Problem Set

Kalyan Cherukuri

March 2024

1 Introduction

Hello! I look forward to writing these problems, I am a freshman at Metea Valley High School, who has always been interested in math competitions. You can often find my name floating around in association with numerous larger competitions, such as Math Kangaroo, AMC 10/AIME, Chicago ARML, or at a local scale if you live in Aurora, with competitions like the North Suburban Math League or ICTM competitions at the regional and state level. You can contact me at kalyan.cherukuri5@gmail.com for any appeals/questions or such.

2 PROBLEMS START ON THE NEXT PAGE

3 Problems

3.1 Pyramid Maximization

Let w denote the largest pyramid that can be inscribed inside the largest cone that can be inscribed in a sphere radius 12. Find the volume of pyramid w.

3.2 2024 Trouble

Denote m and n as two distinct positive integers such that $m \neq n$ and $m(m^2 + n^2) = 2024$. Find the sum of all the possible positive values of n^2 given that m is a positive integer factor of 2024.

3.3 Taxicab

A famous story of G.H. Hardy and Srinivasa Ramanujan goes as follows: Hardy was there to meet his good friend Ramanujan in the hospital and referenced how dull his Taxi number was. The number was 1729. Ramanujan famously remarked, "No, it is a very interesting number." This number was, in fact, the first positive integer that could be written as the sum of two cubes in two different ways! Let me elaborate: $1729 = 10^3 + 9^3 = 12^3 + 1^3$. Find the next number that can be written as the sum of 2 cubes in two different ways.

3.4 Bakery Store

Lamar has 59 dollars. What is the number of cupcakes and cookies she will be able to purchase, given that cupcakes cost 7 dollars and cookies cost 6 dollars, and the shop doesn't sell partial products? The answer can be expressed as (x, y), where x denotes the number of cupcakes and y denotes the number of cookies Lamar can buy. Compute $x + y$, given that Lamar spends all of her money.

3.5 Chinese Remainder Theorem

If the remainder when a number is divided by 13 is 2, divided by 17 is 3, and divided by 9 is 1, find the smallest number that satisfies these conditions, given that it has a factor of 16.

3.6 Quadratic

Let $f(x)$ denote a quadratic polynomial that has the following properties:

- Has one zero
- Concave down
- These points lie on the curve: $(1, 0)$, $(2, -3)$, $(-1, -12)$

The answer can be written as $ax^2 + bx + c$. Compute the sum of a, b, c.

3.7 Floor Trouble

Let X denote an integer over the interval $(0, 100]$, compute the number of values of X given that at least one of $\lfloor \frac{X}{3} \rfloor$, $\lfloor \frac{X}{4} \rfloor$, $\lfloor \frac{X}{5} \rfloor$ is a perfect square. Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. For example, $\lfloor 13.21 \rfloor$'s floor would be 13.

3.8 Nested Function

Denote a function $w(n)$, where it returns the sum of (sides) + (vertices) + (diagonals) of a polyhedron with n sides. Compute the sum of the digits of $w(w(35))$.

3.9 Graph Theory

Denote $\frac{x^2}{16} + \frac{y^2}{25} = 8$ and $\frac{x^2}{16} \cdot \frac{y^2}{25} = 16$ in the Cartesian plane. If these two functions intersect exactly 4 times, find the area of the quadrilateral in the Cartesian plane using these 4 points as vertices.

3.10 Arithmetic Sequences

How many ways are there to write 17 as the last term of an arithmetic sequence, with a positive integer difference? Note the following: Each sequence must start with a positive integer, and the common difference must be a non-negative integer. For example: If we wanted 5 as the last number we could have the following sequences: 1,3,5 or 2,5 or 5.

3.11 "13-8" Special Pairs

A number in base 13 is equivalent to another number's representation in base 8. For example, 14 and 9 would be two numbers that satisfy this condition since their representation in base 13 and 8 are equivalent, respectively (11 in base 13 is 14, 11 in base 8 is 9, so their pair is $(14, 9)$). Let's name all such pairs as "13-8" special. Find the sum of the largest "13-8" special pair, given that the product of the pair doesn't exceed 1000 when evaluated. Note: The pair can be expressed as (x, y), compute $x + y$.

3.12 Expected Number

Bob wants you to analyze his grades. Today is the day of his finals in 7 classes, and unsurprisingly he is failing all of his classes! However, if he passes the final exam, he can pass the class, surprisingly! The probability he fails is 50% on the first exam. For each subsequent exam, he is expected to pass with a 25% probability if he failed his prior exam, otherwise, it is 60%. What is the expected number of classes he is expected to pass? Your answer can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

3.13 Simple Geometry

An equilateral triangle is inscribed into a circle of radius 5. The circle has diameter AD, with center O. The equilateral triangle has points ABC lying on distinct points on circle O. The triangle has altitude AF. If two triangles can be drawn, from this. Triangle BDO, and CDO. If the ratio of the areas of these two triangles, to the area of the circle can be expressed as $\frac{a\sqrt{b}\pi}{c}$. Compute a+b+c.

3.14 Floor Probability

Given the equation $\lfloor 7x \rfloor + \lfloor 3x \rfloor = 30$, where x is a rational number in the range $b \leq x < a$, and a and b are rational numbers. We want to compute the probability that any random rational number over the domain $[0, 4]$ lies in the inequality for solutions for x.

3.15 Gravity

It is commonly known that the gravity and trajectory of a projectile can be represented in this form: $h = -16t^2 + v_0 t + h_0$. Note: t represents the time passed, v_0 represents the initial velocity, h_0 represents the initial launch height, and h represents the overall height in feet after t time. Bob fires a cannonball at an unknown height at a speed of 40 meters per second. Bob attached a sensor on the cannonball to track the maximum height of the cannonball. However, due to his poor vision he isn't completely sure the exact height. Instead he remembers that the maximum height of the cannonball lies between this continuous interval, $[32, 67]$. If the range of values for h_0, can be expressed over this continuous interval $[a, b]$, where a and b are positive integers, compute $a + b$.