Lengstu vaxandi hlutrunur (LIS)

Bergur Snorrason

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- Við höfum nú þegar skoðað hvernig finna megi lengstu vaxandi hlutrunur með tæmandi leit.
- Sjáum núna hvernig við getum gert það með kvikri bestun.

Hlutrunur

- Látum $(a_1, \ldots, a_n)$ vera runu af tölum.
- Runa $(b_1, \ldots, b_k)$ er *hlutruna* rununnar ef til eru heiltölur $j_1 < \cdots < j_k$ þannig að $a_{j_m} = b_m$.
- Runan $(1, 2, 3, 4, 5)$ hefur, til dæmis, hlutrunurnar $(1, 2, 5)$, $(1, 3, 5)$ og (4).
- En aftur á móti eru $(2, 1)$ og $(1, 6)$ ekki hlutrunur.
- Við viljum nú, fyrir gefna runu, finna lengdina á lengstu vaxandi hlutrunum rununnar.
- Það er oft talað um að finna „lengstu vaxandi hlutrununa" þó það geti verið margar jafn langar „lengstu vaxandi hlutrunur".

- Látum $(a_1, \ldots, a_n)$ vera n talna runu.
- Látum $f(x)$ tákna lengd lengstu vaxandi hlutrunu $(a_1, \ldots, a_x)$, sem endar í a_x.
- Þá fáum við

$$
f(x) = \begin{cases} 1, & \text{ef } x = 1, \\ \max_{\substack{1 \le k < x \\ a_k \le a_x}} f(k) + 1, & \text{annars.} \end{cases}
$$

```c
 6  int v[MAXN], d[MAXN];
 7  int dp_lookup(int x)
 8  {
 9      if (d[x] != -1) return d[x];
10      if (x == 0) return 1;
11      int i;
12      for (d[x] = 1, i = 0; i < x; i++) if (v[i] <= v[x])
13          d[x] = max(d[x], dp_lookup(i) + 1);
14      return d[x];
15  }
16
17  int lis(int *a, int n)
18  {
19      int i, r = 1;
20      for (i = 0; i < n; i++) d[i] = -1, v[i] = a[i];
21      for (i = 0; i < n; i++) r = max(r, dp_lookup(i));
22      return r;
23  }
```

- Við erum með einvítt stöðurúm af stærð n og hver uppfærsla er í $\mathcal{O}(n)$.
- Heildartímaflækjan er því $\mathcal{O}(n^2)$.
- Við getum bætt þetta, en fyrst þurfum við að leysa þetta aftur með kvikri bestun, en annarri rakningarformúlu.

- Látum nú $f(x, y)$ tákna minnsta mögulega aftasta stak vaxandi hlutrunu $(a_1, \ldots, a_x)$ af lengd y.
- Við fáum því

$$f(x, y) = \begin{cases} -\infty, & \text{ef } x = 0 \text{ og } y = 0, \\ \infty, & \text{annars, ef } x = 0, \\ f(x - 1, y), & \text{ef } a_x < f(x - 1, y - 1), \\ \min(f(x - 1, y), a_x), & \text{annars.} \end{cases}$$

- Grunntilvikin eru til að auðvelda útfærlsu.
- Lengd lengstu vaxandi hlutrunanna er þá stærsta ℓ þannig að $f(n, \ell) < \infty$.

```
 7  int v[MAXN], d[MAXN][MAXN];
 8  int dp_lookup(int x, int y)
 9  {
10      if (x == -1) return y == 0 ? -INF : INF;
11      if (d[x][y] != -1) return d[x][y];
12      if (v[x] < dp_lookup(x - 1, y - 1)) return d[x][y] = dp_lookup(x - 1, y);
13      return d[x][y] = min(dp_lookup(x - 1, y), v[x]);
14  }
15
16  int lis(int *a, int n)
17  {
18      int i, j, x;
19      for (i = 0; i < n; i++) v[i] = a[i];
20      for (i = 0; i < n; i++) for (j = 0; j < n + 1; j++) d[i][j] = -1;
21      for (x = n; dp_lookup(n - 1, x) == INF; x--);
22      return x;
23  }
```

- Við erum með tvívítt stöðurúm af stærð $(n+1) \cdot (n+1)$ og hver uppfærsla er í $\mathcal{O}(1)$.
- Heildartímaflækjan er því $\mathcal{O}(n^2)$.
- En þetta er sama tímaflækja og við fengum áðan.
- Prófum að útfæra þetta neðansækið og sjáum hvað við fáum.

▶ Rifjum upp að

$$
f(x, y) = \begin{cases}
-\infty, & \text{ef } x = 0 \text{ og } y = 0, \\
\infty, & \text{annars, ef } x = 0, \\
f(x - 1, y), & \text{ef } a_x < f(x - 1, y - 1), \\
\min(f(x - 1, y), a_x), & \text{annars.}
\end{cases}
$$

▶ Sjáum að fallgildi $f(x, y)$ ákvarðast af $f(x - 1, y)$ og $f(x - 1, y - 1)$.

▶ Svo við getum fyllt í töfluna í lestrarröð.

```
 7  int lis(int *a, int *b, int n)
 8  {
 9      int i, j, x, y;
10      int d[n + 1][n + 1];
11      for (i = 0; i < n + 1; i++) d[0][i] = INF;
12      for (i = 0; i < n + 1; i++) d[i][0] = −INF;
13      for (i = 1; i < n + 1; i++) for (j = 0; j < n + 1; j++)
14      {
15          if (a[i − 1] < d[i − 1][j − 1]) d[i][j] = d[i − 1][j];
16          else d[i][j] = min(d[i − 1][j], a[i − 1]);
17      }
18      for (x = n; d[n][x] == INF; x−−);
19      for (i = 0; i < x; i++) b[i] = d[n − x + i + 1][i + 1];
20      return x;
21  }
```

- Heildartímaflækjan er sú sama og áðan, svo hún er $\mathcal{O}(n^2)$.
- Skoðum núna hvernig uppfærslan á töflunni fer fram og sjáum hvort við getum bætt tímaflækjuna.

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  -INF   2   6 INF INF INF INF INF INF INF
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3, 6, 2, 1, 5, 8, 4, 7, 9
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3, 6, 2, 1, 5, 8, 4, 7, 9

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  -INF    1    4    7  INF  INF  INF  INF  INF  INF
  -INF    1    4    7    9  INF  INF  INF  INF  INF
```

- ▶ Tökum eftir að við þurfum í raun alltaf bara að geyma síðustu línuna sem við kláruðum.
- ▶ Annað sem við sjáum er að línurnar eru allar vaxandi.
- ▶ Að lokum sjáum við að hver lína er aðeins uppfærð einu sinni í hverri ítrun.
- ▶ En afhverju gildir þetta?

Afhverju er allar línurnar raðaðar

- ▶ Rifjum upp að

$$f(x, y) = \begin{cases} -\infty, & \text{ef } x = 0 \text{ og } y = 0, \\ \infty, & \text{annars, ef } x = 0, \\ f(x-1, y), & \text{ef } a_x < f(x-1, y-1), \\ \min(f(x-1, y), a_x), & \text{annars.} \end{cases}$$

- ▶ Þetta fæst með einfaldri þrepun.
- ▶ Fyrst línan er alltaf $-\infty, \infty, \ldots, \infty$, sem er röðuð.
- ▶ Gerum nú ráð fyrir að við séum í línu x og við þurfum að uppfæra y.
- ▶ Þá þarf $a_x \geq f(x-1, y-1)$ því uppfærlsa á sér aðeins stað í síðasta tilfellinu.
- ▶ En einnig þarf $a_x < f(x-1, y)$.
- ▶ Svo $f(x-1, y-1) \leq f(x, y) < f(x-1, y)$.
- ▶ En þá fæst líka að $f(x, y) < f(x-1, y) \leq f(x, y+1)$.

Afhverju uppfærum við bara einu sinni í hverri ítrun?

- Rifjum upp að

$$f(x, y) = \begin{cases} -\infty, & \text{ef } x = 0 \text{ og } y = 0, \\ \infty, & \text{annars, ef } x = 0, \\ f(x-1, y), & \text{ef } a_x < f(x-1, y-1), \\ \min(f(x-1, y), a_x), & \text{annars.} \end{cases}$$

- Gerum ráð fyrir að við séum í línu x og við þurfum að uppfæra y.
- Eins og við sáum áðan þá er $f(x-1, y-1) \leq a_x < f(x-1, y)$ nauðsynlegt skilyrði fyrir uppfærslu.
- Þetta getur þó bara átt sér stað, í mesta lagi, einu sinni í röðuðum lista.

Afhverju uppfærum við bara einu sinni í hverri ítrun?

- ▶ Við getum því lýst uppfærslu y-tu línunnar með:
    - ▶ Finnum x þannig að $f(x - 1, y - 1) \leq a_x < f(x - 1, y)$.
    - ▶ Látum $f(x, y) = a_x$.
    - ▶ Uppfærum næstu línu.
- ▶ Í neðansæknu útfærslunni framkvæmum við línulega leit í fyrsta skrefinu.
- ▶ En þar sem línurnar eru allar raðaðar getum við í staðinn framkvæmt helmingunarleit.

```
10  int lis(int *a, int *b, int n)
11  {
12      int i, j, x, y;
13      int d[n + 1], e[n];
14      for (i = 0; i < n + 1; i++) d[i] = i == 0 ? -INF : INF;
15      for (i = 0; i < n; i++)
16      {
17          int r = 0, s = n + 1;
18          while (r < s)
19          {
20              int m = (r + s)/2;
21              if (d[m] <= a[i]) r = m + 1;
22              else s = m;
23          }
24          d[r] = a[i], e[i] = d[r - 1];
25      }
26      for (x = n; d[x] == INF; x--);
27      for (i = n - 1, j = x - 1, y = d[x]; j >= 0; i--) if (a[i] == y)
28          y = e[i], b[j--] = a[i];
29      return x;
30  }
```

- Nú er forritið einföld `for`-lykkja af lengd n sem framkvæmir helmingunarleit í hverri ítrun.
- Helmingunarleitin er $\mathcal{O}(\log n)$.
- Heildartímaflækan er því $\mathcal{O}(n \log n)$.