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Convex inequalities and the Hahn–Banach Theorem

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CONTENTS

Introduction

The main purpose of this paper is to show that many important propositions involving convex functions and convex sets are in fact nothing but equivalent forms of the Hahn–Banach Theorem and can all be derived in a simple, unified manner, from a general proposition on inconsistent systems of convex inequalities.

In Section 1, devoted to finite systems of convex inequalities we shall prove a proposition which might be used for the foundation of the most interesting results known in this field. In particular it yields as a consequence a feasibility theorem for linear programming problems.

In Section 2 the results are extended to infinite systems. The basic theorem to be established here, though formally equivalent to the Eidelheit Separation Theorem (geometric equivalent of the Hahn–Banach Theorem), provides a natural way to derive in the next Section 3, many known propositions, such as: the Hahn–Banach Theorem, Krein's Theorem on the extension of positive linear functionals, the Farkas–Minkowski lemma on linear inequalities (abstract form), Dubovitski–Milyutin propositions on non-intersecting condition for a finite system of convex cones, and on conjugate sets, the Kuhn–Tucker saddle-point theorem for convex programming (generalized by Hurwicz), Golshtein's recent results on duality in mathematical programming, the Fenchel–Moreau proposition on biconjugate functions, and the Moreau–Rockafellar proposition on the convolution of conjugate functions. On the basis of this theorem some known results can be sharpened or generalized; others can be obtained in their strongest known forms.

In the last section (§ 4) we shall point out the formal equivalence of all the above-mentioned propositions. It is rather surprising that this connection has not been noticed in earlier works. For example, little is known as to how the Farkas–Minkowski lemma is related to the main results in the theory of conjugate functions, as developed by Fenchel, Moreau, Rockafellar and others ([16]); and though relations among various aspects of duality have been discussed in some works ([14], [16], ...), up to now the Kuhn–Tucker–Hurwicz theorem, and the mentioned result of Moreau–Rockafellar or that of Dubovitski–Milyutin, are often presented in literature as quite unrelated matters.

Throughout this paper, all underlying spaces are supposed to be real linear. Most of the results will be formulated for the case, when underlying spaces are linear topological. If they are to be applied to linear spaces without a topology, one has only to endow the latter with the finest locally convex topology, i.e. the topology in which every convex absorbing subset of the space is a neighbourhood of the origin. As is well known, in this topology:

1. The concepts of internal point and interior point are equivalent for convex sets ([3] or [6]). Since we shall always deal with convex sets, this enables us to denote the interior and the set of internal points of a set N by the same symbol $\overset{0}{N}$.

2. Every sublinear (in particular, every linear) functional is continuous. Therefore, the algebraic dual coincides with the topological dual ([6] or [13]).

§ 1. Finite systems of convex inequalities

1. A preliminary result. We begin with proving a proposition which includes as a special case the famous Farkas–Minkowski lemma and permits a straightforward derivation of much of the fundamental facts concerning finite systems of convex inequalities.

Let us consider a linear space X and the product space $Z = X \times R^n$ every element of which is thus of the form $z = (x, y)$ with $x \in X$, $y = (y_1, y_2, \ldots, y_n) \in R^n$. For any prescribed set $K \subset \{1, 2, \ldots, n\}$ we shall say that a convex subset C of Z fulfils the condition (S_K) if the projection of C on the subspace $Y_K = \{y \in R^n\colon y_i = 0 \ (i \notin K)\}$ contains the origin in its relative interior, or, equivalently, if

$$\big(\forall (x, y) \in C\big)\big(\exists (x', y') \in C\big)(\exists \lambda > 0)\colon\ (\forall k \in K)\ y'_k = -\lambda y_k.$$

Let us write $p \leqslant q$ whenever $p_j \leqslant q_j$ $(j = 1, 2, \ldots, n)$ and introduce the notation

$$\langle p, q, y\rangle = \sum_{y_j \leqslant 0} p_j y_j + \sum_{y_j > 0} q_j y_j,$$

$$\langle t, y\rangle = \langle t, t, y\rangle = \sum_{j=1}^{n} t_j y_j.$$

THEOREM 1.1. *Let C be a non-void convex subset of $Z = X \times R^n$, $f(z)$ a finite convex function on Z, and $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n)$ two vectors such that $p \leqslant q$ and $p_j < +\infty$, $-\infty < q_j$ $(j = 1, 2, \ldots, n)$. Suppose furthermore that C fulfils the condition (S_K) for $K = \{k\colon p_k = -\infty$*

or $q_k = +\infty\}$. *Then there exists a vector* $t \in R^n$ *satisfying*

$$p \leqslant t \leqslant q \quad (\forall z = (x, y) \in C)\ \langle t, y\rangle + f(z) \geqslant 0 \tag{1.1}$$

if and only if

$$(\forall z = (x, y) \in C)\ \langle p, q, y\rangle + f(z) \geqslant 0. \tag{1.2}$$

Proof. The proof is essentially the same as that given for a less general proposition in our earlier paper [18] (see also [19]).

Since the necessity of the condition is trivial, we need only prove its sufficiency. Without loss of generality it can be assumed that $k > j$ for any $k \in K$, $j \notin K$ (if $K \neq \emptyset$).

The proposition is obvious for $n = 0$ (i.e. $Y = \{0\}$). Assuming that it is true for $n = m-1$, consider the case when $n = m$. If $y_m = 0$ for all $z = (x, y_1, \ldots, y_m) \in C$ then C can be identified with a set in $X \times R^{m-1}$ satisfying the condition $(\mathrm{S}_{K\setminus\{m\}})$, and the proposition is true by the induction assumption; in the contrary case, let $C_m^+ = \{z \in C\colon y_m > 0\}$, $C_m^- = \{z \in C\colon y_m < 0\}$ and let us set for each $z \in C_m \cup C_m^-$

$$a(z) = -\frac{1}{y_m}[\langle \bar{p}, \bar{q}, \bar{y}\rangle + f(z)],$$

where $\bar{p} = (p_1, \ldots, p_{m-1})$, $\bar{q} = (q_1, \ldots, q_{m-1})$, $\bar{y} = (y_1, \ldots, y_{m-1})$.

From (1.2), we deduce $a(z) \leqslant q_m$ for $z \in C_m^+$, and $a(z) \geqslant p_m$ for $z \in C_m^-$. Hence putting

$$a = \begin{cases} \sup\limits_{z \in C_m^+} a(z) & \text{if} \quad C_m^+ \neq \emptyset, \\ -\infty & \text{else;} \end{cases} \qquad \beta = \begin{cases} \inf\limits_{z \in C_m^-} a(z) & \text{if} \quad C_m^- \neq \emptyset, \\ +\infty & \text{else,} \end{cases}$$

we obtain

$$a \leqslant q_m, \qquad \beta \geqslant p_m. \tag{1.3}$$

On the other hand if $C_m^+ = \emptyset$ or $C_m^- = \emptyset$, then obviously $a \leqslant \beta$. If these sets are both non-void, then for any $z \in C_m^+$, $z' \in C_m^-$ we have

$$\lambda\left(\frac{z}{y_m} - \frac{z'}{y'_m}\right) \in C_m,$$

where $\lambda = -y_m y'_m (y_m - y'_m)^{-1} > 0$ and $C_m = \{z \in C\colon y_m = 0\}$.

But the set C_m considered as a set in $X \times R^{m-1}$ fulfils the condition $(\mathrm{S}_{K\setminus\{m\}})$, and by the induction assumption a vector $\bar{t} \in R^{m-1}$ can be found such that $\bar{p} \leqslant \bar{t} \leqslant \bar{q}$ and $\langle \bar{t}, \bar{y}\rangle + f(z) \geqslant 0$ for every $z \in C_m$. Therefore

$$\lambda\left\langle \bar{t}, \frac{\bar{y}}{y_m} - \frac{\bar{y}'}{y'_m}\right\rangle + f\left(\lambda\frac{z}{y_m} - \lambda\frac{z'}{y'_m}\right) \geqslant 0.$$

Taking into account the convexity of f, which implies

$$f\left(\lambda\frac{z}{y_m}-\lambda\frac{z'}{y'_m}\right)\leqslant\frac{\lambda}{y_m}f(z)-\frac{\lambda}{y'_m}f(z'),$$

and noting also the fact that $\langle\bar{t},\bar{y}\rangle\leqslant\langle\bar{p},\bar{q},\bar{y}\rangle$, we can deduce from the above inequality

$$(1.4)\qquad a(z)\leqslant-\frac{1}{y_m}[\langle\bar{t},\bar{y}\rangle+f(z)]\leqslant-\frac{1}{y'_m}[\langle\bar{t},\bar{y}'\rangle+f(z')]\leqslant a(z').$$

Hence in any case, we have $a\leqslant\beta$ and then the relations (1.3) together with the condition $p_m\leqslant q_m$ yield

$$(1.5)\qquad \max\{a,p_m\}\leqslant\min\{\beta,q_m\}.$$

According to the hypothesis, either both p_m and q_m are finite numbers, or both sets C_m^+ and C_m^- are non-void; but in the latter case, relations (1.4) show that $a<+\infty$, $\beta>-\infty$. Hence from (1.5) follows the existence of at least one finite number t_m such that

$$p_m\leqslant t_m\leqslant q_m,\qquad a\leqslant t_m\leqslant\beta.$$

If we set $\bar{f}(z)=t_my_m+f(z)$ then

$$(\forall z\in C)\ \langle\bar{p},\bar{q},\bar{y}\rangle+\bar{f}(z)\geqslant 0$$

and again by the induction assumption, we can find a vector $(t_1,\ldots,t_{m-1})\in R^{m-1}$ such that $p_j\leqslant t_j\leqslant q_j$ $(j=1,2,\ldots,m-1)$,

$$\sum_{j=1}^{m-1}t_jy_j+t_my_m+f(z)\geqslant 0\qquad\text{for all } z\in C.$$

Then $t=(t_1,\ldots,t_{m-1},t_m)$ is the sought vector, and the proof is complete.

Remark. Let us call *positive basis* of a set D any set $B\subset D$ such that each point $y\in D$ may be represented as a positive linear combination of elements of B. By a *schema* of a set $D\subset R^n$ we shall mean any set $E\subset D$ which contains for each $J\subset\{1,2,\ldots,n\}$ some positive basis of the set $\{y\in D\colon y_j=0\ (j\in J)\}$. Using the above argument, but replacing C_m^+ and C_m^- by $E_m^+=\{z\in E\colon y_m>0\}$, $E_m^-=\{z\in E\colon y_m<0\}$ we might prove for the case $X=\{0\}$ a stronger proposition, namely:

Let C be a convex subset of R^n, E a schema for C, $f(y)$ a convex function on C. If we have

$$(\forall y\in E)\ \langle p,q,y\rangle+f(y)\geqslant 0,$$

there exists a vector $t\in R^n$ satisfying:

$$p\leqslant t\leqslant q,\qquad(\forall y\in C)\ \langle t,y\rangle+f(y)\geqslant 0.$$

2. Applications. Several propositions on convex sets and convex inequalities could be obtained as special cases of Theorem 1.1. Let us first mention the following proposition, known as the Farkas–Minkowski Lemma.

COROLLARY 1.1. *Let $f(x), g_i(x)$ $(i = 1, 2, \ldots, n)$ be (finite) convex functions on a convex subset D of a linear space X. Suppose that functions $g_i(x), i \in I_1 \subset \{1, 2, \ldots, n\}$ are affine (i.e. $g_i(x) - g_i(0)$ are linear) and that there exists at least one point $\bar{x}$ such that*

$$\bar{x} \in \overset{0}{D}, \quad g_i(\bar{x}) \leqslant 0 \ (i \in I_1), \quad g_i(\bar{x}) < 0 \ (i \notin I_1). \tag{1.6}$$

Then the system

$$x \in D, \quad g_i(x) \leqslant 0 \ (i = 1, 2, \ldots, n), \quad f(x) < 0 \tag{1.7}$$

is inconsistent if and only if there exists a vector $t \in R^n$, $t \geqslant 0$, such that:

$$(\forall x \in D)\ f(x) + \sum_{i=1}^{n} t_i g_i(x) \geqslant 0. \tag{1.8}$$

Proof. We may assume $g_i(\bar{x}) = 0$ $(i \in I_1)$ for otherwise I_1 could be replaced by $I_1' = \{i\colon g_i(\bar{x}) = 0\}$. Define C to be the set of all $z = (x, y) \in X \times R^n$ such that

$$x \in D, \quad g_i(x) = y_i \ (i \in I_1), \quad g_i(x) \leqslant y_i \ (i \notin I_1).$$

Clearly, C is a convex set and by hypothesis $(\bar{x}, 0) \in C$. For any given $(x, y) \in C$ we shall have $\big(-\lambda x + (1+\lambda)\bar{x}, -\lambda y\big) \in C$ provided $\lambda > 0$ be small enough. Indeed, since the functions

$$\gamma_i(\lambda) = g_i\big(-\lambda x + (1+\lambda)\bar{x}\big) \quad (i \in I_1)$$

are affine, we may write

$$g_i\big(-\lambda x + (1+\lambda)\bar{x}\big) = -\lambda g_i(x) + (1+\lambda) g_i(\bar{x}) = -\lambda y_i$$

and since the functions $\gamma_i(\lambda)$ $(i \notin I_1)$ are convex, hence continuous on every open interval, we shall have, taking into account the fact that $\bar{x} \in \overset{0}{D}$ and $\gamma_i(0) = g_i(\bar{x}) < 0$, $\ -\lambda x + (1+\lambda)\bar{x} \in D$, $g_i\big(-\lambda x + (1+\lambda)\bar{x}\big) \leqslant -\lambda y_i$ $(i \notin I_1)$ when $\lambda > 0$ is small enough. That is, the set C fulfils the condition $(S_{\{1,2,\ldots,n\}})$. If the system (1.7) is inconsistent, then, putting $p_i = 0$, $q_i = +\infty$ $(i = 1, 2, \ldots, n)$ we may write

$$\langle p, q, y\rangle + f(x) \geqslant 0 \quad \text{for all } (x, y) \in C.$$

Hence, by Theorem 1.1, a vector $t \in R^n$, $t \geqslant 0$, may be found satisfying $\big(\forall (x, y) \in C\big)\ f(x) + \sum_{i=1}^{n} t_i y_i \geqslant 0$, which yields (1.8) by taking $x \in D$, $y_i = g_i(x)$.

This proof is much shorter than any other to the knowledge of the author (see for instance the proof in [2]).

It is worthwhile to notice that the condition $\bar{x} \in \overset{0}{D}$ in (1.6) may be replaced by $\bar{x} \in D$ if *either D is a polyhedral set* (the functions $f(x)$ and $g_i(x)$ being then supposed finite and convex on the whole space X), *or the set I_1 is empty.*

In fact, in the first case, if the set D is defined by the system of inequalities $g_i(x) \leqslant 0$ $(i = n+1, \ldots, N)$, with $g_i(x)$ affine functions, then Corollary 1.1 asserts the existence of a vector $t \in R^N$, $t \geqslant 0$, such that

$$(\forall x \in X)\ f(x) + \sum_{i=1}^{n} t_i g_i(x) \geqslant 0,$$

hence

$$(\forall x \in D)\ f(x) + \sum_{i=1}^{n} t_i g_i(x) \geqslant - \sum_{i=n+1}^{N} t_i g_i(x) \geqslant 0.$$

In the second case, for any given $(x, y) \in C$ we have $(\bar{x}, -\lambda y) \in C$, provided $\lambda > 0$ is small enough to ensure $g_i(\bar{x}) < -\lambda y_i$ $(i = 1, 2, \ldots, n)$, which is possible because $g_i(\bar{x}) < 0$. Thus the set C still fulfils the condition $(S_{\{1,2,\ldots,n\}})$ and the proof given above remains valid. In this case the proposition can be stated also as follows:

Let $g_i(x)$ $(i = 1, 2, \ldots, n)$ be convex functions on a convex set $D \subset X$. If the system

$$x \in D, \quad g_i(x) < 0 \ \ (i = 1, 2, \ldots, n)$$

is inconsistent, then there exists $t \in R^n$, $t \geqslant 0$, $t \neq 0$, such that

$$(\forall x \in D) \ \sum_{i=1}^{n} t_i g_i(x) \geqslant 0.$$

If, in addition, the system

$$x \in D, \quad g_i(x) < 0 \ \ (i = 1, 2, \ldots, m)$$

is consistent, then one at least of the t_i $(i > m)$ is positive.

As one might remark, the essential point in the proof of Corollary 1.1 is to show that the set C fulfils the condition $(S_{\{1,2,\ldots,n\}})$. This fact suggests the following way to generalize further Corollary 1.1.

Let $X = X_1 \times X_2$, so that each point of X may be written as $x = (x_1, x_2)$, $x_1 \in X_1$, $x_2 \in X_2$. Then the conclusions of Corollary 1.1 hold if the following condition (S) is satisfied:

(S) *The system*

$$x \in D, \quad g_i(x) \leqslant 0 \ \ (i \in I_1), \qquad g_i(x) < 0 \ \ (i \notin I_1) \tag{1.9}$$

has a solution $\bar{x} = (\bar{x}_1, \bar{x}_2)$ *such that for every* $\{y_i, i \in I_1\}$ *satisfying* $g_i(x) \leqslant y_i$ *for some* $x \in D$ *there exists* $x' = (x'_1, \bar{x}_2) \in D$ *satisfying* $g_i(x') \leqslant -\theta y_i$ $(i \in I_1)$ *for some* $\theta > 0$ (depending on $\{y_i, i \in I_1\}$).

To prove this, consider the set

$$C = \{z = (x, y) \in X \times R^n \colon x \in D,\ g_i(x) \leqslant y_i\ (i = 1, 2, \ldots, n)\}$$

which is obviously convex and contains $(\bar{x}, 0)$. For any $(x, y) \in C$ there exists by hypothesis some $(a, \bar{x}_2) \in D$ such that $g_i(a, \bar{x}_2) \leqslant -\theta y_i$ $(i \in I_1)$ for some $\theta > 0$. Since

$$\begin{aligned} g_i\big(\alpha a + (1-\alpha)\bar{x}_1, \bar{x}_2\big) &\leqslant \alpha g_i(a, \bar{x}_2) + (1-\alpha) g_i(\bar{x}_1, \bar{x}_2) \\ &\leqslant \alpha g_i(a, \bar{x}_2) \leqslant -\alpha\theta y_i\ (i \in I_1) \end{aligned}$$

we shall have

$$\big(\alpha a + (1-\alpha)\bar{x}_1, \bar{x}_2\big) \in D, \qquad g_i\big(\alpha a + (1-\alpha)\bar{x}_1, \bar{x}_2\big) \leqslant -\lambda y_i\ (i \in I_1)$$

for $\lambda = \alpha\theta > 0$ when $\alpha > 0$ is small enough. On the other hand, since $g_i(\bar{x}) < 0$ $(i \notin I_1)$ we shall have also

$$g_i\big(\alpha a + (1-\alpha)\bar{x}_1, \bar{x}_2\big) \leqslant -\lambda y_i\ (i \notin I_1)$$

when $\alpha > 0$ is small enough. Thus $\big(\alpha a + (1-\alpha)\bar{x}_1, \bar{x}_2, -\lambda y\big) \in C$ for some $\alpha > 0$, $\lambda > 0$, i.e. the set C fulfils the condition $(S_{\{1,2,\ldots,n\}})$. The remaining part of the proof is trivial.

Clearly the condition (S) is fulfilled if for example the system (1.9) has a solution $\bar{x} = (\bar{x}_1, \bar{x}_2)$ *such that* $\bar{x}_1$ *is an interior point of* $D_1 = \{x_1 \colon (x_1, \bar{x}_2) \in D\}$ *and the vector-function* $\{g_i(x_1, \bar{x}_2), i \in I_1\}$ *is affine and surjective.*

Among the many consequences of Corollary 1.1 let us mention the next proposition, a sharpening of a well-known theorem due to Kuhn and Tucker ([10], [1] and also [2], [14]):

Let D *be a convex subset of a linear space* X, *let* $f(x)$ *and* $g_i(x)$ $(i = 1, 2, \ldots, n)$ *be convex functions on* D, *satisfying the condition* (S). *Then* $x^0 \in D$ *is an optimal solution to the convex programming problem:*

$$\min\{f(x) \colon x \in D,\ g_i(x) \leqslant 0\ (i = 1, 2, \ldots, n)\}$$

if and only if there exists a vector $t^0 \in R^n$, $t^0 \geqslant 0$, *such that*

$$(\forall x \in D)\ (\forall t \geqslant 0)\ F(x, t^0) \geqslant F(x^0, t)$$

where

$$F(x, t) = f(x) + \sum_{i=1}^{n} t_i g_i(x).$$

The sufficiency of this condition is almost plain. The necessity follows from Corollary 1.1 and from the fact that $x^0 \in D$ is an optimal solution only if the system $x \in D$, $g_i(x) \leqslant 0$ $(i = 1, 2, \ldots, n)$, $f(x) - f(x^0) < 0$ is inconsistent.

COROLLARY 1.2. *For any convex subset C of R^n, not containing the origin, there exists a vector $t \in R^n$, $t \neq 0$, such that*

$$(\forall x \in C) \ \langle t, x\rangle \geqslant 0.$$

If C is closed, t can be taken so as to have

$$(\forall x \in C) \ \langle t, x\rangle \geqslant a > 0.$$

The proof commonly produced in the literature for this classical result is tiresome enough (see for example [2]). Theorem 1.1 enables us to derive it easily.

For $n = 1$ the proposition is obvious. Assuming that it is true for R^{n-1}, we find for the set $C_n = \{x \in C\colon x_n = 0\}$ a vector $(\bar{t}_1, \ldots, \bar{t}_{n-1}) \neq 0$ such that $\sum_{j=1}^{n-1} \bar{t}_j x_j \geqslant 0$ for all $x \in C_n$. But the sets $C_n^+ = \{x \in C\colon x_n > 0\}$ and $C_n^- = \{x \in C\colon x_n < 0\}$ may be supposed non-void, because if for example $x_n \geqslant 0$ for all $x \in C$ then $(0, \ldots, 0, 1)$ will be the vector required. Thus C satisfies the condition $(S_{\{n\}})$ and, putting $p_j = q_j = \bar{t}_j$ $(j = 1, 2, \ldots, n-1)$, $p_n = -\infty$, $q_n = +\infty$ we have $\langle p, q, x\rangle \geqslant 0$ for all $x \in C$. Hence there exists $t \in R^n$ such that $t_j = \bar{t}_j$ $(j = 1, 2, \ldots, n-1)$ (which implies $t \neq 0$) and $(\forall x \in C)$ $\langle t, x\rangle \geqslant 0$.

The proof is still shorter when C is closed. In this case, taking $p_j = -1$, $q_j = +1$ $(j = 1, 2, \ldots, n)$, we have for some $a > 0$

$$(\forall x \in C) \ \langle p, q, x\rangle = \sum_{j=1}^{n} |x_j| \geqslant a$$

and so there exists $t \in R^n$, $-1 \leqslant t_j \leqslant 1$ $(j = 1, 2, \ldots, n)$ such that $(\forall x \in C)$ $\langle t, x\rangle \geqslant a > 0$.

To end this section, let us derive one more corollary of Theorem 1.1, which appears to be useful in some questions of linear programming:

COROLLARY 1.3. *Let $A = (a_{ij})$ be a m by n matrix, A^* its transpose, b a m-vector. The system*

$$x \in R^n, \quad p \leqslant x \leqslant q, \quad Ax \geqslant b \tag{1.10}$$

is consistent if and only if

$$(\forall u \in R^m, u \geqslant 0) \ \langle p, q, A^* u\rangle \geqslant \langle b, u\rangle. \tag{1.11}$$

Here p_j, q_j $(j = 1, 2, \ldots, n)$ can be equal, respectively, to $-\infty$, $+\infty$.

Proof. Let us consider the space

$$C = \{(u, y)\colon u \in R^m, \ y \in R^n, \ y = A^* u\},$$

and let $p_i' = -\infty$, $q_i' = 0$ $(i = 1, 2, \ldots, m)$. Statement (1.11) is then equivalent to

$$(\forall (u, y) \in C) \ \langle p', q', u\rangle + \langle p, q, y\rangle - \langle b, u\rangle \geqslant 0$$

which, by Theorem 1.1, is equivalent to the existence of two vectors x', x such that $p' \leqslant x' \leqslant q'$, $p \leqslant x \leqslant q$ and that

$$(\forall (u, y) \in C) \ \langle x', u\rangle + \langle x, y\rangle - \langle b, u\rangle \geqslant 0,$$

or

$$(\forall u \geqslant 0) \ \langle Ax, u\rangle - \langle b, u\rangle \geqslant 0.$$

Since the latter condition is satisfied if and only if $Ax \geqslant b$, the proof is complete.

If in the system (1.10) we have $Ax = b$ instead of $Ax \geqslant b$, then, writing $Ax \geqslant b$, $-Ax \geqslant -b$, we see that the condition $u \geqslant 0$ in (1.11) must be removed.

Furthermore, if $a_{ij} a_{kj} \geqslant 0$, for all i, k, j then condition (1.11) reduces merely to

$$\langle p, q, A_i\rangle \geqslant b_i \qquad (i = 1, 2, \ldots, m)$$

where A_i are the rows of A, and b_i the components of b.

More generally, suppose we know a schema E of the cone

$$D = \{y \in R^n \colon \ y = A^* u \quad \text{for some } u \geqslant 0\}$$

and for every $t \in E$ let $t = A^* u^{(t)}$, $u^{(t)} \geqslant 0$. Assuming in addition that $Av = b$ for some v, we have $\langle b, u\rangle = \langle v, A^* u\rangle$, so that the condition (1.11) may be written $(\forall y \in D) \ \langle p, q, y\rangle - \langle v, y\rangle \geqslant 0$. Then, by making use of the remark to Theorem 1.1, one can assert that the condition (1.11) will be satisfied, hence the system (1.10) will be consistent, if

$$(\forall t \in E) \ \langle p, q, t\rangle \geqslant \langle b, u^{(t)}\rangle.$$

Since the set of all elementary cocycles of a graph (in the terminology of [2]) is a schema for the set of all cocycles (tensions) of this graph (i.e. the linear space spanned by the rows A_i of the incidence matrix), Corollary 1.3 leads to the following proposition which reduces to a known result of J. Hoffman when $b = 0$ (see for instance [2]):

Let A be the incidence matrix of a graph with m vertices and n arcs. Suppose that the system $Ax = b$ has a solution. Then the system

$$x \in R^n, \quad p \leqslant x \leqslant q, \quad Ax = b$$

is consistent if and only if for every elementary cocycle $t = \sum u_i A_i$ of the graph we have

$$\langle p, q, t\rangle \geqslant \sum_{u_i>0} b_i - \sum_{u_i<0} b_i \geqslant \langle q, p, t\rangle.$$

§ 2. Infinite systems of convex inequalities

We proceed now to extend to infinite systems the main results established in the foregoing section.

1. The separation theorem. For the sake of completeness, let us first provide a direct proof of the classical Eidelheit theorem, based upon our Theorem 1.1.

LEMMA 2.1. *Let C be a convex subset of a linear space X. If C has a non-void interior and does not contain the origin, then there exists a linear form $t(x)$ on X such that $t(x) \not\equiv 0$ and $(\forall x \in C)$ $t(x) \geqslant 0$.*

Proof. Denote by $\mathfrak{C}$ the set of all pairs (M, t) such that M is a subspace of X containing a fixed interior point a of C, and $t(x)$ a linear form on M satisfying $(\forall x \in M \cap C)$ $t(x) \geqslant 0$. Clearly $\mathfrak{C} \neq \emptyset$ (Corollary 1.2), and if we order $\mathfrak{C}$ by agreeing that $(M, t) \prec (M', t')$ whenever $M \subset M'$, $(\forall x \in M)$ $t(x) = t'(x)$, then $\mathfrak{C}$ is an inductive set.

By Zorn's lemma it has a maximal element $(\bar{M}, \bar{t})$ and it is not hard to see that $\bar{M} = X$. Indeed, suppose $x_0 \notin \bar{M}$ and let $\hat{M}$ denote the subspace generated by $\bar{M}$ and x_0, which may be identified with $\bar{M} \times R^1$. Since a is an interior point of C, the convex set

$$\hat{C} = \{(x, \lambda) \in C:\ x \in \bar{M},\ \lambda \in R^1\}$$

fulfils the condition $(S_{\{1\}})$. Putting $p = -\infty$, $q = +\infty$ we have $\big(\forall (x, \lambda) \in \hat{C}\big)$ $\bar{t}(x) + \langle p, q, \lambda\rangle \geqslant 0$. Hence by Theorem 1.1 there exists a number t_0 such that $\big(\forall (x, \lambda) \in \hat{C}\big)$ $\bar{t}(x) + \lambda t_0 \geqslant 0$ and so $(\bar{M}, \bar{t}) \prec \big(\hat{M}, \bar{t}(x) + \lambda t_0\big)$, which contradicts the maximality of $(\bar{M}, \bar{t})$.

We see that the proof is essentially the same as that of Corollary 1.2, the only difference being the use of transfinite induction instead of ordinary induction.

2. Two general propositions on inconsistency. Given a linear space Y and a convex cone $N \subset Y$, containing 0, if we agree to write $y_1 \preceq y_2$ whenever $y_2 - y_1 \in N$, then, as is well-known, the relation $\preceq$ is a quasi-ordering compatible with the linear space structure of Y. For brevity, N will be called the *positive cone* of Y.

If Y is a linear topological space and $\overset{0}{N} \neq \emptyset$, we shall also agree to write $y_1 \prec y_2$ whenever $y_2 - y_1 \in \overset{0}{N}$. It is at once verified that the relation $\prec$ is equally transitive and that

1) $y_1 \prec y_2$ implies $y_1+y \prec y_2+y$ for each $y \in Y$,

2) $y \prec 0$ implies $\lambda y \prec 0$ for each scalar $\lambda > 0$.

A linear form $g(y)$ on Y is said to be *positive* if $g(y) \geqslant 0$ for each $y \succeq 0$, or, what amounts to the same, if $g(y) \leqslant 0$ for each $y \preceq 0$. A map T of a convex subset D of a linear space X into Y is said to be *convex* if

$$T\big(\alpha x_1 + (1-\alpha)x_2\big) \preceq \alpha T x_1 + (1-\alpha) T x_2$$

for each pair $x_1, x_2 \in D$ and for each α, $0 \leqslant \alpha \leqslant 1$.

Throughout this section we shall assume that: Y, Z are two linear topological spaces, N, M are their respective positive cones, D is a non-void convex subset of a linear space X, T is a convex map of D into Y, and S is a convex map of D into Z.

THEOREM 2.1. I. *Suppose that the sets N, M have non-void interiors and that the system*

$$x \in D, \quad Tx \prec 0 \tag{2.1}$$

is consistent. The system

$$x \in D, \quad Tx \prec 0, \quad Sx \prec 0 \tag{2.2}$$

is inconsistent only if there exist two continuous positive linear forms $g(y)$ on Y and $h(z)$ on Z, such that $h(z) \not\equiv 0$ and

$$(\nabla x \in D) \;\; g(Tx) + h(Sx) \geqslant 0. \tag{2.3}$$

II. *Conversely if M has a non-void interior and if there exist two linear forms $g(y)$, $h(z)$ with the stated properties then the system*

$$x \in D, \quad Tx \preceq 0, \quad Sx \prec 0 \tag{2.2'}$$

is inconsistent.

The first part of this proposition will be obtained as a special case of a more general theorem to be proved later. The second part is quite easy to verify. Indeed, if there exist two linear forms $g(y)$, $h(z)$ with the stated properties, and if the system (2.2′) has a solution x_0, then, on account of (2.3),

$$g(Tx_0) + h(Sx_0) \geqslant 0;$$

on the other hand, since $Tx_0 \preceq 0$, $Sx_0 \prec 0$ and since g and h are positive, we have

$$g(Tx_0) \leqslant 0, \quad h(Sx_0) \leqslant 0.$$

Hence $h(-Sx_0) = 0$ and, since $-Sx_0$ is an interior point of M, the linear form $h(z)$ which is by hypothesis, positive on M, must vanish identically. This contradiction proves the second part of our theorem.

In some applications the condition that cones N, M have non-void interiors may happen to be too restrictive. Motivated by this fact, we introduce the following definition.

A subset V_0 of Y is called *qualified* for the positive cone N of Y if 0 belongs to the closure of V_0 and if the set $N+V_0$ is convex; it is called a *regularizing* set for N if, in addition, $N+V_0$ has a non-void interior. We shall then write $y_1 \underset{0}{\prec} y_2+(V_0)$ ($y_1 \prec y_2+(V_0)$, resp.) whenever $y_2-y_1 \in N+V_0$ ($y_2-y_1 \in \widehat{N+V_0}$, resp.), and shall say that $\bar{x}$ is a V_0*-solution* of the system (2.1) if

$$\bar{x} \in D, \quad T\bar{x} \prec 0+(V_0).$$

Similarly, V_0, W_0 being regularizing sets for N and M (resp.), we shall say that x is a (V_0, W_0)*-solution* of the system (2.2) if

$$\bar{x} \in D, \quad T\bar{x} \prec 0+(V_0), \quad S\bar{x} \prec 0+(W_0).$$

The system (2.1) (or (2.2)) is said to be V_0*-consistent* ((V_0, W_0)*-consistent*) if it has a V_0-solution ((V_0, W_0)-solution).

THEOREM 2.2. *Suppose that V_0 and W_0 are regularizing sets for cones N and M (resp.) and that the system (2.1) is V_0-consistent. The system (2.2) is (V_0, W_0)-inconsistent only if there exist two linear forms $g(y)$ and $h(z)$ with the properties stated in Theorem 2.1.*

We shall omit the proof, since this proposition is only a special case of Theorem 2.3 to be proved below.

Let us point out an important corollary of the above theorem.

A net $\{x_\varepsilon, \varepsilon \in E\} \subset X$, i.e. a function x_ε, defined on a directed set E, with values in X, is called a *weak solution* of the system

$$(2.2'') \qquad x \in D, \quad Tx \underset{0}{\prec} 0, \quad Sx \prec 0$$

if there exist two nets $\{V_\varepsilon, \varepsilon \in E\} \subset Y$, $\{W_\varepsilon, \varepsilon \in E\} \subset Z$ converging to 0, such that for each $\varepsilon \in E$ the element x_ε is a $(V_\varepsilon, W_\varepsilon)$-solution of (2.2), i.e.

$$x_\varepsilon \in D, \quad Tx_\varepsilon \underset{0}{\prec} 0+(V_\varepsilon), \quad Sx \prec 0+(W_\varepsilon)$$

(V_ε, W_ε being regularizing sets, for N and M resp.). The system is said to be *weakly consistent* if it has a weak solution.

In an analogous manner we define the concepts of weak solution and weak consistency for the system

$$(2.1') \qquad x \in D, \quad Tx \underset{0}{\prec} 0.$$

Clearly the weak inconsistency of (2.2″) implies that for any two nets $\{V_\varepsilon, \varepsilon \in E\} \subset Y$, $\{W_\varepsilon, \varepsilon \in E\} \subset Z$ converging to 0, where V_ε and W_ε are regularizing sets for N and M, there exists ε_0 such that the system (2.2) is $(V_{\varepsilon_0}, W_{\varepsilon_0})$-inconsistent. Hence:

Corollary 2.1. *If the system* (2.1′) *is weakly consistent and the system* (2.2″) *weakly inconsistent, then there exist two linear forms* $g(y)$, $h(z)$ *with the properties stated in Theorem* 2.1.

Taking $Y = N = X$, $Tx \equiv x$, so that $g \equiv 0$ we have also:

Corollary 2.2. *If* $\overset{0}{M} \neq \emptyset$ *and if the system*

$$x \in D, \quad Sx \precsim 0 \tag{2.4}$$

is inconsistent, then there exists a continuous positive linear form $h(z) \not\equiv 0$ *on* Z, *such that*

$$(\forall x \in D) \ h(Sx) \geqslant 0.$$

Corollary 2.3. *If* W_0 *is a regularizing set for the cone* M, *and if the system* (2.4) *is* W_0*-inconsistent, then there exists a linear form* $h(z)$ *with the above-stated properties.*

Corollary 2.4. *If the system*

$$x \in D, \quad Sx \precsim 0$$

is weakly inconsistent, then there exists a linear form $h(z)$ *with the above-stated properties.*

3. The basic theorem. For some purposes in applications it is desirable to weaken the conditions $\overset{0}{\widehat{N+V_0}} \neq \emptyset$, $\overset{0}{\widehat{M+W_0}} \neq \emptyset$.

Let us denote by B_Y (B_Z) a local base at the origin in the linear topological space Y (Z), all consisting of balanced neighborhoods.

First we notice the following:

Lemma 2.2. *Let* V_0, W_0 *be qualified sets for cones* N, M. *The following two conditions are equivalent*:

1) V_0, W_0 *are regularizing sets for* N, M *and the system* (2.1) *is* V_0*-consistent*;

2) *There exist* $V \in B_Y$, $W \in B_Z$ *and a point* $\bar{z} \in Z$ *such that for every* $v \in V$ *and every* $w \in W$ *the system*

$$x \in D, \quad Tx \precsim v + (V_0), \quad Sx \precsim \bar{z} + w + (W_0)$$

is consistent.

Proof. Let condition 1) be satisfied, and denote by $\bar{x}$ a V_0-solution of (2.1) and by z_0 an interior point of $M + W_0$. If V is a neighborhood in Y and W a neighborhood in Z such that $V - T\bar{x} \subset \overset{0}{\widehat{N+V_0}}$, $W + z_0 \subset \overset{0}{\widehat{M+W_0}}$ then for every $v \in V$ and every $w \in W$ we have

$$T\bar{x} \precsim v + (V_0), \quad S\bar{x} \precsim (S\bar{x} + z_0 + w) + (W_0)$$

so that condition 2) is satisfied with $\bar{z} = S\bar{x} + z_0$.

Conversely if condition 2) is satisfied, then obviously

$$\overset{0}{\widehat{N+V_0}} \neq \emptyset, \qquad \overset{0}{\widehat{M+W_0}} \neq \emptyset$$

and taking $v = 0$ we see that the system (2.1) is V_0-consistent.

The preceding lemma motivates the next definition. We shall say that the *condition* (GS) *with respect to qualified sets* V_0, W_0 (or condition $(\mathrm{GS})_0$, when $V_0 = \overset{0}{N}$, $W_0 = \overset{0}{M}$) is satisfied if there exist $V \in B_Y$, $W \in B_Z$ and $\bar{z} \in Z$ such that for every $v \in V$ and every $w \in W$ the system

$$x \in D, \qquad Tx \preceq v + (V_0), \qquad Sx \preceq \bar{z} + w + (W_0)$$

is consistent.

Clearly condition (GS) implies the consistency of the system $x \in D$, $Tx \preceq 0 + (V_0)$. If A is the set definited by

$$A = \{(y, z) \in Y \times Z\colon\ (\exists x \in D)\, Tx \preceq y + (V_0),\ Sx \preceq z + (W_0)\} \tag{2.5}$$

it is readily verified that condition (GS) is equivalent to requiring the existence of a point $(0, \bar{z})$ belonging to the interior of A.

THEOREM 2.3. *Suppose the condition* (GS) *with respect to* V_0, W_0 *is satisfied. The system*

$$x \in D, \qquad Tx \prec 0 + (V_0), \qquad Sx \prec 0 + (W_0) \tag{2.6}$$

is inconsistent only if there exist two continuous positive linear forms $g(y)$ *on* Y *and* $h(z)$ *on* Z, *such that* $h(z) \not\equiv 0$ *and*

$$(\forall x \in D)\ g(Tx) + h(Sx) \geqslant 0. \tag{2.7}$$

Before proceeding to the proof, let us remark that Theorem 2.3 includes as a special case Theorem 2.2, which itself contains as a special case the first part of Theorem 2.1. Indeed, Theorem 2.2 reduces obviously to the first part of Theorem 2.1 when $V_0 = \{0\}$, $W_0 = \{0\}$ (or $V_0 = \overset{0}{N}$, $W_0 = \overset{0}{M}$). On the other hand, noting that $N + \overset{0}{\widehat{N+V_0}} = \overset{0}{\widehat{N+V_0}}$ we may rewrite any relation of the form $y_1 \prec y_2 + (V_0)$ into the form $y_1 \preceq y_2 + (V_0')$, with $V_0' = \overset{0}{\widehat{N+V_0}}$ and so Theorem 2.2 appears as a particular case of Theorem 2.3, when we take $V_0' = \overset{0}{\widehat{N+V_0}}$, $W_0' = \overset{0}{\widehat{M+W_0}}$ in the role of V_0 and W_0.

Proof of Theorem 2.3. Since D, $N+V_0$, $M+W_0$ are convex sets and T, S are convex maps, the set A defined by (2.5) is convex, and since the system (2.6) is inconsistent, A does not contain the origin $(0, 0)$. But on account of condition (GS), this set has an interior point $(0, \bar{z})$. Hence, according to the Eidelheit theorem, there exists a continuous linear form

$F(y, z) \not\equiv 0$ on $Y \times Z$ satisfying $F(y, z) \geqslant 0$ for every $(y, z) \in A$. Clearly $Tx-(Tx+y_0) \precsim 0+(V_0)$, $Sx-(Sx+z_0) \precsim 0+(W_0)$ for all $x \in D$, $y_0 \succsim 0+(V_0)$, $z_0 \succsim 0+(W_0)$. Consequently, $(Tx+y_0, Sx+z_0) \in A$ and hence, putting $F(y, 0) = g(y)$, $F(0, z) = h(z)$, we have

$$g(Tx+y_0)+h(Sx+z_0) \geqslant 0,$$

which yields (2.7) by making $y_0 \to 0$, $z_0 \to 0$. (We recall that 0 in Y (in Z) belongs to the closure of $N+V_0$ (of $M+W_0$).)

Further, since

$$Tx-(Tx+\lambda y+y_0) \precsim 0+(V_0), \qquad Sx-(Sx+\mu z+z_0) \precsim 0+(W_0)$$

for all $y \in N$, $z \in M$, $\lambda > 0$, $\mu > 0$, we have

$$(Tx+\lambda y+y_0,\; Sx+\mu z+z_0) \in A$$

so that

$$g(Tx+\lambda y+y_0)+h(Sx+\mu z+z_0) \geqslant 0$$

or

$$g(Tx)+\lambda g(y)+g(y_0)+h(Sx)+\mu h(z)+h(z_0) \geqslant 0,$$

which yields $g(y) \geqslant 0$ by making $\lambda \to +\infty$ and $h(z) \geqslant 0$ by making $\mu \to +\infty$. Thus g and h are positive linear forms. On the other hand, if $h(z) \equiv 0$, then $g(y) \geqslant 0$ for all $(y, z) \in A$ and consequently, since $(0, \bar{z})$ is an interior point of A, $g(y) \geqslant 0$ for all y in some neighborhood of 0 (in Y). This may occur only if $g(y) \equiv 0$, for a linear form which takes only non-negative values everywhere in some neighborhood of the origin must vanish identically. Thus $g(y) \equiv 0$, $h(z) \equiv 0$, in contradiction with $F(y, z) \not\equiv 0$. Hence $h(z) \not\equiv 0$, which completes the proof.

4. The regularity condition (GS). By Lemma 2.2, the condition (GS) with respect to V_0, W_0 is satisfied if, V_0, W_0 being regularizing sets for N, M, there exists a point x such that $x \in D$, $Tx \precsim 0+(V_0)$.

In particular, taking $V_0 = \overset{0}{N}$, $W_0 = \overset{0}{M}$, we see that condition $(\mathrm{GS})_0$ is satisfied if there exists a solution x to the system $x \in D$, $Tx \precsim 0$.

Let us discuss other cases, in which condition (GS) with respect to prescribed qualified sets V_0, W_0 is satisfied.

Let X be a linear topological space. The map T (of $D \subset X$ into Y) will be called *sub-open* at a point $\bar{x} \in \overset{0}{D}$ if

(2.8) $(\forall U \in B_X, \bar{x}+U \subset D)\,(\exists V \in B_Y)\,(\forall v \in V)\,(\exists x \in U+\bar{x})\; Tx \precsim T\bar{x}+v.$

The map S (of $D \subset X$ into Z) will be called *sub-continuous* at $\bar{x}$ if

(2.9) $(\forall W \in B_Z)\,(\exists U \in B_X)\,(\forall x \in U+\bar{x})\,(\exists w \in W)\; Sx \precsim S\bar{x}+w.$

THEOREM 2.4. *Suppose that:*

1) $\overset{0}{\widehat{M+W_0}} \neq \emptyset$;

2) *the system* $x \in \overset{0}{D}$, $Tx \precsim 0+(V_0)$ *has a solution* $\bar{x}$;

3) *the map* T *is sub-open at* $\bar{x}$ *and the map* S *is sub-continuous at* $\bar{x}$.

Then condition (GS) *with respect to* V_0, W_0 *is satisfied.*

Proof. It suffices to show that for every $U \in B_X$ verifying $\bar{x}+U \subset D$ there exist $V \in B_Y$ and $W \in B_Z$ such that

$$(\forall v \in V)\,(\forall w \in W)\,(\exists x \in U+\bar{x})\;\; Tx-v \precsim 0+(V_0),\quad Sx-(\bar{z}+w) \precsim 0+(W_0),$$

where $\bar{z} = S\bar{x}+z_0$, z_0 being any fixed interior point of $M+W_0$. For this purpose, let $W \in B_Z$ be a neighborhood such that $W+W+z_0 \subset \overset{0}{\widehat{M+W_0}}$. Since S is sub-continuous at $\bar{x}$, a neighborhood $U' \in B_X$ can be found such that

$$U' \subset U, \qquad (\forall x \in U'+\bar{x})\,(\exists w' \in W)\;\; Sx \precsim S\bar{x}+w'.$$

Since, further, T is sub-open at $\bar{x}$ there exists $V \in B_Y$ satisfying $(\forall v \in V)$ $(\exists x \in U'+\bar{x})$ $Tx \precsim T\bar{x}+v$, hence $Tx-v \precsim 0+(V_0)$ for $T\bar{x} \precsim 0+(V_0)$.

The sets V and W are those required, because $x \in U'+\bar{x}$ implies $Sx = S\bar{x}+w'-z$ for some $w' \in W$ and some $z \in M$, hence

$$(\forall w \in W)\;(S\bar{x}+z_0+w)-Sx$$
$$= (S\bar{x}+z_0+w)-(S\bar{x}+w'-z) \subset W+W+z_0+z \subset \overset{0}{\widehat{M+W_0}}+z \subset \overset{0}{\widehat{M+W_0}},$$

i.e.

$$Sx-(\bar{z}+w) \precsim 0+(W_0).$$

A sharper form of the previous theorem is the following

THEOREM 2.5. *Let* $X = X_1 \times X_2$, $N = N_1 \times N_2 \subset Y_1 \times Y_2 = Y$, $V_0 = V_{01} \times V_{02}$, *where* V_{01} *and* V_{02} *are qualified sets for* N_1 *and* N_2 *respectively,* $Tx = (T_1x, T_2x)$, *where* $T_1x \in Y_1$, $T_2x \in Y_2$.

Suppose that

1) $\overset{0}{\widehat{N_2+V_{02}}} \neq \emptyset$, $\overset{0}{\widehat{M+W_0}} \neq \emptyset$;

2) *the system* $x \in D$, $T_1x \precsim 0+(V_{01})$, $T_2x \precsim 0+(V_{02})$ *has a solution* $\bar{x} = (\bar{x}_1, \bar{x}_2)$ *such that* $\bar{x}_1$ *is an interior point (in* X_1*) of the set* $D_1 = \{x_1 \in X_1\colon (x_1, \bar{x}_2) \in D\}$;

3) *the map* $T_1(x_1, \bar{x}_2)$ *of* D_1 *into* Y_1 *is sub-open at* $\bar{x}_1$, *and the maps* $T_2(x_1, \bar{x}_2)$, $S(x_1, \bar{x}_2)$ *of* D_1 *into* Y_2 *and* Z *resp., are sub-continuous at* $\bar{x}_1$.

Then condition (GS) *with respect to* V_0, W_0 *is satisfied.*

Proof. Setting $y_0 = -T_2\bar{x}$ we have $y_0 \precsim 0+(V_{02})$. Let z_0 denote an interior point of $M+W_0$, $W \in B_Z$ a neighborhood such that $W+W+z_0$

$\subset \overbrace{M+W_0}^{0}$. From the sub-continuity of $S(x_1, \bar{x}_2)$ at $\bar{x}_1$ follows the existence of some $U_1 \in B_{X_1}$, $U_1+\bar{x}_1 \subset D_1$ such that

$$(\forall x_1 \in U_1+\bar{x}_1)\,(\exists w' \in W)\;\; S(x_1, \bar{x}_2) \preceq S(\bar{x}_1, \bar{x}_2)+w'.$$

On the other hand, by the same argument as that used in the demonstration of Theorem 2.4, we can show that to U_1 there correspond $V_1 \in B_{Y_1}$ and $V_2 \in B_{Y_2}$ such that $(\forall v_1 \in V_1)\,(\forall v_2 \in V_2)\;\;(\exists x_1 \in U_1+\bar{x}_1)\;\; T_1(x_1, \bar{x}_2)-v_1 \preceq 0+(V_{01})$, $T_2(x_1, \bar{x}_2)-(T_2\bar{x}+y_0+v_2) \prec 0+(V_{02})$. From the relation $T_2\bar{x}+y_0 = 0$ it follows then $T_2(x_1, \bar{x}_2)-v_2 \prec 0+(V_{02})$. Since $x_1 \in U_1+\bar{x}_1$ implies $S(x_1, \bar{x}_2) = S\bar{x}+w'-z$ for some $w' \in W$ and some $z \in M$, we may write for each $w \in W$:

$$(S\bar{x}+z_0+w)-S(x_1, \bar{x}_2) \subset W+W+z_0+z \subset \overbrace{M+W_0}^{0}+z \subset \overbrace{M+W_0}^{0},$$

so that $(\forall v \in V)\,(\forall w \in W)\,\big(\exists x = (x_1, \bar{x}_2) \in D\big)\;\; Tx-v \preceq 0+(V_0)$, $Sx-(S\bar{x}+$ $+z_0+w) \prec 0+(W_0)$, which is the desired result.

Remarks. 1. Clearly an open map is sub-open at each point and a continuous map is sub-continuous at each point. Hence:

Theorem 2.4 is valid if the map T is open and the map S is continuous, all other conditions being supposed satisfied.

Note also that in case $Z = R^1$ the sub-continuity defined by (2.9) is equivalent (for convex functions) to the continuity, because if (2.9) is fulfilled the function Sx is majored in the neighborhood $U+\bar{x}$ of $\bar{x}$, and it is a known fact that this implies the continuity of Sx (cf. [4], prop. 21).

2. A map T of X into Y is said to be *affine* if the map $x \mapsto Tx-T(0)$ is linear, or, equivalently, if

$$T\big(\alpha x_1+(1-\alpha)x_2\big) = \alpha Tx_1+(1-\alpha)Tx_2$$

for each pair $x_1, x_2 \in X$ and for each scalar α.

According to a classical Banach theorem, generalized by A. Robertson and W. Robertson [12], if X is a F-space, Y a barreled space, then every continuous linear map of X onto Y is open. Hence:

Theorem 2.5 is valid if X_1 is a F-space, Y_1 a barreled space and if $T_1(x_1, \bar{x}_2)$ can be extended to a continuous affine map of X_1 onto Y_1.

3. Suppose that the topology under consideration in each space X, Z is the finest locally convex one. Then it is easy to see that at each point $\bar{x} \in \overset{0}{D}$ any convex map S of D into Z is sub-continuous, and any convex map T of X onto Y is sub-open.

Indeed, if $\bar{x} \in \overset{0}{D}$, $\bar{z} = S\bar{x}$, then it follows from the inequality

$$S\big(\alpha x+(1-\alpha)\bar{x}\big) \preceq \alpha z+(1-\alpha)\bar{z},$$

which holds whenever $x \in D$, $z = Sx$, $0 \leqslant a \leqslant 1$, that for every $W \in B_Z$ the set of all x such that $Sx \precsim S\bar{x} + w$ for some $w \in W$ is convex and absorbing, hence is a neighborhood of $\bar{x}$. In an analogous manner for every $U \in B_X$ the set of all v such that $Tx \precsim T\bar{x} + v$ for some $x \in U + \bar{x}$ is a neighborhood of $T\bar{x}$. Hence:

If the topology of each space X_1, Y_1, Y_2, Z is the finest locally convex one (in particular, if these spaces are finite-dimensional) then condition 3) *in Theorem* 2.5 *is always fulfilled, provided $T_1(x_1, \bar{x}_2)$ can be extended to a convex map of X_1 onto Y_1.*

4. Finally, let us notice that (2.8) holds (i.e. T is sub-open at $\bar{x}$), provided for *some bounded set* $U \subset D$ there exists $V \in B_Y$ such that

$$(\forall v \in V)\,(\exists x \in U + \bar{x})\;\; Tx \precsim T\bar{x} + v.$$

Indeed let $U' \in B_X$. Since U is bounded we have $aU \subset U'$ for some $a > 0$. If $a \geqslant 1$ then $U \subset U'$ because U' is convex. If $a < 1$ then for each $v \in V$ and the element $x \in U + \bar{x}$ such that $Tx \precsim T\bar{x} + v$ we may write

$$T\big(ax + (1-a)\bar{x}\big) \precsim aTx + (1-a)T\bar{x} \precsim T\bar{x} + av.$$

Taking $V' = aV$ and noting that $ax + (1-a)\bar{x} = \bar{x} + a(x - \bar{x}) \in U' + \bar{x}$ we then have

$$(\forall v' \in V')\,(\exists x' \in U' + \bar{x})\, Tx' \precsim T\bar{x} + v'.$$

The same remark can be made about relation (2.9).

§ 3. The Hahn-Banach Theorem and related propositions as corollaries of inconsistency theorems

The main theorems established in the previous section enable us to derive by a simple, unified method many known propositions concerning convex inequalities, and convex sets. It turns out that each of these propositions corresponds merely to a special condition imposed on the structure of spaces X, Y, Z and on the maps T, S involved in the above-mentioned theorems.

1. The Hahn–Banach Theorem. First we note that the following sharp form of the Hahn–Banach Theorem is a direct consequence of Theorem 2.1.

THEOREM 3.1. *Let X be a linear topological space, Y a linear space, let D be a non-void convex subset of the linear space $X \times Y$, let $\varphi(x)$ be a finite continuous sublinear function on X, $f(x, y)$ a finite convex function on D. There exists a continuous linear form $t(x)$ on X satisfying*

$$(\forall x \in X)\;\; t(x) \leqslant \varphi(x), \tag{3.1}$$

$$\big(\forall (x, y) \in D\big)\;\; t(x) + f(x, y) \geqslant 0 \tag{3.2}$$

if and only if

$$(3.3) \qquad \big(\forall (x, y) \in D\big)\ \varphi(x) + f(x, y) \geqslant 0.$$

Proof. It is enough to show that the condition is sufficient. Let

$$\tilde{X} = X \times Y \times R^1, \quad \tilde{Y} = X \times R^1, \quad Z = R^1,$$

$$N = \{(x, a) \in X \times R^1 \colon \varphi(x) < a\}, \quad M = \{a \in R^1 \colon a > 0\},$$

$$T(x, y, a) \equiv -(x, a), \quad S(x, y, a) = a + f(x, y).$$

It is then readily verified that conditions of Theorem 2.1 are satisfied. Hence one can find a linear form on $\tilde{Y} = X \times R^1$, i.e. a linear form $t(x)$ on X and a constant λ, such that:

(a) $t(x) - \lambda a \leqslant 0$ for every (x, a) satisfying $\varphi(x) < a$;

(b) $t(x) - \lambda a + a + f(x, y) \geqslant 0$ for every $(x, y) \in D$, $a \in R^1$.

The latter relation implies $\lambda = 1$, otherwise a could be chosen to verify $t(x) - \lambda a + a + f(x, y) < 0$ (for prescribed x, y). Then (a) yields $t(x) \leqslant \varphi(x)$ for every x and the proof is complete.

Clearly the previous proposition reduces to the Hahn–Banach Theorem when D is a subspace of X, f is a linear function on D, for then condition (3.2) implies $(\forall x \in D)\ t(x) = -f(x)$.

By making use of Theorem 2.4, it is easy to show that the previous proposition remains valid if we replace in it the special conditions upon D, $\varphi(x)$, $f(x, y)$ by the following ones: $\big(\exists (x, y) \in \overset{0}{D}\big)\ \varphi(x) < +\infty$; $f(x, y)$ is continuous on $\overset{0}{D}$.

2. Theorems on positive linear functionals.

THEOREM 3.2 (Krein). *Let Y be a linear topological space with a positive cone N, let X be a subspace of Y, such that $\overset{0}{N} \cap X \neq \emptyset$. Then every linear form $f(x)$ on X, positive with respect to the cone $N \cap X$, can be extended to a continuous positive linear form on Y.*

Indeed, setting $Z = R^1$, $M = \{z \in R^1 \colon z \geqslant 0\}$, $Tx \equiv -x$, $Sx = f(x)$, $D = X$, we obtain, by applying Theorem 2.1, a continuous positive linear form $g(y)$ on Y, such that

$$(\forall x \in X)\ -g(x) + f(x) \geqslant 0,$$

and, consequently

$$(\forall x \in X)\ g(x) = f(x),$$

since X is a subspace.

From Theorem 2.4 follows a more general result:

Let Y be a linear space, with a positive cone N, let X be a subspace of Y, such that for each $y \in Y$ the sets $\{x \in X \colon x \preceq y\}$ and $\{x \in X \colon x \succeq y\}$

are either both empty or both non-empty. Then every linear form on X, positive with respect to the cone $N \cap X$, can be extended to a positive linear form on Y.

Indeed, let us consider the finest locally convex topology in each space X, Y, and let

$$Y_1 = \{y \in Y: (\exists x \in X) \ -x \precsim y\}$$

(because of assumptions Y_1 is a subspace of Y).

For any convex neighborhood $U \in B_X$ the set

$$V_1 = \{y \in Y_1: (\exists x \in U) - x \precsim y\}$$

is convex and absorbing, hence is a neighborhood in Y_1, so that the map $Tx \equiv -x$ is sub-open at $\bar{x} = 0$ and the conditions stated in Theorem 2.4 are fulfilled.

THEOREM 3.3 (Farkas–Minkowski Theorem [9]). *Let X be a linear topological space, D a non-void convex subset of X, $p(x)$ a convex function on D; let Y be a linear topological space equipped with a positive cone N, T a convex map of D into Y, such that $(\exists x \in D)$ $Tx \precsim 0$. If $p(x) \geqslant 0$ for all $x \in D$ satisfying $Tx \precsim 0$, then there exists a continuous positive linear form $g(y)$ on Y, such that*

$$(\forall x \in D) \ p(x) \geqslant -g(Tx). \tag{3.4}$$

This is merely a particular case of Theorem 2.1, when

$$Z = R^1, \quad M = \{z \in R^1: z \geqslant 0\}, \quad Sx = p(x).$$

If both $p(x)$ and $T(x)$ are linear and if $D = X$, it follows from (3.4) that $p(x) \equiv -g(Tx)$, hence $p = -T^* g$ where T^* is the conjugate of T defined by $T^* g(x) \equiv g(Tx)$. Choosing as positive cone in the dual space Y^* (X^*) the set of all linear forms which take only non-negative values on N $\big(T^{-1}(N)\big)$, we can thus state the following proposition, known as the abstract form of the Farkas–Minkowski lemma.

If $p \geqslant 0$ then $p = T^* g$ for some $g \succeq 0$.

It should be noted that, on account of Theorem 2.4 the condition $(\exists x \in D)$ $Tx \precsim 0$ in Theorem 3.5 might be replaced by: $(\exists \bar{x} \in \overset{0}{D})$ $T\bar{x} \precsim 0$ and T is sub-open at $\bar{x}$ (which is always satisfied if X is a F-space, Y is a barrelled space and T is a continuous affine map of X onto Y), whereas $p(x)$ is continuous at $\bar{x}$.

3. Theorems on conjugate sets. Given a convex subset Ω of a linear topological space X, we shall denote by Ω^* the set of all continuous affine functionals $\omega(x)$ such that $(\forall x \in \Omega)$ $\omega(x) \geqslant 0$.

Dubovitski and Milyutin have proved (in a slightly weaker form) the next two propositions which play a basic role in their work [15].

THEOREM 3.4. *Let Ω_i $(i = 1, 2, \ldots, n)$ be convex subsets of a linear topological space X, such that $\overset{0}{\Omega}_i \neq \emptyset$. In order that they have no common interior point in a convex set $\Omega_0 \subset X$ it is necessary and sufficient that there exist continuous affine functionals $\omega_i(x)$ $(i = 0, 1, \ldots, n)$, not all identically zero, such that $\omega_i \in \Omega_i^*$ $(i = 0, 1, \ldots, n)$ and $\sum_{i=0}^{n} \omega_i(x) \equiv 0$.*

When Ω_i are cones, this proposition follows from Corollary 2.2 by taking

$$\underbrace{Z = X \times \quad \times X}_{n}, \quad D = \Omega_0, \quad M = \Omega_1 \times \quad \times \Omega_n, \quad Sx = -\underbrace{(x, \ldots, x)}_{n}.$$

In the general case, it suffices to take $X \times R^1$ instead of X, and $Q_i = \{(x, a): a \geqslant 0, x \in a\Omega_i\}$ instead of Ω_i. Indeed we then obtain continuous linear forms

$$g_i(x) + \lambda_i a \quad (i = 1, 2, \ldots, n)$$

on $X \times R^1$ not all identically zero, such that

$$g_i(x) + \lambda_i a \geqslant 0 \quad \text{for all } (x, a) \in Q_i$$

and

$$\sum_{i=1}^{n} [g_i(x) + \lambda_i a] \geqslant 0 \quad \text{for all } (x, a) \in Q_0$$

which yields the desired result by putting

$$\omega_i(x) = g_i(x) + \lambda_i \ (i = 1, 2, \ldots, n), \quad \omega_0(x) = -\sum_{i=1}^{n} \omega_i(x).$$

THEOREM 3.5. *Let Ω_i $(i = 0, 1, \ldots, n)$ be convex subsets of a linear topological space X, such that the sets Ω_i $(i = 1, 2, \ldots, n)$ have a common interior point in Ω_0. Then*

$$\Big(\bigcap_{i=0}^{n} \Omega_i\Big)^* = \sum_{i=0}^{n} \Omega_i^*.$$

In fact, if Ω_i are cones, and if we consider a continuous linear form $\omega(x)$, then, by taking

$$Y = X^n, \quad Z = R^1, \quad D = \Omega_0, \quad N = \Omega_1 \times \quad \times \Omega_n,$$
$$M = \{z \in R^1: z \geqslant 0\}, \quad Tx = -(x, \ldots, x), \quad Sx = \omega(x)$$

and observing that the statement $\omega \in (\bigcap_{i=0}^{n} \Omega_i)^*$ is equivalent to the statement that the system

$$x \in D, \quad Tx \precsim 0, \quad Sx \prec 0$$

is inconsistent, we see at once, by virtue of Theorem 2.1, that $\omega \in (\bigcap_{i=0}^{n} \Omega_i)^*$ if and only if $\omega \in \sum_{i=0}^{n} \Omega_i^*$.

In the general case, we must only take $X\times R^1$ instead of X, $Q_i = \{(x,\alpha)\colon \alpha\geqslant 0, x\in \alpha\Omega_i\}$ instead of Ω_i $(i=0,1,\ldots,n)$ and $Sx=\omega(x)+ +\lambda\alpha$ instead of $\omega(x)$.

4. Saddle-point theorem in convex programming. Consider the following convex programming problem:

$$\min\{Sx\colon x\in D, Tx \precsim 0\}, \tag{3.5}$$

where D is a non-void convex subset of a linear space X, T and S being convex maps of D into Y and Z resp. An element x_0 is called a *feasible solution* if $x_0\in D$, $Tx_0\precsim 0$; it is an *optimal solution* if, moreover, no feasible solution x exists such that $Sx \prec Sx_0$.

From Theorem 2.1 follows the next proposition, as first proved in a somewhat weaker form by Hurwicz [1].

THEOREM 3.6. *If $x_0\in D$ is an optimal solution of* (3.5) *and if*

$$(\exists x)\ x\in D,\ Tx\prec 0 \tag{3.6}$$

then there exist two continuous linear forms $g_0(y)$ on Y and $h_0(z)$ on Z, such that $h_0(z)\not\equiv 0$ and

$$(\forall x\in D)\,(\forall g\in N^*)\ g_0(Tx)+h_0(Sx)\geqslant g(Tx_0)+h_0(Sx_0). \tag{3.7}$$

Conversely, if for an element $x_0\in D$ there exist two linear forms $g_0(y)$ and $h_0(z)$ with the stated properties, and if the positive cone in Y is closed, Y being a locally convex space, then x_0 is an optimal solution of (3.5).

The first statement is an immediate consequence of Theorem 2.1, because the optimality of x implies the inconsistency of the system

$$x\in D,\quad Tx\prec 0,\quad Sx-Sx_0\prec 0. \tag{3.8}$$

To prove the second statement, we observe that condition (3.7) implies $g_0(Tx_0)\geqslant g(Tx_0)$ for all $g\in N^*$, hence $g(Tx_0)\leqslant 0$ for all $g\in N^*$. If $-Tx_0\notin N$ there would exist an open convex neighborhood U of $-Tx_0$ not meeting N, and for the linear form $g_1\not\equiv 0$ which separates U from N, i.e. such that

$$(\forall y\in N)\,(\forall y'\in U)\ g_1(y)\geqslant 0\geqslant g_1(y'),$$

we would have

$$g_1\in N^*,\quad g_1(-Tx_0)<0.$$

This contradiction shows that $Tx_0\precsim 0$ and so x_0 is a feasible solution, hence an optimal solution, because under the stated conditions the system (3.8) is inconsistent by virtue of the second part of Theorem 2.1.

Condition (3.6) is known as *Slater regularity condition* [1].

If $Z=R^1$, i.e. if Sx is a real-valued function, we may assume $h_0(z)\equiv 1$ and relation (3.7) becomes

$$(\forall x\in D)\,(\forall g\in N^*)\ g_0(Tx)+Sx\geqslant g(Tx_0)+Sx_0.$$

Setting $F(x, g) = g(Tx)+Sx$ we may write

$$(3.9) \qquad (\forall x \in D)\,(\forall g \in N^*)\;\; F(x, g_0) \geqslant F(x_0, g).$$

A pair $(x_0, g_0) \in D \times N^*$ which satisfies (3.9) is often called a *saddle-point* of the function $F(x, g) = g(Tx)+Sx$ in the domain $x \in D$, $g \in N^*$. Thus Theorem 3.6 can be equally formulated as follows:

Let Y be a locally convex space, let the positive cone N in Y be a closed set, and let the Slater regularity condition be satisfied. Then an element $x_0 \in D$ is an optimal solution if and only if there exists a positive linear form $g_0(y)$ on Y such that the pair (x_0, g_0) is a saddle-point of the function $F(x, g) = g(Tx)+Sx$ in the domain $x \in D$, $g \in N^*$

On account of Theorems 2.4 and 2.5 and of remark 3 to Theorem 2.5 the Slater regularity condition may be replaced here by one of the following conditions G_1, G_2, G_3:

G_1. The system $x \in \overset{0}{D}$, $Tx \precsim 0$ has a solution $\bar{x}$ such that the map T is sub-open at $\bar{x}$ and the function S sub-continuous at $\bar{x}$.

This is satisfied, in particular, if $D = X$, S is a continuous convex function, and T is a continuous affine map of X onto Y, X being a F-space, Y a barrelled space. As a special case we thus obtain a recent result of V. L. Levin ([17], Theorem 7).

G_2. $X = X_1 \times X_2$, $N = N_1 \times N_2 \subset Y_1 \times Y_2 = Y$, $Tx = (T_1x, T_2x)$, the system $x \in D$, $T_1x \precsim 0$, $T_2x \prec 0$ has a solution $\bar{x} = (\bar{x}_1, \bar{x}_2)$ such that $\bar{x}_1$ is an interior point of $D_1 = \{x_1 \in X_1 \colon (x_1, \bar{x}_2) \in D\}$ and such that the map $T_1(x_1, \bar{x}_2)$ is sub-open at $\bar{x}_1$, and the maps $T_2(x_1, \bar{x}_2)$, $S(x_1, \bar{x}_2)$ sub-continuous at $\bar{x}_1$.

The latter condition on T_1, T_2, S, is satisfied in particular if $T_2(x_1, \bar{x}_2)$, $S(x_1, \bar{x}_2)$, are continuous functions of x_1 on D_1, and $T_1(x_1, \bar{x}_2)$ is a continuous affine map of X_1 onto Y_1, X_1 being a F-space, Y_1 a barrelled space. As a consequence we thus obtain a result of E. G. Golshtein ([14], Theorem 2.5).

G_3. $X = X_1 \times X_2$, $N = N_1 \times N_2 \subset Y_1 \times Y_2 = Y$, $Tx = (T_1x, T_2x)$, the system $x \in D, T_1x \precsim 0$, $T_2x \prec 0$ has a solution $\bar{x} = (\bar{x}_1, \bar{x}_2)$ such that $\bar{x}_1$ is an interior point of $D_1 = \{x_1 \in X_1 \colon (x_1, \bar{x}_2) \in D\}$, the topology on each space X_1, Y_1, Y_2 is the finest locally convex one, and $T_1(x_1, \bar{x}_2)$ maps X_1 onto Y_1.

Thus, we have the following proposition which reduces to the well-known Kuhn–Tucker theorem when spaces X, Y are finite-dimensional and T_1 is an affine map ([10], see also [2]).

COROLLARY 3.1. *Let $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$ be linear spaces, let $N = N_1 \times N_2$ be the positive cone in Y, let $D = D_1 \times D_2$ be a convex subset of X ($D_1 \subset X_1$, $D_2 \subset X_2$), and let T_1, T_2 be convex maps of D into Y_1 and Y_2 resp., $S(x)$ a convex function on D. Suppose that*

$$(3.10) \qquad (\exists \bar{x})\;\; \bar{x} \in \overset{0}{D}_1 \times D_2,\;\; T_1\bar{x} \precsim 0,\;\; T_2\bar{x} \prec 0$$

and that $T_1(x_1, \bar{x}_2)$ maps X_1 onto Y_1.

Then an element $x_0 \in D$ is an optimal solution if and only if there exists a positive linear form $g_0(y)$ on Y such that the pair (x_0, g_0) is a saddle-point of the function $g(T_1x, T_2x)+S(x)$ in the domain $x \in D$, $g \in N^*$.

5. Duality theorem in convex programming. The notation being the same as before, consider the problem

$$\inf\{S(x):\ x\in D,\ Tx \precsim 0\}, \tag{3.11}$$

where $S(x)$ is a real-valued convex function, i.e. a convex map of D into $Z = R^1$. The lower bound (3.11) is called the *value* of the problem, and the lower bound $\inf \varliminf_{\varepsilon} S(x_\varepsilon)$, where $\{x_\varepsilon, \varepsilon \in E\}$ is a weak solution of the system

$$x \in D, \quad Tx \precsim 0 \tag{3.12}$$

is called its *weak value*. If α denotes the value, α' the weak value, then obviously $\alpha' \leqslant \alpha$.

When Y is a locally convex space and its positive cone is closed, we have

$$\sup_{g\in N^*}[g(Tx)+S(x)] = \begin{cases} S(x) & \text{if } Tx \precsim 0, \\ +\infty & \text{else,} \end{cases} \tag{3.13}$$

because $-Tx \notin N$ implies the existence of a linear form $g \in N^*$ such that $g(Tx) > 0$. Hence, setting

$$\varphi(x) = \sup_{g\in N^*}\{g(Tx)+S(x)\}$$

we have

$$\inf\{\varphi(x):\ x\in D\} = \alpha,$$

and so problem (3.11) is equivalent to the following one

$$\inf\{\varphi(x):\ x\in D\}. \tag{3.14}$$

For this reason the problem

$$\sup\{\psi(g):\ g\in N^*\}, \tag{3.15}$$

where $\psi(g) = \inf\limits_{x\in D}\{g(Tx)+S(x)\}$ is called the *dual* of (3.11).

Let $\tilde{\alpha}$ denote the value of (3.15).

THEOREM 3.7 (Golshtein [14]). *If the constraints* (3.12) *are weakly consistent then the weak value of the primal problem* (3.11) *is equal to the value of the dual problem* (3.15).

This follows from Corollary 2.1. Indeed, if $\alpha' > -\infty$ then for any number $\beta < \alpha'$ the system

$$x\in D, \quad Tx \precsim 0, \quad Sx-\beta \leqslant 0 \tag{3.16}$$

is weakly inconsistent, so that there exists a linear form $g \in N^*$ verifying

$$(\forall x \in D)\ g(Tx) + Sx \geqslant \beta, \tag{3.17}$$

hence $\tilde{\alpha} \geqslant \beta$, and consequently $\tilde{\alpha} \geqslant \alpha'$, because β may be taken arbitrarily near to α'. To prove the converse inequality, observe that for every weak solution $\{x_\varepsilon, \varepsilon \in E\}$ of the system (3.12) we have $x_\varepsilon \in D$, $Tx_\varepsilon + v_\varepsilon \precsim 0$, where $v_\varepsilon \to 0$ and so for each $g \in N^*$:

$$g(Tx_\varepsilon + v_\varepsilon) + S(x_\varepsilon) \leqslant S(x_\varepsilon).$$

That yields

$$\underline{\lim}_{\varepsilon} \{g(Tx_\varepsilon + v_\varepsilon) + S(x_\varepsilon)\} \leqslant \underline{\lim}_{\varepsilon} S(x_\varepsilon),$$

hence

$$\sup \psi(g) \leqslant \inf \underline{\lim}_{\varepsilon} S(x_\varepsilon)$$

as required.

The problem (3.11) is said to be *correctly set* if $\alpha = \alpha'$, i.e. if its value is equal to its weak value, or, equivalently (by Theorem 3.7), to the value of its dual.

COROLLARY 3.2. *Under the assumption that Y is a locally convex space with a closed positive cone, the problem* (3.11) *is correctly set if and only if*

$$\inf_{x \in D} \sup_{g \in N^*} F(x, g) = \sup_{g \in N^*} \inf_{x \in D} F(x, g). \tag{3.18}$$

A sequence $\{x_k, g_k\} \subset D \times N^*$ is called a *weak saddle-point* for $F(x, g)$ on $D \times N^*$ if there exists a sequence $\varepsilon_k \geqslant 0$, $\varepsilon_k \to 0$ such that for every k:

$$(\forall x \in D)\ (\forall g \in N^*)\ F(x, g_k) + \varepsilon_k \geqslant F(x_k, g) - \varepsilon_k. \tag{3.19}$$

A sequence $\{x_k\} \subset D$ is to *solve* problem (3.11) if $Tx_k \precsim 0$ and $Sx_k \to \alpha$.

COROLLARY 3.3. *There exists a weak saddle-point for function $F(x, g)$ on $D \times N^*$ if and only if the problem* (3.11) *is correctly set and has a finite value, and in this case a sequence $\{x_k\} \subset D$ solves the problem* (3.11) *if and only if there exists a sequence $\{g_k\} \subset N^*$ such that $\{x_k, g_k\}$ is a weak saddle-point for $F(x, g)$ on $D \times N^*$.*

Indeed, suppose that $\{x_k, g_k\}$ is a weak saddle-point. From (3.19) it follows that

$$\sup_{g \in N^*} \inf_{x \in D} F(x, g) \geqslant \inf_{x \in D} \sup_{g \in N^*} F(x, g)$$

and since the converse inequality always holds, relation (3.18) is true, which means that the problem (3.11) is correctly set. Since $\inf_{x \in D} F(x, g_k) \geqslant F(x_k, g_k) - 2\varepsilon_k > -\infty$, and similarly $\sup_{g \in N^*} F(x_k, g) \leqslant F(x_k, g_k) + 2\varepsilon_k < +\infty$, the common value α of the two sides of (3.18) is finite. Further, from (3.13) and relation $\varphi(x_k) < +\infty$ it follows that $Tx_k \precsim 0$, hence $g(Tx_k) \leqslant 0$ for each $g \in N^*$ and so

$$\varphi(x_k) = \sup_{g \in N^*} [g(Tx_k) + S(x_k)] = S(x_k).$$

Since

$$\alpha \leqslant \varphi(x_k) = \sup_{g\in N^*} F(x_k, g) \leqslant F(x_k, g_k) + 2\varepsilon_k$$
$$\leqslant \inf_{x\in D} F(x, g_k) + 4\varepsilon_k = \psi(g_k) + 4\varepsilon_k \leqslant \alpha,$$

we have $\varphi(x_k) = S(x_k) \to \alpha$ (and also $\psi(g_k) \to \alpha$), i.e. $\{x_k\}$ solves problem (3.11) (whereas $\{g_k\}$ solves problem (3.15)).

Suppose now the problem (3.11) correctly set, with a finite value α and consider a sequence $\{x_k\} \subset D$ solving (3.11).

Let $\{g_k\} \subset N^*$ be any sequence solving (3.15). Putting

$$\varepsilon_k = \max\{|\alpha - \varphi(x_k)|, |\alpha - \psi(g_k)|\},$$

we have $\varepsilon_k \to 0$ and

$$\varphi(x_k) = \sup_{g\in N^*} F(x_k, g) \leqslant \alpha + \varepsilon_k, \qquad \psi(g_k) = \inf_{x\in D} F(x, g_k) \geqslant \alpha - \varepsilon_k,$$

which implies (3.19). Consequently $\{x_k, g_k\}$ is a weak saddle-point and the proof is complete.

Incidentally, we have shown that if $\{x_k, g_k\}$ is a weak saddle-point, then

(3.20) $$\lim_{k\to\infty} g_k(Tx_k) = 0, \qquad \lim_{k\to\infty} F(x_k, g_k) = \lim_{k\to\infty}\inf_{x\in D} F(x, g_k).$$

These relations follow from $F(x_k, g_k) = g_k(Tx_k) + S(x_k) \to \alpha$, $S(x_k) \to \alpha$, $\psi(g_k) = \inf_{x\in D} F(x, g_k) \to \alpha$.

Conversely if (3.20) hold, then, setting

$$\alpha = \lim_{k\to\infty} F(x_k, g_k), \qquad \varepsilon_k = \max\{|\alpha - \inf_{x\in D} F(x, g_k)|, |\alpha - Sx_k|\},$$

we have $\varepsilon_k \to 0$ and

$$\sup_{g\in N^*} F(x_k, g) = g(Tx_k) + Sx_k \leqslant Sx_k \leqslant \alpha + \varepsilon_k, \qquad \inf_{x\in D} F(x, g_k) \geqslant \alpha - \varepsilon_k,$$

which implies (3.19). Thus

COROLLARY 3.4. *A sequence* $\{x_k, g_k\} \subset D \times N^*$ *is a weak saddle-point for* $F(x, g) = g(Tx) + Sx$ *if and only if the relations* (3.20) *hold.*

6. Theorems on conjugate functions. Let X, Y be two linear spaces in separating duality with respect to a bilinear form $\langle x, y\rangle$. By considering in each of these spaces their respective weak topology, $\sigma(X, Y)$ or $\sigma(Y, X)$, we transform them into two topologically dual separated locally convex spaces. If $f(x)$ is a function on X taking values in $[-\infty, +\infty]$, we define the *conjugate function*

$$g(y) = \sup_{x\in X} \big(\langle x, y\rangle - f(x)\big)$$

and write $g(y) = f^*(y)$.

A function on X, lower semi-continuous in the weak topology of X, will be called *closed*. It is *trivial* if either it is $\equiv +\infty$ or it takes the value $-\infty$ at least at one point.

As a consequence of Corollary 2.1 we note the following result:

THEOREM 3.8 (Fenchel–Moreau [16]). *For a non trivial function $f(x)$ we have $f^{**} = f$ if and only if $f(x)$ is convex and closed.*

Indeed, the relation $f \geqslant f^{**}$ readily follows from the inequality $f(x) \geqslant \langle x, y\rangle - f^*(y)$, which holds for all x and all y. To prove the converse relation $f^{**} \geqslant f$ it suffices to notice that, for every fixed point x_0 and for every number $a < f(x_0)$ ($-\infty < f(x_0)$, since f is non trivial) the system

$$x \in X, \quad x_0 - x \in \{0\}$$

is consistent, whereas the system

$$x \in X, \quad x_0 - x \in \{0\}, \quad f(x) - a \leqslant 0$$

is weakly inconsistent (because f is closed). Consequently there exists $y \in Y$ such that

$$(\forall x \in X)\ \langle x_0 - x, y\rangle + f(x) - a \geqslant 0, \quad \text{or} \quad \langle x_0, y\rangle - a \geqslant \langle x, y\rangle - f(x),$$

hence $\langle x_0, y\rangle - f^*(y) \geqslant a$, which implies $f^{**}(x_0) \geqslant f(x_0)$, since we can make $a \to f(x_0)$.

Given the functions f_1, f_2 we define their convolution by

$$f(x) = \inf_{x_1 + x_2 = x} \big(f_1(x_1) + f_2(x_2)\big) = (f_1 \oplus f_2)(x).$$

THEOREM 3.9 (Moreau–Rockafellar [16]). *If functions $f_1(x), f_2(x)$ are convex and if $f_1(x)$ is continuous at some point $\bar{x}$ where $f_1(\bar{x}) < +\infty$, $f_2(\bar{x}) < +\infty$, then*

$$(f_1 + f_2)^* = f_1^* \oplus f_2^*.$$

Indeed, the inequality $f^* = (f_1 + f_2)^* \leqslant f_1^* \oplus f_2^*$ is true for arbitrary f_1, f_2 and follows from the fact that for every x and for every pair $y_1, y_2 \in Y$ such that $y_1 + y_2 = y$ we have

$$\langle x, y\rangle - f(x) = \langle x, y_1\rangle - f_1(x) + \langle x, y_2\rangle - f_2(x) \leqslant f_1^*(y_1) + f_2^*(y_2).$$

To prove $f^* \geqslant f_1^* \oplus f_2^*$ let $D_i = \{x\colon f_i(x) < \infty\}$ $(i = 1, 2)$. We may assume $(\forall x)\ f_i(x) > -\infty$ $(i = 1, 2)$ for otherwise $f^* \equiv +\infty$ and there should be nothing to prove. Let y be an arbitrary point where $f^*(y) < \infty$. For y_1, y_2 such that $y_1 + y_2 = y$ and for $a > f^*(y)$ the system

$$\begin{cases} (x_1, x_2) \in D_1 \times D_2, \ x_1 - x_2 \in \{0\}, \\ f_1(x_1) + f_2(x_2) - \langle x_1, y_1\rangle - \langle x_2, y_2\rangle + a \leqslant 0 \end{cases}$$

is inconsistent. Since the system

$$(x_1, x_2) \in \overset{0}{D}_1 \times D_2, \quad x_1 - x_2 \in \{0\}$$

has a solution $(\bar{x}, \bar{x})$, since the map $(x_1, \bar{x}) \mapsto x_1 - \bar{x}$ is obviously open, and the map $(x_1, \bar{x}) \mapsto f_1(x_1) + f_2(\bar{x}) - \langle x_1, y_1\rangle - \langle \bar{x}, y_2\rangle + a$ is continuous at $x_1 = \bar{x}$ by hypothesis, it follows from Theorem 2.5 that there exists an element $g \in Y$ satisfying

$$(\forall (x_1, x_2) \in D_1 \times D_2)\ f_1(x_1) + f_2(x_2) - \langle x_1, y_1\rangle - \langle x_2, y_2\rangle + \langle x_1 - x_2, g\rangle + \\ + a \geqslant 0.$$

Setting $y_1' = y_1 - g$, $y_2' = y_2 + g$ and rewriting as follows the above relation

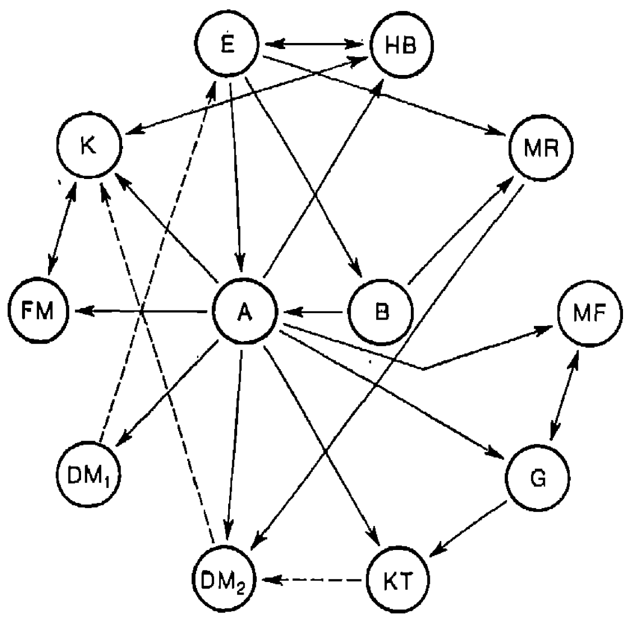
$$\sup_{x_1 \in D_1} \{\langle x_1, y_1'\rangle - f_1(x_1)\} + \sup_{x_2 \in D_2} \{\langle x_2, y_2'\rangle - f_2(x_2)\} \leqslant a,$$

we see that $f_1^*(y_1') + f_2^*(y_2') \leqslant a$ for any pair y_1', y_2' satisfying $y_1' + y_2' = y$. Therefore $(f_1^* \oplus f_2^*)(y) \leqslant a$ and since a can be chosen arbitrarily near to $f^*(y)$ we have $(f_1^* \oplus f_2^*)(y) \leqslant f^*(y)$, which is the desired result.

§ 4. Various equivalent forms of the Hahn–Banach Theorem

We have derived from the inconsistency theorem established in § 2 the most important propositions concerning convex inequalities on convex sets.

Now, to conclude the paper, we are going to show that, in fact, any one of these propositions is equivalent to the Hahn–Banach Theorem.



For the sake of convenience, let us adopt the following abbreviations. (A) — inconsistency Theorem 2.1, (B) — inconsistency Theorem 2.3; (HB) — Hahn–Banach Theorem; (E) — Eidelheit Theorem 3.1; (K) — Krein

Theorem 3.2; (FM) — Farkas–Minkowski Theorem 3.3; (DM_1) — Lubovitski–Milyutin Theorem 3.4; (DM_2) — Dubovitski–Milyutin Theorem 3.5; (KT) — Kuhn–Tucker–Hurwicz Theorem 3.6; (G) — Golshtein Theorem 3.7; (MF) — Fenchel–Moreau Theorem 3.8; (MR) — Moreau–Rockafellar Theorem 3.9.

The equivalence (HB) $\Leftrightarrow$ (E) is a classical fact. It is known also that (K) $\Leftrightarrow$ (FM) and, though in a lesser degree, that (HB) $\Leftrightarrow$ (K) (see for example [3], where (K) is derived from (HB) and [4], where (HB) is derived from (K)).

In [16] it has been proved that (E) $\Rightarrow$ (MR) $\Rightarrow$ (DM_2), and that (MF) $\Leftrightarrow$ (G). The implication (G) $\Rightarrow$ (KT) has been established in [14] (Theorem 2.7).

On the other hand, as shown in the previous section, (A) admits as corollaries: (HB), (K), (FM), (DM_1), (DM_2), (KT), (G), (MF), (MR). Therefore, to establish the equivalence of all the above mentioned propositions it suffices to prove the following implications:

$$(KT) \Rightarrow (DM_2) \Rightarrow (K); \quad (DM_1) \Rightarrow (E).$$

For the sake of completeness we shall show also that (K) $\Leftrightarrow$ (FM).

1) (KT) $\Rightarrow$ (DM_2). Let Ω_i $(i = 0, 1, \ldots, n)$ be convex cones of a linear topological space X, such that the sets Ω_i $(i = 1, 2, \ldots, n)$ have a common interior point in Ω_0, and let $f(x)$ be a continuous linear functional such that $\left(\forall x \in \bigcap_{i=0}^{n} \Omega_i\right) f(x) \geqslant 0$.

Consider the problem:

$$\min\{f(x):\ x \in \Omega_0,\ Tx \in N\}$$

where $Tx \equiv (-x, \ldots, -x) \in X^n$, $N = \Omega_1 \times \quad \times \Omega_n \subset X^n$.

Then $x = 0$ is an optimal solution of this problem and according to Kuhn–Tucker–Hurwicz theorem, there exist linear forms g_i $(i = 1, 2, \ldots, n)$ such that

$$(\forall x \in \Omega_i)\ g_i(x) \geqslant 0 \quad \text{and} \quad (\forall x \in \Omega_0)\ f(x) - \sum_{i=1}^{n} g_i(x) \geqslant 0.$$

Setting $g_0(x) = f(x) - \sum_{i=1}^{n} g_i(x)$, we have

$$f(x) = \sum_{i=0}^{n} g_i(x).$$

Thus

$$f \in \left(\bigcap_{i=0}^{n} \Omega_i\right)^* \Rightarrow f = \sum_{i=0}^{n} g_i, \quad g_i \in \Omega_i^*,$$

and so

$$\left(\bigcap_{i=0}^{n} \Omega_i\right)^* \subset \sum_{i=0}^{n} \Omega_i^*,$$

which establishes theorem (DM_2), since the converse inclusion is obvious.

2) (DM_2) $\Rightarrow$ (K). Let Y be a linear space with a positive cone N, let X be a subspace of Y such that $\overset{0}{N} \cap X \neq \emptyset$ and let $f(x)$ be a linear form on X, positive on the cone $N \cap X$.

Denote by P the subspace complementary to X, so that $Y = X \times P$ and define on Y the linear form $\tilde{f}(y) = f(x)$ for all $y = (x, p)$. Then $\tilde{f} \in (N \cap X)^*$ and since $\overset{0}{N} \cap X \neq \emptyset$ it follows from (DM_2) that $\tilde{f} = F(y) + \bar{f}(y)$ with $F(y) \geqslant 0$ on N, and $\bar{f}(y) \geqslant 0$ on X, hence $\bar{f}(x) \equiv 0$ on X. Thus $F(x) = f(x)$ on X and $F(y)$ is an extension of f, such that $F(y) \geqslant 0$ on N.

If Y is a linear topological space, then $F(y)$ is continuous because it is positive on the cone N, which has a non-void interior (see [4], prop. 17).

3) (DM_1) $\Rightarrow$ (E). This is immediate, for if C and D are non-intersecting convex subsets of a linear space, and if $\overset{0}{C} \neq \emptyset$, then theorem ($DM_1$) asserts the existence of an affine functional $t(x) \not\equiv 0$ such that $(\forall x \in C)\ t(x) \geqslant 0$ and $(\forall x \in D)\ t(x) \leqslant 0$.

4) (K) $\Rightarrow$ (FM). Let X, Y be linear topological spaces, N the positive cone in Y, $p(x)$ a continuous linear form on X, T a convex map of X into Y. Suppose that $(\exists x_0 \in X)\ Tx_0 \prec 0$ and $p(x) \geqslant 0$ for all $x \in X$ verifying $Tx \preceq 0$.

As it can be easily verified, the set

$$Q = \{(x, y) \in X \times Y:\ Tx + y \preceq 0\}$$

is a convex cone with $(x_0, Tx_0) \in \overset{0}{Q}$. We have $T^{-1}(N) = Q \cap X$ and $p(x) \geqslant 0$ for all $x \in T^{-1}(N)$, therefore, according to Krein theorem, $p(x)$ can be extended to a continuous linear form $F(x, y)$ on $X \times Y$, such that $F(x, y) \geqslant 0$ for all $(x, y) \in Q$. Clearly $F(x, y) = p(x) + g(y)$, where g is a positive linear form on Y. Hence

$$(\forall x)\ p(x) + g(Tx) \geqslant 0,$$

which reduces to $p = -T^* g$ when T is linear.

5) (FM) $\Rightarrow$ (K). This is easy to prove, for if $f(x)$ is a linear form defined on a subspace X of Y, positive with respect to $N \cap X$, where N is the positive cone of Y, then setting $Tx \equiv -x$, we have $f(x) \geqslant 0$ for all $x \in X$ verifying $Tx \preceq 0$, and so the implication (FM) $\Rightarrow$ (K) follows readily.

References

[1] K. J. Arrow, L. Hurwicz, and H. Uzawa, *Studies in linear and non-linear programming*, Stanford, California 1958.

[2] C. Berge and A. Ghouila-Houri, *Programmes, jeux et réseaux de transport*, Paris 1962.

[3] N. Bourbaki, *Espaces vectoriels topologiques*, Actualités Sci. Ind. 1189, Paris 1953.

[4] — *Espaces vectoriels topologiques*, Actualités Sci. Ind. 1189, Deuxième édition, Paris 1966.

[5] A. Brondsted, *Conjugate convex functions in topological vector spaces*, Math. fys. Meddelelser udjigate at Det Kongelige Danske Vid. Selskal, Copenhagen 34, 2 (1964).

[6] N. Dunford and J. T. Schwartz, *Linear operators*. Part I: *General theory*, New-York 1953.

[7] W. Fenchel, *On conjugate convex functions*, Canad. Journ. Math. 1 (1949), pp. 73–77.

[8] — *Convex cones, sets and functions*, Princeton Univers. 1953.

[9] L. Hurwicz, *The Minkowski-Farkas lemma for bounded linear transformations in Banach spaces*, Cowles Commission Discussion Papers, Mathematics No 415, July 16, 1952; Mathematics No 416, October 17, 1952; Economics No 2109.

[10] H. W. Kuhn and A. W. Tucker, *Nonlinear programming*, Proc. of the Second Berkeley Symp. on Math. Stat. and Prob. Univ. of Calif., Berkeley 1951, pp. 481–492.

[11] J. J. Moreau, *Sur la fonction polaire d'une fonction semicontinue supérieurement*, Compt. Rendus Acad. Sci. Paris 258 (1964), pp. 1128–1130.

[12] A. P. Robertson and W. Robertson, *Topological vector spaces*, Cambridge 1964.

[13] R. T. Rockafellar, *Extension of Fenchel duality for convex functions*, Duke Math. Journ. 32 (1965), pp. 331–397.

[14] Е. Г. Голштейн, *Двойственные задачи выпуклого и дробно-выпуклого программирования в функциональных пространствах*, Сб. Исследования по математическому программированию, Москва 1968, pp. 10–109.

[15] А. Я. Дубовицкий, А. А. Милютин, *Задачи на экстремум при наличии ограничений*, Ж. вычисл. матем. и матем. физики. 5. Н° 3 (1965), pp. 395–453.

[16] А. Д. Иоффе, В. М. Тихомиров, *Двойственность выпуклых функций и экстремальные задачи*, Успехи матем. наук. 23 (6) (1968), pp. 51–116.

[17] В. Л. Левин, *Условия экстремума в бесконечномерных линейных задачах с операторными ограничениями*, Сб. исследования по математическому программированию, Москва 1968, pp. 159–199.

[18] Хоанг Туй (Hoàng Tuy), *О линейных неравенствах*, Доклады АН СССР 179 (2) (1968).

[19] Hoang Tuy, *Sur les inégalités linéaires*, Colloq. Math. 13 (1964), pp. 107–123.