A GENERAL RUDIN-CARLESON THEOREM

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1. Introduction. Rudin [7] and Carleson [4] have independently proved that if S is a closed set of Lebesgue measure 0 on the unit circle

$$L = \{z: |z| = 1\}$$

and if f is a continuous function on L then there exists a continuous function F on

$$D = \{z: |z| \leqq 1\}$$

which is analytic on $D-L$ such that $F(z)=f(z)$ for all z in S. It is the purpose of this paper to generalize this theorem. Before stating the generalization, we remark that the Rudin-Carleson theorem is closely related to a theorem of F. and M. Riesz, which states that any (finite, complex-valued, Baire) measure on L which is orthogonal to all continuous functions F on D which are analytic on $D-L$ is absolutely continuous with respect to Lebesgue measure $d\theta$ on L. The proofs of the two theorems show that the results of Rudin-Carleson and of F. and M. Riesz are closely related. We shall state an abstract theorem which shows that the Rudin-Carleson theorem is a direct consequence of the F. and M. Riesz theorem. This abstract theorem will permit an automatic generalization of the Rudin-Carleson result to any situation to which the F. and M. Riesz result can be generalized. The theorem to be proved reads as follows:

THEOREM 1. *Let $C(X)$ be the uniformly-normed Banach space of all continuous complex-valued functions on a compact Hausdorff space X. Let B be a closed subspace of $C(X)$. Let $B^{\perp}$ consist of all (finite, complex-valued, Baire) measures μ on X such that $\int f d\mu=0$ for all f in B. Let $\hat{\mu}$ be the regular Borel extension of the Baire measure μ. Let S be a closed subset of X with the property that $\hat{\mu}(T)=0$ for every Borel subset T of S and every μ in $B^{\perp}$. Let f be a continuous complex-valued function on S and Δ a positive function on X such that $|f(x)|<\Delta(x)$ for all x in S. Then there exists F in B with $|F(x)|<\Delta(x)$ for all x in X and $F(x)=f(x)$ for all x in S.*

If X is taken to be the set L defined above and B is taken to be those functions in $C(L)$ which are restrictions to L of functions in

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$C(D)$ analytic on $D-L$, then by the F. and M. Riesz result any closed subset S of L of Lebesgue measure 0 satisfies the hypotheses of Theorem 1.

Therefore from Theorem 1 the Rudin-Carleson theorem follows.

By a general F. and M. Riesz theorem we mean a theorem to the effect that for certain closed subspaces B of certain $C(X)$ there exists a non-negative measure μ_0 on X with respect to which all measures in $B^{\perp}$ are absolutely continuous. From Theorem 1 it follows that to each such general F. and M. Riesz theorem corresponds a general Rudin-Carleson theorem, which states that if S is a closed subset of X with $\bar{\mu}_0(S)=0$ then every continuous function on S is the restriction to S of some function in B.

There are various general F. and M. Riesz theorems which exist in the literature. We mention three of these.

(1) Bochner [**3**] and Helson and Lowdenslager [**5**] have proved an F. and M. Riesz theorem for $X=L\times L$, where B is the subspace of $L\times L$ generated by all functions $z^m w^n$ with (m, n) belonging to a sector of lattice points of opening greater than π. Here μ_0 is Lebesgue measure on $L\times L$.

(2) Bishop [**1**; **2**] has proved a general F. and M. Riesz theorem for the boundary X of a compact set C in the complex plane whose complement is connected. Here B consists of all continuous functions on X which have extensions to C analytic on $C-X$.

(3) Wermer [**9**] and Royden [**6**] (see also Rudin [**8**]) have proved a general F. and M. Riesz theorem for X the boundary of a finite Riemann surface R. Here B again is the set of continuous functions on X which can be extended to be analytic on R and continuous on $X\cup R$.

2. **Proof of Theorem 1.** We first prove a lemma from which the proof of Theorem 1 will be trivial.

LEMMA. *Assume that* $|f(x)| < r < 1$ *for all* x *in* S. *Then under the hypotheses of Theorem* 1 *there exists* F *in* B *with* $\|F\| < 1$ *and* $F(x) = f(x)$ *for all* x *in* S.

PROOF. Let U_r be the set of all g in B of norm less than r. Let ϕ be the restriction mapping of B into $C(S)$. The lemma states that $f\in\phi(U_1)$. We begin by showing that $f\in V_r$, where V_r is the closure of $\phi(U_r)$. Assume otherwise. It follows from one of the many variants of the Hahn-Banach theorem that there exists a bounded linear functional α on $C(S)$ with $\alpha(f)>1$ and $|\alpha(h)|<1$ for all h in V_r. By the Riesz representation theorem, there exists a measure μ_1 on S such that

$$\int g d\mu_1 = \alpha(g)$$

for all g in $C(S)$. Define the linear functional β on B by $\beta(g) = \alpha(\phi(g))$. Since $\phi(g) \in V_r$ for all g in U_r it follows that $|\beta(g)| < 1$ for all g in U_r so that $\|\beta\| \leqq r^{-1}$. By the Hahn-Banach theorem and the Riesz representation theorem it follows that there exists a measure μ_2 on X with $\|\mu_2\| \leqq r^{-1}$ and $\beta(g) = \int g d\mu_2$ for all g in B. Thus the measure

$$\mu = \mu_1 - \mu_2$$

is in $B^\perp$. Also

$$\left| \int_S f d\mu \right| \geqq \int_S f d\mu_1 - r\|\mu_2\| > 1 - rr^{-1} = 0,$$

contradicting the fact that $\hat{\mu}$ vanishes on all subsets of S. This contradiction shows that $f \in V_r$, so that there exists F_1 in B with $\|F_1\| < r$ and $|f(x) - F_1(x)| < \lambda/2$ for all x in S, where $\lambda = 1 - r$. If we write $f_1(x) = f(x) - F_1(x)$, it follows by the result just proved, with f replaced by f_1, that there exists F_2 in B with $\|F_2\| < \lambda/2$ and $|f_1(x) - F_2(x)| < \lambda/4$ for all x in S. Thus by induction we find a sequence $\{F_n\}$ of functions in B with

$$\|F_n\| < 2^{-n+1}\lambda, \qquad n \geqq 2,$$

and

$$\left| f(x) - \sum_{k=1}^{n} F_k(x) \right| < 2^{-n}\lambda, \qquad n \geqq 2, x \in S.$$

We define the function F in $C(X)$ by

$$F = \sum_{n=1}^{\infty} F_n.$$

Thus $F \in B$ and

$$\|F\| < r + \sum_{n=2}^{\infty} 2^{-n+1}\lambda = r + \lambda = 1.$$

Clearly also $F(x) = f(x)$ for all x in S. This proves the lemma.

Proof of Theorem 1. Let B_0 be the closed subspace

$$\{g \colon \Delta g \in B\}$$

of $C(X)$. Then

$$B_0^{\perp} = \{\Delta\mu : \mu \in B^{\perp}\}.$$

Thus $\hat{\nu}(T)=0$ for all Borel sets $T \subset S$ and all ν in $B_0^{\perp}$. It follows from the lemma that there exists F_0 in B_0 with $\|F_0\| < 1$ and $F_0(x) = (\Delta(x))^{-1}f(x)$ for all x in S. Let F be the element ΔF_0 of B. Thus $|F(x)| < |\Delta(x)|$ for all x in X and $F(x) = \Delta(x)F_0(x) = f(x)$ for all x in S. This completes the proof.

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