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Representations of continuous functions

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1. The problem which will be treated in this paper, can quite generally be stated as follows. Let C be a linear class of continuous functions defined on a set S. Let E be a closed subset of S. Under which conditions on E is *every* continuous function on E the restriction to E of some function belonging to the original class C? We shall in section 3 solve this problem in the case when S is the circle $|z|=1$ and C is the class of uniformly continuous analytic functions in $|z|<1$ and in section 4 we shall consider FOURIER-STIELTJES transforms. First, however, we shall in section 2 prove a lemma on BANACH spaces to which our theorems can be reduced.

2. Lemma 1. *Let B be a separable* BANACH *space with norm $\|x\|$. Let $B_1 \subset B$ be a* BANACH *space with the norm $\|x\|_1 \geq \|x\|$. If there exists a sequence of elements x_ν such that for every linear functional $L(x)$ on B*

$$\|L\| \leq M \sup_\nu |L(x_\nu)|, \qquad x_\nu \in B_1, \qquad \|x_\nu\|_1 \leq 1, \tag{1}$$

then

$$B = B_1 \quad \textit{and} \quad \|x\|_1 \leq M \|x\|, \qquad x \in B_1. \tag{2}$$

It is obvious that the constant M in (2) cannot in general be improved.

For the proof, we observe that (1) implies that B_1 is dense in B. Since B is separable (see [1], p. 124), there exists a sequence of linear functionals $\{L_\mu\}_1^\infty$ on B, $\|L_\mu\|=1$, such that every linear functional is the weak limit of linear combinations $\sum c_\mu L_\mu$ *).

We now consider the infinite system of equations

$$L_\mu(x) = \sum_1^\infty \xi_\nu L_\mu(x_\nu), \qquad \sum_1^\infty |\xi_\nu| \leq M', \tag{3}$$

for a fixed element $x \in B$, $\|x\|=1$. By a theorem of F. RIESZ, (3) has a solution if for every sequence $\{c_\mu\}$

$$\left|\sum c_\mu L_\mu(x)\right| \leq M' \sup_\nu \frac{1}{A_\nu} \left|\sum c_\mu L_\mu(x_\nu)\right|, \tag{4}$$

where $A_\nu \geq 1$ and $\lim A_\nu = \infty$. We shall prove that (4) holds provided that $M' > M$. The opposite assumption implies that for every $n \geq 1$ there exists a linear functional L_n^* such that

$$1 = |L_n^*(x)| \geq M' |L_n^*(x_\nu)|, \qquad \nu = 1, 2, \ldots, n, \tag{5}$$

$$|L_n^*(x_\nu)| < \text{const}, \qquad 1 \leq \nu < \infty. \tag{6}$$

*) For the following, compare [2], p. 343.

(6) together with (1) implies that $\|L_n^*\|$ is uniformly bounded. Suppose that $L_{n_i}^*$ converges weakly and let $L(x)$ be the limit. Then, by (5),

$$\|L\| \geq |L(x)| \geq M' |L(x_\nu)|, \qquad 1 \leq \nu < \infty,$$

which implies

$$\|L\| \geq \frac{M'}{M} \|L\|.$$

Hence $L \equiv 0$ which contradicts $L(x) = 1$.

We have thus proved that (3) has a solution if $M' > M$. For such a solution $\{\xi_\nu\}$ we have

$$L_\mu\left(x - \sum_1^\infty \xi_\nu x_\nu\right) = 0, \qquad \mu = 1, 2, \ldots,$$

and, since L_μ is weakly fundamental, $x = \sum\limits_1^\infty \xi_\nu x_\nu$. Hence

$$\|x\|_1 \leq \sum_1^\infty |\xi_\nu| \, \|x_\nu\|_1 \leq \sum_1^\infty |\xi_\nu| \leq M',$$

and the proof is complete.

Let us finally note the following well-known converse of the lemma (see [6], p. 30):

If in the lemma we replace (1) *by the assumption* $B = B_1$, *then* (2) *holds for some constant* M.

3. Let C denote the class of functions $f(z)$ analytic and uniformly continuous in $|z| < 1$. Under the norm $\sup\limits_{|z|<1} |f(z)|$, C is a separable BANACH space. An example of a basis is given by the powers z^n. If E is a closed set on $|z| = 1$, then there exists a non-trivial function $f(z) \in C$, which vanishes on E, if and only if $mE = 0$ (compare below, lemma 2). We shall now prove that this is also the solution of the problem, stated in the introduction.

Theorem 1. *Let E be a closed set on $|z| = 1$ of measure zero and let $\varphi(z)$ be continuous on E. Then there exists a function $f \in C$ with $f(z) = \varphi(z)$ on E. If $mE > 0$, this is not true for every choice of $\varphi(z)$.*

The last statement of the theorem is trivial. If $mE > 0$, we choose $\varphi_1(z)$ and $\varphi_2(z)$ so that $\varphi_1(z) = \varphi_2(z)$ on a set of positive measure but not everywhere. If $f_1(z)$ and $f_2(z)$, corresponding to φ_1 and φ_2 existed, $f_1 - f_2$ would vanish on a set of positive measure and hence everywhere, contradicting our assumption $\varphi_1 \not\equiv \varphi_2$.

In the proof, we shall use the following lemma.

Lemma 2. *Let E_1 and E_2 be two disjoint closed sets on $|z| = 1$ of measure zero. Then there exists a function $f(z) \in C$ with the properties*

(7) $$f(z) = 1, \qquad z \in E_1,$$

(8) $$f(z) = 0, \qquad z \in E_2,$$

(9) $$|f(z)| \leq 2, \qquad |z| \leq 1.$$

Let $h(\vartheta)$ be a continuous function with period 2π satisfying the following conditions: (a) $0 \leq h(\vartheta) \leq 1$; (b) $h(\vartheta) = 0$, $e^{i\vartheta} \in E_1$; (c) $h(\vartheta) = 1$, $e^{i\vartheta} \in E_2$; (d) $h''(\vartheta)$ is continuous; (e) $\int_{-\pi}^{\pi} \log h(\vartheta)\, d\vartheta > -\infty$. It is obvious that such a function exists. If then

$$\log g(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\vartheta} + z}{e^{i\vartheta} - z} \log h(\vartheta)\, d\vartheta, \tag{10}$$

it follows that $|g(e^{i\vartheta})| = h(\vartheta)$ and that g belongs to C. This is a consequence of (b) and (d). Furthermore, by (b), $g(z) = 0$ on E_1, by (d) $|g(z)| = 1$ on E_2 and by (a) $|g(z)| < 1$, $|z| < 1$. Under the mapping $w = g(z)$, the set E_2 is mapped on a set E_2' of measure zero, since $g'(z)$, by the regularity (d) of $h(\vartheta)$, is continuous on E_2. In the same way as above we construct a function $q(z) \in C$ such that $q(0) = 1$, $q(z) = 0$ on E_2' and $|q(z)| \leq 2$, $|z| \leq 1$. The function $q(g(z)) = f(z)$ belongs to C and has the desired properties (7), (8), (9).

Let E be the given closed set of measure zero and divide E in all possible ways into two disjoint closed subsets E_1 and E_2. As is easily seen, this is possible only in a countable number of different ways. For each choice, we construct by lemma 2 a function $f(z)$. We obtain a sequence of elements $\{f_\nu(z)\}_1^\infty$ of C, $|f_\nu| \leq 2$.

In order to be able to apply lemma 1, we introduce as BANACH space B the set of continuous functions φ on E under uniform norm $\|\varphi\|$ and as B_1 the restriction to E of functions $f(z) \in C$ with norm

$$\|\varphi\|_1 = \inf_f \sup_z |f(z)|, \qquad f(z) = \varphi(z) \text{ on } E$$

It is obvious that B_1 is complete under this norm. If $\|\varphi\|_1 \leq M\|\varphi\|$, theorem 1 follows if we observe that the set of polynomials is dense in B.

In lemma 1 we now choose $x_\nu = \frac{1}{2} f_\nu(z)$ and want to prove that for every μ of bounded variation on E

$$m \int_E |d\mu| \leq \sup_\nu \left| \int_E f_\nu(z)\, d\mu(z) \right| \tag{11}$$

with $m > 0$ independant of μ. Since f_ν is real on E we may assume that μ is real and we also assume that the right hand side of (11) $= 1$. Let $A \subset E$ be a BOREL measurable set. Then there is a sequence $\{f_{\nu_i}(z)\}$ such that

$$\lim_{i=\infty} f_{\nu_i}(z) = \begin{cases} 1 & \text{on } A \\ 0 & \text{on } E - A \end{cases}$$

except on sets of variation zero for μ. Hence

$$|\mu(A)| = \left| \lim_{i=\infty} \int_E f_{\nu_i}(z)\, d\mu(z) \right| \leq 1$$

and since ([7], p. 32)

$$\int_E |d\mu| = \sup_{A \subset E} \mu(A) - \inf_{A \subset E} \mu(A) \leq 2,$$

the proof is complete.

It is interesting to observe that theorem 1 contains the following well-known theorem of F. and M. RIESZ:

If

$$\int_{|z|=1} z^n \, d\mu(z) = 0, \qquad n \geq 0,$$

then μ is absolutely continuous.

The proof obtained in this way can be considered as more "natural" than the classical proofs, since it is a direct proof of absolute continuity. Compare also [5].

Let E be a closed set of measure zero and assume $m = \int_E |d\mu| > 0$. By the remark at the end of section 2 and theorem 1, every continuous $\varphi(z)$, $|\varphi| \leq 1$, on E can be represented by $f \in C$, $|f| \leq M$. Let $O \supset E$ be an open set such that $\int_{O-E} |d\mu| < \varepsilon$ and construct by the formula (10) $g(z) \in C$ so that $|g(z)| = 1$ on E; $|g(z)| \leq 1$, $|z| \leq 1$; $|g(z)| < \varepsilon$, $|z| = 1$ and z outside O. Then

$$\int_{|z|=1} z^n g(z) \, d\mu(z) = 0$$

and

$$m = \sup_{\|\varphi\|=1} \int_E \varphi(z) \, g(z) \, d\mu(z) \leq M \int_{O-E} |d\mu| + M \varepsilon \int_{|z|=1} |d\mu| \leq \tfrac{1}{2} m$$

for ε sufficiently small. This contradiction proves the theorem.

4. We now assume that E is a compact set on $(-\infty, \infty)$ and we consider the problem of representing continuous function $\varphi(x)$ on E by FOURIER-STIELTJES transforms

$$\varphi(x) = \int_{-\infty}^{\infty} e^{ixt} \, d\sigma(t), \qquad x \in E. \tag{12}$$

Lemma 1 here gives the following criterion on E.

Theorem 2. *A necessary and sufficient condition on E that every continuous $\varphi(x)$ has a representation* (12) *is that*

$$\sup_t \left| \int_E e^{ixt} \, d\mu(x) \right| \geq m \int_E |d\mu|, \tag{13}$$

where $m > 0$ is independent of μ.

Let B_1 be the set of functions of the form (12) and

$$\|\varphi\|_1 = \inf_\sigma \int_{-\infty}^{\infty} |d\sigma| \qquad \text{for such } \sigma.$$

If (12) holds for every φ, then $\|\varphi\|_1 \leq M \|\varphi\|$, and

$$\int_E |d\mu| = \sup_{\|\varphi\|=1} \int_E \varphi \, d\mu \leq \sup_\sigma \int_E d\mu(x) \int_{-\infty}^{\infty} e^{ixt} \, d\sigma(t), \qquad \int_{-\infty}^{\infty} |d\sigma| \leq M,$$

whence

$$\int_E |d\mu| \leq \sup_t \left| \int_E e^{ixt} \, d\mu(x) \right| \int_{-\infty}^{\infty} |d\sigma|,$$

and (13) holds with $m = \frac{1}{M}$.

To prove the converse, choose in lemma 1 $x_\nu = e^{i r_\nu x}$, where r_ν is dense on $(-\infty, \infty)$. Assumptions (1) and (13) are identical, and $B = B_1$ is the desired result.

Finally we note that we may, if (13) holds, choose σ in (12) in very special ways.

Theorem 3. *If every $\varphi(x)$ has a representation* (12) *and E is a subset of $(-\pi, \pi)$, then $\varphi(x)$ can also be represented in the form*

$$\varphi(x) = \int_{-\infty}^{\infty} e^{ixt} f(t)\, dt, \qquad f \text{ summable}, \tag{14}$$

or

$$\varphi(x) = \sum_{-\infty}^{\infty} a_n e^{inx}, \qquad \sum_{-\infty}^{\infty} |a_n| < \infty. \tag{15}$$

Consider the subspace B_1 of functions of the form (14) under the norm

$$\|\varphi\|_1 = \inf_f \int_{-\infty}^{\infty} |f(t)|\, dt.$$

We choose in lemma 1 as the set $\{x_\nu\}$ the functions

$$\varphi_{\nu k} = e^{i r_\nu x} \cdot \frac{k}{x} \cdot \sin\frac{x}{k}, \qquad \nu = 1, 2, \ldots;\ k = 1, 2, \ldots,$$

where, as before, $\{r_\nu\}$ denotes a dense set on $(-\infty, \infty)$. Since (13) holds, by assumption, it follows that

$$\sup_{\nu, k} \left| \int_E \varphi_{\nu k}(x)\, d\mu(x) \right| \geq m \int_E |d\mu|,$$

whence, by lemma 1, $B = B_1$.—The result can naturally easily be proved directly.

To prove (15), we introduce in an obvious way $\|\varphi\|_1$ and assume that there exists a function μ such that

$$\int_E |d\mu| = 1 \qquad \text{while} \qquad \left| \int_E e^{inx} d\mu(x) \right| \leq \varepsilon, \qquad n = 0, \pm 1, \ldots.$$

By a theorem of M. Cartwright [3], there is a constant C, depending only on E, such that

$$\left| \int_E e^{itx} d\mu(x) \right| \leq C\varepsilon, \qquad -\infty < t < \infty,$$

which is impossible if $\varepsilon < \frac{m}{C}$. We can thus again apply lemma 1 and the proof is complete.

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