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ELEMENTARY PROOF OF THE RUDIN-CARLESON AND THE F. AND M. RIESZ THEOREMS

RAOUF DOSS

ABSTRACT. A very elementary proof is given of the theorem that on a set of measure zero on T, any continuous function is equal to a continuous function of analytic type. The same elementary method proves that a measure of analytic type is absolutely continuous.

A complex Borel measure μ on T, in particular an $f \in L^1(T)$, is said to be of *analytic type* if

$$a_n = (2\pi)^{-1}\int_T e^{-int}\, d\mu(t) = 0, \qquad n = -1, -2, \ldots.$$

The theorems mentioned in the title are:

RUDIN-CARLESON THEOREM. *Let F be a closed subset of T of Lebesgue measure zero. If φ is a continuous function on F, then there is a continuous function f, of analytic type, such that*

$$\begin{aligned} f(t) &= \varphi(t), \qquad t \in F, \\ \sup_{t \in T} |f(t)| &\leqslant M \sup_{t \in F} |\varphi(t)| \end{aligned} \tag{$*$}$$

where M is a constant. (Rudin proves that $M = 1$. See **[8]** and **[1]**.)

THE FIRST F. AND M. RIESZ THEOREM. *If the function f in $L^1(T)$ is of analytic type and if f vanishes on a set S^* of positive measure, then $f = 0$.*

THE SECOND F. AND M. RIESZ THEOREM. *If a complex Borel measure μ on T is of analytic type, then μ is absolutely continuous (with respect to Lebesgue measure).* See **[7]**.

The proofs of these theorems most often use boundary values of functions analytic in the unit disc and the theory of H^p-spaces. For the Second F. and M. Riesz Theorem, for example, see three variants in **[3]**, **[5]** and **[9]**; other proofs of that theorem use Hilbert-space theory: see e.g. **[2]** and **[4]**; a direct short proof is given in **[6]**.

The aim of the present paper is to present a method which gives an elementary proof of all the above theorems.

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LEMMA. *Let F be a closed subset of T of measure zero and φ a continuous function on F. Given $\varepsilon > 0$ and an open set $G \supset F$ there is a continuous function g of analytic type such that*

$$\sup_{t\in F}|g(t) - \varphi(t)| \leqslant \varepsilon \sup_{t\in F}|\varphi(t)|,$$
$$|g(t)| < \varepsilon, \qquad t \notin G, \tag{**}$$
$$\sup_{t\in T}|g(t)| \leqslant 3\sup_{t\in F}|\varphi(t)|.$$

PROOF. Without loss of generality we may assume that $\sup_{t\in F}|\varphi(t)| = 1$ and also that φ is a trigonometric polynomial

$$\varphi(t) = \sum_{|k|<m} \alpha_k e^{ikt}$$

such that

$$|\varphi(t)| < \varepsilon/3, \qquad t \notin G.$$

Let $e^{-A} = \varepsilon$ and let h be a continuous function on T, lying between $-2A$ and 2ε, such that

$$|h(t) + 2A| < \varepsilon, \qquad t \in F.$$

Since $m(F) = 0$ we may take $\|h\|_1$ arbitrarily small and hence we may suppose $\hat{h}(k) = 0$, $|k| \leqslant m$. Take a Fejér sum p of h such that $|p(t) + 2A| < \varepsilon$, $t \in F$. We write

$$p(t) = \sum_{k<-m} \beta_k e^{ikt} + \sum_{k>m} \beta_k e^{ikt} = p^-(t) + p^+(t)$$

where

$$p^+(t) = \sum_{k>m} \beta_k e^{ikt}.$$

We have $\mathrm{Re}(p^+) = p/2 \leqslant \varepsilon$. Put now

$$g(t) = \varphi(t)\left[1 - e^{p^+(t)}\right].$$

The expansion of $[1 - e^{p^+(t)}]$ is of the form $\sum_{k>m} \gamma_k e^{ikt}$. The function g is therefore continuous of analytic type. We have

$$|g(t) - \varphi(t)| = |\varphi(t)|\,|e^{p^+(t)}| \leqslant e^{p/2} < e^{-A+\varepsilon} < 2\varepsilon \qquad (t \in F).$$

Moreover

$$|g(t)| \leqslant |\varphi(t)|\,|1 - e^{p^+(t)}| \leqslant 1 + e^{\varepsilon} < 3 \qquad (t \in T).$$
$$|g(t)| < (\varepsilon/3)3 = \varepsilon \qquad (t \notin G).$$

The Lemma is now proved.

PROOF OF THE RUDIN-CARLESON THEOREM. $\varepsilon < \frac{1}{4}$ being fixed, denote by $\gamma(\varphi)$ any continuous function of analytic type associated to φ by the Lemma. Starting with $\varphi_0 = \varphi$ we put $\varphi_{m+1} = \varphi_m - \gamma(\varphi_m)$. We have

$$\sup_F|\varphi_{m+1}| \leqslant \varepsilon \sup_F|\varphi_m| \leqslant \cdots \leqslant \varepsilon^{m+1}\sup_F|\varphi_0|,$$
$$\sup_T|\gamma(\varphi_m)| \leqslant 3\sup_F|\varphi_m| \leqslant 3\varepsilon^m\sup_F|\varphi_0|.$$

The series $\sum_{m=0}^{\infty} \gamma(\varphi_m)$ is therefore uniformally convergent on T; its sum f is of analytic type and satisfies the relation $f(t) = \varphi(t)$ $(t \in F)$. Moreover

$$\sup_T |f(t)| < 3(1-\varepsilon)^{-1} \sup |\varphi_0| < 4 \sup_F |\varphi|.$$

The theorem is now proved.

REMARK. *The factor M in the estimate* $(*)$ *can easily be reduced to* $1 + \varepsilon$. In fact, given an open set $G \supset F$ and using $(**)$ we can manage to have

$$|f(t)| < \varepsilon \qquad (t \notin G).$$

By the continuity of f, there is an open set $G' \supset F$ such that $G' \subset G$ and $|f(t)| < 1 + \varepsilon$ $(t \in G')$. Thus we can have

$$|f(t)| \geqslant 1 + \varepsilon \quad \text{only if } t \in G \setminus G'.$$

Starting with G' we get f' coinciding with φ on F, bounded by 4 where $|f'(t)| > 1 + \varepsilon$ only if $t \in G' \setminus G''$ for an appropriate $G'' \supset F$, with $G'' \subset G'$. Observing that the sets $G \setminus G'$, $G' \setminus G''$, $G'' \setminus G'''$, . . . are disjoint and taking an arithmetic mean we get a function bounded everywhere by $1 + 2\varepsilon$.

PROOF OF THE FIRST F. AND M. RIESZ THEOREM. It is sufficient to prove that

$$a_0 = (2\pi)^{-1} \int_T f(t)\, dt = 0$$

for, applying the same process to the function $e^{-it}f(t)$, we deduce $a_1 = 0$, and next $a_2 = 0, \ldots$ and finally $f = 0$. We shall follow the same pattern of proof as for the Rudin-Carleson Theorem.

Denote by S the set $\{t \in T: f(t) \neq 0\}$. Given $\varepsilon > 0$ let $e^{-A} = \varepsilon$ and let h be a bounded real function equal to $-2A$ on S and such that $\hat{h}(0) = 0$. There are such functions since $m(S^*) > 0$. Let p_n be the sequence of Fejér polynomials of h. We write as before

$$p_n(t) = \sum_{k<0} \beta_k e^{ikt} + \sum_{k>0} \beta_k e^{ikt} = p_n^-(t) + p_n^+(t)$$

where

$$p_n^+(t) = \sum_{k>0} \beta_k e^{ikt}.$$

Then, boundedly,

$$\mathrm{Re}(p_n^+(t)) = \tfrac{1}{2} p_n(t) \to \tfrac{1}{2} h(t) = -A \quad \text{a.e. on } S.$$

Put now

$$g_n(t) = f(t)\left[1 - e^{p_n^+(t)}\right].$$

The expansion of g_n is of the form $\sum_{k>0} \gamma_k e^{ikt}$ and therefore $\int g_n\, dt = 0$. Hence

$$|2\pi a_0| = \left|\int f\right| = \left|\int (f - g_n)\right| < \left|\int f e^{p_n^+}\right|$$

$$< \int_S |f| e^{p_n/2} \to e^{-A} \int |f| = \varepsilon \|f\|_1.$$

Since ε is arbitrary we have $a_0 = 0$ and the theorem is proved.

PROOF OF THE SECOND F. AND M. RIESZ THEOREM. We may assume $a_0 = 0$. Let F be a closed set of measure zero. Choose a decreasing sequence of open sets $G_n \supset F$ such that $\bigcap G_n = F$, and by the Lemma a sequence of functions g_n of analytic type, such that

$$|1 - g_n(t)| < 1/n, \qquad t \in F,$$
$$|g_n| < 3; \qquad |g_n(t)| < 1/n \quad \text{for } t \notin G_n.$$

Then, boundedly, $g_n \to \chi_F$ (characteristic function of F). Hence $0 = \int g_n \, d\mu \to \int \chi_F \, d\mu = \mu(F)$. This proves that μ is absolutely continuous.

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DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, STONY BROOK, NEW YORK 11794