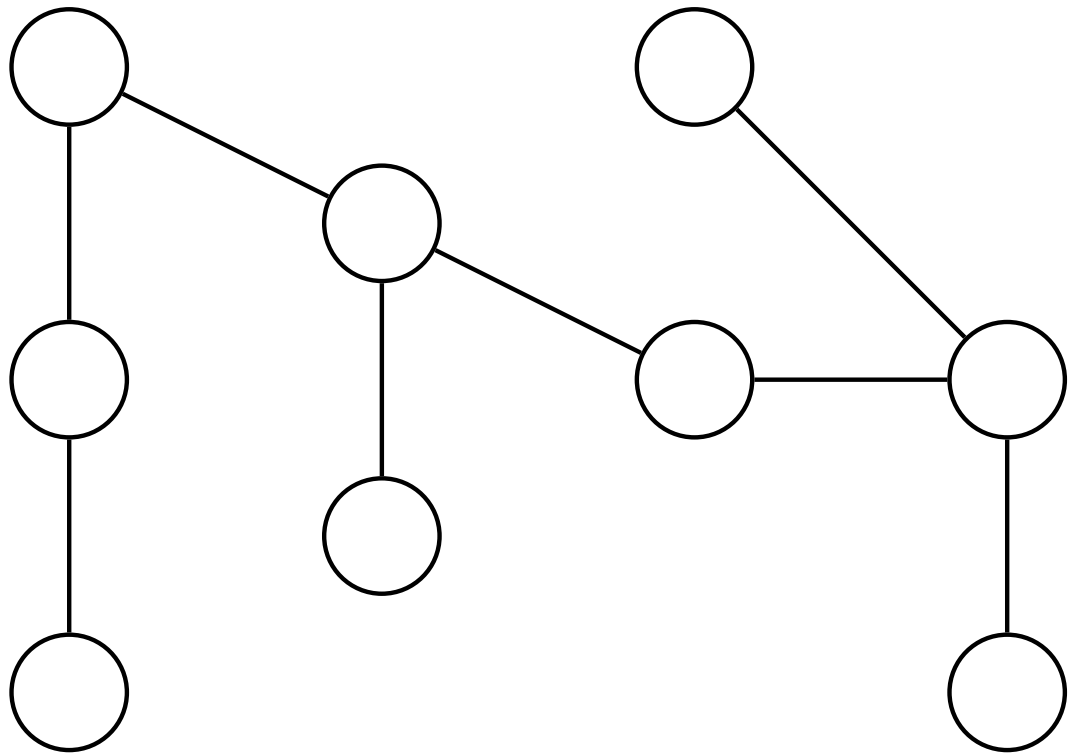
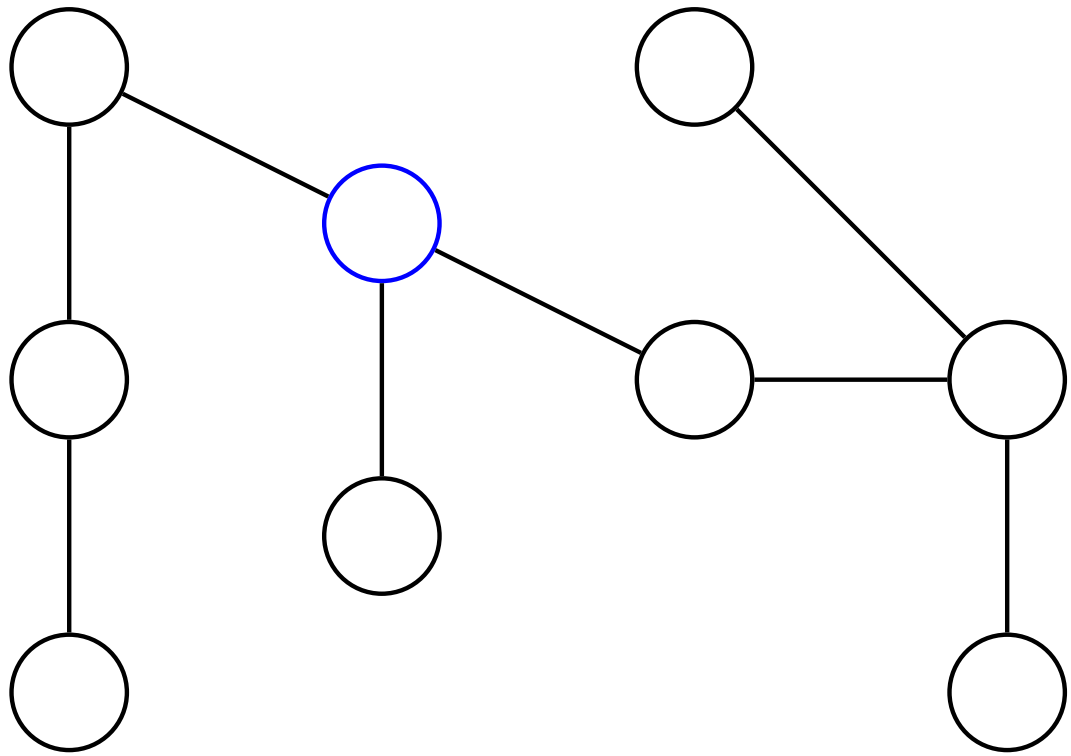
Hæsti sameiginlegi forfaðir

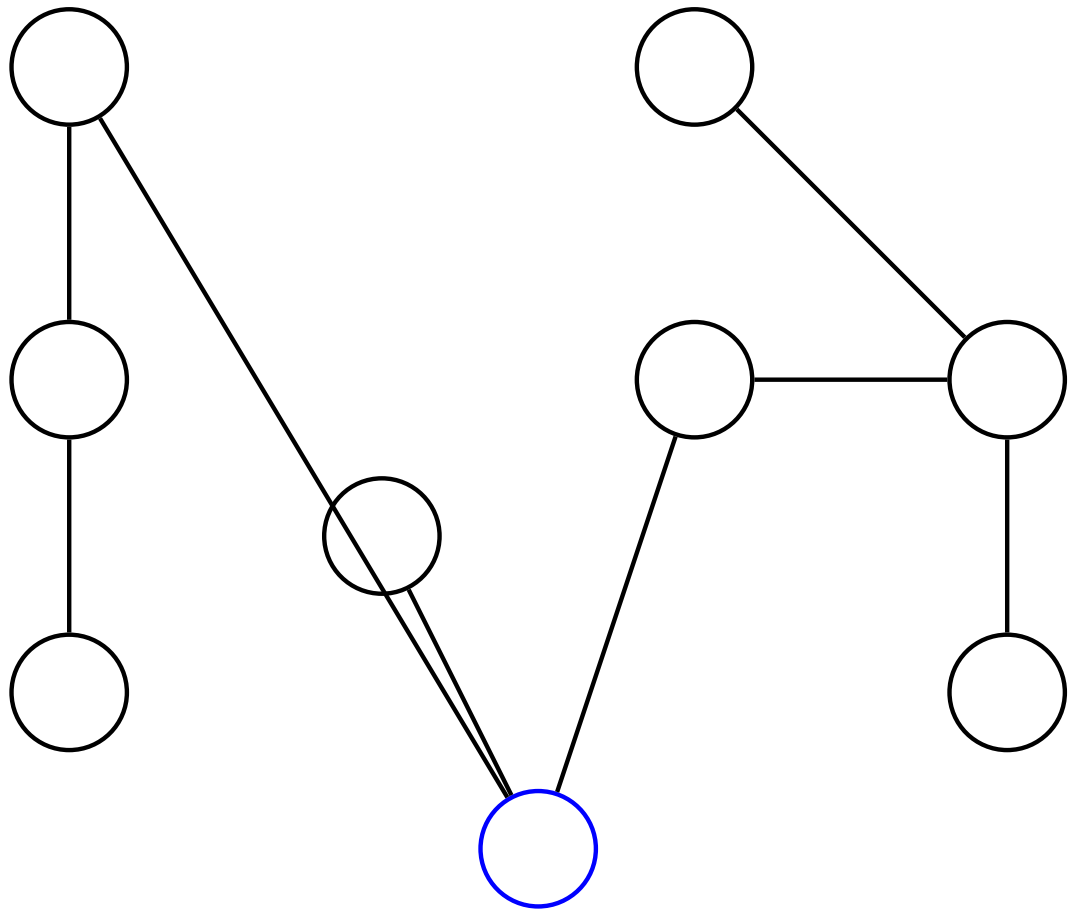
Bergur Snorrason

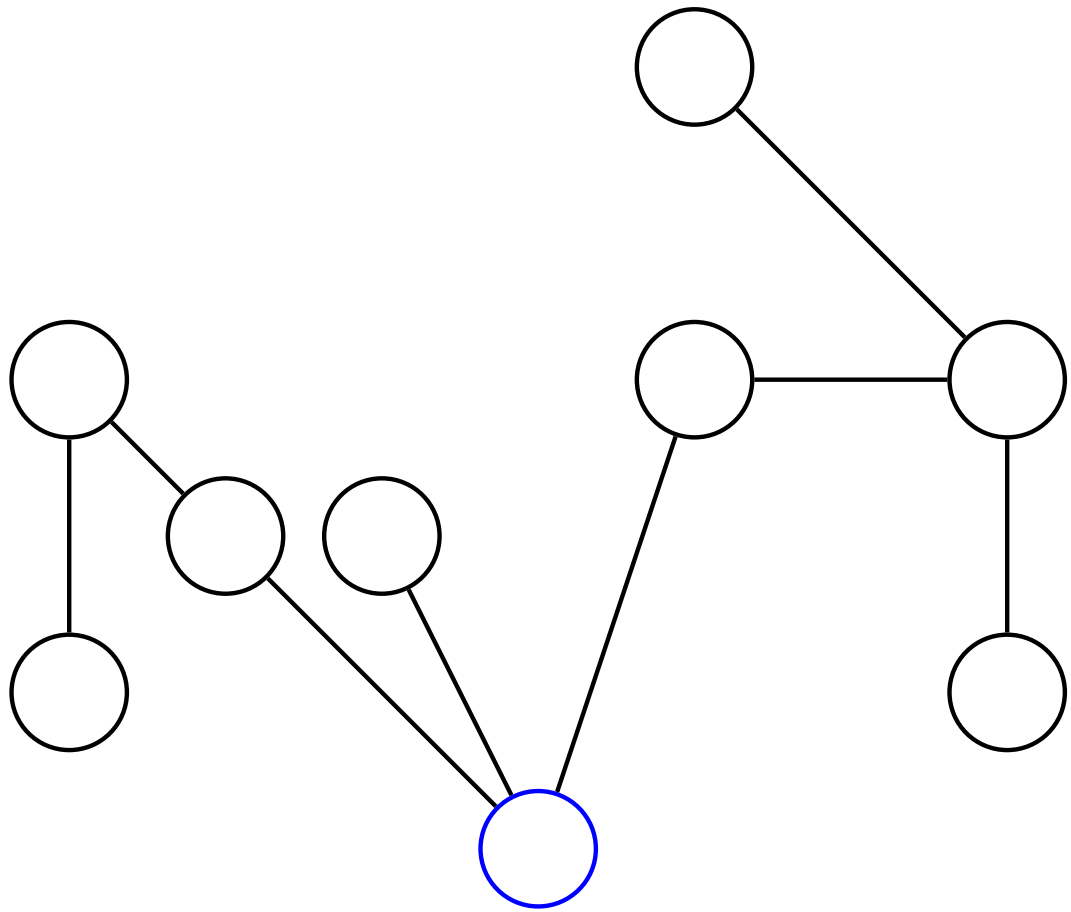
4. apríl 2022

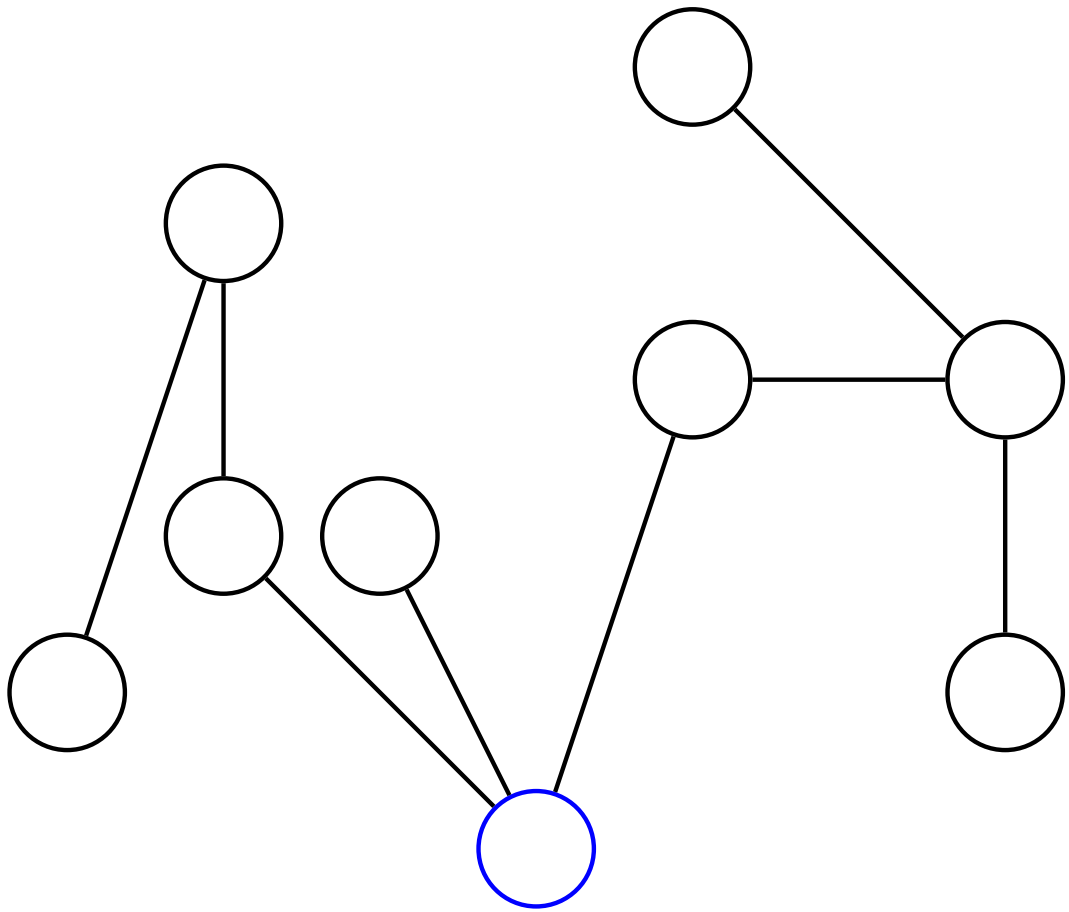
- Gerum ráð fyrir að við séum með tré.
- Látum einn hnút tákna rót trésins.
- Við getum þá talað um *hæð* hnúts í trénu.
- Hæð hnútsins er þá fjarlægðin frá hnútnum í rótina.
- Þar sem við erum í tréi er aðeins einn einfaldur vegur í rótina.
- Ein leið til að finna hæð allra hnúta er með einni dýptarleit.
- Látum hæð hnútsins u vera $h(u)$.
- Við segjum líka að hæð trés sé H ef hnúturinn með hæstu hæð er H.

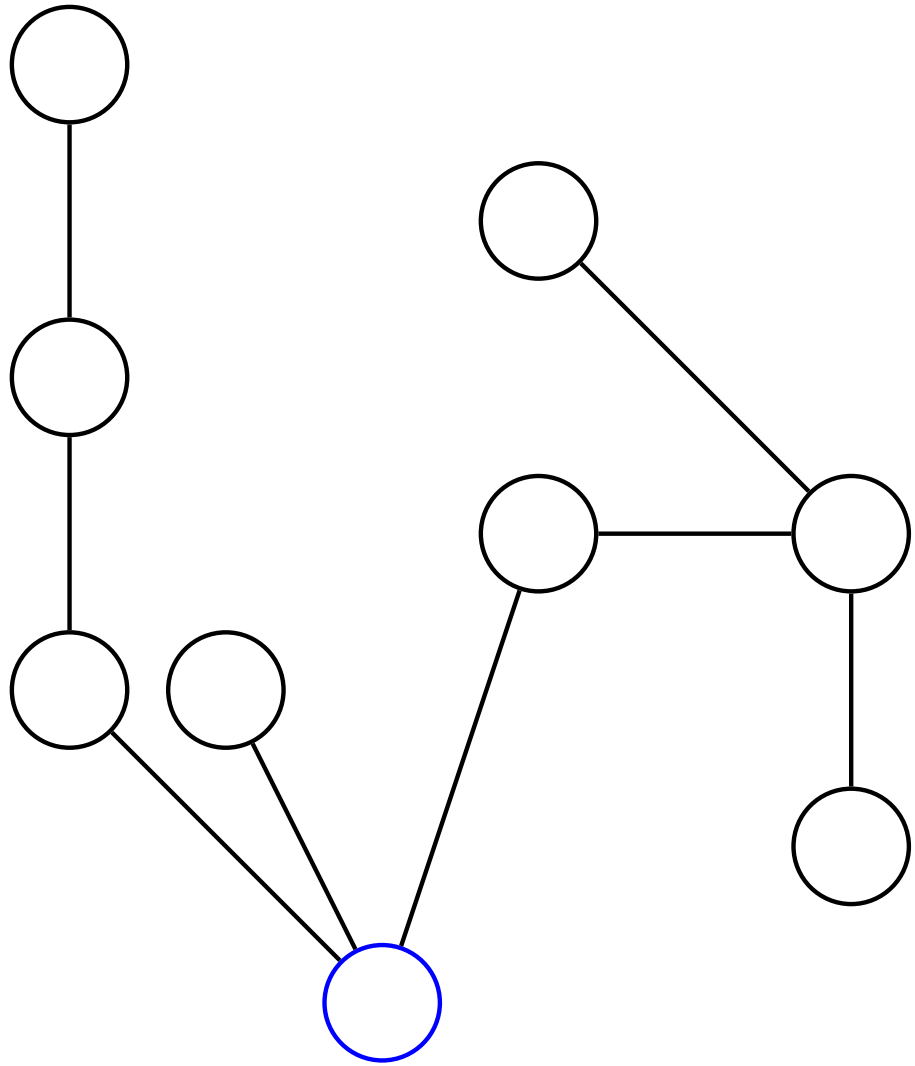


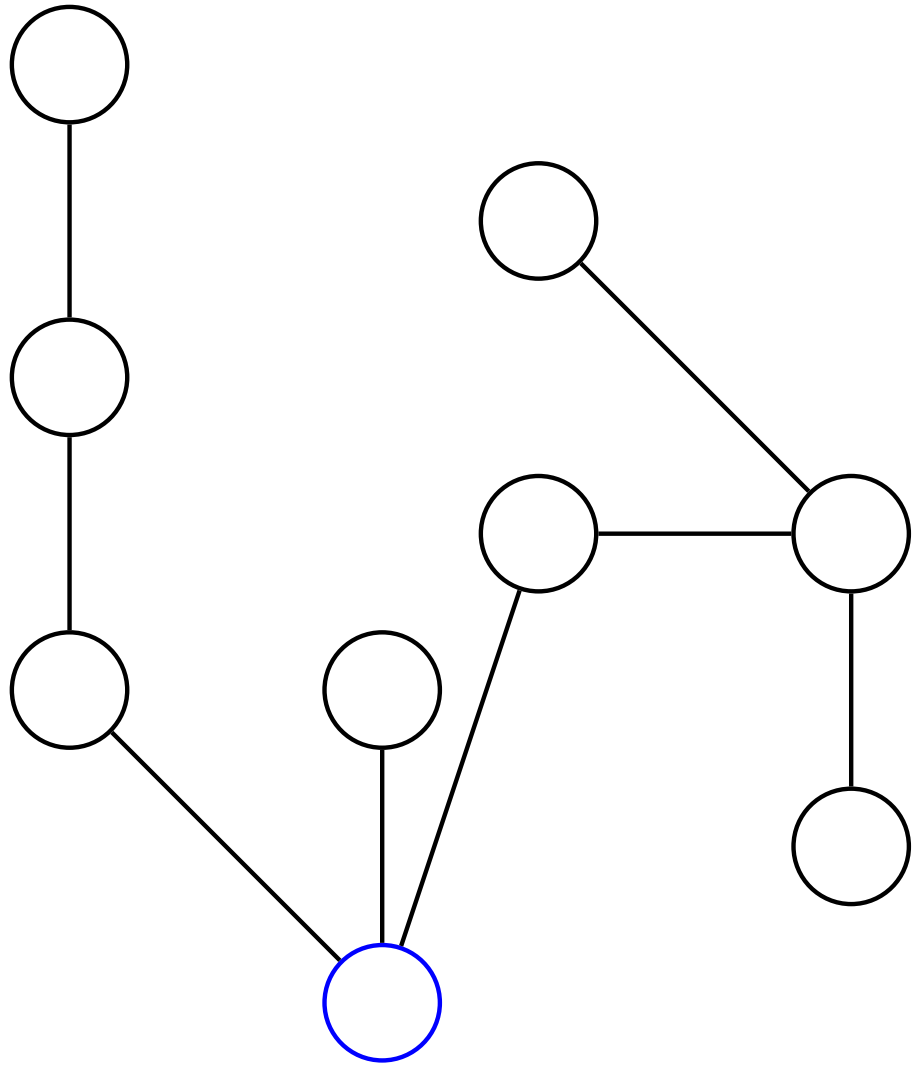


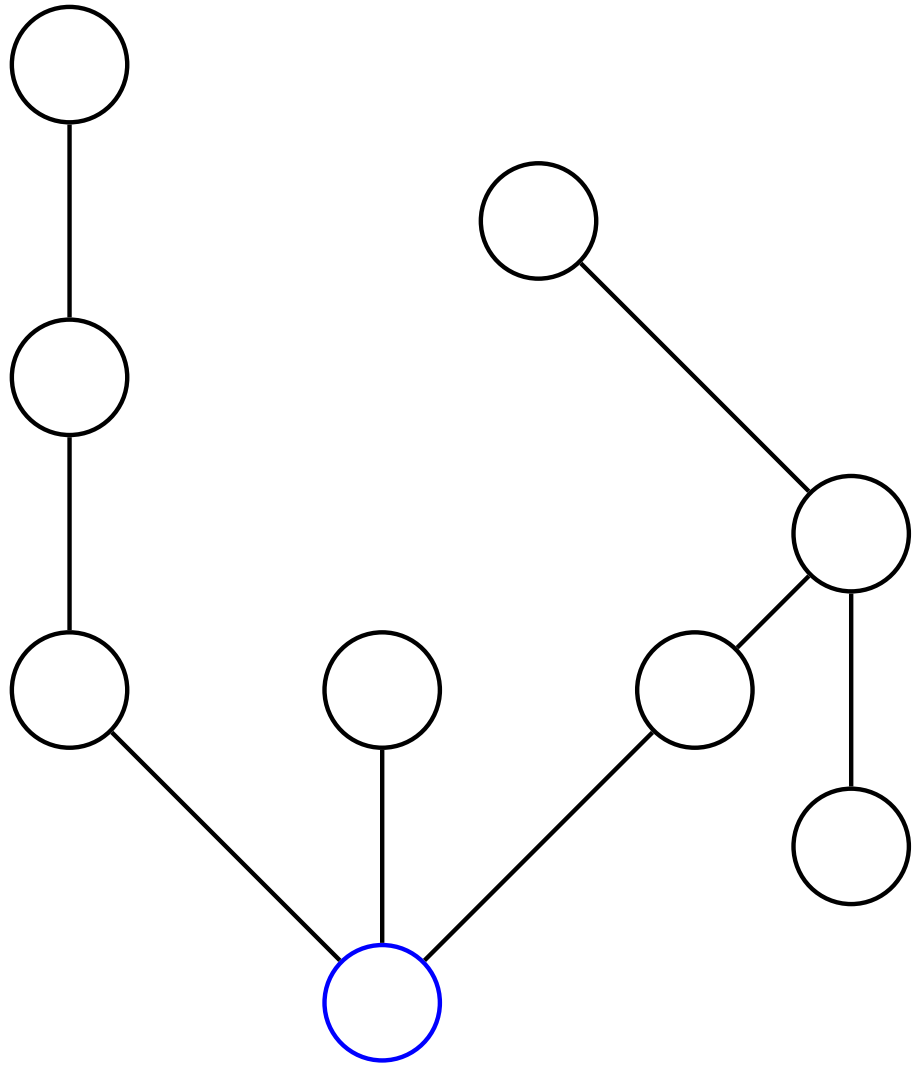


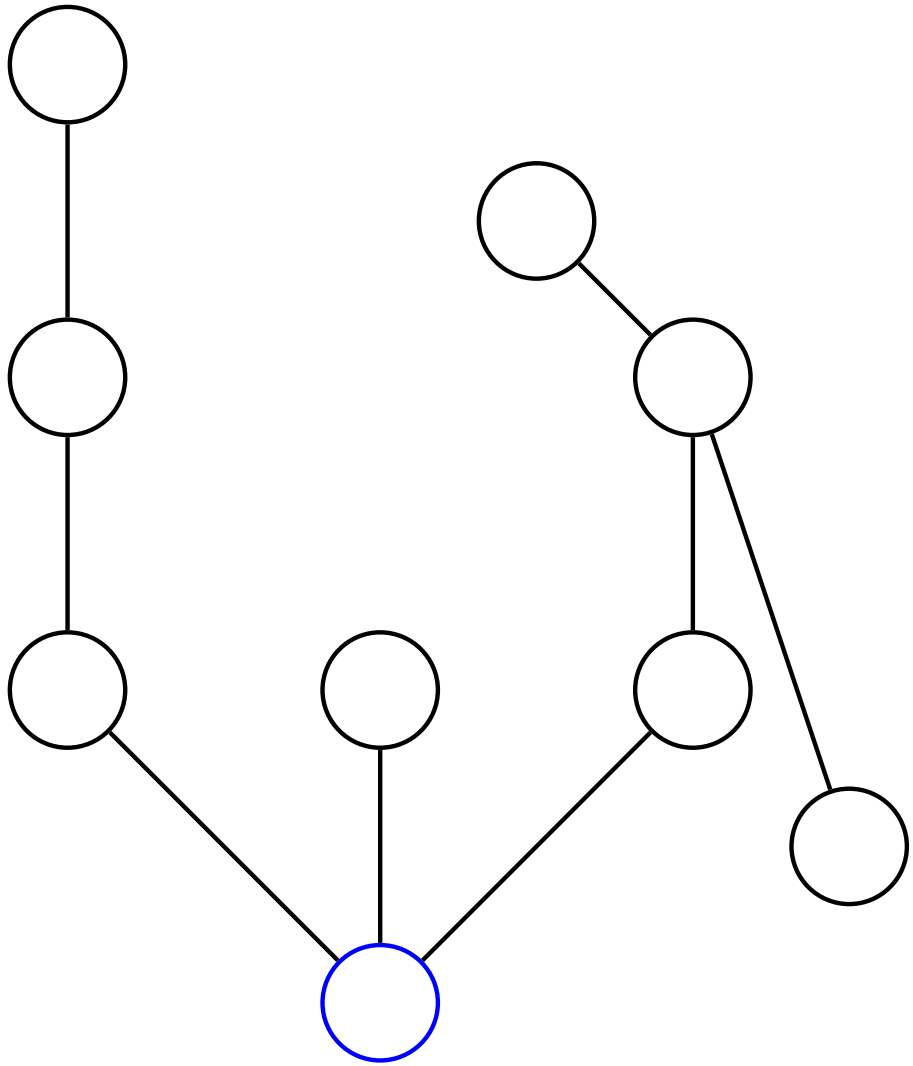


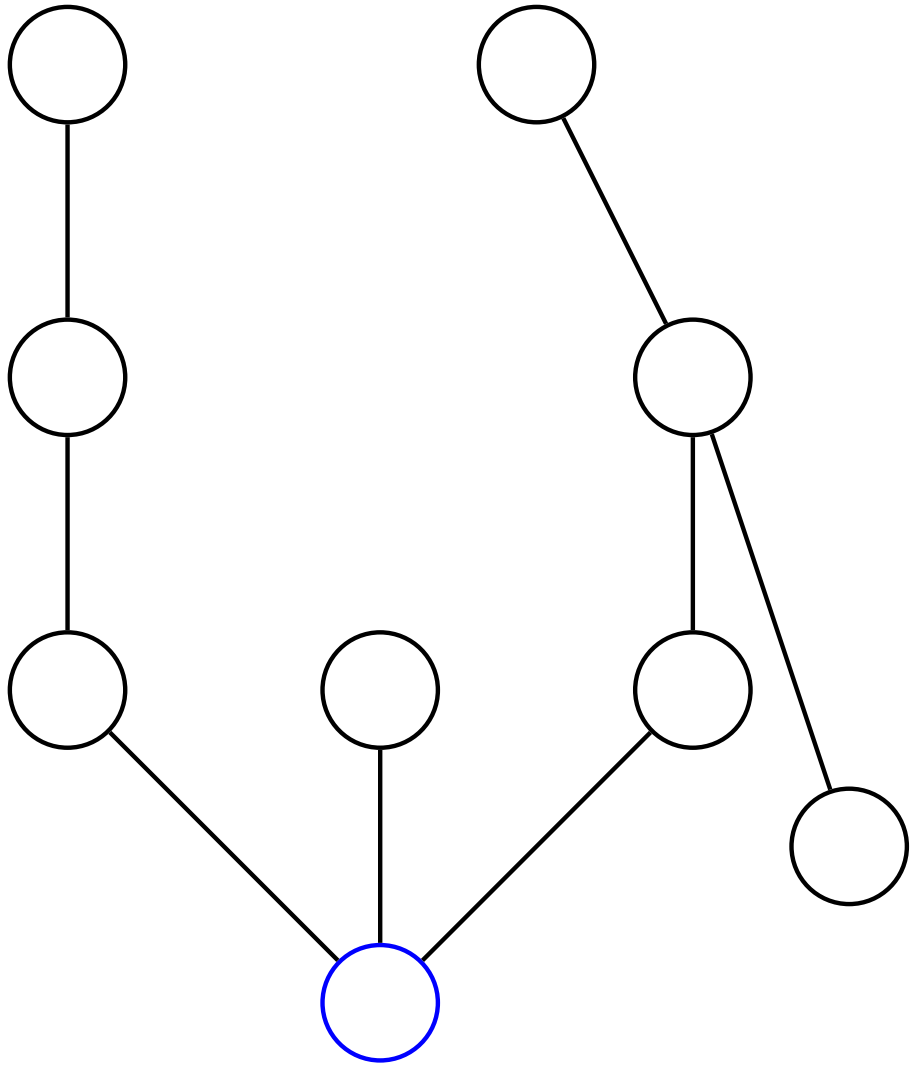


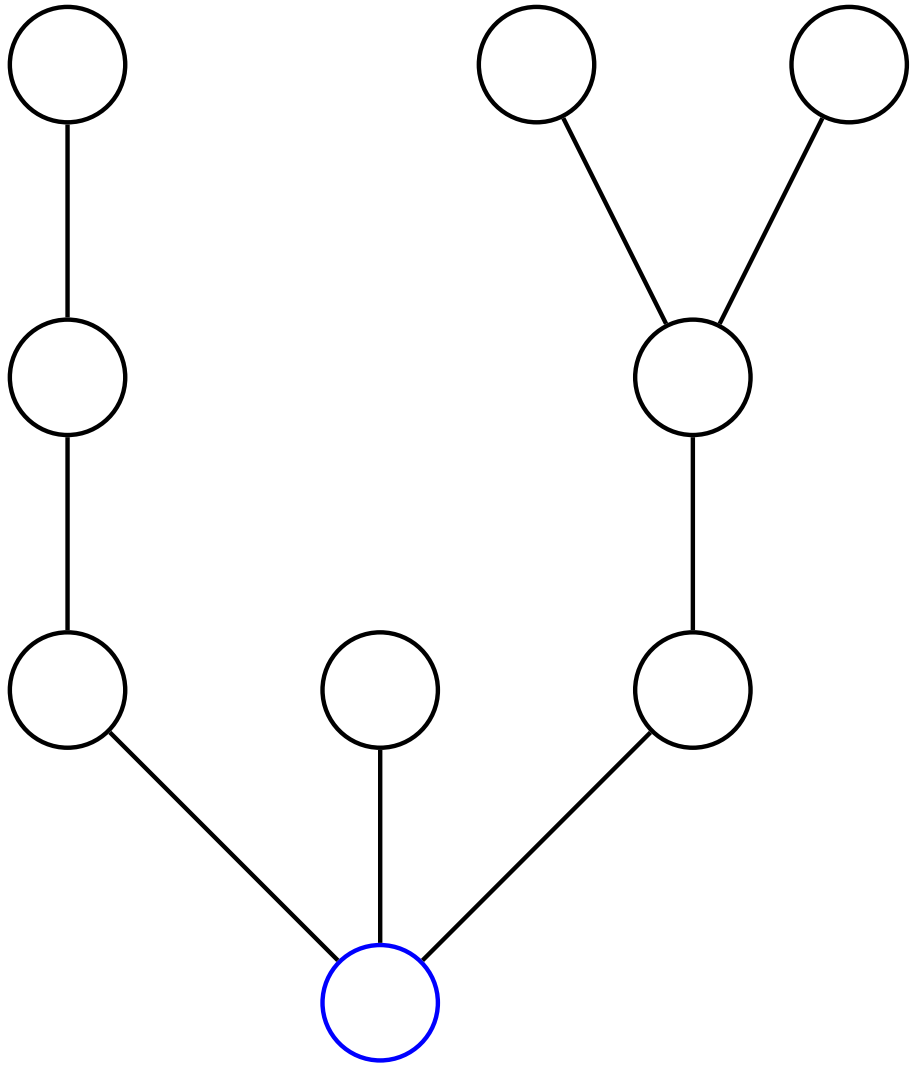


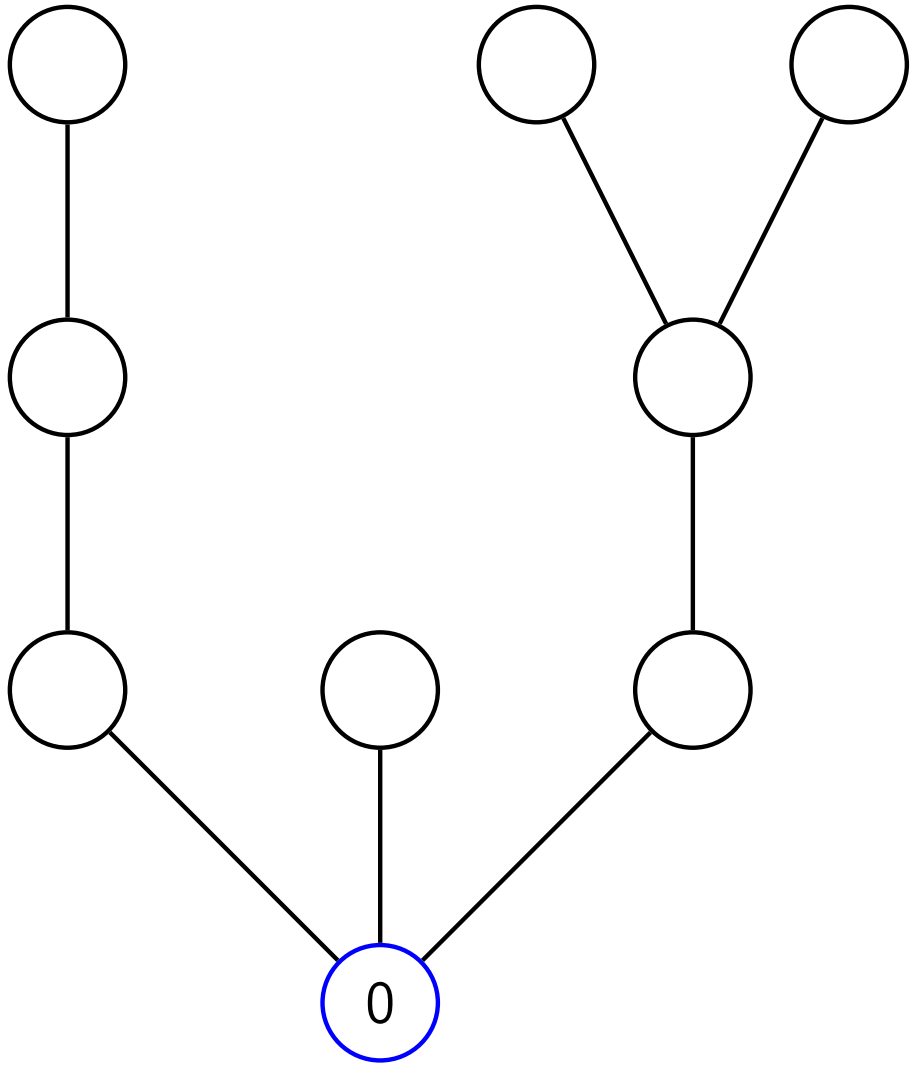








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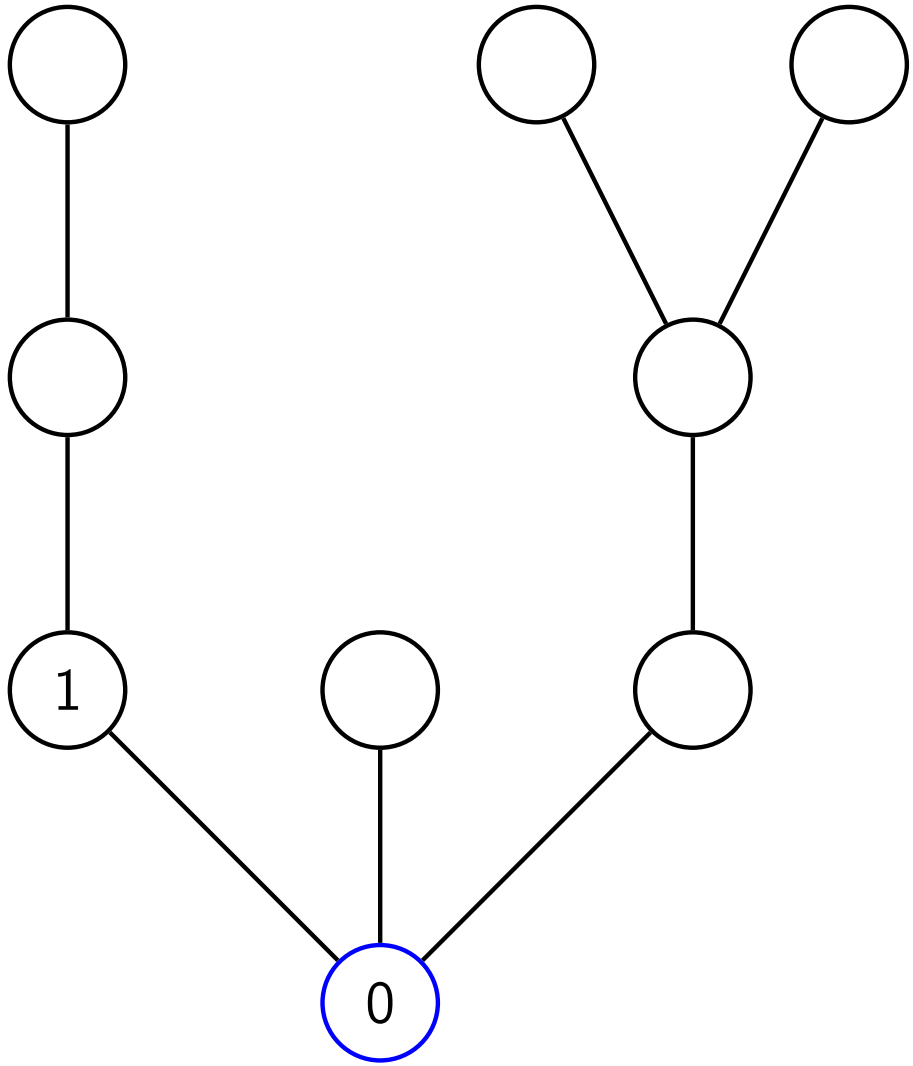
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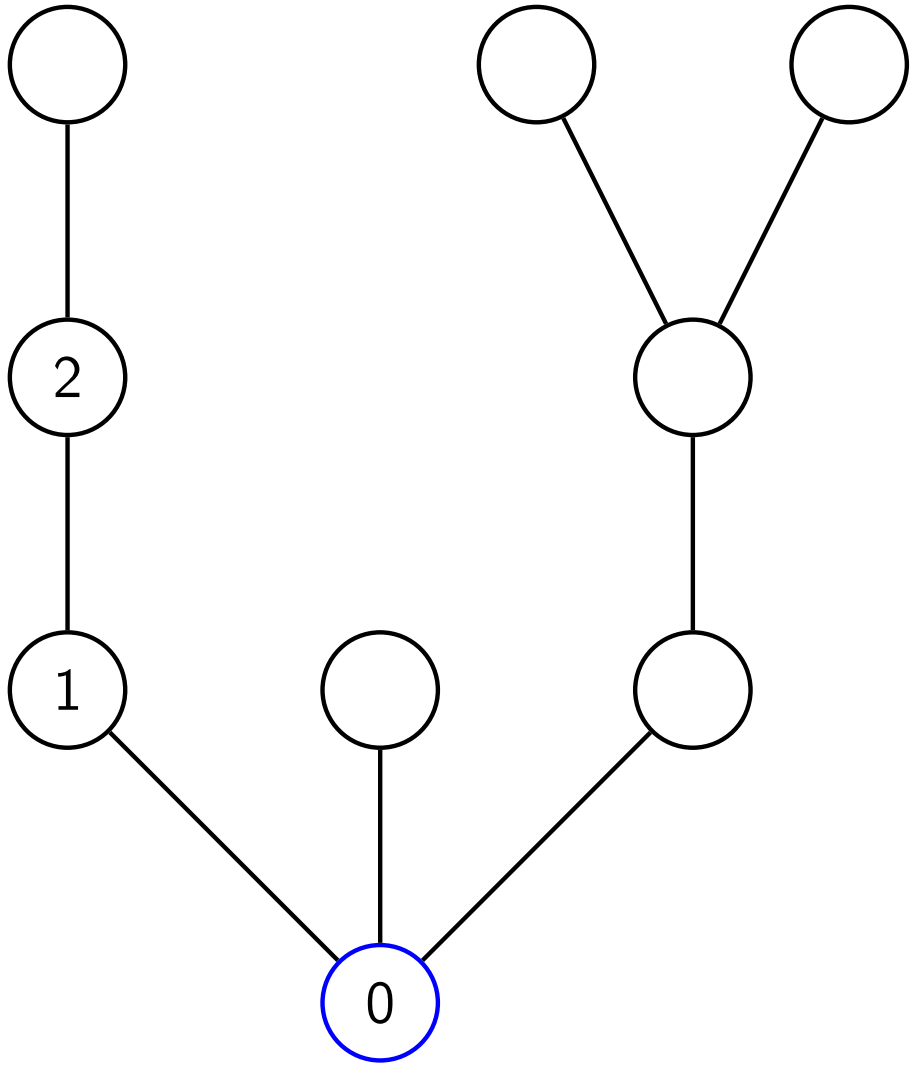
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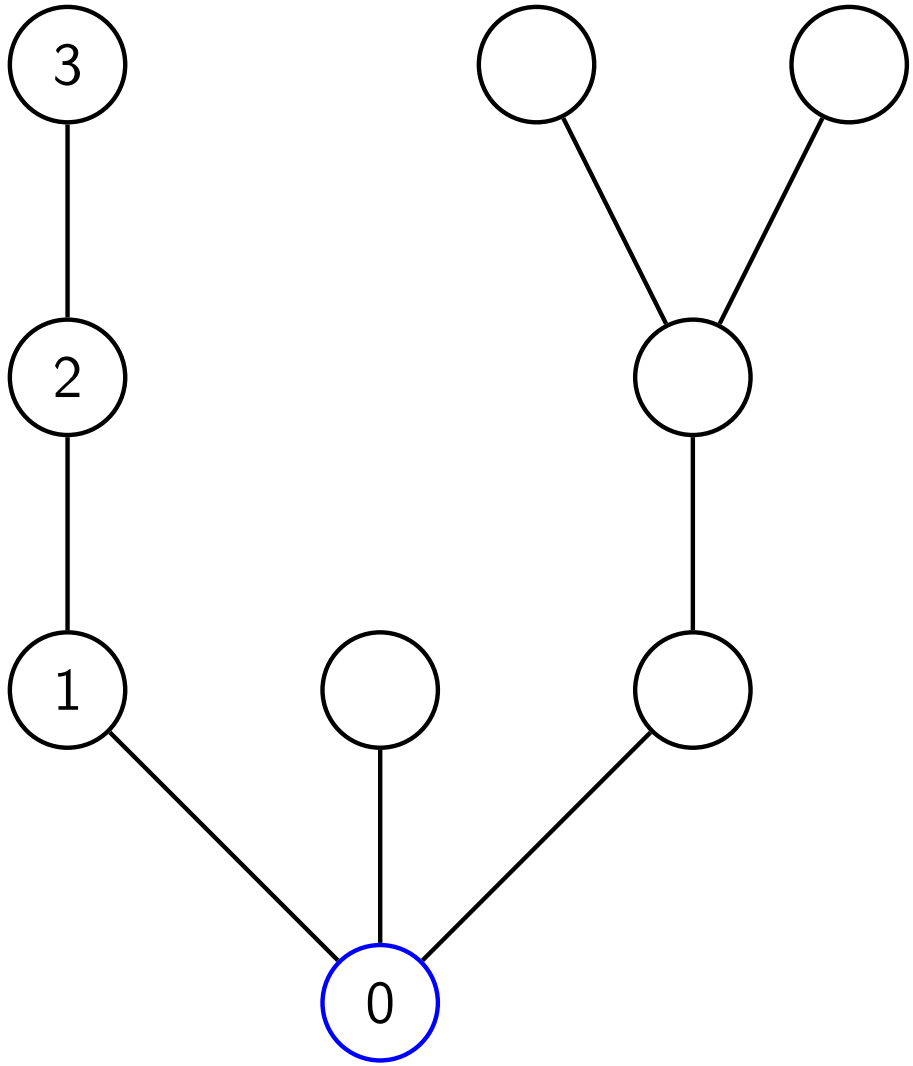
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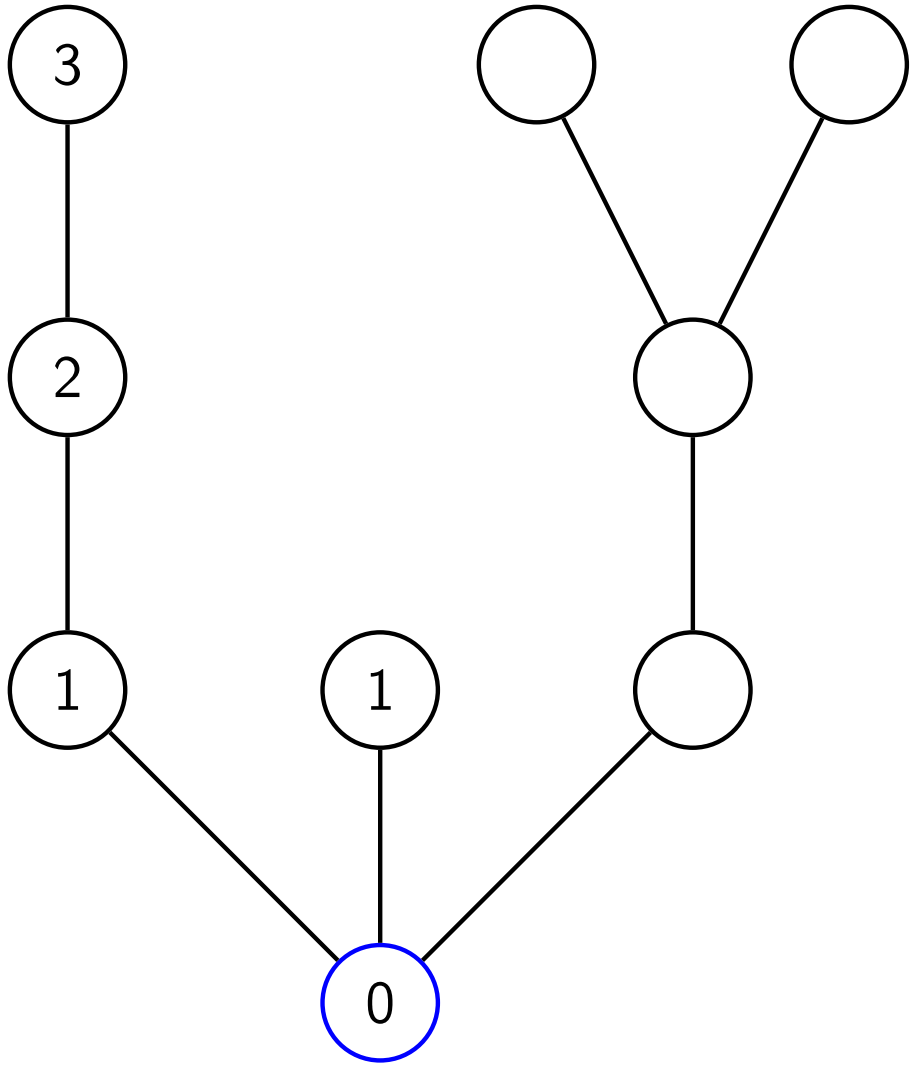
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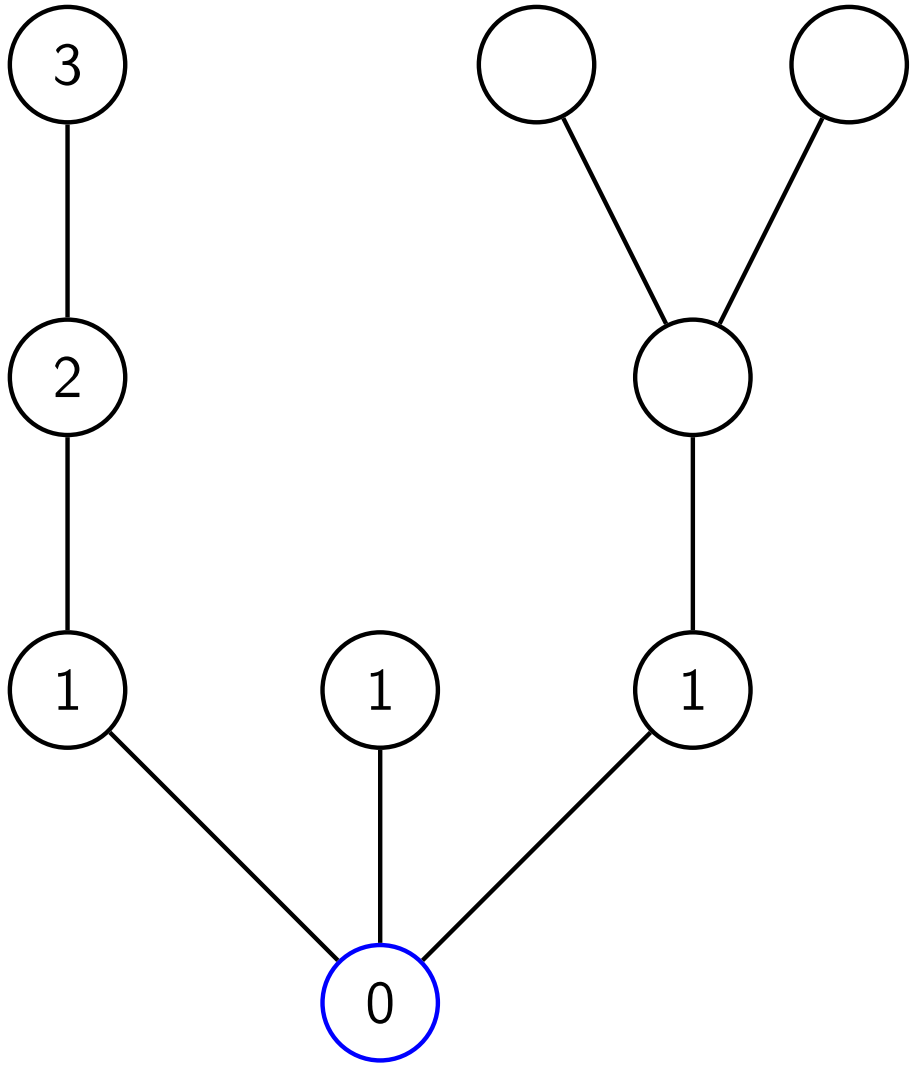
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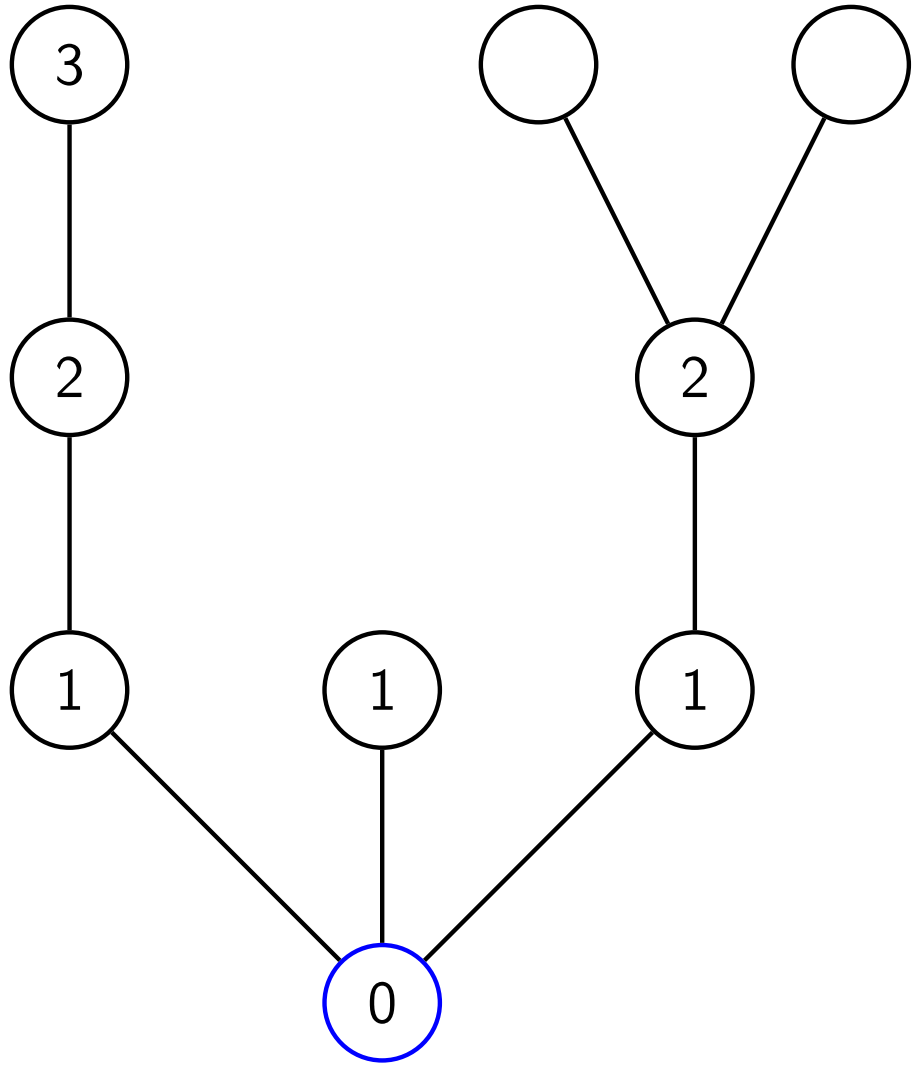
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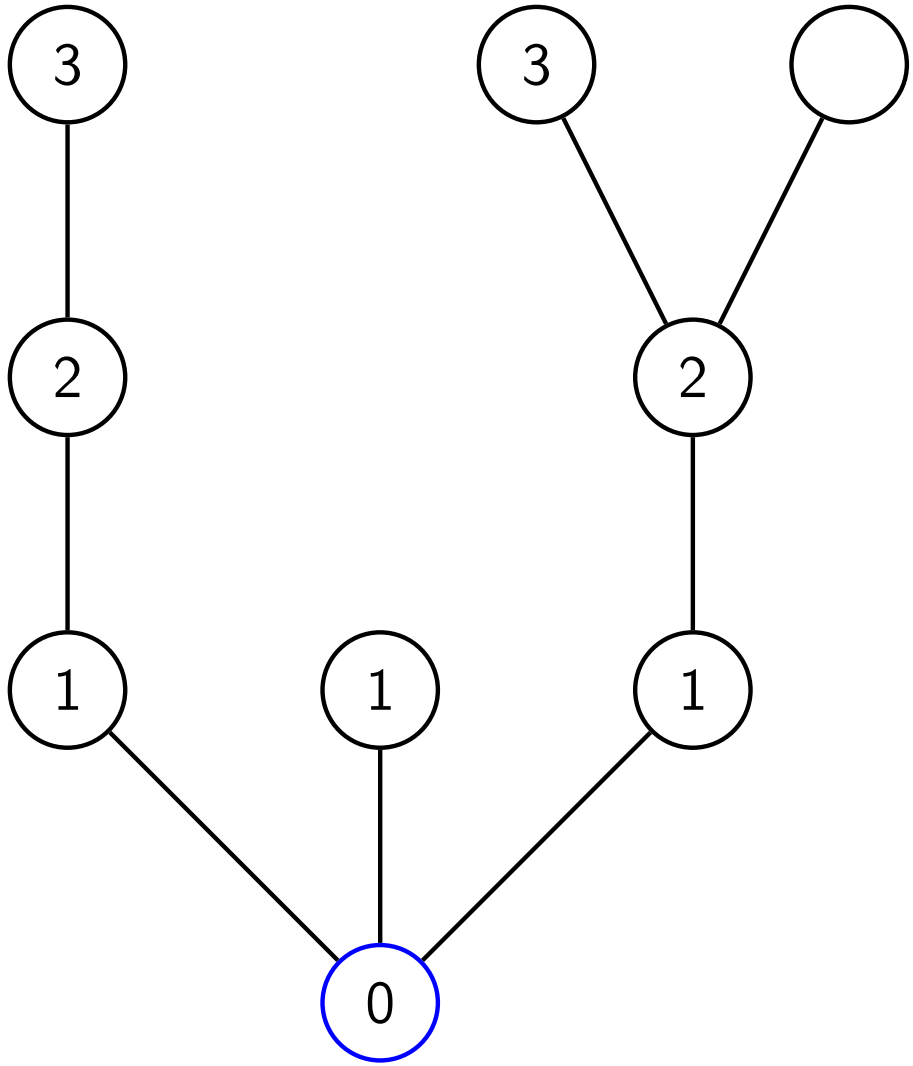
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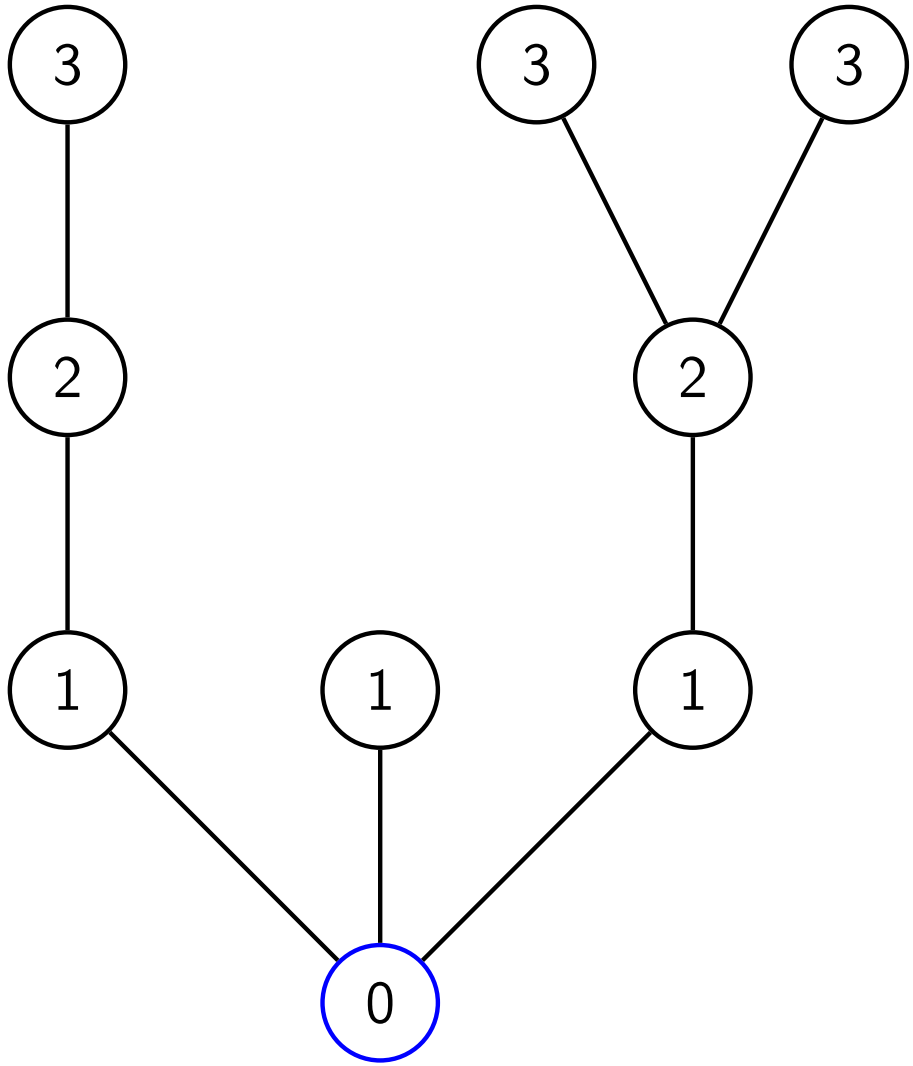
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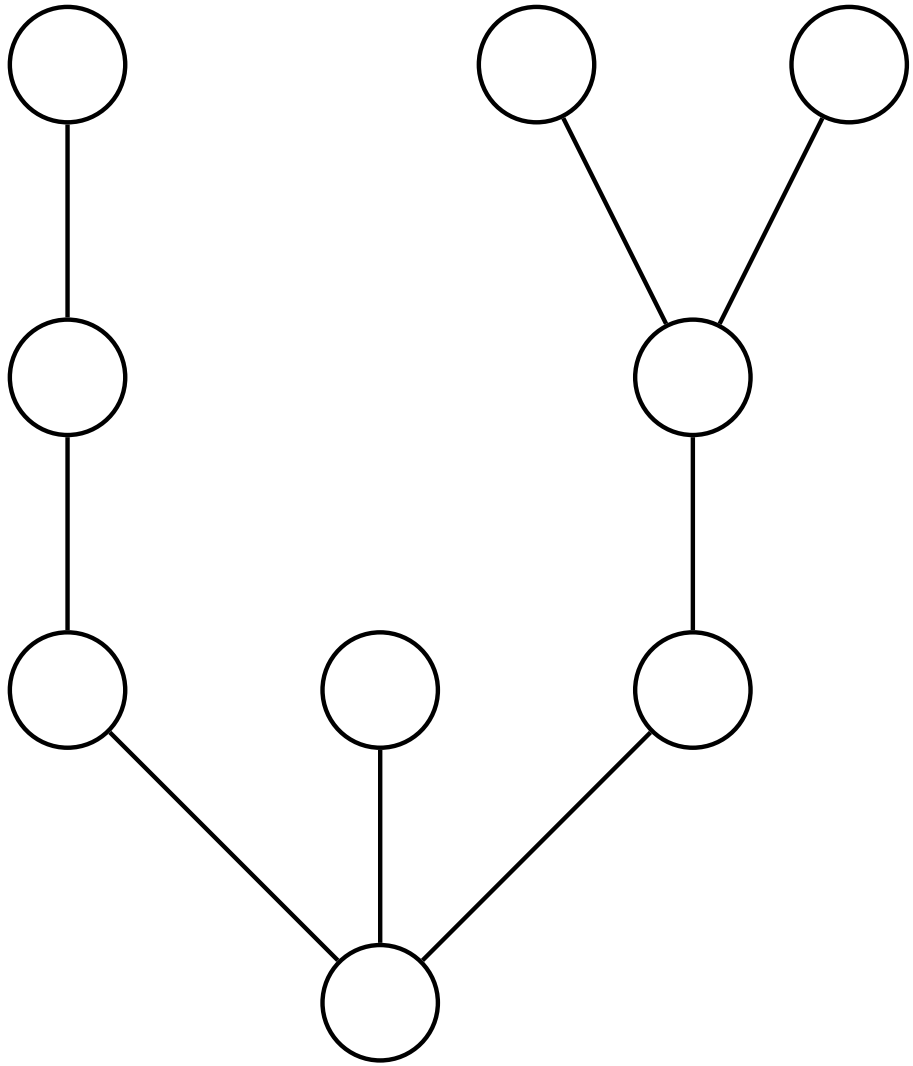
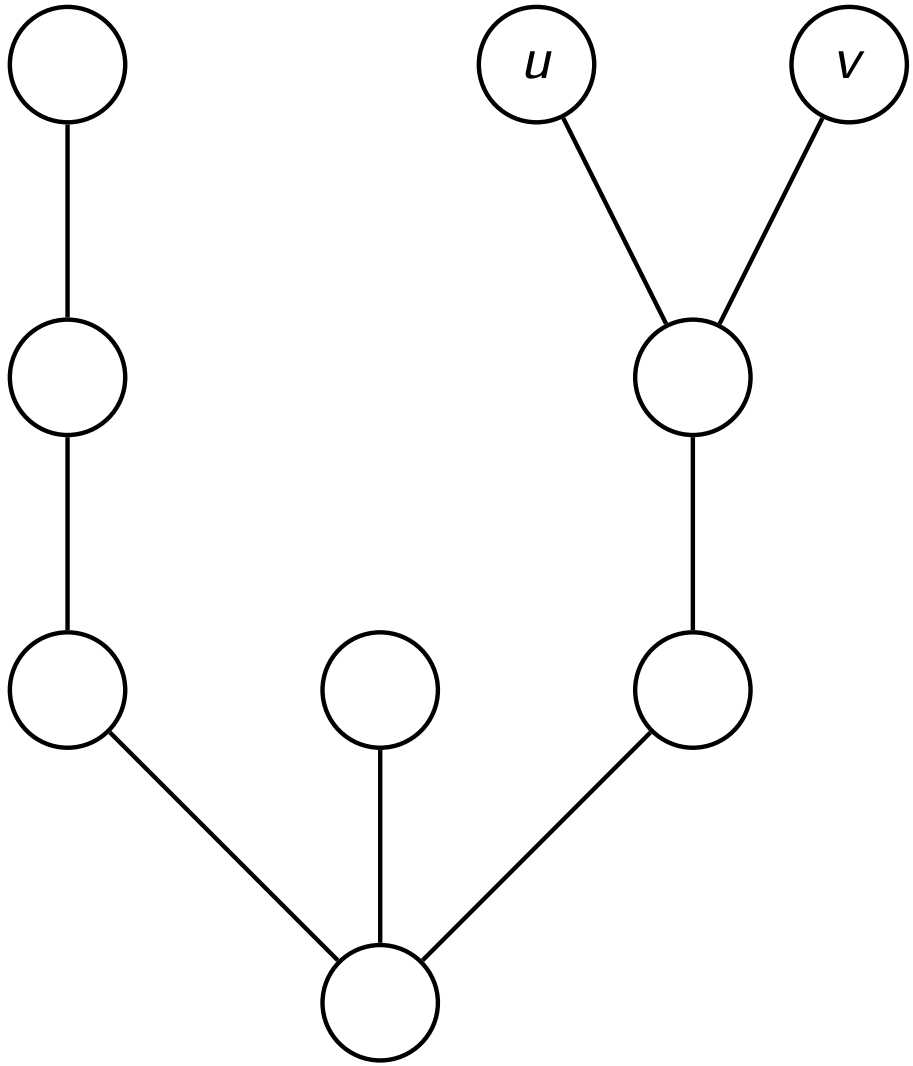
- Við köllum þann nágranna hnúts sem hefur einum lægri hæð *foreldri* hnútsins.
- Við getum þá fundið veginn að rótinni með því að ferðast eftir foreldrum.
- Þeir hnútar sem eru á veginum niður að rótinni kallast *forfeður* hnúts.
- Það er kannski skrýtið, en allir hnútar er forfeður sínir.
- Oft nýtist okkur að vita hvaða sameiginlegi forfaðir tveggja hnúta er hæstur í trénu.

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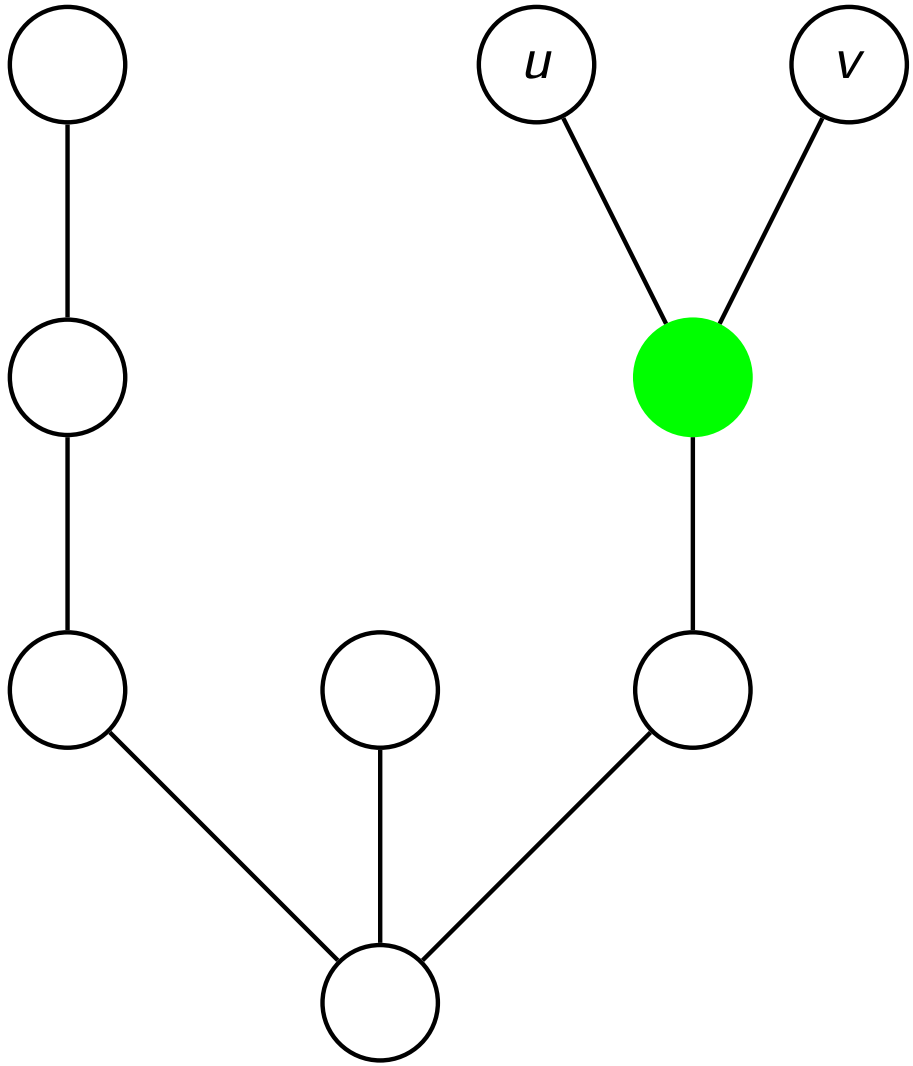
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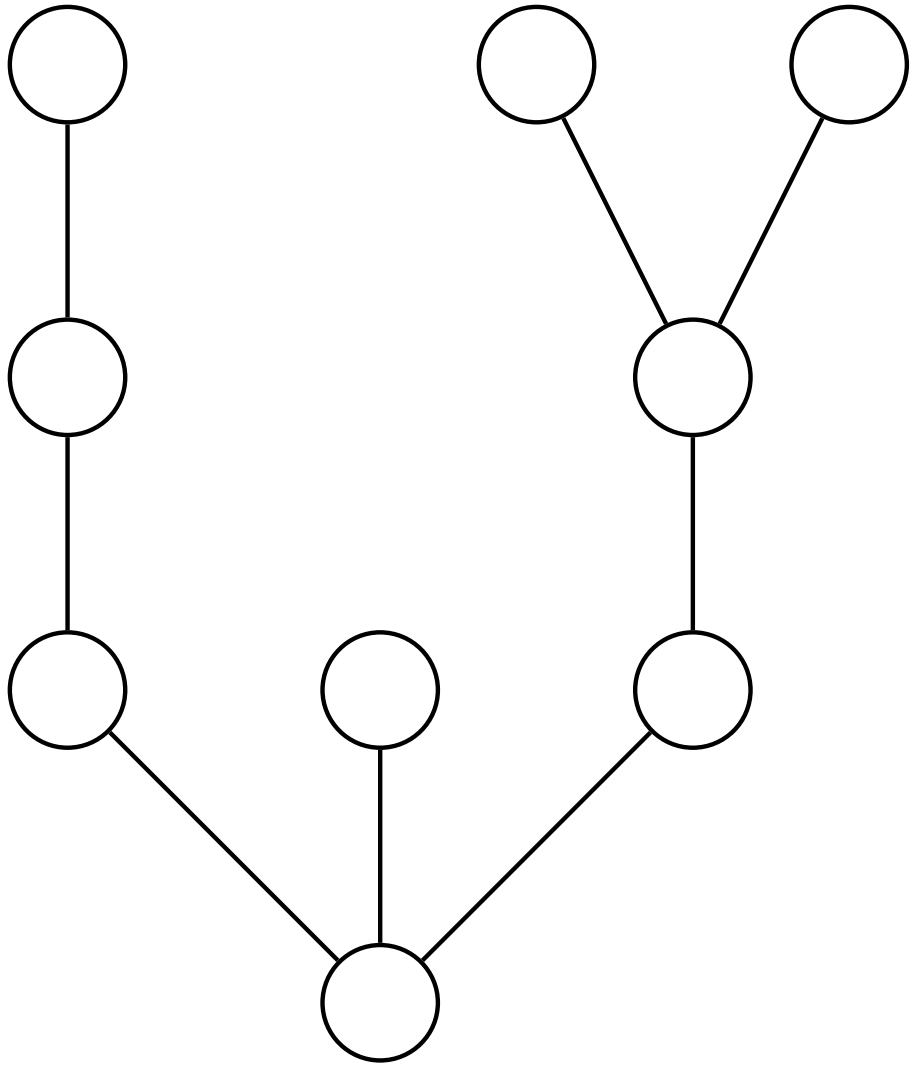

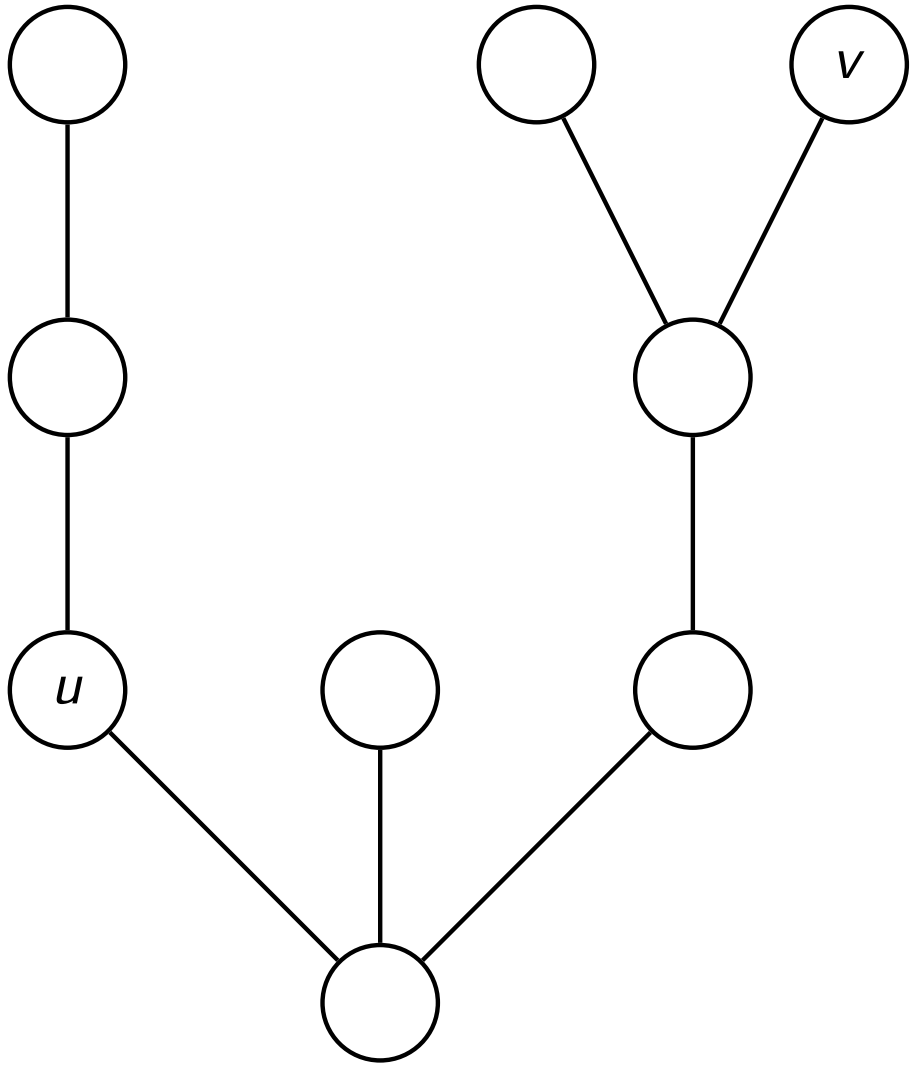
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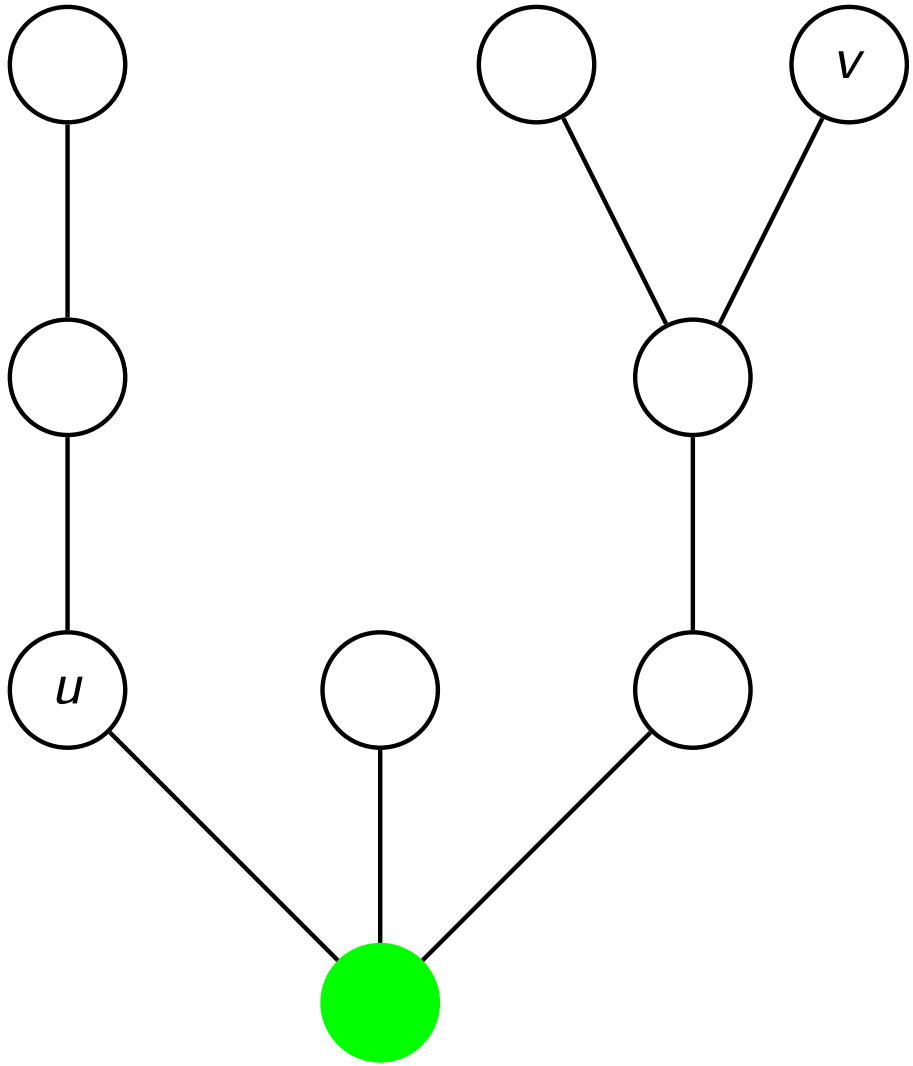
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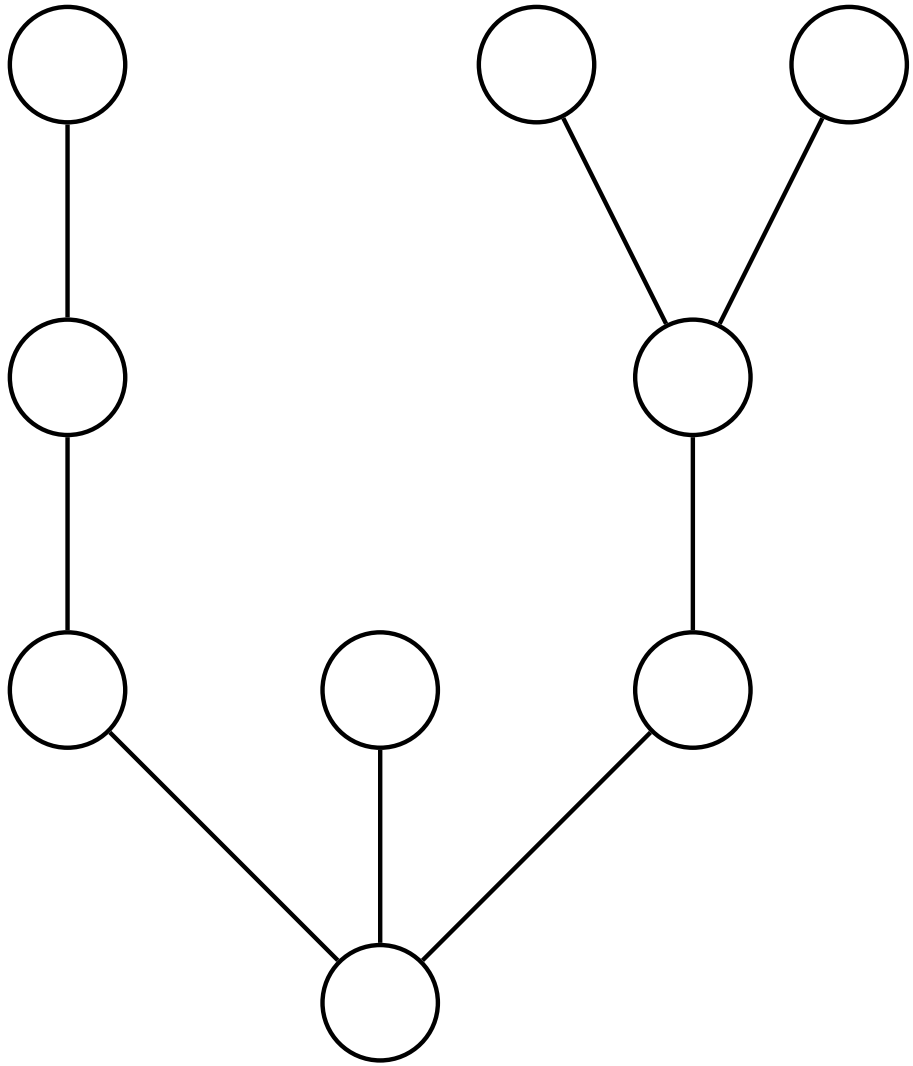

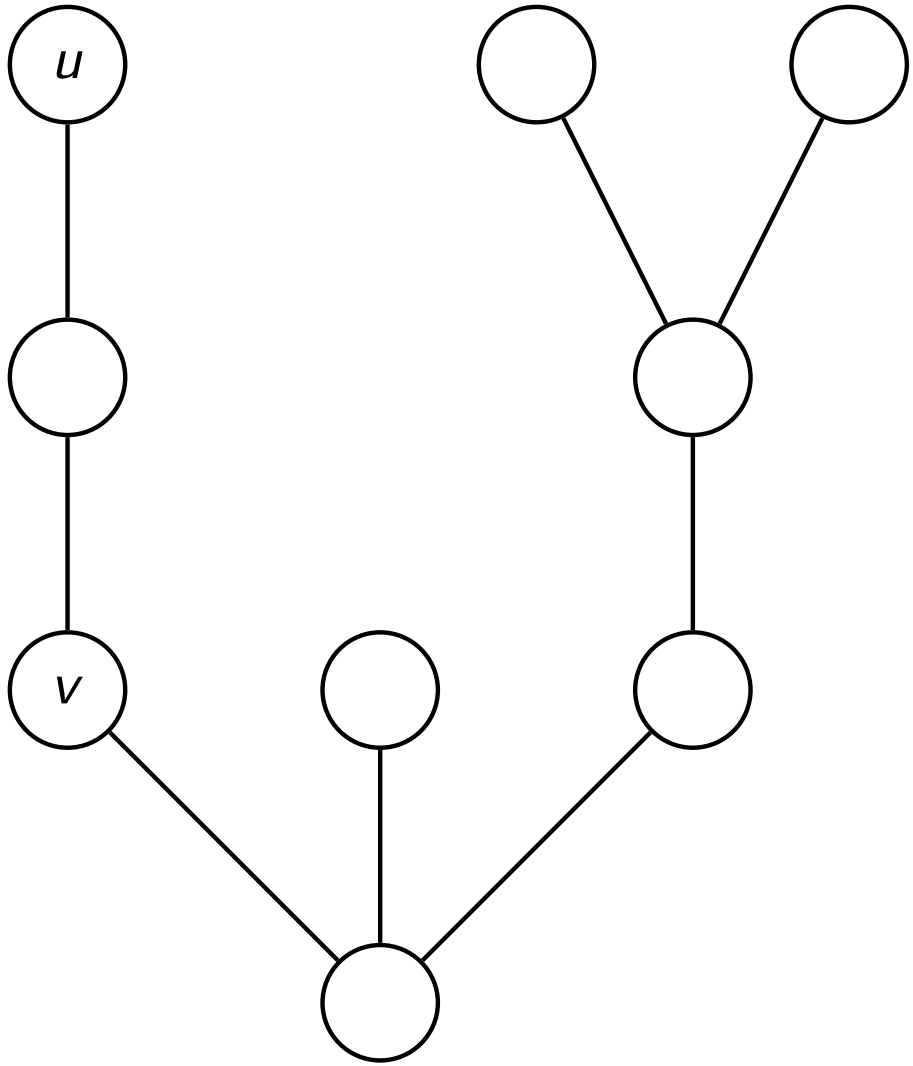
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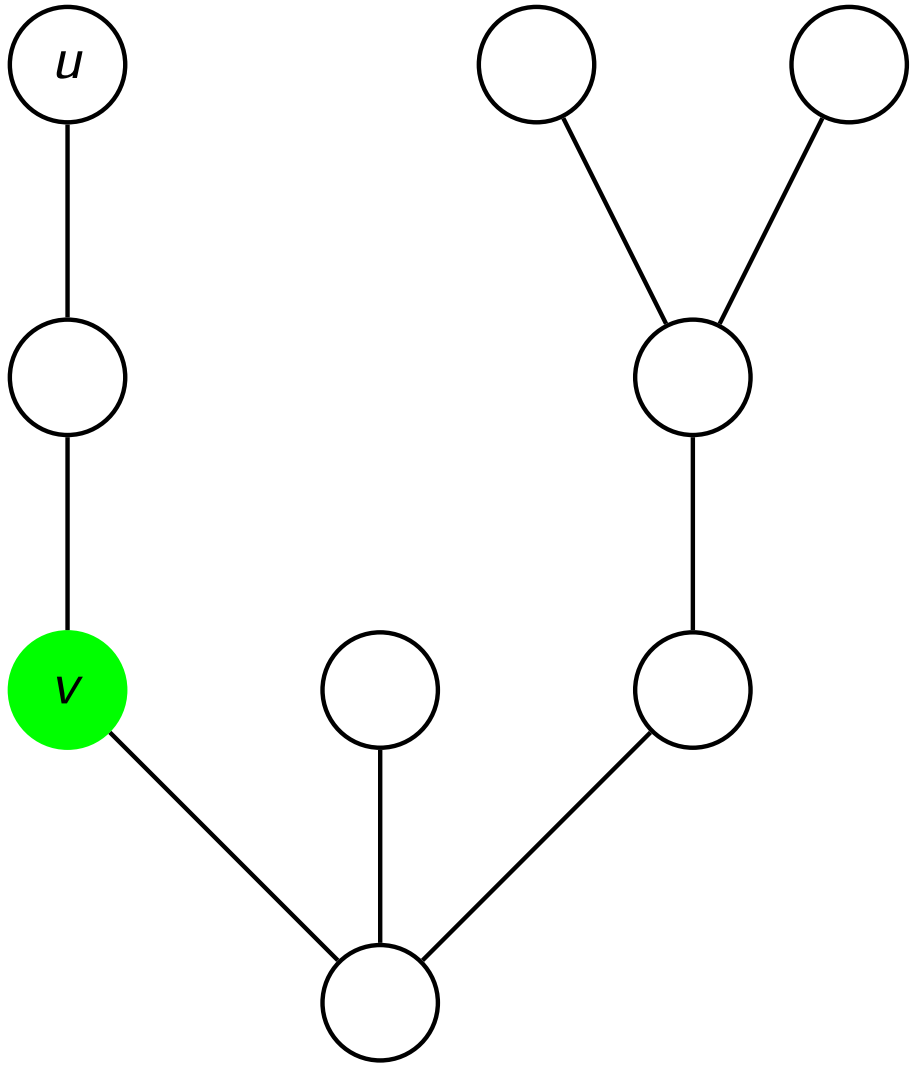
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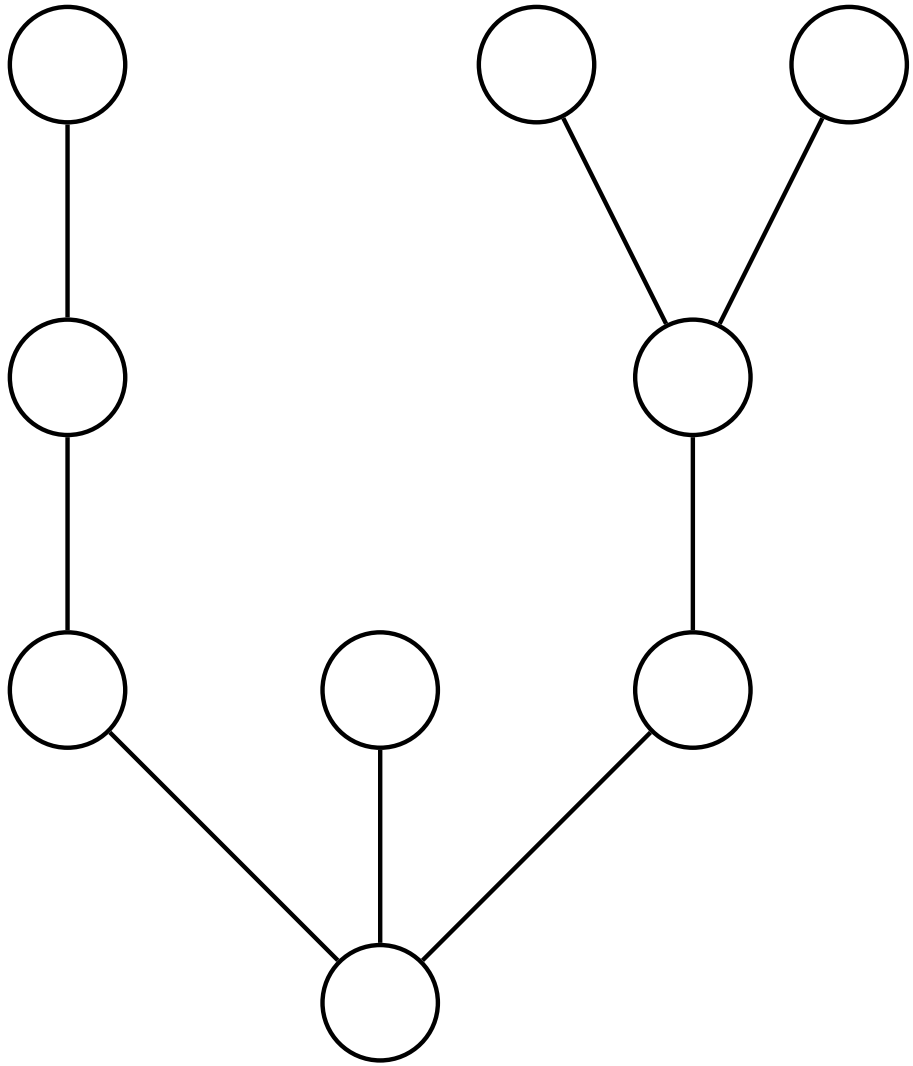
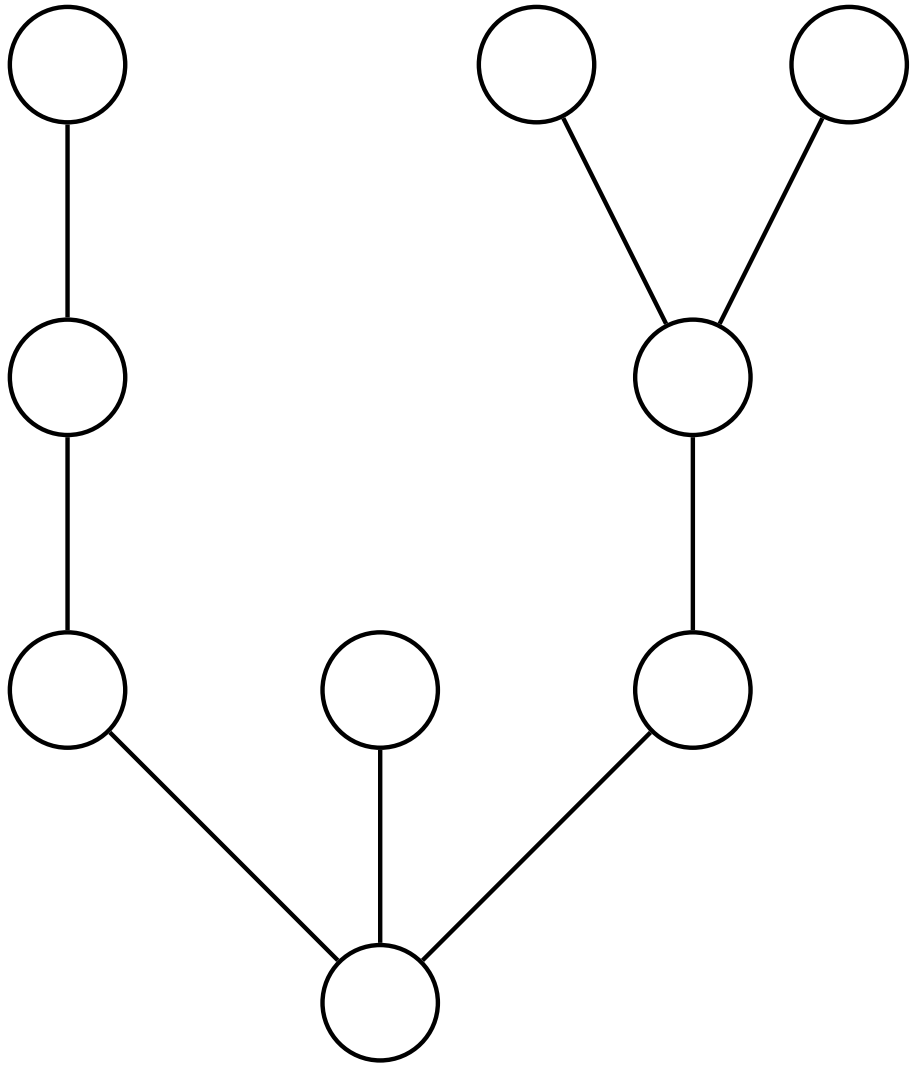
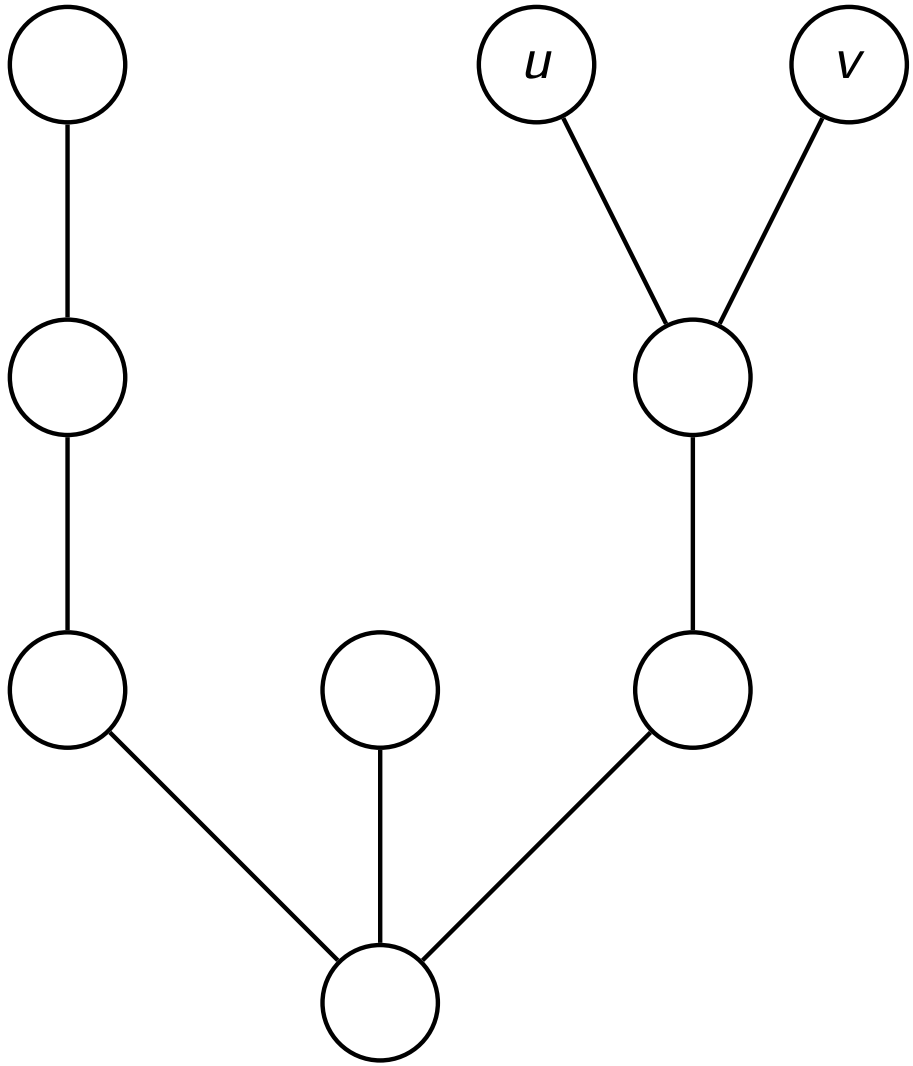


- Hvernig finnum þennan forföður?
- Látum u og v tákna hnútana og x hæsta sameiginlega forföður þeirra.
- Við getum gert ráð fyrir að u sé ofar í trénu, það er að segja $h(u) \geq h(v)$.
- Við vitum að allir forfeður u sem hafa hæð stærri en $h(v)$ eru ekki x.
- Svo við getum ferðast niður tréð frá u þar til við erum komin í sömu hæð og v.
- Við getum svo ferðast eftir foreldrum beggja á sama tíma þangað til við lendum í sama hnútnum.
- Sá hnútur er x.
- Sjáum hvernig þessi aðferð leysir sýnidæmin sem við sáum áðan.

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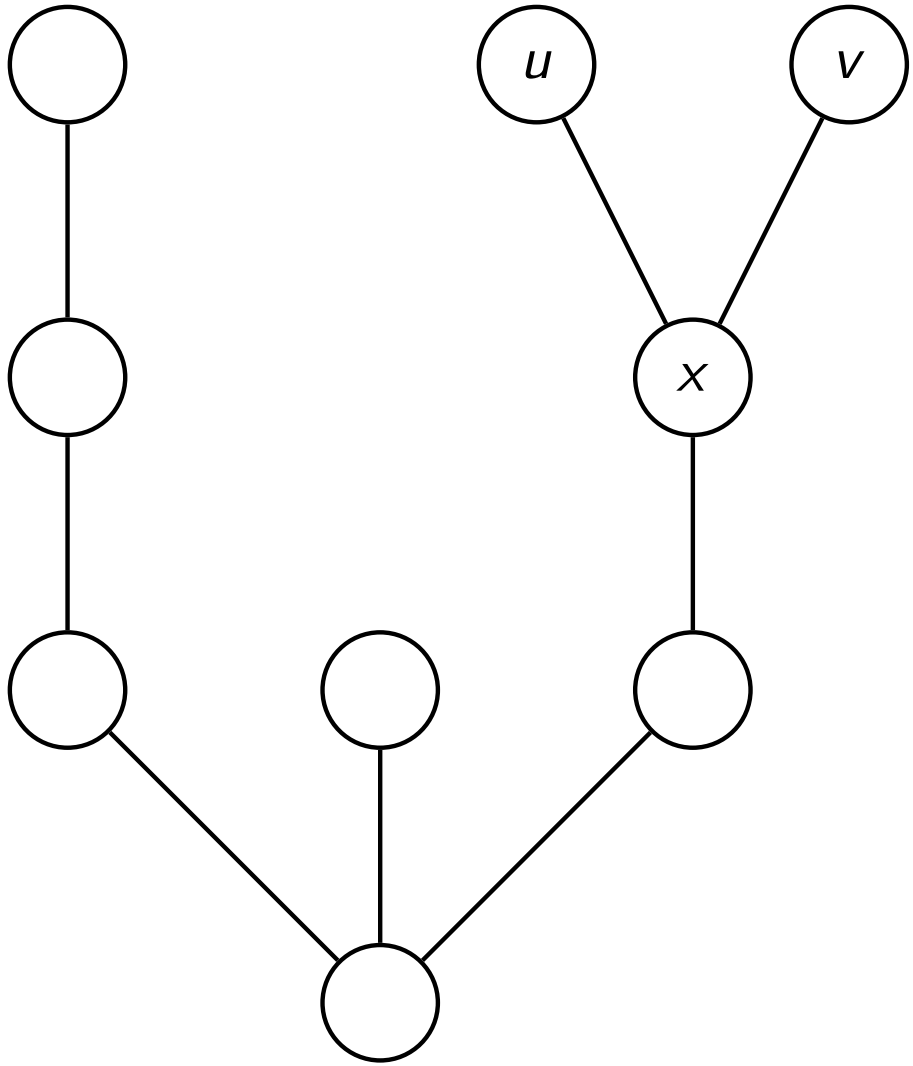
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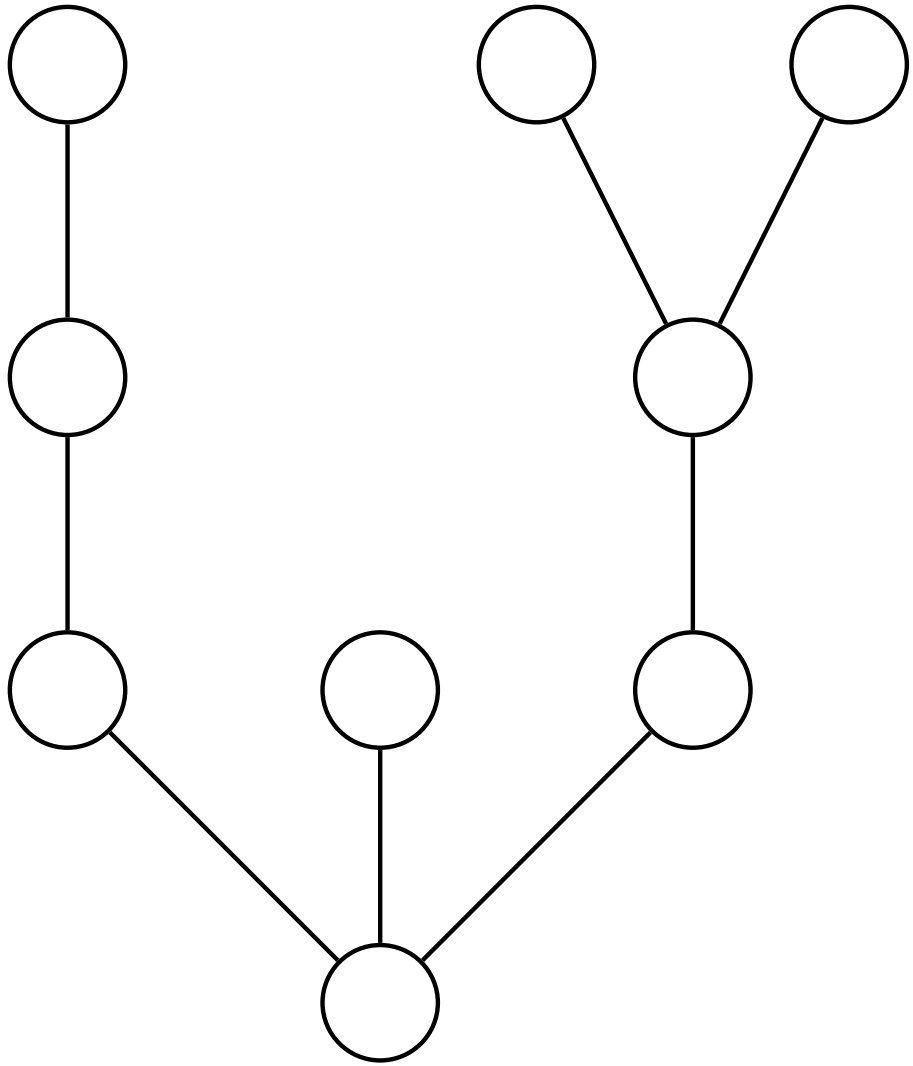

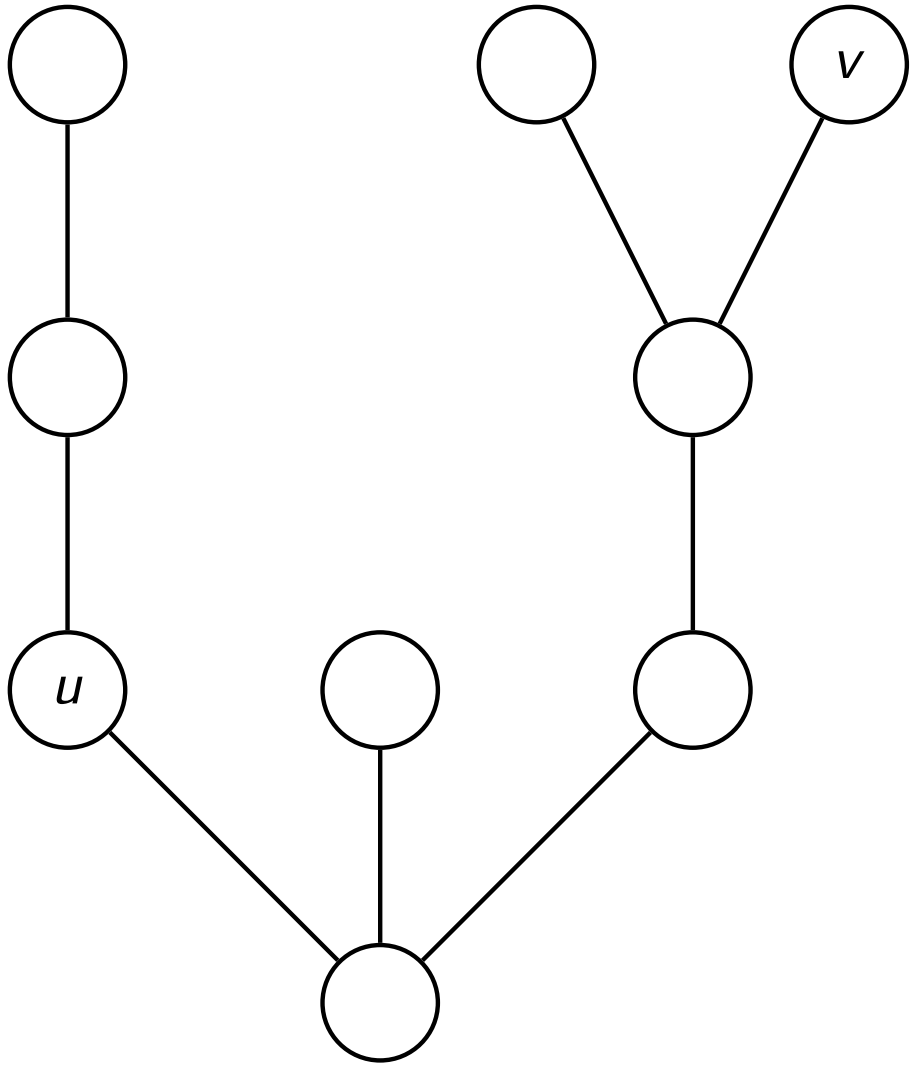
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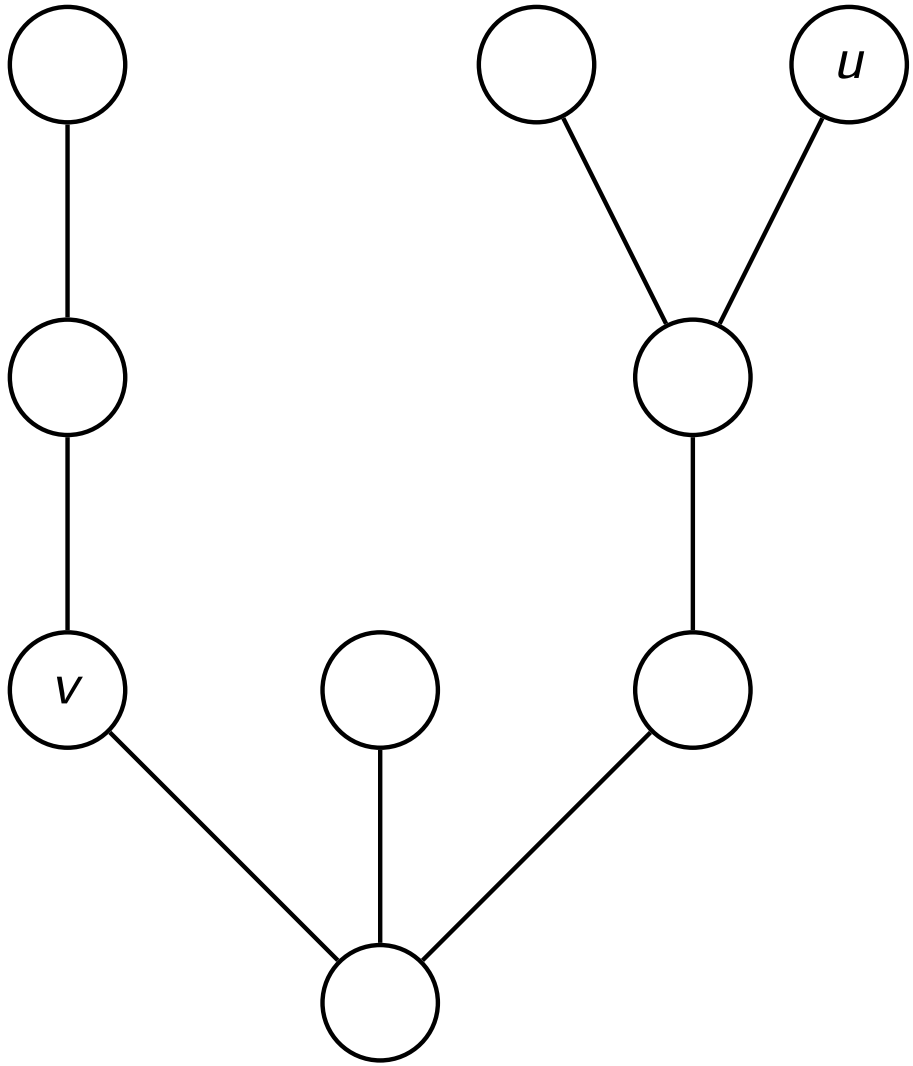
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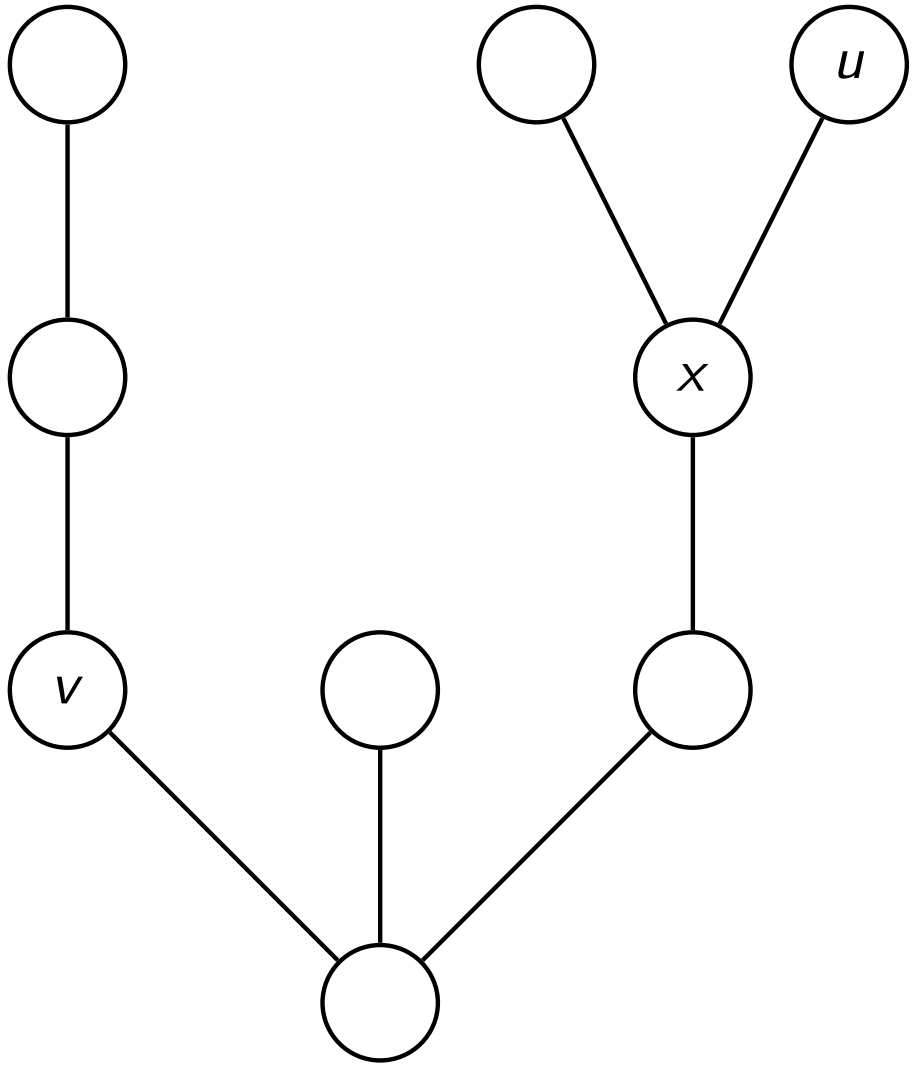
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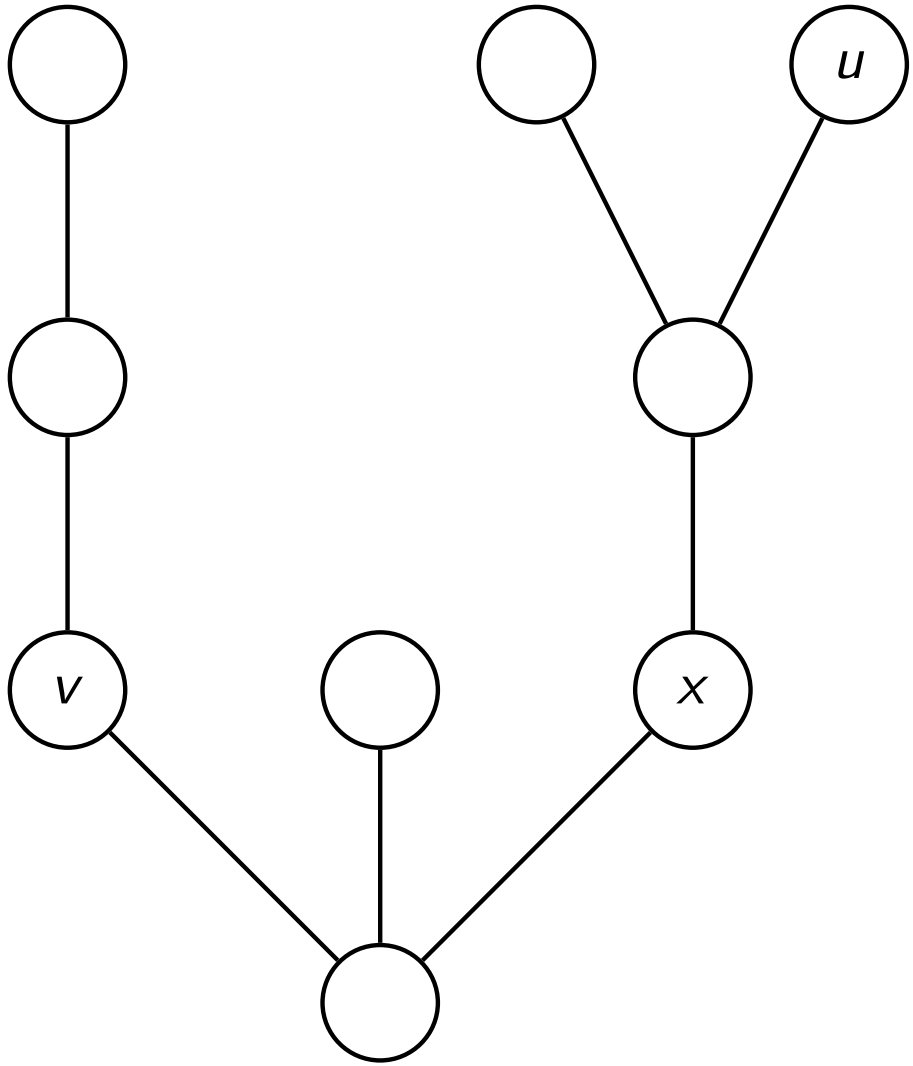
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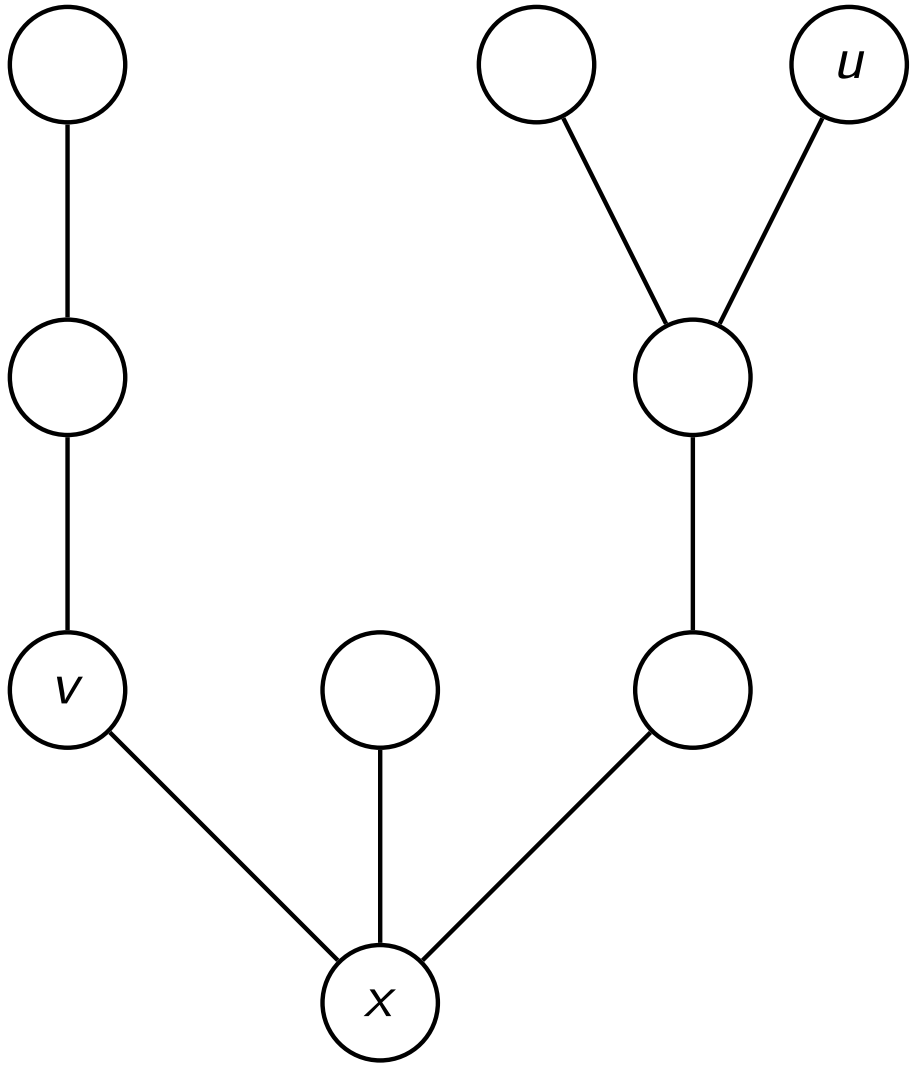
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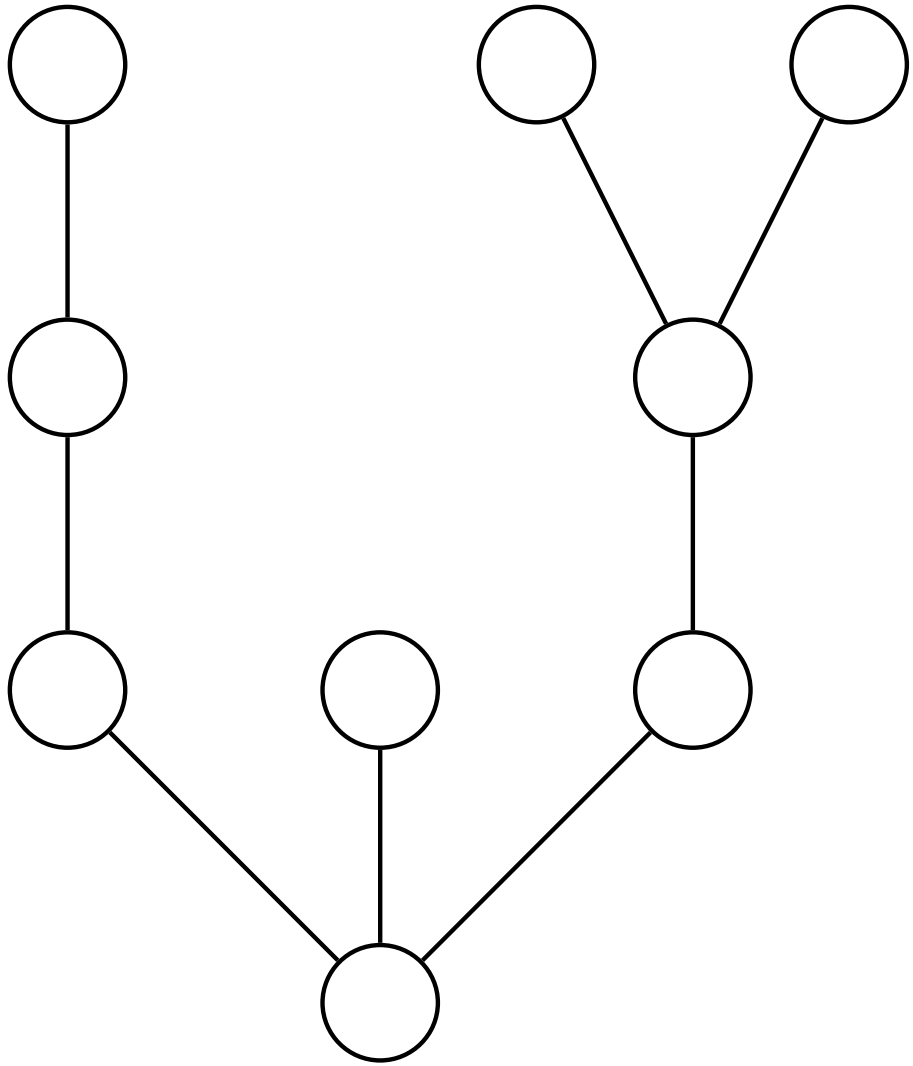

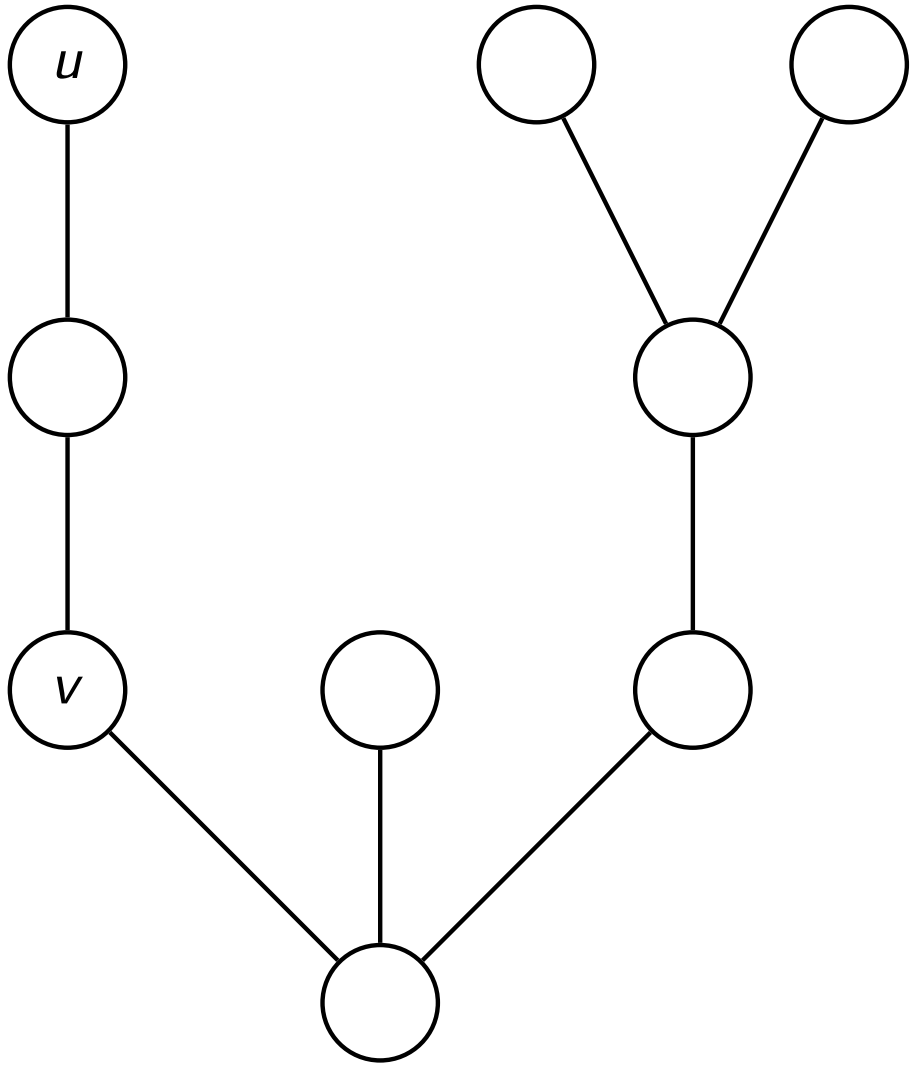
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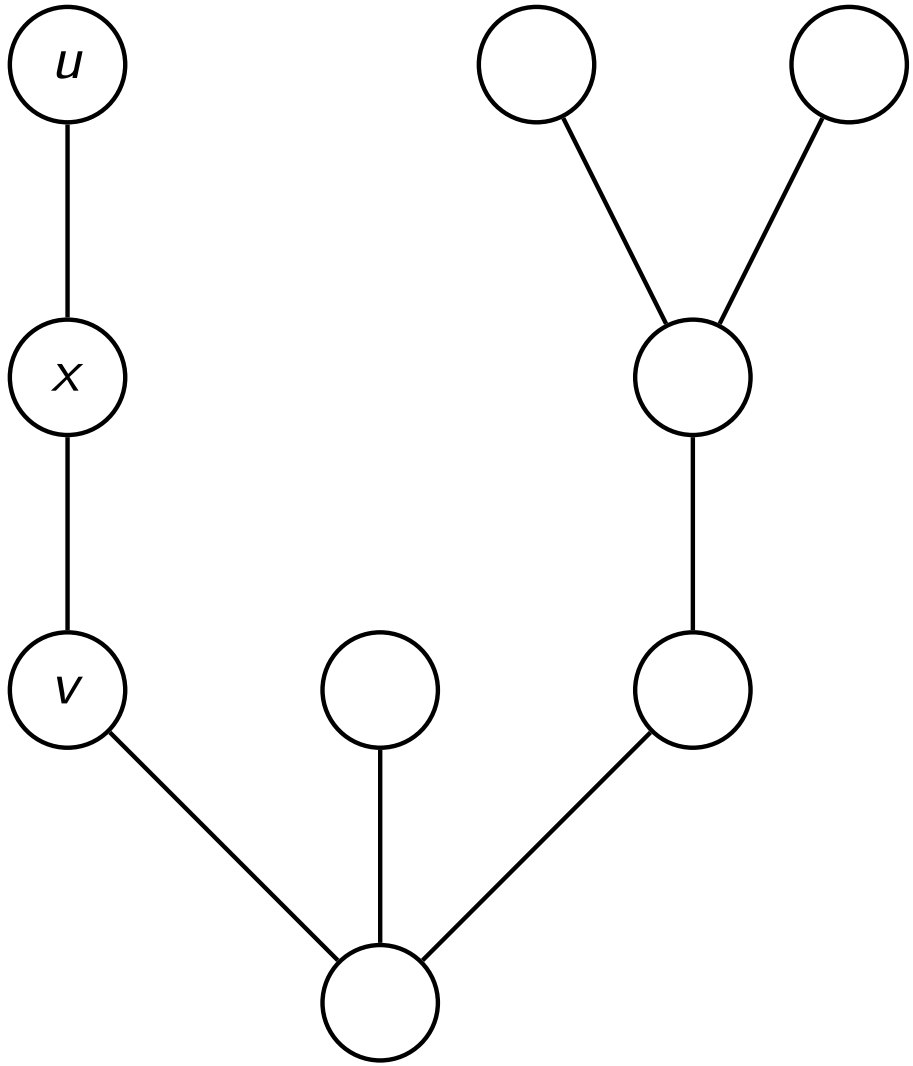
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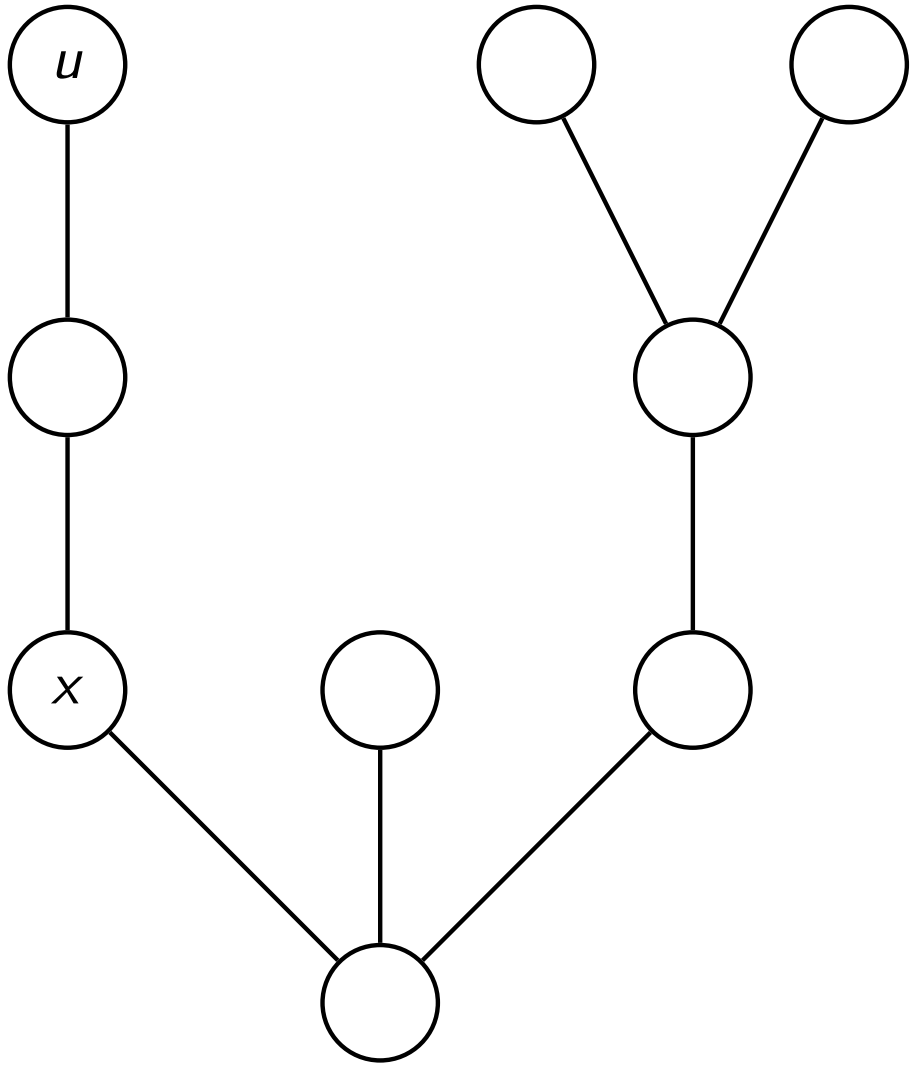
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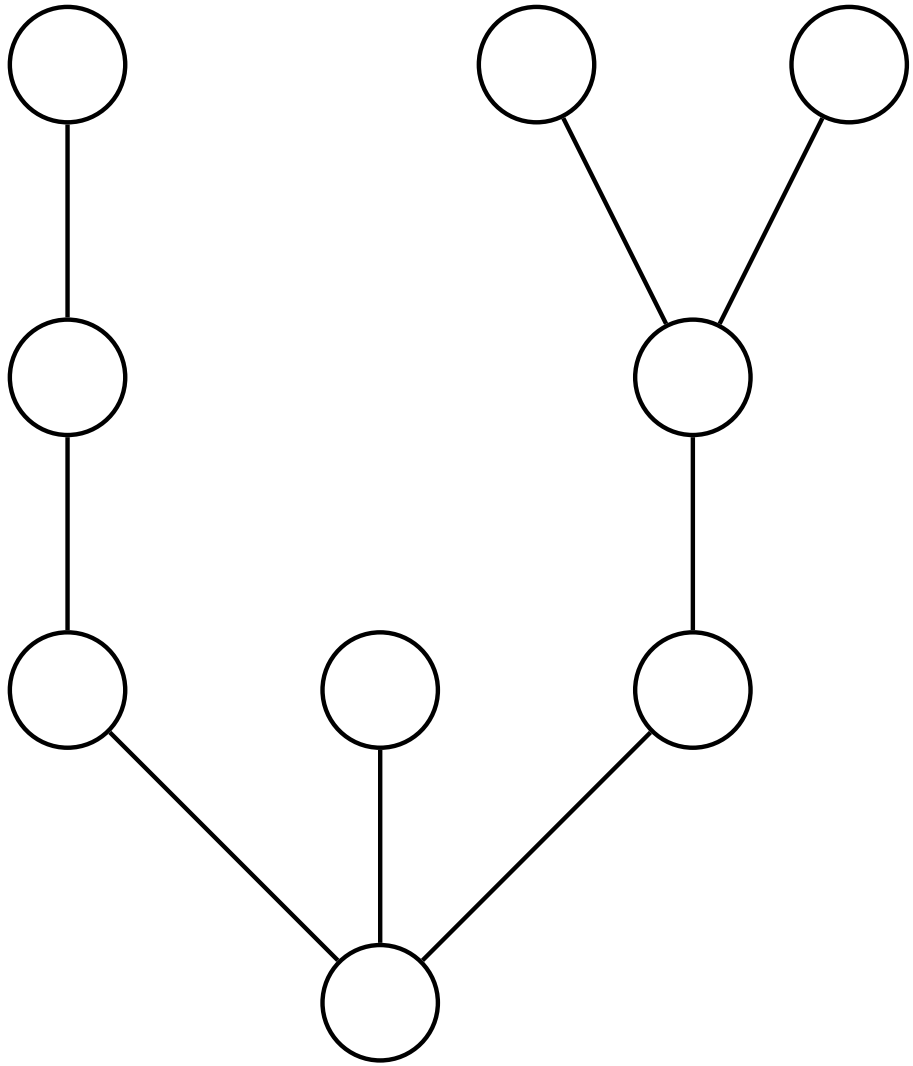

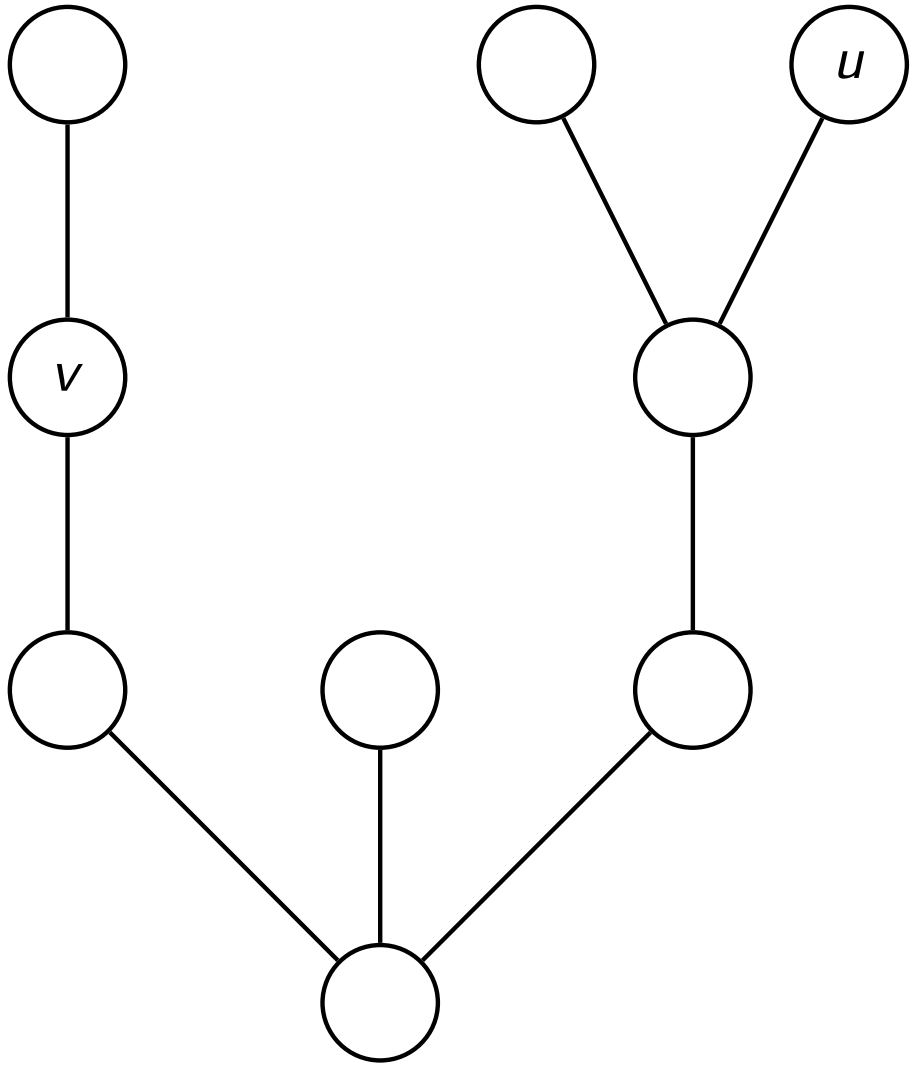
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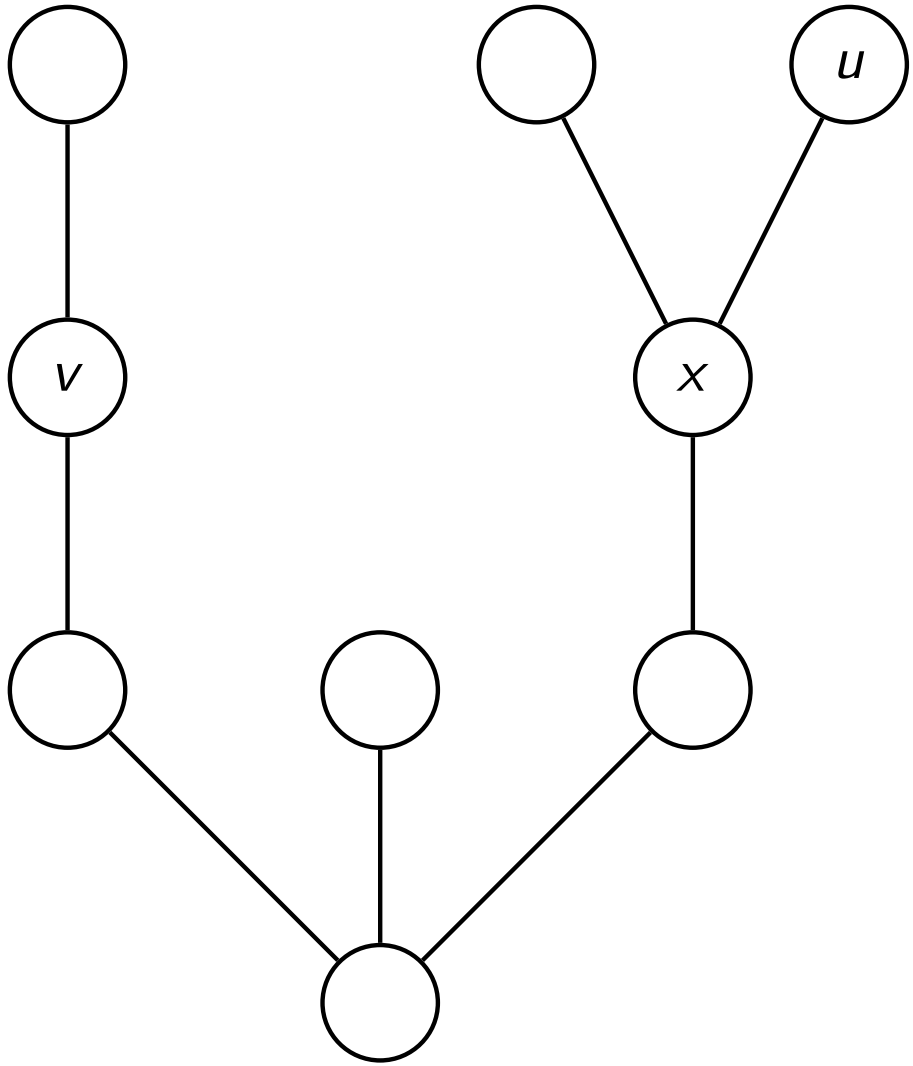
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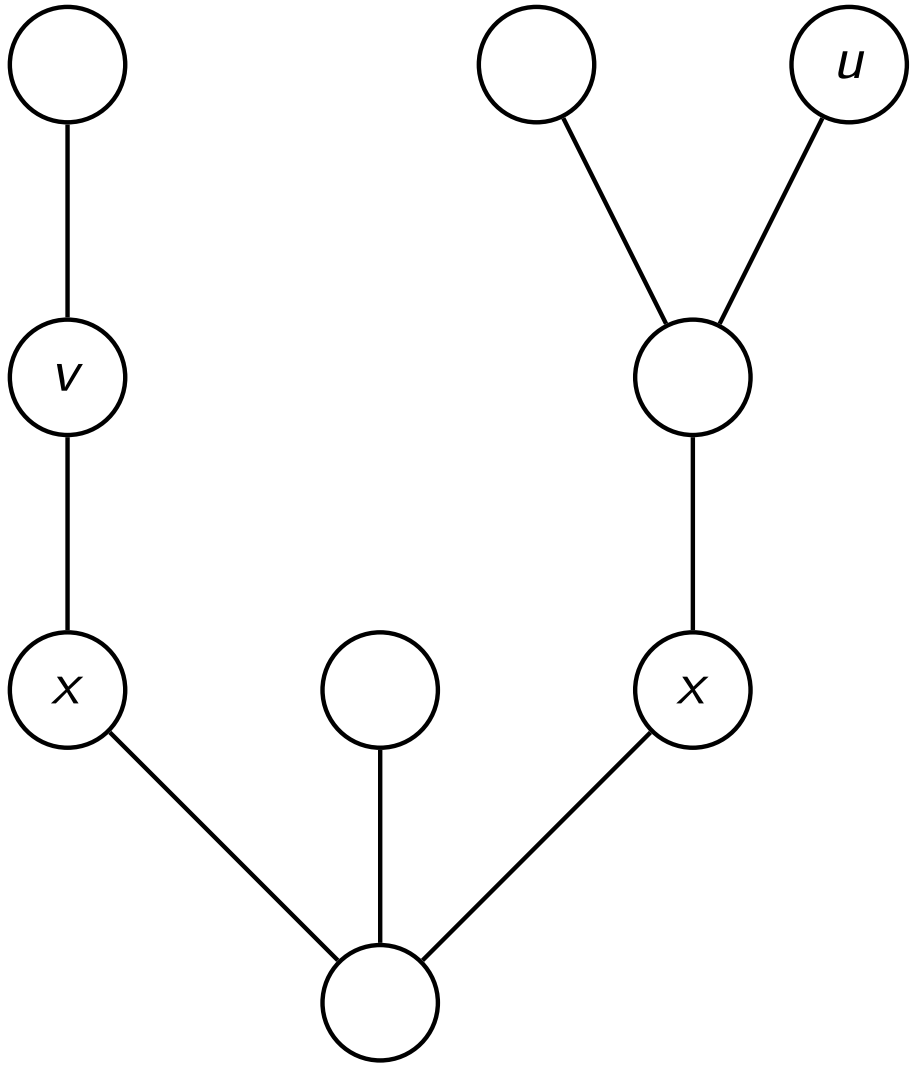
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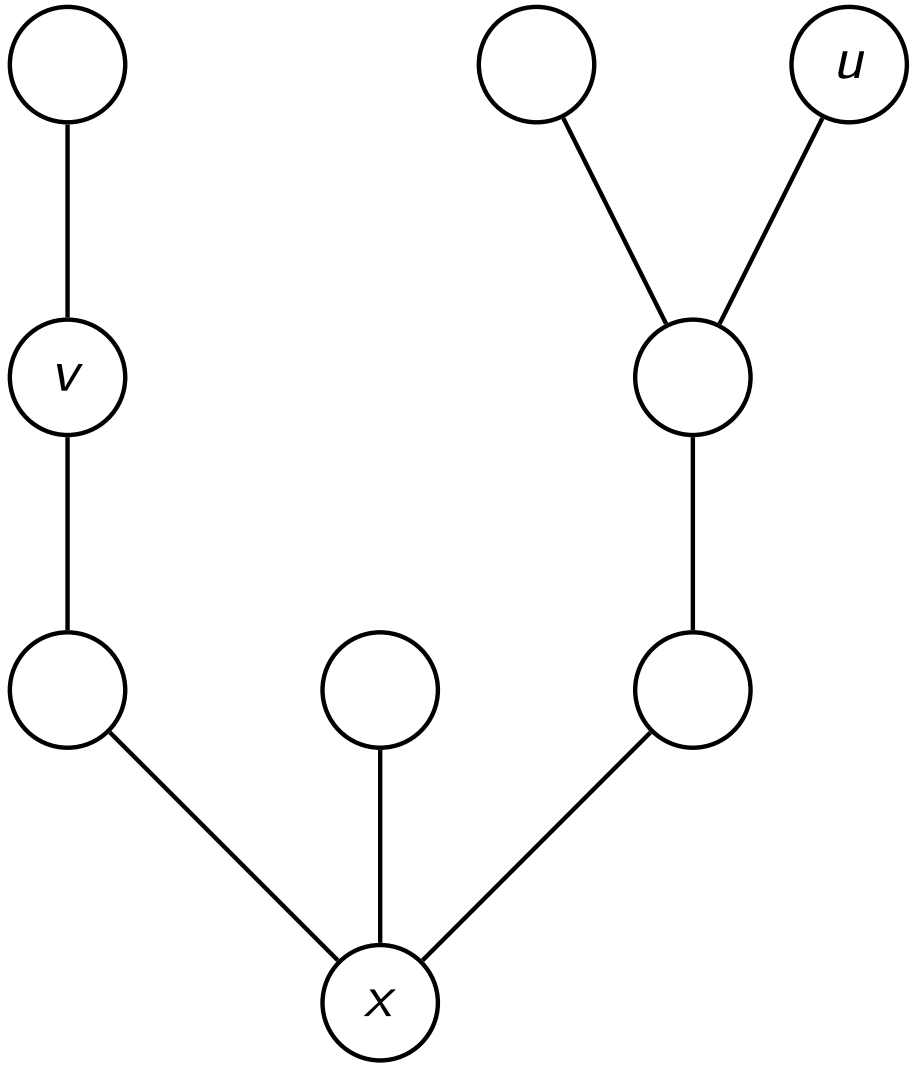
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```
 7 int p[MAXN], d[MAXN];
 8 void lca_dfs(vvi& g, int x, int q, int w)
 9 { // Hjálparfall.
10     int i;
11     d[x] = w, p[x] = q;
12     for (i = 0; i < g[x].size(); i++)
13         if (g[x][i] != q) lca_dfs(g, g[x][i], x, w + 1);
14 }
15
16 void lca_init(vvi& g, int x)
17 { // Upphafstillir fyrir netið g með rót x.
18     lca_dfs(g, 0, x, x);
19 }
20
21 int lca(int u, int v)
22 { // Skilar hæsta sameiginlega forfaðir u og v.
23     if (d[u] < d[v]) swap(u, v);
24     while (d[u] != d[v]) u = p[u];
25     while (u != v) u = p[u], v = p[v];
26     return u;
27 }
```

- Gerum ráð fyrir að hæð trésins sé H.
- Þá er tímaflækjan á þessari aðferð $\mathcal{O}(\, H\,)$.
- Í versta falli er hæð trés með n hnúta $n - 1$.
- Svo tímaflækjan er í versta falli $\mathcal{O}(n)$.
- Við getum þó bætt þetta með því að taka stærri stökk.

- Aðferðin skiptist í tvö skref:
  - Jöfnum hæðina á hnútunum.
  - Löbbum saman niður þangað til við finnum svarið.
- Fyrra skrefinu má lýsa nánar.
- Látum hnútana okkar vera u og v, þannig að $h(u) \geq h(v)$.
- Við viljum því ferðast niður nákvæmlega $h(v) - h(u)$ sinnum.
- Ein leið til að gera þetta hratt er að geyma ekki bara foreldri hvers hnúts, heldur alla hnúta sem eru 2^k fyrir neðan hnútinn í trénu.
- Við þurfum því að geyma $\mathcal{O}(\log n)$ stökk fyrir hvern hnút.
- Táknum með $p(u, k)$ þann hnút sem þú endar í ef þú ferðast 2^k sinnum niður tréð frá u gegnum foreldrin.
- Til dæmis er $p(u, 0)$ foreldri u.
- Til þæginda segjum við að foreldri rótarinnar sé rótin sjálf.
- Við finnum þessi gildi með rakningunni $p(u, i) = p(p(u, i-1), i-1)$.

- Við tökum því eins löng stökk og við getum án þess að $h(u) < h(v)$ þangað til $h(u) = h(v)$.
- Við getum því núna gert ráð fyrir að $h(u) = h(v)$.
- Þá viljum við taka eins löng stökk og við getum þannig að $u \neq v$.
- Að því loknu munu u og v hafa sama foreldri.

```
 9 int p[MAXN][MAXK], d[MAXN];
10 void lca_dfs(vvi& g, int x, int q, int w)
11 { // Hjálparfall.
12     int i;
13     d[x] = w;
14     for (i = 0; i < MAXK; i++) p[x][i] = i == 0 ? q : p[p[x][i - 1]][i - 1];
15     for (i = 0; i < g[x].size(); i++) if (g[x][i] != q)
16         lca_dfs(g, g[x][i], x, w + 1);
17 }
18
19 void lca_init(vvi& g, int x)
20 { // Upphafstillir fyrir netið g með rót x.
21     int i;
22     for (i = 0; i < MAXK; i++) p[x][i] = x;
23     lca_dfs(g, 0, x, x);
24 }
25
26 int lca(int u, int v)
27 { // Skilar hæsta sameiginlega forfaðir u og v.
28     int i;
29     if (d[u] < d[v]) swap(u, v);
30     for (i = MAXK - 1; i >= 0; i--) if (d[p[u][i]] >= d[v]) u = p[u][i];
31     for (i = MAXK - 1; i >= 0; i--) if (p[u][i] != p[v][i])
32         u = p[u][i], v = p[v][i];
33     return u == v ? u : p[u][0];
34 }
```

- Í hverju skrefi þurfum við bara að taka $\mathcal{O}(\log n)$.
- Svo tímaflækjan er $\mathcal{O}(\log n)$ fyrir hverja fyrirspurn.