



XXTitle

Bergur Snorrason



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Bergur Snorrason

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Magister Scientiarum degree in XX

Advisor

XXNN1

XXNN2

Faculty Representative

XXNN3

M.Sc. committee

XXNN4

XXNN5

Faculty of XX

School of Engineering and Natural Sciences

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Faculty of XX

School of Engineering and Natural Sciences

University of Iceland

XXFaculty street address

XXFaculty postal code, Reykjavik, Reykjavik
Iceland

Telephone: 525 4000

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Dedication

Abstract

Útdráttur á ensku sem er að hámarki 250 orð.

Útdráttur

Hér kemur útdráttur á íslensku sem er að hámarki 250 orð. Reynið að koma útdráttum á eina blaðsíðu en ef tvær blaðsíður eru nauðsynlegar á seinni blaðsíða útdráttar að hefjast á oddatölusíðu (hægri síðu).

Preface

Formála má sleppa og skal þá fjarlægja þessa blaðsíðu. Formáli skal hefjast á odd-atölu blaðsíðu og nota skal Section Break (Odd Page).

Ekki birtist blaðsíðutal á þessum fyrstu síðum ritgerðarinnar en blaðsíðurnar teljast með og hafa áhrif á blaðsíðutal sem birtist með rómverskum tölum fyrst á efnisyfirliti.

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Abbreviations

Í þessum kafla mega koma fram listar yfir skammstafanir og/eða breytuheiti. Gefið kaflanum nafn við hæfi, t.d. Skammstafanir eða Breytuheiti. Þessum kafla má sleppa ef hans er ekki þörf.

The section could be titled: Glossary, Variable Names, etc.

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1. Introduction

2. Preliminaries

2.1. Measure theory

2.2. Functional analysis

2.2.1. Hahn-Banach

2.2.2. Riesz representation theorem

3. Rudin-Carleson theorem

Theorem 1. *Let E be a closed subset of \mathbb{T} of Lebesgue-measure 0, let f be a continuous function on E and let T be a simply connected subset of \mathbb{C} such that $f(\overline{\mathbb{D}}) \subset T$. Then there exists an $F \in \mathcal{A}$, such that $F = f$ on E and $F(\overline{\mathbb{D}}) \subset T$. (TODO skilgreina allt)*

We will break the proof into several (TODO how many?) lemmas.

Lemma 1. *Let E be a closed set of Lebesgue-measure 0. Then there exists an integrable function $\mu > 1$ such that μ is continuous on $\mathbb{T} \setminus E$, $\mu = +\infty$ on E , if $w \in E$ then $\mu(z) \xrightarrow{z \rightarrow w} +\infty$, and μ has a bounded derivative on any closed subarc of $\mathbb{T} \setminus E$.*

Proof. The function μ is found by solving the Dirichlet problem. (TODO Thomas Ransford p. 95 4.2.6) (TODO finish this) \square

Lemma 2. *If f is a simple continuous function on E such that $\operatorname{Re} f \geq 0$, then there exists an $F \in \mathcal{A}$ such that $F = f$ on E and $\operatorname{Re} F \geq 0$ on $\overline{\mathbb{D}}$.*

Proof. We will show that this holds in the case where $E = E_0 \cup E_1$, $f = 0$ on E_0 , $f = \alpha \neq 0$ on E_1 and $\operatorname{Re} \alpha \geq 0$. This suffices since simple functions are finite linear combinations of characteristic functions. \square

Proof. TODO \square

Corollary 1 (Fatou). *Let E be a closed subset of \mathbb{T} of Lebesgue-measure 0. There exists a function $f \in \mathcal{A}$ that vanishes on E and nowhere else.*

Proof. It's clear from the theorem (TODO add ref) that there exists a function $f \in \mathcal{A}$ that vanishes on E . TODO \square

4. F. and M. Riesz theorem

In this section we will endeavour to show that the annihilating measures of $\mathcal{A}|_{\mathbb{T}}$ are absolutely continuous with respect to the Lebesgue measure. We will show this to be a corollary of the F. and M. Riesz theorem, which we will prove in the manner of Rudin. To attain the main result of this section we need some lemmas and definitions. To prove one of the lemmas we will also use the following famous theorem:

Definition 1. *Pointwise bounded and equicontinuity.*

Theorem 2 (Ascoli-Arzelà). *Let \mathcal{F} be a pointwise bounded equicontinuous collection of complex functions on a metric space (X, d) , and X contains a countable dense subset. Then every sequence in \mathcal{F} contains a subsequence that converges uniformly on every compact subsets of X .*

Proof. Let E be a countable dense subset of X , $\{f_n\}$ be a series in \mathcal{F} , and x_1, x_2, \dots be an enumeration of E . □

Definition 2. *Poisson kernel, Poisson integral, Poisson integral of a measure.*

Lemma 3. *Let μ be a complex Borel measure, and $u = P[d\mu]$. Then*

$$\|u_r\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})| d\theta \leq \|\mu\|.$$

Proof. TODO □

Lemma 4. *Let $f \in H^1$. Then there exists a $g \in L^1(\mathbb{T})$ such that $f = P[g]$.*

Proof. TODO □

4. F. and M. Riesz theorem

Lemma 5. *Let u be harmonic in \mathbb{D} and*

$$\sup_{0 < r < 1} \|u_r\|_1 = M < \infty.$$

Then there exists a unique complex Borel measure μ on \mathbb{T} such that $u = P[d\mu]$.

We will need the following lemma in the proof of 5:

Lemma 6. *Let X be a separable Banach space, $\{\Gamma_n\}$ be a sequence of linear functionals on X , and $\sup_n \|\Gamma_n\| = M < \infty$. Then there exists a subsequence $\{\Gamma_{n_i}\}$ such that the limit*

$$\Gamma x = \lim_{i \rightarrow \infty} \Gamma_{n_i} x$$

exists for every $x \in X$. We also have that Γ is linear and $\|\Gamma\| \leq M$.

Proof. □

Proof of 5. □

Theorem 3 (F. and M. Riesz theorem). *If μ is a complex Borel measure on \mathbb{T} and*

$$\int e^{-int} d\mu = 0$$

for $n = -1, -2, \dots$, then $\mu \ll m$.

Proof. Let $f = P[d\mu]$. If we set $z = re^{i\theta}$ we get that

$$P(z, e^{it}) = P_r(\theta - t) = \sum_{n \in \mathbb{Z}} r^{|n|} e^{in(\theta-t)} = \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta} e^{-int}.$$

We can use the assumption of the theorem to write f as a power series by

$$\begin{aligned} f(z) &= \int_{\mathbb{T}} P(z, e^{it}) d\mu(e^{it}) \\ &= \int_{\mathbb{T}} \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta} e^{-int} d\mu(e^{it}) \\ &= \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta} \int_{\mathbb{T}} e^{-int} d\mu(e^{it}) \\ &= \sum_{n=0}^{\infty} r^n e^{in\theta} \int_{\mathbb{T}} e^{-int} d\mu(e^{it}) \\ &= \sum_{n=0}^{\infty} \hat{\mu}_n z^n, \end{aligned}$$

where $\hat{\mu}_n$ is the n -th Fourier coefficient of μ . This along with 3 gives us that $f \in H^1$. We can now define a $g \in H^1$, by 4, such that $f = P[g]$. It follows from 5 that $d\mu = f d\sigma$. TODO \square

Corollary 2. *Let A be the closed subspace of $C(\mathbb{T})$ that consists of all functions that are restriction from \mathcal{A} (TODO makes this def global and maybe call it something else). All measures in A^\perp are absolutely continuous with regards to the Lebesgue-measure on \mathbb{T} (TODO define all).*

Proof. Let $\mu \in A^\perp$. By definition we have that

$$\int f d\mu = 0.$$

Now since $t \mapsto e^{-int}$ is entire for $n = -1, -2, \dots$ we have that their restriction to \mathbb{T} are in A . Thus,

$$\int e^{-int} d\mu = 0$$

for all $n = -1, -2, \dots$ and $\mu \in A^\perp$. \square

5. A generalization of the Rudin-Carleson theorem

This borrows from Bishop (reference TODO).

Theorem 4 (General Rudin-Carleson theorem). *Let X be a compact Hausdorff space, $V = (C(X), \|\cdot\|_\infty)$, B be a closed subspace of $C(X)$, B^\perp be the annihilating measures for B , S be a closed subset of X , and f be a continuous function on S . If $\int_S f d\mu = 0$ holds for all $\mu \in B^\perp$ then there exists a function $F \in B$ such that $F = f$ on S .*

Proof. Since f is continuous and S is a closed subset of a compact set, and therefore also compact, f is bounded. So we can, with out loss of generality, assume that $|f| < r < 1$ on S . Let U_r be the subset of B defined by $U_r = \{g; \|g\| < r\}$ and ϕ be the mapping from B to $C(S)$ that sends a member of B to its restriction on S . It suffices to show that $f \in \phi(U_r)$. Let's first show that $f \in \overline{\phi(U_r)} =: V_r$, by assuming otherwise, and showing it leads to a contradiction.

We now assume $f \notin V_r$. By Hahn-Banach (TODO ref) we can define a bounded linear functional α , such that $\alpha(f) > 1$ and $|\alpha(h)| < 1$, for $h \in V_r$. We can then define a measure μ_1 by the Riesz-representation theorem (TODO ref) that fulfills

$$\alpha(g) = \int g d\mu_1$$

for all $g \in C(S)$. We will refer to the associated functional on B by $\beta(g) = \phi(\alpha(g))$. Since $\phi(g) \in V_r$ for all $g \in U_r$ we have that

$$\beta(g) = \alpha(\phi(g)) < 1,$$

for all $g \in U_r$, due to the construction of α . From this we get

$$\begin{aligned} \|\beta\| &= \sup\{|\beta(g)|; |g| < 1\} \\ &= \sup\{(1/r)|\beta(g)|; |g| < r\} \\ &\leq \sup\{(1/r); |g| < r\} \\ &= 1/r. \end{aligned}$$

5. A generalization of the Rudin-Carleson theorem

Let's denote the Riesz representation of β by μ_2 , set $\mu = \mu_1 - \mu_2$ and see that $\mu \in B^\perp$. But

$$0 = \left| \int_S f d\mu \right| \geq \int_S f d\mu_1 - r\|\mu_2\| \geq \int_S f d\mu_1 - r\frac{1}{r} > 1 - r\frac{1}{r} = 0,$$

where the first equality is the assumption in the theorem. This is the contradiction that gives that $f \in V_r$. We can now take a F_1 in U_r , and therefore also in B such that $|f - F_1| < \lambda/2$ on S , with $\lambda := 1 - r$. Remember that $F_1 \in U_r$ implies that $\|F_1\| < r$. Now let $f_1 = f - F_1$ and use the same method as above to obtain an F_2 such that $\|F_2\| < \lambda/2$ and $|f - F_2| < \lambda/4$ on S . Iterating this process yields a series $(F_n)_{n \in \mathbb{N}}$ from B that fulfill $\|F_n\| < 2^{1-n}\lambda$ for $n > 1$ and

$$\left| f - \sum_{k=1}^n F_k \right| < 2^{-n}\lambda$$

on S for $n > 1$. We finally let

$$F = \sum_{k=1}^{\infty} F_k.$$

Now $F \in B$,

$$\|F\| \leq \|F_1\| + \|F - F_1\| = r + \sum_{k=2}^{\infty} 2^{1-k}\lambda = r + \lambda = 1,$$

and $F = f$ on S . (TODO bæta við matinu, svo línan að ofan meiki sens) □

Corollary 3. *Let X be a compact Hausdorff space, $V = (C(X), \|\cdot\|_\infty)$, B be a closed subspace of $C(X)$, B^\perp be the annihilating measures for B , S be a closed subset of X , and f be a continuous function on S . If S is B^\perp -null (TODO define this) then there exists a function $F \in B$ such that $F = f$ on S .*

Proof. If S is B^\perp -null we have that $\int_S f d\mu = 0$ for all $\mu \in B^\perp$ (TODO add ref to above). □

Remark 1. *The corollary is the version of the theorem from Bishop (TODO add ref here). Note also that if we set $X = \mathbb{T}$ and $B = \mathcal{A}$ we can use F and M . Riesz (TODO add ref here) to prove the classical Rudin-Carleson theorem (TODO add ref here).*

It is of course worth noting an applications of (TODO REF to theorem) where the corollary fails.

Example 1. Let $X = \mathbb{T}$, $B = \mathcal{A}$ (TODO define this), E be a closed m -null (TODO define this, an maybe change to m_σ) subset of $\partial\mathbb{T}$ that is not dense in E , $F = \{e^{i\theta}; a \leq \theta \leq b\}$, and choose a and b such that E and F are disjoint and $a \neq b$. The last assumption restricts us to E that are not dense in the \mathbb{E} . Since $a \neq b$ we obtain that $S := E \cup F$ does not fulfill the requirements of the Rudin-Carelsen theorem (TODO add ref here) nor the above corollary (TODO add ref). Let's choose f such that $f = 0$ on F , and f is continuous on S . We now have for all $\mu \in \mathcal{A}^\perp$

$$\begin{aligned} \left| \int_S f d\mu \right| &= \left| \int_E f d\mu + \int_F f d\mu \right| \\ &\leq \left| \int_E f d\mu \right| + \left| \int_F f d\mu \right| \\ &= 0 + \left| \int_F f d\mu \right| \\ &= 0. \end{aligned}$$

The F. and M. Riesz theorem (TODO add ref here) tells us that since E is m -null it is also μ -null, which gives the third step. The final step stems from the fact that f vanishes on F . We now see that X , B , and f are all as in theorem (TODO add ref) so there exists a $F \in B$, such that $F = f$ on S .

A. Annaď