JOURNAL OF FUNCTIONAL ANALYSIS **58**, 175–193 (1984)

Annihilating Measures of the Algebra $R(X)$

DMITRY KHAVINSON

Department of Mathematical Sciences, University of Arkansas at Fayetteville, Fayetteville, Arkansas 72701

Communicated by the Editors

Received December 1983

A special class of "analytic measures" in the totality of measures orthogonal to the algebra of rational functions on a compact set $X \subset \mathbb{C}$ is introduced. It is proved that there always exist nontrivial (i.e., nonzero) analytic measures provided that $R(X) \neq C(X)$. We also give sufficient conditions in order to have the linear span of analytic measures be weak $(*)$ dense in the whole annihilator of the algebra $R(X)$.

I. INTRODUCTION

Let X be an arbitrary compact set in the complex plane and let $R(X)$ be the uniform closure on X of all rational functions with poles outside of X. One of the major problems in the theory of rational approximation is to describe the space of annihilating measures of this algebra supported on the boundary of X. For example, in case X is the unit disk, the answer is given by the celebrated theorem of F. and M. Riesz, which has numerous applications (see [7, 12, 13]).

In a series of papers (see [3–5]), Bishop has investigated possible generalizations of the F. and M. Riesz theorem to the compact set K with a connected complement. He introduced the following concept of "analytic differentials."

Let U be the interior of K. Let $\{\Gamma_i\}$ be a system of curves such that (i) each Γ_i is a finite union of rectifiable closed Jordan curves in U and every connected component of U contains only one of these curves; (ii) every compact set $S \subset U$ belongs to the components bounded by Γ_i for all sufficiently large i. Let $g(z)$ be an analytic function in U. If the total variations of the measures $g(z)\,dz\,|_{\Gamma_n}$ are uniformly bounded, we say that these measures define an analytic differential in K. In that case, there exists a subsequence of these measures which converges weak $(*)$ to the measure μ supported on the boundary of K. It is clear that μ is orthogonal to all polynomials. Then, Bishop has proved that the set of measures defined by such analytic differential coincide with the whole space $(R(K)|_{\partial K})^{\perp}$. (We

0022-1236/84 $3.00

recall that if E_0 is a linear subspace of a topological vector space E, then $E_0^{\perp} = \{f \in E^* : f|_{E_0} \equiv 0\}$. Using this description of $(R(K)|_{\partial K})^{\perp}$, Bishop has obtained a simple proof of a famous theorem of Mergelyan (see [19]). However, his construction of an "analytic differential" cannot be extended directly to the general sets. Moreover, for the compact sets without interior, the above definition does not even make sense.

In this paper we introduce the concept of "analytic measures" related to Bishop's idea of an "analytic differential," but applicable to an arbitrary compact set in $\mathbb{C}$.

Before giving a precise definition, we recall that a function $f(z)$, analytic in a finitely connected region G with a boundary Γ, is said to belong to the class $E_1(G)$ (the Smirnov class) if there exists a sequence of regions $\{G_n\}$ bounded by a finite number of rectifiable curves such that $G_n \subset G_{n+1}$, $n = 1, \ldots, \bigcup_{n=1}^{\infty} \mathring{G}_n = G$ and

$$\|f\|_{E_1(G)} = \limsup_{n \to \infty} \int_{\partial G_n} |f|\,|dz| < +\infty.$$

Note. If the boundary Γ of G is rectifiable, then

$$\|f\|_{E_1(G)} = \int_{\Gamma} |f(\zeta)|\,|d\zeta|,$$

where $f(\zeta)$ stands for angular boundary values of $f(z)$ on Γ. It is known that $f(\zeta)$ exist a.e. on Γ. It is also known that $E_1(G)$ is a Banach space with the norm $\|f\|_{E_1(G)}$. More details on E_p-classes can be found in [7, 12, 22].

Definition. Let X be a compact set in C. Fix a sequence of finitely connected compact sets $\{X_n\}$ bounded by analytic Jordan curves and such that $X_1 \supset X_2 \ldots, \bigcap_{n=1}^{\infty} X_n = X$. Let μ be a complex measure supported on ∂X. We say that μ is an *analytic measure* relative to the given sequence $\{X_n\}$ if there exists a sequence of functions $f_n(z)$ such that $f_n \in E_1(X_n)$, $\|f_n\|_{E_1(X_n)} \leqslant M < +\infty$ for all n and a subsequence of the sequence of measures $\mu_n = f_n(\zeta)\, d\zeta|_{\partial X_n}$ converges to μ in the weak $(*)$ topology of the space of measures on X_1.

The first question arising here is the existence of nontrivial analytic measures on an arbitrary compact set. We investigate this problem using the duality method for the extremal problems of analytic functions. This method has been studied by S. Ya. Khavinson (see [16, 17]) and independently by Rogosinski and Shapiro (see [23]; see also [7]). To state our main result we have to give one more definition.

Definition. Let $f \in C(X)$. We define

$$\lambda_f(X) \overset{\text{def}}{=} \inf_{\phi \in R(X)} \|f - \phi\|_{C(X)}.$$

For $f(\zeta)=\bar{\zeta}|_X$ we shall call $\lambda_{\bar{\zeta}}(X)$ the *rational capacity* of X and denote it by $\lambda(X)$. The main result of the paper is given by Theorem 1. Namely, let h be a function harmonic in the neighborhood of X. Then the following equality holds:

$$\sup_{\substack{\|\mu\|\leqslant 1\\ \mu\perp R(X)\\ \operatorname{supp}\mu\subset\partial X}} \left|\int h\,d\mu\right| = \lambda_h(X)$$

and there exists an analytic measure μ^* for which the supremum in the left-hand side is attained. From this we derive (Corollary 1) that on any X such that $R(X)\neq C(X)$ there exist nontrivial analytic measures.

Let us give a brief description of the contents of this paper. In Section 2 we list the results concerning function theory in finitely connected domains which we use in the other sections. In Section 3 we prove a few auxiliary results on analytic measures and λ_f. In Section 4 we give a proof of Theorem 1 and Corollary 1. Section 5 contains further properties of analytic measures and $\lambda(X)$. In particular, we obtain estimates of $\lambda(X)$ by means of simple geometric characteristics of X (area and perimeter). As an application of these estimates, we obtain the classical isoperimetric inequality with the sharp constants. In Section 6 we study the problem of characterization of annihilating measures of $R(X)$ on an arbitrary compact set. The main result is given by Theorem 3. Namely, let $H(X)$ denote the uniform closure on X of all functions harmonic in a neighborhood of X. If $H(X)|_{\partial X}=C(\partial X)$, then the weak $(*)$ closure of the linear span of all analytic measures coincides with $(R(X)|_{\partial X})^{\perp}$.

Notation. Everywhere in this paper m_2 denotes the area measure in $\mathbb{C}$, m_1 is the 1-dimensional Hausdorff measure in $\mathbb{C}$.

For $p\geqslant 1$, $L^p=\{f\colon \mathbb{C}\to\mathbb{C}$ such that $\|f\|_p=(\int_{\mathbb{C}}|f|^p\,dm_2)^{1/p}<\infty\}$.

When we consider the spaces L^p with respect to other measures, we will always specify those measures, e.g.,

$$L^1(d\zeta,\{z\colon|z|=1\}) = \left\{f\colon\{z\colon|z|=1\}\to\mathbb{C},\ \|f\|_{L^1(d\zeta)}=\int_{|z|=1}|f|\,|d\zeta|<\infty\right\};$$

$C_0^\infty=\{f\colon f\in C^\infty$ and f has a compact support in $\mathbb{C}\}$.

If μ is a complex, compactly supported Borel measure, then by the Cauchy transform of μ we understand the function

$$\hat{\mu}(z)=\int_{\mathbb{C}}\frac{d\mu(\zeta)}{\zeta-z}.$$

Clearly $\hat{\mu}(z) \in L^1_{\mathrm{loc}}$ (i.e., $\|\hat{\mu}\|_{L^1(K)} < \infty$ for any compact set K) and $\hat{\mu}(z)$ is analytic outside of $\operatorname{supp} \mu$. The detailed survey on Cauchy transforms can be found in [11].

II. Preliminaries

The following two propositions are the well-known corollaries of the Hahn–Banach and Banach–Alaoglu theorems. For the proofs, see [6, 8, 17, 24]. S. Ya. Khavinson's papers [16, 17] contain also many applications of these results to the function theory in multiply-connected domains.

Proposition 1. *Let E be a Banach space and let $E_0 \subset E$ be a subspace. Let $l_0 \in E^*$. Then,*

$$\sup_{\substack{f \in E_0 \\ \|f\| \leqslant 1}} |l_0(f)| = \inf_{l \in E_0^{\perp}} \|l_0 - l\| \tag{1}$$

and there exists $l^* \in E_0^{\perp}$ for which the infinum in (1) is attained.

Proposition 2. *Let E_1, E_0 be the same as in Proposition 1. Let $\omega \in E$. Then,*

$$\inf_{f \in E_0} \|\omega - f\|_E = \sup_{\substack{l \in E_0^{\perp} \\ \|l\| \leqslant 1}} |l(\omega)|. \tag{2}$$

Moreover, there exists $l^* \in E_0^{\perp}$ for which the supremum in (2) is attained.

The following Proposition is the well-known generalization of the F. and M. Riesz theorem to finitely connected domains. We refer to [7] and [18] for the proof and further discussion on the function theory in finitely connected domains.

Proposition 3. *Let G be a finitely connected domain with a rectifiable Jordan boundary Γ. Let μ be a complex measure supported on Γ such that $\mu \perp R(\bar{G})$. Then, there exists a function $f(z) \in E_1(G)$ such that*

$$\mu = f(\zeta)\, d\zeta\,|_{\Gamma},$$

and

$$\hat{\mu}(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{\Gamma} \frac{d\mu(\zeta)}{\zeta - z} = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)\, d\zeta}{\zeta - z} = \begin{cases} f(z), & z \in G, \\ 0, & z \notin \bar{G}. \end{cases} \tag{3}$$

Conversely, if $f \in E_1(G)$, then (3) holds.

We recall that if G is a finitely connected domain, then $H^\infty(G)$ denotes the Banach space of all bounded analytic functions in G with the norm

$$\|f\|_{H^\infty} = \|f\|_{L^\infty(G)} = \|f\|_{L^\infty(d\zeta, \partial G)}.$$

Details concerning the definitions and properties of Hardy classes in multiply-connected domains can be found in [7, 14,16, 18].

The following proposition is also known (see [16, 17]).

PROPOSITION 4. *Let X be a finitely connected compact set and let ∂X consist of analytic closed Jordan curves. Let $H^\infty(X) = \oplus_{i=1}^n H^\infty(X_i)$, where X_i are connected components of X, $\|\cdot\|_{H^\infty(X)} = \max_{1 \leqslant i \leqslant n} \|\cdot\|_{H^\infty(X)}$, $E_1(X) = \oplus_{i=1}^n E_1(X_i)$, $\|\cdot\|_{E_1(X)} = \sum_{i=1}^n \|\cdot\|_{E_1(X_i)}$. Let $\omega(\zeta) \in C(\partial X)$. Then,*

$$\inf_{\phi \in H^\infty(X)} \|\omega - \phi\|_{L^\infty(\partial X)} = \inf_{\phi \in R(X)} \|\omega - \phi\|_{C(\partial X)} = \sup_{\substack{f \in E_1(X) \\ \|f\|_{E_1(X)} \leqslant 1}} \left| \int_{\partial X} f(\zeta)\, \omega(\zeta)\, d\zeta \right|.$$

Moreover, $\exists f^* \in E_1(X)$, for which the supremum is attained.

Proof. For the reader's convenience, we assume that X has only one component. All generalizations for the case $n > 1$ are straightforward and we omit them. Put $E = L^1(d\zeta, \partial X)$, $E_0 = E_1(X)|_{\partial X}$. Since $(L^1)^* = L^\infty$, from Proposition 1, we obtain

$$\sup_{\substack{f \in E_1(X) \\ \|f\|_{E_1(X)} \leqslant 1}} \left| \int_{\partial X} f(\zeta)\, \omega(\zeta)\, d\zeta \right| = \inf_{\substack{\phi \in L^\infty(d\zeta, \partial X) \\ \phi \perp E_1(X)}} \|\omega(\zeta) - \phi(\zeta)\|_{L^\infty(\partial X)}.$$

Let $\phi(\zeta) \in L^\infty(d\zeta, \partial X)$ and $\phi \perp E_1(X)$. Then, the measure $\phi(\zeta)\, d\zeta|_{\partial X}$ is orthogonal to $R(X)$. According to Proposition 3 this implies that $\phi(\zeta)$ represents boundary values of a function $\phi(z) \in E_1(X)$. Since $\phi(\zeta) \in L^\infty(d\zeta, \partial X)$, $\phi(z) \in H^\infty(X)$ (see [18]). So, we have proved the equality

$$\inf_{\phi \in H^\infty(X)} \|\omega - \phi\|_{L^\infty(d\zeta, \partial X)} = \sup_{\substack{f \in E_1(X) \\ \|f\|_{E_1(X)} \leqslant 1}} \left| \int_{\partial X} f(\zeta)\, \omega(\zeta)\, d\zeta \right|.$$

Since $\omega \in C(\partial X)$, from F. Riesz' theorem and Proposition 2, putting $E = C(\partial X)$, $E_0 = R(X)|_{\partial X}$, we get that

$$\inf_{\phi \in R(X)} \|\omega - \phi\|_{C(\partial X)} = \sup_{\substack{\mu \perp R(X) \\ \operatorname{supp} \mu \subset \partial X}} \left| \int_{\partial X} \omega\, d\mu \right|.$$

According to Proposition 3 the above supremum is equal to

$$\sup_{\substack{f \in E_1(X) \\ \|f\|_{E_1} \leqslant 1}} \left| \int_{\partial X} \omega(\zeta) f(\zeta)\, d\zeta) \right|.$$

Also, in view of Propositions 2 and 3 $\exists f^* \in E_1(X)$ for which the supremum is attained. The proof is complete.

3. Properties of Analytic Measures and λ_f.

From now on $\|\cdot\|_X$ denotes the uniform norm on a compact set X.

PROPOSITION 5. *Let μ be an analytic measure. Then, $\mu \perp R(X)$.*

Proof. Take $z \in C\backslash X$. Then $1/(\zeta - z)$, $\zeta \in X$, is continuous in the neighborhood of X. Let $\{f_n\}^\infty$ be the sequence of $E_1(X_n)$-functions defining μ. Then, according to Proposition 3,

$$\hat{\mu}(z) = \int_X \frac{d\mu(\zeta)}{\zeta - z} = \lim_{k\to\infty} \int_{\partial X_{n_k}} \frac{f_{n_k}(\zeta)\, d\zeta}{\zeta - z} = 0,$$

since $z \notin X_{n_k}$ for all n_k sufficiently large. So, $\hat{\mu} \equiv 0$ on $\mathbb{C}\backslash X$. Therefore, $\mu \perp R(X)$.

Recall that $\lambda(X) = \lambda_{\bar{\zeta}}(X)$. The following proposition is a simple corollary of the Stone–Weierstrass theorem (see [10, 24]).

PROPOSITION 6. $\lambda(X) = 0 \Leftrightarrow R(X) = C(X)$.

PROPOSITION 7. *Let $h \in H(X)$. Then, $\lambda_h(X) = \inf_{\phi \in R(X)} \|h - \phi\|_{\partial X}$.*

Proof. Fix $\phi_0 \in R(X)$. The function $h - \phi_0$ is a harmonic function on $\dot{X}$. Then, the function $|h - \phi_0|$ is subharmonic on $\dot{X}$. Therefore, according to the maximal principle, we have

$$\|h - \phi_0\|_X = \sup_{z\in\partial X} |h(z) - \phi_0(z)| = \|h - \phi_0\|_{\partial X}.$$

Hence, $\lambda_h(X) = \inf_{\phi_0 \in R(X)} \|h - \phi_0\|_{\partial X}$.

PROPOSITION 8. *Let $X = \bigcap_{i=1}^\infty X_i$, where $\{X_i\}_{i=1}^\infty$ is a decreasing sequence of finitely connected sets bounded by analytic Jordan curves. Let $h(\zeta)$ be a harmonic function in the neighborhood of X_1. Then,*

$$\lambda_h(X_1) \geqslant \lambda_h(X_2) \geqslant \cdots \geqslant \lambda_h(X),$$

and (4)

$$\lim_{n\to\infty} \lambda_h(X_n) = \lambda_h(X).$$

Proof. Fix n. Take any $\phi_0 \in R(X_n)$. Then

$$\lambda_h(X_{n+1}) \overset{\text{def}}{=} \inf_{\phi \in R(X_{n+1})} \|h-\phi\|_{X_{n+1}} \leqslant \|h-\phi_0\|_{X_{n+1}} \leqslant \|h-\phi_0\|_{X_n}.$$

Taking an infimum over $\phi_0 \in R(X_n)$, we obtain

$$\lambda_h(X_{n+1}) \leqslant \lambda_h(X_n).$$

The same argument shows that for any n, $\lambda_h(X_n) \geqslant \lambda_h(X)$. Now, fix $\varepsilon > 0$. Choose ϕ_1 analytic in the neighborhood of X such that

$$\|h-\phi_1\|_X = \max_{\zeta \in \partial X} |h(\zeta)-\phi_1(\zeta)| \leqslant \lambda_h(X)+\varepsilon.$$

As $h-\phi_1$ is uniformly continuous near ∂X, we can find a neighborhood U of X such that $\phi_1 \in R(\bar{U})$ and

$$\|h-\phi_1\|_X \leqslant \|h-\phi_1\|_{\bar{U}} \leqslant \|h-\phi_1\|_X + \varepsilon.$$

We can choose $n_0 : \forall n > n_0$, $\partial X_n \subset U$. Then, $\phi_1(\zeta) \in R(X_{n_0})$. Moreover, according to the choice of U and ϕ_1, we have for $n > n_0$,

$$\|h-\phi_1\|_{X_n} = \|h-\phi_1\|_{\partial X_n} \leqslant \|h-\phi_1\|_{\bar{U}} \leqslant \|h-\phi_1\|_X + \varepsilon \leqslant \lambda_h(X) + 2\varepsilon.$$

Hence,

$$\lambda_h(X_n) \leqslant \|h-\phi_1\|_{X_n} \leqslant \lambda_h(X) + 2\varepsilon.$$

Therefore,

$$\lim_{n\to\infty} \lambda_h(X_n) \leqslant \lambda_h(X) + 2\varepsilon.$$

Since ε was arbitrary, we obtain that

$$\lim_{n\to\infty} \lambda_h(X_n) \leqslant \lambda_h(X).$$

Above we have shown that

$$\lambda_h(X_n) \geqslant \lambda_h(X) \qquad \text{for all } n.$$

Thus,

$$\lim_{n\to\infty} \lambda_h(X_n) = \lambda_h(X).$$

The following Proposition summarizes some of the information in Sections 2 and 3.

Proposition 9. *Let $f \in C(X)$. Then,*

$$\sup_{\substack{\mu \perp R(X) \\ \|\mu\| \leqslant 1}} \left| \int_X f\,d\mu \right| = \lambda_f(X) \overset{\text{def}}{=} \inf_{\phi \in R(X)} \|f - \phi\|_X. \tag{5}$$

Moreover, if $f \in H(X)$, then

$$\sup_{\substack{\mu \perp R(X) \\ \|\mu\| \leqslant 1 \\ \operatorname{supp} \mu \subset \partial X}} \left| \int_{\partial X} f\,d\mu \right| = \inf_{\phi \in R(X)} \|f - \phi\|_{\partial X} = \lambda_f(X). \tag{6}$$

There always exists an extremal measure μ^*, $\|\mu^*\| = 1$ for which the supremum in (5) (*or in* (6)) *is attained.*

Proof. Equation (5) directly from Proposition 2 and F. Riesz' representation theorem for $C(X)^*$ if we put $E = C(X)$, $E_0 = R(X)$. Equation (6) also follows immediately from Propositions 2 and 7 if we set $E = C(\partial X)$, $E_0 = R(X)|_{\partial X}$.

4. Analytic Measures as Solutions of Extremal Problems

Theorem 1. *Let X be an arbitrary compact set. Let $\{X_n\}_1^\infty$ be a decreasing sequence of finitely connected compact sets with analytic boundaries such that $\bigcap_{n=1}^\infty X_n = X$. Let $h(\zeta)$ be a function harmonic in the neighborhood of X. Then,*

$$\sup_{\substack{\|\mu\| \leqslant 1 \\ \mu \perp R(X)}} \left| \int_{\partial X} h\,d\mu \right| = \sup_{\substack{\|\mu\| \leqslant 1 \\ \mu \perp R(X) \\ \operatorname{supp} \mu \subset \partial X}} \left| \int_{\partial X} h\,d\mu \right|$$

$$= \inf_{\phi \in R(X)} \|h - \phi\|_{\partial X} = \lambda_h(X) \tag{7}$$

and there exists the measure μ^* analytic *with respect to the sequence $\{X_n\}$ for which the supremum in* (7) *is attained.*

Proof. Equation (7) follows immediately from Proposition 9, since $h \in H(X)$. So, it remains to show that there exists an *analytic measure μ^** giving the supremum. Without loss of generality we can assume that h is harmonic on X_1. According to Proposition 7

$$\lambda_h(X_n) = \inf_{\phi \in R(X_n)} \|h - \phi\|_{\partial X_n}$$

for all n. As $h \in C(\partial X_n)$, from Proposition 4 it follows that for all n

$$\lambda_h(X_n) = \sup_{\substack{f \in E_1(X_n) \\ \|f\|_{E_1(X_n)} \leqslant 1}} \left| \int_{\partial X_n} f(\zeta)\, h(\zeta)\, d\zeta \right| \tag{8}$$

and $\exists f_n^*(\zeta) \in E_1(X_n)$, $\|f_n^*\|_{E_1(X_n)} = 1$ for which the supremum in (8) is attained. Consider the sequence of measures $\mu_n = f_n^*(\zeta)\, d\zeta\,|_{\partial X_n}$. As $\|\mu_n\| = 1$, there exists a subsequence $\{\mu_{n_k}\}$ converging weak $(*)$ to an analytic measure μ^*. Furthermore, according to our choice of f_n^* and Proposition 8, we obtain

$$\left| \int_{\partial X} h\, d\mu^* \right| = \lim_{k \to \infty} \left| \int_{\partial X_{n_k}} h f_{n_k}^*\, d\zeta \right| = \lim_{k \to \infty} \lambda_h(X_{n_k}) = \lambda_h(X).$$

Thus, μ^* is an extremal measure for the problem (7). The proof is complete.

COROLLARY 1. *Let X be a compact set such that $R(X) \neq C(X)$. Then, there always exist nontrivial* ($\not\equiv 0$) *analytic measures on ∂X orthogonal to $R(X)$.*

Proof. In view of Proposition 6, $\lambda(X) > 0$. According to Theorem 1 there exists an analytic measure μ^*, giving the supremum in (7) for $h = \bar{\zeta}$. Since

$$\left| \int_{\partial X} \bar{\zeta}\, d\mu^* \right| = \lambda(X) > 0,$$

we conclude that $\mu^* \not\equiv 0$.

5. FURTHER REMARKS ON ANALYTIC MEASURES AND $\lambda(X)$

We start out this section with the following remark:

Observation. Unfortunately, the definition of analytic measures depends on a choice of the sequence $\{X_n\}_1^\infty$. Namely, let $\{X_n'\}_1^\infty$ be another decreasing sequence of the finitely connected compact sets with analytic boundaries converging to X (i.e., $X = \bigcap_{n=1}^\infty X_n'$). Let μ be an analytic measure on ∂X defined by the sequence of the analytic differentials $\mu_n = f_n\, d\zeta\,|_{\partial X_n}$, $f_n \in E_1(X_n)$. Then, there is no guarantee that the measures $\mu_n' = f_n\, d\zeta\,|_{\partial X_n'}$ (assuming $X_{n+1} \supset X_n' \supset X_n$) even have uniformly bounded total variations. Thus we cannot talk about the weak $(*)$ convergence of μ_n' to μ. At the same time in the following Proposition we discuss the situation when some information can be obtained.

We need the following lemma.

LEMMA 1. *Let $\{\mu_n\}_1^\infty$ be a sequence of finite Borel measures which converges in the weak $(*)$ topology to the measure μ. For the sake of* *simplicity we assume that* $\hat{\mu}_n \equiv 0$ *for all* n *outside of a fixed compact set* K. *Then,* $\hat{\mu}_n \to \hat{\mu}$ *in the weak topology of* L^1.

Proof. Fix $\phi \in L^\infty$. Then

$$\int_K \phi\hat{\mu}_n = \int_K \phi \int_{\mathbb{C}} \frac{d\mu_n(\zeta)}{\zeta - z}\, dx\, dy = \left[\left(\frac{1}{z} * \mu_n\right) * \phi\right](0)$$
$$= \left[\mu_n * \left(\frac{1}{z} * \phi\right)\right](0).$$

Since $1/z \in L^1_{\mathrm{loc}}$ and $\phi \in L^\infty$, $(1/z) * \phi$ is continuous in $\mathbb{C}$. Hence,

$$\lim_{n\to\infty} \int_K \hat{\mu}_n \phi\, dx\, dy = \lim_{n\to\infty} \int_K \left(\frac{1}{z} * \phi\right) d\mu_n = \int_K \left(\frac{1}{z} * \phi\right) d\mu = \int_K \hat{\mu}\phi\, dx\, dy.$$

The lemma is proved.

PROPOSITION 10. *Let* X, $\{X_n\}_1^\infty$, $\{X_n'\}_1^\infty$, μ, $\{f_n\}_1^\infty$, μ_n, μ_n' *be as above. Let* $\mu_n \to \mu$ *weak* $(*)$. *Assume* $\|\mu_n'\| = \int_{\partial X_n'} |f_n|\,|d\zeta| < +\infty$. *Then, there exists a subsequence* $\{\mu_{n_k}'\}$ *such* $\mu_{n_k}' = f_{n_k}\, d\zeta\,|_{\partial X_{n_k}'}$ *converges weak* $(*)$ *to the measure* μ. *Moreover, if any subsequence* $\{\mu_{n_k}'\}$ *converges weak* $(*)$ *to a certain measure* μ', *then* $\mu' = \mu$.

Proof. Choose a subsequence $\{\mu_{n_k}'\}$ converging weak $(*)$ to a measure μ' as $k \to \infty$. (We can do it since $\|\mu_n'\|$ are uniformly bounded.) It remains to be shown that $\mu' \equiv \mu$. Let $\sigma \equiv \mu' - \mu$. Then $\hat{\sigma}(z) \equiv 0$, $z \in C\backslash X$, since μ, μ' are orthogonal to $R(X)$. At the same time, for each $z \in X$ we have, according to Proposition 3,

$$\frac{1}{2\pi i}\int_{\partial X_{n_k}'} \frac{f_{n_k}\, d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\partial X_{n_k}} \frac{f_{n_k}\, d\zeta}{\zeta - z} = f_{n_k}(z) - f_{n_k}(z) \equiv 0.$$

So $\hat{\mu}_{n_k}(z) - \hat{\mu}_{n_k}'(z) \equiv 0$ on X for all n_k. From Lemma 1 it follows that $\hat{\mu}_{n_k}(z) - \hat{\mu}_{n_k}'(z) \to \hat{\sigma}(z)$ in the weak topology of L^1. Hence, $\hat{\sigma}(z) \equiv 0$ a.e. on X. So $\hat{\sigma}(z) \equiv 0$ a.e. This implies that $\sigma \equiv 0$, i.e., $\mu \equiv \mu'$. The Proposition is proved.

We recall that

$$\frac{\partial}{\partial \bar{z}}\hat{\mu} = -\pi\mu$$

in the distribution sense (see [11]).

PROPOSITION 11. *Let* μ *be an analytic measure on* ∂X *defined by the sequence* $\{f_n\}_1^\infty$, $f_n \in E_1(X_n)$, $\operatorname{supp} f_n \subset X_n$. *Then, the following statements hold.*

(i) $\operatorname{supp} \hat{\mu}(z) \subset X$.

(ii) *There exists a subsequence* $\{f_{n_k}\}$ *such that* $f_{n_k} \to \hat{\mu}(z)$ *in the weak topology of* L^1 *on* X.

(iii) *There exists a sequence of functions* $\{\phi_k\}$ *and a sequence of measures* ψ_k *such that* (a) $\phi_k \in L^1$, $\operatorname{supp} \phi_k \subset X_1$; (b) ϕ_k *are analytic in a neighborhood of* X, $\|\phi_k\|_{L^1}$ *are uniformly bounded*; (c) *for all* k, ψ_k *is concentrated on a finite union of the boundary curves of* $\{\partial X_n\}_1^\infty$; $\psi_k|_{\partial X_n}$ *are absolutely continous with respect to* $d\zeta$ *for all* n, $\|\psi_k\|$ *are uniformly bounded and* $\partial \phi_k / \partial \bar{z} = \psi_k$ *in the distribution sense*; (d) $\phi_k \to \hat{\mu}(z)$ *a.e.*

On the other hand, let $f(z) \in L^1$, $\operatorname{supp} f \subset X$. *Then, there exists an analytic measure such that* $(1/2\pi i)\, \hat{\mu}(z) = f(z)$ *a.e. provided that one of the following conditions holds.*

(I) $\exists \{f_n\}_1^\infty$, $f_n \in E_1(X_n)$, $\|f_n\|_{E_1(X_n)} \leqslant M_1 < +\infty$ *and* $f_n \to f$ *weakly in* L^1.

(II) $\exists \{\phi_n\}_1^\infty$, $\phi_n \in E_1(X_n)$, $\|\phi_n\|_{E_1(X_n)} \leqslant M_2 < +\infty$, $\operatorname{supp} \phi_n \subset X_n$ *and* $\phi_n \to f$ *a.e.*

Proof. Part 1. (i) follows from Proposition 5; (ii) follows directly from the definition of analytic measures and Lemma 1. (iii) from (ii) and properties of the weak convergence (see, e.g. [24]) it follows that we can find a sequence of functions

$$\phi_k = \sum_{j=1}^{n_k} \alpha_k^j f_{n_j}, \qquad \text{where} \quad \alpha_k^j \geqslant 0, \quad \sum_{j=1}^{n_k} \alpha_k^j = 1$$

such that $\phi_k \to \hat{\mu}(z)$ in the L^1-norm. Since

$$\phi_k(z) = \frac{1}{2\pi i} \sum_{j=1}^{n_k} \alpha_k^j \int_{\partial X_{n_j}} \frac{f_{n_j}(\zeta)\, d\zeta}{\zeta - z},$$

for any $z \in X_{n_k}$, for all k, then

$$\frac{\partial}{\partial \bar{z}} \phi_k(z) = -2i \sum_{j=1}^{n_k} \alpha_k^j f_{n_j}(\zeta)\, d\zeta\, |_{\partial X_{n_j}},$$

and

$$\left\| \frac{\partial}{\partial \bar{z}} \phi_k(z) \right\| \leqslant 2 \sup_n \|f_n\|_{E_1(X_n)} < +\infty.$$

Taking a subsequence which converges to $\hat{\mu}(z)$ a.e., we finish the proof of Part 1.

Part II. Note that (II) implies (I). In reality let E be a measurble set in $\mathbb{C}$. Let $\{\phi_n\}$ satisfy (II). Then

$$\left|\int_E \phi_n \, dx \, dy\right| = \left|\int_E \left\{\int_{\partial X_n} \frac{1}{2\pi i} \frac{\phi_n(\zeta) \, d\zeta}{\zeta - z} \, dx \, dy\right|\right. \leqslant M_2 \int_E \frac{1}{|z|} \, dx \, dy.$$

Since $1/z \in L^1_{\text{loc}}$, this implies that the integrals $\{\int \phi_n \, dx dy\}$ are uniformly absolutely continuous. Hence, for any $\phi_0 \in L^\infty$, we have

$$\int \phi_n \phi_0 \to \int f\phi_0 \, dx \, dy,$$

i.e., (II) $\Rightarrow$ (I).

Assume that (I) holds. Then, $f_n \to f$ in the distribution sense. Since according to Proposition 3 we have

$$f_n(z) = \frac{1}{2\pi i} \int_{\partial X_n} \frac{f_n(\zeta) \, d\zeta}{\zeta - z}, \qquad z \in X_n,$$

then putting $\mu_n = f_n(\zeta) \, d\zeta \,|_{\partial X_n}$, we obtain $\hat{\mu}_n(z) \to f(z)$ weakly in L^1. Therefore, $(\partial/\partial\bar{z}) \hat{\mu}_n \to (\partial/\partial\bar{z}) f$ in the distribution sense. Also,

$$-2i \frac{\partial}{\partial\bar{z}} \hat{\mu}_n = f_n(\zeta) \, d\zeta \,|_{\partial X_n} = \mu_n.$$

As $\|\mu_n\| \leqslant M_1 < +\infty$ we can choose a subsequence $\{\mu_{n_k}\}$ converging weak $(*)$ to the measure μ. Then, $\{\mu_{n_k}\}$ converges to μ in the distribution sense. Hence, $(i/2) \mu \equiv (\partial/\partial\bar{z}) f$. This implies (see [11]) that $f(z) = \hat{\mu}(z)/2\pi i$ a.e. The proof is complete.

For the following we need to recall the concept of the sets with a finite perimeter due to De Giorgi and Federer (see [8, 9, 11, 15, 21]).

We shall call the compact set X a set of *finite perimeter* if there exists a measure μ such that

$$\frac{\partial}{\partial\bar{z}} \chi_X = \mu, \qquad \chi_X \equiv \begin{cases} 1, & \text{on} \quad X, \\ 0, & \text{on} \quad \mathbb{C} \setminus X. \end{cases}$$

From the general theory of sets with a finite perimeter contained in [8], it follows that if X has a finite perimeter, then there exists a m_1-measurable set $B_X \subset X$ contained in a countable union of rectifiable curves, and such that

$$\mu \equiv \frac{1}{2\pi i} \, d\zeta \,|_{B_X}.$$

We will call $2\pi \|\mu\|$ the *perimeter* of X and denote it by $P(X)$.

PROPOSITION 12. *Let X have a finite perimeter. Then*

$$\lambda(X) \geqslant 2\,\frac{m_2(X)}{P(X)}.$$

The inequality is sharp since for $X = \{z: |z| \leqslant 1\}$ it becomes an equality.

Proof. Let ϕ be a function analytic in the neighborhood of X. Then

$$\begin{aligned} P(X)\cdot\|\bar{\zeta}-\phi\|_X &\geqslant \sup_{B_X} |\bar{\zeta}-\phi(\zeta)|\, P(X) \\ &= \int_{B_X} \sup_{B_X} |\bar{\zeta}-\phi(\zeta)|\,|dz| \geqslant \left| \int_{B_X} (\bar{z}-\phi(z))\, dz \right|. \end{aligned} \tag{8}$$

Since $(1/2\pi i)\, dz\,|_{B_X} = \chi_X$, $dz\,|_{B_X} \perp R(X)$. So, from (8) we obtain

$$|\bar{\zeta}-\phi(\zeta)|\cdot P(X) \geqslant \left| \int_{B_X} \bar{z}\, dz \right|.$$

According to the Gauss–Green formula for sets with a finite perimeter (see [8]), we obtain

$$\left| \int_{B_X} \bar{z}\, dz \right| = 2 \left| \int_X \frac{\partial}{\partial \bar{z}}\, \bar{z}\, dx\, dy \right| = 2m_2(X).$$

So, $\|\zeta - \phi(\zeta)\|_X \geqslant 2(m_2(X)/P(X))$ for all $\phi \in R(X)$. Taking an infimum, we conclude that

$$\lambda(X) \geqslant 2\,\frac{m_2(X)}{P(X)}.$$

If X is the unit disk, then $m_2(X) = \pi$, $P(X) = 2\pi$. Also,

$$\lambda(X) \leqslant \|\zeta - 0\|_X = 1.$$

Therefore, $\lambda(X) = 1$. The proof is complete.

PROPOSITION 13. *Let X be a compact set in $\mathbb{C}$. Then,*

$$\lambda(X) = \frac{1}{\pi} \sup_{\substack{\mu \perp R(X) \\ \|\mu\| \leqslant 1}} \left| \int_X \hat{\mu}(z)\, dx\, dy \right|.$$

Moreover, there exists an analytic measure μ^* for which supremum is attained.

Proof. According to Proposition 9, we have

$$\lambda(\chi) = \sup_{\substack{\mu \perp R(X) \\ \|\mu\| \leqslant 1}} \left| \int_X \bar{\zeta}\, d\mu \right|.$$

Fix $\mu \perp R(X)$. We can regard $\bar{\zeta}|_X$ as a restriction of a C_0^∞-function ψ on X. Then, according to Green's formula and the fact that $\hat{\mu} \equiv 0$ outside of X, we obtain

$$\begin{aligned}\int_X \bar{\zeta}\, d\mu &= \frac{1}{\pi} \int_X \left\{ \int_{\mathbb{C}} \int \frac{1}{z-\zeta} \frac{\partial \psi}{\partial \bar{z}}\, dx\, dy \right\} d\mu(\zeta) \\ &= \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \psi}{\partial \bar{z}} \hat{\mu}(z)\, dx\, dy = \frac{1}{\pi} \int_X \hat{\mu}(z)\, dx\, dy.\end{aligned}$$

Hence,

$$\lambda(X) = \frac{1}{\pi} \sup_{\substack{\mu \perp R(X) \\ \|\mu\| \leqslant 1}} \left| \int_X \hat{\mu}(z)\, dx\, dy \right|.$$

Taking into account that $\bar{\zeta}$ is harmonic in the neighborhood of X and applying Theorem 1, we finsih the proof.

Remark. Proposition 13 explains why we called $\lambda(X)$ the rational capacity of X. According to this proposition, $\lambda(X)$ shows "how much" of the Cauchy transform of measures orthogonal to $R(X)$ can be accumulated on X.

The following inequality has been observed by Alexander (see [2]).

PROPOSITION 14. *Let X be any compact set in $\mathbb{C}$. Then,*

$$\sqrt{m_2(X)/\pi} \geqslant \lambda(X).$$

The inequality is sharp since for the unit disk it becomes equality.

We shall indicate the proof. According to Proposition 13, we have

$$\lambda(X) = \frac{1}{\pi} \sup_{\substack{\mu \perp R(X) \\ \|\mu\| \leqslant 1}} \left| \int_X \hat{\mu}(z)\, dx\, dy \right|.$$

Fix $\mu \perp R(X)$, $\|\mu\| \leqslant 1$. Then, applying Fubini's theorem, we obtain

$$\left| \int_X \hat{\mu}(z)\, dx\, dy \right| = \left| \int_X d\mu(\zeta) \int_X \frac{dx\, dy}{z-\zeta} \right| \leqslant \max_{\zeta \in X} \left| \int_X \frac{dx\, dy}{z-\zeta} \right|.$$

A beautiful argument given in the classical paper of Ahlfors and Beurling [1] provides the estimate

$$\max_{\zeta \in X} \left| \int_X \frac{dx\,dy}{z-\zeta} \right| \leqslant \sqrt{\pi m_2(X)},$$

from which our statement easily follows.

COROLLARY 2. (Isoperimetric inequality—cf. [8]) *Let X be a compact set in $\mathbb{C}$. Then*

$$P(X)^2 \geqslant 4\pi m_2(X).$$

The inequality is sharp since it becomes an inequality for the unit disk.

Proof. The result follows immediately from Propositions 12 and 14.

6. ANNIHILATING MEASURES OF $R(X)$ AND THE SPACE OF ANALYTIC MEASURES

Let X be an arbitrary compact set. Let $\{X_n\}_1^\infty$, $X_1 \supset \cdots$, $\bigcap_{n=1}^\infty X_n = X$ be a fixed sequence of finitely connected compact sets with Jordan analytic boundaries converging to X.

Let $M = M(X, \{X_n\})$ be a weak $(*)$ closure of the linear span of all analytic measures relative to the sequence $\{X_n\}$.

THEOREM 2. *Let $f \in H(X)$. Then, there exists a measure $\mu^* \in M$, $\|\mu^*\| = 1$ such that μ^* is an extremal measure in the left hand side of problem* (7).

Proof. Let $\{h_n\}$ be a sequence of functions such that each h_n is harmonic in a certain neighborhood of X and $\|f - h_n\|_{\partial X} < 1/n$. Put

$$\sup_{\substack{\|\mu\| \leqslant 1 \\ \mu \perp R(X)}} \left| \int_X h_n\, d\mu \right| = L_n.$$

Let μ_n^* be analytic measures such that $\|\mu_n^*\| = 1$ and μ_n^* is an extremal measure for this extremal problem. The existence of μ_n^* is guaranteed by Theorem 1. By Proposition 9 there exists an extremal measure in problem (7) for f. We denote this measure by μ_0^*. Applying Proposition 9 and Theorem 1 again, we obtain

$$
\begin{aligned}
|\lambda_f(X)-L_n| &= \left| \sup_{\substack{\|\mu\|\leqslant 1\\ \mu\perp R(X)}} \left|\int_X f\,d\mu\right| - \left|\int_X h_n\,d\mu_n^*\right| \right| \\
&= \left| \left|\int_X f\,d\mu_0^*\right| - \left|\int_X h_n\,d\mu_n^*\right| \right| \\
&\leqslant \begin{cases} \left|\int_X f\,d\mu_0^*\right| - \left|\int_X h_n\,d\mu_0^*\right|, & \text{if } \lambda_f(X)\geqslant \lambda_{h_n}(X), \\ \left|\int_X h_n\,d\mu_n^*\right| - \left|\int_X f\,d\mu_n^*\right|, & \text{if } \lambda_f(X)<\lambda_{h_n}(X) \end{cases} \\
&\leqslant \|f-h_n\|_X = 1/n \to 0 \qquad \text{as} \quad n\to\infty.
\end{aligned}
$$

At the same time, since $\|\mu_n^*\|=1$, there exists a subsequence $\{\mu_{n_k}^*\}$ converging weak $(*)$ to a measure μ^*. Therefore,

$$
\left|\int_X f\,d\mu^*\right| = \lim_{k\to\infty}\left|\int_X f\,d\mu_{n_k}^*\right| = \lim_{k\to\infty}\left|\int_X h_{n_k}\,d\mu_{n_k}^*\right| = \lim_{k\to\infty} L_{n_k} = \lambda_f(X).
$$

Hence μ^* is an extremal measure. Since $\mu^* = \text{weak } (*) \lim_{k\to\infty}\mu_{n_k}^*$ and $\mu_{n_k}^*\in M(X, \{X_n\})$, $\mu^*\in M$. The theorem is proved.

We recall that if E_1 is a subspace of the space E^* dual to the linear topological space E, then ${}^\perp E_1 = \{x\in E\colon \forall f\in E_1, f(x)=0\}$.

THEOREM 3. *Let X, $\{X_n\}_1^\infty$ be as above. If $H(X)|_{\partial X} = C(\partial X)$, then $M=(R(X)|_{\partial X})^\perp$.*

Proof. It is clear that $M\subset (R(X)|_{\partial X})^\perp$. Assume that the theorem is false. Then, there exists a measure μ_0 such that $\operatorname{supp}\mu_0\subset\partial X$, $\mu_0\perp (R(X)|_{\partial X})$ and $\mu_0\notin M$. Since M is weak $(*)$ closed, it is closed in the strong (normed) topology in $C(\partial X)^*$. Hence,

$$
\inf_{\mu\in M}\|\mu_0-\mu\|>0. \tag{9}
$$

Consider the following extremal problem in $C(\partial X)$.

$$
\sup_{\substack{f\in{}^\perp M\\ \|f\|_{\partial X}\leqslant 1}} \left|\int_{\partial X} f\,d\mu_0\right|.
$$

According to Proposition 1, we have

$$
\sup_{\substack{f\in{}^\perp M\\ \|f\|_{\partial X}\leqslant 1}} \left|\int_X f\,d\mu_0\right| = \inf_{\mu\in({}^\perp M)^\perp}\|\mu_0-\mu\|.
$$

As is known (see [24]), $({}^{\perp}M)^{\perp} = M$, since M is weak $(*)$ closed. So, from (9), we obtain

$$\sup_{\substack{f \in {}^{\perp}M \\ \|f\| \leq 1}} \left| \int_{\partial X} f \, d\mu_0 \right| = \inf_{\mu \in M} \|\mu_0 - \mu\| > 0.$$

Therefore, there exists $f_0 \in {}^{\perp}M$ such that

$$\left| \int_{\partial X} f_0 \, d\mu_0 \right| > 0. \tag{10}$$

Since $\mu_0 \in (R(X)|_{\partial X})^{\perp}$, (10) implies that $f_0 \notin R(X)|_{\partial X}$. At the same time, $f_0 \in H(X)|_{\partial X}$. Then, according to Proposition 7,

$$\inf_{\phi \in R(X)} \|f_0 - \phi\|_{\partial X} = \inf_{\phi \in R(X)} \|f_0 - \phi\|_X \stackrel{\text{def}}{=} \lambda_{f_0}(X) > 0.$$

Hence, in view of Theorem 2, there exists the measure $\mu^* \in M$ such that $\|\mu^*\| = 1$ and

$$\left| \int_{\partial X} f_0 \, d\mu^* \right| = \lambda_{f_0}(X) > 0.$$

But this is impossible, since $f_0 \in {}^{\perp}M$. This contradiction proves our theorem.

COROLLARY 3. *If $H(X)|_{\partial X} = C(\partial X)$, then the space $M(X, \{X_n\})$ does not depend on the sequence $\{X_n\}$.*

Remark. We point out that we never made any assumptions concerning the existence of the interior in X. So, Theorem 3 holds in particular for nowhere dense sets X for which $H(X) = C(X)$. Also, we did not put any restrictions on the connectivity of $\mathbb{C} \setminus X$. The latter was assumed in Bishop's papers [3 – 5] and in the further generalization of his results due to Oksendal (see [20]).

Now, we state the corollary giving certain geometric conditions on the set X under which one can apply Theorem 3.

We recall that the point $z \in \partial X$ is said to satisfy the Lebesgue condition if

$$\int_S \frac{dr}{r} = \infty,$$

where $S = \{r: 0 < r < 1, \{\zeta: |z - \zeta\} = r\} \cap X^c \neq \varnothing\}\}$.

COROLLARY 4. *Let X be a compact set in $\mathbb{C}$ such that each point on ∂X satisfies the Lebesgue condition. Then, $M(X, \{X_n\}) = ((R(X)|_{\partial X})^{\perp}$ for any sequence $\{X_n\}$ converging to X.*

The statement follows immediately from Theorem 3 and the well-known fact that for such X, $H(\bar{X})|_{\partial X} = C(\partial X)$ (see [10], Chap. II, Lemma 3.2).

We want to finish this section with a list of problems related to the subject of this paper.

PROBLEM 1. Let $\mathscr{M}(X)$ denote the weak $(*)$ closure of a linear span of the spaces $M_\alpha(X, \{X_n^\alpha\})$, where $\{X_n^\alpha\}_1^\infty$, $\alpha \in I$ are all possible sequences of finitely connected compact sets converging decreasingly to X. Is $\mathscr{M}(X) = (R(X)|_{\partial X})^\perp$? It is possible that this is false. Then, what are the necessary and sufficient conditions on X for this statement still to be true?

PROBLEM 2. To construct an example of a compact set X and a sequence of finitely connected compact sets $\{X_n\}$ converging to X such that

$$M(X, \{X_n\}) \neq (R(X)|_{\partial X})^\perp.$$

PROBLEM 3. Let $\{X_n^{(1)}\}$, $\{X_n^{(2)}\}$ be two sequences of finitely connected compact sets converging to X. What are necessary and sufficient conditions on $\{\partial X_n^{(1)}\}$ and $\{\partial X_n^{(2)}\}$ for $M(X, \{X_n^{(1)}\}) = M(X, \{X_n^{(2)}\})$? Furthermore, if any decreasing sequence $\{X_n\}$ leads to the same space $M(X, \{X_n\}) \subset (R(X)|_{\partial X})^\perp$, then what could we say about the geometry of X?

ACKNOWLEDGMENTS

This work is a part of the author's Ph.D. thesis submitted to Brown University. The author is indebted to his research supervisor, Professor John Wermer, for his invaluable advice, support, and encouragement.

REFERENCES

1. L. AHLFORS AND A. BEURLING, Conformal invariants and function theoretic null sets, *Acta Math.* **83** (1950), 101–129.
2. H. ALEXANDER, Projections of Polynomial hulls, *J. Funct. Anal.* **3** (1973), 13–19.
3. E. BISHOP, The structure of certain measures, *Duke Math. J.* **25**(2) (1958), 283–289.
4. E. BISHOP, Subalgebras of functions on a Riemann surface, *Pacific J. Math.* **8** (1958), 29–50.
5. E. BISHOP, Boundary measures of analytic differentials, *Duke Math. J.* **27** (3) (1960), 331–340.
6. N. DUNFORD AND J. T. SCHWARTZ, "Linear operators. Part I, General Theory," Interscience, New York/London, 1958.
7. P. DUREN, "Theory of H^p-Spaces," Academic Press, New York, 1970.
8. H. FEDERER, "Geometric Measure Theory," Springer-Verlag, Heidelberg/New York, 1969.

9. W. H. FLEMING, Functions whose partial derivatives are measure, *Illinois J. Math.* **4** (3) (1960), 452–478.
10. T. GAMELIN, "Uniform Algebras," Prentice–Hall, Englewood Cliffs, N.J., 1969.
11. J. GARNETT, "Analytic Capacity and Measure," Lecture Notes in Mathematics, Vol. 291, Springer-Verlag, Berlin/New York, 1972.
12. G. M. GOLUZIN, "Geometric Theory of Functions of a Complex Variable," Amer. Math. Soc., Providence, R. I., 1969; translated from Russian.
13. K. HOFFMAN, "Banach Spaces of Analytic Functions," Prentice–Hall, Englewood Cliffs, N.J., 1962.
14. D. KHAVINSON, Factorization theorems for different classes of analytic functions in multiply connected domains, *Pacific J. of Math.* **108** (2) (1983), 295–318.
15. D. KHAVINSON, "On a Geometric Approach to Problems Concerning Cauchy Integrals and Rational Approximation," Ph.D. dissertation, Brown University, June 1983.
16. S. YA. KHAVINSON, Extremal problems for certain classes of analytic functions in finitely connected domains, *Mat. Sb.* **26** (18) (3) (1955), 445–478. [Russian]
17. S. YA. KHAVINSON, "Principals of the Theory of Extremal Problems for Bounded Analytic Functions and Their Certain Generalizations," Ministry of high and school education SSSR, Moscow Institute of Constructional Engineering, Moscow, 1981. [Russian]
18. S. YA. KHAVINSON AND G. C. TUMARKIN, Classes of analytic functions in multiply connected domains, *in* "Issledovaniga po Sovremennym Problemam Teorii Funkcii Kompleksnogo Peremennogo," pp. 45–71, Moscow, 1960. [Russian] French transl. *in* "Fonctions d'une Variable Complexe, Problèmes Contemporains," pp. 37–71, Gauthiers–Villars, Paris, 1962.
19. S. N. MERGELYAN, Uniform approximation to functions of a complex variable (*Uspehi Mat. Nauk.*, **VII** (2) (48) (1952), 31–122. [Russian]
20. B. K. ØKSENDAL, $R(X)$ as a Dirichlet algebra and representation of orthogonal measures by differentials, *Math. Scand.* **29** (1971), 87–103.
21. K. PIETZ, Cauchy transforms and characteristic functions, *Pacific J. Math.* **58** (2) (1976), 197–200.
22. I. I. PRIVALOV, "Randeigenschaften Analytischer Funktionen," Deutscher Verlag der Wiss., East Berlin, 1956.
23. W. ROGOSINSKI AND H. SHAPIRO, On certain extremal problems for analytic functions. *Acta Math.* **90**(3) (1954), 287–318.
24. W. RUDIN, "Functional Analysis," McGraw–Hill, New York/San Francisco/Toronto/London, 1973.