Homework 2

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1 Question 1: Sec 1.6 #30

W_1: Assume A a $2 * 2$ matrix of the form $[;\frac{\bar{\Xi}}{\Xi};]$ $\begin{pmatrix} a & b \\ c & a \end{pmatrix}$and B = $\begin{pmatrix} a' & b' \\ c' & a' \end{pmatrix}$, then $A+B = \begin{pmatrix} a & b \\ c & a \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & a' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & a+a' \end{pmatrix} = \begin{pmatrix} a'' & b'' \\ c'' & a'' \end{pmatrix}$ which is in W_1 as a'', b'', and $c'' \in F$
as the result of the addition of two scalars is a scalar. Also, take xA where $x \in F$
$xA = \begin{pmatrix} xa & xb \\ xc & xa \end{pmatrix}$ which is of the form highlighted above, as the multiplication of two scalars yields a scalar, and is, thus, in W_1. Thus, W_1 is a subspace of V.
W_2: Assume A = $\begin{pmatrix} 0 & a \\ -a & b \end{pmatrix}$and B = $\begin{pmatrix} 0 & a' \\ -a' & b' \end{pmatrix}$, then $A+B = \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} + \begin{pmatrix} 0 & a' \\ -a' & b' \end{pmatrix} = \begin{pmatrix} 0 & a+a' \\ -a-a' & b+b' \end{pmatrix}$ and, as noted above addition of two scalars results in a scalar (and the addition of two negative scalars gives a negative scalar), this yields the matrix $\begin{pmatrix} 0 & a'' \\ -a'' & b'' \end{pmatrix}$ $(-a-a' = -1(a+a') = -1(a'') = -a'')$ which is in W_2 Therefore, the set is closed under addition, to show it is closed under scalar multiplication, consider xA where $x \in F = \begin{pmatrix} x0 & xa \\ -xa & xb \end{pmatrix}$ since multiplication by a scalar results in scalars, we get $\begin{pmatrix} 0 & a' \\ -a' & b' \end{pmatrix}$which is in W_2, therefore W_2 is a vector space of V. Dim$(W_1) = 3$, as $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ is a basis for W_1 as it is Linearly Independent and spans all of W_1 Dim$(W_2) = 2$ as$\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis for W_2.
$W_2 \cap W_1 = \left\{ \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \right\}$ as this matrix can be generated in W_1 by taking $a \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + -a \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ and in W_2 by tak-

ing $a\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ where, in both cases, a is any scalar. Therefore, $\dim(W_1 \cap W_2) = 1$.
Therefore, $\dim(W_1+W_2) = dim(W_1)+dim(W_2)-dim(W_1\cap W_2) = 3+2-1 = 4$

2 Question 2: Sec 2.1: #14b

Let V and W be vector spaces and $T : V \to W$ be linear:
Suppose that T is injective and S is a subset of V. Prove that S is linearly independent iff $T(S)$ is linearly independent.
Need to prove a $\Leftrightarrow$ statement means need to prove both ways.
($\Rightarrow$) Assume S is linearly independent, then no element in S can be rewritten as a linear combination of other elements in S. Therefore, since T is injective, every element in T(S) maps to one element in S, so that for $u, v \in S$ $T(u) = T(v) \Rightarrow u = v$ therefore, the only element in S that can map to the zero vector in T(S) is the zero vector from S, as the zero vector always maps to the zero vector and since it is assumed that T is injective, no other vector in S can map to the zero vector in T(S) and since S is linearly independent, no combination of vectors in S can give the zero vector, therefore nullity(T(S)) = 1 ($null(T(S) = \{0_s\})$). Therefore, for S, $a_1v_1 + \cdots + a_nv_n = 0 \Rightarrow a_1 = \cdots = a_n = 0$ Therefore, $T(a_1v_1 + \cdots + a_nv_n) = T(0) = 0$, but by definition of a linear transformation, $T(a_1v_1 + \cdots + a_nv_n) = a_1T(v_1) + \ldots a_nT(v_n)$ and it was shown that $a_1 = \cdots = a_n = 0$, therefore, since $T(v_1) \ldots T(v_n)$ are vectors in T(S), T(S) is linearly independent.
($\Leftarrow$) Assume T(S) is linearly independent and that T is an injective linear transformation, therefore every element in T(S) maps to exactly one element in S and only the zero vector in T(S) maps to 0_s, again by the injectivity of T. Therefore, no linear combination of vectors in T(S) can create the zero vector, i.e. $a_1T(v_1) + \cdots + a_nT(v_n) = 0$ if $a_1 = \cdots = a_n = 0$ which is only true if $a_1v_1 + \cdots + a_nv_n = 0 \Rightarrow a_1 = \cdots = a_n = 0$ as $a_iT(v_i) = T(a_iv_i)$ and therefore, S is linearly independent.

3 Question 3: Sec 2.1: #17

Let V and W be finite vector spaces and $T : V \to W$ be linear.
a) Prove that if $dim(V) < dim(W)$ then T cannot be surjective.
By dimension theorem, we have that $rank(T)+nullity(T) = dim(V)$ which gives that $rank(T) = dim(V) - nullity(T)$, therefore $rank(T) \leq dim(V)$ as rank(T) at most can be equal to dim(V). It is assumed, though that $dim(V) < dim(W)$ with a strict inequality, therefore $rank(T) < dim(W)$. Therefore T cannot be surjective onto W, as there are bases in W that are not in T.
b) Prove that if $dim(V) > dim(W)$ then T cannot be injective.
By same theorem as above, we get that $nullity(T) = dim(V) - rank(T)$, therefore $nullity(T) \leq dim(V)$ and $dim(V) > dim(W)$ subtracting dim(W) yields $dim(V) - dim(W) > 0$ since rank(T), however, can, at most, be equal

to $dim(W)$, then $dim(V) - rank(T) \geq dim(V) - dim(W)$ as rank(T) cannot be greater than $dim(W)$ as T maps to W $dim(V) - rank(T) = nullity(T)$, however. Therefore, $nullity(T) \geq dim(V) - dim(W) > 0$, thus we have that $nullity(T) \neq 0$, therefore, by theorem 2.5, T cannot be injective.