Problem Set

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1 Problems

1.1 Pyramid Maximization

Let w denote the largest pyramid that can be inscribed inside the largest cone that can be inscribed in a sphere radius 12. Find the volume of pyramid w.

1.2 2024 Trouble

Denote m and n as two distinct positive integers such that $m \neq n$ and $m(m^2 + n^2) = 2024$. Find the sum of all the possible positive values of n^2 given that m is a positive integer factor of 2024.

1.3 Taxicab

A famous story of G.H. Hardy and Srinivasa Ramanujan goes as follows: Hardy was there to meet his good friend Ramanujan in the hospital and referenced how dull his Taxi number was. The number was 1729. Ramanujan famously remarked, "No, it is a very interesting number." This number was, in fact, the first positive integer that could be written as the sum of two cubes in two different ways! Let me elaborate: $1729 = 10^3 + 9^3 = 12^3 + 1^3$. Find the next number that can be written as the sum of 2 cubes in two different ways.

1.4 Bakery Store

Lamar has 59 dollars. What is the number of cupcakes and cookies she will be able to purchase, given that cupcakes cost 7 dollars and cookies cost 6 dollars, and the shop doesn't sell partial products? The answer can be expressed as (x, y), where x denotes the number of cupcakes and y denotes the number of cookies Lamar can buy. Compute $x + y$, given that Lamar spends all of her money.

1.5 Chinese Remainder Theorem

If the remainder when a number is divided by 13 is 2, divided by 17 is 3, and divided by 9 is 1, find the smallest number that satisfies these conditions, given that it has a factor of 16.

1.6 Quadratic

Let $f(x)$ denote a quadratic polynomial that has the following properties:

- Has one zero
- Concave down
- These points lie on the curve: $(1, 0)$, $(2, -3)$, $(-1, -12)$

The answer can be written as $ax^2 + bx + c$. Compute the sum of a, b, c.

1.7 Floor Trouble

Let X denote an integer over the interval $(0, 100]$, compute the number of values of X given that at least one of $\lfloor \frac{X}{3} \rfloor$, $\lfloor \frac{X}{4} \rfloor$, $\lfloor \frac{X}{5} \rfloor$ is a perfect square. Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. For example, $\lfloor 13.21 \rfloor$'s floor would be 13.

1.8 Nested Function

Denote a function $w(n)$, where it returns the sum of (sides) + (vertices) + (diagonals) of a polyhedron with n sides. Compute the sum of the digits of $w(w(35))$.

1.9 Graph Theory

Denote $\frac{x^2}{16} + \frac{y^2}{25} = 8$ and $\frac{x^2}{16} \cdot \frac{y^2}{25} = 16$ in the Cartesian plane. If these two functions intersect exactly 4 times, find the area of the quadrilateral in the Cartesian plane using these 4 points as vertices.

1.10 Arithmetic Sequences

How many ways are there to write 17 as the last term of an arithmetic sequence, with a positive integer difference? Note the following: Each sequence must start with a positive integer, and the common difference must be a non-negative integer. For example: If we wanted 5 as the last number we could have the following sequences: 1,3,5 or 2,5 or 5.

1.11 "13-8" Special Pairs

A number in base 13 is equivalent to another number's representation in base 8. For example, 14 and 9 would be two numbers that satisfy this condition since their representation in base 13 and 8 are equivalent, respectively (11 in base 13 is 14, 11 in base 8 is 9, so their pair is $(14, 9)$). Let's name all such pairs as "13-8" special. Find the sum of the largest "13-8" special pair, given that the product of the pair doesn't exceed 1000 when evaluated. Note: The pair can be expressed as (x, y), compute $x + y$.

1.12 Expected Number

Abhi wants you to analyze his grades. Today is the day of his finals in 7 classes, and unsurprisingly he is failing all of his classes! However, if he passes the final exam, he can pass the class, surprisingly! The probability he fails is 50% on the first exam. For each subsequent exam, he is expected to pass with a 25% probability if he failed his prior exam, otherwise, it is 60%. What is the expected number of classes he is expected to pass? Your answer can be expressed as $\frac{m}{n}$, where m and n are relatively prime integers. Compute $m + n$.

1.13 Coordinate Geometry

1.14 Cement Troubles

1.15 Gravity

2 Answer Key

2.1 1296

2.2 2262

2.3 4104

2.4 9

2.5 496

2.6 0

2.7 47

2.8 36

2.9 320

2.10 65

2.11

2.12 51

2.13

2.14

2.15