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# A GENERALIZATION OF THE RUDIN-CARLESON THEOREM

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ABSTRACT. The purpose of this paper is to prove a common generalization of a theorem due to T. W. Gamelin [3] and a theorem due to Z. Semadeni [5]. Both these results are generalizations of E. Bishop's abstract version of the well-known Rudin-Carleson theorem [2].

In the following $X$ denotes a compact Hausdorff space, $F$ a closed subset of $X$ and $C(X)$ and $C(F)$ denote the spaces of all complex-valued functions on the topological spaces $X$ and $F$ respectively. $A$ denotes a closed linear subspace of $C(X)$ with respect to the sup norm topology, and $A|F$ denotes the set of all restrictions of the elements of $A$ to $F$. For $\tilde{f} \in A, \tilde{f}|F$ denotes the restriction of $\tilde{f}$ to $F$ and for $\mu \in M(X)$, the set of all complex Radon measures on $X$, $\mu_F$ denotes the restriction of $\mu$ to $F$. By $A^\perp$ we understand the set of all elements $\mu \in M(X)$ with the property that $\int_X \tilde{f}\, d\mu = 0$ for all $\tilde{f} \in A$.

Our purpose in this paper is to present a common generalization of the following two theorems:

THEOREM 1 (SEMADENI). *Assume that the condition,*

$$(*) \qquad \mu \in A^\perp \Rightarrow \mu_F = 0 \quad \textit{for all } \mu \in M(X),$$

*is satisfied. Let $a_0 \in C(F)$ and let $\psi: X \to (0, \infty]$ be a lower semicontinuous function such that $|a_0(x)| \leqq \psi(x)$ for all $x \in F$. Then there exists an $\tilde{a} \in A$ such that $\tilde{a}|F = a_0$ and $|\tilde{a}(x)| \leqq \psi(x)$ for all $x \in X$.*

THEOREM 2 (GAMELIN). *Assume that the condition,*

$$(**) \qquad \mu \in A^\perp \Rightarrow \mu_F \in A^\perp \quad \textit{for all } \mu \in M(X),$$

*is satisfied. Let $a_0 \in A|F$ and let $p: X \to (0, \infty)$ be a continuous function such that $|a_0(x)| \leqq p(x)$ for all $x \in F$. Then there exists an $\tilde{a} \in A$ such that $\tilde{a}|F = a_0$ and $|\tilde{a}(x)| \leqq p(x)$ for all $x \in X$.*

Our theorem is the following:

THEOREM 3. *Assume that condition $(**)$ is satisfied. Let $a_0 \in A|F$ and let $\psi: X \to (0, \infty]$ be a lower semicontinuous function such that $|a_0(x)| \leqq \psi(x)$*

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*for all* $x \in F$. *Then there exists an* $\tilde{a} \in A$ *such that* $\tilde{a}|F = a_0$ *and* $|\tilde{a}(x)| \leqq \psi(x)$ *for all* $x \in X$.

Observe that condition (**) is weaker than condition (*). Observe also that the conclusion in Theorem 1 is a stronger one than the conclusion in Theorem 2. (The fact that any $a_0 \in C(F)$ can be extended to an element in $A$ in Theorem 1 follows immediately from the well-known fact that condition (*) implies that $F$ is an interpolation set for $A$.) Hence our theorem, dealing with the weaker condition and the stronger conclusion, generalizes both these theorems.

To prove Theorem 3 we need the following lemma:

LEMMA. *Assume that condition* (**) *is satisfied. Let* $a_0 \in A|F$ *and let* $\phi: X \to (0, \infty]$ *be lower semicontinuous and such that* $|a_0(x)| \leqq \phi(x)$ *for all* $x \in F$. *Then for each* $\varepsilon > 0$ *there exists an* $\tilde{a}_\varepsilon \in A$ *such that* $\tilde{a}_\varepsilon|F = a_0$ *and* $|\tilde{a}_\varepsilon(x)| \leqq \phi(x)(1+\varepsilon)$ *for all* $x \in X$.

PROOF OF LEMMA. Here and in the proof of Theorem 3 we can assume without loss of generality that $\phi$ is bounded. [If this is not the case, we introduce the function $\phi_0 = \phi \wedge (|\tilde{a}| \vee \min \phi)$ instead of $\phi$, where $\tilde{a}$ is an arbitrary extension of $a_0$ in $A$.]

Next we observe that $\phi$, being lower semicontinuous and strictly positive, attains a minimum $m > 0$. Choose $\varepsilon > 0$ and define $\varepsilon' = m \cdot \varepsilon$.

We claim that there exists a continuous function $p: X \to (0, \infty)$ such that

$$|a_0(x)| < p(x) \leqq \phi(x) + \varepsilon' \quad \text{for all } x \in F$$

and such that

$$p(x) \leqq \phi(x) + \varepsilon' \quad \text{for all } x \in X.$$

To prove this claim we use the fact that there exists a monotone increasing sequence $\{f_n\}_{n=1}^\infty$ of continuous real-valued functions on $X$ such that

$$\lim_{n \to \infty} f_n(x) = \phi(x) + \varepsilon' \quad \text{for all } x \in X.$$

We introduce the sets $K_n = \{x \in F; f_n(x) \leqq |a_0(x)|\}$ for all $n$. We observe that $\{K_n\}_{n=1}^\infty$ is a monotone decreasing sequence of compact subsets of $F$ with $\bigcap_{n=1}^\infty K_n = \varnothing$. Hence there exists an $n = n_1$ such that $K_{n_1} = \varnothing$. This implies that $f_{n_1}(x) > |a_0(x)|$ for all $x \in F$. Obviously $f_{n_1}(x) \leqq \phi(x) + \varepsilon'$ for all $x \in X$. By a similar argument it follows that there exists an $n = n_2$ such that $f_{n_2}(x) > 0$ for all $x \in X$. Let $n_0 = \max(n_1, n_2)$. Now choose $p = f_{n_0}$.

To complete the proof of the lemma we now apply Theorem 2 and conclude that there exists an $\tilde{a}_\varepsilon \in A$ such that $\tilde{a}_\varepsilon|F = a_0$ and $|\tilde{a}_\varepsilon(x)| \leqq p(x)$ for all $x \in X$. But this implies that

$$|\tilde{a}_\varepsilon(x)| \leqq \phi(x) + \varepsilon' \leqq \phi(x) + \varepsilon \cdot \phi(x) \quad \text{for all } x \in X. \quad \square$$

PROOF OF THEOREM 3. Choose $\varepsilon=\frac{1}{4}$ in the lemma. Then we know that there exists a function $\tilde{g}_1 \in A$ such that $\tilde{g}_1|F=a_0$ and $|\tilde{g}_1(x)|\leqq 5\psi(x)/4$ for all $x \in X$.

Assume, as induction hypothesis, the existence of the functions $\tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_{n-1} \in A$ with $\tilde{g}_j|F=a_0$ and such that

$$|\tilde{g}_j(x)| \leqq \psi(x)(1 + 1/2^{j+1}) \quad \text{for all } x \in X$$

and

$$|\tilde{g}_j(x)| \leqq \tfrac{1}{2}\psi(x) \quad \text{for all } x \in X\setminus U_j,$$

where $U_1=X$ and

$$U_j = \{x \in X; |\tilde{g}_k(x)| < \psi(x)(1 + 1/2^{j+1}), 1 \leqq k \leqq j-1\}$$

for $j \in \{2, 3, \cdots, n-1\}$.

We next define the set

$$U_n = \{x \in X; |\tilde{g}_k(x)| < \psi(x)(1 + 1/2^{n+1}), 1 \leqq k \leqq n-1\}.$$

Since any function of the form $c\psi-|\tilde{g}_j|$, where $c$ is a positive constant, is lower semicontinuous, it follows that the sets $U_j$, $j=1, 2, \cdots, n$, are all open. Furthermore, $F\subseteq U_j$ for each $j$.

By Tietze's theorem there exists a continuous function $h_n: X\to \boldsymbol{R}$ such that

$$2^{-1} \cdot (1 + 1/2^{n+1})^{-1} \leqq h_n(x) \leqq 1 \quad \text{for all } x \in X$$

and $h_n(x)=1$ for $x \in F$ and $h_n(x)=2^{-1}(1+1/2^{n+1})^{-1}$ for $x \in X\setminus U_n$. The function $\psi_n=h_n \cdot \psi$ is therefore strictly positive and lower semicontinuous and such that $|a_0(x)|\leqq\psi_n(x)$ for all $x \in F$. Using the lemma, we know that there exists a $\tilde{g}_n \in A$ such that

$$\tilde{g}_n|F = a_0 \quad \text{and} \quad \tilde{g}_n(x) \leqq \psi_n(x)(1 + 1/2^{n+1}) \quad \text{for all } x \in X.$$

From this follows

$$|\tilde{g}_n(x)| \leqq \psi(x)(1 + 1/2^{n+1}) \quad \text{for all } x \in X$$

and

$$|\tilde{g}_n(x)| \leqq \tfrac{1}{2}\psi(x) \quad \text{for all } x \in X\setminus U_n.$$

We now define:

$$\tilde{a}(x) = \sum_{n=1}^{\infty} \frac{1}{2^n}\tilde{g}_n(x) \quad \text{for all } x \in X.$$

Since $\psi$ is bounded and $A$ is a Banach space, it follows that the Cauchy sequence $\tilde{s}_N=\sum_{n=1}^{N} 2^{-n}\tilde{g}_n$ converges in sup norm. Hence $\tilde{a} \in A$. Furthermore $\tilde{a}|F=a_0$ since $\tilde{g}_n|F=a_0$ for all $n$.

It remains to prove that $|\tilde{a}(x)| \leqq \psi(x)$ for all $x \in X$. Assume first that $x \in U_n$ but $x \notin U_{n+1}$, $n \in \{1, 2, \cdots\}$. Then we have

$$|\tilde{g}_j(x)| < \psi(x)(1 + 1/2^{n+1}) \quad \text{for } 1 \leqq j \leqq n$$

and

$$|\tilde{g}_j(x)| \leqq \tfrac{1}{2}\psi(x) \quad \text{for } n < j.$$

(Observe that $U_{k+1} \subseteq U_k$ for all $k \in \{1, 2, \cdots\}$.) From this follows:

$$|\tilde{a}(x)| \leqq \psi(x)\left(1 + \frac{1}{2^{n+1}}\right) \sum_{j=1}^{n} \frac{1}{2^j} + \frac{\psi(x)}{2} \sum_{j=n+1}^{\infty} \frac{1}{2^j} = \left(1 - \frac{1}{2^{2n+1}}\right) \psi(x) < \psi(x).$$

If $x \in U_n$ for all $n$, we have $|\tilde{g}_n(x)| \leqq \psi(x)$ for all $n$ and therefore $|\tilde{a}(x)| \leqq \psi(x)$. $\square$

Remark. The proof of the lemma is based on an idea communicated to me by Professor B. A. Taylor of the University of Michigan. In **[4]** a more cumbersome proof of this lemma is given. This latter proof is based on an idea due to Semadeni **[5]**, used in his proof of Theorem 1. The same proof is also in Alfsen-Hirsberg **[1]**.

The proof of Theorem 3 from the lemma is a modification of the method used by Gamelin **[3]** in his proof of Theorem 2.

## References

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