Rudin-Carleson theorems

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Extensions

Definition

If $f\colon A \to X$, $g\colon B \to X$, $A \subset B$ and $f = g$ on A, then we say g extends f.

We will use the following to denote common sets in $\mathbb{C}$.

- $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$
- $\overline{\mathbb{D}} = \{z \in \mathbb{C} \colon |z| \leqslant 1\}$
- $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$

Tietze

Theorem (Tietze)

Let X be a normal space and A be a subset of X. For any continuous function $f\colon A \to \mathbb{R}$ there exists a $g\colon X \to \mathbb{R}$ that extends it.

Theorem (Tietze)

Let X be a normal space and A be a subset of X. For any continuous function $f\colon A \to [a, b]$ there exists a $g\colon X \to [a, b]$ that extends it.

An example of a normal space is any metric space.

The Dirichlet problem (on the unit disk in $\mathbb{R}^2$)

Given a continuous function $f\colon \{x \in \mathbb{R}^2;\ \|x\| = 1\} \to \mathbb{R}$ can you find a continuous function u on the closed unit disk that's harmonic on the open unit disk and extends f?

Rudin-Carelson

Theorem (Rudin-Carelson)

Let $E \subset \mathbb{T}$ be a closed set such that $m(E) = 0$, where m is the Lebesgue-measure on $\mathbb{T}$, $f : E \to \mathbb{C}$ be continuous, and T be a subset of $\mathbb{C}$ homeomorphic to $\overline{\mathbb{D}}$ such that $f(E) \subset T$. There exists a continuous function $g : \overline{\mathbb{D}} \to \mathbb{C}$ that extends f, is holomorphic on $\mathbb{D}$ and $g(\overline{\mathbb{D}}) \subset T$.

Bishop

Theorem (Bishop)

Let

1. X *be a compact Hausdorff space,*
2. B *be a closed subspace of* $(C(X), \|\cdot\|_\infty)$*,*
3. S *be a closed subset of* X *that is* $B^\perp$*-null,*
4. f *be a continuous function on* S*,*
5. $\Xi : X \to [0, +\infty[$ *be a continuous function such that* $|f| < \Xi$ *on* S*.*

Then there exists a function $F \in B$ *that extends* f *and* $|F| < \Xi$ *on* X*.*

Note that for S *to be* $B^\perp$*-null it has to be a null set with regards to all complex measures* μ *such that*

$$\int f \, d\mu = 0 \qquad \text{for all } f \text{ in } B.$$

Rudin-Carleson as a corollary of Bishop's theorem

Theorem (F. and M. Riesz)

Let μ be a complex measure on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} e^{-int} \, d\mu = 0$$

holds for $n = -1, -2, \ldots$. Then $\mu \ll m$. That is, if $E \subset \mathbb{T}$ is m-null then E is also μ-null.

Regarding the proof of Bishop's theorem

When proving Bishop's theorem we use that

$$\int_S f \; d\mu = 0$$

holds generally for μ in $B^\perp$ and

$$\int_S (f - F) \; d\mu = 0$$

holds generally for μ in $B^\perp$ and F in B.

Alternative version of Bishop

Theorem

Let X and B be as in Bishop's theorem, S be a closed subset of X and f be a continuous function on S. If

$$\int_S f \, d\mu = 0$$

holds for all μ in $B^\perp$ and

$$\int_S G \, d\mu = 0$$

holds for all μ in $B^\perp$ and all G in B then there exists a function F in B that extends f.

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