Homework 5

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1 5.1 Problem 14

Prove that for any square matirx A, A and A^t have the same characteristic polynomial and eigenvalues.

Proof. For a matrix, A, $char(A) = det(A - \lambda I)$, therefore $char(A^t) = det(A^t - \lambda I)$, furthermore, we know that $det(A) = det(A^t)$, therefore $det(A - \lambda I) = det((A - \lambda I)^t) = det(A^t - \lambda I^t)$, but $I^t = I$, therefore $det(A^t - \lambda I^t) = det(A^t - \lambda I) = char(A^t)$, therefore $char(A) = char(A^t)$ and if two matrices have the same characteristic polynomial then it follows, from the definition of Eigenvalue, that they have the same eigenvalues. $\square$

2 5.2 Problem 18a

Prove that if T and U are simultaneously diagonalizable operators, then T and U commute ($TU = UT$)

Proof. Let T, U simultaneously diagonalizable by way of β, some ordered basis of V, then $[T]_\beta$ and $[U]_\beta$ are diagonal matrices. And since diagonal matrices commute, we have that $[T]_\beta[U]_\beta = [U]_\beta[T]_\beta$, therefore this implies that the operators generating $[U]_\beta$ and $[T]_\beta$, T, U commute. $\square$

3 5.4 problem 17

Let A be an $n \times n$ matrix, prove that $dim(span(\{I_n, A, A^2, \dots\})) \leq n$

Proof. Consider the characteristic polynomial $f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_0$, then for the set, we have that $f(A) = (-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_0 I = 0$, by the Cayley-Hamilton Theorem, which implies that A^n is a linear combination of $I, A, \dots, A^{n-1}$, $(-1)^n A^n = -a_{n-1}A^{n-1} - \cdots - a_0 I$, therefore any further power, k, of A greater than n is also a linear combination of $I, A, \dots, A^{n-1}$, therefore $\{I, A, \dots, A^{n-1}\}$ is the largest linearly independent set of matrices, meaning it is a basis of $\left\{I_n, A, A^2, \dots\right\}$ and its dimension is n, therefore $dim(span(\left\{I_n, A, A^2, \dots\right\})) = n$ and is not greater than n, because otherwise, there would be a basis larger than $\left\{I, A, \dots, A^{n-1}\right\}$. $\square$