University of Houston

Midterm Review

COSC 4393
Digital Image Processing

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March 7, 2019

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For an $M \times N$ image I, the **Discrete Fourier Transform** at a pixel u, v is given by

$$\tilde{I}(u,v) = \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} I(i,j)e^{-2\pi\sqrt{-1}\left(\frac{ui}{M}+\frac{vj}{N}\right)}$$

For a square image, we have $N = M$ and

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

A complex-valued function f is **Conjugate Symmetric** or **Hermitian** if $f^*(z) = f(-z)$. In the case of the DFT, we have

$$\begin{aligned}
\tilde{I}(-u,-v) &= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)e^{-\sqrt{-1}\frac{2\pi}{N}(-ui+-vj)} \\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\left(\cos\left(-\frac{2\pi}{N}(-ui+-vj)\right) + \sqrt{-1}\sin\left(-\frac{2\pi}{N}(-ui+-vj)\right)\right) \\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\left(\cos\left(-\frac{2\pi}{N}(ui+vj)\right) - \sqrt{-1}\sin\left(-\frac{2\pi}{N}(-ui+-vj)\right)\right) \\
&= \tilde{I}^*(u,v)
\end{aligned}$$

Equivalently, a complex valued function is Conjugate Symmetric if and only if its real part is even and its imaginary part is odd. The real part of the DFT is just

$$\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\cos\left(-\sqrt{-1}\frac{2\pi}{N}(ui+vj)\right)$$

and the imaginary part is

$$\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\sin\left(-\sqrt{-1}\frac{2\pi}{N}(ui+vj)\right)$$

which are clearly even and odd, respectively.

The **magnitude** of a complex number $z = x + iy$ is given by $|z| = x^2 + y^2$. To see that the magnitude of the DFT is symmetric, evaluate

$$\left|\tilde{I}(u,v)\right| = \left|\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}\right|$$

For example, to compute the DFT of the following matrix:

$$\begin{bmatrix} 5 & 7 \\ 8 & 3 \end{bmatrix}$$

We evaluate

$$\begin{aligned}
\tilde{I}(0,0) &= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)e^{-\sqrt{-1}\frac{2\pi}{2}(0i+0j)} \\
&= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j) \\
&= I(0,0) + I(0,1) + I(1,0) + I(1,1) \\
&= 5 + 7 + 8 + 3 \\
&= 23
\end{aligned}$$

$$
\begin{aligned}
\tilde{I}(0,1) &= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)e^{-\sqrt{-1}\frac{2\pi}{2}(0i+j)} \\
&= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)e^{-\sqrt{-1}\pi j} \\
&= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)\left(\cos\left(-\pi j\right)+\sqrt{-1}\sin\left(-\pi j\right)\right)
\end{aligned}
$$

Plugging in the values of i and j:

$$
\begin{aligned}
\tilde{I}(0,1) &= I(0,0)\left(\cos\left(-\pi\cdot 0\right)+\sqrt{-1}\sin\left(-\pi\cdot 0\right)\right) \\
&\quad + I(0,1)\left(\cos\left(-\pi\cdot 1\right)+\sqrt{-1}\sin\left(-\pi\cdot 1\right)\right) \\
&\quad + I(1,0)\left(\cos\left(-\pi\cdot 0\right)+\sqrt{-1}\sin\left(-\pi\cdot 0\right)\right) \\
&\quad + I(1,1)\left(\cos\left(-\pi\cdot 1\right)+\sqrt{-1}\sin\left(-\pi\cdot 1\right)\right) \\
&= 5\left(1+0i\right)+7\left(-1-0i\right)+8\left(1+0i\right)+3\left(-1-0i\right) \\
&= 5-7+8-3 \\
&= 3
\end{aligned}
$$

$$
\begin{aligned}
\tilde{I}(1,0) &= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)e^{-\sqrt{-1}\frac{2\pi}{2}(i+0j)} \\
&= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)e^{-\sqrt{-1}\pi i} \\
&= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)\left(\cos\left(-\pi i\right)+\sqrt{-1}\sin\left(-\pi i\right)\right)
\end{aligned}
$$

Plugging in the values of i and j:

$$
\begin{aligned}
\tilde{I}(1,0) &= I(0,0)\left(\cos\left(-\pi\cdot 0\right)+\sqrt{-1}\sin\left(-\pi\cdot 0\right)\right) \\
&\quad + I(0,1)\left(\cos\left(-\pi\cdot 0\right)+\sqrt{-1}\sin\left(-\pi\cdot 0\right)\right) \\
&\quad + I(1,0)\left(\cos\left(-\pi\cdot 1\right)+\sqrt{-1}\sin\left(-\pi\cdot 1\right)\right) \\
&\quad + I(1,1)\left(\cos\left(-\pi\cdot 1\right)+\sqrt{-1}\sin\left(-\pi\cdot 1\right)\right) \\
&= 5\left(1+0i\right)+7\left(1-0i\right)+8\left(-1+0i\right)+3\left(-1-0i\right) \\
&= 5+7-8-3 \\
&= 1
\end{aligned}
$$

Finally:

$$
\begin{aligned}
\tilde{I}(1,1) &= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)e^{-\sqrt{-1}\frac{2\pi}{2}(i+j)} \\
&= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)e^{-\sqrt{-1}\pi(i+j)} \\
&= \sum_{i=0}^{1}\sum_{j=0}^{1} I(i,j)\left(\cos\left(-\pi(i+j)\right)+\sqrt{-1}\sin\left(-\pi(i+j)\right)\right)
\end{aligned}
$$

Plugging in the values of i and j:

$$\begin{aligned}
\tilde{I}(0,1) &= I(0,0)\left(\cos\left(-\pi\cdot(0+0)\right)+\sqrt{-1}\sin\left(-\pi\cdot(0+0)\right)\right) \\
&+ I(0,1)\left(\cos\left(-\pi\cdot(0+1)\right)+\sqrt{-1}\sin\left(-\pi\cdot(0+1)\right)\right) \\
&+ I(1,0)\left(\cos\left(-\pi\cdot(1+0)\right)+\sqrt{-1}\sin\left(-\pi\cdot(1+0)\right)\right) \\
&+ I(1,1)\left(\cos\left(-\pi\cdot(1+1)\right)+\sqrt{-1}\sin\left(-\pi\cdot(1+1)\right)\right) \\
&= 5\,(1+0i)+7\,(-1+0i)+8\,(-1+0i)+3\,(1+0i) \\
&= 5-7-8+3 \\
&= -7
\end{aligned}$$

This gives the DFT Matrix

$$\begin{bmatrix} 21 & 3 \\ 1 & -7 \end{bmatrix}$$