Factorial Problem

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Definition 1. For any $a > 1$, let $\tau(a)$ denote the smallest integer n such that

$$a^n < n!$$

Theorem 1. $\tau(a)$ is well-defined, i.e., for any $a > 1$, there exists an n such that

$$a^n < n!$$

Proof. Consider the sequence $h(n) = \frac{a^n}{n!}$. We see that

$$\begin{aligned} \lim_{n\to\infty} \frac{h(n+1)}{h(n)} &= \lim_{n\to\infty} \frac{a}{n+1} \\ &= 0 \end{aligned}$$

By the ratio test, $\sum_{n=0}^{\infty} h(n)$ converges. Thus, the sequence $h(n) = \frac{a^n}{n!}$ converges to 0. Then, for every $\epsilon > 0$, there exists an n_0 such that, for all $n \geq n_0$,

$$\left| \frac{a^n}{n!} \right| < \epsilon$$

In particular, taking $\epsilon = 1$, we have $a^n < n!$ for all $n \geq n_0$. □

Lemma 1. For all $n > 1$, $n! < n^n$.

Proof. Proceed by induction on n. The base case is $2! = 2 < 4 = 2^2$. Suppose that, for some $k > 1$, $k! < k^k$. Then

$$\begin{aligned} (k+1)! &= k! \cdot (k+1) \\ &< k^k \cdot (k+1) \text{ by the induction hypothesis} \\ &< (k+1)^k \cdot (k+1) \\ &= (k+1)^{k+1} \end{aligned}$$

□

Theorem 2. For any integer $n > 1$, $\tau(n) > n$.

Proof. This follows directly from the above lemma: since $n! < n^n$, $\tau(n) > n$. □

In fact, we can tighten this lower bound to $\tau(n) > 2n$. First, we show the following lemma:

Lemma 2. For any $n > 0$, $n^{2n} \left(4 - \frac{2}{n+1}\right) < (n+1)^{2n}$.

Proof. Proceed again by induction. The base case is $1^{2\cdot 1}(4 - \frac{2}{2}) = 3 < 4 = 2^2$. □

Theorem 3. For any integer $n > 2$, $\tau(n) > 2n$.

Proof. We shall show by induction that $n^{2n} > (2n)!$. Begin with the base case:

$$(3 \cdot 2)! = 6! = 720 < 729 = 3^6 = 3^{2\cdot 3}$$

Suppose that, for some $k > 2$, $(2k)! < k^{2k}$. Then, for $k + 1$, we have

$$\begin{aligned} (2(k+1))! &= (2k+2)! \\ &= (2k)!(2k+1)(2k+2) \\ &< k^{2k}(2k+1)(2k+2) \text{ by our Induction Hypothesis} \\ &= k^{2k}(k+1)^2 \left(4 - \frac{2}{k+1}\right) \\ &< (k+1)^{2k}(k+1)^2 \text{ by Lemma 2} \\ &= (k+1)^{2(k+1)} \end{aligned}$$

□