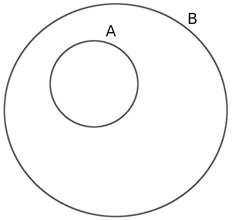
Rudin-Carleson theorems

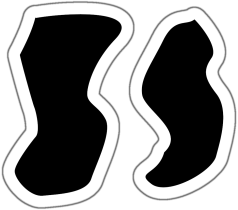
Bergur Snorrason

3. júní 2020

- The focus of the thesis is the Rudin-Carleson theorem and its variations.
- These theorems are what we call extension theorems, that is, they tell us when we can extend a function to a larger set while maintaining some properties.
- Concretely, if $f\colon A \to X$, $g\colon B \to X$, $A \subset B$ and $f = g$ on A, then we say g *extends* f.
- We will use the following to denote common sets in $\mathbb{C}$.
  - $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$
  - $\overline{\mathbb{D}} = \{z \in \mathbb{C} \colon |z| \leqslant 1\}$
  - $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$



- We will look at two famous examples of extension theorems before looking at the Rudin-Carleson theorem
- The first one is Tietze's extension theorem, from topology.
- We say a topological space is *normal* if all disjoint closed sets can be separated by open neighbourhoods and if the singletons are closed.



- All metric spaces are normal and so are compact Hausdorff spaces.

Theorem (Tietze)

Let X be a normal space and A be a closed subset of X. For any continuous function $f\colon A \to \mathbb{R}$ there exists a $g\colon X \to \mathbb{R}$ that extends it.

Theorem (Tietze)

Let X be a normal space and A be a closed subset of X. For any continuous function $f\colon A \to [a, b]$ there exists a $g\colon X \to [a, b]$ that extends it.

- Recall that a function defined on an open subset of $\mathbb{C}$ is *harmonic* if
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

- We define the *Poisson kernel* on $\mathbb{D}$ by
$$P(z, e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2}.$$

- If f is an integrable function defined on $\mathbb{T}$ then we define its *Poisson integral* by
$$P[f](z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(z, e^{it}) f(e^{it}) \, dt.$$

- Given a continuous function f on the unit circle $\mathbb{T}$ can you find a continuous function u on $\overline{\mathbb{D}}$ that's harmonic on $\mathbb{D}$ and agrees with f on $\mathbb{T}$.
- This is the famous Dirichlet problem on the unit disk.
- We can define such a u by

$$u(z) = \begin{cases} f(z), & |z| = 1 \\ P[f](z), & |z| < 1 \end{cases}$$

  to solve this problem.
- In other words, we can use the Poisson kernel to extend f harmonically into $\mathbb{D}$.

- The Dirichlet problem is about extending harmonically, but can we extend holomorphically?
- The set of all continuous functions on the closed unit disk $\overline{\mathbb{D}}$ which are holomorphic on the open unit disk $\mathbb{D}$ is called the *disk algebra* and we denote it by $\mathcal{A}$.
- A theorem from the study of Hardy spaces tells us that if a f is a function in $\mathcal{A}$ and

$$m(\{z \in \mathbb{T} \colon f(z) = 0\}) > 0,$$

  where m is the arc length measure on $\mathbb{T}$, then $f = 0$.
- So not all continuous functions on the unit circle $\mathbb{T}$ can be extended to $\mathcal{A}$.
- We can always extend, however, if we limit ourselves to a sufficiently small subset of $\mathbb{T}$.

Theorem (Rudin-Carleson (1956-1957))

Let E be a closed subset of $\mathbb{T}$ such that $m(E) = 0$, $f : E \to \mathbb{C}$ be continuous, and T be a subset of $\mathbb{C}$ homeomorphic to $\overline{\mathbb{D}}$ such that $f(E) \subset T$. There exists a function g in $\mathcal{A}$ that extends f and $g(\overline{\mathbb{D}}) \subset T$.

- This is proved twice in the thesis, first in the same manner Rudin did originally and then as consequence of Bishop's theorem and the F. and M. Riesz theorem.
- Both of these proofs, at some point, use the Poisson integral solution of the Dirichlet problem mentioned earlier.
- To discuss Bishop's theorem we first have to find a way to classify a set as sufficiently small (like E in the Rudin-Carlson theorem) with regards to the family of functions we want to extend into.

- Let B be a family of measurable functions defined on X.
- We say a measure μ is an *annihilating measure of* B if

$$\int_X g \, d\mu = 0$$

  for all g in B.
- We denote by $B^{\perp}$ the family of all the annihilating measures of B.
- A set E is said to be $B^{\perp}$-null if it is μ-null for all μ in $B^{\perp}$.
- We can intuitively think of it like this: Increasing the size of B, means you have fewer annihilating measures, which leads to more $B^{\perp}$-null sets.

Theorem (Bishop (1962))

Let

1. X *be a compact Hausdorff space,*
2. B *be a closed subspace of* $(C(X), \|\cdot\|_\infty)$*,*
3. S *be a closed subset of* X *that is* $B^\perp$*-null,*
4. f *be a continuous function on* S*,*
5. $\Psi : X \to [0, +\infty[$ *be a continuous function such that* $|f| < \Psi$ *on* S*.*

Then there exists a function $F \in B$ *that extends* f *and* $|F| < \Psi$ *on* X*.*

- To show that this is a generalized version of the Rudin-Carleson theorem we need a connection between the annihilating measures of $\mathcal{A}$ and the arc length measure on $\mathbb{T}$.
- That is, if E is a closed subset of $\mathbb{T}$ that is m-null and μ is an annihilating measure of $\mathcal{A}$ then we need to show that E is also μ-null.

Theorem (F. and M. Riesz (1916))

Let μ be a measure on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} e^{-int}\, d\mu(t) = 0$$

holds for $n = -1, -2, \ldots$. Then $\mu \ll m$. That is, if $E \subset \mathbb{T}$ is m-null then E is also μ-null.

- Let's look at a few examples of how we can use Bishop's theorem.
- Let $X = \mathbb{T}$, $B = \mathcal{A}$ and E be a closed subset of $\mathbb{T}$ such that $m(E) = 0$.
- If μ is in $B^{\perp}$ then it satisfies the condition of the F. and M. Riesz theorem so E is μ-null.
- So E is $B^{\perp}$-null.
- Bishop's theorem then tells us that all continuous functions on E can be extended with a function in $\mathcal{A}$.

- Let $B = C(X)$.
- If μ is in $B^{\perp}$ then, as a consequence of the Riesz representation theorem, $\mu = 0$.
- That is $\mu = 0$ is the only annihilating measure of B.
- So all subsets of X are $B^{\perp}$-null.
- This result agrees with Tietze's extension theorem.

- Let B include only the zero function.
- Then every measure is an annihilating measure of B.
- Subsequently, no subset of X is $B^{\perp}$-null, and Bishop's theorem gives us nothing.
- This doesn't mean that no function can be extended, we can extend the zero function defined on any subset of X.
- This is a motivating idea behind the alternative version of Bishop's theorem.

- Let's look at a sketch of the proof of Bishop's theorem.
- The first step of the proof is to show that f is in the closure of the image of the restriction mapping $G \mapsto G|_S$.
- To do this we assume that it doesn't hold.
- We construct a measure μ in $B^\perp$ by Hahn-Banach and the Riesz representation theorem and show that

$$0 = \left| \int_S f \, d\mu \right| > 0.$$

- The second step of the proof is finding a sequence of function in B with limit f.
- This is done by applying the first step on

$$f - \sum_{k=0}^{n} F_k$$

where $(F_n)_{n \in \mathbb{N}}$ is the sequence in B we want to find.

- To summarize:
- We need
$$\int_S f \, d\mu = 0$$
to hold generally for μ in $B^\perp$ and $f - G$ has to satisfy the condition of the theorem, for all G in B, to allow us to inductively create our sequence.
- If we chose these condition to characterize S we get the alternative version of Bishop's theorem.

Theorem

Let X and B be as in Bishop's theorem, S be a closed subset of X and f be a continuous function on S. If

$$\int_S f \, d\mu = 0$$

holds for all μ in $B^\perp$ and

$$\int_S G \, d\mu = 0$$

holds for all μ in $B^\perp$ and all G in B then there exists a function F in B that extends f.

- The difference between these two theorems is that the first one states that all continuous function on S may be extended while this alternative version fixes a continuous function and gives a condition on S dependent on f.

Acknowledgments

- Benedikt Steinar Magnússon
- Ragnar Sigurðsson
- Alti, Eyleifur, Garðar, Hjörtur, Sandra, and Þórarinn.