University of Houston

Midterm Review

COSC 4393
Digital Image Processing

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For an $M \times N$ image I, the **Discrete Fourier Transform** at a pixel u, v is given by

$$\tilde{I}(u,v) = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} I(i,j) e^{-2\pi\sqrt{-1}\left(\frac{ui}{M}+\frac{vj}{N}\right)}$$

For a square image, we have $N = M$ and

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j) e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

A complex-valued function f is **Conjugate Symmetric** or **Hermitian** if $f^*(z) = f(-z)$. In the case of the DFT, we have

$$\begin{aligned}
\tilde{I}(-u,-v) &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j) e^{-\sqrt{-1}\frac{2\pi}{N}(-ui+-vj)} \\
&= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j) \left(\cos\left(-\frac{2\pi}{N}(-ui+-vj)\right) + \sqrt{-1}\sin\left(-\frac{2\pi}{N}(-ui+-vj)\right)\right) \\
&= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j) \left(\cos\left(-\frac{2\pi}{N}(ui+vj)\right) - \sqrt{-1}\sin\left(-\frac{2\pi}{N}(ui+vj)\right)\right) \\
&= \tilde{I}^*(u,v)
\end{aligned}$$

Equivalently, a complex valued function is Conjugate Symmetric if and only if its real part is even and its imaginary part is odd. The real part of the DFT is just

$$\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j) \cos\left(-\sqrt{-1}\frac{2\pi}{N}(ui+vj)\right)$$

and the imaginary part is

$$\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j) \sin\left(-\sqrt{-1}\frac{2\pi}{N}(ui+vj)\right)$$

which are clearly even and odd, respectively.

The **magnitude** of a complex number $z = x + iy$ is given by $|z| = x^2 + y^2$. To see that the magnitude of the DFT is symmetric, note that $|z| = |z*|$ for any complex number z. Since $\tilde{I}^*(u,v) = -u\tilde{,}-v$,

$$\left|\tilde{I}^*(u,v)\right| = \left|\tilde{I}(-u,-v)\right|$$

To see that the DFT is periodic, show that $\tilde{I}(u+kN, v+lN) = \tilde{I}(u,v)$. We write

$$W_N^{ui+vj} = e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

$$\begin{aligned}
\tilde{I}(u+kN, v+lN) &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j) W_N^{(u+kN)i+(v+lN)j} \\
&= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j) W_N^{ui+vj+N(ki+lj)} \\
&= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j) W_N^{ui+vj} \cdot W_N^{N(ki+lj)}
\end{aligned}$$

Then, we only need to show that $W_N^{N(ki+lj)} = 1$. This is straightforward:

$$
\begin{aligned}
W_N^{N(ki+lj)} &= e^{-\sqrt{-1}\frac{2\pi}{N}(N(ki+lj)} \\
&= e^{-\sqrt{-1}\cdot 2\pi(ki+lj)} \\
&= \cos\left(2\pi(ki+lj)\right) + \sqrt{-1}\sin\left(2\pi(ki+lj)\right) \\
&= 1
\end{aligned}
$$

To compute the DFT of the following matrix:

$$
\begin{bmatrix} 5 & 7 \\ 8 & 3 \end{bmatrix}
$$

We evaluate

$$
\begin{aligned}
\tilde{I}(0,0) &= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)e^{-\sqrt{-1}\frac{2\pi}{2}(0i+0j)} \\
&= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j) \\
&= I(0,0) + I(0,1) + I(1,0) + I(1,1) \\
&= 5+7+8+3 \\
&= 23
\end{aligned}
$$

$$
\begin{aligned}
\tilde{I}(0,1) &= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)e^{-\sqrt{-1}\frac{2\pi}{2}(0i+j)} \\
&= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)e^{-\sqrt{-1}\pi j} \\
&= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)\left(\cos(-\pi j) + \sqrt{-1}\sin(-\pi j)\right)
\end{aligned}
$$

Plugging in the values of i and j:

$$
\begin{aligned}
\tilde{I}(0,1) = {} & I(0,0)\left(\cos(-\pi\cdot 0) + \sqrt{-1}\sin(-\pi\cdot 0)\right) \\
& + I(0,1)\left(\cos(-\pi\cdot 1) + \sqrt{-1}\sin(-\pi\cdot 1)\right) \\
& + I(1,0)\left(\cos(-\pi\cdot 0) + \sqrt{-1}\sin(-\pi\cdot 0)\right) \\
& + I(1,1)\left(\cos(-\pi\cdot 1) + \sqrt{-1}\sin(-\pi\cdot 1)\right) \\
& = 5\,(1+0i) + 7\,(-1-0i) + 8\,(1+0i) + 3\,(-1-0i) \\
& = 5-7+8-3 \\
& = 3
\end{aligned}
$$

$$
\begin{aligned}
\tilde{I}(1,0) &= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)e^{-\sqrt{-1}\frac{2\pi}{2}(i+0j)} \\
&= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)e^{-\sqrt{-1}\pi i} \\
&= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)\left(\cos(-\pi i) + \sqrt{-1}\sin(-\pi i)\right)
\end{aligned}
$$

Plugging in the values of i and j:

$$\begin{aligned}\tilde{I}(1,0) &= I(0,0)\left(\cos(-\pi\cdot 0)+\sqrt{-1}\sin(-\pi\cdot 0)\right)\\ &+ I(0,1)\left(\cos(-\pi\cdot 0)+\sqrt{-1}\sin(-\pi\cdot 0)\right)\\ &+ I(1,0)\left(\cos(-\pi\cdot 1)+\sqrt{-1}\sin(-\pi\cdot 1)\right)\\ &+ I(1,1)\left(\cos(-\pi\cdot 1)+\sqrt{-1}\sin(-\pi\cdot 1)\right)\\ &= 5(1+0i)+7(1-0i)+8(-1+0i)+3(-1-0i)\\ &= 5+7-8-3\\ &= 1\end{aligned}$$

Finally:

$$\begin{aligned}\tilde{I}(1,1) &= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)e^{-\sqrt{-1}\frac{2\pi}{2}(i+j)}\\ &= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)e^{-\sqrt{-1}\pi(i+j)}\\ &= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)\left(\cos(-\pi(i+j))+\sqrt{-1}\sin(-\pi(i+j))\right)\end{aligned}$$

Plugging in the values of i and j:

$$\begin{aligned}\tilde{I}(0,1) &= I(0,0)\left(\cos(-\pi\cdot(0+0))+\sqrt{-1}\sin(-\pi\cdot(0+0))\right)\\ &+ I(0,1)\left(\cos(-\pi\cdot(0+1))+\sqrt{-1}\sin(-\pi\cdot(0+1))\right)\\ &+ I(1,0)\left(\cos(-\pi\cdot(1+0))+\sqrt{-1}\sin(-\pi\cdot(1+0))\right)\\ &+ I(1,1)\left(\cos(-\pi\cdot(1+1))+\sqrt{-1}\sin(-\pi\cdot(1+1))\right)\\ &= 5(1+0i)+7(-1+0i)+8(-1+0i)+3(1+0i)\\ &= 5-7-8+3\\ &= -7\end{aligned}$$

This gives the DFT Matrix

$$\begin{bmatrix}21 & 3\\ 1 & -7\end{bmatrix}$$

The **Inverse Discrete Fourier Transform** is given by

$$I(i,j) = \sum_{u=0}^{N-1}\sum_{v=0}^{N-1}\tilde{I}(u,v)e^{\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

Its periodicity can be shown directly, as done earlier previously.
The **convolution** of two functions f and g is denoted $f * g$ or $f \otimes g$

$$(f\otimes g)(n) = \sum_{m=-\infty}^{\infty} f(m)g(n-m)$$

If $\mathcal{F}(f)$ denotes the Discrete Fourier Transformation of F, then

$$\mathcal{F}(f\otimes g) = \mathcal{F}(f)\times\mathcal{F}(g)$$

since

$$
\begin{aligned}
\mathcal{F}(f \otimes g)(\mu) &= \mathcal{F}\left(\sum_{m=-\infty}^{\infty} f(m)g(n-m)\right) \\
&= \sum_{t=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} f(m)g(n-m)\right) e^{-2\pi i \mu t} \\
&= \sum_{m=-\infty}^{\infty} f(n) \left(\sum_{t=-\infty}^{\infty} g(n-m)e^{-2\pi i \mu t}\right) \\
&= \sum_{m=-\infty}^{\infty} f(n) \left(\sum_{t=-\infty}^{\infty} g(n-m)e^{-2\pi i \mu (t-m)}\right) e^{-2\pi i m \mu} \\
&= \sum_{m=-\infty}^{\infty} f(n) \left(\mathcal{F}(g)(\mu)\right) e^{-2\pi i m \mu} \\
&= \mathcal{F}(g)(\mu) \left(\sum_{m=-\infty}^{\infty} f(n)e^{-2\pi i m \mu}\right) \\
&= \mathcal{F}(g)(\mu)\mathcal{F}(f)(\mu) \\
&= \mathcal{F}(f)(\mu)\mathcal{F}(g)(\mu)
\end{aligned}
$$