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SHORTER NOTES

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THE RUDIN-CARLESON THEOREM FOR VECTOR-VALUED FUNCTIONS

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ABSTRACT. The following generalization of the Rudin-Carleson theorem is proved. Let X be a complex Banach space and let $f: F \to X$ be a continuous function, where F is a closed subset of the unit circle in C of Lebesgue measure zero. There exists a continuous function g from the closed unit disc to X which is analytic on the open unit disc and satisfies (i) $g|F = f$, (ii) $\max_{|z| \leq 1}\|g(z)\| = \max_{z \in F}\|f(z)\|$.

Throughout, Δ (resp. $\overline{\Delta}$) is the open (resp. closed) unit disc in C. Given a complex Banach space X and a compact space K, we denote by $C(K, X)$ the (Banach) space of all continuous functions from K to X with sup norm. We denote by $A(\Delta, X)$ the (Banach) space of all continuous functions from $\overline{\Delta}$ to X which are analytic on Δ, with sup norm. We write $C(K)$, $A(\Delta)$ for $C(K, C)$, $A(\Delta, C)$, respectively.

We prove the following generalization of the well-known Rudin-Carleson theorem (see [2], [3], [6]).

Theorem. *Let X be a complex Banach space and F a closed set of Lebesgue measure zero on the unit circle in C. Let $f \in C(F, X)$. There exists $g \in A(\Delta, X)$ such that $g|F = f$ and $\|g\|_{A(\Delta, X)} = \|f\|_{C(F, X)}$.*

Let K be a compact space and let $\mathfrak{A}$ be a closed subalgebra of $C(K)$. A set $F \subset K$ is called a peak set for $\mathfrak{A}$ if there exists $g \in \mathfrak{A}$ such that $g(z) = 1$ $(z \in F)$ and $|g(z)| < 1$ $(z \in K \setminus F)$. Let X be a complex Banach space, A a closed subspace of $C(K, X)$ and F a closed subset of K. Define $kF = \{f \in A: f|F = 0\}$. kF is a closed subspace of A, hence A/kF with norm $\|f + kF\| = \inf_{g \in kF}\|f + g\|_{C(K, X)}$ is a Banach space. It is easy to see that T, defined by $T(f|F) = f + kF$, is a one-to-one linear operator from $A|F = \{f|F: f \in A\}$ onto A/kF (see [5, p. 163]).

Lemma. *Let X be a complex Banach space, K a compact space, A a closed subspace of $C(K, X)$, and $\mathfrak{A}$ a closed subalgebra of $C(K)$. Denote by*

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$\mathfrak{A}A$ the set of all functions of the form $z \mapsto \phi(z)f(z)$ where $\phi \in \mathfrak{A}$ and $f \in A$ and suppose that $\mathfrak{A}A \subset A$. Let $F \subset K$ be a peak set for $\mathfrak{A}$. Then

(i) *$T: f|F \mapsto f + kF$ is an isometry from $A|F$ onto A/kF and, consequently, $A|F$ is closed in $C(F, X)$;*

(ii) *for each $f \in A|F$ there exists $g \in A$ satisfying $g|F = f$ and $\|g\|_{C(K, X)} = \|f\|_{C(F, X)}$.*

To prove (i) and (ii) observe that the corresponding proofs for scalar-valued functions ([5, p. 163, Theorem 3(c)], [5, p. 164, Theorem 4]) work for vector-valued functions as well.

Proof of theorem. $A(\Delta)$ is a closed subalgebra of $C(\overline{\Delta})$ and it is easy to see that $A(\Delta, X)$ is a closed subspace of $C(\overline{\Delta}, X)$. Given $\phi \in A(\Delta)$ and $f \in A(\Delta, X)$, it is also easy to see that the function $z \mapsto \phi(z)f(z)$ is again continuous on $\overline{\Delta}$ and analytic on Δ, hence $A(\Delta)A(\Delta, X) \subset A(\Delta, X)$. By [3, p. 81] F is a peak set for $A(\Delta)$, and by (i) of the Lemma it follows that $A(\Delta, X)|F$ is closed in $C(F, X)$. Further, by the Mergelyan theorem for vector-valued functions (see [1]), every function in $C(F, X)$ is the uniform limit of a sequence of polynomials which means that $A(\Delta, X)|F$ is dense in $C(F, X)$. By the preceding discussion it follows that $A(\Delta, X)|F = C(F, X)$. Now the assertion follows by (ii) of the Lemma. Q.E.D.

Remark. The generalization of the Rudin-Carleson theorem to vector-valued functions was motivated by the following problem posed by D. Patil at the Conference on Infinite Dimensional Holomorphy, University of Kentucky, May 1973. Let X be a complex separable Banach space. Does there exist an analytic function $f: \Delta \to X$ such that the convex hull of $f(\Delta)$ is contained and dense in the unit ball of X? Patil's problem will be discussed in a separate paper. Note that the Theorem gives a solution to this problem in the case when X is finite dimensional. To see this, let F be a Cantor set of Lebesgue measure zero on the unit circle in C. Since every compact metric space is a continuous image of F (see [4, p. 166]), there exists a continuous function f from F onto the closed unit ball B of a finite-dimensional X by the compactness of B. Then the extension $g \in A(\Delta, X)$ given by the Theorem has the property that $g(\Delta)$ is contained and dense in B.

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Added in proof. When the present paper was already in print the author found that a more general theorem than the theorem above was proved by E. L. Stout, *On some restriction algebras* (pp. 6–11 of *Function algebras*, Scott-Foresman, Chicago, Ill., 1966. MR 35 #3447).

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