Factorial Problem

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Definition 1. For any $a > 1$, let $\tau(a)$ denote the smallest integer n such that

$$a^n < n!$$

Theorem 1. $\tau(a)$ is well-defined, i.e., for any $a > 1$, there exists an n such that

$$a^n < n!$$

Proof. Consider the sequence $h(n) = \frac{a^n}{n!}$. We see that

$$\begin{aligned}\lim_{n\to\infty} \frac{h(n+1)}{h(n)} &= \lim_{n\to\infty} \frac{a}{n+1} \\ &= 0\end{aligned}$$

By the ratio test, $\sum_{n=0}^{\infty} h(n)$ converges. Thus, the sequence $h(n) = \frac{a^n}{n!}$ converges to 0. Then, for every $\epsilon > 0$, there exists an n_0 such that, for all $n \geq n_0$,

$$\left|\frac{a^n}{n!}\right| < \epsilon$$

In particular, taking $\epsilon = 1$, we have $a^n < n!$ for all $n \geq n_0$. $\square$

Lemma 1. For all $n > 1$, $n! < n^n$.

Proof. Proceed by induction on n. The base case is $2! = 2 < 4 = 2^2$. Suppose that, for some $k > 1$, $k! < k^k$. Then

$$\begin{aligned}(k+1)! &= k! \cdot (k+1) \\ &< k^k \cdot (k+1) \text{ by the induction hypothesis} \\ &< (k+1)^k \cdot (k+1) \\ &= (k+1)^{k+1}\end{aligned}$$

$\square$

Theorem 2. For any integer $n > 1$, $\tau(n) > n$.

Proof. This follows directly from the above lemma: since $n! < n^n$, $\tau(n) > n$. $\square$

In fact, we can tighten this lower bound to $\tau(n) > 2n$. First, we show the following lemma:

Lemma 2. For any $n > 0$, $n^{2n}\left(4 - \frac{2}{n+1}\right) < (n+1)^{2n}$.

Proof. This is equivalent to showing that

$$4 - \frac{2}{n+1} < \left(\frac{n+1}{n}\right)^{2n}$$

However, the right hand side of the above inequality is simply the square of the sequence $\left(1 + \frac{1}{n}\right)^n$, which is an increasing sequence that converges to e. It suffices, therefore, to find show that the inequality holds for $n = 1$, since the left-hand-side is bounded above by 4:

$$4 - \frac{2}{1+1} = 4 - 1 = 3 < 4 = \left(\frac{1+1}{1}\right)^{2\cdot 1}$$

$\square$

Theorem 3. For any integer $n > 2$, $\tau(n) > 2n$.

Proof. We shall show by induction that $n^{2n} > (2n)!$. Begin with the base case:

$$(2 \cdot 3)! = 6! = 720 < 729 = 3^6 = 3^{2 \cdot 3}$$

Suppose that, for some $k > 2$, $(2k)! < k^{2k}$. Then, for $k+1$, we have

$$\begin{aligned}
(2(k+1))! &= (2k+2)! \\
&= (2k)!(2k+1)(2k+2) \\
&< k^{2k}(2k+1)(2k+2) \text{ by our Induction Hypothesis} \\
&= k^{2k}(k+1)^2 \left(4 - \frac{2}{k+1}\right) \\
&< (k+1)^{2k}(k+1)^2 \text{ by Lemma 2} \\
&= (k+1)^{2(k+1)}
\end{aligned}$$

Thus, $\tau(n) > 2n$. $\square$