Exercise 1: Implement PCA

(implementation exercise)

Exercise 2: Implement kernel PCA

(implementation exercise)

Exercise 3: Exactly recovery from a PCA is impossible

(a) We have already a vector $x \in \mathbb{R}^n$ and would like to fine a compression matrix W s.t. $Wx \in \mathbb{R}^d, \quad d < n$.

$$C = X^T X$$
$$C = VDV'$$

$V \in \mathbb{R}^{n \times n}$ is the eigenvetor of x and $D \in \mathbb{R}^{n \times n}$ is the eigenvalue of x.

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ \cdots & & & \cdots \\ & & & \end{bmatrix}, \qquad D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

As we learned by lecture slide, define V_d as a matrix containing the l largest eigenvectors.(i.e., the first d columns of V if the eigs in D are ordered decreasingly).
So

$$V_d = \begin{bmatrix} v_1 & v_2 & \cdots & v_d \\ \cdots & & & \cdots \\ & & & \end{bmatrix} \quad \in \mathbb{R}^{n \times d}$$
$$V_d^T x \to x \in \mathbb{R}^{d \times 1}$$
$$V_d^T = W$$

Then if we say

$$Wx = \begin{bmatrix} w_1 \\ w_2 \\ \cdots \\ w_d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \quad s.t. \quad w_{j1} = wj2 = \alpha$$
$$= \begin{bmatrix} < W_1, x >, < w_2, x >, \cdots, < w_d, x > \\ \cdots \end{bmatrix} = Wx$$

And if we also say

$$u^T = [1, 2, a_3, \cdots, a_n]$$
$$v^T = [1, 2, a_3, \cdots, a_n]$$

Then

$$\begin{bmatrix} \alpha + 2\alpha + a_3 w_{13} + \cdots + a_n w_{1n} \\ \cdots \\ \alpha + 2\alpha + a_3 w_{13} + \cdots + a_n w_{dn} \end{bmatrix} = Wu$$
$$\begin{bmatrix} 2\alpha + \alpha + a_3 w_{13} + \cdots + a_n w_{1n} \\ \cdots \\ 2\alpha + \alpha + a_3 w_{13} + \cdots + a_n w_{dn} \end{bmatrix} = Wv$$

Therefore we can say $Wu = Wv \quad (u \neq v)$.

(b) As we said at (a), we can also say about U

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_d \\ \cdots & & & \cdots \\ & & & \end{bmatrix} \in \mathbb{R}^{d\times d}, \qquad D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_d \end{bmatrix} \in \mathbb{R}^{d\times d}$$
$$V_n = \begin{bmatrix} v_1 & v_2 & \cdots & v_d & v_{d+1} & v_{d+2} & \cdots v_n \\ \cdots & & & & & & \cdots \\ & & & & & & \end{bmatrix} \quad \in \mathbb{R}^{d\times n}$$
$$V_n^T x \to x \in \mathbb{R}^{n\times 1}$$
$$V_n^T = U$$

And

$$x \to Wx$$

So we say

$$UWx \to x$$

But this means

$$UW = I$$
$$U = W^T$$

It is impossible.
Therefore, once we delete the dimensions, we are unable to guess these dimensions back.