



# XXTitle

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# XXTITLE

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*Dedication*



# Abstract

Útdráttur á ensku sem er að hámarki 250 orð.

# Útdráttur

Hér kemur útdráttur á íslensku sem er að hámarki 250 orð. Reynið að koma útdráttum á eina blaðsíðu en ef tvær blaðsíður eru nauðsynlegar á seinni blaðsíða útdráttar að hefjast á oddatölusíðu (hægri síðu).





# Preface

Formála má sleppa og skal þá fjarlægja þessa blaðsíðu. Formáli skal hefjast á odd-atölu blaðsíðu og nota skal Section Break (Odd Page).

Ekki birtist blaðsíðutal á þessum fyrstu síðum ritgerðarinnar en blaðsíðurnar teljast með og hafa áhrif á blaðsíðutal sem birtist með rómverskum tölum fyrst á efnisyfirliti.



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# Abbreviations

Í þessum kafla mega koma fram listar yfir skammstafanir og/eða breytuheiti. Gefið kaflanum nafn við hæfi, t.d. Skammstafanir eða Breytuheiti. Þessum kafla má sleppa ef hans er ekki þörf.

The section could be titled: Glossary, Variable Names, etc.



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# 1. Introduction

$\mathcal{O}(U)$  for open  $U \subset \mathbb{C}$  is the family of functions holomorphic on  $U$ .  $C(X)$  for open topological space  $X$  is the family of continuous functions from  $X$  to  $\mathbb{C}$ . We will use  $\mathbb{D}$  to refer to the open unit disk  $\{z \in \mathbb{C}; |z| < 1\}$ . The closed unit disk will get no special notation, but will be referred to by  $\overline{\mathbb{D}}$ . We will use  $\mathbb{T}$  to refer to the open unit circle  $\{z \in \mathbb{C}; |z| = 1\}$ .



## 2. Preliminaries

### 2.1. Measure theory

### 2.2. Complex analysis in one variable

#### 2.2.1. The disk algebra

**Definition 1.** A function is said to be in the disk algebra if it is holomorphic on  $\mathbb{D}$  and continuous on  $\overline{\mathbb{D}}$ . We will refer to this family of function  $\mathcal{A}$ .

When studying a family of function it is helpful to know when the limit of a sequence in the family is also in the family.

**Lemma 1.** Let  $K$  be a compact subset of  $\mathbb{C}$  and  $(f_n)_{n \in \mathbb{N}}$  be a sequence of continuous functions on  $K$  with limit  $f$ . Then  $f$  is the uniform limit of  $(f_n)_{n \in \mathbb{N}}$ .

**Lemma 2.** Let  $U$  be a subset of  $\mathbb{C}$  and  $(f_n)_{n \in \mathbb{N}}$  be a sequence of continuous functions on  $K$  with uniform limit  $f$ . Then  $f$  is the continuous.

#### 2.2.2. Carathéodory

**Definition 2.** A continuous function  $\gamma : [0, 1] \rightarrow \mathbb{C}$  is said to be a Jordan curve if  $\gamma(0) = \gamma(1)$  and

$$\gamma(s) = \gamma(t) \implies s = t \quad \text{for all } s, t \in ]0, 1[.$$

The definition above can be restated as: A Jordan curve is a closed simple curve. The term ‘simple’ here means that the curve is not self-intersecting. The name stems from a famous result by Camille Jordan stating that  $\mathbb{C} \setminus \gamma([0, 1])$  has two connected

## 2. Preliminaries

component, one of which is simply connected. The simply connected component will be called *the domain bounded by  $\gamma$* . The proof of this result is rather technical and outside the scope of this thesis [Greene and Krantz, 2006, Whyburn, 1958]. The result is however necessary to make statement such as ‘let  $U$  be the domain bounded by  $\gamma$ ’. An example of this is the following theorem:

**Theorem 1** (Carathéodory). *Let  $\Omega_1$  and  $\Omega_2$  be domains in  $\mathbb{C}$  each bounded by a Jordan curve and  $\Phi : \Omega_1 \rightarrow \Omega_2$  be a conformal mapping. There exists a continuous injection  $\hat{\Phi} : \overline{\Omega_1} \rightarrow \overline{\Omega_2}$  that extends  $\Phi$ .*

A proof of this can be found in section 13.2 of [Greene and Krantz, 2006].

### 2.2.3. The Riemann Mapping Theorem

We will start of with a definition.

**Definition 3.** *A map  $f : U \rightarrow V$ , with open  $U, V \subset \mathbb{C}$  is said to be conformal if it is holomorphic, bijective, and its inverse is holomorphic.*

**Remark 1.** *The fact that the inverse is homomorphic is actually redundant. It can be shown [Greene and Krantz, 2006] that if  $h$  is holomorphic and  $h'$  vanishes to order  $k$  at  $z_0$  then  $h$  is  $(k + 1)$ -to-one in a neighbourhood of  $z_0$ . So if  $h$  is bijective then  $h'$  vanishes nowhere.*

**Theorem 2** (Riemann mapping theorem). *If  $U \subset \mathbb{C}$ ,  $U \neq \mathbb{C}$  is homeomorphic to  $\mathbb{D}$  then there exists a conformal mapping from  $\mathbb{D}$  to  $U$ .*

*Proof.* A proof of this is rather involved and can be found in [Greene and Krantz, 2006]. □

**Corollary 1.** *If  $U$  and  $V$  are both homeomorphic to  $\mathbb{D}$  then there exists a conformal mapping from  $U$  to  $V$ .*

*Proof.* The Riemann mapping theorem gives us  $\Phi_1$ , a conformal mapping from  $\mathbb{D}$  to  $U$ , and  $\Phi_2$ , a conformal mapping from  $\mathbb{D}$  to  $V$ . The desired conformal mapping from  $U$  to  $V$  is then  $\Phi_2 \circ \Phi_1^{-1}$ . □



In this thesis the disk algebra  $\mathcal{A}$  is of special interest so a version of the Riemann mapping theorem that considers continuity at the boundary is desirable. We can combine the Riemann mapping theorem and Carathéodory's theorem to achieve the desired theorem. The only thing to prove is that a homeomorphism  $f$  maps the boundary of a bounded set  $U$  to the boundary of  $f(U)$  and that a Jordan curve under a homeomorphism is still a Jordan curve.

**Corollary 2.** *If  $K$  is homeomorphic to  $\overline{\mathbb{D}}$  then there exists a continuous, injective  $\Phi : \overline{\mathbb{D}} \rightarrow K$  such that its restriction to  $\mathbb{D}$  is a conformal mapping.*

*Proof.* Firstly, let  $U$  be bounded, open set in  $\mathbb{C}$ ,  $f : \overline{U} \rightarrow \mathbb{C}$  be an injective continuous map, and let's show that  $\partial f(U) = f(\partial U)$ . Let  $p \in \partial U$  and  $B_r(z) = \{z \in \mathbb{C}; |r - z| < r\}$ . Let's also assume that  $f(p) \in f(U)$  and show that it leads to a contradiction. There exists an  $r > 0$  such that  $B_r(p) \subset f(U)$ , because  $U$  is open and therefore  $f(U)$  as well. This gives us as an open neighbourhood  $f^{-1}(B_r(p)) \subset U$  of  $p$ , but  $p \in \partial U$  implies that on such neighbourhood exists. So  $f(p)$  is not in  $f(U)$  but it is in  $\overline{f(U)}$ , implying  $\partial f(U) \supset f(\partial U)$ . We can use the same argument to show that  $\partial f(U) \subset f(\partial U)$ , since  $f$  is bijective if we consider it as a map from  $\overline{U}$  into  $f(\overline{U})$ . So  $f(\partial U) = \partial f(U)$ .

Secondly, let  $U$  and  $V$  be open sets in  $\mathbb{C}$ ,  $\gamma : [0, 1] \rightarrow \partial \mathbb{D}$  Jordan curve,  $f : U \rightarrow V$  be homeomorphism, Let  $\lambda = f \circ \gamma$  and  $s, t \in ]0, 1[$  such that  $\lambda(s) = \lambda(t)$ . It suffices to show that  $s = t$ , since  $\lambda(0) = \lambda(1)$  obviously holds. We have that  $f(z) = f(w)$  implies  $z = w$ , since  $f$  is bijective, and therefore injective. So  $\lambda(s) = (f \circ \gamma)(s) = (f \circ \gamma)(t) = \lambda(t)$  implies that  $\gamma(s) = \gamma(t)$ . But  $\gamma$  is a Jordan curve, so  $\gamma(s) = \gamma(t)$  implies  $s = t$ . So  $\lambda$  is also a Jordan curve.

Finally, we can prove the corollary. Let  $f$  be a homeomorphism from  $\overline{\mathbb{D}}$  to  $K$ . The Riemann mapping theorem also gives us a conformal map  $\Phi : \overline{\mathbb{D}} \rightarrow K$ . We know that  $\overline{\mathbb{D}}$  is compact, so  $K = f(\overline{\mathbb{D}})$  is also compact, since the image of a compact set under a continuous mapping is also compact. So  $K$  is also bounded. So  $f(\partial \mathbb{D} = \partial f(\mathbb{D}) = \partial K$  according to our first step and  $\partial K$  is a Jordan curve according to the second step, since  $\partial \mathbb{D}$  is a Jordan curve. So  $\Phi$  is a conformal map between two domains bounded, each bounded by a Jordan curve. This allows us to use Carathéodory's theorem to extend  $\Phi$  continuously and injectively to  $\overline{\mathbb{D}}$ , concluding the proof.  $\square$

The Riemann mapping theorem is a strong tool when analyzing holomorphic functions on simply connected domains. We can often solve things for the unit disk (or unit square as in ??) and then map that solution to a general simply connected domain.

## *2. Preliminaries*

### **2.2.4. Weierstrass M-test**

## **2.3. Functional analysis**

### **2.3.1. Hahn-Banach**

### **2.3.2. Riesz representation theorem**

### **2.4. Miscellaneous**

## 3. Rudin-Carleson theorem

### 3.1. Rudin-Carleson theorem

**Theorem 3** (Rudin-Carleson theorem). *Let  $E$  be a closed subset of  $\mathbb{T}$  of Lebesgue-measure 0, let  $f$  be a continuous function on  $E$  and let  $T$  be a simply connected subset of  $\mathbb{C}$  such that  $f(\overline{\mathbb{D}}) \subset T$ . Then there exists an  $F \in \mathcal{A}$ , such that  $F = f$  on  $E$  and  $F(\overline{\mathbb{D}}) \subset T$ .*

We will break the proof into several lemmas.

**Lemma 3.** *Let  $H$  be a closed set of Lebesgue-measure 0. Then there exists an integrable function  $\mu > 1$  such that  $\mu$  is continuous on  $\mathbb{T} \setminus H$ ,  $\mu = +\infty$  on  $H$ , if  $w \in H$  then  $\mu(z) \xrightarrow{z \rightarrow w} +\infty$ , and  $\mu$  has a bounded derivative on any closed subarc of  $\mathbb{T} \setminus H$ .*

*Proof.* TODO □

**Lemma 4.** *If  $f$  is a simple continuous function on  $E$  such that  $\operatorname{Re} f \geq 0$ , then there exists an  $F \in \mathcal{A}$  such that  $F = f$  on  $E$  and  $\operatorname{Re} F \geq 0$  on  $\overline{\mathbb{D}}$ .*

*Proof.* It suffices to show that this holds if  $f$  takes only two values on  $E$ , since simple functions are finite linear combinations of characteristic functions. Let's assume these values are 0 and  $\alpha \neq 0$ , with  $\operatorname{Re} \alpha \geq 0$ ,  $E_0 = f^{-1}(0)$ , and  $E_1 = f^{-1}(\alpha)$ . Our assumption that  $f$  only takes two values then implies that  $E_0 \cap E_1 = E$ .

Let  $u_H(z)$  be the Poisson integral of the function from the above lemma with  $H$  as  $E$ . This function is continuous on  $\mathbb{T} \setminus H$ ,  $u_H|_H = \infty$ , and  $\lim_{z \rightarrow w} u_H(z) = \infty$  for  $w \in H$ . ... We now define

$$g_H(z) = \begin{cases} u_H(z) + iv_H(z), & z \in \mathbb{D} \setminus H \\ \infty, & \text{otherwise} \end{cases}.$$

### 3. Rudin-Carleson theorem

By our construction of  $u_H$  we see the  $\operatorname{Re} g > 1$ , so it has a well defined square root. Let's call it  $h_H$  and define

$$q = \frac{h_{E_1}}{h_{E_0} + h_{E_1}}.$$

Note that  $|\arg h_H(z)| < \pi/4$  since if a  $w \in \mathbb{C}$  had an argument outside of this range then its square would have an argument outside of the range  $[-\pi/2, \pi/2]$  meaning  $\operatorname{Re} w^2 < 0$ . Also,  $q(z) = 0$  if and only if  $h_{E_0} = \infty$ , so  $q$  is zero only on  $E_0$ , and  $q(z) = 1$  if and only if  $h_{E_1} = \infty$ , so  $q$  is one only on  $E_1$ . We now want to show that  $0 \leq \operatorname{Re} q \leq 1$ . We will let  $z, w \in \mathbb{C}$ , with  $|\arg z|, |\arg w| < \pi/4$  and  $\operatorname{Re} z, \operatorname{Re} w > 1$ , and show that  $0 < \operatorname{Re} z/(w + z) < 1$ .

Firstly note that

$$\frac{z}{w + z} = \frac{1}{w/z + 1}$$

so

$$\arg \frac{z}{w + z} = -\arg \left( \frac{w}{z} + 1 \right)$$

and

$$|\arg w/z| = |\arg w - \arg z| \leq |\arg w| + |\arg z| < \pi/4 + \pi/4 = \pi/2.$$

So  $w/z$  is in the right halfplane and, since  $\arctan(y/x)$  is decreasing in  $x$  for positive  $y$ , we get

$$\arg \frac{z}{w + z} = -\arg \left( \frac{w}{z} + 1 \right) < -\arg \frac{w}{z}$$

and thus

$$\left| \arg \frac{z}{w + z} \right| < \left| \arg \frac{w}{z} \right| < \pi/2.$$

We know  $\arctan(y/x)$  is decreasing in  $x$  for positive  $y$  because

$$\frac{d}{dx} \arctan \left( \frac{y}{x} \right) = -\frac{y}{x^2 + y^2} < 0.$$

So  $0 < \operatorname{Re} z/(w + z)$ . Note that  $0 < \operatorname{Re} z/(w + z) \implies 0 > \operatorname{Re} -w/(w + z)$  due to  $z$  and  $w$  being constrained in the same manner. So

$$\begin{aligned} 0 &> \operatorname{Re} \frac{-w}{w + z} \\ &= \operatorname{Re} \frac{z - (z + w)}{w + z} \\ &= \operatorname{Re} \left( \frac{z}{w + z} - 1 \right) \\ &= \operatorname{Re} \frac{z}{w + z} - 1 \\ \implies 1 &> \operatorname{Re} \frac{z}{z + w}. \end{aligned}$$

So we have shown that

$$0 < \operatorname{Re} \frac{z}{z+w} < 1.$$

So we have constructed a function  $q$  that maps  $\overline{D}$  to the ribbon  $\{z; 0 \leq \operatorname{Re} z \leq 1\}$ . We then let  $\Phi$  be the conformal mapping from the ribbon  $\{z; 0 \leq \operatorname{Re} z \leq 1\}$  to  $\{z; 0 \leq \operatorname{Re} z \leq \operatorname{Re} \alpha\}$ . We will also choose  $\Phi$  such that  $\Phi(0) = 0$  and  $\Phi(1) = \alpha$ . We can then let  $F = \Phi \circ q$  and conclude the proof. □

**Lemma 5.** *If  $f$  is a simple continuous function on  $E$  that maps  $E$  into a simply connected  $T$ , then there exists a  $F \in \mathcal{A}$ , such that  $F = f$  on  $E$  and  $F$  maps  $\overline{D}$  into  $T$ .*

*Proof.* TODO □

**Lemma 6.** *If  $f$  is a continue function on  $E$  which maps  $E$  into*

$$S = \{z; |\max(\operatorname{Re} z, \operatorname{Im} z)| \leq 1\},$$

*then there exists a sequence  $(f_n)_{n \in \mathbb{N}}$  of simple continuous function on  $E$  such that*

$$f(x) = \sum_{n \in \mathbb{N}} f_n(z),$$

$$f_n(E) \subset 2^{-n}S.$$

*Proof.* We will set  $f_0 = 0$  and construct  $f_n$  iteratively. Assuming  $f_0, f_1, \dots, f_{n-1}$  have been constructed such that

$$\lambda_{n-1}(E) \subset 2^{1-n}S$$

with  $\lambda_{n-1} = f - \sum_{k=0}^{n-1} f_k$ . According to ?? we have can write  $E$  as the union of disjoint closed sets  $E_1, E_2, \dots, E_p$  such that the oscillation of  $\lambda_{n-1}$  is less than  $2^{-n}$  on each  $E_k$ . So we can define  $Q_k \subset 2^{1-n}S$  for  $k = 1, 2, \dots, p$  such that  $Q_k = 2^{-n}S + a_k$  for some  $a_k \in S$ . We can choose  $c_k \in Q_k \cap 2^{-n}$  since  $Q_k$  has side length  $2^{-n}$  and is a subset of  $2^{-n+1}$  and both are closed. We can now define

$$f_n(z) = c_k, \quad z \in E_k, \quad k = 1, 2, \dots, p.$$

If we then look at  $\lambda_n = f - \sum_{k=0}^n f_k = \lambda_{n-1} - f_n$  we see that  $\lambda_n(E) \subset 2^{-n}S$  due to the way we decomposed  $E$  into  $E_1, E_2, \dots, E_p$  using the oscialltions of  $\lambda_{n-1}$ . This means we can continue the process. □

### 3. Rudin-Carleson theorem

*Proof of ??.* Let's first show the result for  $T = S$ .

Let  $f_n$  be the functions from Lemma ??. According to Lemma ?? we have functions  $g_n \in \mathcal{A}$  which extend  $f_n$  and map  $\overline{\mathbb{D}}$  into  $2^{-n}S$ . We then define

$$F = \sum_{n \in \mathbb{N}} g_n$$

on  $\overline{\mathbb{D}}$ . To show that  $F$  is in  $\mathcal{A}$  we have to show it is holomorphic on  $\mathbb{D}$  and continuous on  $\overline{\mathbb{D}}$ . Both of these things can be shown by demonstrating that the series converges uniformly. Let  $M_n = 2^{-n+1}$  and note that  $|g_n(z)| \leq \sqrt{2} \cdot 2^{-n} < 2^{-n+1} = M_n$  and

$$\sum_{n \in \mathbb{N}} M_n = \sqrt{2} \sum_{n \in \mathbb{N}} 2^{-n} = \sqrt{2} < \infty$$

so the Weierstrass M-test tells us that  $F$  converges uniformly. This in turn tells that  $F$  is both holomorphic on  $\mathbb{D}$  and continuous on  $\overline{\mathbb{D}}$  so it is in  $\mathcal{A}$ . We also have that

$$\operatorname{Re} F = \sum_{n \in \mathbb{N}} \operatorname{Re} g_n \leq \sum_{n \in \mathbb{N}} 2^{-n} = 1.$$

It can be shown in the same manner that  $\operatorname{Im} F \leq 1$ , so  $F$  maps into  $S$ . Lastly, for  $z \in E$  we have that

$$F(z) = \sum_{n \in \mathbb{N}} g_n(z) = \sum_{n \in \mathbb{N}} f_n(z) = f(z)$$

so  $F$  is an extension of  $f$ .

To proof the result for a general  $T$  we first let  $\Phi$  a the conformal mapping from  $S$  to  $T$ , its existence provided by the Riemann mapping theorem. We will also let  $g = f \circ \Phi^{-1}$ . Note that it maps  $E$  into  $S$ , so we can use what we showed above to find  $G \in \mathcal{A}$  that extends  $g$  and maps into  $S$ . We finally set  $F = G \circ \Phi$ .  $\square$

**Corollary 3** (Fatou). *Let  $E$  be a closed subset of  $\mathbb{T}$  of Lebesgue-measure 0. There exists a function  $f \in \mathcal{A}$  that vanishes on  $E$  and nowhere else.*

*Proof.* It's clear from the theorem that there exists a function  $f \in \mathcal{A}$  that vanishes on  $E$ .  $\square$

## 3.2. F. and M. Riesz theorem

In this section we will endeavour to show that the annihilating measures of  $\mathcal{A}|_{\mathbb{T}}$  are absolutely continuous with respect to the Lebesgue measure. We will show this to

be a corollary of the F. and M. Riesz theorem, which we will prove in the manner of Rudin. To attain the main result of this section we need some lemmas and definitions. To prove one of the lemmas we will also use the following two famous theorems:

**Definition 4.** Let  $\mathcal{F}$  be a family of complex functions on a metric space  $(X, d)$ .

We say that the family is pointwise bounded if for all  $x \in X$  there exists a constant  $M < \infty$  such that

$$|f(x)| < M, \text{ for all } f \in \mathcal{F}.$$

Note that  $M$  is dependent on  $x$ .

We say that the family is equicontinuous if for all  $\varepsilon > 0$  there exists a  $\delta > 0$  such that

$$|f(x) - f(y)| < \varepsilon, \text{ for all } f \in \mathcal{F} \text{ and } x, y \in X \text{ such that } d(x, y) < \delta.$$

Note here that  $\delta$  is globally defined and only dependent on  $\varepsilon$ .

**Theorem 4** (Bolzano-Weierstrass). Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of numbers in  $\mathbb{R}^n$ , such that  $|a_n| < M < \infty$ , for all  $k \in \mathbb{N}$ . There than exists and infinite  $S \subset \mathbb{N}$  such that  $(a_n)_{n \in S}$  is convergent.

*Proof.* Let's first assume that the sequence is in  $\mathbb{R}$ , that no element in it is repeated infinitely often (there is nothing to prove in that case), and that  $a_n \in ]0, 1[$  for all  $n \in \mathbb{N}$ . The last assumption can be done with out loss of generality by studying the sequence  $((a_n + M)/(2M))_{n \in \mathbb{N}}$  instead. We will obtain the subsequence by a diagonal process. Let  $S_0 = \mathbb{N}$ ,  $S_0^- = \{n \in S_0; a_n < 1/2\}$ , and  $S_0^+ = \{n \in S_0; a_n > 1/2\}$ . We then set  $S_1 = S_0^-$  if it is infinite, but  $S_1 = S_0^+$  otherwise. This gives us a subsequence  $(a_n)_{n \in S_1}$  such that

$$\sup_{n \in S_1} a_n - \inf_{n \in S_1} a_n < 1/2.$$

We can then repeat this to get a sequence of sets  $(S_n)_{n \in \mathbb{N}}$  such that  $S_0 \supset S_1 \supset S_2 \supset \dots$  and

$$\sup_{n \in S_k} a_n - \inf_{n \in S_k} a_n < 2^{1-k},$$

for all  $k \in \mathbb{N}$ . Specifically, if we have  $S_k$  we set

$$U = m2^{-k}, \quad L = (m+1)2^{-k}$$

$S_k^- = \{n \in S_k; a_n < (U+L)/2\}$ , and  $S_k^+ = \{n \in S_k; a_n > (U+L)/2\}$ . We now set  $S_{k+1} = S_k^-$  if it has infinitely many elements, otherwise we set  $S_{k+1} = S_k^+$ . We conclude our construction by setting

$$S = \bigcup_{n \in \mathbb{N}} r_n,$$

### 3. Rudin-Carleson theorem

where  $r_n$  is the  $n$ -th smallest element of  $S_n$ . This gives us the convergent sequence  $(a_n)_{n \in S}$  with limit

$$\sum_{k=0}^{\infty} \delta_k 2^{-k}$$

where

$$\delta_k = \begin{cases} 0, & \text{if we chose } S_k^- \\ 1, & \text{if we chose } S_k^+ \end{cases}.$$

To show the result for  $\mathbb{R}^n$  we can start by finding a subsequence such that the first coordinate is convergent. We can then chose a subsequence thereof such that the second coordinate is also convergent. Now the first two coordinates are convergent. If we do this  $n - 2$  more times we get a desired subsequence.  $\square$

**Remark 2.** *The theorem above clearly holds for sequences in  $\mathbb{C}$  as well.*

**Theorem 5** (Ascoli-Arzelà). *Let  $\mathcal{F}$  be a pointwise bounded equicontinuous collection of complex functions on a metric space  $(X, d)$ , and  $X$  contains a countable dense subset. Then every sequence in  $\mathcal{F}$  contains a subsequence that converges uniformly on every compact subsets of  $X$ .*

*Proof.* Let  $E$  be a countable dense subset of  $X$ ,  $(f_n)_{n \in \mathbb{N}}$  be a series in  $\mathcal{F}$ , and  $x_1, x_2, \dots$  be an enumeration of  $E$ . We will prove the theorem in two steps. The first step is finding a subsequence of  $(f_n)_{n \in \mathbb{N}}$  that's pointwise convergent on  $E$  using the point wise boundedness along with Bolzano-Weierstrass. The second step is using the equicontinuity to show that this gives us uniform continuity on compact subsets.

Let's first set  $S_0 = \mathbb{N}$ . Pointwise boundedness gives us that the sequence  $(f_n(x_1))_{n \in S_0}$  has a convergent subsequence. Let  $S_1$  index that subsequence. We can use this process to generate sets  $S_0 \supset S_1 \supset \dots$  such that  $(f_n(x_k))_{n \in S_k}$  is convergent. We then set

$$S = \bigcup_{k \in \mathbb{N}} r_k$$

where  $r_n$  is the  $k$ -th smallest element of  $S_k$ . We now have concluded the first step of the proof.

We will now assume the  $(f_n)_{n \in \mathbb{N}}$  is pointwise convergent on  $E$ , let  $K$  be a compact subset of  $X$ , and  $\varepsilon > 0$ . Equicontinuity gives us a  $\delta > 0$  such that  $d(x, y) < \delta$  implies that  $|f_n(x) - f_n(y)| < \varepsilon/3$ , for all  $n$ . Let's now cover  $K$  with  $m$  balls of radius  $\delta/2$  and call the  $k$ -th ball  $B_k$ . We can now set  $p_k$  as a point in  $B_k$ . This point exists because  $E$  is dense in  $X$ . Pointwise convergence on  $E$  let's us chose an  $N$  such that  $|f_{n_1}(p_k) - f_{n_2}(p_k)| < \varepsilon/3$  for  $k = 1, 2, \dots, m$  and all  $n_1, n_2 > N$ . Let's conclude by



setting  $x \in K$ . Then there is a  $k$  such that  $x \in B_k$  and thus  $d(x, p_k) < \delta$ . The choice of  $\delta$  and  $N$  then gives us that

$$\begin{aligned} |f_{n_1}(x) - f_{n_2}(x)| &\leq |f_{n_1}(x) - f_{n_1}(p_k)| + |f_{n_1}(p_k) - f_{n_2}(p_k)| + |f_{n_2}(p_k) - f_{n_2}(x)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon. \end{aligned}$$

□

**Definition 5.** *Poisson kernel, Poisson integral, Poisson integral of a measure.*

**Lemma 7.** *Let  $\mu$  be a complex Borel measure, and  $u = P[d\mu]$ . Then*

$$\|u_r\|_1 \leq \|\mu\|.$$

*Proof.* First, we need to see that, if  $n \neq 0$

$$in \int_{-\pi}^{\pi} e^{int} dt = (e^{in\pi} - e^{-in\pi}) = (e^{in\pi} - e^{-i(2n\pi - n\pi)}) = (e^{in\pi} - e^{in\pi}) = 0,$$

so

$$\begin{aligned} \int_{-\pi}^{\pi} P_r(t) dt &= \int_{-\pi}^{\pi} \sum_{n \in \mathbb{Z}} r^{|n|} e^{int} dt \\ &= \sum_{n \in \mathbb{Z}} \int_{-\pi}^{\pi} r^{|n|} e^{int} dt \\ &= \int_{-\pi}^{\pi} dt \\ &= 2\pi. \end{aligned}$$

Fubini let's us

$$\begin{aligned} \|u\|_1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})| d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \int_{\mathbb{T}} P(re^{i\theta}, e^{it}) d\mu(e^{it}) \right| d\theta \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{\mathbb{T}} P(re^{i\theta}, e^{it}) d|\mu(e^{it})| d\theta \\ &= \int_{\mathbb{T}} \frac{1}{2\pi} \int_{-\pi}^{\pi} P(re^{i\theta}, e^{it}) d\theta d|\mu(e^{it})| \\ &= \int_{\mathbb{T}} d|\mu(e^{it})| \\ &= |\mu|(\mathbb{T}) \\ &= \|\mu\|. \end{aligned}$$

□

### 3. Rudin-Carleson theorem

**Lemma 8.** *Let  $f \in H^1$ . Then there exists a  $g \in L^1(\mathbb{T})$  such that  $f = P[g]$ .*

*Proof.* □

**Lemma 9.** *Let  $u$  be harmonic in  $\mathbb{D}$  and*

$$\sup_{0 < r < 1} \|u_r\|_1 = M < \infty.$$

*Then there exists a unique complex Borel measure  $\mu$  on  $\mathbb{T}$  such that  $u = P[d\mu]$ .*

We will need the following lemma in the proof of 9:

**Lemma 10.** *Let  $X$  be a separable Banach space,  $(\Gamma_n)_{n \in \mathbb{N}}$  be a sequence of linear functionals on  $X$ , and  $\sup_n \|\Gamma_n\| = M < \infty$ . Then there exists a subsequence  $\{\Gamma_{n_i}\}$  such that the limit*

$$\Gamma x = \lim_{k \rightarrow \infty} \Gamma_{n_k} x$$

*exists for every  $x \in X$ . We also have that  $\Gamma$  is linear and  $\|\Gamma\| \leq M$ .*

*Proof.* We have that  $|\Gamma_n x| \leq M\|x\|$  and

$$\begin{aligned} |\Gamma_n x - \Gamma_n y| &= |\Gamma_n(x - y)| \\ &\leq M\|x - y\|. \end{aligned}$$

The first inequality gives us pointwise boundedness and the second gives us equicontinuity. Now, since singletons are compact, Ascoli-Arzelà gives us a subsequence, let's index it by  $S$ , such that  $(\Gamma_n x)_{n \in S}$  is convergent for all  $x \in X$ . Let's now define  $\Gamma$  by

$$\Gamma(x) = \lim_{k \in S} \Gamma_k x,$$

see the

$$\begin{aligned} \Gamma(x) + \Gamma(y) &= \lim_{k \in S} \Gamma_k x + \lim_{k \in S} \Gamma_k y \\ &= \lim_{k \in S} (\Gamma_k x + \Gamma_k y) \\ &= \lim_{k \in S} \Gamma_k(x + y) \\ &= \Gamma(x + y), \end{aligned}$$

where the third equality holds because addition is continuous, and  $a\Gamma(x) = \Gamma(ax)$  obviously holds. So  $\Gamma$  is linear. Lastly

$$\begin{aligned}\|\Gamma\| &= \sup\{|\Gamma x|; \|x\| \leq 1\} \\ &= \sup\left\{\left|\lim_{n \in S} \Gamma_n x\right|; \|x\| \leq 1\right\} \\ &\leq \sup\{M; \|x\| \leq 1\} \\ &= M.\end{aligned}$$

□

*Proof of 9.* Let  $\Gamma_r$ , for  $r \in [0, 1[$ , be linear functionals on  $C(\mathbb{T})$  defined by

$$\Gamma_r g = \int_{\mathbb{T}} g u_r d\sigma.$$

If  $\|g\| \leq 1$  is assumed we get that

$$\Gamma_r g = \int_{\mathbb{T}} g u_r d\sigma \leq \int_{\mathbb{T}} u_r d\sigma = \|u_r\|_1 \leq M.$$

so

$$\|\Gamma_r\| \leq M.$$

By the above lemma and the Riezs representation theorem we get a measure  $\mu$  on  $\mathbb{T}$  with  $\|\mu\| \leq M$ , and a sequence  $(r_n)_{n \in \mathbb{N}}$  on  $[0, 1[$  with limit 1, such that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}} g u_{r_n} d\sigma = \int_{\mathbb{T}} g d\mu \quad (3.1)$$

for all  $g \in C(\mathbb{T})$ . Let's now define functions  $h_k$  on  $\overline{\mathbb{B}}$  by  $h_k(z) = u(r_k z)$ . We get that, since  $u$  is harmonic on  $r\mathbb{B}$  for  $r \in ]0, 1[$ , the functions  $h_k$  are harmonic on  $\mathbb{B}$  and continuous on  $\overline{\mathbb{B}}$ . So each of them can be represented by the Poisson integral of their restriction to  $\mathbb{T}$ , according to Ramsford. Note that  $h_k(e^{it}) = u_{r_k}(e^{it})$ , so

$$\begin{aligned}u(z) &= \lim_{n \rightarrow \infty} u(r_n z) \\ &= \lim_{n \rightarrow \infty} h_n(z) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{T}} P(z, e^{it}) h_n(e^{it}) d\sigma(e^{it}) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{T}} P(z, e^{it}) u_{r_n}(e^{it}) d\sigma(e^{it}) \\ &= \int_{\mathbb{T}} P(z, e^{it}) d\mu(e^{it}) \\ &= P[d\mu](z),\end{aligned}$$

### 3. Rudin-Carleson theorem

where the fifth equality is achieved by putting  $g = P(z, e^{it})$  into 3.1. This concludes the proof of existence.

Let's assume that  $P[d\mu] = 0$ , and let  $f \in C(\mathbb{T})$ ,  $u = P[f]$  and  $v = P[d\mu]$ . We firstly have the symmetry

$$P(re^{i\theta}, e^{it}) = P(re^{it}, e^{i\theta}).$$

This symmetry is due to

$$|e^{it} - re^{i\theta}| = |1 - re^{i(\theta-t)}| = |1 - re^{i(t-\theta)}| = |e^{i\theta} - re^{it}|,$$

which is geometrically intuitive. The first and last equalities hold because the euclidian metric is rotationally invariant, and the second equality holds because the distance from  $z$  to a real number  $a$  is the same distance from  $\bar{z}$  to  $a$ . We now obtain

$$\begin{aligned} \int_{\mathbb{T}} u_r d\mu &= \int_{\mathbb{T}} \int_{-\pi}^{\pi} P(re^{i\theta}, e^{it}) f(e^{i\theta}) d\theta d\mu(e^{it}) \\ &= \int_{-\pi}^{\pi} f(e^{i\theta}) \int_{\mathbb{T}} P(re^{it}, e^{i\theta}) d\mu(e^{it}) d\theta \\ &= \int_{-\pi}^{\pi} f(e^{i\theta}) v_r d\theta \\ &= \int_{\mathbb{T}} f v_r d\sigma. \end{aligned}$$

If we let  $r \rightarrow 1$  we get

$$\int_{\mathbb{T}} f d\mu = 0.$$

This holds for all  $f \in C(\mathbb{T})$ , so the measure  $\mu$  represents zero in the dual of  $C(\mathbb{T})$ . The Riesz representation theorem then tells us that  $|\mu|(\mathbb{T}) = 0$ , so  $\mu = 0$ .

Now let  $\lambda$  and  $\nu$  be measures on  $\mathbb{T}$  such that  $P[d\lambda] = P[d\nu]$ . We have that  $P[d(\lambda - \nu)] = 0$ , so, as shown above  $\lambda - \nu = 0$ . Moreover  $\lambda = \nu$ , which concludes the proof of uniqueness.  $\square$

**Theorem 6** (F. and M. Riesz theorem). *If  $\mu$  is a complex Borel measure on  $\mathbb{T}$  and*

$$\int e^{-int} d\mu = 0$$

*for  $n = -1, -2, \dots$ , then  $\mu \ll m$ .*

*Proof.* Let  $f = P[d\mu]$ . If we set  $z = re^{i\theta}$  we get that

$$P(z, e^{it}) = P_r(\theta - t) = \sum_{n \in \mathbb{Z}} r^{|n|} e^{in(\theta-t)} = \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta} e^{-int}.$$

### 3.3. A generalization of the Rudin-Carleson theorem

We can use the assumption of the theorem to write  $f$  as a power series by

$$\begin{aligned}
 f(z) &= \int_{\mathbb{T}} P(z, e^{it}) d\mu(e^{it}) \\
 &= \int_{\mathbb{T}} \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta} e^{-int} d\mu(e^{it}) \\
 &= \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta} \int_{\mathbb{T}} e^{-int} d\mu(e^{it}) \\
 &= \sum_{n=0}^{\infty} r^n e^{in\theta} \int_{\mathbb{T}} e^{-int} d\mu(e^{it}) \\
 &= \sum_{n=0}^{\infty} \hat{\mu}_n z^n,
 \end{aligned}$$

where  $\hat{\mu}_n$  is the  $n$ -th Fourier coefficient of  $\mu$ . This along with 7 gives us that  $f \in H^1$ . We can now define a  $g \in H^1$ , by 8, such that  $f = P[g]$ . It follows from 9 that  $d\mu = f d\sigma$ . TODO  $\square$

**Corollary 4.** *Let  $A$  be the closed subspace of  $C(\mathbb{T})$  that consists of all functions that are restriction from  $\mathcal{A}$ . All measures in  $A^\perp$  are absolutely continuous with regards to the Lebesgue-measure on  $\mathbb{T}$ .*

*Proof.* Let  $\mu \in A^\perp$ . By definition we have that

$$\int f d\mu = 0.$$

Now since  $t \mapsto e^{-int}$  is entire for  $n = -1, -2, \dots$  we have that their restriction to  $\mathbb{T}$  are in  $A$ . Thus,

$$\int e^{-int} d\mu = 0$$

for all  $n = -1, -2, \dots$  and  $\mu \ll m$ .  $\square$

### 3.3. A generalization of the Rudin-Carleson theorem

This borrows from [Bishop, 1962].

**Theorem 7** (General Rudin-Carleson theorem). *Let  $X$  be a compact Hausdorff space,  $V = (C(X), \|\cdot\|_\infty)$ ,  $B$  be a closed subspace of  $C(X)$ ,  $B^\perp$  be the annihilating*

### 3. Rudin-Carleson theorem

measures for  $B$ ,  $S$  be a closed subset of  $X$ , and  $f$  be a continuous function on  $S$ . If  $\int_S f d\mu = 0$  holds for all  $\mu \in B^\perp$  then there exists a function  $F \in B$  such that  $F = f$  on  $S$ .

*Proof.* Since  $f$  is continuous and  $S$  is a closed subset of a compact set, and therefore also compact,  $f$  is bounded. So we can, with out loss of generality, assume that  $|f| < r < 1$  on  $S$ . Let  $U_r$  be the subset of  $B$  defined by  $U_r = \{g; \|g\| < r\}$  and  $\phi$  be the mapping from  $B$  to  $C(S)$  that sends a member of  $B$  to its restriction on  $S$ . It suffices to show that  $f \in \phi(U_r)$ . Let's first show that  $f \in \overline{\phi(U_r)} =: V_r$ , by assuming otherwise, and showing it leads to a contradiction.

We now assume  $f \notin V_r$ . By Hahn-Banach we can define a bounded linear functional  $\alpha$ , such that  $\alpha(f) > 1$  and  $|\alpha(h)| < 1$ , for  $h \in V_r$ . We can then define a measure  $\mu_1$  by the Riesz-representation theorem that fulfills

$$\alpha(g) = \int g d\mu_1$$

for all  $g \in C(S)$ . We will refer to the associated functional on  $B$  by  $\beta(g) = \phi(\alpha(g))$ . Since  $\phi(g) \in V_r$  for all  $g \in U_r$  we have that

$$\beta(g) = \alpha(\phi(g)) < 1,$$

for all  $g \in U_r$ , due to the construction of  $\alpha$ . From this we get

$$\begin{aligned} \|\beta\| &= \sup\{|\beta(g)|; |g| < 1\} \\ &= \sup\{(1/r)|\beta(g)|; |g| < r\} \\ &\leq \sup\{(1/r); |g| < r\} \\ &= 1/r. \end{aligned}$$

Let's denote the Riesz representation of  $\beta$  by  $\mu_2$ , set  $\mu = \mu_1 - \mu_2$  and see that  $\mu \in B^\perp$ . But

$$0 = \left| \int_S f d\mu \right| \geq \int_S f d\mu_1 - r\|\mu_2\| \geq \int_S f d\mu_1 - r\frac{1}{r} > 1 - r\frac{1}{r} = 0,$$

where the first equality is the assumption in the theorem. This is the contradiction that gives that  $f \in V_r$ . We can now take a  $F_1$  in  $U_r$ , and therefore also in  $B$  such that  $|f - F_1| < \lambda/2$  on  $S$ , with  $\lambda := 1 - r$ . Remember that  $F_1 \in U_r$  implies that  $\|F_1\| < r$ . Now let  $f_1 = f - F_1$  and use the same method as above to obtain an  $F_2$  such that  $\|F_2\| < \lambda/2$  and  $|f - F_2| < \lambda/4$  on  $S$ . Iterating this process yields a series  $(F_n)_{n \in \mathbb{N}}$  from  $B$  that fulfill  $\|F_n\| < 2^{1-n}\lambda$  for  $n > 1$  and

$$\left| f - \sum_{k=1}^n F_k \right| < 2^{-n}\lambda$$

### 3.3. A generalization of the Rudin-Carleson theorem

on  $S$  for  $n > 1$ . We finally let

$$F = \sum_{k=1}^{\infty} F_k.$$

Now  $F \in B$ ,

$$\|F\| \leq \|F_1\| + \|F - F_1\| = r + \sum_{k=2}^{\infty} 2^{1-n} \lambda = r + \lambda = 1,$$

and  $F = f$  on  $S$ . □

**Corollary 5.** *Let  $X$  be a compact Hausdorff space,  $V = (C(X), \|\cdot\|_{\infty})$ ,  $B$  be a closed subspace of  $C(X)$ ,  $B^{\perp}$  be the annihilating measures for  $B$ ,  $S$  be a closed subset of  $X$ , and  $f$  be a continuous function on  $S$ . If  $S$  is  $B^{\perp}$ -null then there exists a function  $F \in B$  such that  $F = f$  on  $S$ .*

*Proof.* If  $S$  is  $B^{\perp}$ -null we have that  $\int_S f d\mu = 0$  for all  $\mu \in B^{\perp}$ . □

**Remark 3.** *The corollary is the version of the theorem from Bishop. Note also that if we set  $X = \mathbb{T}$  and  $B = \mathcal{A}$  we can use  $F$  and  $M$ . Riesz to prove the classical Rudin-Carleson theorem.*

It is of course worth noting an applications of where the corollary fails.

**Example 1.** *Let  $X = \mathbb{T}$ ,  $B = \mathcal{A}$ ,  $E$  be a closed  $m$ -null subset of  $\partial\mathbb{T}$  that is not dense in  $E$ ,  $F = \{e^{i\theta}; a \leq \theta \leq b\}$ , and choose  $a$  and  $b$  such that  $E$  and  $F$  are disjoint and  $a \neq b$ . The last assumption restricts us to  $E$  that are not dense in the  $\mathbb{E}$ . Since  $a \neq b$  we obtain that  $S := E \cup F$  does not fulfill the requirements of the Rudin-Carleson theorem nor the above corollary. Let's choose  $f$  such that  $f = 0$  on  $F$ , and  $f$  is continuous on  $S$ . We now have for all  $\mu \in \mathcal{A}^{\perp}$*

$$\begin{aligned} \left| \int_S f d\mu \right| &= \left| \int_E f d\mu + \int_F f d\mu \right| \\ &\leq \left| \int_E f d\mu \right| + \left| \int_F f d\mu \right| \\ &= 0 + \left| \int_F f d\mu \right| \\ &= 0. \end{aligned}$$

*The  $F$  and  $M$ . Riesz theorem tells us the since  $E$  is  $m$ -null it is also  $\mu$ -null, which gives the third step. The final step stems from the fact that  $f$  vanishes on  $F$ . We now see the  $X$ ,  $B$ , and  $f$  are all as in theorem so there exists a  $F \in B$ , such that  $F = f$  on  $S$ .*





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