

Lengstu vaxandi hlutrunur (LIS)

Bergur Snorrason

17. febrúar 2022

- ▶ Við höfum nú þegar skoðað hvernig finna megí lengstu vaxandi hlutrunur með tæmandi leit.
- ▶ Sjáum núna hvernig við getum gert það með kvikri bestun.

Hlutrunur

- ▶ Látum (a_1, \dots, a_n) vera runu af tölum.
- ▶ Runa (b_1, \dots, b_k) er *hlutruna* rununnar ef til eru heiltölur $j_1 < \dots < j_k$ þannig að $a_{j_m} = b_m$.
- ▶ Runan $(1, 2, 3, 4, 5)$ hefur, til dæmis, hlutrunurnar $(1, 2, 5)$, $(1, 3, 5)$ og (4) .
- ▶ En aftur á móti eru $(2, 1)$ og $(1, 6)$ ekki hlutrunur.
- ▶ Við viljum nú, fyrir gefna runu, finna lengdina á lengstu vaxandi hlutrunum rununnar.
- ▶ Það er oft talað um að finna „lengstu vaxandi hlutrununa“ þó það geti verið margar jafn langar „lengstu vaxandi hlutrunur“.

- ▶ Látum (a_1, \dots, a_n) vera n talna runu.
- ▶ Látum $f(x)$ tákna lengd lengstu vaxandi hlutrunu (a_1, \dots, a_x) , sem endar í a_x .
- ▶ Þá fáum við

$$f(x) = \begin{cases} 1, & \text{ef } x = 1, \\ \max_{\substack{1 \leq k < x \\ a_k \leq a_x}} f(k) + 1, & \text{annars.} \end{cases}$$

```

6 int v[MAXN], d[MAXN];
7 int dp_lookup(int x)
8 { // Finnur lengd lengstu vaxandi hlutruna a sem enda í staki x.
9     if (d[x] != -1) return d[x];
10    if (x == 0) return 1;
11    int i;
12    for (d[x] = 1, i = 0; i < x; i++) if (v[i] <= v[x])
13        d[x] = max(d[x], dp_lookup(i) + 1);
14    return d[x];
15 }
16
17 int lis(int *a, int n)
18 { // Finnur lengd lengstu vaxandi hlutruna a.
19     int i, r = 1;
20     for (i = 0; i < n; i++) d[i] = -1, v[i] = a[i];
21     for (i = 0; i < n; i++) r = max(r, dp_lookup(i));
22     return r;
23 }

```

- ▶ Við erum með einvítt stöðurúm af stærð n og hver uppfærsla er í $\mathcal{O}(n)$.
- ▶ Heildartímaflækjan er því $\mathcal{O}(n^2)$.
- ▶ Við getum bætt þetta, en fyrst þurfum við að leysa þetta aftur með kvikri bestun, en annarri rakningarformúlu.

- ▶ Látum nú $f(x, y)$ tákna minnsta mögulega aftasta stak vaxandi hlutrunu (a_1, \dots, a_x) af lengd y .
- ▶ Við fáum því

$$f(x, y) = \begin{cases} -\infty, & \text{ef } x = 0 \text{ og } y = 0, \\ \infty, & \text{annars, ef } x = 0, \\ f(x-1, y), & \text{ef } a_x < f(x-1, y-1), \\ \min(f(x-1, y), a_x), & \text{annars.} \end{cases}$$

- ▶ Grunntilvikin eru til að auðvelda útfærslu.
- ▶ Lengd lengstu vaxandi hlutrunanna er þá stærsta ℓ þannig að $f(n, \ell) < \infty$.

```

7 int v[MAXN], d[MAXN][MAXN];
8 int dp_lookup(int x, int y)
9 { // Finnur minnsta aftasta stak vaxandi hlutruna a[0:x] af lengd y.
10     if (x == -1) return y == 0 ? -INF : INF;
11     if (d[x][y] != -1) return d[x][y];
12     if (v[x] < dp_lookup(x - 1, y - 1)) return d[x][y] = dp_lookup(x - 1, y);
13     return d[x][y] = min(dp_lookup(x - 1, y), v[x]);
14 }
15
16 int lis(int *a, int n)
17 { // Finnur lengd lengstu vaxandi hlutruna a.
18     int i, j, x;
19     for (i = 0; i < n; i++) v[i] = a[i];
20     for (i = 0; i < n; i++) for (j = 0; j < n + 1; j++) d[i][j] = -1;
21     for (x = n; dp_lookup(n - 1, x) == INF; x--);
22     return x;
23 }

```


- ▶ Við erum með tvívítt stöðurúm af stærð $(n + 1) \cdot (n + 1)$ og hver uppfærsla er í $\mathcal{O}(1)$.
- ▶ Heildartímaflækjan er því $\mathcal{O}(n^2)$.
- ▶ En þetta er sama tímaflækja og við fengum áðan.
- ▶ Prófum að útfæra þetta neðansækið og sjáum hvað við fáum.

- Rifjum upp að

$$f(x, y) = \begin{cases} -\infty, & \text{ef } x = 0 \text{ og } y = 0, \\ \infty, & \text{annars, ef } x = 0, \\ f(x-1, y), & \text{ef } a_x < f(x-1, y-1), \\ \min(f(x-1, y), a_x), & \text{annars.} \end{cases}$$

- Sjáum að fallgildi $f(x, y)$ ákvarðast af $f(x-1, y)$ og $f(x-1, y-1)$.
- Svo við getum fyllt í töfluna í lestrarröð.

```

7 int lis(int *a, int *b, int n)
8 { // Finnur eina af lengstu vaxandi hlutrunum a.
9     int i, j, x, y;
10    int d[n + 1][n + 1];
11    for (i = 0; i < n + 1; i++) d[0][i] = INF;
12    for (i = 0; i < n + 1; i++) d[i][0] = -INF;
13    for (i = 1; i < n + 1; i++) for (j = 0; j < n + 1; j++)
14    {
15        if (a[i - 1] < d[i - 1][j - 1]) d[i][j] = d[i - 1][j];
16        else d[i][j] = min(d[i - 1][j], a[i - 1]);
17    }
18    for (x = n; d[n][x] == INF; x--);
19    for (i = 0; i < x; i++) b[i] = d[n - x + i + 1][i + 1];
20    return x;
21 }

```

- ▶ Heildartímaflækjan er sú sama og áðan, svo hún er $\mathcal{O}(n^2)$.
- ▶ Skoðum núna hvernig uppfærslan á töflunni fer fram og sjáum hvort við getum bætt tímaflækjuna.

3, 6, 2, 1, 5, 8, 4, 7, 9

-INF INF INF INF INF INF INF INF INF INF

-INF

-INF

-INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF INF INF INF INF INF INF INF INF INF

> -INF

-INF

-INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF INF INF INF INF INF INF INF INF INF

> -INF 3

-INF

-INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]
$$> -\text{INF} \quad 3$$

-INF

-INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

```

-1NF  1NF  1NF  1NF  1NF  1NF  1NF  1NF  1NF
> -1NF    3  1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF

```

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

```

-1NF  1NF  1NF  1NF  1NF  1NF  1NF  1NF  1NF
> -1NF    3  1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF

```


3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

```

-1NF    1NF    1NF    1NF    1NF    1NF    1NF    1NF    1NF
> -1NF      3    1NF    1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF

```

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

```

-1NF      1NF      1NF      1NF      1NF      1NF      1NF      1NF      1NF
> -1NF      3      1NF      1NF      1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF

```

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

```

-1NF      1NF      1NF      1NF      1NF      1NF      1NF      1NF      1NF
> -1NF      3      1NF      1NF      1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF
-1NF

```

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
------	-----	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

> -INF

-INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF INF INF INF INF INF INF INF INF INF

-INF 3 INF INF INF INF INF INF INF INF

> -INF 3

-INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF INF INF INF INF INF INF INF INF INF

-INF 3 INF INF INF INF INF INF INF INF

> -INF 3

-INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF INF INF INF INF INF INF INF INF INF

-INF 3 INF INF INF INF INF INF INF INF

> -INF 3 6

-INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF INF INF INF INF INF INF INF INF INF

-INF 3 INF INF INF INF INF INF INF INF

> -INF 3 6

-INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF INF INF INF INF INF INF INF INF INF

-INF 3 INF INF INF INF INF INF INF INF

> -INF 3 6 INF

-INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF INF INF INF INF INF INF INF INF INF

-INF 3 INF INF INF INF INF INF INF INF

> -INF 3 6 INF

-INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF INF INF INF INF INF INF INF INF INF

-INF 3 INF INF INF INF INF INF INF INF

> -INF 3 6 INF

-INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF INF INF INF INF INF INF INF INF INF

-INF 3 INF INF INF INF INF INF INF INF

> -INF 3 6 INF INF

-INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

```

-INF      3  INF  INF  INF  INF  INF  INF  INF  INF

```

```
> -INF      3      6    INF    INF
```

–INF

-INF

–INF

–INF

–INF

–INF

–INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

-INF INF INF INF INF INF INF INF INF INF

```

-INF      3  INF  INF  INF  INF  INF  INF  INF  INF

```

```
> -INF      3      6    INF    INF    INF
```

–INF

–INF

–INF

–INF

–INF

–INF

–INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

```

-INF      3  INF  INF  INF  INF  INF  INF  INF  INF

```

```
> -INF      3      6    INF    INF    INF
```

–INF

–INF

–INF

–INF

–INF

–INF

–INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

```

-INF      3  INF  INF  INF  INF  INF  INF  INF  INF

```

```
> -INF      3      6  INF  INF  INF  INF
```

–INF

-INF

–INF

–INF

–INF

–INF

–INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

```

-INF      3  INF  INF  INF  INF  INF  INF  INF  INF

```

```
> -INF      3      6  INF  INF  INF  INF
```

-INF

-INF

-INF

-INF

-INF

–INF

–INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

```

-INF      3  INF  INF  INF  INF  INF  INF  INF  INF

```

```
> -INF      3      6  INF  INF  INF  INF  INF
```

-INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

```

-INF      3  INF  INF  INF  INF  INF  INF  INF  INF

```

```
> -INF      3      6  INF  INF  INF  INF  INF
```

-INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF INF INF INF INF INF INF INF INF INF

-INF 3 INF INF INF INF INF INF INF INF

> -INF 3 6 INF INF INF INF INF INF

-INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

[illegible]

```

-INF      3  INF  INF  INF  INF  INF  INF  INF  INF

```

```
> -INF      3      6  INF  INF  INF  INF  INF  INF  INF
```

–INF

–INF

–INF

–INF

–INF

–INF

–INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
------	-----	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

> -INF

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
------	-----	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

> -INF 2

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
------	-----	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

> -INF 2

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF INF INF INF INF INF INF INF INF INF

-INF 3 INF INF INF INF INF INF INF INF

-INF 3 6 INF INF INF INF INF INF INF

> -INF 2 6

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

$$> -\infty \quad 2 \quad 6$$

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

-INF INF INF INF INF INF INF INF INF INF

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

$$> -\infty \quad 2 \quad 6 \quad \infty$$

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

-INF INF INF INF INF INF INF INF INF INF

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

$$> -\infty \quad 2 \quad 6 \quad \infty$$

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      2      6    INF    INF
```

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      2      6    INF    INF    INF
```

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      2      6  INF  INF  INF  INF
```

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      2      6  INF  INF  INF  INF  INF
```

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      2      6  INF  INF  INF  INF  INF
```

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      2      6  INF  INF  INF  INF  INF  INF
```

-INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
------	-----	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

> -INF

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
------	-----	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

> -INF 1

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

$$> -\text{INF} \quad 1$$

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

$$> -\infty \quad 1 \quad 6$$

-INF

- INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

$$> -\infty \quad 1 \quad 6$$

-INF

- INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

$$> -\text{INF} \quad 1 \quad 6 \quad \text{INF}$$

-INF

-INF

-INF

-INF

- INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      1      6  INF  INF
```

-INF

- INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

-INF INF INF INF INF INF INF INF INF INF

[illegible]

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      1      6  INF  INF  INF
```

-INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
>	-INF	1	6	INF	INF	INF	INF		
	-INF								
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
>	-INF	1	6	INF	INF	INF	INF		
	-INF								
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
>	-INF	1	6	INF	INF	INF	INF		
	-INF								
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
>	-INF	1	6	INF	INF	INF	INF		
	-INF								
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
>	-INF	1	6	INF	INF	INF	INF	INF	
	-INF								
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
>	-INF	1	6	INF	INF	INF	INF	INF	
	-INF								
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
>	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF								
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
------	-----	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

> -INF

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
------	-----	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

> -INF 1

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
------	-----	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

> -INF 1

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

-INF INF INF INF INF INF INF INF INF INF

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      1      5
```

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

-INF INF INF INF INF INF INF INF INF INF

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

$$> -\infty \quad 1 \quad 5$$

-INF

-INF

-INF

-INF

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
>	-INF	1	5	INF					
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

>	-INF	1	5	INF
---	------	---	---	-----

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
>	-INF	1	5	INF	INF				
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
>	-INF	1	5	INF	INF				
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
>	-INF	1	5	INF	INF	INF			
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

-INF INF INF INF INF INF INF INF INF INF

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      1      5  INF  INF  INF
```

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      1      5  INF  INF  INF  INF
```

-INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
>	-INF	1	5	INF	INF	INF	INF		
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
>	-INF	1	5	INF	INF	INF	INF		
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
>	-INF	1	5	INF	INF	INF	INF		
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
>	-INF	1	5	INF	INF	INF	INF	INF	
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
>	-INF	1	5	INF	INF	INF	INF	INF	
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
>	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF								
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF

> -INF

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF INF INF INF INF INF INF INF INF INF

-INF 3 INF INF INF INF INF INF INF INF

-INF 3 6 INF INF INF INF INF INF INF

-INF 2 6 INF INF INF INF INF INF INF

-INF 1 6 INF INF INF INF INF INF INF

-INF 1 5 INF INF INF INF INF INF INF

> -INF 1

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
------	-----	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	5	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

> -INF 1

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
------	-----	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	5	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

> -INF 1 5

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	5	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      1      5
```

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

-INF INF INF INF INF INF INF INF INF INF

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	5	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      1      5      8      INF
```

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	5	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      1      5      8      INF
```

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	5	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      1      5      8  INF  INF
```

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

-INF INF INF INF INF INF INF INF INF INF

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	5	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      1      5      8  INF  INF  INF
```

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	5	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      1      5      8  INF  INF  INF  INF
```

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
>	-INF	1	5	8	INF	INF	INF		
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
>	-INF	1	5	8	INF	INF	INF	INF	
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	5	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

```
> -INF      1      5      8  INF  INF  INF  INF  INF
```

-INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
>	-INF	1	5	8	INF	INF	INF	INF	INF
	-INF								
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF

> -INF

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF INF INF INF INF INF INF INF INF INF

-INF 3 INF INF INF INF INF INF INF INF

-INF 3 6 INF INF INF INF INF INF INF

-INF 2 6 INF INF INF INF INF INF INF

-INF 1 6 INF INF INF INF INF INF INF

-INF 1 5 INF INF INF INF INF INF INF

-INF 1 5 8 INF INF INF INF INF INF

> -INF 1

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

[illegible]

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	5	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	5	8	INF	INF	INF	INF	INF	INF
------	---	---	---	-----	-----	-----	-----	-----	-----

$$> -\text{INF} \quad 1$$

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
------	-----	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
------	---	-----	-----	-----	-----	-----	-----	-----	-----

-INF	3	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	2	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	6	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	5	INF	INF	INF	INF	INF	INF	INF
------	---	---	-----	-----	-----	-----	-----	-----	-----

-INF	1	5	8	INF	INF	INF	INF	INF	INF
------	---	---	---	-----	-----	-----	-----	-----	-----

> -INF 1 4

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF INF INF INF INF INF INF INF INF INF

-INF 3 INF INF INF INF INF INF INF INF

-INF 3 6 INF INF INF INF INF INF INF

-INF 2 6 INF INF INF INF INF INF INF

-INF 1 6 INF INF INF INF INF INF INF

-INF 1 5 INF INF INF INF INF INF INF

-INF 1 5 8 INF INF INF INF INF INF

> -INF 1 4 8

-INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
>	-INF	1	4	8	INF				
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
>	-INF	1	4	8	INF				
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
>	-INF	1	4	8	INF	INF			
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
>	-INF	1	4	8	INF	INF			
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
>	-INF	1	4	8	INF	INF	INF		
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
>	-INF	1	4	8	INF	INF	INF		
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
>	-INF	1	4	8	INF	INF	INF	INF	
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
>	-INF	1	4	8	INF	INF	INF	INF	
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
>	-INF	1	4	8	INF	INF	INF	INF	
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
>	-INF	1	4	8	INF	INF	INF	INF	INF
	-INF								
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF

> -INF

-INF

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
	-INF	1	4	8	INF	INF	INF	INF	INF
>	-INF	1							
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
	-INF	1	4	8	INF	INF	INF	INF	INF
>	-INF	1	4	7					
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
	-INF	1	4	8	INF	INF	INF	INF	INF
>	-INF	1	4	7	INF				
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
	-INF	1	4	8	INF	INF	INF	INF	INF
>	-INF	1	4	7	INF				
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
	-INF	1	4	8	INF	INF	INF	INF	INF
>	-INF	1	4	7	INF	INF			
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
	-INF	1	4	8	INF	INF	INF	INF	INF
>	-INF	1	4	7	INF	INF			
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
	-INF	1	4	8	INF	INF	INF	INF	INF
>	-INF	1	4	7	INF	INF	INF		
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
	-INF	1	4	8	INF	INF	INF	INF	INF
>	-INF	1	4	7	INF	INF	INF		
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
	-INF	1	4	8	INF	INF	INF	INF	INF
>	-INF	1	4	7	INF	INF	INF		
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
	-INF	1	4	8	INF	INF	INF	INF	INF
>	-INF	1	4	7	INF	INF	INF	INF	
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
	-INF	1	4	8	INF	INF	INF	INF	INF
>	-INF	1	4	7	INF	INF	INF	INF	
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

1

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
	-INF	1	4	8	INF	INF	INF	INF	INF
>	-INF	1	4	7	INF	INF	INF	INF	INF
	-INF								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF

> -INF

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF

$$> -\text{INF} \quad 1$$

3, 6, 2, 1, 5, 8, 4, 7, 9

—

1

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF
> -INF	1								

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF
> -INF	1	4							

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF
> -INF	1	4							

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF
> -INF	1	4	7						

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF
> -INF	1	4	7						

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF
> -INF	1	4	7	9					

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF
> -INF	1	4	7	9					

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF
> -INF	1	4	7	9	INF				

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF
> -INF	1	4	7	9	INF				

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF
> -INF	1	4	7	9	INF	INF			

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF
> -INF	1	4	7	9	INF	INF			

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF
> -INF	1	4	7	9	INF	INF	INF		

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF
> -INF	1	4	7	9	INF	INF	INF		

3, 6, 2, 1, 5, 8, 4, 7, 9

^

|

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF
> -INF	1	4	7	9	INF	INF	INF	INF	

3, 6, 2, 1, 5, 8, 4, 7, 9

^

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
	-INF	1	4	8	INF	INF	INF	INF	INF
	-INF	1	4	7	INF	INF	INF	INF	INF
>	-INF	1	4	7	9	INF	INF	INF	

3, 6, 2, 1, 5, 8, 4, 7, 9

^

	-INF	INF	INF	INF	INF	INF	INF	INF	INF
	-INF	3	INF	INF	INF	INF	INF	INF	INF
	-INF	3	6	INF	INF	INF	INF	INF	INF
	-INF	2	6	INF	INF	INF	INF	INF	INF
	-INF	1	6	INF	INF	INF	INF	INF	INF
	-INF	1	5	INF	INF	INF	INF	INF	INF
	-INF	1	5	8	INF	INF	INF	INF	INF
	-INF	1	4	8	INF	INF	INF	INF	INF
	-INF	1	4	7	INF	INF	INF	INF	INF
>	-INF	1	4	7	9	INF	INF	INF	INF

3, 6, 2, 1, 5, 8, 4, 7, 9

-INF	INF	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	INF	INF	INF	INF	INF	INF	INF	INF
-INF	3	6	INF	INF	INF	INF	INF	INF	INF
-INF	2	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	6	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	INF	INF	INF	INF	INF	INF	INF
-INF	1	5	8	INF	INF	INF	INF	INF	INF
-INF	1	4	8	INF	INF	INF	INF	INF	INF
-INF	1	4	7	INF	INF	INF	INF	INF	INF
-INF	1	4	7	9	INF	INF	INF	INF	INF

- ▶ Tökum eftir að við þurfum í raun alltaf bara að geyma síðustu línuna sem við kláruðum.
- ▶ Annað sem við sjáum er að línurnar eru allar vaxandi.
- ▶ Að lokum sjáum við að hver lína er aðeins uppfærð einu sinni í hverri ítrun.
- ▶ En afhverju gildir þetta?

Afhverju er allar línurnar raðaðar

- Rifjum upp að

$$f(x, y) = \begin{cases} -\infty, & \text{ef } x = 0 \text{ og } y = 0, \\ \infty, & \text{annars, ef } x = 0, \\ f(x-1, y), & \text{ef } a_x < f(x-1, y-1), \\ \min(f(x-1, y), a_x), & \text{annars.} \end{cases}$$

- Þetta fæst með einfaldri þrepun.
- Fyrst línan er alltaf $-\infty, \infty, \dots, \infty$, sem er röðuð.
- Gerum nú ráð fyrir að við séum í línu x og við þurfum að uppfæra y .
- Þá þarf $a_x \geq f(x-1, y-1)$ því uppfærsla á sér aðeins stað í síðasta tilfellinu.
- En einnig þarf $a_x < f(x-1, y)$.
- Svo $f(x-1, y-1) \leq f(x, y) < f(x-1, y)$.
- En þá fæst líka að $f(x, y) < f(x-1, y) \leq f(x, y+1)$.

Afhverju uppfærum við bara einu sinni í hverri ítrun?

- Rifjum upp að

$$f(x, y) = \begin{cases} -\infty, & \text{ef } x = 0 \text{ og } y = 0, \\ \infty, & \text{annars, ef } x = 0, \\ f(x-1, y), & \text{ef } a_x < f(x-1, y-1), \\ \min(f(x-1, y), a_x), & \text{annars.} \end{cases}$$

- Gerum ráð fyrir að við séum í línu x og við þurfum að uppfæra y .
- Eins og við sáum áðan þá er $f(x-1, y-1) \leq a_x < f(x-1, y)$ nauðsynlegt skilyrði fyrir uppfærslu.
- Þetta getur þó bara átt sér stað, í mesta lagi, einu sinni í röðuðum lista.

Afhverju uppfærum við bara einu sinni í hverri ítrun?

- ▶ Við getum því lýst uppfærslu y -tu línunnar með:
 - ▶ Finnum x þannig að $f(x-1, y-1) \leq a_x < f(x-1, y)$.
 - ▶ Látum $f(x, y) = a_x$.
 - ▶ Uppfærum næstu línu.
- ▶ Í neðansæknu útfærslunni framkvæmum við línulega leit í fyrsta skrefinu.
- ▶ En þar sem línurnar eru allar raðaðar getum við í staðinn framkvæmt helmingunarleit.

```

10 int lis(int *a, int *b, int n)
11 { // Finnur eina af lengstu vaxandi hlutrunum a.
12     int i, j, x, y;
13     int d[n + 1], e[n];
14     for (i = 0; i < n + 1; i++) d[i] = i == 0 ? -INF : INF;
15     for (i = 0; i < n; i++)
16     {
17         int r = 0, s = n + 1;
18         while (r < s)
19         {
20             int m = (r + s)/2;
21             if (d[m] <= a[i]) r = m + 1;
22             else s = m;
23         }
24         d[r] = a[i], e[i] = d[r - 1];
25     }
26     for (x = n; d[x] == INF; x--);
27     for (i = n - 1, j = x - 1, y = d[x]; j >= 0; i--) if (a[i] == y)
28         y = e[i], b[j--] = a[i];
29     return x;
30 }

```

- ▶ Nú er forritið einföld for-lykkja af lengd n sem framkvæmir helmingunarleit í hverri ítrun.
- ▶ Helmingunarleitin er $\mathcal{O}(\log n)$.
- ▶ Heildartímaflækan er því $\mathcal{O}(n \log n)$.

