XXTitle

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Dedication

Abstract

Útdráttur á ensku sem er að hámarki 250 orð.

Útdráttur

Hér kemur útdráttur á íslensku sem er að hámarki 250 orð. Reynið að koma útdráttum á eina blaðsíðu en ef tvær blaðsíður eru nauðsynlegar á seinni blaðsíða útdráttar að hefjast á oddatölusíðu (hægri síðu).

Preface

Formála má sleppa og skal þá fjarlægja þessa blaðsíðu. Formáli skal hefjast á odd-atölu blaðsíðu og nota skal Section Break (Odd Page).

Ekki birtist blaðsíðutal á þessum fyrstu síðum ritgerðarinnar en blaðsíðurnar teljast með og hafa áhrif á blaðsíðutal sem birtist með rómverskum tölum fyrst á efnisyfirliti.

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Abbreviations

Í þessum kafla mega koma fram listar yfir skammstafanir og/eða breytuheiti. Gefið kaflanum nafn við hæfi, t.d. Skammstafanir eða Breytuheiti. Þessum kafla má sleppa ef hans er ekki þörf.
The section could be titled: Glossary, Variable Names, etc.

Acknowledgments

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1. Introduction

2. Preliminaries

2.1. Measure theory

2.2. Functional analysis

2.2.1. Hahn-Banach

2.2.2. Riesz representation theorem

3. Rudin-Carleson theorem

Theorem 1. *Let E be a closed subset of $\mathbb{T}$ of Lebesgue-measure 0, let f be a continuous function on E and let T be a simply connected subset of $\mathbb{C}$ such that $f(\overline{\mathbb{D}}) \subset T$. Then there exists an $F \in \mathcal{A}$, such that $F = f$ on E and $F(\overline{\mathbb{D}}) \subset T$. (TODO skilgreina allt)*

We will break the proof into several (TODO how many?) lemmas.

Lemma 1. *Let E be a clsoed set of Lebesbue-measure 0. Then there exists an integrable function $\mu > 1$ such that μ is continuous on $\mathbb{T}\backslash E$, $\mu = +\infty$ on E, if $w \in E$ then $\mu(z) \xrightarrow{z \to w} +\infty$, and μ has a bounded derivative on any closed subarc of $\mathbb{T}\backslash E$.*

Proof. The function μ is found by solving the Dirichlet problem. (TODO Thomas Ransford p. 95 4.2.6) (TODO finish this) □

Lemma 2. *If f is a simple continuous function on E such that Re $f \geqslant 0$, then there exists an $F \in \mathcal{A}$ such that $F = f$ on E and Re $F \geqslant 0$ on $\overline{\mathbb{D}}$.*

Proof. We will show that this holds in the case where $E = E_0 \cup E_1$, $f = 0$ on E_0, $f = \alpha \neq 0$ on E_1 and Re $\alpha \geqslant 0$. This suffices since simple functions are finite linear combinations of characteristic functions. □

Proof. TODO □

Corollary 1 (Fatou). *Let E be a closed subset of $\mathbb{T}$ of Lebesgue-measure 0. There exists a function $f \in \mathcal{A}$ that vanishes on E and nowhere else.*

Proof. It's clear from the theorem (TODO add ref) that there exists a function $f \in \mathcal{A}$ that vanishes on E. TODO □

4. F. and M. Riesz theorem

Theorem 2. *All measures in $\mathcal{A}^{\perp}$ are absolutely continuous with regards to the Lebesgue-measure on $\mathbb{T}$. (TODO define all)*

Proof. TODO $\square$

5. A generalization of the Rudin-Carleson theorem

This borrows from Bishop (reference TODO).

Theorem 3 (General Rudin-Carleson theorem)**.** *Let X be a compact Hausdorff space, $V = (C(X), \|\cdot\|_\infty)$, B be a closed subspace of $C(X)$, $B^\perp$ be the annihilating measures for B, S be a closed subset of X, and f be a continues function on S. If $\int_S f d\mu = 0$ holds for all $\mu \in B^\perp$ then there exists a function $F \in B$ such that $F = f$ on S.*

Proof. Since f is continuous and S is a closed subset of a compact set, and therefore also compact, f is bounded. So we can, with out loss of generality, assume that $|f| < r < 1$ on S. Let U_r be the subset of B defined by $U_r = \{g;\ \|g\| < r\}$ and ϕ be the mapping from B to $C(S)$ that sends a member of B to its restriction on S. It suffices to show that $f \in \phi(U_r)$. Let's first show that $f \in \overline{\phi(U_r)} =: V_r$, by assuming otherwise, and showing it leads to a contradiction.

We now assume $f \notin V_r$. By Hahn-Banach (TODO ref) we can define a bounded linear functional α, such that $\alpha(f) > 1$ and $|\alpha(h)| < 1$, for $h \in V_r$. We can then define a measure μ_1 by the Riesz-representation theorem (TODO ref) that fulfills

$$\alpha(g) = \int g d\mu_1$$

for all $g \in C(S)$. We will refer to the associated functional on B by $\beta(g) = \phi(\alpha(g))$. Since $\phi(g) \in V_r$ for all $g \in U_r$ we have that

$$\beta(g) = \alpha(\phi(g)) < 1,$$

for all $g \in U_r$, due to the construction of α. From this we get

$$\begin{aligned}\|\beta\| &= \sup\{|\beta(g)|;\ |g| < 1\} \\ &= \sup\{(1/r)|\beta(g)|;\ |g| < r\} \\ &\leqslant \sup\{(1/r);\ |g| < r\} \\ &= 1/r.\end{aligned}$$

Let's denote the Riesz representation of β by μ_2, set $\mu = \mu_1 - \mu_2$ and see that $\mu \in B^\perp$. But

$$0 = \left| \int_S f d\mu \right| \geqslant \int_S f d\mu_1 - r\|\mu_2\| \geqslant \int_S f d\mu_1 - r\frac{1}{r} > 1 - r\frac{1}{r} = 0,$$

where the first equality is the assumption in the theorem. This is the contradiction that gives that $f \in V_r$. We can now take a F_1 in U_r, and therefore also in B such that $|f - F_1| < \lambda/2$ on S, with $\lambda := 1 - r$. Remember that F_1 inU_r implies that $\|F_1\| < r$. Now let $f_1 = f - F_1$ and use the same method as above to obtain an F_2 such that $\|F_2\| < \lambda/2$ and $|f - F_2| < \lambda/4$ on S. Iterating this process yields a series $(F_n)_{n\in\mathbb{N}}$ from B that fulfill $\|F_n\| < 2^{1-n}\lambda$ for $n > 1$ and

$$\left| f - \sum_{k=1}^{n} F_k \right| < 2^{-n}\lambda$$

on S for $n > 1$. We finally let

$$F = \sum_{k=1}^{\infty} F_k.$$

Now $F \in B$,

$$\|F\| \leqslant \|F_1\| + \|F - F_1\| = r + \sum_{k=2}^{\infty} 2^{1-n}\lambda = r + \lambda = 1,$$

and $F = f$ on S. (TODO bæta við matinu, svo línan að ofan meiki sens) □

Corollary 2. *Let X be a compact Hausdorff space, $V = (C(X), \|\cdot\|_\infty)$, B be a closed subspace of $C(X)$, $B^\perp$ be the annihilating measures for B, S be a closed subset of X, and f be a continues function on S. If S is $B^\perp$-null (TODO define this) then there exists a function $F \in B$ such that $F = f$ on S.*

Proof. If S is $B^\perp$-null we have that $\int_S f d\mu = 0$ for all $\mu \in B^\perp$ (TODO add ref to above). □

Remark 1. *The corollary is the version of the theorem from Bishop (TODO add ref here). Note also that if we set $X = \mathbb{T}$ and $B = \mathcal{A}$ we can use F. and M. Riesz (TODO add ref here) to prove the classical Rudin-Carleson theorem (TODO add ref here).*

It is of course worth noting an applications of (TODO REF to theorem) where the corollary fails.

Example 1. *Let $X = \mathbb{T}$, $B = \mathcal{A}$ (TODO define this), E be a closed m-null (TODO define this, an maybe change to m_σ) subset of $\partial\mathbb{T}$ that is not dense in E, $F = \{e^{i\theta};\ a \leqslant \theta \leqslant b\}$, and choose a and such that E and F are disjoint and $a \neq b$. The last assumption restricts us to E that are not dense in the $\mathbb{E}$. Since $a \neq b$ we obtain that $S := E \cup F$ does not fulfill the requirements of the Rudin-Carelson theorem (TODO add ref here) nor the above corollary (TODO add ref). Let's choose f such that $f = 0$ on F, and f is continues on S. We now have for all $\mu \in \mathcal{A}^\perp$*

$$\begin{aligned}\left|\int_S f d\mu\right| &= \left|\int_E f d\mu + \int_F f d\mu\right| \\ &\leqslant \left|\int_E f d\mu\right| + \left|\int_F f d\mu\right| \\ &= 0 + \left|\int_F f d\mu\right| \\ &= 0.\end{aligned}$$

The F. and M. Riesz theorem (TODO add ref here) tells us the since E is m-null it is also μ-null, which gives the third step. The final step stems from the fact that f vanishes on F. We now see the X, B, and f are all as in theorem (TODO add ref) so there exists a $F \in B$, such that $F = f$ on S.

A. Annað