Homework 4

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1 4.1 problem 9

Prove that $det(AB) = det(A)det(B)$ for any $A, B \in M_{2x2}(F)$.

Proof. Let A $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and B $= \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, then $det(A)det(B) = (ad - bc) * (eh - fg) = adeh - adfg - bceh + bcfg$ and $AB = \begin{pmatrix} ea + bg & af + bh \\ ec + dg & cf + dh \end{pmatrix}$, therefore $det(AB) = (ea + bg)(cf + dh) - (af + bh)(ec + dg) = eacf + eadh + bgcf + bgdh - (eacf + afdg + bhec + bhdg) = eadh - afdg - bhec + bgcf$ note that this is equal to $adeh - adfg - bceh + bcfg$, therefore $det(AB) = det(A)det(B)$ for any $A, B \in M_{2x2}(F)$ □

2 4.2 problem 25

prove that $det(kA) = k^n det(A)$ for any $A \in M_{nxn}(F)$.

Proof. For any matrix A, kA is equivalent to every row of A multiplied by k. Since the determinant is linear in every row, if you leave the other rows fixed, if you multiply k by the first row of A, then $det(A)$ becomes $kdet(A)$ and similarly for every row, so multiplying all n rows of A by k yields that $det(A) = k * k * k * \cdots * kdet(A)$ n times, for every row in A, which yields that $det(A) = k^n det(A)$ □

3 4.3 problem 12

A matrix $Q \in M_{n \times n}$ is called orthogonal if $QQ^t = I$ prove that if Q is orthogonal, then $det(Q) = \pm 1$.

Proof. Assume Q is orthogonal, then $det(QQ^t) = det(Q)det(Q^t) = det(I) = 1$ and since $det(Q^t) = det(Q)$ as the transpose only flips the rows and columns and the definition of the determinant does not depend on whether expansion happens on rows or columns as they are equivalent (Expansion on rows for Q is equivalent to expansion on columns for Q^t), therefore the value of the determinant does not change, then $det(Q)^2 = 1$. Therefore $det(Q) = \sqrt{1} = \pm 1$ $\square$