Approximation of Holomorphic Functions in the Complex Plane

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60 ECTS thesis submitted in partial fulfillment of a
Magister Scientiarum degree in Mathematics

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Abstract

This thesis is a study of approximation of holomorphic functions, by polynomials, rational functions or entire functions. Hörmander's L^2 existence theorem for the Cauchy-Riemann operator is proved and used to prove a generalization of the Bernstein-Walsh theorem, which describes the equivalence between possible holomorphic continuation of a function f in a neighbourhood of a compact set K in terms of the decay of the sequence $(d_n(f, K))$ of best approximations of f by polynomials of degree less than or equal to n. The generalization uses the best approximation of f by rational functions with poles in a prescribed set instead of polynomials. A theorem of Vitushkin is proved. It characterizes in terms of analytic capacity the compact sets K having the property that every function f, continuous on K and holomorphic in the interior of K, can be approximated uniformly on K by rational functions. Finally, Mergelyan's theorem is used to prove a generalization of Arakelian's theorem, which describes uniform approximation of holomorphic functions on possibly unbounded sets by entire functions.

Útdráttur

Þessi ritgerð fjallar um nálganir á fáguðum föllum, með margliðum, ræðum föllum eða heilum föllum. L^2-tilvistarsetning Hörmanders fyrir Cauchy-Riemann virkjann er sönnuð og hún er notuð til þess að sanna alhæfingu á setningu Bernstein-Walsh, sem lýsir jafngildi milli mögulegrar fágaðar framlengingar á falli f á opinni grennd við þjappað hlutmengi K og runu bestu nálgana $(d_n(f, K))$ á f með margliðum af stigi minna eða jöfnu n. Alhæfingin notar bestu nálganir á f með ræðum föllum með skaut í gefnu mengi. Setning Vitushkins er sönnuð, en hún lýsir hvernig fáguð rýmd mengis er notuð til þess að auðkenna þjöppuð mengi K með þann eiginleika að sérhvert fall f, samfellt á K og fágað á innmengi K, megi nálga í jöfnu mæli á K með ræðum föllum. Að lokum er setning Mergelyans beitt til þess að sanna alhæfingu á setningu Arakelians, sem lýsir nálgun í jöfnum mæli á fáguðum föllum á ótakmörkuðum mengjum með heilum föllum.

Contents

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1 Introduction

When studying polynomial approximation of functions defined on $\mathbb{R}^n$ the Weierstraß approximation theorem gives a very strong result.

Theorem 1.0.1 (Weierstraß)**.** *Let K be a compact subset of $\mathbb{R}^n$ and $f \in C(K)$. For every $\epsilon > 0$ there exists a polynomial P such that*

$$\sup_{x \in K} |P(x) - f(x)| < \epsilon.$$

When we look at functions defined on $\mathbb{C}$, the results become more complicated. Since all polynomials in the variable z are holomorphic the following theorem implies that if a function f is to be approximated uniformly by polynomials on an open set it must be holomorphic.

Theorem 1.0.2 (Cor. 1.2.6. of [6])**.** *If $\Omega \subset \mathbb{C}$ is an open set, $f_n \in \mathcal{O}(\Omega)$ for all $n \in \mathbb{N}$ and $f_n \to f$ uniformly on compact subsets of Ω then $f \in \mathcal{O}(\Omega)$.*

Here $\mathcal{O}(\Omega)$ is the set of functions holomorphic on Ω. The following question now arises:

Can we find some necessary and/or sufficent conditions such that a function f on a compact set K can be approximated by polynomials?

From Theorem 1.0.2 it follows that the following conditions are necessary on f

- f is continuous on K.
- f is holomorphic on $\mathrm{int}(K)$.

We denote the set of functions which satisfy these conditions by $A(K)$. But can we find some sufficient or necessary conditions on the set K so that every function in $A(K)$ can be approximated in K? Suppose its compliment, K^c, has a bounded component U and assume we can approximate every function $g \in A(K)$ uniformly in K by polynomials. We choose some $\zeta \in U$ and consider the function on K given by $g(z) = (z - \zeta)^{-1}$. By assumption we can find a sequence $(g_n)_{n \in \mathbb{N}}$ of polynomials such that $g_n \to g$ uniformly on K. By the maximum principle we have

$$\sup_{z \in \overline{U}} |g_n(z) - g_m(z)| \leq \sup_{z \in \partial U} |g_n(z) - g_m(z)| \leq \sup_{z \in K} |g_n(z) - g_m(z)|$$

so g_n converges uniformly in $\overline{U}$ to some limit $G \in A(\overline{U})$. We notice that $G(z)(z - \zeta) = 1$ on ∂U, so $G(z)(z - \zeta) = 1$ on all of U. This gives a contradiction when $z = \zeta$. Our conclusion is:

- If every function $f \in A(K)$ can be approximated by polynomials on K, then K^c is connected.

To continue this discussion we need to define a new set of functions.

Definition 1.0.3. *Let $E \subset \mathbb{C}$ be an arbitrary set and f be a function on E. We say that $f \in \mathcal{O}(E)$ if there exists some open set $\Omega \subset \mathbb{C}$ and a holomorphic function, $g \in \mathcal{O}(\Omega)$ such that $E \subseteq \Omega$ and $g|_E = f$.*

In 1885, Runge proved a considerably general result concerning the question stated above:

Theorem 1.0.4 (Runge, Cor. 1.3.2 of [6])**.** *Let K be a compact set. Every function in $\mathcal{O}(K)$ can be uniformly approximated by polynomials on K if and only if the set K^c is connected.*

Quite obviously we have $\mathcal{O}(K) \subseteq A(K)$. Therefore it is rather natural to wonder if Runge's theorem can be extended to the set $A(K)$. The answer to that question is quite satisfyingly yes, and was proved by Mergelyan in 1952. It is startling that a period of sixty-seven years elapsed between the appearance of Runge's theorem and that of Mergelyan's theorem. Especially if one examines the large number of papers written on this subject during those years. The explanation is perhaps that people thought that Mergelyan's theorem was too good to be true.

Theorem 1.0.5 (Mergelyan, Th. 12.2.1 of [4])**.** *Let K be a compact set. Every function in $A(K)$ can be uniformly approximated on K by polynomials if and only if the set K^c is connected.*

Let $K \subset \mathbb{C}$ be compact and $f : K \to \mathbb{C}$ be any function. For $n \geq 1$, we define

$$d_n(f, K) := \inf\{\|f - p\|_K \,; p \text{ is a polynomial of degree at most } n\}.$$

By the preceding discussion we see that $\lim_{n\to\infty} d_n(f, K) = 0$ if and only if $f \in A(K)$. Quite simple result, but for $f \in A(K)$ it can be of great interest to study the decay of the sequence $(d_n(f, K))_{n\in\mathbb{N}}$ because it helps us giving answer to an important question: What is the largest open set to which f can be extended holomorphically? To describe the equivalence between the decay of $(d_n(f, K))_{n\in\mathbb{N}}$ and the possible analytic extension of f, the so called Green's function g_K, for the set $\mathbb{C} \setminus K$ is introduced, which will be discussed in Section 2.5. For the Green's function to be defined, K must be regular, which means that the Dirichlet problem has a solution on $\mathbb{C} \setminus K$. For a real number $R > 1$ we define $K_R = \{z \in \mathbb{C};\ g_K(z) < \log(R)\}$ for the following result.

Theorem 1.0.6 (Bernstein-Walsh, Th. BW1 in [8])**.** *Let $K \subset \mathbb{C}$ be regular and f be a function on K. Let g_K and K_R be as above, then*

$$\limsup_{n\to\infty} (d_n(f, K))^{1/n} < \frac{1}{R}$$

if and only if f is the restriction to K of a function holomorphic on K_R.

In Chapter 4 we will give a proof of a generalized version of the theorem of Bernstein-Walsh, considering compact sets K without assuming K^c to be connected. Of course, without assuming K^c to be connected we cannot hope to approximate functions in $\mathcal{O}(K)$ with polynomials. Therefore we will rather consider approximation by rational functions. We immediadely run into some complications, the most obvious one being: How should one define a similar quantity to $d_n(f,K)$ for rational functions? We will define the notation $Rd(f,K,\mathbf{a},\mathbf{n})$, where $\mathbf{a}$ is a vector, describing the position of poles of the rational functions and $\mathbf{n}$ describes the maximum degree of each pole. We will have to define a Green's function for each of the components of K^c, but another complication then arises. Previously we needed just one real number R to describe the holomorphic extension of f, but now we need many. It is simple enough to use a vector $\mathbf{R}$ rather than a real number, but how is one to relate those numbers to the decay of $Rd(f,K,\mathbf{a},\mathbf{n})$? Since $Rd(f,K,\mathbf{a},\mathbf{n})$ doesn't even define a sequence, as $d_n(f,K)$ does in the variable n, one will have to ask: what is the best way to measure the decay of $Rd(f,K,\mathbf{a},\mathbf{n})$? Those questions will be answered in Chapter 4. In the proof of the generalization we rely heavily on Hörmander's L^2 theorem, which gives an estimate on the solution of the inhomogeneous $\bar{\partial}$-equation. In Chapter 3 we will discuss and prove this theorem. Similar methods are considered in Claveras [2].

In light of the discussion it should be clear that Runge's theorem extends to an arbitrary compact K by considering approximation by rational functions rather than polynomials.

Theorem 1.0.7 (Runge, Th. 12.1.1 of [4]). *Let K be a compact set and A be a set which contains at least one point from each bounded component of K^c. Then every function in $\mathcal{O}(K)$ can be uniformly approximated on K by rational functions with poles in A.*

Mergelyan's theorem only extends to a certain degree though.

Theorem 1.0.8 (Mergelyan, Th. 12.2.7 of [4]). *Let K be a compact such that K^c has only finitely many components. Let A be a set which contains at least one point from every bounded component of K^c. Then every function in $A(K)$ can be uniformly approximated on K by rational functions with poles in A.*

Notice the extra condition in Mergelyan's case, that K^c may only have finitely many components. This extra condition can definitely not be abolished completely as shown by the following counter-example, sometimes called the Swiss-cheese example, by Alice Roth [13]. We denote the open disc with center z and radius r with $\Delta(z,r)$.

Example. Consider the open unit disc $\Delta = \Delta(0,1)$. Let $\overline{\Delta_j} = \overline{\Delta(a_j,r_j)}$ be a sequence of closed discs satisfying the following conditions:

1. They are pairwise disjoint.

2. The union $\bigcup_{j\in\mathbb{N}} \Delta_j$ is dense in $\overline{\Delta}$.
3. $\sum_{j\in\mathbb{N}} r_j < 1$

Assuming for the moment such sequences $(a_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ exist, we let $K = \overline{\Delta} \setminus \left(\bigcup_{j\in\mathbb{N}} \Delta_j\right)$. Notice that $A(K) = C(K)$ since K has no interior. Consider the measure μ on K given by

$$\int_K f d\mu = \int_{\partial\Delta} f(z)dz - \sum_{j\in\mathbb{N}} \int_{\partial\Delta_j} f(z)dz, \qquad f \in C(K).$$

We see that if R is a rational function with poles in $\mathbb{C} \setminus K$ then $\int_K R d\mu = 0$ since by the residue theorem all the residues cancel out. Considering the function $f(z) = \bar{z}$, we get

$$\int_K f d\mu = \int_{\partial\Delta} \bar{z}dz - \sum_{j\in\mathbb{N}} \int_{\partial\Delta_j} \bar{z}dz = 2\pi i - \sum_{j\in\mathbb{N}} r_j 2\pi i = 2\pi i(1 - \sum_{j\in\mathbb{N}} r_j) \neq 0.$$

Therefore $f(z) = \bar{z}$ can not be approximated by rational functions on K. But $f \in A(K)$, so not every function in $A(K)$ can be approximated by rational functions.

Lemma 1.0.9. *Sequences $(a_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ satisfying 1.-3. in the example above exist.*

Proof. For $z \in \mathbb{C}$ and $E \subset \mathbb{C}$, we denote by $d(z, E)$ the distance between z and E. Let $(s_n)_{n\in\mathbb{N}}$ be any sequence of positive numbers such that $\sum_{n\in\mathbb{N}} s_n < 1$. Let $K_0 = K$, and define by induction $K_{n+1} = K_n \setminus \Delta(a_n, r_n)$ where a_n is equal to a point where the function $d(z, K_n^c)$ gains its maximum (considered as a function of z) and $r_n = \min\{s_n, d(a_n, K_n^c)/2\}$. Note that since K_n is compact, a_n exists.

It is obvious that $(a_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ satisfy 1. and 3. above. Suppose they do not satisfy 2. Then there exists an open disc $\Delta(b, \epsilon) \subset \overline{\Delta} \setminus \left(\bigcup_{j\in\mathbb{N}} \Delta_j\right)$. By the way a_n is chosen this means that $|a_n - a_m| \geq \epsilon$ for all $n, m \in \mathbb{N}$, $n \neq m$. Since $\bar{\Delta}$ is compact this is a contradiction. $\square$

Even though the condition that K^c must have only finitely many components can't be discarded altogether, it can be lightened. There does in fact exist a compact set K such that the set K^c has infinitely many components and every function in $A(K)$ can be approximated by rational functions. In 1967 Vitushkin [15] gave necessary and sufficient conditions on K for that to be the case. So far the conditions on K that we have considered have been purely topological. Vitushkin's conditions are on the other hand not topological but depend rather on a concept of analytic capacity. We will devote Chapter 5 to a discussion of Vitushkin's theorem.

Finally, consider a closed set $E \subset \mathbb{C}$, possibly unbounded, and assume that E^c is connected. Since non-constant polynomials are unbounded we cannot expext to be able to approximate every $f \in A(E)$ uniformly on E by polynomials, therefore we rather consider approximation by entire functions.

Theorem 1.0.10 (Arakelian, [12]). *Let $E \subset \mathbb{C}$ be a closed set satisfying the following condition: E^c is connected and for every closed disc $\bar{\Delta} \subset \mathbb{C}$ the union of the bounded components of the set $\mathbb{C} \setminus (E \cup \bar{\Delta})$ is a bounded set. Then every $f \in A(E)$ can be approximated uniformly on E by entire functions.*

In Chapter 6 we use Mergelyan's theorem to give a generalization of this theorem for sets E for which E^c isn't required to be connected. For that to be possible we need to consider approximations by functions in $\mathcal{O}(\mathbb{C} \setminus A)$, where A is a closed set containing at least one element from every bounded component of E^c.

2 Preliminaries

In this chapter we collect some basic facts necessary for the rest of the thesis. Most of the theorems are be stated without proof, but sometimes we give a short reasoning instead. We start by looking at some fundemental results concerning complex analysis. More explicitly we discuss holomorphic functions, harmonic and subharmonic functions, the Dirichlet problem and Green's functions. We also discuss a few basic results from distribution theory and functional analysis.

2.1 Holomorphic functions

Throughout this discussion we identify the complex plane, $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way. Let Ω be an open set in the complex plane and let f be a complex valued function in $C^1(\Omega)$. If the real coordinates are denoted by x and y, then we set $z = x + iy$ and $\bar{z} = x - iy$. Also we have

$$x = \frac{z + \bar{z}}{2} \qquad \text{and} \qquad y = \frac{z - \bar{z}}{2i}.$$

We define partial differential operators in the following way

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$

Now, the differential of f can be expressed as a linear combination of dz and $d\bar{z}$:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}.$$

Definition 2.1.1. *A function $f \in C^1(\Omega)$ is said to be holomorphic (analytic) if $\partial f/\partial \bar{z} = 0$ in Ω, or equivalently if df is proportional to dz. If the function f is holomorphic we write f' rather then $\partial f/\partial z$. We denote the set of all holomorphic functions on Ω by $\mathcal{O}(\Omega)$.*

Recall the classic Green-formula for functions defined in $\mathbb{R}^2$.

$$\int_{\partial\Omega} Pdx + Qdy = \int_{\Omega}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

By applying it with $P = f$ and $Q = if$ we get the following result:

Theorem 2.1.2 (Green's theorem)**.** *Let Ω be a bounded open set in $\mathbb{C}$ such that the boundary consists of a finite number of regular C^1 curves. If $f \in C^1(U)$ for some open set U such that $\overline{\Omega} \subset U$, then*

$$\int_{\partial\Omega} f dz = 2i \int_{\Omega} \frac{\partial f}{\partial \bar{z}} dx \wedge dy = -\int_{\Omega} \frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z}.$$

Corollary 2.1.3. *If $f \in C^1(U)$ is holomorphic and Ω is as before, then $\displaystyle\int_{\partial\Omega} f dz = 0$.*

It is rather spectacular to see how one can manipulate Green's theorem for complex functions to derive countless statements in the theory of complex analysis, as we will see.

Let f, Ω and U be as before. By fixing some point $\zeta \in \Omega$ and applying Theorem 2.1.2 to the function $f(z)/(z-\zeta)$, noting that the function $1/(z-\zeta)$ is holomorphic on the set $U \setminus \overline{\Delta(\zeta, \epsilon/2)}$, we get the following

$$-\int_{\Omega\setminus\overline{\Delta(\zeta,\epsilon)}} \frac{\partial f/\partial \bar{z}}{(z-\zeta)} dz \wedge d\bar{z} = \int_{\partial(\Omega\setminus\overline{\Delta(\zeta,\epsilon)})} \frac{f(z)}{(z-\zeta)} dz = \int_{\partial\Omega} \frac{f(z)}{(z-\zeta)} dz - \int_0^{2\pi} if(\zeta+\epsilon e^{i\theta}) d\theta.$$

Since $(z-\zeta)^{-1}$ is integrable over Ω and f is continuous at ζ we can let ϵ tend to zero to get the following result:

Theorem 2.1.4 (Cauchy-Pompeiu's formula, Th. 1.2.1 of [6])**.** *If $f \in C^1(U)$, and Ω is as before, then*

$$f(\zeta) = \frac{1}{2\pi i}\left(\int_{\partial\Omega} \frac{f(z)}{(z-\zeta)} dz + \int_{\Omega} \frac{\partial f/\partial \bar{z}}{z-\zeta} dz \wedge d\bar{z}\right), \qquad \zeta \in \Omega.$$

Corollary 2.1.5. *Let U and Ω be as before.*

(i) If $f \in \mathcal{O}(U)$, then $f(\zeta) = \dfrac{1}{2\pi i}\displaystyle\int_{\partial\Omega} \frac{f(z)}{(z-\zeta)} dz$ for all $\zeta \in \Omega$.

(ii) If $f \in C^1_c(\Omega)$, then $f(\zeta) = \dfrac{1}{2\pi i}\displaystyle\int_{\Omega} \frac{\partial f/\partial \bar{z}}{z-\zeta} dz \wedge d\bar{z}$ for all $\zeta \in \mathbb{C}$.

Now, consider some function $g \in C^1_c(\mathbb{C})$ and define

$$u(\zeta) := \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g(z)}{(z-\zeta)} dz \wedge d\bar{z} = \frac{-1}{2\pi i}\int_{\mathbb{C}} \frac{g(\zeta - z)}{z} dz \wedge d\bar{z}.$$

Since z^{-1} is integrable over every compact set, it is legitimate to differentiate under the sign of integration to get

$$\frac{\partial u}{\partial \bar{\zeta}}(\zeta) = \frac{-1}{2\pi i}\int_{\mathbb{C}} \frac{(\partial g/\partial \bar{\zeta})(\zeta - z)}{z} dz \wedge d\bar{z} = \int_{\mathbb{C}} \frac{\partial g/\partial \bar{z}}{z-\zeta} dz \wedge d\bar{z} = g(\zeta)$$

Hence u is the solution to the partial differential equation $\partial u/\partial \bar{z} = g$. A more general result is stated in the following theorem:

Theorem 2.1.6 (Th. 1.2.2 of [6]). *If μ is a complex measure with compact support in $\mathbb{C}$, the integral*

$$u(\zeta) = \int_{\mathbb{C}} \frac{d\mu(z)}{z - \zeta}$$

defines a holomorphic C^∞ function outside the support of μ. If $d\mu = (2\pi i)^{-1} g dz \wedge d\bar{z}$ for some $g \in C_c^k(\mathbb{C})$, $k \geq 1$, we have $u \in C^k(\mathbb{C})$ and $\partial u/\partial \bar{z} = g$.

By applying Theorem 2.1.4 and Theorem 2.1.6 we get the following corollary:

Corollary 2.1.7 (Cor. 1.2.3 of [6]). *Every $u \in \mathcal{O}(\Omega)$ is in $C^\infty(\Omega)$. Hence $u' \in \mathcal{O}(\Omega)$ if $u \in \mathcal{O}(\Omega)$.*

It now makes sense to define by induction the notation $u^{(1)} = u'$ and $u^{(n+1)} = (u^{(n)})'$.

Consider some function $u \in \mathcal{O}(\Omega)$ and let $K \subset \Omega$ be compact. Choose some $\psi \in C_c^\infty(\Omega)$ such that $\psi = 1$ in some neighbourhood of K. Since u is holomorphic we notice that $\partial(\psi u)/\partial \bar{z} = u \partial \psi/\partial \bar{z}$ and by Theorem 2.1.4, we get for every $\zeta \in K$ that

$$\psi(\zeta) u(\zeta) = \frac{1}{2\pi i} \int_\Omega \frac{u(z) \partial \psi/\partial \bar{z}}{z - \zeta} dz \wedge d\bar{z}.$$

Since $\partial \psi/\partial \bar{z} = 0$ in K, the integrand is bounded so differentiation under the integral sign is allowed, which yields the following result:

Theorem 2.1.8 (Th. 1.2.4 of [6]). *For every compact set $K \subset \Omega$ and every open neighbourhood $\omega \subset \Omega$ of K there are constants C_j, $j \in \mathbb{N}$ such that*

$$\sup_{z \in K} |u^{(j)}(z)| \leq C_j \|u\|_{L^1(\omega)}, \qquad u \in \mathcal{O}(\Omega).$$

Consider a sequence, $(u_n)_{n \in \mathbb{N}}$, of holomorphic functions on Ω, which is uniformly bounded on some compact set $K \subset \Omega$. The preceding theorem implies that the sequence $(u_n')_{n \in \mathbb{N}}$ is also uniformly bounded on K. By applying Ascoli's theorem we come to the following conclusion:

Theorem 2.1.9 (Montel, Th. in Ch. 7.1.1 of [11]). *If $u_n \in \mathcal{O}(\Omega)$ and the sequence $|u_n|$ is uniformly bounded on every compact subset of Ω, there is a subsequence u_{n_j} converging uniformly on every compact subset of Ω to a limit $u \in \mathcal{O}(\Omega)$.*

Finally, we state the Riemann mapping theorem, which is applied frequently in the thesis.

Theorem 2.1.10 (Riemann, Th. 6.4.2 of [4]). *If $U \subset \mathbb{C}$ is open , simply connected and $U \neq \mathbb{C}$, then there is an bijective holomorphic map from U to the open unit disc.*

2.2 Harmonic functions

We start with a formal definition.

Definition 2.2.1. *Let U be an open subset of $\mathbb{C}$. A function $h : U \to \mathbb{R}$ is said to be harmonic if $h \in C^2(\Omega)$ and*

$$\Delta h := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) h = 0.$$

The following basic result allows us to construct numerous examples of harmonic functions. It also provides a useful tool for deriving their elementary properties from those of analytic functions.

Theorem 2.2.2 (Th. 1.1.2 of [10])**.** *Let D be a domain in $\mathbb{C}$.*

(i) If f is holomorphic on D and $h = Re\ f$, then h is harmonic on D.

(ii) If h is harmonic on D, and if D is simply connected, then $h = Re\ f$ for some holomorphic f on D. Moreover f is unique up to adding a constant.

Notice that the condition that D should be simply connected is crucial to the second part of the theorem. To see that we can consider the function $h(z) = \log|z|$ which is clearly harmonic on the set $\mathbb{C}\setminus\{0\}$. It cannot be the real part of an analytic function g on $\mathbb{C}\setminus\{0\}$ because if it were then $|\exp(g(z))/z| = 1$ on $\mathbb{C}\setminus\{0\}$ so by the maximum principle the analytic function $\exp(g(z))/z$ is equal to a constant C. That would mean that $g'(z) = 1/z$ which in turn would give

$$0 = \int_{|z|=1} g'(z)dz = \int_{|z|=1} \frac{1}{z}dz = 2\pi i.$$

However, since discs are simply connected, every harmonic function is locally the real part of an analytic function which leads to the following corollary.

Corollary 2.2.3 (Cor. 1.1.4 of [10])**.** *If h is a harmonic function on an open subset U of $\mathbb{C}$, then $h \in C^\infty(U)$.*

A simple consequence of Theorem 2.2.2 and Cauchy's integral formula is the following theorem.

Theorem 2.2.4 (Mean-value property, Th. 1.1.6 of [10])**.** *Let h be a function harmonic on an open neighbourhood of the disc $\overline{\Delta}(z,\rho)$, then*

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} h(z + \rho e^{i\theta})d\theta.$$

This section ends with two further ways in which harmonic functions behave like analytic ones, an identity principle and a maximum principle.

Theorem 2.2.5 (Identity principle, Th. 1.1.7 of [10]). *Let h and k be harmonic functions on a domain D in $\mathbb{C}$. If $h = k$ on a non-empty open subset U of D, then $h = k$ on D.*

Theorem 2.2.6 (Maximum principle, Th. 1.1.8 of [10]). *Let h be a harmonic function on a domain D in $\mathbb{C}$.*

(i) If h attains a local maximum on D then h is constant.

(ii) If h extends continuously to $\overline{D}$ and $h \leq 0$ on ∂D, then $h \leq 0$ on D.

2.3 Subharmonic functions

The analogue of the Laplacian for functions of one real variable is the differential operator d^2/dx^2. Thus the analogue of harmonic functions on $\mathbb{R}$ is the null space of this operator, which consists of the affine functions. A convex function, $f: \mathbb{R} \to \mathbb{R}$ is a function whose graph satisfies the following property: If l is a line in the plane which cuts the graph of f in the points $(a, f(a))$ and $(b, f(b))$ $(a < b)$ then the graph of f on the interval $[a, b]$ lies below l. Subharmonic functions satisfy the complex analogue of this property. We start by defining semicontinuous functions.

Definition 2.3.1. *Let X be a topological space. We say that a function $u : X \to [-\infty, \infty[$ is upper semicontinuous if the set $\{x \in X : u(x) < \alpha\}$ is open in X for each $\alpha \in \mathbb{R}$. We say that $u : X \to]-\infty, \infty]$ is lower semicontinuous if $-u$ is upper semicontinuous.*

Let $f \in C^2(\Omega)$, for some open set $\Omega \subset \mathbb{C}$. Then f is subharmonic if $\Delta f \geq 0$. This could seem like a suitable definition for subharmonicity, but it turns out that it is way to strict. One of the great virtues of subharmonic functions is their flexibility, and this would be lost if we were to assume they were smooth.

Definition 2.3.2. *Let U be an open subset of $\mathbb{C}$. A function $u : U \to [-\infty, \infty[$ is called subharmonic if it satisfies the following properties:*

(i) It is upper semicontinuous.

(ii) For every $z \in U$ and $r > 0$ such that $\overline{\Delta(z, r)} \subset U$, and for every continuous function h on $\overline{\Delta(z, r)}$ which is harmonic on $\Delta(z, r)$ and satisfies $h \geq u$ on $\partial\Delta(z, r)$, it holds that $h \geq u$ on $\Delta(z, r)$.

Almost immediately from definition and the mean-value property of harmonic function we can derive the following theorem, which some textbooks prefer as the original definition:

Theorem 2.3.3 (Submean inequality, special case of Th. 3.2.3(ii) of [7]). *A function $u : U \to [-\infty, \infty[$ is subharmonic if and only if it is upper semicontinuous and for every $z \in U$ and every $r > 0$ such that $\overline{\Delta(z, r)} \subset U$ we have*

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it}) dt.$$

Even though subharmonic functions need not be smooth, they can never the less be approximated by others which are.

Theorem 2.3.4 (Th. 2.7.2 of [10]). *Let $u : U \to [-\infty, \infty[$ be subharmonic. There exist open sets $U_1 \subset U_2 \subset U_3 ...$ increasing to U and functions $u_n \in C^\infty(U_n)$ such that u_n is subharmonic on U_n for all n and $u_n \searrow u$.*

We gain numerous examples of subharmonic functions by noticing that for a holomorphic function f, the function $\log |f|$ is subharmonic. Further examples can be generated from the following result:

Theorem 2.3.5 (Th. 3.2.2 and Cor. 2.4.3 of [7]).

(i) If u, v are functions subharmonic on U and $\alpha, \beta \geq 0$, then $\alpha u + \beta v$ is subharmonic on U.

(ii) If $(u)_{i \in I}$ is a family of subharmonic functions on U and the function $\sup_{i \in I} u_i$ is upper semicontinuous on U and takes values in $[-\infty, \infty[$, then $\sup_{i \in I} u_i$ is subharmonic on U.

(iii) If $f : U_1 \to U_2$ is a conformal mapping between open subsets U_1 and U_2 of $\mathbb{C}$, and if u is subharmonic on U_2, then $u \circ f$ is subharmonic on U_1.

Using *(iii)* in the preceding theorem, we can extend the definition of subharmonicity to the Riemann sphere. Given a function u defined on a neighbourhood U of ∞, we say u is subharmonic on U if $u \circ g^{-1}$ is subharmonic on $g(U)$, where g is a conformal mapping of U onto an open subset of $\mathbb{C}$. It can be seen that it does not matter which map g is chosen.

Theorem 2.3.6 (Maximum principle, Th. 2.3.1 of [10]). *Let u be a subharmonic function on a domain D in $\mathbb{C}$.*

(i) If u attains a global maximum in D, then u is constant.

(ii) If D is bounded and $\limsup_{z \to \zeta} u(z) \leq 0$ for all $\zeta \in \partial D$, then $u \leq 0$ on D.

2.4 The Dirichlet problem

Let $U \subset \mathbb{C}$ be an open set, $U \neq \mathbb{C}$. Let f be a given continuous function on ∂U. Does there exist a continuous function u on $\overline{U}$ such that $u = f$ on ∂U and u is harmonic on U? If u exists, is it unique? These two questions taken together are called the Dirichlet problem for the domain U. It has many motivations from physics, for instance the distribution of heat in a heat-conducting, thin film in thermal equilibrium is known to be a harmonic function. Suppose one knew the temperature at the edge of such a film, then by solving the Dirichlet problem one could calculate the heat at every point of that film.

The uniqueness part of the question is easily answered. Suppose u_1 and u_2 were two functions solving the Dirichlet problem discussed above. Then $u_1 - u_2$ would be harmonic on U and equal to 0 on ∂U. By the maximum principle we get $u_1 = u_2$. We thus see that if such a function exists it is unique, it is much harder to determine whether such a function exists.

2.4.1 The Perron method

The key idea of solving the Dirichlet problem will be enshrined in the following definition. This method of gaining the solution is often called the Perron method.

Definition 2.4.1. *Let D be a proper subdomain of the Riemann sphere, $\widehat{\mathbb{C}}$, and let $\phi : \partial D \to \mathbb{R}$ be a bounded function. The associated Perron function $H_D\phi : D \to \mathbb{R}$ is defined by*

$$H_D\phi(z) = \sup_{u \in \mathcal{U}} u(z) \qquad z \in D$$

Where $\mathcal{U}$ denotes the family of all subharmonic functions u on D which satisfy $\limsup_{z \to z_0} u(z) \leq \phi(z_0)$ for each $z_0 \in \partial D$.

The motivation for this definition is that if the Dirichlet problem has a solution at all, then $H_D\phi$ is the solution. Indeed, if h is such a solution then $h \in \mathcal{U}$ and so $h \leq H_D\phi$. On the other hand, by the maximum principle, if $u \in \mathcal{U}$, then $u \leq h$ on D, and so $H_D\phi \leq h$. Therefore $h = H_D\phi$. The only question left to answer is: when does a solution exist?

Consider for a moment the domain $D = \{z \in \mathbb{C} : \ 0 < |z| < 1\}$ and the continuous function $\phi : \ \partial D \to \mathbb{R}$ given by

$$\phi(z) = \begin{cases} 0 & \text{if } |z| = 1, \\ 1 & \text{if } |z| = 0. \end{cases}$$

If $u \in \mathcal{U}$ as in the notation above, then by defining $u(0) =: \limsup_{z \to 0} u(z)$ it can be seen that u is subharmonic on all $\Delta := \Delta(0,1)$. Hence the Perron function $H_D\phi$ is be the same as the one if we were considering the Dirichlet problem on Δ with constant boundary function 0. But the solution to that is the zero function hence

$$H_D\phi = 0,$$

which is obviously does not solve the Dirichlet problem on D with boundary function ϕ because of the discontinuity at 0. We see that the Dirichlet problem for $D = \{z \in \mathbb{C} : \ 0 < |z| < 1\}$ is unsolvable. The reason for that is that the isolated boundary point 0 lacks sufficient "influence" on the subharmonic functions in $\mathcal{U}$.

Definition 2.4.2. *Let D be a proper subdomain of $\widehat{\mathbb{C}}$ and let $\zeta_0 \in \partial D$. A barrier at ζ_0 is a subharmonic function b defnied on $D \cap N$ where N is an open neighbourhood of ζ_0 satisfying*

$$b < 0 \qquad \text{on} \qquad D \cap N$$

and

$$\lim_{z \to \zeta_0} b(z) = 0.$$

A boundary point at which a barrier exists is called regular, otherwise irregular. If every $\zeta \in \partial D$ is regular, then D is called a regular domain.

Now for the main result:

Theorem 2.4.3 (Cor. 4.1.8 of [10])**.** *Let D be a regular domain, and let $\phi : \partial D \to \mathbb{R}$ be a continuous function. Then there exists a unique harmonic function h on D such that $\lim_{z \to \zeta} h(z) = \phi(\zeta)$ for all $\zeta \in \partial D$.*

There is also a converse to the preceding theorem, which means that regularity is not only sufficient to guarantee the existence of a solution but also necessary:

Theorem 2.4.4 (Discussed after Cor. 4.1.8 of [10])**.** *Let D be a domain which is not regular. Then there exists a continuous function $\phi : \partial D \to \mathbb{R}$ such that no function $h \in C(\overline{D})$ which is harmonic on D is equal to ϕ on the boundary.*

2.4.2 Criteria for regularity

Although the results of the previous section seem to solve the Dirichlet problem completely, it leaves a certain question unanswered. How can we tell if a boundary point of a set D is regular?

Theorem 2.4.5. *If D is a simply connected domain such that $\widehat{\mathbb{C}} \setminus D$ contains at least two points, then D is a regular domain.*

Proof. We need to show that every boundary point is regular. Given $z_0 \in \partial D$, pick $z_1 \in \partial D \setminus \{z_0\}$. Applying a conformal mapping of the sphere if necessery, we can assume that $z_0 = 0$ and $z_1 = \infty$. Then D is a simply connected subdomain of $\mathbb{C} \setminus \{0\}$ and therefore there exists a branch of the logarithm on D, which we call log. Put $N = \Delta(0, 1)$ and define a function b on $N \cap D$ by

$$b(z) = \operatorname{Re}\left(\frac{1}{\log(z)}\right).$$

Then clearly b satisfies the conditions to be a barrier at 0. □

Since regularity is clearly a local property, but not a global one, it is easy to generalize this theorem.

Corollary 2.4.6. *Let D be a subdomain of $\widehat{\mathbb{C}}$, let $z_0 \in \partial D$ and C be the component of ∂D which contains z_0. If $C \neq \{z_0\}$ then z_0 is a regular boundary point of D.*

2.5 Green's functions

In essence, a Green's function for D is a family of fundamental solutions of the Laplacian. In Chapter 4 they become very useful in constructing holomorphic functions with a prescribed growth rate.

Definition 2.5.1. *Let D be a proper regular subdomain of $\widehat{\mathbb{C}}$. A Green's function for D is a map $g_D : D \times D \to]-\infty, \infty]$ such that for each $w \in D$:*

(i) $g_D(\cdot, w)$ *is harmonic on $D \setminus \{w\}$ and bounded outside each neighbourhood of w.*

(ii) $\lim_{z \to w} g_D(z, w) = \infty$, *more explicitly*

$$g_D(z, w) = \begin{cases} \log|z| + O(1) & \text{if } w = \infty, \\ -\log|z - w| + O(1) & \text{if } w \neq \infty. \end{cases}$$

(iii) $\lim_{z \to \zeta} g_D(z, w) = 0$ *for every $\zeta \in \partial D$.*

For example, a Green's function for the unit disc $\Delta = \Delta(0, 1)$ is

$$g_\Delta(z, w) = \log \left| \frac{1 - z\bar{w}}{z - w} \right|.$$

Having solved the Dirichlet problem, the construction of a Green's function becomes very easy. For a domain D and $w \in D$, $w \neq \infty$ we solve the Dirichlet problem on D with a boundary function $\phi : \partial D \to \mathbb{R}$ given by $\phi(z) = \log|z - w|$. Say the solution is h then we have

$$g_D(z, w) = h(z) - \log|z - w|.$$

If $w = \infty$ the Green's function is constructed in a similar way.

From the construction and the uniqueness of the solution to the Dirichlet problem it should be clear that the Green's function for a given domain is unique. Therefore we usually talk about the Green's function with a definite article, rather than a Green's function. Some of the most basic properties of Green's functions are the following.

Theorem 2.5.2 (Theorems 4.4.3, 4.4.5 and 4.4.8 of [10])**.** *Let D_1, D_2 be a regular domains in $\widehat{\mathbb{C}}$ such that $D_1 \subset D_2$.*

(i) $g_{D_1}(z, w) > 0 \qquad z, w \in D_1$.

(ii) $g_{D_1}(z, w) = g_{D_1}(w, z) \qquad z, w \in D_1$.

(iii) $g_{D_1}(z, w) \leq g_{D_2}(z, w) \qquad z, w \in D_1$.

2.6 Distributions

Various difficulties in the theory of partial differential equations and Fourier analysis are some of the main reasons to extend the space of continuous functions and explore the theory of distributions.

2.6.1 Definitions

Let X be an open set in $\mathbb{R}^n$. A *distribution* u in X is a linear form on $C_c^\infty(X)$ such that for every compact set $K \subset X$ there exist constants C and k such that

$$|u(\phi)| \leq C \sum_{|\alpha| \leq k} \sup |\partial^\alpha \phi|, \qquad \phi \in C_c^\infty(K).$$

The set of distributions in X is denoted by $\mathscr{D}'(X)$. If the same integer k can be used in the estimate above for every compact K we say that u is of order $\leq k$ and the set of such distributions is denoted by $\mathscr{D}'^k(X)$. The union

$$\mathscr{D}'_F(X) = \bigcup_{k \in \mathbb{N}} \mathscr{D}'^k(X)$$

is the space of distributions of finite order. Some people choose to define distributions in a different but equivalent way. We state the equivalence of the two definitions as a theorem.

Theorem 2.6.1 (Th. 2.1.4 of [5])**.** *A linear form u on C_c^∞ is a distribution if and only if $u(\phi_j) \to 0$ when $j \to \infty$ for every sequence $\phi_j \in C_c^\infty(X)$ converging to 0 in the sense that there exists some compact $K \subset X$ such that supp $\phi_j \subset K$ for every j and $\partial^\alpha \phi_j \to 0$ uniformly for every α.*

Now, for each $f \in L^1_{\text{loc}}(X)$ we can define a distribution in X with

$$\phi \to \int_X f(x)\phi(x)dx, \qquad \phi \in C_c^\infty(X).$$

It is easily seen that if $f, g \in L^1_{\text{loc}}(X)$ give the same distribution in that way then f and g are equal almost everywhere. Therefore we can identify the space $L^1_{\text{loc}}(X)$ as a subspace of $\mathscr{D}'(X)$, if it is considered as a space of equvialence classes, and we write $f \in \mathscr{D}'(X)$ meaning

$$f(\phi) = \int_X f(x)\phi(x)dx.$$

If $f \in L^1_{\text{loc}}(X)$ then f has order 0 since

$$|f(\phi)| \leq \int_K |f(x)|dx \cdot \sup |\phi|, \qquad \phi \in C_c^\infty(X).$$

2.6.2 Differential equations in the sense of distributions

If f is a continuous function on X such that $\partial_k f$ is defined everywhere on X and is continuous, integration by parts gives

$$\int_X \partial_k f(x)\phi(x)dx = -\int_X f(x)\partial_k\phi(x)dx.$$

In the sense of distributions, this means $(\partial_k f)(\phi) = -f(\partial_k\phi)$. It is therefore reasonable to define differentiation on the set of distributions in a similar way. For $u \in \mathscr{D}'(X)$ we define

$$(\partial_k u)(\phi) := -u(\partial_k\phi), \qquad \phi \in C_c^\infty(X).$$

For similar reasons we can define the product of distributions and infinitely differentiable functions. If $u \in \mathscr{D}'(X)$ and $f \in C^\infty(X)$ we define

$$(fu)(\phi) := u(f\phi), \qquad \phi \in C_c^\infty(X).$$

It is not quite obvious that the classical rules of differentiation will carry over to the space of distributions.

Theorem 2.6.2 (Discussed in Ch. 3.1 of [5])**.** *Let $u \in \mathscr{D}'(X)$ and $f \in C^\infty(X)$. Then*

$$\partial_k(\partial_j u) = \partial_j(\partial_k u).$$
$$\partial_k(fu) = (\partial_k f)u + f(\partial_k u).$$

Extending the definition of differentiation to distributions in such a way can sometimes be useful when solving classical partial differential equations.

Theorem 2.6.3 (Cor. 3.1.6 of [5])**.** *If $X \subset \mathbb{R}$ is open, $u \in \mathscr{D}'(X)$ and if*

$$u^{(m)} + a_{m-1}u^{(m-1)} + ... + a_0 u = f,$$

where $f \in C(X)$ and the coefficients $a_j \in C^\infty(X)$, then $u \in C^m(X)$ so the equation is fulfilled in the classical sense.

The function $E(z) = \frac{1}{\pi z}$ is a fundemental solution of the $\bar{\partial}$-operator, which means that $\partial E/\partial\bar{z} = \delta_0$ in the sense of distributions. The singular support of E, i.e. the complement of the open set where E is C^∞, is the set $\operatorname{singsupp}(E) = \{0\}$. The following theorem shows that every solution of $\partial u/\partial\bar{z} = f$ with f a C^∞ function is in fact a C^∞ function and a solution in the classical sense.

Theorem 2.6.4 (Th. 4.4.1 of [5])**.** *If P has a fundamental solution E, with* $\operatorname{singsupp}(E) = \{0\}$ *and X is any open set in $\mathbb{R}^n$, then*

$$\operatorname{singsupp}(u) = \operatorname{singsupp}(Pu), \qquad u \in \mathscr{D}'(X).$$

2.7 Results from functional analysis

Let X be some vector space over a scalar field $\mathbb{F}$. A functional f on X is a linear map from X to $\mathbb{F}$. In other words, a scalar valued function on X such that

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \qquad \alpha, \beta \in \mathbb{F}, \qquad x, y \in X.$$

If the vector space X is normed it is easily seen that the following statements are equivalent for a functional f:

1. f is continuous.
2. f is continuous at 0.
3. $f(B)$ is bounded, where B is the unit ball in X.
4. There exists a real number $C \geq 0$ such that $|f(x)| \leq C\|x\|$ for all $x \in X$.

A functional f is said to be bounded if it satisfies one of the preceding statements. In light of this it makes sense to define a norm on the set of bounded functionals by

$$\|f\| := \sup\{|f(x)| \; ; \|x\| \leq 1\}.$$

We now state two fundamental results from functional analysis.

Theorem 2.7.1 (Hahn-Banach, Th. 8.9 of [9])**.** *Let f_0 be a bounded functional defined on a linear subspace M of a real or complex normed vector space X. Then f_0 has an extension to a bounded functional f on X such that $\|f\| = \|f_0\|$.*

Theorem 2.7.2 (Riesz, Th. 12.10 of [9])**.** *Let H be an Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let f be a bounded functional on H. Then there exists an $a \in H$ such that $f(x) = \langle x, a \rangle$ for all $x \in H$, further $\|a\| = \|f\|$.*

3 Hörmander's L^2 theorem

The goal of this chapter is to prove Hörmander's theorem for the solution of the $\bar{\partial}$-equation with L^2-estimates in the case of one complex variable. As an application we prove the Brunn-Minkowski inequality. The proof will be based on the one given in notes by Bo Berndtsson [1]. For the rest of the thesis we will let $d\lambda$ denote the usual Lebesgue-measure on $\mathbb{C}$.

3.1 Hörmander

Hörmander's theorem is as follows:

Theorem 3.1.1 (Hörmander's Theorem)**.** *Let Ω be a domain in $\mathbb{C}$ and $\phi \in C^2(\Omega)$ be a real function such that $\Delta\phi = 4\frac{\partial^2}{\partial z \partial \bar{z}}\phi > 0$. Let $f \in L^2_{loc}(\Omega)$, then there is a function $u \in L^2_{loc}(\Omega)$ such that*

$$\frac{\partial u}{\partial \bar{z}} = f$$

in the sense of distributions and

$$\int_\Omega |u|^2 e^{-\phi} d\lambda \leq 4 \int_\Omega \frac{|f|^2}{\Delta\phi} e^{-\phi} d\lambda.$$

The proof of this theorem needs some preparation. Since we are estimating the norm of u considered as an element of the Hilbert space $L^2_\phi(\Omega) = \{g \in L^2_{\mathrm{loc}}(\Omega) : \int_\Omega |g|^2 e^{-\phi} d\lambda < \infty\}$ it will be useful to know the adjoint of the $\bar{\partial}$-operator in this space:

Lemma 3.1.2. *The adjoint of the $\bar{\partial}$-operator with respect to the inner product*

$$\langle f, g\rangle_\phi = \int_\Omega f\bar{g}e^{-\phi} d\lambda$$

is given by the equation

$$\bar{\partial}^*_\phi \alpha = -e^\phi \frac{\partial}{\partial z}(e^{-\phi}\alpha).$$

Proof. The set $C^\infty_c(\Omega)$ is dense in $L^2_\phi(\Omega)$. Therefore it is sufficient to show that

$$\langle f, \bar{\partial}^*_\phi \alpha\rangle_\phi = \langle \frac{\partial f}{\partial \bar{z}}, \alpha\rangle_\phi$$

for every $\alpha \in C_c^\infty(\Omega)$:

$$\langle f, \bar{\partial}_\phi^* \alpha \rangle_\phi = \int_\Omega f \overline{\bar{\partial}_\phi^* \alpha} e^{-\phi} d\lambda = \int_\Omega f e^{-\phi} \overline{(-e^{\phi} \frac{\partial}{\partial z}(e^{-\phi}\alpha))} d\lambda = -\int_\Omega f \frac{\partial}{\partial \bar{z}}(e^{-\phi}\bar{\alpha}) d\lambda$$
$$= \int_\Omega (\frac{\partial}{\partial \bar{z}} f) \bar{\alpha} e^{-\phi} d\lambda = \langle \frac{\partial f}{\partial \bar{z}}, \alpha \rangle_\phi .$$

□

The following proposition is the key idea of the proof of Hörmander's theorem. It reduces the proof of an existence statement to the proof of an inequality.

Proposition 3.1.3. *Let f and ϕ be as in Theorem 3.1.1. The following two statements are equivalent:*

(i) There exists a function u which satisfies the equation $\frac{\partial u}{\partial \bar{z}} = f$ and the estimate

$$\int_\Omega |u|^2 e^{-\phi} d\lambda \leq C.$$

(ii) The inequality

$$|\int_\Omega f \bar{\alpha} e^{-\phi} d\lambda|^2 \leq C \int_\Omega |\bar{\partial}_\phi^* \alpha|^2 e^{-\phi} d\lambda$$

holds true for all $\alpha \in C_c^2(\Omega)$.

Proof. Let's begin assuming that (i) is true. Interpreted in the sense of distributions the equation $\frac{\partial u}{\partial \bar{z}} = f$ implies

$$-\int_\Omega u \frac{\partial \alpha}{\partial \bar{z}} d\lambda = \int_\Omega f \alpha d\lambda, \qquad \alpha \in C_c^2(\Omega). \tag{1}$$

If we interchange the function α for the function $\bar{\alpha} e^{-\phi}$ we get

$$\int_\Omega u \overline{\bar{\partial}_\phi^* \alpha} e^{-\phi} d\lambda = \int_\Omega f \bar{\alpha} e^{-\phi} d\lambda, \qquad \alpha \in C_c^2(\Omega). \tag{2}$$

Now by combining equations of statement (i) and equation (2) and applying the Cauchy-Schwarz inequality, we get

$$|\int_\Omega f \bar{\alpha} e^{-\phi} d\lambda|^2 = |\int_\Omega u \overline{\bar{\partial}_\phi^* \alpha} e^{-\phi} d\lambda|^2 \leq (\int_\Omega |u|^2 e^{-\phi} d\lambda)(\int_\Omega |\overline{\bar{\partial}_\phi^* \alpha}|^2 e^{-\phi} d\lambda)$$
$$\leq C \int_\Omega |\bar{\partial}_\phi^* \alpha|^2 e^{-\phi} d\lambda$$

for all $\alpha \in C_c^2(\Omega)$. Hence (ii) is true.

Now, let's assume that (ii) is true. Let

$$E := \{\bar{\partial}_\phi^* \alpha; \alpha \in C_c^2(\Omega)\},$$

and consider E as a subspace of

$$L^2_\phi(\Omega) = \{g \in L^2_{\text{loc}}(\Omega); \int_\Omega |g|^2 e^{-\phi} d\lambda < \infty\}.$$

We define an antilinear functional on E by

$$H(\bar{\partial}^*_\phi \alpha) = \int_\Omega f\bar{\alpha} e^{-\phi} d\lambda.$$

Notice that the operator $\bar{\partial}^*_\phi$ is one-to-one on the space of C^2-functions with compact support so H is well defined. Also, according to assumption, the functional H is continuous and of norm not exceeding C. The Hahn-Banach extension theorem now guarantees an antilinear extension of H to $L^2_\phi(\Omega)$ with the same norm. The Riesz representation theorem then implies that there is some element $u \in L^2_\phi(\Omega)$ with norm not exceeding C such that

$$H(g) = \int_\Omega u\bar{g} e^{-\phi} d\lambda$$

for all $g \in L^2_\phi(\Omega)$. If we set $g = \bar{\partial}^*_\phi \alpha$ then

$$\int_\Omega u e^{-\phi} \overline{\bar{\partial}^*_\phi \alpha} d\lambda = \int_\Omega f\bar{\alpha} e^{-\phi} d\lambda,$$

which according to equation (2) implies that $\frac{\partial u}{\partial \bar{z}} = f$ in the sense of distributions. $\square$

Corollary 3.1.4. *Let $\mu > 0$ be in $L^1_{loc}(\Omega)$. The following two statements are equivalent.*

(i) Statement (ii) of Proposition 3.1.3 is fulfilled for every f which satisfies the estimate

$$\int_\Omega \frac{|f|^2}{\mu} e^{-\phi} d\lambda \leq C.$$

(ii) The inequality

$$\int_\Omega \mu |\alpha|^2 e^{-\phi} d\lambda \leq \int_\Omega |\bar{\partial}^*_\phi \alpha|^2 e^{-\phi} d\lambda$$

holds true for all $\alpha \in C^2_c(\Omega)$.

Proof. Assume (ii) is true. Then the Cauchy-Schwarz inequality gives

$$|\int_\Omega f\bar{\alpha} e^{-\phi} d\lambda|^2 = |\int_\Omega \frac{f}{\sqrt{\mu}} \overline{\sqrt{\mu}\alpha} e^{-\phi} d\lambda|^2 \leq \int_\Omega \frac{|f|^2}{\mu} e^{-\phi} d\lambda \int_\Omega \mu |\alpha|^2 e^{-\phi} d\lambda \leq C \int_\Omega |\bar{\partial}^*_\phi \alpha|^2 e^{-\phi} d\lambda,$$

hence (i) holds.

Now assume (i) is true. For any function $h \in L^2_\phi(\Omega)$ with norm less than C we can set $f = h\sqrt{\mu}$ and then we get by assumption that

$$|\int_\Omega h\bar{\alpha}\sqrt{\mu}e^{-\phi}d\lambda|^2 \leq C\int_\Omega |\bar{\partial}^*_\phi\alpha|^2 e^{-\phi}d\lambda.$$

By the converse Hölder's inequality, we get (ii). $\square$

To complete the proof of Hörmander's theorem it is enough to prove an inequality as in statement (ii) of Proposition 3.1.3. This will be accomplished by the following integral identity.

Proposition 3.1.5. *Let Ω be a domain in $\mathbb{C}$, $\phi \in C^2(\Omega)$ and $\alpha \in C^2_c(\Omega)$, then*

$$\int_\Omega \frac{\Delta\phi}{4}|\alpha|^2 e^{-\phi}d\lambda + \int_\Omega \left|\frac{\partial\alpha}{\partial\bar{z}}\right|^2 e^{-\phi}d\lambda = \int_\Omega |\bar{\partial}^*_\phi\alpha|^2 e^{-\phi}d\lambda.$$

Proof. Applying Lemma 3.1.2, we get

$$\int_\Omega |\bar{\partial}^*_\phi\alpha|^2 e^{-\phi}d\lambda = \int_\Omega (\bar{\partial}^*_\phi\alpha)(\overline{\bar{\partial}^*_\phi\alpha})e^{-\phi}d\lambda = \int_\Omega (\frac{\partial}{\partial\bar{z}}\bar{\partial}^*_\phi\alpha)\bar{\alpha}e^{-\phi}d\lambda.$$

also

$$\bar{\partial}^*_\phi\alpha = -e^\phi\frac{\partial}{\partial z}(e^{-\phi}\alpha) = -e^\phi(-\frac{\partial\phi}{\partial z}e^{-\phi}\alpha + e^{-\phi}\frac{\partial\alpha}{\partial z}) = -\frac{\partial\alpha}{\partial z} + \frac{\partial\phi}{\partial z}\alpha.$$

so

$$\frac{\partial}{\partial\bar{z}}\bar{\partial}^*_\phi\alpha = -\frac{\Delta\alpha}{4} + \frac{\partial\phi}{\partial z}\cdot\frac{\partial\alpha}{\partial\bar{z}} + \frac{\Delta\phi}{4}\alpha = \bar{\partial}^*_\phi\left(\frac{\partial\alpha}{\partial\bar{z}}\right) + \frac{\Delta\phi}{4}\alpha.$$

Therefore we get

$$\begin{aligned}\int_\Omega |\bar{\partial}^*_\phi\alpha|^2 e^{-\phi}d\lambda &= \int_\Omega (\frac{\partial}{\partial\bar{z}}\bar{\partial}^*_\phi\alpha)\bar{\alpha}e^{-\phi}d\lambda = \int_\Omega \bar{\partial}^*_\phi\left(\frac{\partial\alpha}{\partial\bar{z}}\right)\bar{\alpha}e^{-\phi}d\lambda + \int_\Omega \frac{\Delta\phi}{4}|\alpha|^2 e^{-\phi}d\lambda \\ &= \int_\Omega \frac{\partial\alpha}{\partial\bar{z}}\overline{\frac{\partial\alpha}{\partial\bar{z}}}e^{-\phi}d\lambda + \int_\Omega \frac{\Delta\phi}{4}|\alpha|^2 e^{-\phi}d\lambda \\ &= \int_\Omega \left|\frac{\partial\alpha}{\partial\bar{z}}\right|^2 e^{-\phi}d\lambda + \int_\Omega \frac{\Delta\phi}{4}|\alpha|^2 e^{-\phi}d\lambda.\end{aligned}$$

$\square$

From the last two propositions the proof of Hörmander's theorem follows almost immediately.

Proof (Hörmander's Theorem, 3.1.1). From Proposition 3.1.5 we get

$$\int_\Omega \frac{\Delta\phi}{4}|\alpha|^2 e^{-\phi}d\lambda \leq \int_\Omega |\bar{\partial}^*_\phi\alpha|^2 e^{-\phi}d\lambda, \qquad \alpha \in C^2_c(\Omega).$$

According to Corollary 3.1.4

$$|\int_\Omega f\bar{\alpha}e^{-\phi}d\lambda|^2 \leq C\int_\Omega |\bar{\partial}_\phi^*\alpha|^2 e^{-\phi}d\lambda.$$

Where $C = 4\int_\Omega \frac{|f|^2}{\Delta\phi}e^{-\phi}d\lambda$. From Corollary 3.1.3 we see that there exists a function u such that $\frac{\partial}{\partial\bar{z}}u = f$ and

$$\int_\Omega |u|^2 e^{-\phi}d\lambda \leq 4\int_\Omega \frac{|f|^2}{\Delta\phi}e^{-\phi}d\lambda.$$

□

Corollary 3.1.6. *Suppose $\phi \in C^2(\Omega)$ satisfies $\Delta\phi > 0$. Let u be a C^1 function in a domain Ω such that*

$$\int_\Omega u\bar{h}e^{-\phi}d\lambda = 0$$

for every holomorphic function h in $L^2_\phi(\Omega)$. Then

$$\int_\Omega |u|^2 e^{-\phi}d\lambda \leq 4\int_\Omega \frac{|\partial u/\partial\bar{z}|^2}{\Delta\phi}e^{-\phi}d\lambda. \tag{3}$$

Proof. Hörmander's theorem states that the equation

$$\frac{\partial v}{\partial\bar{z}} = \frac{\partial u}{\partial\bar{z}}$$

has some solution v satisfying (3) with u replaced by v. But, the condition that u is orthogonal to all holomorphic functions means that u is the solution to this equation which has minimal norm in $L^2_\phi(\Omega)$. Hence u satisfies the estimate as well which is what the corollary claims. □

Now, lets consider the weight function ϕ in Hörmander's theorem for a moment. We notice that the Laplacian of the weight function has to maintain a strict inequality on all of Ω. That can be very inconvenient since we might want to use some subharmonic function as a weight function for which the Laplacian is definitely not guaranteed to be strictly positive. Therefore we shall state and prove a second version of Hörmander's theorem:

Theorem 3.1.7 (Hörmander's Theorem (Second Version))**.** *Let $f \in L^2_{\text{loc}}(\Omega)$. If $a > 0$ and $\phi \in C^2(\Omega)$ is subharmonic, then there is a function $u \in L^2_{loc}(\Omega)$ such that $\frac{\partial u}{\partial\bar{z}} = f$ in the sense of distributions and*

$$a\int_\Omega |u(z)|^2 e^{-\phi(z)}(1+|z|^2)^{-a}d\lambda(z) \leq \int_\Omega |f(z)|^2 e^{-\phi(z)}(1+|z|^2)^{2-a}d\lambda(z). \tag{4}$$

Proof. Let $\psi(z) = \phi(z) + a\log(1+|z|^2)$, let $r = |z|$ and get

$$\Delta\psi(z) \geq a\Delta\log(1+|z|^2) = ar^{-1}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}\log(1+r^2)) = \frac{4a}{(1+r^2)^2} > 0.$$

Hörmanders Theorem therefore applies to ψ so there exists some u such that $\frac{\partial u}{\partial \bar{z}} = f$ and

$$\int_\Omega |u|^2 e^{-\psi} d\lambda \leq 4\int_\Omega \frac{|f|^2}{\Delta\psi} e^{-\psi} d\lambda.$$

Now we have

$$e^{-\psi} = e^{-\phi}(1+|z|^2)^{-a}$$

and

$$\frac{4e^{-\psi}}{\Delta\psi} \leq \frac{4e^{-\phi}(1+|z|^2)^{-a}}{4a/(1+|z|^2)^2} = \frac{e^{-\phi}(1+|z|^2)^{2-a}}{a},$$

which implies (4). □

Still there are some inconvenient restrictions for the weight function ϕ, it is required to be in $C^2(\Omega)$. In fact we can do better:

Theorem 3.1.8. *The preceding theorem holds true for any subharmonic weight function ϕ which is not constantly equal to $-\infty$ in any component of Ω.*

Proof. According to Theorem 2.3.4 there are open sets, $Y_1 \subset Y_2 \subset Y_3 \subset ...$ increasing to Ω and a sequence of subharmonic functions $\phi_j \in C^\infty(Y_j)$ with $\phi_{j+1} \leq \phi_j$ in Y_j for all j and $\lim_{j\to\infty}\phi_j = \phi$ in X. According to Theorem 3.1.7 there exists a function u_j, such that $\frac{\partial u_j}{\partial \bar{z}} = f$ in Y_j and

$$a\int_{Y_j} |u_j(z)|^2 e^{-\phi_j(z)}(1+|z|^2)^{-a} d\lambda(z) \leq \int_\Omega |f(z)|^2 e^{-\phi(z)}(1+|z|^2)^{2-a} d\lambda(z) \qquad (5)$$

where on the right-hand side we have extended the domain of integration and increased the weight. We see that the sequence $(u_n)_{n\geq j}$ is bounded in Y_j for every j, if it is considered as a sequence in the space $L^2_\phi(Y_j)$. Since the unit ball is compact in the weak topology, we can therefore find a subsequence u_{n_k} which is weakly convergent in $L^2_\phi(Y_j)$ for every j to some limit u. Since differential operators are continuous in the weak topology of $\mathscr{D}'(X)$, we have $\partial u/\partial\bar{z} = f$. Also, by monotone convergence, letting $j \to \infty$ we see that equation (4) holds true with u_j replaced with u and Y_j replaced with X. □

3.2 An application: Brunn-Minkowski type inequalities

In this section we give an application of Corollary 3.1.6. It will be applied in a proof of a functional form of the Brunn-Minkowski inequality, which can in fact be considered a generalization of it.

Proposition 3.2.1. *Let $\phi \in C^2(\mathbb{R})$ be a convex function with $\phi'' > 0$. Let $u \in C^1(\mathbb{R})$ be a function such that*

$$\int_{\mathbb{R}} u(x) e^{-\phi(x)} dx = 0.$$

Then

$$\int_{\mathbb{R}} |u(x)|^2 e^{-\phi(x)} dx \leq \int_{\mathbb{R}} \frac{|u'(x)|}{\phi''(x)} dx.$$

This statement is very similar to the one of Corollary 3.1.6. In a sense just a real version of it. We have replaced subharmonic ϕ by a convex ϕ and we require u to be orthogonal to constants, which is the kernel of the regular differential operator, rather than to be orthogonal to the space of holomorphic functions, which is the kernel of the $\partial/\partial\bar{z}$ operator. We omit a formal proof which is very similar to the proof of the complex version. A generalized version of the Brunn-Minkowski theorem is as follows:

Theorem 3.2.2. *Let $\phi(t,x) \in C^2(\mathbb{R}^m \times \mathbb{R}^n)$ be a convex function, (this is to be interpreted as $t \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$), and assume that*

$$\Delta_x \phi(t,x) > 0 \qquad t \in \mathbb{R}^m, x \in \mathbb{R}^n,$$

where $\Delta_x \phi$ is the Laplacian of ϕ when considered as a function of $x \in \mathbb{R}^n$ with t constant. Define a function $\tilde{\phi}$ on $\mathbb{R}^m$ by

$$\tilde{\phi}(t) = -\log\left(\int_{\mathbb{R}^n} e^{-\phi(t,x)} dx\right), \qquad t \in \mathbb{R}^m.$$

Then $\tilde{\phi}$ is a convex function.

Proof. We start by making some reductions. By Fubini's theorem we may assume that $n = 1$, then the case $n > 1$ follows by a simple induction argument. Since convexity means convexity on any line, we may also assume that $m = 1$. In the following argument we write $\phi_t := \partial\phi/\partial t$ and $\phi_x := \partial\phi/\partial x$.
We have

$$\tilde{\phi}(t) = -\log\left(\int_{\mathbb{R}} e^{-\phi(t,x)} dx\right).$$

Differentiating with respect to t we get

$$\tilde{\phi}'(t) = \frac{\int \phi_t(t,x) e^{-\phi(t,x)} dx}{\int e^{-\phi(t,x)} dx},$$

and differentiating once more gives

$$\tilde{\phi}'' = \frac{\left(\int (\phi_{tt} - (\phi_t)^2) e^{-\phi} dx\right)\left(\int e^{-\phi} dx\right) + \left(\int \phi_t e^{-\phi} dx\right)^2}{\left(\int e^{-\phi} dx\right)^2}.$$

We simplify:

$$\begin{aligned}
\tilde{\phi}'' &= \frac{\int (\phi_{tt} - (\phi_t)^2) e^{-\phi} dx}{\int e^{-\phi} dx} + \left(\frac{\int \phi_t e^{-\phi} dx}{\int e^{-\phi} dx} \right)^2 = \frac{\int (\phi_{tt} - (\phi_t)^2) e^{-\phi} dx}{\int e^{-\phi} dx} + (\tilde{\phi}')^2 \\
&= \frac{\int (\phi_{tt} - (\phi_t)^2) e^{-\phi} dx}{\int e^{-\phi} dx} + \frac{(-(\tilde{\phi}')^2 + 2(\tilde{\phi}')^2) \int e^{-\phi} dx}{\int e^{-\phi} dx} \\
&= \frac{\int (\phi_{tt} - (\phi_t)^2) e^{-\phi} dx - \int (\tilde{\phi}')^2 e^{-\phi} dx + 2\tilde{\phi}' \int \phi_t e^{-\phi} dx}{\int e^{-\phi} dx} \\
&= \frac{\int (\phi_{tt} - (\phi_t)^2 - (\tilde{\phi}')^2 + 2\tilde{\phi}' \phi_t) e^{-\phi} dx}{\int e^{-\phi} dx} = \frac{\int (\phi_{tt} - (\phi_t - \tilde{\phi}')^2) e^{-\phi} dx}{\int e^{-\phi} dx}.
\end{aligned}$$

Now, define the function $u(t,x) := \phi_t(t,x) - \tilde{\phi}'(t)$. We notice that

$$\int u(t,x) e^{-\phi(t,x)} dx = 0 \qquad t \in \mathbb{R}.$$

Since by assumption $\phi_{xx} > 0$ we can use Proposition 3.2.1. Note that $u_x = \phi_{tx}$.

$$\int (\phi_t(t,x) - \tilde{\phi}'(t))^2 e^{-\phi(t,x)} dx \leq \int \frac{(\phi_{tx}(t,x))^2}{\phi_{xx}(t,x)} dx.$$

Thus we get

$$\tilde{\phi}'' = \frac{\int (\phi_{tt} - (\phi_t - \tilde{\phi}')^2) e^{-\phi} dx}{\int e^{-\phi} dx} \geq \frac{\int (\phi_{tt} - (\phi_{tx})^2 / \phi_{xx}) e^{-\phi} dx}{\int e^{-\phi} dx}.$$

Since the function ϕ is convex, the determinant of its Hessian matrix is non-negative and we have

$$\phi_{tt} - \frac{(\phi_{tx})^2}{\phi_{xx}} = \frac{\phi_{tt}\phi_{xx} - (\phi_{tx})^2}{\phi_{xx}} \geq 0,$$

and thus

$$\tilde{\phi}'' \geq 0.$$

□

The preceding theorem is a functional form of the Brunn-Minkowski inequality, which can be stated as follows.

Theorem 3.2.3. *Let D be a convex open set in $\mathbb{R}^m \times \mathbb{R}^n$. For any $t \in \mathbb{R}^m$, let D_t be a slice of D*

$$D_t := \{x \in \mathbb{R}^n : \ (t,x) \in D\}.$$

Let $|D_t|$ be the Lebesgue measure of D_t. Then the function

$$t \longrightarrow -\log |D_t|$$

is convex.

Proof. Take ϕ to be the function that equals 0 in D, and ∞ outside D. This function can be written as an increasing limit of smooth convex functions which satisfy the criteria in the theorem above. Therefore the function

$$\tilde{\phi}(t) = -\log\left(\int_{\mathbb{R}^n} e^{-\phi(t,x)}dx\right) = -\log|D_t|$$

is convex. □

There is another, perhaps more common, way of stating the Brunn-Minkowski inequality:

Theorem 3.2.4 (Brunn-Minkowski). *Let D_0 and D_1 be open convex sets in $\mathbb{R}^n$. For any $t \in [0,1]$ we have*

$$|tD_1 + (1-t)D_0|^{\frac{1}{n}} \geq t|D_1|^{\frac{1}{n}} + (1-t)|D_0|^{\frac{1}{n}}.$$

Proof. We define a set $D \in \mathbb{R} \times \mathbb{R}^n$ with

$$D := \{(t,x);\ x \in tD_1 + (1-t)D_0,\ t \in [0,1]\}.$$

Then

$$D_t = tD_1 + (1-t)D_0, \qquad t \in [0,1],$$

where as before, D_t were the slices of D given by $D_t := \{x \in \mathbb{R}^n :\ (t,x) \in D\}$. It can be seen that D is convex. Theorem 3.2.3 then implies that $|D_t| \geq \min\{|D_0|, |D_1|\}$. To see that assume $|D_t| < \min\{|D_0|, |D_1|\}$. Then $-\log|D_t| > \max\{-\log|D_0|, -\log|D_1|\}$ which is a contradiction because $-\log|D_t|$ is a convex function of t.

Now, let A and B be convex sets in $\mathbb{R}^n$ and $t \in]0,1[$.

$$|A+B|^{\frac{1}{n}} = \left|\frac{tA}{t} + \frac{(1-t)B}{1-t}\right|^{\frac{1}{n}} \geq \min\{\frac{|A|^{\frac{1}{n}}}{t}, \frac{|B|^{\frac{1}{n}}}{(1-t)}\}.$$

Choosing

$$t = \frac{|A|^{\frac{1}{n}}}{|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}}$$

we get

$$|A+B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}.$$

By substituting $D_0 := \frac{B}{1-t}$ and $D_1 := \frac{A}{t}$ we obtain

$$|tD_1 + (1-t)D_0|^{\frac{1}{n}} \geq t|D_1|^{\frac{1}{n}} + (1-t)|D_0|^{\frac{1}{n}},$$

which is the desired result. □

4 A generalized version of the Bernstein-Walsh theorem

From now on, for a given set $E \subset \mathbb{C}$, any bounded component of the set E^c will be refered to as a hole of E. The polynomial hull of a compact set K is the set

$$\hat{K} := \{z \in \mathbb{C};\ |f(z)| \leq \sup_{z \in K} |f(z)| \text{ for all } f \in \mathcal{O}(\mathbb{C})\}$$

and it is known that $\hat{K}$ is the union of K and all its holes.

Runge's theorem states that for a compact $K \subset \mathbb{C}$, a function $f \in \mathcal{O}(K)$, and $\epsilon > 0$ there exists a rational function $R = p/q$ with poles outside of K such that $\|f - R\|_K < \epsilon$. In practice, it could be beneficial to know the least possible degree of the polynomials p and q for such an estimate to be possible. The goal of this chapter is to estimate this least possible degree of p and q with respect to ϵ and prove a theorem similar to the Bernstein-Walsh theorem. We start by some definitions.

4.1 Definitions

Let $\mathbf{a} = (a_1, a_2, ..., a_m) \in \mathbb{C}^m$ and $\mathbf{n} = (n_0, n_1, n_2, ..., n_m) \in \mathbb{N}^{m+1}$. We let $R(\mathbf{a}, \mathbf{n})$ denote the set of all rational functions $r = \frac{p}{q}$ which satisfy the following conditions:

- If a is a zero of the polynomial q, then $a = a_k$ for some $k \in \{1, 2, ..., m\}$.
- If a_k is a zero of q its multiplicity is at most n_k.
- The degree of the polynomial p is at most

$$m + 2 + \sum_{k=0}^{m} n_k.$$

Now let K be a compact subset of $\mathbb{C}$ and assume that $a_k \notin K$ for all $k \in \{1, 2, ..., m\}$. Then we set

$$Rd(f, K, \mathbf{a}, \mathbf{n}) = \inf\{\|f - r\|_K : r \in R(\mathbf{a}, \mathbf{n})\}.$$

The goal of this chapter is to estimate $Rd(f, K, \mathbf{a}, \mathbf{n})$ as well as possible. Giving precise information on the degree of each pole of the optimal approximating rational function.

Now suppose that $K \subset \mathbb{C}$ is compact and $f \in \mathcal{O}(K)$. Let U be a neighbourhood of K on which f is holomorphic. The unbounded component of K^c will be denoted by Ω_0 and the holes of K not coverd by U will be denoted by Ω_1, Ω_2,...,Ω_m. There are only finitely many such holes, as stated in the following lemma.

Lemma 4.1.1. *Let K be a compact set and U be a neighbourhood of K. Let $(\Omega_i)_{i\in I}$ be the family of holes in K. Then $\Omega_i \subseteq U$ for all but finitely many $i \in I$.*

Proof. Note that since K is compact, $\hat{K}$ is as well. The family $(\Omega_i)_{i\in I}$ along with the open set U is an open cover of $\hat{K}$. Since $\hat{K}$ is compact there is a finite subcover. $\square$

Continuing our discussion, we assume that the vector $\mathbf{a}$ contains precisely one point from each of the holes Ω_1,...,Ω_m. If $\mathbf{a}$ didn't contain a point from some hole then we could not expect to approximate f with functions from $\cup_{\mathbf{n}} R(\mathbf{a}, \mathbf{n})$. On the other hand, if some hole of K would contain more than one point of $\mathbf{a}$ one could just pick one and omit the others. By redefining the holes we can assume that $a_k \in \Omega_k$ for all $k \in \{1, 2, ..., m\}$.

We assume that K is regular, since if it was not we could find a slightly bigger K' which is regular and $K \subset K' \subset U$. We define

$$G_0(z) := \begin{cases} g_{\Omega_0}(z, \infty) & \text{if } z \in \Omega_0, \\ 0 & \text{if } z \in \mathbb{C} \setminus \Omega_0 \end{cases}$$

and for $k \in \{1, 2, ..., m\}$

$$G_k(z) := \begin{cases} g_{\Omega_k}(z, a_k) & \text{if } z \in \Omega_k, \\ 0 & \text{if } z \in \mathbb{C} \setminus \Omega_k. \end{cases}$$

Where g_D is the Green's function for the regular domain D. Notice that by the definition of the Green's function, $G_k \in C(\mathbb{C} \setminus \{a_k\})$ for all k.

At last we define level sets of the set K, with respect to those Green's functions with the following notation. For a vector $\mathbf{r} = (r_0, r_1, r_2, ..., r_m) \in \mathbb{R}^{m+1}$ we set

$$K_{\mathbf{r}} = \{z \in \mathbb{C} : \ G_k(z) \leq \log(r_k) \text{ for } k \in \{0, 1, ..., m\}\}.$$

We see that if $r_k < 1$ for some k then $K_{\mathbf{r}} = \emptyset$ because the Green's functions are positive. We let $\mathbf{R} = (R_0, R_1, R_2, ...R_m) \in \mathbb{R}^{m+1}$ be a vector such that $R_k > 1$ for all k and $f \in A(K_{\mathbf{R}})$. Since f is holomorphic on a whole neighbourhood of K we know such an $\mathbf{R}$ exists.

4.2 The estimation procedure

In this section we use Hörmander's L^2-estimates to construct a holomorphic function $F \in R(\mathbf{a}, \mathbf{n})$. Later we see that this function estimates the function f properly.

4.2.1 The construction of F

First we state a lemma:

Lemma 4.2.1. *The function G_0 as constructed above is subharmonic on $\mathbb{C}$. The function G_k as constructed above is subharmonic on $\mathbb{C} \setminus \{a_k\}$ for $k \in \{1, 2, ..., m\}$.*

Proof. Let $k \neq 0$. By definition G_k is harmonic on $\Omega_k \setminus \{a_k\}$, it is also harmonic on $\mathbb{C} \setminus \overline{\Omega_k}$ since there it is constant. In particular it is subharmonic on those sets. We are only left to show that it is subharmonic on $\partial\Omega_k$.

For any $z_0 \in \partial\Omega_k$ and $\epsilon > 0$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} G_k(z_0 + \epsilon e^{i\theta})d\theta \geq 0 = G_k(z_0).$$

Therefore, by the sub-mean inequality theorem, G_k is subharmonic in z_0.

The proof for G_0 is similar. □

Now to the actual construction. For all $k \in \{0, 1, 2, ..., m\}$ we choose some $r_k \in]1, R_k[$ and define $\mathbf{r} = (r_0, r_1, ..., r_m)$. Let $\beta_k \in C^1(\mathbb{R})$ be functions which satisfy the following

- $0 \leq \beta_k(x) \leq 1$ for all $x \in \mathbb{R}$.
- $\beta_k(x) = 0$ if $x \geq \log(R_k)$.
- $\beta_k(x) = 1$ if $x \leq \log(r_k)$.
- $$\|\beta_k'\|_{\mathbb{R}} \leq \frac{3}{2(\log(R_k) - \log(r_k))} = \frac{3}{2\log(R_k/r_k)}.$$

We set $\chi_k(z) := \beta_k(G_k(z))$. Then we have:

- $0 \leq \chi_k \leq 1$,
- $\chi_k(z) = 1$ if $G_k(z) \leq \log(r_k)$,
- $\chi_k(z) = 0$ if $G_k(z) \geq \log(R_k)$.

For any z such that $\log(r_k) \leq G_k(z) \leq \log(R_k)$ we then get

$$|\frac{\partial}{\partial \bar{z}}\chi_k(z)| = |\beta_k'(G(z))||\frac{\partial}{\partial \bar{z}}G_k(z)| \leq \frac{3M_k}{2\log(R_k/r_k)}.$$

Where we define M_k by

$$M_k := \sup_{\log(r_k) \leq G_k(z) \leq \log(R_k)} |\frac{\partial}{\partial \bar{z}}G_k(z)|.$$

Finally we define the global cutoff function with

$$\chi(z) := \begin{cases} 1 & \text{if } z \in K, \\ \chi_k(z) & \text{if } z \in \Omega_k. \end{cases}$$

Take particular notice that $\chi \in C^1(\mathbb{C})$. This cutoff function is designed such that the product $f\chi$ is defined and differentiable on all $\mathbb{C}$, holomorphic in a neighbourhood of K and has a compact support. Now define

$$G := \sum_{k=0}^{m} 2n_k G_k.$$

Since each of the G_k is subharmonic on $\mathbb{C} \setminus \{a_k\}$, we see that G is subharmonic on $\mathbb{C} \setminus \{a_1, a_2, ..., a_m\}$. Therefore we can use it as a weight function in Theorem 3.1.8.

By Hörmander's theorem we can find a function $u \in L^2_{\text{loc}}(\mathbb{C} \setminus \{a_1, ..., a_m\})$ such that

$$\frac{\partial}{\partial \bar{z}} u = \frac{\partial}{\partial \bar{z}} (f\chi) = f \left(\frac{\partial}{\partial \bar{z}} \chi \right) \tag{1}$$

in the sense of distributions and

$$\int_{\mathbb{C}\setminus\{a_1,a_2,...,a_m\}} |u(z)|^2 (1+|z|^2)^{-2} e^{-G(z)} d\lambda(z) \leq \int_{\mathbb{C}\setminus\{a_1,a_2,...,a_m\}} |f(z) \frac{\partial}{\partial \bar{z}} \chi(z)|^2 e^{-G(z)} d\lambda(z).$$

By Theorem 2.6.4 we see that $u \in C^\infty$ and (1) holds in the classical sense. Since the set $\{a_1, a_2, ..., a_m\}$ has measure zero, we can consider $\mathbb{C}$ to be the domain of integration, instead of $\mathbb{C} \setminus \{a_1, a_2, ..., a_m\}$ in the following calculations. We now define our approximation function with

$$F := f\chi - u.$$

Then F is holomorphic on $\mathbb{C} \setminus \{a_1, a_2, ..., a_m\}$ since by the definition of u

$$\frac{\partial}{\partial \bar{z}} F = \frac{\partial}{\partial \bar{z}} (f\chi - u) = \frac{\partial}{\partial \bar{z}} (f\chi) - \frac{\partial}{\partial \bar{z}} (f\chi) = 0.$$

Next we are going to show that F is in fact a member of the set $R(\mathbf{a}, \mathbf{n})$, and after that we show that F approximates f properly on K.

4.2.2 Showing that $F \in R(\mathbf{a}, \mathbf{n})$

Notice that since the function $f\chi$ has compact support $\frac{\partial}{\partial \bar{z}}(f\chi)$ does as well. Therefore the function u is holomorphic near ∞, and we can use the mean value theorem for holomorphic functions. Now we estimate the growth rate of F at ∞ and each of the points of $\{a_1, a_2, ...a_m\}$. We start at ∞. For a large enough $z \in \mathbb{C}$ we get

$$|F(z)| = |u(z)| \leq \frac{1}{\pi} \int_{\Delta(z,1)} |u(\zeta)| d\lambda(\zeta).$$

Using the inequality of Cauchy-Schwarz and estimating one of the integrals by integrating over $\mathbb{C}$ rather than $\Delta(z,1)$ we get

$$|F(z)| \leq \frac{1}{\pi}\left(\int_{\Delta(z,1)}(1+|\zeta|^2)^2 e^{G(\zeta)}d\lambda(\zeta)\right)^{\frac{1}{2}}\left(\int_{\mathbb{C}}|u(\zeta)|^2(1+|\zeta|^2)^{-2}e^{-G(\zeta)}d\lambda(\zeta)\right)^{\frac{1}{2}}$$

Notice that the second integral in the product is independent of the variable z. In the next subsection we see that it is actually finite. We can therefore estimate it by some constant C_1. Since z is large $G(z)$ is equal to $2n_0G_0(z) = 2n_0g_{\Omega_0}(z,\infty)$. We get the following

$$|F(z)| \leq \frac{C_1}{\pi}\left(\int_{\Delta(z,1)}(1+|\zeta|^2)^2 e^{2n_0 g_{\Omega_0}(\zeta,\infty)}d\lambda(\zeta)\right)^{\frac{1}{2}} \leq \frac{C_1}{\pi}\left(\pi \sup_{\zeta\in\Delta(z,1)}(1+|\zeta|^2)^2 e^{2n_0 g_{\Omega_0}(\zeta,\infty)}\right)^{\frac{1}{2}}.$$

By definition of the Green's function we have

$$g_{\Omega_0}(z,\infty) = \log|z| + O(1) \qquad \text{as} \qquad z\to\infty.$$

In other words, there is a constant C_2 such that for large enough z we get

$$|g_{\Omega_0}(z,\infty)| \leq \log|z| + C_2.$$

Therefore we have

$$|F(z)| \leq \frac{C_1}{\pi^{1/2}}\left((1+|z+1|^2)^2 e^{2n_0(\log|z+1|+C_2)}\right)^{\frac{1}{2}} = \frac{C_1}{\pi^{1/2}}e^{C_2 n_0}(1+|z+1|^2)|z+1|^{n_0}.$$

This last function has growth rate equal to a polynomial of degree n_0+2. In other words, we have shown that there is a constant C such that

$$|F(z)| \leq C|z|^{n_0+2},$$

for z large enough. That is $F(z) = O(z^{n_0+2})$ as $z\to\infty$.

The estimation for the growth rate at each of the points in $\{a_1, a_2, ..., a_m\}$ is similar. For z close enough to a_k, $F = u$ on $\Delta(z, |a_k - z|/2)$, also u is holomorphic there. Let $\epsilon = \frac{|a_k - z|}{2}$ and get

$$\begin{aligned}
|F(z)| = |u(z)| &\overset{(i)}{\leq} \frac{4|a_k - z|^{-2}}{\pi}\int_{\Delta(z,\epsilon)}|u(\zeta)|d\lambda(\zeta)\\
&\overset{(ii)}{\leq} C_1|a_k - z|^{-2}\left(\int_{\Delta(z,\epsilon)}(1+|\zeta|^2)^2 e^{G(\zeta)}d\lambda(\zeta)\right)^{\frac{1}{2}}\\
&\overset{(iii)}{\leq} C_1|a_k - z|^{-2}\left(\frac{|a_k - z|^2}{4}\sup_{\zeta\in B(z,\epsilon)}(1+|\zeta|^2)^2 e^{-2n_k G_k(\zeta)}\right)^{\frac{1}{2}}\\
&\overset{(iv)}{\leq} C_3|a_k - z|^{-1}(e^{-2n_k(\log(\frac{|a_k - z|}{2})+C_2)})^{\frac{1}{2}}\\
&\leq C|a_k - z|^{-n_k - 1}.
\end{aligned}$$

In (i) we used the mean value theorem, in (ii) we used Cauchy-Schwarz and estimated one integral by a constant as before, in (iii) we estimated the integral by it's supremum and in (iv) we used the definition of the Green's function. This implies

$$F(z) = O(|z - a_k|^{-n_i-1}) \qquad \text{as} \qquad z \to a_k.$$

In other words, F has a pole of degree at most $n_k + 1$ at a_k. Therefore there is function P holomorphic in $\mathbb{C}$ such that

$$F(z) = P(z)(z - a_1)^{-(n_1+1)}(z - a_2)^{-(n_2+1)} \cdots (z - a_m)^{-(n_k+1)}.$$

Now, since the growth rate of F at infinity is $O(z^{n_0+2})$ the growth rate of P at infinity is $O(z^{m+2+\sum n_k})$. Therefore by Liouville's theorem, P is a polynomial of degree, at most $m + 2 + \sum n_k$ and so F is a member of the set $R(\mathbf{a}, \mathbf{n})$.

The only thing left to show now is that F approximates f properly on K.

4.2.3 Estimation of $\|F - f\|_K$

Let $r \in \mathbb{R}$ be such that $1 < r < \min\{r_0, r_1, ..., r_m\}$ and $n = \max\{n_0, n_1, ..., n_m\}$. Let s be a constant such that $G_k(z) < \log(r)$ for all $z \in K + \Delta(0, s)$ and all $k \in \{0, 1, .., m\}$. Then for any $z \in K$, u is holomorphic on $\Delta(z, s)$, because there we have $\chi = 1$ and so we get $\partial u/\partial \bar{z} = \partial(f\chi)/\partial \bar{z} = \partial f/\partial \bar{z} = 0$. Therefore we can use the mean value theorem on u and so we have

$$|F(z) - f(z)| = |u(z)| \le \frac{1}{s^2\pi}\int_{\Delta(z,s)} |u(\zeta)| d\lambda.$$

Using Cauchy-Schwarz, and then integrating over $\mathbb{C}$ rather than $\Delta(z, s)$ in the second integral of the product we have

$$|F(z)-f(z)| \le \frac{1}{s^2\pi}\Big(\int_{\Delta(z,s)} (1+|\zeta|^2)^2 e^{G(\zeta)} d\lambda(\zeta)\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{C}} |u(\zeta)|^2(1+|\zeta|^2)^{-2} e^{-G(\zeta)} d\lambda(\zeta)\Big)^{\frac{1}{2}}.$$

Now, u was constructed by Hörmander's theorem so that the second integral of this product could be estimated. The first integral of the product is estimated by the supremum of the integrand,

$$|F(z)-f(z)| \le \frac{1}{s^2\pi}\left(s^2\pi \sup_{\zeta\in\Delta(z,s)} (1 + |\zeta|^2)^2 e^{G(\zeta)}\right)^{\frac{1}{2}} \left(\int_{\mathbb{C}} |f(\zeta)\frac{\partial\chi}{\partial\bar{\zeta}}(\zeta)|^2 e^{-G(\zeta)} d\lambda(\zeta)\right)^{\frac{1}{2}}.$$

By the choice of s and the definition of G, the supremum of e^G in the first factor can be estimated by r^{2n}. The integral in the second factor can be taken over the support of $\partial\chi/\partial\bar{z}$ rather than $\mathbb{C}$.

$$|F(z) - f(z)| \le \frac{r^n}{s\pi^{\frac{1}{2}}} \sup_{\zeta\in K+\Delta(0,s)} (1 + |\zeta|^2) \left(\int_{\operatorname{supp}(\partial\chi/\partial\bar{\zeta})} |f(\zeta)\frac{\partial\chi}{\partial\bar{\zeta}}(\zeta)|^2 e^{-G(\zeta)} d\lambda(\zeta)\right)^{\frac{1}{2}}.$$

Now, by the definition of χ we see that the sets $\mathrm{supp}(\partial\chi_k/\partial\bar{z})$ form a partion of $\mathrm{supp}(\partial\chi/\partial\bar{z})$. Therefore we can write the integral as a sum of smaller integrals

$$|F(z)-f(z)| \leq \frac{r^n}{s\pi^{\frac{1}{2}}} \sup_{\zeta\in K+\Delta(0,s)} (1+|\zeta|^2) \left(\sum_{k=0}^{m} \int_{\mathrm{supp}(\partial\chi_k/\partial\bar{\zeta})} |f(\zeta)\frac{\partial\chi_k}{\partial\bar{\zeta}}(\zeta)|^2 e^{-2n_kG_k(\zeta)} d\lambda(\zeta) \right)^{\frac{1}{2}}.$$

We have also changed G to $2n_kG_k$ because $\mathrm{supp}(\partial\chi_k/\partial\bar{z}) \subset \Omega_k$ and $G = 2n_kG_k$ on Ω_k. By defintion of χ_k we have that $G_k(z) \leq \log(r_k)$ on $\mathrm{supp}(\partial\chi_k/\partial\bar{z})$, hence we can estimate $e^{-2n_kG_k}$ in the integral by $r_k^{-2n_k}$. Above we have already estimated that

$$\left\| \frac{\partial\chi_k}{\partial\bar{z}} \right\|_{\mathbb{C}} \leq \frac{3M_k^2}{2(\log^2(R_k/r_k))}.$$

Where M_k was defined by

$$M_k = \sup_{\log(r_k)<G_k(z)\leq\log(R_k)} |\frac{\partial}{\partial\bar{z}} G_k(z)|.$$

so we get

$$|F(z)-f(z)| \leq \frac{r^n}{s\pi^{\frac{1}{2}}} \sup_{\zeta\in K+\Delta(0,s)} (1+|\zeta|^2) \left(\sum_{k=0}^{m} \left(\frac{3M_k}{2(\log(R_k/r_k))} \right)^2 r_k^{-2n_k} \int_{K_{\mathbf{R}}} |f(\zeta)|^2 d\lambda(\zeta) \right)^{\frac{1}{2}}.$$

Here we have also estimated the integral of $|f|^2$ over $\mathrm{supp}(\partial\chi/\partial\bar{z})$ by integrating over a larger set $K_{\mathbf{R}}$. Simplifying we get

$$|F(z)-f(z)| = \frac{\|f\|_{L_2,K_{\mathbf{R}}}}{s\pi^{\frac{1}{2}}} \sup_{\zeta\in K+\Delta(0,s)} (1+|\zeta|^2) \left(\sum_{k=0}^{m} \left(\frac{3M_k}{2(\log(R_k/r_k))} \right)^2 \frac{r^{2n}}{r_k^{2n_k}} \right)^{\frac{1}{2}}.$$

Now, remembering that the real numbers r_k were chosen arbitrarily from the set $]1, R_k[$, we could choose them to minimize this expression. By differentiating

$$\frac{r_k^{-n_k}}{\log(R_k/r_k)}$$

with respect to r_k we find that the minimal value is at $r_k = R_ke^{-n_k}$. Putting those values in the above expression gives us the estimate

$$\|F-f\|_K \leq \frac{3e\|f\|_{L_2,K_{\mathbf{R}}}}{2s\pi^{\frac{1}{2}}} \sup_{\zeta\in K+\Delta(0,s)} (1+|\zeta|^2) \left(\sum_{k=0}^{m} (M_kn_k)^2 \frac{r^{2n}}{R_k^{2n_k}} \right)^{\frac{1}{2}}.$$

But since we have already shown that $F \in R(\mathbf{a}, \mathbf{n})$ we see that $Rd(f, K, \mathbf{a}, \mathbf{n})$ can be estimated by the same expression.

At last it can be easily seen that $Rd(f, K, \mathbf{a}, \mathbf{n})$ is invariant under shifting in the plane. That is to say, if for any $w \in \mathbb{C}$ we set $K_w = K + w = \{z + w; z \in K\}$, $f_w(z) = f(z - w)$ and $\mathbf{a_w} = (a_1 + w, a_2 + w, ..., a_m + w)$ then

$$Rd(f, K, \mathbf{a}, \mathbf{n}) = Rd(f_w, K_w, \mathbf{a}_w, \mathbf{n}).$$

Thus we can estimate the expression $\sup(1 + |\zeta|^2)$ by the diameter of K, getting the following result

$$Rd(f, K, \mathbf{a}, \mathbf{n}) \leq \frac{3e\|f\|_{L_2, K_{\mathbf{R}}}}{2s\pi^{\frac{1}{2}}}(1 + (\mathrm{diam}(K) + s)^2)\left(\sum_{k=0}^{m}(M_k n_k)^2 \frac{r^{2n}}{R_k^{2n_k}}\right)^{\frac{1}{2}}.$$

4.3 Discussion on the estimate

The estimate on $R(f, K, \mathbf{a}, \mathbf{n})$ given above includes a lot of constants which are very hard to calculate explicitly, but the expression still gives us some information about the approximation of f by rational functions. The first thing to check would be whether this estimate guaranties that f can be approximated arbitrary well by rational functions. It does indeed! Remember that $n = \max\{n_0, n_1, ..., n_m\}$, and that by definition $r < R_k$ for all k. Therefore by choosing $n = n_0 = n_1 = ... = n_m$ we see that

$$\sum_{k=0}^{m}\left((M_k n)^2 \frac{r^{2n}}{R_k^{2n}}\right) \longrightarrow 0 \qquad \text{as} \qquad n \to \infty,$$

because each of the terms of the sum tends to zero. Thus we can make $Rd(f, K, \mathbf{a}, \mathbf{n})$ arbitrarily small by letting $\mathbf{n}$ be large enough. At first we explore what happens if we let $\mathbf{n}$ tend to infinity in the most simple way.

Theorem 4.3.1. *Let $K, m, f, \mathbf{a}$ be as before. For any $n \in \mathbb{N}$ we define $\mathbf{n}^* := (n, n, n, ..., n) \in \mathbb{N}^{m+1}$. If $\mathbf{R} = (R_0, R_1, ..., R_m) \in \mathbb{R}^{m+1}$ is a vector such that $f \in \mathcal{O}(\mathrm{int}(K_{\mathbf{R}}))$, then*

$$\limsup_{n\to\infty}(Rd(f, K, \mathbf{a}, \mathbf{n}^*))^{1/n} \leq (\min\{R_0, R_1, ..., R_m\})^{-1}.$$

We skip the proof of this theorem momenteraly since it can be considered as a corollary to the next one. By letting the vector $\mathbf{n}$ tend to infinity in a more complicated manner we get different results. For any $x \in \mathbb{R}$ we denote the smallest integer larger than or equal to x by $\lceil x \rceil$.

Theorem 4.3.2. *Let $K, m, f, \mathbf{a}$ be as before and let $\mathbf{l} = (l_0, l_1, ..., l_m) \in \mathbb{R}_+^{m+1}$ be an arbitrary vector. For any $n \in \mathbb{N}$ define $\mathbf{n}_\mathbf{l}^* = (\lceil l_0 n \rceil, \lceil l_1 n \rceil, ..., \lceil l_m, n \rceil)$. Let $\mathbf{R} = (R_0, R_1, ..., R_m) \in \mathbb{R}^{m+1}$ be a vector such that $f \in A(K_{\mathbf{R}})$. Then*

$$\limsup_{n\to\infty}(Rd(f, K, \mathbf{a}, \mathbf{n}_\mathbf{l}^*))^{\frac{1}{n}} \leq (\min\{R_0^{l_0}, R_1^{l_1}, ..., R_m^{l_m}\})^{-1}.$$

Proof. Let $l = \max\{l_0, l_1, ..., l_m\}$. From the calculations above we have seen that there are constants A, M_k and $r < \min\{R_0, R_1, ..., R_m\}$ such that

$$Rd(f, K, \mathbf{a}, \mathbf{n}_\mathbf{l}^*) \leq A \left(\sum_{k=0}^{m} (M_k \lceil nl_k \rceil)^2 \frac{r^{2\lceil nl \rceil}}{R_k^{2\lceil nl_k \rceil}} \right)^{\frac{1}{2}}.$$

Therefore we see that

$$\begin{aligned}\limsup_{n\to\infty} (Rd(f, K, \mathbf{a}, \mathbf{n}_\mathbf{l}^*))^{\frac{1}{n}} &\leq \limsup_{n\to\infty} A^{\frac{1}{n}} \left(\sum_{k=0}^{m} (M_k n l_k)^2 \left(\frac{r^l}{R_k^{l_k}} \right)^{2n} \right)^{\frac{1}{2n}} \\ &= \max_{k\in\{0,1,...,m\}} \left(\frac{r^l}{R_k^{l_k}} \right) \\ &= r^l (\min\{R_0^{l_0}, R_1^{l_1}, ..., R_m^{l_m}\})^{-1}.\end{aligned}$$

Now, we see in the calculations above that r could have been chosen arbitrarily close to 1. Thus

$$\limsup_{n\to\infty} (Rd(f, K, \mathbf{a}, \mathbf{n}_\mathbf{l}^*))^{\frac{1}{n}} \leq (\min\{R_0^{l_0}, R_1^{l_1}, ..., R_m^{l_m}\})^{-1}.$$

□

We can generalize the last theorem a little bit more:

Theorem 4.3.3. *Theorem 4.3.2 holds true for any vector* $\mathbf{R} = (R_0, R_1, ..., R_m) \in \mathbb{R}^{m+1}$ *such that* $f \in \mathcal{O}(\text{int}(K_\mathbf{R}))$

Proof. If $f \in \mathcal{O}(\text{int}(K_\mathbf{R}))$ then $f \in A(K_\mathbf{r})$ for any $\mathbf{r} = (r_0, r_1, ..., r_m)$ such that $r_k < R_k$ for all K. Then

$$\limsup_{n\to\infty} (Rd(f, K, \mathbf{a}, \mathbf{n}_\mathbf{l}^*))^{\frac{1}{n}} \leq (\min\{r_0^{l_0}, r_1^{l_1}, ..., r_m^{l_m}\})^{-1}.$$

Letting $\mathbf{r}$ tend to $\mathbf{R}$ in an obvious manner, we get the result. □

Notice that by setting $\mathbf{l} = (1, 1, ..., 1)$ in the above theorems we get Theorem 4.3.1. We can also prove a sort of converse to the preceding theorem.

Theorem 4.3.4. *Let* K, m *and* $\mathbf{a}$ *be as before and* $f \in C(K)$. *Let* $\mathbf{R} = (R_0, R_1, ..., R_m) \in]1, \infty[^{m+1}$ *be a vector such that*

$$\limsup_{n\to\infty} (Rd(f, K, \mathbf{a}, \mathbf{n}_\mathbf{l}^*))^{\frac{1}{n}} \leq (\min\{R_0^{l_0}, R_1^{l_1}, ..., R_m^{l_m}\})^{-1}$$

for every $\mathbf{l} = (l_0, l_1, ..., l_m) \in \mathbb{R}_+^{m+1}$. *Then* f *is the restriction to* K *of a holomorphic function on* $\text{int}(K_\mathbf{R})$.

Proof. Note that $\mathrm{int}(K_{\mathbf{R}}) = \{z \in \mathbb{C};\ G_k(z) < \log(R_k),\ k = 0, 1, ..., m\}$. For an arbitrary $r = p/q \in \cup_{\mathbf{n}} R(\mathbf{a}, \mathbf{n})$ we define

$$V_{K,r}(z) := \log \frac{|r(z)|}{\|r\|_K} - (n_0 - \sum_{k=1}^{m} n_k) G_0(z) - \sum_{k=1}^{m} n_k G_k(z),$$

where $n_0 = \deg(p)$ and n_k is the degree of a_k as a root of q. Notice that $V_{K,r}$ is subharmonic and bounded on $\mathbb{C} \setminus \{a_1, a_2, ..., a_m\}$. Since $V_{K,r} \leq 0$ on ∂K, by the maximum principle, $V_{K,r} \leq 0$ on $\mathbb{C} \setminus K$. Obviously we have $V_{K,r} \leq 0$ on K, thus we have seen that

$$(n_0 - \sum_{k=1}^{m} n_k) G_0(z) + \sum_{k=1}^{m} n_k G_k(z) \geq \log \frac{|r(z)|}{\|r\|_K}, \qquad z \in \mathbb{C}.$$

Taking the exponential on both sides gives

$$|r(z)| \leq \|r\|_K \exp((n_0 - \sum_{k=1}^{m} n_k) G_0(z) + \sum_{k=1}^{m} n_k G_k(z)), \qquad z \in \mathbb{C}.$$

Let Ω_k be the k-th hole of K as before, and Ω_0 the unbounded component of $\mathbb{C} \setminus K$. It is sufficient to show that f is the restriction to K of a function holomorphic on $(\Omega_k \cap \mathrm{int}(K_{\mathbf{R}})) \cup U$, for some neighbourhood U of K, for every k. Let $k \geq 1$ be fixed and $\mathbf{l} = (l_1, l_2, ..., l_n)$ be such that $\min\{R_0^{l_0}, R_1^{l_1}, ..., R_m^{l_m}\} = R_k^{l_k}$. By assumption we have

$$\limsup_{n \to \infty} (Rd(f, K, \mathbf{a}, \mathbf{n}_{\mathbf{l}}^*))^{\frac{1}{n}} \leq \frac{1}{R_k^{l_k}}$$

so there is a constant M such that

$$Rd(f, K, \mathbf{a}, \mathbf{n}_{\mathbf{l}}^*) < \frac{M}{R_k^{nl_k}}, \qquad n \in \mathbb{N}.$$

Now, pick $r_n \in R(\mathbf{a}, \mathbf{n}_{\mathbf{l}}^*)$ such that $\|r_n - f\|_K < M/R^{nl_k}$ for all n. We claim that the series $r_0 + \sum_{k=1}^{\infty}(r_k - r_{k-1})$ converges uniformly on compact subsets of $(\Omega_k \cap \mathrm{int}(K_{\mathbf{R}})) \cup U$, where U is some neighbourhood of K, to a holomorphic function F which agrees with f on K. By the definition of the sequence $(r_n)_{n \in \mathbb{N}}$, this series converges on some neighbourhood U of K. Let $1 < \rho < R_k$ and define $B_\rho := \Omega_k \cap \{z \in \mathbb{C};\ G_k(z) \leq \rho\} \subset \Omega_k \cap \mathrm{int}(K_{\mathbf{R}})$. Note that if $j \neq k$ then $G_j = 0$ on B_ρ. Therefore

$$\begin{aligned}
&\sup_{z \in B_\rho} |r_n(z) - r_{n-1}(z)| \\
&\leq \sup_{z \in B_\rho} \left(\|r_n - r_{n-1}\|_K \exp((\lceil nl_0 \rceil + 2 + m) G_0(z) + \sum_{k=1}^{m} \lceil nl_k \rceil G_k(z)) \right) \\
&\leq \sup_{z \in B_\rho} ((\|r_n - f\|_K + \|f - r_{n-1}\|_K) \exp(\lceil nl_k \rceil G_k(z))) \\
&\leq \frac{2M}{R^{(n-1)l_k}} \rho^{\lceil nl_k \rceil} \leq 2MR\rho \left(\frac{\rho^{l_k}}{R^{l_k}} \right)^n .
\end{aligned}$$

So the series converges.

The case when $k = 0$ is very similar.

□

The preceding theorems are very similar to the Bernstein-Walsh theorem, which can actually be considered as a corollary to Theorems 4.3.1 and 4.3.4.

Definition 4.3.5. *Let K be a compact subset of the complex plane and f be a function on K. Then we define*

$$d_n(f, K) := \inf\{\|f - p\|_K : \ p \text{ is a polynomial}\}.$$

By noticing that $d_n(f, K) = Rd(f, K, \emptyset, (n))$, the following corollary is just a special case of the above theorem.

Corollary 4.3.6 (Bernstein-Walsh)**.** *Let K be a compact subset of $\mathbb{C}$ which is regular and has no holes. Let g_K be the Green's function for $\mathbb{C} \setminus K$ with the pole at infinity and let $K_R = \{z \in \mathbb{C};\ g_K(z) < \log(R)\}$. Let $f \in C(K)$. Then*

$$\limsup_{n\to\infty} (d_n(f, K))^{1/n} \leq \frac{1}{R}$$

if and only if f is the restriction to K of a holomorphic function on K_R.

For a given function f holomorphic on a neighbourhood U of a compact set K and some $\epsilon > 0$, it can sometimes be useful to know some upper limit to the degree of poles of a rational function r necessary so that $\|r - f\|_K < \epsilon$. We shall state the problem more explicitly.

Let $K, m, \mathbf{a}, f, \mathbf{R}, \mathbf{l}$ and $\mathbf{n}_\mathbf{l}^*$ be as before. Given some $\epsilon > 0$ we want to find $N \in \mathbb{N}$ such that

$$Rd(f, K, \mathbf{a}, \mathbf{n}_\mathbf{l}^*) < \epsilon, \qquad n \geq N.$$

We have already seen that there are constants A, M_k and $r \in]1, \min\{R_0, ..., R_m\}[$ such that

$$Rd(f, K, \mathbf{a}, \mathbf{n}_\mathbf{l}^*) \leq A \left(\sum_{k=0}^{m} (M_k \lceil nl_k \rceil)^2 \frac{r^{2\lceil nl \rceil}}{R_k^{2\lceil nl_k \rceil}} \right)^{\frac{1}{2}}.$$

We put this expression less than ϵ and solve for a lower bound on n,

$$\sum_{k=0}^{m} (M_k \lceil nl_k \rceil)^2 \frac{r^{2\lceil nl \rceil}}{R_k^{2\lceil nl_k \rceil}} < \left(\frac{\epsilon}{A}\right)^2.$$

We demand every term in this sum to be small enough,

$$(M_k \lceil nl_k \rceil)^2 \frac{r^{2\lceil nl \rceil}}{R_k^{2\lceil nl_k \rceil}} < \left(\frac{\epsilon}{A}\right)^2 (m+1)^{-1}, \qquad k \in \{0, 1, ..., m\}.$$

The expression on the left is decreasing with n so we can estimate $\lceil nl \rceil$ with nl.

$$n\left(\frac{r^l}{R_k^{l_k}}\right)^n < \frac{\epsilon}{A(m+1)^{1/2}M_k l_k}, \qquad k \in \{0,1,...,m\}.$$

Remembering that r could have been chosen arbitarily close to 1, we can assume that $(r^l/R_k^{l_k}) < 1$. By taking both sides to the power -1, and then taking the logarithm yields

$$n + B_k \log\left(\frac{1}{n}\right) > B_k \log\left(\frac{A(m+1)^{1/2}M_k l_k}{\epsilon}\right), \qquad k \in \{0,1,...,m\}.$$

For some constants B_k dependent on $r^l/R_k^{l_k}$. It is clear that as n grows, the term $B_k \log(1/n)$ will be neglectable compared to n. Thus by changing the constant B_k to some other constant B_k', it suffices to choose

$$n > B_k' \log\left(\frac{A(m+1)^{1/2}M_k l_k}{\epsilon}\right), \qquad k \in \{0,1,...,m\}.$$

By taking $B = \max\{B_0',...,B_m'\}$ and $A' = \max\{A(m+1)^{1/2}M_k l_k;\ k = 0,1,...,m\}$ we get the following theorem.

Theorem 4.3.7. *Let $K, m, \mathbf{a}, f, \mathbf{l}$ and $\mathbf{n}_\mathbf{l}^*$ be as before. There are constants A and B such that if $n > B\log(A/\epsilon)$ then*

$$Rd(f, K, \mathbf{a}, \mathbf{n}_\mathbf{l}^*) < \epsilon.$$

5 Vitushkin's Theorem

In this chapter we prove Vitushkin's theorem, which characterizes the compact sets K for which $A(K) = R(K)$. Before we can even state this theorem we must define some new concepts, the ones of analytic capacity and continuous analytic capacity. We discuss some properties of those concepts and introduce a third, the analytic diameter, before moving on to the proof of the main theorem. (The proof given here follows the steps of [3].)

5.1 Introduction

For an arbitrary subset E of the complex plane, we say that a function f is holomorphic off E if there is a compact set $K \subseteq E$ such that f is holomorphic on $\mathbb{C} \setminus K$.

Let K be a compact set in the complex plane and f be a function defined on some subset of the Riemann sphere $\widehat{\mathbb{C}}$. We say that the function f is *admissible for* K if f is holomorphic on $\widehat{\mathbb{C}} \setminus K$ and if f satisfies

$$\|f\|_{\widehat{\mathbb{C}}\setminus K} \leq 1 \qquad \text{and} \qquad f(\infty) = 0.$$

For an arbitrary bounded subset E of the complex plane we say that the function f is *admissible for* E if it is admissible for some compact subset of E. The set of functions which are admissible for E is denoted by $\mathscr{A}(E)$. The set of functions which are admissible for E and are defined and continuous on the whole Riemann-sphere is denoted by $\mathscr{C}\mathscr{A}(E)$. In other words

$$\mathscr{C}\mathscr{A}(E) = C(\widehat{\mathbb{C}}) \cap \mathscr{A}(E).$$

The *analytic capacity* of a set E is defined by

$$\gamma(E) := \sup\{|f'(\infty)| : \ f \in \mathscr{A}(E)\}$$

and the *continuous analytic capacity* of a set E is defined by

$$\alpha(E) := \sup\{|f'(\infty)| : \ f \in \mathscr{C}\mathscr{A}(E)\},$$

where the derivative of f at infinity is defined by

$$f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty)).$$

Observe that $f'(\infty)$ is equal to the derivative of the function $g(z) := f(1/z)$, $g(0) = f(\infty)$ at the origin.

With those concepts at hand, we can now state the main theorem of this chapter.

Theorem 5.1.1 (Vitushkin's theorem)**.** *Let K be a compact set. The following statemens are equivalent.*

(i) $R(K) = A(K)$

(ii) *For every bounded open set D, we have $\alpha(D \setminus K) = \alpha(D \setminus \mathrm{int}(K))$.*

(iii) *For every $z \in \mathbb{C}$, $r > 1$ and $\delta > 0$, we have $\alpha(\Delta(z,\delta) \setminus \mathrm{int}(K)) \leq \alpha(\Delta(z, r\delta) \setminus K)$.*

(iv) *There exist $r \geq 1$, $c > 0$ and $\delta_0 > 0$ such that for every $z \in \mathbb{C}$ and $\delta < \delta_0$, we have $\alpha(\Delta(z,\delta) \setminus \mathrm{int}(K)) \leq c\alpha(\Delta(z, r\delta) \setminus K)$.*

(v) *For every $z \in \partial K$, there exists $r \geq 1$ such that*

$$\limsup_{\delta \to 0} \frac{\alpha(\Delta(z,\delta) \setminus \mathrm{int}(K))}{\alpha(\Delta(z, r\delta) \setminus K)} < \infty.$$

Before we can prove the theorem we must explore some properties of analytic capacity and also introduce analytic diameter.

Any function f which is holomorphic at ∞ can be represented near ∞ as a Laurent series

$$f(z) = a_0 + \frac{a_1}{z - z_0} + \frac{a_2}{(z - z_0)^2} + \ldots = \sum_{k=0}^{\infty} a_k (z - z_0)^{-k},$$

where z_0 is any given complex number. It is easy to see that the first two coefficients, a_0 and a_1, are independent of the choice of z_0. In fact

$$a_0 = f(\infty) \qquad \text{and} \qquad a_1 = f'(\infty).$$

The third coefficient, a_2, depends on z_0 though and we set

$$\beta(f, z_0) := a_2.$$

It is easy to see that if $R > 0$ is large, then

$$\beta(f, z_0) = \frac{1}{2\pi i} \int_{|z|=R} f(z)(z - z_0) dz.$$

Since the analytic capacity of a set E was defined as the supremum of the second coefficient of the Laurent series of an admissible function on E, it is reasonable to do something similar for the third coefficient. For each bounded set E we define

$$\beta(E, z) = \begin{cases} \frac{\sup_{f \in \mathscr{A}(E)} |\beta(f,z)|}{\gamma(E)} & \text{if } \gamma(E) > 0, \\ 0 & \text{if } \gamma(E) = 0 \end{cases}$$

and we define the *analytic diameter* of E by

$$\beta(E) = \inf_{z \in \mathbb{C}} \beta(E, z).$$

Any point w_0 such that $\beta(E, w_0) = \beta(E)$ is called the *analytic centre of* E. In fact, each bounded set has an analytic centre. To show that we need only to show that $\beta(E, z)$ is a continuous function in the variable z and if $\beta(E, z)$ is not equal to the constant 0 then $\lim_{z \to \infty} \beta(E, z) = \infty$.

Lemma 5.1.2. *The function $z \to \beta(E, z)$ is continuous for every bounded set E.*

Proof. Simple calculations show that

$$\beta(f, z_1) = \beta(f, z_0) + (z_0 - z_1) f'(\infty). \tag{1}$$

From this it follows that for an admissible function f

$$|\beta(f, z_1)| \leq \beta(E, z_0)\gamma(E) + |z_1 - z_0|\gamma(E).$$

Taking the supremum over admissible functions we get

$$\beta(E, z_1) - \beta(E, z_0) \leq |z_0 - z_1|.$$

Since z_0 and z_1 are interchangeable

$$|\beta(E, z_1) - \beta(E, z_0)| \leq |z_0 - z_1|.$$

In particular $\beta(E, z)$ is a Lipschitz-continuous function of z. □

Lemma 5.1.3. *If $\gamma(E) > 0$ then* $\lim_{z \to \infty} \beta(E, z) = +\infty$.

Proof. Suppose $\gamma(E) > 0$. Let f be an admissible function for E such that $f'(\infty) > 0$. From the equation (1), we obtain

$$\gamma(E)\beta(E, z_1) \geq |\beta(f, z_1)| \geq |z_0 - z_1||f'(\infty)| - |\beta(f, z_0)|.$$

By fixing z_0 and letting z_1 tend to infinity we see that

$$\lim_{z \to \infty} \beta(E, z) = +\infty.$$

□

5.2 Theorems and estimates of analytic capacity

Some important theorems concerning analytic capacity and diameter follow almost instantly from definitions:

Theorem 5.2.1.

(i) For every set E we have $\alpha(E) \leq \gamma(E) \leq \beta(E)$.

(ii) If U is an open set then $\gamma(U) = \alpha(U)$.

(iii) If $E \subset F$, then $\gamma(E) \leq \gamma(F)$ and $\alpha(E) \leq \alpha(F)$.

(iv) If a and z_0 are complex, then $\gamma(z_0 + aE) = |a|\gamma(E)$ and $\alpha(z_0 + aE) = |a|\alpha(E)$.

(v) The analytic capacity of a compact set K depends only on the boundary of $\widehat{K}$. That is $\gamma(K) = \gamma(\partial K) = \gamma(\widehat{K}) = \gamma(\partial \widehat{K})$.

Proof. **(i)** Since $\mathscr{C}\mathscr{A}(E) \subset \mathscr{A}(E)$, we have $\alpha(E) \leq \gamma(E)$.

For $\epsilon > 0$, choose an admissible function f for E such that $f(\infty) > \gamma(E) - \epsilon$. Then $\beta(f^2, z) > (\gamma(E) - \epsilon)^2$ for all z. Consequently

$$\beta(E, z) = \sup_{f \in \mathscr{A}(E)} |\beta(f, z)|/\gamma(E) \geq \gamma(E)$$

for all z, and $\beta(E) \geq \gamma(E)$.

(ii) From (i) we have $\alpha(U) \leq \gamma(U)$.

If f is an admissible function for U there is a compact set $K \subset U$ such that f is admissible for K. Choose some compact $J \subset U$ such that $K \subset \mathrm{int}(J)$. Now, by Tietze's extension theorem, we can find some function $f^* \in \mathscr{C}\mathscr{A}(U)$ such that $f^* = f$ on $\widehat{\mathbb{C}} \setminus J$. In particular those functions agree at infinity so $\alpha(U) \geq \gamma(U)$. Hence $\alpha(U) = \gamma(U)$.

(iii) If $E \subset F$ then $\mathscr{A}(E) \subset \mathscr{A}(F)$ and $\mathscr{C}\mathscr{A}(E) \subset \mathscr{C}\mathscr{A}(F)$.

(iv) If f is an admissible function for E, then $g(z) := f(z_0 + az)$ is an admissible function for $z_0 + aE$.

(v) If f is an admissible function for $\widehat{K}$, then the function g defined with $g(z) = f(z)$ on $\widehat{\mathbb{C}} \setminus \widehat{K}$ and $g(z) = 0$ on $\widehat{K}$ is admissible for any of the other sets. $\square$

Now let K be a compact connected subset of the plane. It turns out that the problem of calculating the analytic capacity of such a set can be reduced to finding the Riemann-mapping from $\widehat{\mathbb{C}} \setminus \widehat{K}$ to the open unit disc.

Theorem 5.2.2. *Let K be a compact connected subset of the complex plane which contains more than one point. Let g be the conformal map of $\widehat{\mathbb{C}} \setminus \widehat{K}$ onto the interior of the unit disc, satisfying $g(\infty) = 0$ and $g'(\infty) \in \mathbb{R}$, $g'(\infty) > 0$. Then $\gamma(K) = g'(\infty)$.*

Proof. Since $\gamma(K) = \gamma(\widehat{K})$ we can assume that $K = \widehat{K}$. Since g is admissible for K, $g'(\infty) \leq \gamma(K)$. If h is also admissible, then $h \circ g^{-1}$ is a holomorphic function on $\Delta(0,1)$ satisfying $(h \circ g^{-1})(0) = 0$ and $|h \circ g^{-1}| \leq 1$. By Schwarz's lemma $|h \circ g^{-1}(w)| \leq |w|$, which gives $|h(z)| \leq |g(z)|$ for all $z \in \widehat{C} \setminus K$. Consequently $|h'(\infty)| = \lim_{z\to\infty} |zh(z)| \leq \lim_{z\to\infty} |zg(z)| = g'(\infty)$ and $\gamma(K) \leq g'(\infty)$. □

This theorem allows us to calculate the analytic capacity of any set for which this Riemann-mapping is known.

Theorem 5.2.3. *The analytic capacity of a disc is equal to its radius. The analytic capacity of a line segment of length L is $L/4$.*

Proof. The function
$$g(z) = \frac{r}{z - z_0}$$
is a conformal mapping of $\Delta(z_0, r)$ onto the unit disc which satisfies $g(\infty) = 0$. We obtain $\gamma(\Delta(z_0, r)) = g'(\infty) = r$.

For the second assertion, we consider the Joukowsky map
$$h(z) = \frac{L}{4}\left(z + \frac{1}{z}\right),$$
which maps the unit disc conformally onto $\widehat{\mathbb{C}} \setminus [-L/2, L/2]$, with $h(0) = \infty$. We obtain
$$\gamma([-L/2, L/2]) = (h^{-1})'(\infty) = \lim_{z\to\infty} zh^{-1}(z) = \lim_{z\to\infty} \frac{L}{4}\left(h^{-1}(z) + \frac{1}{h^{-1}(z)}\right) h^{-1}(z)$$
$$= \lim_{z\to\infty} \frac{L}{4}((h^{-1}(z))^2 + 1) = \frac{L}{4}.$$
□

Calculations of capacity of further sets are not necessary for the upcoming proof of Vitushkin's theorem. We include a list of examples though, for the interested reader (Table 5.1. of [10])[1].

[1] The table in the book by Ransford refers to logarithmic capacity but not analytic capacity. For a connected compact set this is the same thing. That can by seen by comparing Theorem 5.2.9 of this thesis with Theorem 5.2.1. of [10].

K	$\gamma(K)$
disc, radius r	r
line segment, length L	$\frac{L}{4}$
ellipse, semi axes a, b	$\frac{a+b}{2}$
equilateral triangle, side h	$\frac{3^{1/2}\Gamma^3(1/3)}{8\pi^2} \cdot h$
isosceles right triangle, short side h	$\frac{3^{3/4}\Gamma^2(1/4)}{2^{7/2}\pi^{3/2}} \cdot h$
square, side h	$\frac{\Gamma^2(1/4)}{4\pi^{3/2}} \cdot h$
regular n-gon, side h	$\frac{\Gamma(1/n)}{2^{1+2/n}\pi^{1/2}\Gamma(1/2+1/n)} \cdot h$
circular arc, radius r, angle α	$r\sin(\alpha/4)$
half-disc, radius r	$\frac{4}{3^{3/2}} \cdot r$

For a disconnected compact set K a conformal mapping from $\widehat{\mathbb{C}} \setminus K$ to the unit disc of course does not exist, but there is still a function for which the supremum is gained.

Theorem 5.2.4. *Let $K \subset \mathbb{C}$ be compact and holomorphically convex. Then there is a unique admissible function g for K satisfying $g'(\infty) = \gamma(K)$.*

Proof. The existence of such a function is always assured by the fact that $\mathscr{A}(K)$ is a normal family of holomorphic functions on $\widehat{\mathbb{C}} \setminus K$.

Suppose $g_0, g_1 \in \mathscr{A}(K)$ both have derivatives at ∞ equal to $\gamma(K)$. Then the function $g = (g_0 + g_1)/2 \in \mathscr{A}(K)$ also satisfies $g'(\infty) = \gamma(K)$. Let $h = (g - g_0)/2$. Then $g_1 = g + h$ and $g_0 = g - h$. It suffices to show that $h = 0$. Since by definition we have $|g_0|, |g_1| \leq 1$, we get

$$1 \geq |g \pm h|^2 = |g|^2 + |h|^2 \pm 2\mathrm{Re}(g\bar{h})$$

and obtain

$$|g|^2 + |h|^2 \leq 1.$$

Let $k = h^2/2$. Then

$$|k| \leq \frac{1-|g|^2}{2} = \frac{(1-|g|)(1+|g|)}{2} \leq \frac{(1-|g|)(1+1)}{2} = 1 - |g|,$$

so

$$|g| + |k| \leq 1.$$

If k is not zero, we can write its Laurent series near ∞ as

$$k(z) = \frac{a_n}{z^n} + \frac{a_{n+1}}{z^{n+1}} + ...$$

where $a_n \neq 0$. Since $h(\infty) = h'(\infty) = 0$, also $k(\infty) = k'(\infty) = 0$ and $n > 1$. Consequently for $\epsilon > 0$ small enough, $\epsilon |a_n||z|^{n-1} \leq 1$ in a neighbourhood of K. Let

$$f = g + \epsilon \bar{a}_n z^{n-1} k.$$

Then $f(\infty) = 0$. In a neighbourhood of K we get

$$|f| \leq |g| + \epsilon |a_n||z|^{n-1}|k| \leq |g| + |k| \leq 1.$$

So by the maximum principle we have $|f| \leq 1$ on $\widehat{\mathbb{C}} \setminus K$. Therefore, f is admissible for K. However

$$f'(\infty) = g'(\infty) + \epsilon |a_n|^2 > \gamma(K).$$

This contradiction shows that $k = 0$. Hence also $h = 0$, and the theorem is established. □

The function g mentioned in the theorem above is called the *Ahlfors function* for K. Notice the condition $K = \widehat{K}$ in the theorem. It is quite easy to look past this condition since for an arbitrary compact K we lose the uniqueness property only because an admissible function can change in the bounded components of K^c without effecting the behavior in the unbounded one. We are only interested in its behaviour in the unbounded part. Therefore, for an arbitrary K we say that the Ahlfors function for $\widehat{K}$ is also the Ahlfors function for K.

Now we turn to estimates for admissible functions in terms of analytic capacity.

Theorem 5.2.5. *Let K be compact and let f be admissible for K.*

(i) If $z \in \widehat{\mathbb{C}} \setminus \widehat{K}$, then $|f(z)| \leq \dfrac{\gamma(K)}{d(z,K)}$.

(ii) If $z \in \widehat{\mathbb{C}} \setminus \widehat{K}$, then $|f'(z)| \leq \dfrac{4\gamma(K)}{d(z,K)^2}$.

(iii) If $K \subset \Delta(0,R)$ and $f(z) = \displaystyle\sum_{n=1}^{\infty} \frac{a_n}{z^n}$ represents f near ∞, then $|a_n| \leq enR^{n-1}\gamma(K)$.

Proof. **(i)** For $z \in \widehat{\mathbb{C}} \setminus \widehat{K}$ fixed and $a < d(z,K)$, let

$$h(\zeta) := \left(\frac{a}{\zeta - z}\right)\left(\frac{f(\zeta) - f(z)}{1 - \overline{f(z)}f(\zeta)}\right).$$

We have $|f(\zeta) - f(z)|/|1 - \overline{f(z)}f(\zeta)| < 1$ for all $\zeta \in \widehat{\mathbb{C}} \setminus \widehat{K}$, so $|h(\zeta)| \leq 1$ whenever $\zeta \in \widehat{\mathbb{C}} \setminus \widehat{K}$ satisfies $|\zeta - z| \geq a$. Since h is analytic in the disc $\overline{\Delta(z,a)}$, we obtain $|h(z)| \leq 1$ for all $\zeta \in \widehat{\mathbb{C}} \setminus \widehat{K}$ and therefore h is admissible for K. We get

$$\gamma(K) \geq |h'(\infty)| = \lim_{\zeta \to \infty} |\zeta h(\zeta)| = \lim_{\zeta \to \infty} \left| \frac{\zeta a}{\zeta - z} \right| \left| \frac{f(\zeta) - f(z)}{1 - \overline{f(z)}f(\zeta)} \right| = a|f(z)|.$$

Hence $|f(z)| \leq \gamma(K)/a$. Since a is an arbitrary positive number greater than $d(z,K)$ we have the desired result.

(ii) Let $d = d(z,K)$. Again, fixing $z \in \widehat{\mathbb{C}} \setminus \widehat{K}$, for any ζ such that $|\zeta - z| = d/2$ by (i) we have $|f(\zeta)| \leq 2\gamma(K)/d(z,K)$. Hence

$$|f'(z)| = \left| \frac{1}{2\pi i} \int_{|\zeta - z| = d/2} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \right| \leq \frac{4\gamma(K)}{d(z,K)^2}.$$

(iii) Fix R_0 so that $R_0 > R$. By (i) we have $|f(z)| \leq \gamma(K)/(R_0 - R)$ if $|z| = R_0$. Hence

$$|a_n| = \left| \frac{1}{2\pi i} \int_{|z| = R_0} f(z) z^{n-1} dz \right| \leq \frac{\gamma(K) R_0^n}{R_0 - R}.$$

Setting $R_0 = nR/(n-1)$, and noting that $(n/(n-1))^{n-1} \leq e$, we get the desired result. $\square$

Even though direct calculations of analytic capacity are usually extremely difficult we have some estimates which sometimes are sufficient:

Theorem 5.2.6.

(i) If K is compact and connected then $\gamma(K) \leq \operatorname{diam}(K) \leq 4\gamma(K)$.

(ii) If E is a bounded measurable subset of the complex plane, then $\operatorname{Area}(E) \leq 4\pi\alpha(E)^2 \leq 4\pi\gamma(E)^2$.

(iii) If E is bounded, then $\beta(E) \leq 6 \operatorname{diam}(E)$.

(iv) If $z_0 \in E$ and w_0 is the analytic centre of E then $|z_0 - w_0| \leq 24 \operatorname{diam}(E)$.

Before we prove this theorem we must mention a known result from complex analysis, which we state without proof.

Theorem 5.2.7 (Koebe's 1/4-Theorem, Th. 14.14 of [14])**.** *If $f \in \mathcal{O}(\Delta(0,1))$ is injective, $f(0) = 0$ and $f'(0) = 1$ then $\Delta(0, 1/4)$ is contained in the image of f.*

Proof of Theorem 5.2.6. **(i)** Any bounded set E is contained in a disc Δ with radius $\operatorname{diam}(E)$. So the inequality $\gamma(E) \leq \gamma(\Delta) = \operatorname{diam}(E)$ holds for an arbitrary E.

For the second inequality, first assume that K contains only one point. Any admissible function for K has a removable singularity at the point of K. Since it is bounded on the Riemann sphere it is equal to a constant and its derivative at infinity is 0. We obtain $\gamma(K) = 0$.

Now assume that K contains at least two points. Let g be the Ahlfors function of K. Let $z_1 \in K$ be fixed. Define $f(z) := \gamma(K)/(g^{-1}(z) - z_1)$. Then f is an injective function on the unit disc, $f(0) = 0$ and $f'(0) = 1$. If $z_2 \in K$ is not equal to z_1 then $\gamma(E)/(z_2 - z_1)$ does not belong to the range of f. By Theorem 5.2.7 we have $\gamma(K)/|z_2 - z_1| \geq 1/4$ or $|z_1 - z_2| \leq 4\gamma(K)$. Since $z_1, z_2 \in K$ were arbitrary, we obtain $\mathrm{diam}(K) \leq 4\gamma(K)$.

(ii) It suffices to prove the theorem for compact sets K. We consider the convolution

$$f(\zeta) = \iint_K \frac{dxdy}{z - \zeta},$$

of the locally integrable function $1/z$ with the characteristic function of K. The function f is continuous on the Riemann sphere, and f is holomorphic off K. Also $f(\infty) = 0$ and

$$f'(\infty) = \lim_{\zeta \to \infty} \zeta f(\zeta) = \iint_K \lim_{\zeta \to \infty} \frac{\zeta}{z - \zeta} dxdy = \iint_K 1 dxdy = \mathrm{Area}(K).$$

Let $a = (\mathrm{Area}(K)/\pi)^{\frac{1}{2}}$. For ζ fixed, the disc $\Delta(\zeta, a)$ has the same area as K. Since $1/|z - \zeta|$ is a decreasing function of $|z - \zeta|$, we obtain

$$|f(\zeta)| \leq \iint_K \frac{dxdy}{|z - \zeta|} \leq \iint_{\Delta(\zeta, a)} \frac{dxdy}{|z - \zeta|} = \int_0^{2\pi} \int_0^a \frac{rdrd\theta}{r} = 2\pi a.$$

It follows that $f/(2\pi a)$ is an admissible function for K, continuous on $\widehat{\mathbb{C}}$, and

$$\alpha(K) \geq \frac{f'(\infty)}{2\pi a} = \frac{(\mathrm{Area}(K))^{1/2}}{2\sqrt{\pi}}.$$

This is the desired inequality.

(iii) Let $z_0 \in E$, then $E \subset \Delta(z_0, \mathrm{diam}(E))$. By a simple shifting in the plane and Theorem 5.2.5 *(iii)* we have $|\beta(f, z_0)| \leq 6 \cdot \mathrm{diam}(E)\gamma(E)$ for any function f which is admissible for E. Thus we have $\beta(E, z_0) \leq 6 \cdot \mathrm{diam}(E)$ for all $z_0 \in E$ and we get $\beta(E) = \inf_z \beta(E, z) \leq 6 \cdot \mathrm{diam}(E)$.

(iv) As in the proof of Lemma 5.1.3, for any $z_1 \in \mathbb{C}$ and any admissible function f we have

$$\gamma(E)\beta(E, z_1) \geq |z_0 - z_1||f'(\infty)| - |\beta(f, z_0)|.$$

By substituting $z_1 = w_0$ and letting f be an admissible function such that $|f'(\infty)| \geq \gamma(E)/2$ we get

$$\gamma(E)\beta(E) \geq |z_0 - w_0|\frac{\gamma(E)}{2} - |\beta(f, z_0)|.$$

By *(iii)* we have $\beta(E) \leq 6 \cdot \mathrm{diam}(E)$ and as in the proof of *(iii)* we have $|\beta(f, z_0)| \leq 6 \cdot \mathrm{diam}(E)\gamma(E)$ so

$$6\gamma(E)\mathrm{diam}(E) \geq |z_0 - w_0|\frac{\gamma(E)}{2} - 6\gamma(E)\mathrm{diam}(E).$$

Simplifying, we get the desired result. □

The next theorem is an existence theorem, guaranteeing the existence of a function f satisfying certain capacity estimates. This theorem will be extremely important in the upcoming proof of Vitushkin's Theorem since it will be used in the actual approximating step.

Theorem 5.2.8. *Let E be bounded and let w_0 be the analytic centre of E. If a and b are complex numbers such that $|a| \leq \gamma(E)$ and $|b| \leq \beta(E)\gamma(E)$, then there is a function f such that $f/20$ is admissible for E, $f'(\infty) = a$ and $\beta(f, w_0) = b$.*

Proof. Let h be an admissible function for E, such that $h'(\infty) = \gamma(E)/2$. Let $z_0 = w_0 + 2\beta(h, w_0)/\gamma(E)$. By equation (1) in the proof of Lemma 5.1.2 we have

$$\begin{aligned}\beta(h, z_0) &= \beta(h, w_0) + (w_0 - z_0)h'(\infty) \\ &= \beta(h, w_0) + (w_0 - w_0 - 2\beta(h, w_0)/\gamma(E))\frac{\gamma(E)}{2} = 0.\end{aligned}$$

Also $|w_0 - z_0| \leq 2\beta(E)$.

Let f_0 be an admissible function for E such that $\beta(f_0, z_0) = \gamma(E)\beta(E)/2$. Let $f_1 = 2f_0 - 4f_0'(\infty)h/\gamma(E)$. Then $f_1(\infty) = f_1'(\infty) = 0$,

$$\begin{aligned}\beta(f_1, w_0) &= \beta(f_1, z_0) + (z_0 - w_0)f_1'(\infty) = \beta((2f_0 - 4f_0'(\infty)h/\gamma(E)), z_0) \\ &= 2\beta(f_0, z_0) - 4f'(\infty)\beta(h, z_0)/\gamma(E) = \gamma(E)\beta(E)\end{aligned}$$

and $|f_1| \leq 2|f_0| + 4|h| \leq 6$.

Set $f_2 = 2h + (w_0 - z_0)f_1/\beta(E)$. Then $f_2(\infty) = 0$, $f_2'(\infty) = \gamma(E)$,

$$\begin{aligned}\beta(f_2, w_0) &= 2\beta(h, w_0) + (w_0 - z_0)\frac{\beta(f_1, w_0)}{\beta(E)} \\ &= -2(w_0 - z_0)h'(\infty) + (w_0 - z_0)\gamma(E) = 0\end{aligned}$$

and $|f_2| \leq 2|h| + |f_1||w_0 - z_0|/\beta(E) \leq 2 + 6 \cdot 2 = 14$. Now

$$f := a\frac{f_1}{\gamma(E)} + b\frac{f_2}{\beta(E)\gamma(E)}$$

is the desired function. □

5.2.1 Green's functions and analytic capacity

Green's functions can be helpful in the computation of analytic capacity. Let K be a connected compact set of the plane, and suppose f is the Ahlfors function of K. (Since K is connected f is essentially just the Riemann mapping of $\widehat{\mathbb{C}} \setminus \widehat{K}$ to the unit disc such that $f(\infty) = 0$, $f'(\infty) > 0$). We notice that the function $g := -\log|f|$ is harmonic on $\mathbb{C} \setminus \widehat{K}$. In fact we see that g satisfies every condition a Green's function with pole at ∞ should have. Therefore by the uniqueness property of the Green's function we see that

$$g_{\widehat{\mathbb{C}} \setminus \widehat{K}}(z, \infty) = -\log|f|.$$

Now, f can be expressed as a Laurent series near infinity as $f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z^k}$. Therefore we see that

$$g_{\widehat{\mathbb{C}}\setminus K}(z,\infty) = -\log|\sum_{k=1}^{\infty} \frac{a_k}{z^k}| = \log|z| - \log|a_1| + o(1) \qquad \text{as } z \to \infty.$$

Now remembering that $a_1 = \gamma(K)$ we see the following

Theorem 5.2.9. *Let K be a connected compact set and D be the component of $\widehat{\mathbb{C}} \setminus K$ which contains ∞. Then*

$$g_D(z,\infty) = \log|z| - \log|\gamma(K)| + o(1) \qquad \textit{as } z \to \infty.$$

5.3 Rational approximation

Let $K \subset \mathbb{C}$ be compact and suppose $f \in A(K)$. Now we develope an approximation procedure, to approximate f with some function which extends analytically beyond the interior of K. Since f is continuous to the boundary of K we can extend it to be continuous on the whole complex plane and with compact support. That is to say, we can assume that $f \in C_c(\mathbb{C})$.

5.3.1 Partition of f

Our first step is to write f as a finite sum of functions, each one easier to approximate than the original function f. First we find a suitable partition of unity:

Theorem 5.3.1. *Let $\delta > 0$. There exist discs $\Delta_{k,\delta} = \Delta(z_k, \delta)$ which cover the complex plane, and continuously differentiable functions $g_{k,\delta}$ such that*

(i) $g_{k,\delta}$ *is supported on* $\Delta_{k,\delta}$.

(ii) $\sum_{k\in\mathbb{N}} g_{k,\delta} = 1$.

(iii) $\left\| \frac{\partial g_{k,\delta}}{\partial \bar{z}} \right\|_{\infty} \leq \frac{4}{\delta}$.

(iv) No point z is contained in more than 25 of the disks $\Delta_{k,\delta}$.

Proof. Let $g \in C_c^1(\Delta(0, 1/2))$ be a function such that $g \geq 0$, $\iint g(z)dxdy = 1$ and $|\partial g/\partial \bar{z}| \leq 16$. Let $\{E_k\}_{k\in\mathbb{N}}$ be a partition of the plane into squares whose sides have length $1/2$, and set $g_{k,1} = g * \chi_{E_k}$, where χ_{E_k} is the characteristic function for the set E_k. Then each $g_{k,1}$ is supported on the disc $\Delta_{k,1} = \Delta(z_k, 1)$ where z_k is the centre of E_k. Now, let $g_{k,\delta}(z) = g_{k,1}(z/\delta)$ and $\Delta_{k,\delta} = \Delta(\delta z_k, \delta)$. $\square$

Now we let $\delta > 0$ and $g_{k,\delta}$ be a partition of unity as mentioned in the theorem above, each supported in $\Delta_{k,\delta}$. We define for each k an new function

$$G_{k,\delta}(\zeta) := \frac{1}{\pi}\iint \frac{f(z)-f(\zeta)}{z-\zeta}\frac{\partial g_{k,\delta}}{\partial \bar{z}}dxdy = f(\zeta)g_{k,\delta}(\zeta) + \frac{1}{\pi}\iint \frac{f(z)}{z-\zeta}\frac{\partial g_{k,\delta}}{\partial \bar{z}}dxdy.$$

These functions have many good properties as stated in the next theorem.

Theorem 5.3.2. *Let g be a continuously differentiable function supported on the disc $\Delta(z_0,\delta)$. Let $f \in C_c(\mathbb{C})$ be holomorphic on some open set U and*

$$G(\zeta) := \frac{1}{\pi}\iint \frac{f(z)-f(\zeta)}{z-\zeta}\frac{\partial g}{\partial \bar{z}}dxdy, \qquad \zeta \in \mathbb{C}.$$

Then $G \in C(\widehat{\mathbb{C}})$ and:

(i) G is holomorphic on U and off the disc $\Delta(z_0,\delta)$.

(ii) $G(\infty) = 0$.

(iii) $G'(\infty) = -\frac{1}{\pi}\iint f(z)\frac{\partial g}{\partial \bar{z}}dxdy$.

(iv) $\beta(G,w) = -\frac{1}{\pi}\iint f(z)(z-w)\frac{\partial g}{\partial \bar{z}}dxdy$.

(v) $\|G\|_\infty \leq 2\delta\omega(f;2\delta)\left\|\frac{\partial g}{\partial \bar{z}}\right\|_\infty$.

(vi) $|G'(\infty)| \leq 2\delta\omega(f;2\delta)\left\|\frac{\partial g}{\partial \bar{z}}\right\|_\infty \alpha(\Delta(z_0,\delta)\setminus U)$.

(vii) $\frac{1}{\pi}\left|\iint f(z)(z-z_0)\frac{\partial g}{\partial \bar{z}}dxdy\right| \leq 4\delta^2\omega(f;2\delta)\left\|\frac{\partial g}{\partial \bar{z}}\right\|_\infty \alpha(\Delta(z_0,\delta)\setminus U)$.

where $\omega(f;\cdot)$ is the modulus of continuity of f.

Proof. **(i)** Since $\partial g/\partial \bar{z}$ has support in $\Delta(z_0,\delta)$ we see that G is holomorphic off $\Delta(z_0,\delta)$.

If f is holomorphic in a neighbourhood of ζ we get

$$\begin{aligned}\frac{\partial G}{\partial \bar{\zeta}}(\zeta) &= \frac{\partial}{\partial \bar{\zeta}}\left(f(\zeta)g(\zeta) + \frac{1}{\pi}\iint \frac{f(z)}{z-\zeta}\frac{\partial g}{\partial \bar{z}}dxdy\right)\\ &= f(\zeta)\frac{\partial g}{\partial \bar{\zeta}}(\zeta) - f(\zeta)\frac{\partial g}{\partial \bar{\zeta}}(\zeta) = 0.\end{aligned}$$

So G is holomorphic wherever f is and off the support of g.

(ii)

$$\begin{aligned}G(\infty) &= \lim_{\zeta\to\infty} G(z)\\ &= \frac{1}{\pi}\iint \lim_{\zeta\to\infty}\left(\frac{f(z)-f(\zeta)}{z-\zeta}\frac{\partial g}{\partial \bar{z}}\right)dxdy = \frac{1}{\pi}\iint 0dxdy = 0.\end{aligned}$$

(iii)

$$\begin{aligned} G'(\infty) &= \lim_{\zeta\to\infty} \zeta G(\zeta) \\ &= \frac{1}{\pi}\iint \lim_{\zeta\to\infty}\left(\frac{\zeta(f(z)-f(\zeta))}{z-\zeta}\frac{\partial g}{\partial \bar z}\right) dxdy = -\frac{1}{\pi}\iint f(z)\frac{\partial g}{\partial \bar z} dxdy. \end{aligned}$$

(iv) We have

$$\begin{aligned} G(\zeta) - \frac{G'(\infty)}{\zeta - w} &= \frac{1}{\pi}\iint \frac{f(z)-f(\zeta)}{z-\zeta}\frac{\partial g}{\partial \bar z} dxdy + \frac{1}{\pi}\iint \frac{f(z)}{(\zeta-w)}\frac{\partial g}{\partial \bar z} dxdy \\ &= \frac{1}{\pi}\iint \frac{f(z)(z-w)+f(\zeta)(w-\zeta)}{(\zeta-w)(z-\zeta)}\frac{\partial g}{\partial \bar z} dxdy, \end{aligned}$$

so, remembering that f has compact support, we get

$$\begin{aligned} \beta(G,w) &= \lim_{\zeta\to\infty}(G(\zeta) - \frac{G'(\infty)}{\zeta-w})(\zeta-w)^2 \\ &= \frac{1}{\pi}\iint \lim_{\zeta\to\infty} \frac{(\zeta-w)(f(z)(z-w)+f(\zeta)(w-\zeta))}{(z-\zeta)}\frac{\partial g}{\partial \bar z} dxdy \\ &= -\frac{1}{\pi}\iint f(z)(z-w)\frac{\partial g}{\partial \bar z} dxdy. \end{aligned}$$

(v) Let $\zeta \in \Delta(z_0,\delta)$, since $\mathrm{supp}(g) \subseteq \Delta(z_0,\delta)$ we have

$$|G(\zeta)| \le \frac{1}{\pi}\iint \frac{|f(z)-f(\zeta)|}{|z-\zeta|}\left|\frac{\partial g}{\partial \bar z}\right| dxdy \le \frac{\omega(f;2\delta)}{\pi}\left\|\frac{\partial g}{\partial \bar z}\right\|_\infty \iint_{\Delta(z_0,\delta)} \frac{dxdy}{|z-\zeta|}.$$

The last integral is maximized when $\zeta = z_0$, in which case the value is computed to be $2\pi\delta$. Therefore $|G(\zeta)| \le 2\delta\omega(f;2\delta)\left\|\frac{\partial g}{\partial \bar z}\right\|_\infty$. Since G is holomorphic off $\Delta(z_0,\delta)$, by the maximum principle we have

$$\|G\|_\infty \le 2\delta\omega(f;2\delta)\left\|\frac{\partial g}{\partial \bar z}\right\|_\infty.$$

(vi) G is continuous on $\widehat{\mathbb{C}}$ and holomorphic on U and off $\Delta(z_0,\delta)$. By the definition of continuous analytic capacity we have $|G'(\infty)| \le \|G\|_\infty \alpha(\Delta(z_0,\delta)\setminus U)$.

(vii) Very similar to the proof of (v) and (vi) with $f(z)$ replaced by $f(z) = f(z)(z-z_0)$. If $\zeta \in \Delta(z_0,\delta)$, we can evaluate $|f(z)(z-z_0) - f(\zeta)(\zeta-z_0)|$ as in the integral in the proof of (v) by $2\delta\omega(f;2\delta)$ rather than $\omega(f;2\delta)$. □

Now, let the largest open set on which f is holomorphic be called U. Since f has compact support U^c is bounded. Therefore only finitely many of the discs $\Delta_{k,\delta}$ intersect U^c.

Theorem 5.3.2 gives that the function $G_{k,\delta}$ is holomorphic on U and off $\Delta_{k,\delta}$, it also states that $G_{k,\delta}(\infty) = 0$. Consequently from Liouville's theorem we see that if

$\Delta_{k,\delta} \cap U^c = \emptyset$, then $G_{k,\delta} = 0$. Therefore the function $G_{k,\delta}$ is equal to the constant 0 for all but finitely many k.

At last we notice that for every $\zeta \in \mathbb{C}$

$$\begin{aligned} f(\zeta) - \sum G_{k,\delta}(\zeta) &= f(\zeta) - \sum \left(f(\zeta) g_{k,\delta}(\zeta) + \frac{1}{\pi} \iint \frac{f(z)}{z-\zeta} \frac{\partial g_{k,\delta}}{\partial \overline{z}} dxdy \right) \\ &= f(\zeta) - f(\zeta) \left(\sum g_{k,\delta}(\zeta) \right) - \frac{1}{\pi} \iint \frac{f(z)}{z-\zeta} \frac{\partial}{\partial \overline{z}} \left(\sum g_{k,\delta} \right) dxdy \\ &= 0. \end{aligned}$$

The conclusion is

$$f = \sum G_{k,\delta}$$

where the sum is only over the finitely many non-zero terms. Observe that the number of terms depends on δ. Now, if we only approximate each of the $G_{k,\delta}$ suitably well then f will be approximated.

Theorem 5.3.3. *Let K be compact, and let $f \in A(K)$. For each $\delta > 0$ let $\{g_{k,\delta}\}$ be a partition of unity as discussed above, and let $G_{k,\delta}$ be as above. Suppose there is a constant $r \geq 1$ independent of δ, a function $a(\delta)$ and functions $H_{k,\delta} \in C(\widehat{\mathbb{C}})$ such that*

(i) $\lim_{\delta \to 0} a(\delta) = 0$,

(ii) $\|H_{k,\delta}\| \leq a(\delta)$,

(iii) $H_{k,\delta}$ *is holomorphic off* $\Delta(z_k, r\delta) \setminus K$,

(iv) $H_{k,\delta} - G_{k,\delta}$ *has a triple zero at* ∞.

Then $f \in R(K)$.

Proof. Let δ be fixed. As before, we can assume that f has compact support, and that $f = \sum_k G_{k,\delta}$ is a finite sum. By assumption (iii) $H_{k,\delta}$ is holomorphic in a neighbourhood of K. Therefore Runge's Theorem states that $H_{k,\delta} \in R(K)$ and it is sufficient to show that

$$\| \sum_k G_{k,\delta} - H_{k,\delta} \|_K \leq b(\delta),$$

where b is some function such that $b(\delta) \to 0$ as $\delta \to 0$. By Theorems 5.3.1 (iii) and 5.3.2 (v) we have $\|G_{k,\delta} - H_{k,\delta}\| \leq \|G_{k,\delta}\| + \|H_{k,\delta}\| \leq 8\omega(f; 2\delta) + a(\delta)$.

By assumption (iii) and (iv) the function $(z-z_k)^3(G_{k,\delta} - H_{k,\delta})/(r^3\delta^3)$ is holomorphic in a neighbourhood of $\widehat{\mathbb{C}} \setminus \Delta(z_k, r\delta)$. On the boundary of the disc $\Delta(z_k, r\delta)$ we get

$$\frac{|z-z_k|^3 |G_{k,\delta} - H_{k,\delta}|}{(r^3\delta^3)} \leq |G_{k,\delta} - H_{k,\delta}| \leq |H_{k,\delta}| + |G_{k,\delta}| \leq a(\delta) + 8\omega(f; 2\delta).$$

By the maximum principle this inequality then holds when $|z - z_k| \geq r\delta$. This inequality holds trivially when $|z - z_k| \leq r\delta$. Consequently

$$|G_{k,\delta}(z) - H_{k,\delta}(z)| \leq \frac{r^3\delta^3(a(\delta) + 8\omega(f; 2\delta))}{|z - z_k|^3}, \qquad z \in \mathbb{C}.$$

Now fix some complex number $z_0 \in K$. Each disc $\Delta_{k,\delta} = \Delta(z_k, \delta)$ passes through at least one, and at most two of the circles $C_n := \{|z - z_0| = n\delta\}$, $n \in \mathbb{N}$. Let $N(n)$ be the number of discs which meet C_n. If $n \geq 2$ and the disc $\Delta_{k,\delta}$ meets C_n it is contained in the annulus $A_n = \{(n-2)\delta \leq |z - z_0| \leq (n+2)\delta\}$. If $\Delta_{k,\delta}$ meets C_1, it is contained in $A_1 = \{|z - z_0| \leq 3\delta\}$. The area of A_n is $8n\delta^2\pi$ if $n \geq 2$ and the area of A_1 is $9\delta^2\pi$. Since, by Theorem 5.3.1, each point is contained in at most 25 discs the sum of the areas of the discs meeting C_n is less than $9n\delta^2\pi \cdot 25$. The area of each disc is $\delta^2\pi$ so we see that

$$N(n) \leq \frac{9n\delta^2\pi \cdot 25}{\delta^2\pi} = 225n.$$

If $n \geq 2$ and $\Delta_{k,\delta}$ meets C_n, then $|z_0 - z_k| \geq (n-1)\delta$ so

$$|G_{k,\delta}(z_0) - H_{k,\delta}(z_0)| \leq \frac{r^3(a(\delta) + 8\omega(f; 2\delta))}{(n-1)^3}.$$

Using the estimate $|G_{k,\delta}(z_0) - H_{k,\delta}(z_0)| \leq a(\delta) + 8\omega(f; 2\delta)$ when $\Delta_{k,\delta}$ meets C_1, we obtain

$$\begin{aligned}\sum_k |G_{k,\delta}(z_0) - H_{k,\delta}(z_0)| &\leq N(1)(a(\delta) + 8\omega(f; 2\delta)) + \sum_{n=2}^{\infty} \frac{N(n)r^3(a(\delta) + 8\omega(f; 2\delta))}{(n-1)^3} \\ &\leq 225(a(\delta) + 8\omega(f; 2\delta)) \left(1 + r^3 \sum_{n=2}^{\infty} \frac{n}{(n-1)^3}\right).\end{aligned}$$

Hence $\| \sum_k (G_{k,\delta} - H_{k,\delta})\| \leq b(\delta)$, where

$$b(\delta) = 225(a(\delta) + 8\omega(f; 2\delta)) \left(1 + r^3 \sum_{n=2}^{\infty} \frac{n}{(n-1)^3}\right).$$

We see that $b(\delta) \to 0$ as $\delta \to 0$. □

The existance of functions H_k satisfying the conditions in Theorem 5.3.3 is highly dependant on analytic capacity.

Theorem 5.3.4. *Suppose $K, f, \delta, g_{k,\delta}$ and $G_{k,\delta}$ are as above. Suppose there is a function $c(\delta) > 0$ and a constant $r \geq 1$ independent of δ such that*

(i) $\lim_{\delta \to 0} c(\delta) = 0$,

(ii) $|G'_{k,\delta}(\infty)| \leq c(\delta)\gamma(\Delta(z_k, r\delta) \setminus K)$,

(iii) $|\beta(G_{k,\delta}, w_k)| \leq c(\delta)\gamma(\Delta(z_k, r\delta) \setminus K)\beta(\Delta(z_k, r\delta) \setminus K)$, *where* w_k *is the analytic centre of* $\Delta(z_k, r\delta) \setminus K$.

Then $f \in R(K)$.

Proof. By Theorem 5.2.8 there is a function $H_{k,\delta}$ holomorphic off $\Delta(z_k, r\delta) \setminus K$ such that $H_{k,\delta}(\infty) = 0$, $H'_{k,\delta}(\infty) = G'_{k,\delta}(\infty)$, $\beta(H_{k,\delta}, w_k) = \beta(G_{k,\delta}, w_k)$ and $\|H_{k,\delta}\|_\infty \leq 20c(\delta)$. Since $\Delta(z_k, r\delta) \setminus K$ is open we can arrange so that $H_{k,\delta} \in C(\widehat{\mathbb{C}})$. Now $H_{k,\delta} - G_{k,\delta}$ has a triple zero at ∞ so we can apply Theorem 5.3.3. $\square$

Now, using the theory which we have discussed the proof of Mergelyan's classical result is easy.

Theorem 5.3.5. *Suppose* K *is compact and the set* K^c *has only finitely many components. Then every function* $f \in A(K)$ *can be approximated uniformly on* K *by rational functions.*

Proof. It is sufficient to approximate f with a function which has an analytic extension to a neighbourhood of K. Extend f continuously to $\widehat{\mathbb{C}}$, so that f vanishes near ∞ and construct $g_{k,\delta}$ and $G_{k,\delta}$ as before. The indices can be arranged so that $\Delta_{k,\delta}$ meets ∂K if and only if $1 \leq k \leq m$. Since $G_{k,\delta}$ is holomorphic in a neighbourhood of K for $k > m$ it is sufficient to find functions $H_{1,\delta}, H_{2,\delta}, ..., H_{m,\delta}$, holomorphic in a neighbourhood of K such that

$$\left\| \sum_{k=1}^{m} (G_{k,\delta} - H_{k,\delta}) \right\| \leq b(\delta).$$

Where b is some function such that $\lim_{\delta \to 0} b(\delta) = 0$. The desired approximating function is then

$$f_\delta = f + \sum_{k=1}^{m} (H_{k,\delta} - G_{k,\delta}) = \sum_{k=1}^{m} H_{k,\delta} + \sum_{k=m+1}^{\infty} G_{k,\delta},$$

where of course the last sum can be considered as finite since $G_k = 0$ for all but finitely many k. Suppose that $1 \leq k \leq m$ and that δ is suitably small. More exactly, suppose δ is smaller than half the diameter of every bounded component of K^c. Let $E = \Delta(z_k, 3\delta) \setminus K$. Then E contains an arc in K^c whose diameter exceeds δ. By Theorems 5.2.6 and 5.2.1 we get

$$\beta(E) \geq \gamma(E) \geq \frac{\delta}{4}.$$

Estimating $\alpha(\Delta(z_k, \delta) \setminus K) \leq \delta$ and $\left\| \dfrac{\partial g_{k,\delta}}{\partial \overline{z}} \right\|_\infty \leq \dfrac{4}{\delta}$ as in Theorem 5.3.1, from Theorem 5.3.2 we get

$$|G'_{k,\delta}(\infty)| \leq 8\delta\omega(f; 2\delta) \leq 32\omega(f; 2\delta)\gamma(E) \qquad \text{and}$$

$$|\beta(G_{k,\delta}, z_k)| \le 16\delta^2\omega(f;2\delta) \le 256\omega(f;2\delta)\gamma(E)\beta(E).$$

If w_k is the analytic centre of $E = \Delta(z_k, 3\delta) \setminus K$ then by Theorem 5.2.6

$$|w_k - z_k||G'_{k,\delta}(\infty)| \le 72\delta|G'_{k,\delta}(\infty)| \le 10^4\omega(f;2\delta)\gamma(E)\beta(E).$$

From the equality $\beta(G_{k,\delta}, w_k) = \beta(G_{k,\delta}, z_k) + (w_k - z_k)G_{k,\delta}(\infty)$ discussed late in section 5.1 we get

$$|\beta(G_{k,\delta}, w_k)| \le 10^5\omega(f;2\delta)\gamma(E)\beta(E).$$

Now we have verified the estimates of Theorem 5.3.4 for the functions $G_{1,\delta}, G_{2,\delta}, ..., G_{m,\delta}$ with $c(\delta) = 10^5\omega(f;2\delta)$ and $r = 3$.

□

5.3.2 Proof of Vitushkin's Theorem

According to Vitushkin's theorem the question whether $A(K)$ equals $R(K)$ or not depends only on some estimates of continuous analytic capacity, but not on analytic diameter. The final steps of Vitushkin's proof are essentially methods to obtain the estimate of $|\beta(G_{k,\delta}, w_k)|$ as in Theorem 5.3.4 from the estimate of $|G'_{k,\delta}(\infty)|$ in the same theorem. First we need two lemmas.

Lemma 5.3.6. *Let $p > 0$ be an integer. Let $\{c_{n,k} : 1 \le k \le pn, 1 \le n < \infty\}$ be a double sequence such that $0 \le c_{n,k} \le 1$ for all n, k. Then*

$$\left(\sum_{n,k} \frac{c_{n,k}}{n}\right)^2 \le 2p \sum_{n,k} c_{n,k},$$

assuming that the left sum converges.

Proof. Let $a_n = \sum_{k=1}^{pn} c_{n,k}/n$. Then $a_n \ge 0$ and $a_n \le \sum_{k=1}^{pn} 1/n = pn/n = p$. By induction it is easily seen that for all N we have

$$\left(\sum_{n=1}^{N} a_n\right)^2 \le 2p\left(\sum_{n=1}^{N} na_n\right).$$

Letting $N \to \infty$ we get the desired result. □

Lemma 5.3.7. *Let $p > 0$ be an integer and let E be a bounded set. Let $\{E_j\}$ be a family of subsets of E such that every disc $\Delta(z, \gamma(E))$ meets at most p of the sets E_j. Then*

$$\sum_j \gamma(E_j) \le 200p\gamma(E).$$

If f_j is an admissible function for E_j for all j, then

$$\sum_j |f_j(z)| \le 100p,$$

where the series is being summed over those indices of j for which f_j is defined at z.

Proof. Let z_0 be fixed and let $N(n)$ be the number of the sets E_j which meet $\Delta(z_0,(n+1)\gamma(E))$ but not $\Delta(z_0,n\gamma(E))$. We renumber the sets with double indices $E_{n,1},E_{n,2}...E_{n,N(n)}$, $n=1,2,3,...N$, such that $E_{n,k}$ meets $\Delta(z_0,(n+1)\gamma(E))$ but not $\Delta(z_0,n\gamma(E))$ for all n,k. We take for granted that for $n\geq 1$ the annulus $A_n := \{n\gamma(E)\leq |z-z_0|\leq (n+1)\gamma(E)\}$ can be covered by $20n$ discs of radius $\gamma(E)$. Therefore $N(n)\leq 20pn$ if $n\geq 1$. Also we have

$$n\leq \frac{d(z_0,E_{n,k})}{\gamma(E)}\leq n+1.$$

Let $f_{n,k}$ be an admissible function for $E_{n,k}$ for all n,k. By Theorem 5.2.5 (i) and the estimate above we have

$$|f_{n,k}(z_0)|\leq \frac{\gamma(E_{n,k})}{d(z_0,E_{n,k})}\leq \frac{\gamma(E_{n,k})}{n\gamma(E)},\qquad n\geq 1.$$

Using Lemma 5.3.6 and estimating $|f_{0,k}(z_0)|\leq 1$, we obtain

$$\sum_{n,k}|f_{n,k}(z_0)|\leq p+\sum_{n\neq 0,k}\frac{\gamma(E_{n,k})}{n\gamma(E)}\leq p+\left(40p\sum_{n,k}\frac{\gamma(E_{n,k})}{\gamma(E)}\right)^{\frac{1}{2}}.$$

Let $a=\sum\gamma(E_j)/\gamma(E)$. The estimate becomes

$$\sum_j |f_j(z_0)|\leq p+\sqrt{40pa}.$$

If h_j is an admissible function for E_j such that $h_j'(\infty)=\gamma(E_j)/2$, then the function

$$h:=\frac{\sum_j h_j}{p+\sqrt{40pa}}$$

is admissible for E. So

$$\gamma(E)\geq h'(\infty)=\frac{\sum_j\gamma(E_j)}{2(p+\sqrt{40pa})},$$

which implies

$$a\leq 2(p+\sqrt{40pa}).$$

Solving for a we get $a\leq (82+8\sqrt{105})p\leq 200p$, which proves the first inequality. Also,

$$\sum_j |f_j(z_0)|\leq p+\sqrt{40pa}\leq p+\sqrt{8000p^2}\leq 100p.$$

□

Before we prove Vitushkins theorem, which gives a description of the compact sets K such that $R(K)=A(K)$, we completely describe those functions $f\in C(K)$ that satisfy $f\in R(K)$.

Theorem 5.3.8. *Let K be compact. The following statements are equivalent for $f \in C(\widehat{\mathbb{C}})$.*

(i) $f \in R(K)$.

(ii) If $g \in C_c^1(\Delta(z_0, \delta))$, then

$$\left| \iint f(z) \frac{\partial g}{\partial \bar{z}} dxdy \right| \leq 2\pi\delta\omega(f; 2\delta) \left\| \frac{\partial g}{\partial \bar{z}} \right\|_\infty \gamma(\Delta(z_0, \delta) \setminus K).$$

(iii) There exists $r \geq 1$ and a function $a(\delta)$ which tends to zero as $\delta \to 0$, such that

$$\left| \iint f(z) \frac{\partial g}{\partial \bar{z}} dxdy \right| \leq \delta a(\delta) \left(\left\| \frac{\partial g}{\partial x} \right\|_\infty + \left\| \frac{\partial g}{\partial y} \right\|_\infty \right) \gamma(\Delta(z_0, r\delta) \setminus K)$$

whenever $g \in C_c^1(\Delta(z_0, \delta))$.

Proof. (**i**) $\Rightarrow$ (**ii**). Let $g \in C_c^1(\Delta(z_0, \delta))$. First we shall assume that $f \in R(K)$ is holomorphic in a neighbourhood of K. Then, according to Theorem 5.3.2, the function

$$G(\zeta) = \frac{1}{\pi} \iint \frac{f(\zeta) - f(z)}{\zeta - z} \frac{\partial g}{\partial \bar{z}} dxdy$$

is holomorphic off a compact subset of $\Delta(z_0, \delta) \setminus K$ and is bounded by $2\delta\omega(f, 2\delta)\|\partial g/\partial \bar{z}\|_\infty$. Since

$$G'(\infty) = -\frac{1}{\pi} \iint f(z) \frac{\partial g}{\partial \bar{z}} dxdy,$$

we obtain the estimate in *(ii)*.

Now let's assume that $f \in R(K)$ is arbitrary. Let q be a rational function such that $\|f - q\|_K < \epsilon/2$ and let $U = \{z \in \mathbb{C};\ |f(z) - q(z)| < \epsilon\}$. Now let $\psi \in C_c^\infty(U)$ be a cutoff function such that $\psi = 1$ in a neighbourhood of K, and define $h := \psi q + (1 - \psi) f$. Then $h \in C(\widehat{\mathbb{C}}) \cap \mathcal{O}(K)$ and $\|f - h\|_{\widehat{\mathbb{C}}} < \epsilon$. We have

$$\begin{aligned} \left| \iint f(z) \frac{\partial g}{\partial \bar{z}} dxdy \right| &\leq \left| \iint (f(z) - h(z)) \frac{\partial g}{\partial \bar{z}} dxdy \right| + \left| \iint h(z) \frac{\partial g}{\partial \bar{z}} dxdy \right| \\ &\leq \epsilon \iint \left| \frac{\partial g}{\partial \bar{z}} \right| dxdy + 2\pi\delta\omega(h; 2\delta) \left\| \frac{\partial g}{\partial \bar{z}} \right\|_\infty \gamma(\Delta(z_0, \delta) \setminus K). \end{aligned}$$

Letting $\epsilon \to 0$ and noting that $\omega(h; 2\delta) \leq \omega(f; 2\delta) + 2\epsilon$ we obtain the estimate in *(ii)*.

(**ii**) $\Rightarrow$ (**iii**).This is obvious.

(**iii**) $\Rightarrow$ (**i**). Suppose f satisfies the estimate in *(iii)*. We verify the estimates of Theorem 5.3.4, for an appropriate function $c(\delta)$.

Fix $\delta > 0$ and k, and set $E = \Delta(z_k, (r + 2)\delta) \setminus K$. By Theorems 5.3.2 and 5.3.4, it suffices to show that

$$\left| \iint f(z) \frac{\partial g_{g,\delta}}{\partial \bar{z}} dxdy \right| \leq c(\delta)\gamma(E)$$

and
$$\left|\iint f(z)(z-w_k)\frac{\partial g_{k,\delta}}{\partial \bar z}dxdy\right| \le c(\delta)\gamma(E)\beta(E).$$
The former estimate follows easily from (iii) since
$$\left\|\frac{\partial g_{k,\delta}}{\partial x}\right\|_\infty + \left\|\frac{\partial g_{k,\delta}}{\partial y}\right\|_\infty \le 4\left\|\frac{\partial g_{k,\delta}}{\partial \bar z}\right\|_\infty \le \frac{16}{\delta}$$
and $\gamma(\Delta(z_0,r\delta)\setminus K)\le\gamma(E)$, providing only $c(\delta)\ge 16a(\delta)$. Obtaining the second estimate is much harder.

Set $\beta_0 := \min\{\delta,\beta(E)\}$. Note that $\gamma(\Delta(z_k,(r+2)\delta)) = (r+2)\delta$ and by Theorem 5.2.1 (i) we have $\gamma(E)\le\beta(E)\le(r+2)\beta(E)$. Therefore
$$\gamma(E)\le(r+2)\beta_0.$$
Let $h_j = g_{j,\beta_0}$ be a partition of unity like the one in Theorem 5.3.1 associated with the discs $\Delta(t_j,\beta_0)$. From the properties of $g_{k,\delta}$ and h_j, we obtian
$$\left\|\frac{\partial}{\partial x}(g_{k,\delta}h_j)\right\|_\infty \le \left\|\frac{\partial g_{k,\delta}}{\partial x}\right\|_\infty + \left\|\frac{\partial h_j}{\partial x}\right\|_\infty \le \frac{8}{\delta}+\frac{8}{\beta_0}\le\frac{16}{\beta_0},$$
and
$$\left\|\frac{\partial}{\partial x}((z-t_j)g_kh_j)\right\|_\infty \le \|g_kh_j\|_\infty + \beta_0\left\|\frac{\partial}{\partial x}(g_{k,\delta}h_j)\right\|_\infty \le 17.$$
There are similar estimates for the derivative with respect to y. From (iii) we now obtain
$$\begin{aligned}&\left|\iint f(z)\frac{\partial}{\partial \bar z}(g_{k,\delta}h_j)dxdy\right|\\ &\le \beta_0 a(\beta_0)\left(\left\|\frac{\partial}{\partial x}(g_{k,\delta}h_j)\right\|_\infty + \left\|\frac{\partial}{\partial y}(g_{k,\delta}h_j)\right\|_\infty\right)\gamma(\Delta(t_j,r\beta_0)\setminus K)\\ &\le \beta_0 a(\beta_0)\left(\frac{16}{\beta_0}+\frac{16}{\beta_0}\right)\gamma(\Delta(t_j,r\beta_0)\setminus K) = 32a(\beta_0)\gamma(\Delta(t_j,r\beta_0)\setminus K)\end{aligned}$$
and similarly
$$\begin{aligned}&\left|\iint f(z)(z-t_j)\frac{\partial}{\partial \bar z}(g_{k,\delta}h_j)dxdy\right|\\ &= \left|\iint f(z)\frac{\partial}{\partial \bar z}((z-t_j)g_{k,\delta}h_j)dxdy\right|\\ &\le 34\beta_0 a(\beta_0)\gamma(\Delta(t_j,r\beta_0)\setminus K).\end{aligned}$$

Now let J be the set of those indices j such that $\Delta(t_j, \beta_0)$ meets $\Delta(z_k, \delta)$. Note that any such disc is contained in $\Delta(z_k, (r+2)\delta)$. From the preceding estimates we get:

$$\begin{aligned}
&\left|\iint f(z)(z-w_k)\frac{\partial g_{k,\delta}}{\partial \bar z}dxdy\right| \\
\le&\sum_{j\in J}\left|\iint f(z)(z-w_k)\frac{\partial}{\partial \bar z}(g_{k,\delta}h_j)dxdy\right| \\
\le&\sum_{j\in J}\left|\iint f(z)(z-t_j)\frac{\partial}{\partial \bar z}(g_{k,\delta}h_j)dxdy\right| + \sum_{j\in J}|t_j-w_k|\left|\iint f(z)\frac{\partial}{\partial \bar z}(g_{k,\delta}h_j)dxdy\right| \\
\le&34\beta_0 a(\beta_0)\sum_{j\in J}\gamma(\Delta(t_j, r\beta_0)\setminus K) + 32a(\beta_0)\sum_{j\in J}|t_j-w_k|\gamma(\Delta(t_j, r\beta_0)\setminus K) \\
\le&\mathbf{A}+\mathbf{B}.
\end{aligned}$$

We estimate $\mathbf{A}$ and $\mathbf{B}$ separately.

By construction, each z is contained in at most 25 of the discs $\Delta(t_j, \beta_0)$ (Theorem 5.3.1 (iv)). It follows that every disc of radius $(r+2)\beta_0$ meets at most $M(r)$ of the discs $\Delta(t_j, r\beta_0)$,where M is some function of r independent of δ and β_0. Since $\gamma(E) \le (r+2)\beta_0$, every disc of radius $\gamma(E)$ meets at most $M(r)$ of the sets $\Delta(t_j, r\beta_0)\setminus K$. By Lemma 5.3.7 we have

$$\mathbf{A} \le 34\beta_0 a(\beta_0)(200M(r)\gamma(E)) \le 6800M(r)a(\beta_0)\gamma(E)\beta(E).$$

Note that $\beta_0 \le \delta$ so that takes care of $\mathbf{A}$. $\mathbf{B}$ is harder.

Choose an admissible function f_j for $\Delta(t_j, r\beta_0)\setminus K$ such that

$$(t_j-w_k)f_j'(\infty) = \frac{|t_j-w_k|}{2}\gamma(\Delta(t_j, r\beta_0)\setminus K).$$

By Lemma 5.3.7, the function

$$F := \sum_{j\in J}\frac{f_j}{100M(r)}$$

is admissible for E. For R large we get

$$\begin{aligned}
100M(r)\gamma(E)\beta(E) &= 100M(r)\gamma(E)\beta(E, w_k) \ge 100M(r)|\beta(F, w_k)| \\
&= \frac{100M(r)}{2\pi}\left|\int_{|z|=R}F(z)(z-w_k)dz\right| \\
&= \frac{1}{2\pi}\left|\sum_{j\in J}(t_j-w_k)\int_{|z|=R}f_j(z)dz + \sum_{j\in J}\int_{|z|=R}(z-t_j)f_j(z)dz\right| \\
&= \frac{1}{2\pi}\left|\sum_{j\in J}\frac{|t_j-w_k|}{2}\gamma(\Delta(t_j, r\beta_0)\setminus K) + \sum_{j\in J}\int_{|z|=R}(z-t_j)f_j(z)dz\right| \\
&\ge \frac{1}{4\pi}\sum_{j\in J}|t_j-w_k|\gamma(\Delta(t_j, r\beta_0)\setminus K) - \sum_{j\in J}\left|\frac{1}{2\pi}\int_{|z|=R}(z-t_j)f_j(z)dz\right|.
\end{aligned}$$

From the estimate of a_2 in Theorem 5.2.5 *(iii)*, we obtain

$$\begin{aligned}\sum_{j\in J}\left|\frac{1}{2\pi}\int_{|z|=R}(z-t_j)f_j(z)dz\right| &\le 2er\beta_0\sum_{j\in J}\gamma(\Delta(t_j,r\beta_0)\setminus K)\\ &\le 400erM(r)\gamma(E)\beta(E).\end{aligned}$$

Where the second inequality comes from Lemma 5.3.7 as before. Consequently

$$\begin{aligned}&\frac{1}{4\pi}\sum_{j\in J}|t_j-w_k|\gamma(\Delta(t_j,r\beta_0)\setminus K)\\ \le&100M(r)\gamma(E)\beta(E)+400erM(r)\gamma(E)\beta(E).\end{aligned}$$

Which leads to

$$\mathbf{B}=32a(\beta_0)\sum_{j\in J}|t_j-w_k|\gamma(\Delta(t_j,r\beta_0)\setminus K)\le 10^6rM(r)a(\beta_0)\gamma(E)\beta(E).$$

Combining **A** and **B** we obtain the required estimate:

$$\left|\iint f(z)(z-w_k)\frac{\partial g_{k,\delta}}{\partial\bar z}dxdy\right|\le c(\delta)\gamma(E)\beta(E),$$

with

$$c(\delta)=10^7rM(r)\sup_{x\le\delta}a(x).$$

□

In his paper, Vitushkin gives other criteria necessary and sufficient for a function $f\in C(\widehat{\mathbb{C}})$ to belong to $R(K)$. Since they are not necessary for the proof of Vitushkin's main theorem we state them without proof. We denote the square with centre z_0 and side length δ with $S(z_0,\delta)$.

Theorem 5.3.9 (Th. IV.2.2. of [15])**.** *The following statements are equivalent to the ones given in the preceding theorem.*

(iv) There is a constant c such that

$$\left|\int_{\partial S(z_0,\delta)}f(z)dz\right|\le c\omega(f,2\delta)\gamma(S(z_0,\delta)\setminus K),$$

for all squares $S(z_0,\delta)$.

(v) There exists $r\ge 1$ and a function $c(\delta)$ which tends to zero as $\delta\to 0$, such that

$$\left|\int_{\partial S(z_0,\delta)}f(z)dz\right|\le c(\delta)\gamma(S(z_0,r\delta)\setminus K),$$

for all squares $S(z_0,\delta)$.

We are finally ready to prove Vitushkin's theorem

Proof of Theorem 5.1.1. (**iv**) $\Rightarrow$ (**i**). Suppose *(iv)* is true and let $f \in C(\widehat{\mathbb{C}}) \cap A(K)$. If $g \in C_c^1(\Delta(z_0, \delta))$, then

$$G(\zeta) = \frac{1}{\pi} \iint \frac{f(\zeta) - f(z)}{\zeta - z} \frac{\partial g}{\partial \bar{z}} dxdy$$

is holomorphic off $\Delta(z_0, \delta)$ and on $\mathrm{int}(K)$. G is bounded by $2\delta\omega(f; 2\delta)\|\partial g/\partial \bar{z}\|_\infty$. Hence

$$\begin{aligned} \left| \iint f(z) \frac{\partial g}{\partial \bar{z}} dxdy \right| &= \pi |G'(\infty)| \leq 2\pi\delta\omega(f; 2\delta) \left\| \frac{\partial g}{\partial \bar{z}} \right\|_\infty \alpha(\Delta(z_0, \delta) \setminus \mathrm{int}(K)) \\ &\leq 2\pi c \delta \omega(f; 2\delta) \left\| \frac{\partial g}{\partial \bar{z}} \right\|_\infty \gamma(\Delta(z_0, r\delta) \setminus K). \end{aligned}$$

This shows that statement *(iii)* in Theorem 5.3.8 holds with $c(\delta) = 2\pi c \omega(f; 2\delta)$. Therefore $f \in R(K)$ and $A(K) = R(K)$.

(**i**) $\Rightarrow$ (**ii**). Suppose $R(K) = A(K)$ and let D be an open set. Since α is monotone it is sufficient to show that $\alpha(D \setminus K) \geq \alpha(D \setminus \mathrm{int}(K))$. Let $\epsilon > 0$ and choose $f \in C(\widehat{\mathbb{C}})$ which is admissible for $D \setminus \mathrm{int}(K)$ and $|f'(\infty)| \geq \alpha(D \setminus \mathrm{int}(K)) - \epsilon$. Then there is some compact set $L \subset D \setminus \mathrm{int}(K)$ such that f is holomorphic on $\widehat{\mathbb{C}} \setminus L$. Let $\chi \in C_c^\infty(D)$ be a cutoff function such that $\|\chi\|_\infty \leq 1$ and $\chi = 1$ in a neighbourhood of L. Let $M \geq 1$ be a constant such that

$$M \geq \iint |\partial \chi / \partial \bar{w}| d\lambda(w) \qquad \text{and} \qquad M \geq \iint \frac{|\partial \chi / \partial \bar{w}|}{|z - w|} d\lambda(w)$$

for all z. Since $f \in A(K) = R(K)$ we can find $Q \in R(K)$ such that

$$\|f - Q\|_K < \frac{\epsilon}{M}.$$

Define $U := \{z \in \widehat{\mathbb{C}};\ |f(z) - Q(z)| < \frac{2\epsilon}{M}\}$. Then U is an open neighbourhood of K. Define

$$r(z) := \frac{1}{\pi} \iint_U \frac{f(w) - Q(w)}{z - w} \frac{\partial \chi}{\partial \bar{w}}(w) d\lambda(w).$$

Since $\partial \chi / \partial \bar{z}$ is supported on $D \setminus L$, r is holomorphic off that set. Also

$$|r(z)| \leq \frac{2\epsilon}{M} \iint_{\mathbb{C}} \frac{|\partial \chi / \partial \bar{w}|}{|z - w|} d\lambda(w) \leq 2\epsilon,$$

for all z so $\|r\|_\infty \leq 2\epsilon$. Let $g := \chi Q + (1 - \chi) f + r$. Then g is holomorphic off D since f and r are and $\mathrm{supp}(\chi) \subset D$. Also, g is holomorphic on U because there we have

$$\frac{\partial g}{\partial \bar{z}} = (Q - f) \frac{\partial \chi}{\partial \bar{z}} + \frac{\partial r}{\partial \bar{z}} = 0.$$

If $z \in \widehat{\mathbb{C}} \setminus \operatorname{supp}(\chi)$ then $|g(z)| \le |f(z)| + |r(z)| \le 1 + 2\epsilon$. If $z \in U$, then

$$|g(z)| \le |f(z)| + |\chi(z)||Q(z) - f(z)| + |r(z)| \le 1 + 3\epsilon.$$

We have seen that $g < 1 + 3\epsilon$ off D and on U. Since it is also holomorphic there we see that $g/(1+3\epsilon)$ is admissible for $D \setminus K$ and therefore

$$\begin{aligned}(1+3\epsilon)\alpha(D \setminus K) &\ge |g'(\infty)| = |f'(\infty) + r'(\infty)| \\ &\ge \alpha(D \setminus \operatorname{int}(K)) - \epsilon - \left|\lim_{z\to\infty} zr(z)\right| \\ &\ge \alpha(D \setminus \operatorname{int}(K)) - \epsilon - \left|\frac{1}{\pi}\iint_U (f(w) - Q(w))\frac{\partial \chi}{\partial \bar{w}} d\lambda(w)\right| \\ &\ge \alpha(D \setminus \operatorname{int}(K)) - \epsilon - \frac{2\epsilon}{M}\iint |\partial\chi/\partial\bar{w}| d\lambda(w) \\ &\ge \alpha(D \setminus \operatorname{int}(K)) - 3\epsilon.\end{aligned}$$

By letting $\epsilon \to 0$ we see that $\alpha(D \setminus K) \ge \alpha(D \setminus \operatorname{int}(K))$.

(ii) $\Rightarrow$ **(iii)** Assuming (ii) and using the monotonicity of α we get

$$\alpha(\Delta(z_0, \delta) \setminus \operatorname{int}(K)) = \alpha(\Delta(z_0, \delta) \setminus K) \le \alpha(\Delta(z_0, r\delta) \setminus K).$$

(iii) $\Rightarrow$ **(iv)** and **(iv)** $\Rightarrow$ **(v)** are obvious.

We have seen that conditions $(i) - (iv)$ are equivalent and that they imply (v). In the last step we show that if the equivalent statements $(i) - (iv)$ fail, then (v) also fails.

Suppose $R(K) \ne A(K)$. We construct three sequences, $(K_n)_{n=0}^\infty$, $(\delta_n)_{n=0}^\infty$ and $(z_n)_{n=1}^\infty$ of compact sets, positive real numbers and complex numbers respectively. Set $K_0 = K$ and $\delta_0 = 1$. For $n \ge 1$, if $A(K_{n-1}) \ne R(K_{n-1})$ it follows from (iv) (by taking $c = n$, $r = 4n$ and $\delta_0 = \frac{\delta_{n-1}}{2}$) that there exist z_n and $\delta_n < \frac{\delta_{n-1}}{2}$ such that

$$\alpha(\Delta(z_n, \delta_n) \setminus \operatorname{int}(K_{n-1})) > n\alpha(\Delta(z_n, 4n\delta_n) \setminus K_{n-1}).$$

Let $K_n = K_{n-1} \cap \overline{\Delta(z_n, \delta_n)}$. Then $\Delta(z_n, \delta_n) \setminus \operatorname{int}(K_{n-1}) = \Delta(z_n, \delta_n) \setminus \operatorname{int}(K_n)$, $\Delta(z_n, \delta_n) \setminus K_{n-1} = \Delta(z_n, \delta_n) \setminus K_n$ and since α is monotone we have

$$\begin{aligned}\alpha(\Delta(z_n, \delta_n) \setminus \operatorname{int}(K_n)) &= \alpha(\Delta(z_n, \delta_n) \setminus \operatorname{int}(K_{n-1})) \\ &> n\alpha(\Delta(z_n, 4n\delta_n) \setminus K_{n-1}) \ge \alpha(\Delta(z_n, \delta_n) \setminus K_{n-1}) \\ &= \alpha(\Delta(z_n, \delta_n) \setminus K_n).\end{aligned}$$

Therefore (ii) fails for K_n and $A(K_n) \ne R(K_n)$. By induction we see that this holds true for all n. Next we show that $\Delta(z_n, \delta_n) \subset \Delta(z_{n-1}, \delta_{n-1})$. Suppose on the contrary that $\Delta(z_n, \delta_n) \setminus \Delta(z_{n-1}, \delta_{n-1}) \ne \emptyset$. Then the set $\Delta(z_n, 4n\delta_n) \setminus K_{n-1}$ contains as a subset a disc $\Delta(w, \delta_n)$ (for some $w \in \mathbb{C}$) and

$$\begin{aligned}\delta_n = \alpha(\Delta(z_n, \delta_n)) &\ge \alpha(\Delta(z_n, \delta_n) \setminus \operatorname{int}(K_{n-1})) \\ &> n\alpha(\Delta(z_n, 4n\delta_n) \setminus K_{n-1}) \ge \alpha(\Delta(w, \delta_n)) = \delta_n.\end{aligned}$$

Therefore

$$\Delta(z_n, \delta_n) \subset \Delta(z_{n-1}, \delta_{n-1}) \qquad n \geq 1.$$

Consequently, the sequence $(z_n)_{n=1}^{\infty}$ has a limit z, also $K_n = K \cap \overline{\Delta(z_n, \delta_n)}$ for all n. For any $r \geq 1$ and n sufficiently large we have

$$\Delta(z_n, \delta_n) \subset \delta(z, 2\delta_n) \subset \Delta(z, 2r\delta_n) \subset \Delta(z, 4n\delta_n).$$

Therefore, for n large enough, we obtain

$$\begin{aligned}
\alpha(\Delta(z, 2\delta_n) \setminus \mathrm{int}(K)) &\geq \alpha(\Delta(z_n, \delta_n) \setminus \mathrm{int}(K)) \\
&= \alpha(\Delta(z_n, \delta_n) \setminus \mathrm{int}(K \cap \overline{\Delta(z_{n-1}, \delta_{n-1})})) \\
&= \alpha(\Delta(z_n, \delta_n) \setminus \mathrm{int}(K_{n-1})) \\
&> n\alpha(\Delta(z_n, 4n\delta_n) \setminus K_{n-1}) \\
&\geq n\alpha(\Delta(z_n, 4n\delta_n) \setminus K) \\
&\geq n\alpha(\Delta(z, 2r\delta_n) \setminus K).
\end{aligned}$$

and

$$\begin{aligned}
\limsup_{\delta \to 0} \frac{\alpha(\Delta(z, \delta) \setminus \mathrm{int}(K))}{\alpha(\Delta(z, r\delta) \setminus K)} &\geq \lim_{n \to \infty} \frac{\alpha(\Delta(z, 2\delta_n) \setminus \mathrm{int}(K))}{\alpha(\Delta(z, 2r\delta_n) \setminus K)} \\
&\geq \lim_{n \to \infty} n = \infty.
\end{aligned}$$

Thus (v) fails at z. Evidently $z \in \partial K$.

□

For an arbitrary compact set K it can be very hard to tell if it satisfies the above criteria and determine if $A(K) = R(K)$. The following few corollaries give some descriptions on K which are easier to visualize. Setting $r = 1$ in Theorem 5.1.1 (v), reversing the fraction and using the estimate $\alpha(\Delta(z, \delta) \setminus \mathrm{int}(K)) \leq \delta$, we get the following corollary

Corollary 5.3.10. *If K is compact, and*

$$\liminf_{\delta \to 0} \frac{\gamma(\Delta(z, \delta) \setminus K)}{\delta} > 0$$

for all $z \in \partial K$, then $R(K) = A(K)$.

Here we have interchanged α for γ since it is the same on open sets. The next corollary includes Mergelyan's theorem.

Corollary 5.3.11. *If every point on the boundary of the compact set K belongs to the boundary of some component of K^c, then*

$$R(K) = A(K)$$

.

Proof. Let $z \in \partial K$. For sufficiently small $\delta > 0$, the interior of the set $\Delta(z, \delta) \setminus K$ contains an arc whose diameter exceeds $\delta/2$. By Theorem 5.2.6 $\gamma(\Delta(z, \delta) \setminus K) \geq \delta/8$. By Corollary 5.3.10 $R(K) = A(K)$. $\square$

For some set E, and $z \in \mathbb{C}$ we define the lower Lebesgue density at the point z to be

$$\liminf_{\delta \to 0} \frac{\text{Area}(\Delta(z, \delta) \cap E)}{\pi \delta^2}.$$

In fact it is not very hard to visualize what the lower Lebesgue density tells you about a set locally. For example, every interior point of E has obviusly lower Lebesgue density equal to one.

Corollary 5.3.12. *If K is compact and K^c has positive lower Lebesgue density at every point of ∂K, then $A(K) = R(K)$.*

Proof. The lower Lebesgue density of K at the point $z \in \partial K$ is given with

$$\liminf_{\delta \to 0} \frac{\text{Area}(\Delta(z, \delta) \setminus K)}{\pi \delta^2}.$$

By Theorem 5.1.1 and estimating by Theorem 5.2.6, we obtain the conclusion. $\square$

6 Arakelian's Theorem

6.1 Arakelian

Arakelian's theorem concerns uniform approximation by entire functions on possibly unbounded closed subsets E of the complex plane $\mathbb{C}$. To motivate the following definition, notice that if E is a closed set with no holes and $\overline{\Delta}$ is a closed disc in $\mathbb{C}$ the intersection $\overline{\Delta} \cap E$ obviously has no holes, but the union $\overline{\Delta} \cup E$ may well have some holes, even infinitely many.

Definition 6.1.1. *A set $E \subset \mathbb{C}$ is called an Arakelian set if it is closed, E^c is conneced, and for every closed disc $\overline{\Delta} \subset \mathbb{C}$ the union of all holes of the set $E \cup \overline{\Delta}$ is a bounded set.*

The classical Arakelian theorem is as follows

Theorem 6.1.2 (Arakelian [12])**.** *Let $E \subset \mathbb{C}$ be an Arakelian set and f be a continuous function on E which is holomorphic on the interior of E. Then for every $\epsilon > 0$ there is an entire function h such that*

$$|f(z) - h(z)| < \epsilon, \qquad z \in E.$$

In Chapter 1 we discussed how easily approximation theorems can be generalized to a bigger variety of sets by approximating by rational functions rather than polynomials. In this section we prove a slight generalization of the Arakelian theorem concerning sets with holes. This proof is based on the one given by Jean-Pierre Rosay and Walter Rudin [12], changed a little because of the generalization.

We define a new kind of an Arakelian set.

Definition 6.1.3. *A closed set $E \subset \mathbb{C}$ is called an Arakelian set with holes if $\widehat{E}$ is an Arakelian set and if for any closed disc $\overline{\Delta}$ the intersection $\overline{\Delta} \cap E$ has finitely many holes.*

Our generalized version of the theorem is as follows:

Theorem 6.1.4. *Let E be an Arakelian set with holes and f be a continuous function on E which is holomorphic on the interior of E. Let A be a closed set which contains at least one point from every hole of E. Then there exists a function $h \in \mathcal{O}(\mathbb{C} \setminus A)$ such that*

$$|f(z) - h(z)| < \epsilon, \qquad z \in E.$$

Proof. For convenience we can assume that A contains exactly one point from every hole of E. We denote the holes of E by $\Omega_1, \Omega_2, \ldots$. We let m denote the number of holes, note that $m \in \mathbb{N} \cup \{\infty\}$. The point in $\Omega_k \cap A$ (which is a singleton by assumption) is denoted by a_k.

For every $k \in \mathbb{N}$ we let $(r_n^{(k)})$ be a sequence such that $\overline{\Delta}(a_k, r_1^{(k)}) \subset \Omega_k$ and $r_n^{(k)} \searrow 0$ as $n \to \infty$. Let

$$P_n := \bigcup_{1 \le k \le m} \Delta(a_k, r_n^{(k)}).$$

Let R_n be a sequence such that $R_n \nearrow \infty$ and $\overline{\Delta}(0, R_n) \cup \overline{H_n} \subset \Delta(0, R_{n+1})$ where H_n is defined to be the union of all holes of $\overline{\Delta}(0, R_n) \cup \widehat{E}$. This sequence exists because E is an Arakelian set with holes. Now define

$$E_0 = E \qquad \text{and} \qquad E_n = (\widehat{E} \cup \overline{\Delta}(0, R_n) \cup \overline{H_n}) \setminus P_n.$$

Notice that $E_n \subset E_{n+1}$ for all n and

$$\bigcup_{n \in \mathbb{N}} E_n = \mathbb{C} \setminus A.$$

We define now a sequence of holomorphic functions h_n such that $h_n \in A(E_n)$ for all n and $h_n \to h$ uniformly on compact subsets of $\mathbb{C} \setminus A$, for some function $h \in \mathcal{O}(\mathbb{C} \setminus A)$.

Let $h_0 = f$ and assume that $h_{n-1} \in A(E_{n-1})$. Let $\psi_n \in C_c^\infty(\mathbb{C})$ be such that:

- $0 \le \psi_n(z) \le 1$ for all $z \in \mathbb{C}$.
- $\psi_n = 1$ in some neighbourhood U_n of $(\overline{\Delta}(0, R_n) \cup \overline{H_n}) \setminus P_n$.
- $\mathrm{supp}(\psi_n) \subset \overline{\Delta(0, R_{n+1})} \setminus P_{n+1}$.

Since the function $z \to \frac{1}{\pi} \int \frac{\partial \psi_n / \partial \bar{w}}{z - w} d\lambda(w)$ is continuous and goes to zero at ∞ it is bounded by some constant $M_n \ge 1$. Since E is an Arakelian set with holes, the set $E_{n-1} \cap \overline{\Delta}(0, R_{n+1})$ has finitely many holes and Mergelyan's theorem states that there is a rational function Q_n with poles in A such that

$$|h_{n-1}(z) - Q_n(z)| \le \frac{\epsilon}{2^{n+1} M_n}, \qquad z \in E_{n-1} \cap \overline{\Delta}(0, R_{n+1}).$$

Define

$$r_n(z) := \frac{1}{\pi} \int_{E_{n-1}} (h_{n-1}(w) - Q_n(w)) \frac{\partial \psi_n}{\partial \bar{z}}(w)(z - w)^{-1} d\lambda(w).$$

Note that since $\mathrm{supp}(\frac{\partial \psi_n}{\partial \bar{z}}) \subset \overline{\Delta}(0, R_{n+1}) \setminus U_n$ we see that r_n is holomorphic on U_n. Finally we let

$$h_n := \psi_n Q_n + (1 - \psi_n) h_{n-1} + r_n.$$

Then h_n is defined and continuous on $U_n \cup E_{n-1}$. Since $\psi(z) = 1$ on U_n we have $h_n(z) = Q_n(z) + r_n(z)$ on U_n. Hence h_n is holomorphic on U_n. On the interior of E_{n-1} the Cauchy-Pompeiu's formula gives

$$\frac{\partial h_n}{\partial \bar{z}} = Q_n \frac{\partial \psi_n}{\partial \bar{z}} - h_{n-1} \frac{\partial \psi_n}{\partial \bar{z}} + \frac{\partial r_n}{\partial \bar{z}} = (Q_n - h_{n-1}) \frac{\partial \psi_n}{\partial \bar{z}} + (h_{n-1} - Q_n) \frac{\partial \psi_n}{\partial \bar{z}} = 0.$$

Since $E_n \subset E_{n-1} \cup U_n$ we see that $h_n \in A(E_n)$. We have constructed a sequence of functions $(h_n)_{n\in\mathbb{N}}$ such that $h_n \in A(E_n)$ for all $n \in \mathbb{N}$. Next we note that

$$|r_n(z)| \leq \sup_{\zeta\in E_{n-1}} |h_{n-1}(\zeta) - Q_n(\zeta)| \frac{1}{\pi} \int_{E_{n-1}} \left| \frac{\bar{\partial}\psi_n(w)}{z-w} \right| d\lambda(w) \leq \frac{\epsilon}{2^{n+1}},$$

so for $z \in E_{n-1}$ we have

$$|h_n(z) - h_{n-1}(z)| \leq |r_n(z)| + \psi_n(z)|Q_n(z) - h_{n-1}(z)| < \frac{\epsilon}{2^n}.$$

This shows that the sequence $(h_m)_{m\geq(n-1)}$ converges uniformly to a holomorphic function h on E_{n-1}. This is true for all n so we see that the sequence (h_n) converges uniformly to a holomorphic function h on compact subsets of $\mathbb{C} \setminus A$. Therefore $h \in \mathcal{O}(\mathbb{C} \setminus A)$.

On E we have

$$h = h_0 + \sum_{n=1}^{\infty} (h_n - h_{n-1}),$$

and since $h_0 = f$ by definition, we get

$$|f(z) - h(z)| \leq \sum_{n=1}^{\infty} |h_n(z) - h_{n-1}(z)| \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon, \qquad z \in E.$$

□

On sets with no interior a considerably stronger version of the theorem can be derived without much extra effort.

Corollary 6.1.5. *Let E and A be as in the previous theorem and assume further that the interior of E is empty. Given a continuous function $\omega : E \to \mathbb{R}_+$ and $f \in C(E)$, there is a function $h \in \mathcal{O}(\mathbb{C} \setminus A)$ such that*

$$|h(z) - f(z)| \leq \omega(z), \qquad z \in E.$$

Proof. From the previous theorem there is a function $g_1 \in \mathcal{O}(\mathbb{C} \setminus A)$ such that

$$|g_1(z) - \log(\omega(z))| < 1 \qquad z \in E.$$

Let $g_2(z) = g_1(z) - 1$, we have

$$\operatorname{Re}(g_2(z)) = \operatorname{Re}(g_1(z)) - 1 < \log(\omega(z)) \qquad z \in E.$$

By the same theorem we can find $g_3 \in \mathcal{O}(\mathbb{C} \setminus A)$, such that

$$|g_3(z) - f(z)\exp(-g_2(z))| < 1, \qquad z \in E.$$

Hence

$$|g_3(z)\exp(g_2(z)) - f(z)| < |\exp(g_2(z))| < \omega(z),$$

which concludes the proof of the corollary. □

Bibliography

[1] Bo Berndtsson. An introduction to things $\overline{\partial}$. In *Analytic and algebraic geometry*, volume 17 of *IAS/Park City Math. Ser.*, pages 7–76. Amer. Math. Soc., Providence, RI, 2010.

[2] Daniel Lear Claveras. L^2-estimates of the inhomogeneous Cauchy-Riemann equation and applications. Master's thesis, Universitat de Barcelona, Barcelona, 2013.

[3] Theodore W. Gamelin. *Uniform algebras*. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1969.

[4] Robert E. Greene and Steven G. Krantz. *Function theory of one complex variable*, volume 40 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2002.

[5] Lars Hörmander. *The analysis of linear partial differential operators. I*. Springer Study Edition. Springer-Verlag, Berlin, second edition, 1990. Distribution theory and Fourier analysis.

[6] Lars Hörmander. *An introduction to complex analysis in several variables*, volume 7 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[7] Lars Hörmander. *Notions of convexity*, volume 127 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1994.

[8] Norm Levenberg. Approximation in $\mathbb{C}^N$. *Surv. Approx. Theory*, 2:92–140, 2006.

[9] J. D. Pryce. *Basic methods of linear functional analysis*. Hutchinson & Co., Ltd., London, 1973. Hutchinson University Library.

[10] Thomas Ransford. *Potential theory in the complex plane*, volume 28 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995.

[11] Reinhold Remmert. *Classical topics in complex function theory*, volume 172 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998. Translated from the German by Leslie Kay.

[12] Jean-Pierre Rosay and Walter Rudin. Arakelian's approximation theorem. *Amer. Math. Monthly*, 96(5):432–434, 1989.

[13] Alice Roth. Approximationseigenschaften und Strahlengrenzwerte meromorpher und ganzer Funktionen. *Comment. Math. Helv.*, 11(1):77–125, 1938.

[14] Walter Rudin. *Real and complex analysis.* McGraw-Hill Book Co., New York, third edition, 1987.

[15] A. G. Vituškin. Analytic capacity of sets in problems of approximation theory. *Uspehi Mat. Nauk*, 22(6 (138)):141–199, 1967.