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Theorem 0.1. *Let A denote the uniform algebra of continuous functions on $\overline{\mathbb{B}}$ which are holomorphic on $\mathbb{B}$. Assume $E = E_1 \cup E_2$ is a disjoint union of a interpolation set E_1 and a set E_2 such that for every $\mu \in A^{\perp}$ and every $g \in A$*

$$\int_{E_2} g \, d\mu = 0.$$

Then for $f \in C(E)$ satisfying

$$\int_{E_2} f \, d\mu = 0 \qquad \text{for every } \mu \in A^{\perp}. \tag{1}$$

there exists $F \in A$ such that $F|_E = f$.

Proposition 0.2. *If $E_2 = \mathbb{S} \cap M$ is the intersection of the unit sphere and an analytic subset [1, Chapter I.8] M of $\{|z| < r\}$, $r > 1$ then E_2 satisfies (1).*

Proof. First note that since $\{|z| < r\}$ is a domain of holomorphy the set M can be defined by global functions. I.e. there are $f_1, \ldots, f_m \in \mathcal{O}(\{|z| < r\})$ such that

$$M = \{z; f_1(z) = f_2(z) = \cdots = f_m(z) = 0\},$$

and since $r > 1$ the f_j's are in A. Fix $\mu \in A^{\perp}$ and define $\mu_j = f_j \mu$ for $j = 1, \ldots, m$. Note that μ_j is in $A^{\perp}$ since for $g \in A$,

$$\int g \, d\mu_j = \int g \cdot f_j \, d\mu = 0$$

because $g \cdot f_j \in A$. Define $\mu_M = \mu - \sum_{j=1}^{m} \mu_j$ and note firstly that $\mu_M \in A^{\perp}$, since μ and μ_j are in $A^\perp$, and secondly that $\mu_M(E_2) = 0$.

$$
\begin{aligned}
&\int_{E_2} g\, d\mu \\
=&\int_{E_2} g - g\left(\sum_{j=1}^{m} f_j\right) d\mu && \text{(since the } f_j\text{'s are 0 on } E_2) \\
=&\int_{\mathbb{S}} g - g\left(\sum_{j=1}^{m} f_j\right) d\mu - \int_{\mathbb{S}\setminus E_2} g - g\left(\sum_{j=1}^{m} f_j\right) d\mu \\
=&-\int_{\mathbb{S}\setminus E_2} g - g\left(\sum_{j=1}^{m} f_j\right) d\mu && \left(\text{since } g - g\sum f_j \in A\right) \\
=&-\int_{\mathbb{S}\setminus E_2} g\left(1 - \sum_{j=1}^{m} f_j\right) d\mu \\
=&-\int_{\mathbb{S}\setminus E_2} g\, d\mu_M \\
=&-\int_{\mathbb{S}} g\, d\mu_M && (\text{since } \mu_M(E_2) = 0) \\
=&0 && (\text{since } \mu_M \in A^\perp).
\end{aligned}
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□

References

[1] K. Fritzsche and H. Grauert, *From holomorphic functions to complex manifolds*, vol. 213 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2002.