

UNIVERSITY OF HOUSTON

MIDTERM REVIEW

COSC 4393
Digital Image Processing

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For an $M \times N$ image I , the **Discrete Fourier Transform** at a pixel u, v is given by

$$\tilde{I}(u, v) = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} I(i, j) e^{-2\pi \sqrt{-1} \left(\frac{ui}{M} + \frac{vj}{N} \right)}$$

For a square image, we have $N = M$ and

$$\tilde{I}(u, v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j) e^{-\sqrt{-1} \frac{2\pi}{N} (ui + vj)}$$

A complex-valued function f is **Conjugate Symmetric** or **Hermitian** if $f^*(z) = f(-z)$. In the case of the DFT, we have

$$\begin{aligned} \tilde{I}(-u, -v) &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j) e^{-\sqrt{-1} \frac{2\pi}{N} (-ui - vj)} \\ &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j) \left(\cos \left(-\frac{2\pi}{N} (-ui - vj) \right) + \sqrt{-1} \sin \left(-\frac{2\pi}{N} (-ui - vj) \right) \right) \\ &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j) \left(\cos \left(-\frac{2\pi}{N} (ui + vj) \right) - \sqrt{-1} \sin \left(-\frac{2\pi}{N} (ui + vj) \right) \right) \\ &= \tilde{I}^*(u, v) \end{aligned}$$

Equivalently, a complex valued function is Conjugate Symmetric if and only if its real part is even and its imaginary part is odd. The real part of the DFT is just

$$\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j) \cos \left(-\sqrt{-1} \frac{2\pi}{N} (ui + vj) \right)$$

and the imaginary part is

$$\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j) \sin \left(-\sqrt{-1} \frac{2\pi}{N} (ui + vj) \right)$$

which are clearly even and odd, respectively.

The **magnitude** of a complex number $z = x + iy$ is given by $|z| = \sqrt{x^2 + y^2}$. To see that the magnitude of the DFT is symmetric, note that $|z| = |z^*|$ for any complex number z . Since $\tilde{I}^*(u, v) = \tilde{I}(-u, -v)$,

$$|\tilde{I}(u, v)| = |\tilde{I}^*(u, v)| = |\tilde{I}(-u, -v)|$$

To see that the DFT is periodic, show that $\tilde{I}(u + kN, v + lN) = \tilde{I}(u, v)$. We write

$$W_N^{ui + vj} = e^{-\sqrt{-1} \frac{2\pi}{N} (ui + vj)}$$

$$\begin{aligned} \tilde{I}(u + kN, v + lN) &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j) W_N^{(u+kN)i + (v+lN)j} \\ &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j) W_N^{ui + vj + N(ki + lj)} \\ &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j) W_N^{ui + vj} \cdot W_N^{N(ki + lj)} \end{aligned}$$

Then, we only need to show that $W_N^{N(ki + lj)} = 1$. This is straightforward:

$$\begin{aligned}
W_N^{N(ki+lj)} &= e^{-\sqrt{-1}\frac{2\pi}{N}(N(ki+lj))} \\
&= e^{-\sqrt{-1}\cdot 2\pi(ki+lj)} \\
&= \cos(2\pi(ki+lj)) + \sqrt{-1}\sin(2\pi(ki+lj)) \\
&= 1
\end{aligned}$$

To compute the DFT of the following matrix:

$$\begin{bmatrix} 5 & 7 \\ 8 & 3 \end{bmatrix}$$

We evaluate

$$\begin{aligned}
\tilde{I}(0,0) &= \sum_{i=0}^1 \sum_{j=0}^1 I(i,j) e^{-\sqrt{-1}\frac{2\pi}{2}(0i+0j)} \\
&= \sum_{i=0}^1 \sum_{j=0}^1 I(i,j) \\
&= I(0,0) + I(0,1) + I(1,0) + I(1,1) \\
&= 5 + 7 + 8 + 3 \\
&= 23
\end{aligned}$$

$$\begin{aligned}
\tilde{I}(0,1) &= \sum_{i=0}^1 \sum_{j=0}^1 I(i,j) e^{-\sqrt{-1}\frac{2\pi}{2}(0i+j)} \\
&= \sum_{i=0}^1 \sum_{j=0}^1 I(i,j) e^{-\sqrt{-1}\pi j} \\
&= \sum_{i=0}^1 \sum_{j=0}^1 I(i,j) (\cos(-\pi j) + \sqrt{-1}\sin(-\pi j))
\end{aligned}$$

Plugging in the values of i and j :

$$\begin{aligned}
\tilde{I}(0,1) &= I(0,0) (\cos(-\pi \cdot 0) + \sqrt{-1}\sin(-\pi \cdot 0)) \\
&\quad + I(0,1) (\cos(-\pi \cdot 1) + \sqrt{-1}\sin(-\pi \cdot 1)) \\
&\quad + I(1,0) (\cos(-\pi \cdot 0) + \sqrt{-1}\sin(-\pi \cdot 0)) \\
&\quad + I(1,1) (\cos(-\pi \cdot 1) + \sqrt{-1}\sin(-\pi \cdot 1)) \\
&= 5(1+0i) + 7(-1-0i) + 8(1+0i) + 3(-1-0i) \\
&= 5 - 7 + 8 - 3 \\
&= 3
\end{aligned}$$

$$\begin{aligned}
\tilde{I}(1,0) &= \sum_{i=0}^1 \sum_{j=0}^1 I(i,j) e^{-\sqrt{-1}\frac{2\pi}{2}(i+0j)} \\
&= \sum_{i=0}^1 \sum_{j=0}^1 I(i,j) e^{-\sqrt{-1}\pi i} \\
&= \sum_{i=0}^1 \sum_{j=0}^1 I(i,j) (\cos(-\pi i) + \sqrt{-1}\sin(-\pi i))
\end{aligned}$$

Plugging in the values of i and j :

$$\begin{aligned}
\tilde{I}(1, 0) &= I(0, 0) (\cos(-\pi \cdot 0) + \sqrt{-1} \sin(-\pi \cdot 0)) \\
&\quad + I(0, 1) (\cos(-\pi \cdot 0) + \sqrt{-1} \sin(-\pi \cdot 0)) \\
&\quad + I(1, 0) (\cos(-\pi \cdot 1) + \sqrt{-1} \sin(-\pi \cdot 1)) \\
&\quad + I(1, 1) (\cos(-\pi \cdot 1) + \sqrt{-1} \sin(-\pi \cdot 1)) \\
&= 5(1 + 0i) + 7(1 - 0i) + 8(-1 + 0i) + 3(-1 - 0i) \\
&= 5 + 7 - 8 - 3 \\
&= 1
\end{aligned}$$

Finally:

$$\begin{aligned}
\tilde{I}(1, 1) &= \sum_{i=0}^1 \sum_{j=0}^1 I(i, j) e^{-\sqrt{-1} \frac{2\pi}{2} (i+j)} \\
&= \sum_{i=0}^1 \sum_{j=0}^1 I(i, j) e^{-\sqrt{-1} \pi (i+j)} \\
&= \sum_{i=0}^1 \sum_{j=0}^1 I(i, j) (\cos(-\pi(i+j)) + \sqrt{-1} \sin(-\pi(i+j)))
\end{aligned}$$

Plugging in the values of i and j :

$$\begin{aligned}
\tilde{I}(0, 1) &= I(0, 0) (\cos(-\pi \cdot (0+0)) + \sqrt{-1} \sin(-\pi \cdot (0+0))) \\
&\quad + I(0, 1) (\cos(-\pi \cdot (0+1)) + \sqrt{-1} \sin(-\pi \cdot (0+1))) \\
&\quad + I(1, 0) (\cos(-\pi \cdot (1+0)) + \sqrt{-1} \sin(-\pi \cdot (1+0))) \\
&\quad + I(1, 1) (\cos(-\pi \cdot (1+1)) + \sqrt{-1} \sin(-\pi \cdot (1+1))) \\
&= 5(1 + 0i) + 7(-1 + 0i) + 8(-1 + 0i) + 3(1 + 0i) \\
&= 5 - 7 - 8 + 3 \\
&= -7
\end{aligned}$$

This gives the DFT Matrix

$$\begin{bmatrix} 21 & 3 \\ 1 & -7 \end{bmatrix}$$

The **Inverse Discrete Fourier Transform** is given by

$$I(i, j) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{I}(u, v) e^{\sqrt{-1} \frac{2\pi}{N} (ui+vj)}$$

Its periodicity can be shown directly, as done previously.

The **convolution** of two functions f and g is denoted $f * g$ or $f \otimes g$

$$(f \otimes g)(n) = \sum_{m=-\infty}^{\infty} f(m)g(n-m)$$

If $\mathcal{F}(f)$ denotes the Discrete Fourier Transformation of F , then

$$\mathcal{F}(f \otimes g) = \mathcal{F}(f) \times \mathcal{F}(g)$$

since

$$\begin{aligned}
\mathcal{F}(f \otimes g)(\mu) &= \mathcal{F}\left(\sum_{m=-\infty}^{\infty} f(m)g(n-m)\right) \\
&= \sum_{t=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} f(m)g(n-m)\right) e^{-2\pi i \mu t} \\
&= \sum_{m=-\infty}^{\infty} f(n) \left(\sum_{t=-\infty}^{\infty} g(n-m) e^{-2\pi i \mu t}\right) \\
&= \sum_{m=-\infty}^{\infty} f(n) \left(\sum_{t=-\infty}^{\infty} g(n-m) e^{-2\pi i \mu (t-m)}\right) e^{-2\pi i m \mu} \\
&= \sum_{m=-\infty}^{\infty} f(n) (\mathcal{F}(g)(\mu)) e^{-2\pi i m \mu} \\
&= \mathcal{F}(g)(\mu) \left(\sum_{m=-\infty}^{\infty} f(n) e^{-2\pi i m \mu}\right) \\
&= \mathcal{F}(g)(\mu) \mathcal{F}(f)(\mu) \\
&= \mathcal{F}(f)(\mu) \mathcal{F}(g)(\mu)
\end{aligned}$$