μ be a Baire measure on Y. Let μ_X, ν_X, and ν be the same as in the proof of Theorem 4.1. We have to prove that ν is an extension of μ. To do this, let $B \in \mathrm{Ba}(Y)$. For each $i \in N$, by the regularity of ν, there exist a closed set F_i and an open set G_i in Y such that $F_i \subset B \subset G_i$ and

$$\nu(B) - 1/i < \nu(F_i) \leq \nu(G_i) < \nu(B) + 1/i. \tag{1}$$

Similarly, we can choose $Z_i \in Z(Y)$ and $U_i \in \mathrm{Coz}(Y)$ such that $Z_i \subset B \subset U_i$ and

$$\mu(B) - 1/i < \mu(Z_i) \leq \mu(U_i) < \mu(B) + 1/i. \tag{1'}$$

We may assume that $\{F_i\}$ and $\{Z_i\}$ are increasing and $\{G_i\}$ and $\{U_i\}$ are decreasing. For each $i \in N$, there exist a closed set E_i in X with $E_i \cap Y = F_i \cup Z_i$ and an open set H_i in X with $H_i \cap Y = G_i \cap U_i$. Since Y is a generalized Baire set, there exists $J_i \in \mathrm{Ba}(X)$ such that $Y \subset J_i$ and $J_i \cap (E_i - H_i) = \varnothing$. Let us set $J = \bigcap_{i \in N} J_i$. Then $Y \subset J \in \mathrm{Ba}(X)$, so $\mu_X(J) = \mu(Y)$. For each $i \in N$, take $K_i \in Z(X)$ such that $K_i \subset J$ and $\mu_X(J) - 1/i < \mu_X(K_i)$, and define $K = \bigcup_{i \in N} K_i$. Then, since $\mu_X(J - K) = 0$, $\mu(Y - K) = 0$. On the other hand, since

$$\nu(Y - K) \leq \nu_X(X - K_i) = \mu_X(X - K_i) < 1/i$$

for each $i \in N$, $\nu(Y - K) = 0$. Thus

$$\nu(Y - K) = \mu(Y - K) = 0. \tag{2}$$

Since $E_i \cap K \subset H_i \cap K$, by Lemma 4.5 there exists $A_i \in \mathrm{Ba}(X)$ such that $E_i \cap K \subset A_i$ and $A_i \cap (K - H_i) = \varnothing$. Then, for each $i \in N$,

$$(F_i \cup Z_i) \cap K \subset A_i \cap K \cap Y \subset (G_i \cap U_i) \cap K.$$

Define $A_* = \bigcup_{j\in N}(\bigcap_{i\geq j}(A_i \cap K \cap Y))$ and $A^* = \bigcap_{j\in N}(\bigcup_{i\geq j}(A_i \cap K \cap Y))$; then

$$\left(\bigcup_{i\in N} F_i\right) \cap K \subset A_* \subset A^* \subset \left(\bigcap_{i\in N} G_i\right) \cap K.$$

Since $\nu((\bigcup_{i\in N} F_i) \cap K) = \nu(B) = \nu((\bigcap_{i\in N} G_i) \cap K)$ by (1) and (2),

$$\begin{aligned}\nu(B) &\leq \nu(A_*) \leq \liminf \nu(A_i \cap K \cap Y)\\ &\leq \limsup \nu(A_i \cap K \cap Y) \leq \nu(A^*) \leq \nu(B),\end{aligned}$$

and hence

$$\nu(B) = \lim_{i\to\infty} \nu(A_i \cap K \cap Y).$$

Similarly,

$$\mu(B) = \lim_{i\to\infty} \mu(A_i \cap K \cap Y).$$

Since $A_i \cap K \in \mathrm{Ba}(X)$, it follows from Lemma 4.2 that $\nu(A_i \cap K \cap Y) = \mu(A_i \cap K \cap Y)$ for each $i \in N$. Consequently, $\nu(B) = \mu(B)$, which completes the proof. □

Corollary 4.7. *Let X be a Baire-separated, Mařík space, and let Y be either an F_σ-set or a Baire set of X. Then Y is a Mařík space.*

Remarks 4.8. (1) In Theorem 4.6, "generalized Baire" cannot be replaced by "open." Consider a locally compact, non-Mařík space X (see Example 3.6) as a subspace of βX.

(2) A Baire set of a Baire-separated, Mařík space need not be Baire-embedded, so Theorem 4.6 cannot be reduced to Theorem 4.1. To see this, let M be the Michael line defined in §2 and P the subspace of irrational numbers. Being paracompact, M is a Baire-separated, Mařík space, and $P \in \mathrm{Ba}(M)$. We show that P is not Baire-embedded in M. Let $\{P_1, P_2\}$ be a partition of P with the usual topology such that $U \cap P_i \in \mathscr{C}_2(P)$ for each nonempty open set $U \subset P$ and $i = 1, 2$. The existence of such a partition can be shown similarly to Claim 1 in the proof of Theorem 2.5. Since P is discrete in M, P_1 is a Baire set of P in M. Suppose that there exists $A \in \mathrm{Ba}(M)$ with $P_1 = A \cap P$. Then, since $P_1 \subset A$, it follows from Corollary 2.9 that $P - A \in \mathscr{C}_1(P)$. But $P - A = P_2 \in \mathscr{C}_2(P)$, a contradiction. This also shows that, in Lemma 4.5, "F_σ-set" cannot be replaced by "Baire-set."

We now turn to the preservation under taking unions.

Theorem 4.9. *Assume that $X = \bigcup_{n\in N} X_n$, and each X_n is a Mařík space and is a Baire-embedded, generalized Baire set of X. Then X is a Mařík space.*

Proof. Let μ be a Baire measure on X. For each $n \in N$ and each $B \in \mathrm{Ba}(X_n)$, define

$$\mu_n(B) = \inf\{\mu(U) : B \subset U \in \mathrm{Coz}(X)\}.$$

Since X_n is Baire-embedded, μ_n is a Baire-measure on X_n, and hence μ_n extends to a regular Borel measure ξ_n on X_n. For each $i \in N$, take $J_{ni} \in \mathrm{Coz}(X)$ such that $X_n \subset J_{ni}$ and $\mu(J_{ni}) < \mu_n(X_n) + 1/i$, and define $J_n =$

$\bigcap_{i\in N} J_{ni}$. Then $X_n \subset J_n \in \mathrm{Ba}(X)$ and $\mu(J_n) = \mu_n(X_n)$. For each $C \in \mathrm{Bo}(J_n)$, define $\nu_n(C) = \xi_n(C \cap X_n)$. Then ν_n is a Borel measure on J_n.

Claim 1. For each $A \in \mathrm{Ba}(X)$, $\nu_n(A \cap J_n) = \mu(A \cap J_n)$.

Proof. It is easily checked that $\nu_n(A \cap J_n) \le \mu(A \cap J_n)$. To prove the converse, let $\varepsilon > 0$. Then there exists $V \in \mathrm{Coz}(X)$ such that $A \cap X_n \subset V$ and $\mu(V) < \mu_n(A \cap X_n) + \varepsilon = \nu_n(A \cap J_n) + \varepsilon$. Let $W = (A \cap J_n) - V$. Then $\mu(W) = 0$ by the definition of J_n. Since $A \cap J_n \subset V \cup W$, $\mu(A \cap J_n) \le \mu(V \cup W) = \mu(V)$, and hence $\mu(A \cap J_n) \le \nu_n(A \cap J_n)$. □

Claim 2. For each $D \in \mathrm{Bo}(J_n)$, $\nu_n(D) = \sup\{\nu_n(F) : F \subset D$ and F is closed in $X\}$.

Proof. We first prove that ν_n is a regular measure on J_n. For this end, by [10, Proposition 6.2], it suffices to prove that for each open set G in J_n,

$$\nu_n(G) = \sup\{\nu_n(H) : H \subset G \text{ and } H \text{ is closed in } J_n\}.$$

Let $\varepsilon > 0$. Since ξ_n is regular, there exists a closed set H' in X_n such that $H' \subset G \cap X_n$ and

$$\xi_n(H') > \xi_n(G \cap X_n) - \varepsilon/2 = \nu_n(G) - \varepsilon/2. \tag{1}$$

Since X_n is a generalized Baire set, there exists $K \in \mathrm{Ba}(X)$ such that $X_n \subset K$ and $K \cap (\mathrm{cl}_X H' - G_0) = \varnothing$, where G_0 is an open set in X with $G_0 \cap J_n = G$. Take $Z \in Z(X)$ such that $Z \subset K$ and $\mu(K - Z) < \varepsilon/2$, and define $H = \mathrm{cl}_X H' \cap Z \cap J_n$. Then H is closed in J_n and $H \subset G$. Since $H' \cap Z = H \cap X_n$ and $H' - Z \subset K - Z$,

$$\begin{aligned} \xi_n(H') &= \xi_n(H' \cap Z) + \xi_n(H' - Z) \\ &\le \xi_n(H \cap X_n) + \mu(K - Z) < \nu_n(H) + \varepsilon/2. \end{aligned} \tag{2}$$

It follows from (1) and (2) that $\nu_n(H) > \nu_n(G) - \varepsilon$. Thus ν_n is proved to be regular. Let $D \in \mathrm{Bo}(J_n)$, and let $\varepsilon > 0$ again. Since ν_n is regular, there exists a closed set F' in J_n such that $F' \subset D$ and $\nu_n(F') > \nu_n(D) - \varepsilon/2$. Take $Z' \in Z(X)$ such that $Z' \subset J_n$ and $\mu(J_n - Z') < \varepsilon/2$, and define $F = F' \cap Z'$. Then F is closed in X, $F \subset D$, and by Claim 1

$$\begin{aligned} \nu_n(F) &\ge \nu_n(F') - \nu_n(J_n - Z') \\ &> (\nu_n(D) - \varepsilon/2) - \mu(J_n - Z') > \nu_n(D) - \varepsilon, \end{aligned}$$

thus proving the claim. □

To complete the proof, let $Y_n = J_n - \bigcup_{i<n} J_i$ for each $i \in N$. Then $X = \bigcup_{n\in N} Y_n$, $Y_n \in \mathrm{Ba}(X)$, and $Y_n \cap Y_m = \varnothing$ if $n \ne m$. For each $E \in \mathrm{Bo}(X)$, define $\nu(E) = \sum_{n\in N} \nu_n(E \cap Y_n)$. Then it follows from Claims 1 and 2 that ν is a regular Borel extension of μ. □

Corollary 4.10. *Assume that $\mathscr{U}$ is a locally finite cover of a space X by cozero-sets such that each $U \in \mathscr{U}$ is a Mařík space and $|\mathscr{U}| < m_r$. Then X is a Mařík space.*

Proof. By [21, Theorem 1.2], $\mathscr{U}$ has a refinement $\mathscr{V} = \bigcup_{n\in N}\mathscr{V}_n$ by cozero-sets such that each $\mathscr{V}_n = \{V_\lambda : \lambda \in \Lambda_n\}$ is discrete. For each $n \in N$, let $X_n = \bigcup_{\lambda\in\Lambda_n} V_\lambda$; then $X = \bigcup_{n\in N} X_n$ and $X_n \in \mathrm{Coz}(X)$. By Theorem 4.9, it suffices to prove that each X_n is a Mařík space. For this end, let μ be a Baire measure on X_n. Since μ is finite, there exists a countable set $M \subset \Lambda_n$ such that $\mu(V_\lambda) = 0$ if $\lambda \in \Lambda_n - M$. Define $Y_n = \bigcup_{\lambda\in M} V_\lambda$. Then $Y_n \in \mathrm{Coz}(X)$ and Y_n is a Mařík space by Theorem 4.9, so $\mu|\mathrm{Ba}(Y_n)$ extends to a regular Borel measure ν_n on Y_n. Since $|\Lambda_n - M| < m_r$, $\mu(X_n - Y_n) = 0$. Consequently, if we define $\nu(B) = \nu_n(B \cap Y_n)$ for each $B \in \mathrm{Bo}(X_n)$, then ν is a regular Borel extension of μ. □

Corollary 4.11. *Assume that* $X = \bigcup_{n\in N} X_n$ *is a Baire-separated space, and each* X_n *is a Mařík space and is either a closed set or a Baire set of* X. *Then* X *is a Mařík space.*

Proof. By the proof of Theorem 4.9, it suffices to prove that for each $n \in N$, there exist $J_n \in \mathrm{Ba}(X)$ with $X_n \subset J_n$ and a Borel measure ν_n on J_n satisfying the Claims 1 and 2. In case X_n is closed, we can define such J_n and ν_n quite similarly since X_n is then a Baire-embedded, generalized Baire set by Lemma 4.5. In case $X_n \in \mathrm{Ba}(X)$, define $J_n = X_n$. For each $i \in N$, take $K_i \in Z(X)$ such that $K_i \subset J_n$ and $\mu(K_i) > \mu(J_n) - 1/i$, and define $K = \bigcup_{i\in N} K_i$. Since K is Baire-embedded by Lemma 4.5, it follows from Theorem 4.1 that K is a Mařík space. Hence, if we consider K instead of X_n, then we can define ν_n on J_n similarly to the proof of Theorem 4.9. □

Remarks 4.12. (1) The space Y defined in the proof of Example 3.6 shows that the union of two Mařík spaces need not be a quasi-Mařík space even if one is a cozero-set and the other is a zero-set.

(2) We do not know whether the assumption in Theorem 4.9 that each X_n is a generalized Baire set can be removed. The assumption was used only to ensure the regularity of the extension ν. Thus X is a quasi-Mařík space even if each X_n is only assumed to be Baire-embedded.

Question 4.13. Let $X = Y \cup K$ be the union of a Mařík space Y with a compact space K. Then is X is Mařík space?

Question 4.14. Let $X = \bigoplus_{\lambda\in\Lambda} X_\lambda$ be the disjoint sum of Mařík spaces X_λ, $\lambda \in \Lambda$. Then is X a Mařík space even if $|\Lambda|$ is real-valued measurable?

Finally we are concerned with the preservation under maps and products. Examples 3.6 and 3.7 show that the image and the preimage of Mařík spaces under perfect maps need not be quasi-Mařík spaces, respectively. If we make some additional assumptions, then Mařík spaces are preserved under perfect maps. To show this, we need a theorem due to Bachman and Sultan [4]. Before stating their theorem, let us agree on some terminology. Let $\mathscr{L}_1 \subset \mathscr{L}_2$ be two lattices, closed under countable intersections, of subsets of a set X, and let $\mathscr{A}(\mathscr{L}_i)$ denote the smallest algebra containing $\mathscr{L}_i$, $i = 0, 1$. Then $\mathscr{L}_2$ is said to be $\mathscr{A}(\mathscr{L}_1)$*-countably paracompact* if $A_n \downarrow \varnothing$ in $\mathscr{L}_2$ implies the existence of a sequence $\{B_n\}_{n\in\omega} \subset \mathscr{A}(\mathscr{L}_1)$ such that $A_n \subset B_n$ and $B_n \downarrow \emptyset$. A (finitely additive) measure μ defined on $\mathscr{A}(\mathscr{L}_i)$ is called *$\mathscr{L}_i$-regular* if for each $B \in \mathscr{A}(\mathscr{L}_i)$, $\mu(B) = \sup\{\mu(A) : A \subset B, A \in \mathscr{L}_i\}$.

Bachman-Sultan's extension theorem. *Let $\mathscr{L}_1 \subset \mathscr{L}_2$ be the same as above. Then every finitely additive, $\mathscr{L}_1$-regular measure μ defined on $\mathscr{A}(\mathscr{L}_1)$ can be extended to a finitely additive, $\mathscr{L}_2$-regular measure ν defined on $\mathscr{A}(\mathscr{L}_2)$. If $\mathscr{L}_2$ is $\mathscr{A}(\mathscr{L}_1)$-countably paracompact and if μ is σ-additive, then so is ν.*

For a space X, $\mathrm{Ba}^\omega(X)$ $(\mathrm{Bo}^\omega(X))$ denotes the smallest algebra containing all zero-sets (closed sets) of X. It is well known (cf. [13, 10.36]) that every measure μ defined on $\mathrm{Ba}^\omega(X)$ $(\mathrm{Bo}^\omega(X))$ can be extended to a unique Baire (Borel) measure ν on X, and if μ is regular, then so is ν.

Theorem 4.15. *Let f be a continuous map from a Mařík space X onto a space Y such that for each $Z \in Z(X)$, $f(Z) \in Z(Y)$, and such that for each $y \in Y$, $f^{-1}(y)$ is relatively pseudocompact in X; i.e., each $g \in C(X)$ is bounded on $f^{-1}(y)$. Then Y is a Mařík space.*

Proof. Let μ be a Baire measure on Y. Define $\mathscr{L}_1 = \{f^{-1}(Z) : Z \in Z(Y)\}$ and $\mathscr{L}_2 = Z(X)$. Then $\mathscr{L}_1 \subset \mathscr{L}_2$ and $\mathscr{A}(\mathscr{L}_1) = \{f^{-1}(B) : B \in \mathrm{Ba}^\omega(Y)\}$. For each $B \in \mathrm{Ba}^\omega(Y)$, define $\lambda(f^{-1}(B)) = \mu(B)$. Then λ is a $\mathscr{L}_1$-regular measure defined on $\mathscr{A}(\mathscr{L}_1)$. By the condition of f, it is easily checked that $\mathscr{L}_2$ is $\mathscr{A}(\mathscr{L}_1)$-countably paracompact, and hence it follows from Bachman-Sultan's theorem that λ can be extended to an $\mathscr{L}_2$-regular measure λ_1 defined on $\mathscr{A}(\mathscr{L}_2)$ $(= \mathrm{Ba}^\omega(X))$. Since λ_1 extends to a Baire measure on X and X is a Mařík space, λ extends to a regular Borel measure ξ on X. For each $A \in \mathrm{Bo}(Y)$, define $\nu(A) = \xi(f^{-1}(A))$. Then, since $\mu|\mathrm{Ba}^\omega(Y) = \nu|\mathrm{Ba}^\omega(Y)$, $\mu = \nu|\mathrm{Ba}(Y)$, and hence ν is a regular Borel extension of μ. □

Corollary 4.16. *Let f be an open, perfect map from a Mařík space X onto a space Y. Then Y is a Mařík space.*

Proof. By [8, Lemma 3.4], an open perfect map carries a zero-set to a zero-set. Hence this follows from Theorem 4.15. □

Theorem 4.17. *Let f be a closed, continuous map from a space X onto a Baire-separated, Mařík space Y such that for each $y \in Y$, $f^{-1}(y)$ is countably compact. Then X is a Mařík space.*

Proof. Let μ be a Baire measure on X. We have to prove that μ admits a regular Borel extension. For each $B \in \mathrm{Ba}(Y)$, define $\lambda(B) = \mu(f^{-1}(B))$; then λ is a Baire measure on Y. Since Y is a Mařík space, λ extends to a regular Borel measure ξ on Y. On the other hand, by Bachman-Sultan's theorem, $\mu|\mathrm{Ba}^\omega(X)$ extends to a finitely additive, regular measure ν_1 defined on $\mathrm{Bo}^\omega(X)$. For each $A \in \mathrm{Bo}^\omega(Y)$, define $\xi_1(A) = \nu_1(f^{-1}(A))$. Then ξ_1 is regular. We now prove that

$$\xi_1 = \xi|\mathrm{Bo}^\omega(Y). \tag{1}$$

Since both ξ and ξ_1 are regular, it suffices to show that they coincide on open sets. To do this, let $G \subset Y$ be open and let $\varepsilon > 0$. Then there exists a closed set $E \subset G$ such that $\xi(E) > \xi(G) - \varepsilon$. Since Y is Baire-separated, there exists $J \in \mathrm{Ba}(Y)$ with $E \subset J \subset G$. Then

$$\begin{aligned}\xi(J) = \lambda(J) &= \sup\{\lambda(Z) : Z \subset J, Z \in Z(Y)\} \\ &= \sup\{\xi_1(Z) : Z \subset J, Z \in Z(Y)\} \le \xi_1(G),\end{aligned}$$

so $\xi(G) - \varepsilon < \xi(J) \le \xi_1(G)$, and hence $\xi(G) \le \xi_1(G)$. Conversely, let $\varepsilon > 0$ again, and take a closed set $F \subset G$ and $K \in \mathrm{Ba}(Y)$ such that $\xi_1(F) > \xi_1(G) - \varepsilon$ and $F \subset K \subset G$. Then

$$\begin{aligned}\xi_1(F) = \nu_1(f^{-1}(F)) &\le \inf\{\nu_1(V) : f^{-1}(K) \subset V \in \mathrm{Coz}(X)\} \\ &= \mu(f^{-1}(K)) = \lambda(K) = \xi(K),\end{aligned}$$

so $\xi_1(G) - \varepsilon < \xi(K) \le \xi(G)$, and hence $\xi_1(G) \le \xi(G)$. Thus (1) is proved. To see that ν_1 is σ-additive, let $\{F_n\}_{n\in\omega}$ be a sequence of closed sets in X with $F_n \downarrow \varnothing$. By the condition of f, $f(F_n)$ is closed and $f(F_n) \downarrow \varnothing$. Since ξ is σ-additive, it follows from (1) that

$$\lim_{n\to\infty} \nu_1(F_n) \le \lim_{n\to\infty} \nu_1(f^{-1}(f(F_n))) = \lim_{n\to\infty} \xi(f(F_n)) = 0.$$

Consequently, ν_1 is σ-additive, and hence ν_1 can be extended to a regular Borel measure on X, which is a required extension of μ. □

Corollary 4.18. *Let Y be a Baire-separated, Mařík space. Then the absolute $E(Y)$ is a Mařík space.*

Corollary 4.19. *Let X be a Baire-separated, Mařík space and Y a compact space. Then $X \times Y$ is a Mařík space.*

Example 3.7 shows that the assumptions in Theorem 4.17 and Corollary 4.18 that Y is Baire-separated cannot be removed; however, the following questions remain unanswered.

Questions 4.20. Is the product of a Mařík space with a compact space a Mařík space? More generally, is the preimage of a Mařík space under an open, perfect map a Mařík space?

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