Homework 3

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1 Herstein p.117, problem 3

Express as the product of disjoint cycles and find the order:
a) (12357)(2476) = (124)(356) order is the LCM of 3 and 3, which is 3.
b) (12)(13)(14) = (1432) which is of order 4
c) (12345)(12346)(12347) = (1473625) Which is of order 6
d) (123)(132) = (1)(3)(2) whose LCM is 1 and is, thus of order 4
e) $(123)(3579)(123)^{-1}$ = (123)(3579)(321) = (1579)(2)(3) Whose LCM is 4 and is, thus of order 4
f) (12345)(12345)(12345) = (14253) Which is of order 5

2

Show that the subset of $GL_2(\mathbb{R})$ consisting of the matrices of the form: $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is a subgroup.

Proof. To show it's a subgroup, we must show that it is closed under matrix multiplication and inverses. Start with inverses. Take A defined as above, then $A^{-1} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^{-1}$ Since this is a subset of $GL_2(\mathbb{R})$ we know it has an inverse, defined as $\frac{1}{a^2+b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \frac{a}{a^2+b^2} & \frac{-b}{a^2+b^2} \\ \frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{pmatrix}$ define $\frac{a}{a^2+b^2}$ to be a' and $\frac{-b}{a^2+b^2}$ to be b', then the matrix can be represented as $\begin{pmatrix} a' & b' \\ -b' & a' \end{pmatrix}$ which is a part of the subset as it is of the form defined above, therefore the set is closed under inverses. For multiplication, take two matrices $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -bc-ad \\ bc+ad & ac-bd \end{pmatrix}$ this time, let $ac-bd = a'$ and $-bc-ad = b'$, then this matrix is of the form $\begin{pmatrix} a' & b' \\ -b' & a' \end{pmatrix}$ which is of the form highlighted above, therefore the set is closed under matrix multiplication and thus, it is a subgroup with matrix multiplication as its binary operation $\square$

3

Let Q_8 denote the subgroup of $GL_2(\mathbb{C})$ generated by the following three matrices:
$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$
$j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
$k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$
a) Show that Q_8 consists of the following matrices:
I,i,j,k,-I,-i,-j,-k Where I is the identity matrix

For Q_8 to be a subgroup it must be closed under multiplication and inverses. Therefore, the following must be true: $i \times j, j \times i, i \times k, k \times i, j \times k, k \times j, i^2, j^2, k^2 \in Q_8$. These are:
$i \times j = -k$, $i \times k = -j$, $j \times i = -k$, $k \times i = j$, $j \times k = i$, $k \times j = -i$, $i^2 = -I$, $j^2 = -I$, and $k^2 = -I$. Therefore, -j,-i must be in Q_8. Now for inverses. $i^{-1} = -i$, $j^{-1} = -j$, $k^{-1} = -k$, therefore $-k$ must be in the subgroup as the inverse of k. I must be in the subgroup as it is the identity. Therefore, Q_8 consists of I,i,j,k,-I,-i,-j, and -k
b) Write out the multiplication table for Q_8

×	I	i	j	k	-I	-i	-j	-k
I	I	i	j	k	-I	-i	-j	-k
i	i	-I	-k	j	-i	I	k	-j
j	j	-k	-I	i	-j	k	I	-i
k	k	-j	i	-I	-k	j	-i	I
-I	-I	-i	-j	-k	I	i	j	k
-i	-i	I	k	-j	i	-I	-k	j
-j	-j	k	I	i	-j	-k	-I	i
-k	-k	j	i	I	k	-j	i	-I

c)Show that every subgroup of Q_8 is normal.
The subgroups of Q_8 are $< i >$, $< j >$, $< k >$, $< I >$, and $< -I >$. For $< i >$, we must show that gng^{-1} for all $g \in Q_8$, $n \in < i >$.The elements in $< i >$ are i, i^{-1}, and I. Therefore, for i, $iii^{-1} = i$, which is in $< i >$ and $jij^{-1} = k^{-1}j^{-1} = (kj)^{-1} = i^{-1}$ which is also in $< i >$, and $kik^{-1} = jk^{-1} = i$. For i^{-1}, $ji^{-1}j^{-1} = kj^{-1} = i$ and $ki^{-1}k^{-1} = j^{-1}k^{-1} = i$, and $iIi^{-1} = ii^{-1} = I$, $jIj^{-1} = jj^{-1} = I$, $kIk^{-1} = kk^{-1} = I$ therefore $< i >$ is a normal subgroup. $< j >$ follows a similar path, with $jjj^{-1} = j$, $iji^{-1} = k^{-1}i^{-1} = j^{-1}$, and $kjk^{-1} = ik^{-1} = j$. And for j^{-1}, $ij^{-1}i^{-1} = ki^{-1} = j^{-1}$, $kj^{-1}k^{-1} = ik^{-1} = j$

and the identity is the same as for $< i >$ therefore $< j >$ is a normal subgroup. For $< k >$, $kkk^{-1} = k$, $iki^{-1} = j^{-1}i^{-1} = k^{-1}$, $jkj^{-}1 = ij^{-1} = k$, and for k^{-1}, $ik^{-1}i^{-1} = ji^{-1} = k$, and $jk^{-1}j^{-1} = ij^{-1} = k$ with the identity proven the same as for $< i >$, therefore $< k >$ is a normal subgroup. $< I >$ is normal as per above. For $< -I >$, we have $I^{-1}I^{-1}I = I^{-1}I^{-1} = I$, $iI^{-1}i^{-1} = i^{-1}i^{-1} = I^{-1}$, $jI^{-1}j^{-1} = j^{-1}j^{-1} = I^{-1}$, and $kI^{-1}k^{-1} = k^{-1}k^{-1} = I^{-1}$, therefore $< -I >$ is a normal subgroup.

4 Herstein p.73, problem 1

Determine if the given mapping is a homomorphism. If so, identify its kernel and whether or not the mapping is 1-1 or onto.
a) $G = \mathbb{Z}$ under $+$, $G' = \mathbb{Z}_n$, $\phi(a) = [a]$ for $a \in \mathbb{Z}$
b) G group, $\phi : G \to G$ defined by $\phi(a) = a^{-1}$ for $a \in G$
For $a, b \in G$ $\phi(ab) = (ab)^{-1} = a^{-1}b^{-1} = \phi(a)\phi(b)$ therefore ϕ is a homomorphism. Kernel of ϕ is the set of all elements who are their own inverses. ϕ is onto, as every element has an inverse, since G is a group, ϕ is not 1-1, however, as elements that are their own inverses would map to the identity and the identity would map to the identity, breaking injectivity.
c) G abelian group, $\phi : G \to G$ defined by $\phi(a) = a^{-1}$ for $a \in G$
For $a, b \in G$ $\phi(ab) = (ab)^{-1} = a^{-1}b^{-1} = \phi(a)\phi(b)$ Therefore, ϕ is a homomorphism. As above, $ker(\phi)$ is the set of all elements who are their own inverse and ϕ is surjective, but not injective
d) G group of all nonzero real numbers under multiplication, $G' = \{1, -1\}$, $\phi(r) = 1$ if $r > 0$ and $\phi(r) = -1$, if $r < 0$.
Take $a, b > 0$, then $ab > 0$, so $\phi(ab) = 1 = 1 \times 1 = \phi(a)\phi(b)$
Take $b < 0 < a$, then $ab < 0$, so $\phi(ab) = -1 = 1 \times -1 = \phi(a)\phi(b)$
Take $a, b < 0$, then $ab > 0$, so $\phi(ab) = 1 = -1 \times -1 = \phi(a)\phi(b)$
Therefore, ϕ is a homomorphism. ϕ is clearly surjective onto G' and clearly not injective as all positive real numbers map to 1 and all negative real numbers map to -1. $ker(\phi)$ is equivalent to all the positive real numbers, as they map to the identity of G'
e) G an abelian group, $n > 1$ a fixed integer and $\phi : G \to G$ defined by $\phi(a) = a^n$ for $a \in G$
For $a, b \in G$ $\phi(ab) = (ab)^n = a^nb^n = \phi(a)\phi(b)$, so ϕ is a homomorphism. ϕ is not surjective or one-to-one as a finite abelian group with every element as its inverse with $n = 2$ would only go to the identity.

5

Let $2\pi i\mathbb{Z}$ be the subgroup of $\mathbb{C}$ generated by the element $2\pi i$. show that $\mathbb{C}/2\pi i\mathbb{Z}$ is isomorphic to $\mathbb{C}^*$

Proof. By the first isomorphism theorem, to show that $\mathbb{C}/2\pi i\mathbb{Z}$ is isomorphic to $\mathbb{C}^*$, it is sufficient to show a homomorphism $\phi : \mathbb{C} \to \mathbb{C}^*$ such that $ker(\phi) =$

$2\pi i\mathbb{Z}$, where $\phi(2\pi i\mathbb{Z}) = 1$. Define this homomorphism as $\phi(a) = e^a$, this is a homomorphism as $\phi(ab) = e^{ab} = e^a e^b = \phi(a)\phi(b)$. Then, for some $n \in \mathbb{Z}$ $\phi(2n\pi i) = e^{2n\pi i} = cos(2n\pi) + isin(2n\pi) = 1 + 0 = 1$ which is the identity in $\mathbb{C}^*$, therefore $ker(\phi) = 2\pi i\mathbb{Z}$ which, by the first isomorphism theorem, $\mathbb{C}/2\pi i\mathbb{Z} \cong \mathbb{C}^*$ □

6 Bonus Problem A

7 Bonus Problem B

Prove that a finite group has even order if and only if it has an element of order 2.

Proof. ($\Rightarrow$) Assume a group G has even order, then its order can be represented as $2k$, for some int k. Furthermore, if the group is of even order, then some element in the group must be its own inverse, otherwise the group would have $2k + 1$ elements (Every element + every element's inverse + the identity) and not be of even order. Therefore, since there is at least one element, x, that is its own inverse this element has order 2 as it can be placed in a subgroup of order 2, $\{e, x\}$.
($\Leftarrow$) Assume a group G has an element of order 2, then since the order of an element must divide the order of a group, by Lagrange's theorem, the order of the group must be even. □