XXTitle

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2020

XXTITLE

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XX ECTS thesis submitted in partial fulfillment of a
Magister Scientiarum degree in XX

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Faculty of XX
School of Engineering and Natural Sciences
University of Iceland
Reykjavik, XXmonth 2020

XXTitle
XXShort title (50 characters including spaces)
XX ECTS thesis submitted in partial fulfillment of a M.Sc. degree in XX

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Bibliographic information:
Bergur Snorrason, 2020, XXTitle, M.Sc. thesis, Faculty of XX, University of Iceland.

ISBN XX

Printing: Háskólaprent, Fálkagata 2, 107 Reykjavík
Reykjavik, Iceland, XXmonth 2020

Dedication

Abstract

Útdráttur á ensku sem er að hámarki 250 orð.

Útdráttur

Hér kemur útdráttur á íslensku sem er að hámarki 250 orð. Reynið að koma útdráttum á eina blaðsíðu en ef tvær blaðsíður eru nauðsynlegar á seinni blaðsíða útdráttar að hefjast á oddatölusíðu (hægri síðu).

Preface

Formála má sleppa og skal þá fjarlægja þessa blaðsíðu. Formáli skal hefjast á oddatölu blaðsíðu og nota skal Section Break (Odd Page).

Ekki birtist blaðsíðutal á þessum fyrstu síðum ritgerðarinnar en blaðsíðurnar teljast með og hafa áhrif á blaðsíðutal sem birtist með rómverskum tölum fyrst á efnisyfirliti.

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Abbreviations

Í þessum kafla mega koma fram listar yfir skammstafanir og/eða breytuheiti. Gefið kaflanum nafn við hæfi, t.d. Skammstafanir eða Breytuheiti. Þessum kafla má sleppa ef hans er ekki þörf.
The section could be titled: Glossary, Variable Names, etc.

Acknowledgments

Í þessum kafla koma fram þakkir til þeirra sem hafa styrkt rannsóknina með fjárframlögum, aðstöðu eða vinnu. T.d. styrktarsjóðir, fyrirtæki, leiðbeinendur, og aðrir aðilar sem hafa á einhvern hátt aðstoðað við gerð verkefnisins, þ.m.t. vinir og fjölskylda ef við á. Þakkir byrja á oddatölusíðu (hægri síðu).

1. Introduction

$\mathcal{O}(U)$ for open $U \subset \mathbb{C}$ is the family of functions holomorphic on U. $C(X)$ for open topological space X is the family of continuous functions from X to $\mathbb{C}$. We will use $\mathbb{D}$ to refer to the open unit disk $\{z \in \mathbb{C};\ |z| < 1\}$. The closed unit disk will get no special notation, but will be referred to by $\overline{\mathbb{D}}$. We will use $\mathbb{T}$ to refer to the open unit circle $\{z \in \mathbb{C};\ |z| = 1\}$.

2. Preliminaries

2.1. Measure theory

2.2. Complex analysis in one variable

2.2.1. The disk algebra

Definition 2.2.1. A function is said to be in the *disk algebra* if it is holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. We will refer to this family of function $\mathcal{A}$.

When studying a family of function it is helpful to know when the limit of a sequence in the family is also in the family.

Lemma 2.2.2. *Let K be a compact subset of $\mathbb{C}$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions on K with limit f. Then f is the uniform limit of $(f_n)_{n\in\mathbb{N}}$.*

Lemma 2.2.3. *Let U be a subset of $\mathbb{C}$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions on K with uniform limit f. Then f is the continuous.*

2.2.2. Carathéodory

Definition 2.2.4. A continuous function $\gamma : [0,1] \to \mathbb{C}$ is said to be a *Jordan curve* if $\gamma(0) = \gamma(1)$ and

$$\gamma(s) = \gamma(t) \implies s = t \qquad \text{for all } s,t \in]0,1[.$$

The definition above can be restated as: A Jordan curve is a closed simple curve. The term ‘simple’ here means that the curve is not self-intersecting. The name stems from a famous result by Camille Jordan stating that $\mathbb{C}\backslash\gamma([0,1])$ has two connected component, one of which is simply connected. The simply connected component will be called *the domain bounded by* γ. The proof of this result is rather technical and outside the scope of this thesis [Greene and Krantz, 2006, Whyburn, 1958]. The result is however necessary to make statement such as 'let U be the domain bounded by γ'. An example of this is the following theorem:

Theorem 2.2.5 (Carathéodory)**.** *Let Ω_1 and Ω_2 be domains in $\mathbb{C}$ each bounded by a Jordan curve and $\Phi : \Omega_1 \to \Omega_2$ be a conformal mapping. There exists a continuous injection $\hat{\Phi} : \overline{\Omega_1} \to \overline{\Omega_2}$ that extends Φ.*

A proof of this can be found in section 13.2 of [Greene and Krantz, 2006].

2.2.3. The Riemann Mapping Theorem

We will start of with a definition.

Definition 2.2.6. A map $f : U \to V$, with open $U, V \subset \mathbb{C}$ is said to be *conformal* if it is holomorphic, bijective, and its inverse is holomorphic.

Remark 2.2.7. *The fact that the inverse is homomorphic is actually redundant. It can be shown [Greene and Krantz, 2006] that if h is holomorphic and h' vanishes to order k at z_0 then h is $(k+1)$-to-one in a neighbourhood of z_0. So if h is bijective then h' vanishes nowhere.*

Theorem 2.2.8 (Riemann mapping theorem)**.** *If $U \subset \mathbb{C}$, $U \neq \mathbb{C}$ is homeomorphic to $\mathbb{D}$ then there exists a conformal mapping from $\mathbb{D}$ to U.*

Proof. A proof of this is rather involved and can be found in [Greene and Krantz, 2006]. □

Corollary 2.2.9. *If U and V are both homeomorphic to $\mathbb{D}$ then there exists a conformal mapping from U to V.*

Proof. The Riemann mapping theorem gives us Φ_1, a conformal mapping from $\mathbb{D}$ to U, and Φ_2, a conformal mapping from $\mathbb{D}$ to V. The desired conformal mapping from U to V is then $\Phi_2 \circ \Phi_1^{-1}$. □

In this thesis the disk algebra $\mathcal{A}$ is of special interest so a version of the Riemann mapping theorem that considers continuity at the boundary is desirable. We can combine the Riemann mapping theorem and Carathéodory's theorem to achieve the desired theorem. The only thing to prove is that a homeomorphism f maps the boundary of a bounded set U to the boundary of $f(U)$ and that a Jordan curve under a homeomorphism is still a Jordan curve.

Corollary 2.2.10. *If K is homeomorphic to $\overline{\mathbb{D}}$ then there exists a continuous, injective $\Phi : \overline{\mathbb{D}} \to K$ such that its restriction to $\mathbb{D}$ is a conformal mapping.*

Proof. Firstly, let U be bounded, open set in $\mathbb{C}$, $f : \overline{U} \to \mathbb{C}$ be an injective continuous map, and let's show that $\partial f(U) = f(\partial U)$. Let $p \in \partial U$ and $B_r(z) = \{z \in \mathbb{C};\ |r - z| < r\}$. Let's also assume that $f(p) \in f(U)$ and show that it leads to a contradiction. There exists an $r > 0$ such that $B_r(p) \subset f(U)$, because U is open and therefore $f(U)$ as well. This gives us as an open neighbourhood $f^{-1}(B_r(p)) \subset U$ of p, but $p \in \partial U$ implies that on such neighbourhood exists. So $f(p)$ is not in $f(U)$ but it is in $\overline{f(U)}$, implying $\partial f(U) \supset f(\partial U)$. We can use the same argument to show that $\partial f(U) \subset f(\partial U)$, since f is bijective if we consider it as a map from $\overline{U}$ into $f(\overline{U})$. So $f(\partial U) = \partial f(U)$.

Secondly, let U and V be open sets in $\mathbb{C}$, $\gamma : [0, 1] \to \partial\mathbb{D}$ Jordan curve, $f : U \to V$ be homeomorphism, Let $\lambda = f \circ \gamma$ and $s, t \in]0, 1[$ such that $\lambda(s) = \lambda(t)$. It suffices to show that $s = t$, since $\lambda(0) = \lambda(1)$ obviously holds. We have that $f(z) = f(w)$ implies $x = w$, since f is bijective, and therefore injective. So $\lambda(s) = (f \circ \gamma)(s) = (f \circ \gamma)(t) = \lambda(t)$ implies that $\gamma(s) = \gamma(t)$. But γ is a Jordan curve, so $\gamma(s) = \gamma(t)$ implies $s = t$. So λ is also a Jordan curve.

Finally, we can prove the corollary. Let f be a homeomorphism from $\overline{\mathbb{D}}$ to K. The Riemann mapping theorem also gives us a conformal map $\Phi : \overline{\mathbb{D}} \to K$. We know that $\overline{\mathbb{D}}$ is compact, so $K = f(\overline{\mathbb{D}})$ is also compact, since the image of a compact set under a continuous mapping is also compact. So K is also bounded. So $f(\partial\mathbb{D} = \partial f(\mathbb{D}) = \partial K$ according to our first step and ∂K is a Jordan curve according to the second step, since $\partial\mathbb{D}$ is a Jordan curve. So Φ is a conformal map between two domains bounded, each bounded by a Jordan curve. This allows us to use Carathéodory's theorem to extend Φ continuously and injectively to $\overline{\mathbb{D}}$, concluding the proof. □

The Riemann mapping theorem is a strong tool when analyzing holomorphic functions on simply connected domains. We can often solve things for the unit disk (or unit square as in 3.1.1) and then map that solution to a general simply connected domain.

2.3. Functional analysis

Definition 2.3.1. Let X be a topological space. We say that X is *locally compact* if for each $x \in X$ there exists a open set U_x such that $x \in U_x$ and $\overline{U_x}$ is compact.

Definition 2.3.2. Let X be a locally compact space. We say a complex function f *vanishes at infinity* if for all $\varepsilon > 0$ there exists a compact set K such that $|f(x)| < \varepsilon$ for all $x \in X \backslash K$. The family of all such functions is referred as $C_0(X)$.

Let X be compact. Then if $f \in C(X)$ and $\varepsilon > 0$ we set $K = X$ and see that $|f(x)| < \varepsilon$ vacuously holds for all $x \in X \backslash K = \varnothing$. So $C(X)$ and $C_0(X)$ are identical in this case.

Definition 2.3.3. Let α be a linear map from a normed vector space X into a normed vector space Y. We define a its *norm* by

$$\|\alpha\| = \sup\{\|\alpha(x)\|;\ \|x\| < 1\}$$

and say it is *bounded* if said norm is finite.

Theorem 2.3.4. *If α is a linear map from a normed vector space X into a normed vector space Y then the following properties are equivalent:*

(i) *α is bounded.*

(ii) *α is continuous.*

(iii) *α is continuous at $x \in X$.*

We will use the terms ‘bounded’ and ‘continuous’ interchangeably when talking about linear mappings between normed vector spaces, due to this theorem. A proof of this theorem can be found in [Rudin, 1987].

2.3.1. Hahn-Banach

There are many related theorems going by the name ‘Hahn-Banach Theorem’. These are sometime split in two groups, ‘separation theorems’ and ‘extension theorems’. When proving 3.3.1 we need the following Hahn-Banach separation theorem:

Theorem 2.3.5. *Let X be a normed vector space, $A \neq \emptyset$ be a closed convex subset of X, and $p \in X \backslash A$. Then there exists a continuous linear functional f such that* $\sup\{f(x);\ x \in A\} < 1$ *and* $f(p) = 1$.

Corollary 8.15 in [Pryce, 1973] is very similar. It states that, for same X, A, and p, there exists a continuous linear functional f such that $f(p) < \inf\{f(x);\ x \in A\}$. With such a functional given, ... TODOTODOTODO

2.3.2. The Riesz representation theorem

Theorem 2.3.6 (The Riesz reprentation theorem for bounded linear functionals on $C_0(X)$ with locally compact X)**.** *Let X be a locally compact Hausdorff space and α be a bounded linear functional on $C_0(X)$. There then exists a measure μ such that*

(i)

$$\alpha(f) = \int_X f \ d\mu$$

for all $f \in C_0(X)$ and

(ii)

$$\|\alpha\| = |\mu|(X).$$

2.4. Miscellaneous

3. Rudin-Carleson theorem

3.1. Rudin-Carleson theorem

Theorem 3.1.1 (Rudin-Carleson theroem). *Let E be a closed subset of $\mathbb{T}$ of Lebesgue-measure 0, let f be a continuous function on E, and let T be a subset of $\mathbb{C}$ homeomorphic to $\overline{\mathbb{D}}$ such that $f(\overline{\mathbb{D}}) \subset T$. Then there exists an $F \in \mathcal{A}$, such that $F = f$ on E and $F(\overline{\mathbb{D}}) \subset T$.*

We will break the proof into several lemmas.

Lemma 3.1.2. *Let H be a closed set of Lebesbue-measure 0. Then there exists an integrable function $\mu > 1$ such that μ is continuous on $\mathbb{T}\backslash H$, $\mu = +\infty$ on H, if $w \in H$ then $\mu(z) \xrightarrow{z \to w} +\infty$, and μ has a bounded derivative on any closed subarc of $\mathbb{T}\backslash H$.*

Proof. TODO □

Lemma 3.1.3. *If f is a simple continuous function on E such that Re $f \geqslant 0$, then there exists an $F \in \mathcal{A}$ such that $F = f$ on E and Re $F \geqslant 0$ on $\overline{\mathbb{D}}$.*

Proof. It suffices to show that this holds if f takes only two values on E, since simple functions are finite linear combinations of characteristic functions. Let's assume these values are 0 and $\alpha \neq 0$, with Re $\alpha \geqslant 0$, $E_0 = f^{-1}(0)$, and $E_1 = f^{-1}(\alpha)$. Our assumption that f only takes two values then implies that $E_0 \cap E_1 = E$.

Let $u_H(z)$ be the Poisson integral of the function from the above lemma with H as E. This function is continuous on $\mathbb{T}\backslash H$, $u_H|_H = \infty$, and $\lim_{z \to w} u_H(z) = \infty$ for $w \in H$. ... We now define

$$g_H(z) = \begin{cases} u_H(z) + iv_H(z), & z \in \mathbb{D}\backslash H \\ \infty, & \text{otherwise} \end{cases}.$$

By our construction of u_H we see the Re $g > 1$, so it has a well defined square root. Let's call it h_H and define
$$q = \frac{h_{E_1}}{h_{E_0} + h_{E_1}}.$$
Note that $|\arg h_H(z)| < \pi/4$ since if a $w \in \mathbb{C}$ had an argument outside of this range then its square would have and argument outside of the range $[-\pi/2, \pi/2]$ meaning $\operatorname{Re} w^2 < 0$. Also, $q(z) = 0$ if and only if $h_{E_0} = \infty$, so q is zero only on E_0, and $q(z) = 1$ if and only if $h_{E_1} = \infty$, so q is one only on E_1. We now want to show that $0 \leqslant \operatorname{Re} q \leqslant 1$. We will let $z, w \in \mathbb{C}$, with $|\arg z|, |\arg w| < \pi/4$ and $\operatorname{Re} z, \operatorname{Re} w > 1$, and show that $0 < \operatorname{Re} z/(w+z) < 1$.

Firstly note that
$$\frac{z}{w+z} = \frac{1}{w/z+1}$$
so
$$\arg \frac{z}{w+z} = -\arg\left(\frac{w}{z}+1\right)$$
and
$$|\arg w/z| = |\arg w - \arg z| \leqslant |\arg w| + |\arg z| < \pi/4 + \pi/4 = \pi/2.$$
So w/z is in the right halfplane and, since $\arctan(y/x)$ is decreasing in x for positive y, we get
$$\arg \frac{z}{w+z} = -\arg\left(\frac{w}{z}+1\right) < -\arg\frac{w}{z}$$
and thus
$$\left|\arg \frac{z}{w+z}\right| < \left|\arg \frac{w}{z}\right| < \pi/2.$$
We know $\arctan(y/x)$ is decreasing in x for positive y because
$$\frac{d}{dx}\arctan\left(\frac{y}{x}\right) = -\frac{y}{x^2+y^2} < 0.$$
So $0 < \operatorname{Re} z/(w+z)$. Note that $0 < \operatorname{Re} z/(w+z) \implies 0 > \operatorname{Re} -w/(w+z)$ due to z and w being constrained in the same manner. So
$$\begin{aligned}
0 &> \operatorname{Re} \frac{-w}{w+z} \\
&= \operatorname{Re} \frac{z-(z+w)}{w+z} \\
&= \operatorname{Re} \left(\frac{z}{w+z} - 1\right) \\
&= \operatorname{Re} \frac{z}{w+z} - 1 \\
\implies 1 &> \operatorname{Re} \frac{z}{z+w}.
\end{aligned}$$

So we have shown that

$$0 < \text{Re}\ \frac{z}{z+w} < 1.$$

We have now constructed a function q that maps $\overline{\mathbb{D}}$ to the ribbon $\{z;\ 0 \leqslant \text{Re}\ z \leqslant 1\}$. We then let Φ be the conformal mapping from the ribbon $\{z;\ 0 \leqslant \text{Re}\ z \leqslant 1\}$ to $\{z;\ 0 \leqslant \text{Re}\ z \leqslant \text{Re}\ \alpha\}$. We will also choose Φ such that $\Phi(0) = 0$ and $\Phi(1) = \alpha$. We can then let $F = \Phi \circ q$ and conclude the proof.

□

Lemma 3.1.4. *If f is a simple continuous function on E that maps E into $T \subset \mathbb{C}$ homeomorphic to $\overline{\mathbb{D}}$, then there exists a $F \in \mathcal{A}$, such that $F = f$ on E and F maps $\overline{\mathbb{D}}$ into T.*

Proof. Let $z_0 \in T \backslash f(E)$ and Φ be a conformal mapping from the right halfplane to the interior of T such that $\Phi(\infty) = z_0$. There exists a $g \in \mathcal{A}$ that extends $\Phi^{-1} \circ f$, according to Lemma 3.1.3. The desired function in then obtained with $F = \Phi \circ g$. □

Lemma 3.1.5. *If f is a continue function on E which maps E into*

$$S = \{z;\ |\max(Re\ z, Im\ z)| \leqslant 1\},$$

then there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of simple continuous function on E such that

$$f(x) = \sum_{n \in \mathbb{N}} f_n(z) \text{ and } f_n(E) \subset 2^{-n} S.$$

Proof. We will set $f_0 = 0$ and construct f_n iteratively. Assuming $f_0, f_1, ..., f_{n-1}$ have been constructed such that

$$\lambda_{n-1}(E) \subset 2^{1-n} S$$

with $\lambda_{n-1} = f - \sum_{k=0}^{n-1} f_k$. According to **??** we have can write E as the union of disjoint closed sets $E_1, E_2, ..., E_p$ such that the oscillation of λ_{n-1} is less than 2^{-n} on each E_k. So we can define $Q_k \subset 2^{1-n} S$ for $k = 1, 2, ..., p$ such that $Q_k = 2^{-n} S + a_k$ for some $a_k \in S$. We can choose $c_k \in Q_k \cap 2^{-n}$ since Q_k has side length 2^{-n} and is a subset of 2^{-n+1} and both are closed. We can now define

$$f_n(z) = c_k, \qquad z \in E_k,\ k = 1, 2, ..., p.$$

If we then look at $\lambda_n = f - \sum_{k=0}^{n} f_k = \lambda_{n-1} - f_n$ we see that $\lambda_n(E) \subset 2^{-n} S$ due to the way we decomposed E into $E_1, E_2, ..., E_p$ using the oscillations of λ_{n-1}. This means we can continue the process. □

Proof of 3.1.1. Let's first show the result for $T = S$.

Let f_n be the functions from Lemma 3.1.5. According to Lemma 3.1.4 we have functions $g_n \in \mathcal{A}$ which extend f_n and map $\overline{\mathbb{D}}$ into $2^{-n}S$. We then define

$$F = \sum_{n\in\mathbb{N}} g_n$$

on $\overline{\mathbb{D}}$. To show that F is in $\mathcal{A}$ we have to show it is holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Both of these things can be shown by demonstrating that the series converges uniformly. Let $M_n = 2^{-n+1}$ and note that $|g_n(z)| \leqslant \sqrt{2} \cdot 2^{-n} < 2^{-n+1} = M_n$ and

$$\sum_{n\in\mathbb{N}} M_n = \sqrt{2} \sum_{n\in\mathbb{N}} 2^{-n} = \sqrt{2} < \infty$$

so the Weierstrass M-test tells us that F converges uniformly. This in turn tells that F is both holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ so it is in $\mathcal{A}$. We also have that

$$\mathrm{Re}\ F = \sum_{n\in\mathbb{N}} \mathrm{Re}\ g_n \leqslant \sum_{n\in\mathbb{N}} 2^{-n} = 1.$$

It can be shown in the same manner that $\mathrm{Im}\ F \leqslant 1$, so F maps into S. Lastly, for $z \in E$ we have that

$$F(z) = \sum_{n\in\mathbb{N}} g_n(z) = \sum_{n\in\mathbb{N}} f_n(z) = f(z)$$

so F is an extension of f.

To prove the result for a general T we first let $\Phi : T \to S$ be the map provided to us by . We will also let $g = f \circ \Phi$. Note that it maps E into S, so we can use what we showed above to find $G\mathcal{A}$ that extends g and maps into S. We finally set $F = G \circ \Phi^{-1}$. On E we have that

$$F = G \circ \Phi^{-1} = g \circ \Phi^{-1} = f \circ \Phi \circ \Phi^{-1} = f,$$

so F extends f. It is also a composition of functions in $\mathcal{A}$ so it is also in $\mathcal{A}$. □

Corollary 3.1.6 (Fatou). *Let E be a closed subset of $\mathbb{T}$ of Lebesgue-measure 0. There exists a function $f \in \mathcal{A}$ that vanishes on E and nowhere else.*

Proof. It's clear from the Theorem that there exists a function $f \in \mathcal{A}$ that vanishes on E. TODO □

3.2. F. and M. Riesz theorem

The main result of this section is that the annihilating measures of

$$\mathcal{A}|_{\mathbb{T}} = \{f|_{\mathbb{T}};\ f \in \mathcal{A}\}$$

are absolutely continuous with respect to the Lebesgue measure. We will show this to be a corollary of the F. and M. Riesz theorem, which we will prove in the manner of [Rudin, 1987]. To attain the main result of this section we need some lemmas and definitions. To prove one of the lemmas we will also use the following two famous theorems:

Definition 3.2.1. Let $\mathcal{F}$ be a family of complex functions on a metric space (X, d).

We say that the family is *pointwise bounded* if for all $x \in X$ there exists a constant $M < \infty$ such that

$$|f(x)| < M, \text{ for all } f \in \mathcal{F}.$$

Note that M may depend on x.

We say that the family is *equicontinuous* if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \text{ for all } f \in \mathcal{F} \text{ and } x, y \in X \text{ such that } d(x, y) < \delta.$$

Note here that δ is globally defined and only dependent on ε.

Theorem 3.2.2 (Bolzano-Weierstrass)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of numbers in $\mathbb{R}^n$, such that $|a_n| < M < \infty$, for all $k \in \mathbb{N}$. There than exists and infinite $S \subset \mathbb{N}$ such that $(a_n)_{n\in S}$ is convergent.*

Proof. Let's first assume that the sequence is in $\mathbb{R}$, that no element in it is repeated infinitely often (there is nothing to prove in that case), and that $a_n \in]0, 1[$ for all $n \in \mathbb{N}$. The last assumption can be done with out loss of generality by studying the sequence $((a_n + M)/(2M))_{n\in\mathbb{N}}$ instead. We will obtain the subsequence by a diagonal process. Let $S_0 = \mathbb{N}$, $S_0^- = \{n \in S_0;\ a_n < 1/2\}$, and $S_0^+ = \{n \in S_0;\ a_n > 1/2\}$. We then set $S_1 = S_0^-$ if it is infinite, but $S_1 = S_0^+$ otherwise. This gives us a subsequence $(a_n)_{n\in S_1}$ such that

$$\sup_{n\in S_1} a_n - \inf_{n\in S_1} a_n < 1/2.$$

We can then repeat this to get a sequence of sets $(S_n)_{n\in\mathbb{N}}$ such that $S_0 \supset S_1 \supset S_2 \supset ...$ and

$$\sup_{n\in S_k} a_n - \inf_{n\in S_k} a_n < 2^{1-k},$$

for all $k \in \mathbb{N}$. Specifically, if we have S_k we set

$$U = m2^{-k},\ L = (m+1)2^{-k}$$

$S_k^- = \{n \in S_k;\ a_n < (U+L)/2\}$, and $S_k^+ = \{n \in S_k;\ a_n > (U+L)/2\}$. We now set $S_{k+1} = S_k^-$ if it has infinitely many elements, otherwise we set $S_{k+1} = S_k^+$. We conclude our construction by setting

$$S = \bigcup_{n \in \mathbb{N}} r_n,$$

where r_n is the n-th smallest element of S_n. This gives us the convergent sequence $(a_n)_{n \in S}$ with limit

$$\sum_{k=0}^{\infty} \delta_k 2^{-k}$$

where

$$\delta_k = \begin{cases} 0, & \text{if we chose } S_k^- \\ 1, & \text{if we chose } S_k^+ \end{cases}.$$

To show the result for $\mathbb{R}^n$ we can start by finding a subsequence such that the first coordinate is convergent. We can then chose a subsequence thereof such that the second coordinate is also convergent. Now the first two coordinates are convergent. If we do this $n-2$ more times we get a desired subsequence. □

Remark 3.2.3. *The theorem above clearly holds for sequences in $\mathbb{C}$ as well.*

Theorem 3.2.4 (Ascoli-Arzela)**.** *Let $\mathcal{F}$ be a pointwise bounded equicontinuous collection of complex functions on a metric space (X, d), and X contains a countable dense subset. Then every sequence in $\mathcal{F}$ contains a subsequence that converges uniformly on every compact subsets of X.*

Proof. Let E be a countable dense subset of X, $(f_n)_{n \in \mathbb{N}}$ be a series in $\mathcal{F}$, and $x_1, x_2, ...$ be an enumeration of E. We will prove the theorem in two steps. The first step is finding a subsequence of $(f_n)_{n \in \mathbb{N}}$ that's pointwise convergent on E using the point wise boundedness along with Bolzano-Weierstrass. The second step is using the equicontinuity to show that this gives us uniform continuity on compact subsets.

Let's first set $S_0 = \mathbb{N}$. Pointwise boundedness gives us that the sequence $(f_n(x_1))_{n \in S_0}$ has a convergent subsequence. Let S_1 index that subsequence. We can use this process to generate sets $S_0 \supset S_1 \supset ...$ such that $(f_n(x_k))_{n \in S_k}$ is convergent. We then set

$$S = \bigcup_{k \in \mathbb{N}} r_k$$

where r_n is the k-th smallest element of S_k. We now have concluded the first step of the proof.

We will now assume the $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent on E, let K be a compact subset of X, and $\varepsilon > 0$. Equicontinuity gives us a $\delta > 0$ such that $d(x, y) < \delta$ implies that $|f_n(x) - f_n(y)| < \varepsilon/3$, for all n. Let's now cover K with m balls of radius $\delta/2$ and call the k-th ball B_k. We can now set p_k as a point in B_k. This point exists because E is dense in X. Pointwise convergence on E let's us chose an N such that $|f_{n_1}(p_k) - f_{n_2}(p_k)| < \varepsilon/3$ for $k = 1, 2, ..., m$ and all $n_1, n_2 > N$. Let's conclude by setting $x \in K$. Then there is a k such that $x \in B_k$ and thus $d(x, p_k) < \delta$. The choice of δ and N then gives us that

$$\begin{aligned}|f_{n_1}(x) - f_{n_2}(x)| &\leqslant |f_{n_1}(x) - f_{n_1}(p_k)| + |f_{n_1}(p_k) - f_{n_2}(p_k)| + |f_{n_2}(p_k) - f_{n_2}(x)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon.\end{aligned}$$

□

Definition 3.2.5. Poisson kernel, Poisson integral, Poisson integral of a measure.

Lemma 3.2.6. *Let μ be a complex Borel measure, and $u = P[d\mu]$. Then*

$$\|u_r\|_1 \leqslant \|\mu\|.$$

Proof. First, we need to see that, if $n \neq 0$

$$in \int_{-\pi}^{\pi} e^{int}\, dt = (e^{in\pi} - e^{-in\pi}) = (e^{in\pi} - e^{-i(2n\pi - n\pi)}) = (e^{in\pi} - e^{in\pi}) = 0,$$

so

$$\begin{aligned}\int_{-\pi}^{\pi} P_r(t)\, dt &= \int_{-\pi}^{\pi} \sum_{n\in\mathbb{Z}} r^{|n|} e^{int}\, dt \\ &= \sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} r^{|n|} e^{int}\, dt \\ &= \int_{-\pi}^{\pi} dt \\ &= 2\pi.\end{aligned}$$

Fubini let's us

$$\begin{aligned}
\|u\|_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})| d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \int_{\mathbb{T}} P(re^{i\theta}, e^{it})\ d\mu(e^{it}) \right| d\theta \\
&\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \int_{\mathbb{T}} P(re^{i\theta}, e^{it})\ d|\mu(e^{it})| d\theta \\
&= \int_{\mathbb{T}} \frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i\theta}, e^{it})\ d\theta d|\mu(e^{it})| \\
&= \int_{\mathbb{T}} d|\mu(e^{it})| \\
&= |\mu|(\mathbb{T}) \\
&= \|\mu\|.
\end{aligned}$$

□

Lemma 3.2.7. *Let $f \in H^1$. Then there exists a $g \in L^1(\mathbb{T})$ such that $f = P[g]$.*

Proof. □

Lemma 3.2.8. *Let u be harmonic in $\mathbb{D}$ and*

$$\sup_{0<r<1} \|u_r\|_1 = M < \infty.$$

Then there exists a unique complex Borel measure μ on $\mathbb{T}$ such that $u = P[d\mu]$.

We will need the following lemma in the proof of 3.2.8:

Lemma 3.2.9. *Let X be a separable Banach space, $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of linear functionals on X, and $\sup_n \|\Gamma_n\| = M < \infty$. Then there exists a subsequence $\{\Gamma_{n_i}\}$ such that the limit*

$$\Gamma x = \lim_{k\to\infty} \Gamma_{n_k}\ x$$

exists for every $x \in X$. We also have that Γ is linear and $\|\Gamma\| \leqslant M$.

Proof. We have that $|\Gamma_n\ x| \leqslant M\|x\|$ and

$$\begin{aligned}
|\Gamma_n\ x - \Gamma_n\ y| &= |\Gamma_n(x-y)| \\
&\leqslant M\|x-y\|.
\end{aligned}$$

The first inequality gives us pointwise boundedness and the second gives us equicontinuity. Now, since singletons are compact, Ascoli-Arzela gives us a subsequence, let's index it by S, such that $(\Gamma_n x)_{n\in S}$ is convergent for all $x \in X$. Let's now define Γ by

$$\Gamma(x) = \lim_{k\in S} \Gamma_k\, x,$$

see the

$$\begin{aligned}\Gamma(x) + \Gamma(y) &= \lim_{k\in S} \Gamma_k\, x + \lim_{k\in S} \Gamma_k\, y \\ &= \lim_{k\in S} (\Gamma_k\, x + \Gamma_k\, y) \\ &= \lim_{k\in S} \Gamma_k(x+y) \\ &= \Gamma(x+y),\end{aligned}$$

where the third equality holds because addition is continuous, and $a\Gamma(x) = \Gamma(ax)$ obviously holds. So Γ is linear. Lastly

$$\begin{aligned}\|\Gamma\| &= \sup\{|\Gamma x|;\ \|x\| \leqslant 1\} \\ &= \sup\left\{\left|\lim_{n\in S} \Gamma_n x\right|;\ \|x\| \leqslant 1\right\} \\ &\leqslant \sup\{M;\ \|x\| \leqslant 1\} \\ &= M.\end{aligned}$$

□

Proof of 3.2.8. Let Γ_r, for $r \in [0,1[$, be linear functionals on $C(\mathbb{T})$ defined by

$$\Gamma_r g = \int_{\mathbb{T}} g u_r d\sigma.$$

If $\|g\| \leqslant 1$ is assummed we get that

$$\Gamma_r\ g = \int_{\mathbb{T}} g u_r\ d\sigma \leqslant \int_{\mathbb{T}} u_r\ d\sigma = \|u_r\|_1 \leqslant M.$$

so

$$\|\Gamma_r\| \leqslant M.$$

By Lemma 3.2.9 and the Riezs representation theorem we get a measure μ on $\mathbb{T}$ with $\|\mu\| \leqslant M$, and a sequence $(r_n)_{n\in\mathbb{N}}$ on $[0,1[$ with limit 1, such that

$$\lim_{n\to\infty} \int_{\mathbb{T}} g u_{r_n}\ d\sigma = \int_{\mathbb{T}} g\ d\mu \tag{3.1}$$

for all $g \in C(\mathbb{T})$. Let's now define functions h_k on $\overline{\mathbb{B}}$ by $h_k(z) = u(r_k z)$. We get that, since u is harmonic on $r\mathbb{B}$ for $r \in]0,1[$, the functions h_k are harmonic on $\mathbb{B}$

and continuous on $\overline{\mathbb{B}}$. So each of them can be represented by the Poisson integral of their restriction to $\mathbb{T}$. Note that $h_k(e^{it}) = u_{r_k}(e^{it})$, so

$$\begin{aligned}
u(z) &= \lim_{n\to\infty} u(r_n z) \\
&= \lim_{n\to\infty} h_n(z) \\
&= \lim_{n\to\infty} \int_{\mathbb{T}} P(z, e^{it}) h_n(e^{it})\ d\sigma(e^{it}) \\
&= \lim_{n\to\infty} \int_{\mathbb{T}} P(z, e^{it}) u_{r_n}(e^{it})\ d\sigma(e^{it}) \\
&= \int_{\mathbb{T}} P(z, e^{it})\ d\mu(e^{it}) \\
&= P[d\mu](z),
\end{aligned}$$

where the fifth equlity is achived by putting $g = P(z, e^{it})$ into 3.1. This concludes the proof of excistence.

Let's assume that $P[d\mu] = 0$, and let $f \in C(\mathbb{T})$, $u = P[f]$ and $v = P[d\mu]$. We firstly have the symmetry

$$P(re^{i\theta}, e^{it}) = P(re^{it}, e^{i\theta}).$$

This symmetry is due to

$$|e^{it} - re^{i\theta}| = |1 - re^{i(\theta - t)}| = |1 - re^{i(t-\theta)}| = |e^{i\theta} - re^{it}|,$$

which is geometrically intuitive. The first and last equalities hold because the euclidean metric is rotationally invariant, and the second equality holds because the distance from z to a real number a is the same distance from $\overline{z}$ to a. We now obtain

$$\begin{aligned}
\int_{\mathbb{T}} u_r d\mu &= \int_{\mathbb{T}} \int_{-\pi}^{\pi} P(re^{i\theta}, e^{it}) f(e^{i\theta})\ d\theta d\mu(e^{it}) \\
&= \int_{-\pi}^{\pi} f(e^{i\theta}) \int_{\mathbb{T}} P(re^{it}, e^{i\theta})\ d\mu(e^{it}) d\theta \\
&= \int_{-\pi}^{\pi} f(e^{i\theta}) v_r\ d\theta \\
&= \int_{\mathbb{T}} f v_r\ d\sigma.
\end{aligned}$$

If we let $r \to 1$ we get

$$\int_{\mathbb{T}} f\ d\mu = 0.$$

This holds for all $f \in C(\mathbb{T})$, so the measure μ represents zero in the dual of $C(\mathbb{T})$. The Riesz representation theorem then tells us that $|\mu|(\mathbb{T}) = 0$, so $\mu = 0$.

Now let λ and ν be measures on $\mathbb{T}$ such that $P[d\lambda] = P[d\nu]$. We have that $P[d(\lambda - \nu)] = 0$, so, as shown above $\lambda - \nu = 0$. Moreover $\lambda = \nu$, which concludes the proof of uniqueness. □

Theorem 3.2.10 (F. and M. Riesz theorem)**.** *If μ is a complex Borel measure on $\mathbb{T}$ and*

$$\int e^{-int} d\mu = 0$$

for $n = -1, -2, ...,$ then $\mu \ll m$.

Proof. Let $f = P[d\mu]$. If we set $z = re^{i\theta}$ we get that

$$P(z, e^{it}) = P_r(\theta - t) = \sum_{n \in \mathbb{Z}} r^{|n|} e^{in(\theta - t)} = \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta} e^{-int}.$$

We can use the assumption of the theorem to write f as a power series by

$$\begin{aligned} f(z) &= \int_{\mathbb{T}} P(z, e^{it}) d\mu(e^{it}) \\ &= \int_{\mathbb{T}} \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta} e^{-int} d\mu(e^{it}) \\ &= \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta} \int_{\mathbb{T}} e^{-int} d\mu(e^{it}) \\ &= \sum_{n=0}^{\infty} r^n e^{in\theta} \int_{\mathbb{T}} e^{-int} d\mu(e^{it}) \\ &= \sum_{n=0}^{\infty} \hat{\mu}_n z^n, \end{aligned}$$

where $\hat{\mu}_n$ is the n-th Fourier coefficient of μ. This along with 3.2.6 gives us that $f \in H^1$. We can now define a $g \in H^1$, by 3.2.7, such that $f = P[g]$. It follows from 3.2.8 that $d\mu = f d\sigma$. TODO □

Corollary 3.2.11. *Every annihilating measures of $\mathcal{A}|_{\mathbb{T}}$ is absolutely continuous with regards to the Lebesgue-measure on $\mathbb{T}$.*

Proof. Let $\mu \in A^{\perp}$. By definition we have that

$$\int f d\mu = 0.$$

Now since $t \mapsto e^{-int}$ is entire for $n = -1, -2, ...$ we have that their restriction to $\mathbb{T}$ are in $\mathcal{A}|_{\mathbb{T}}$. Thus,

$$\int e^{-int} d\mu = 0$$

for all $n = -1, -2, ...$ and $\mu \ll m$. □

3.3. A generalization of the Rudin-Carleson theorem

This borrows from [Bishop, 1962].

Theorem 3.3.1 (General Rudin-Carleson theorem)**.** *Let X be a compact Hausdorff space, $V = (C(X), \|\cdot\|_\infty)$, B be a closed subspace of $C(X)$, $B^\perp$ be the annihilating measures of B, S be a closed subset of X that is μ-null for all $\mu \in B^\perp$, f be a continues function on S, and $\Xi > 0$ be a function on X such that $|f| < \Xi$ on S. Then there exists a $F \in B$ such that $F = f$ on S and $|F| < \Xi$ on X.*

Let's start with the following lemma:

Lemma 3.3.2. *Assume $|f| < r < 1$ on S. Then there exists a $F \in B$ such that $F = f$ on S and $\|F\| < 1$.*

Proof. Let U_r be the subset of B defined by $U_r = \{g;\ \|g\| < r\}$ and ϕ be the mapping from B to $C(S)$ that sends a member of B to its restriction on S. It suffices to show that $f \in \phi(U_r)$. Let's first show that $f \in \overline{\phi(U_r)} =: V_r$, by assuming otherwise, and showing it leads to a contradiction.

We now assume $f \notin V_r$. By Hahn-Banach we can define a bounded linear functional α, such that $\alpha(f) > 1$ and $|\alpha(h)| < 1$, for $h \in V_r$. We can then define a measure μ_1 by the Riesz representation theorem that fulfills

$$\alpha(g) = \int g d\mu_1$$

for all $g \in C(S)$. We will refer to the associated functional on B by $\beta(g) = \phi(\alpha(g))$. Since $\phi(g) \in V_r$ for all $g \in U_r$ we have that

$$\beta(g) = \alpha(\phi(g)) < 1,$$

for all $g \in U_r$, due to the construction of α. From this we get

$$\begin{aligned} \|\beta\| &= \sup\{|\beta(g)|;\ \|g\| < 1\} \\ &= \sup\{(1/r)|\beta(g)|;\ \|g\| < r\} \\ &\leqslant 1/r. \end{aligned}$$

Let's denote the Riesz representation of β by μ_2, set $\mu = \mu_1 - \mu_2$, and note that $\mu \in B^\perp$. But

$$0 = \left|\int_S f d\mu\right| \geqslant \int_S f d\mu_1 - r\|\mu_2\| \geqslant \int_S f d\mu_1 - r\frac{1}{r} > 1 - r\frac{1}{r} = 0.$$

This is the contradiction that gives that $f \in V_r$. We can now take a F_1 in U_r, and therefore also in B such that $|f - F_1| < \lambda/2$ on S, with $\lambda := 1 - r$. Remember that $F_1 \in U_r$ implies that $\|F_1\| < r$. Now let $f_1 = f - F_1$ and use the same method as above to obtain an F_2 such that $\|F_2\| < \lambda/2$ and $|f - F_2| < \lambda/4$ on S. Iterating this process yields a series $(F_n)_{n\in\mathbb{N}}$ from B that fulfill $\|F_n\| < 2^{1-n}\lambda$ for $n > 1$ and

$$\left| f - \sum_{k=1}^{n} F_k \right| < 2^{-n}\lambda$$

on S for $n > 1$. We finally let

$$F = \sum_{k=1}^{\infty} F_k.$$

Now $F \in B$,

$$\|F\| \leqslant \|F_1\| + \|F - F_1\| \leqslant r + \sum_{k=2}^{\infty} 2^{1-n}\lambda = r + \lambda = 1,$$

and $F = f$ on S. □

Proof of 3.3.1. Let B_0 be the closed subspace of $C(X)$ consisting of function g such that $\Xi \cdot g \in B$. We have that $B_0^{\perp} = B^{\perp}$, since $\Xi > 0$. So we can use Lemma 3.3.2 for B_0 instead of B and f/Ξ instead of f. This gives us a $F_0 \in B_0$ such that $\Xi \cdot F_0 = f$ on S and $\|F_0\| < 1$. We set $F = \Xi \cdot F_0$ which is in B by the construction of B_0. Also note that $\|F\| < \Xi$ on X and

$$F = \Xi \cdot F_0 = \Xi \cdot f/\Xi = f$$

on S. □

Alternate proof of 3.1.1. Let $X = \mathbb{T}$, $B = \mathcal{A}|_{\mathbb{T}}$, and S be a closed set of Lebesgue-measure zero. Then, according to 3.2.11, S is also a $B^{\perp}$-null. So all requirements of 3.3.1 are met □

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