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Notes

COSC 3340
Intro. to Automata and Computability

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1 Formal Languages

1.1 Introduction

Definition 1.1.1. An **Alphabet** is a finite, non-empty set of atomic symbols.

Definition 1.1.2. A **word** or **string** is any finite sequence of symbols from an alphabet.

Definition 1.1.3. The **length** of a string, s, denoted $|s|$, is the number of symbols in s.

Definition 1.1.4. Given strings $s = s_1 s_2 \dots s_n$ and $t = t_1 t_2 \dots t_m$, their **concatenation** is defined

$$s \cdot t = s_1 s_2 \cdots s_n t_1 t_2 \cdots t_m$$

We denote by ε the **empty string**, the unique string of 0 characters.

Definition 1.1.5. Let A be any alphabet. The **Kleene Closure** of A, denoted A^*, is the set of all strings of any length over A.

Theorem 1.1.1. Let A be any finite set. Then A^* is countably infinite.

Proof. That A^* is infinite is straightforward: since A is non-empty, take $a \in A$. Then $\{a, aa, aaa, \dots\} \subseteq A^*$.

To see that it is countable, we first write $|A| = n$. Now, consider the set of all strings of length 0. This is simply $\{\varepsilon\}$. Moreover, there are n strings of length 1, n^2 strings of length 2, n^3 strings of length 3, and so on. Thus, we map ε to 0, the strings of length 1 to $1, 2, \dots, n$, the strings of length 2 to $n+1, n+2, \dots, n+n^2$, the strings of length 3 to $n+n^2+1, n+n^2+2, \dots, n+n^2+n^3$, and so on. This is a bijection from A^* to $\mathbb{N}$, which completes the proof. □

Definition 1.1.6. Given an alphabet A, a **formal language** or simply **language** L is any subset of A^*.

Theorem 1.1.2. Given an alphabet A, the set of languages over A is uncountable.

Proof. Suppose, by way of contradiction, that the set of languages were countable, i.e., that we can enumerate the set as $\{L_1, L_2, L_3, \dots\}$. Consider the set of all strings $\{s_1, s_2, s_3, \dots\}$. Let L be the language defined as follows:

$$s_i \in L \text{ if and only if } s_i \notin L_i$$

To see that L is not in the above list, consider s_i. If s_i is in L, then s_i is not in L_i, by construction, and $L \neq L_i$. Similarly, if s_i is not in L, then s_i must be in L_i, by construction, and $L \neq L_i$. In other words, for all i, $L \neq L_i$. Then L is not in the above list, which is a contradiction. Hence, the set of languages is uncountable. □

All set operations, such as union, intersection, complement, set-difference, etc. can be applied to languages, since languages are simply subsets of a Kleene Closure of an alphabet.

Definition 1.1.7. Given two languages L_1 and L_2, the concatenation $L_1 \cdot L_2$ is given by

$$L_1 \cdot L_2 = \{s \cdot t | s \in L_1 \text{ and } t \in L_2\}$$

Clearly, we have

$$L \cdot \emptyset = \emptyset = \emptyset \cdot L$$
$$L \cdot \{\varepsilon\} = L = \{\varepsilon\} \cdot L$$

Note that $L_1 \cdot L_2$ is not the same as $L_1 \times L_2$. Let $L_1 = L_2 = \{\varepsilon, 0, 00\}$. Then

$$L_1 \times L_2 = \{(\varepsilon,\varepsilon), (\varepsilon,0), (\varepsilon,00), (0,\varepsilon), (0,0), (0,00), (00,\varepsilon), (00,0), (00,00)\}$$

whereas

$$L_1 \cdot L_2 = \{\varepsilon, 0, 00, 000, 0000\}$$

Definition 1.1.8. Given a language L, the **Kleene Closure** of L, L^*, is

$$L^* = \bigcup_{i=0}^{\infty} L^i$$

where

$$L^i = \begin{cases} \{\varepsilon\} & \text{if } i = 0 \\ L \cdot L^{i-1} & \text{otherwise} \end{cases}$$

Note that, while 0^0 is normally left undefined, we define $\emptyset^0 = \{\varepsilon\}$.

Theorem 1.1.3. L^* is finite if and only if $L = \emptyset$ or $L = \{\varepsilon\}$.

Proof. If $L = \emptyset$, then $L^i = \emptyset^i = \emptyset$ for $i > 0$. Then

$$\begin{aligned} \emptyset^* &= \bigcup_{i=0}^{\infty} \emptyset^i \\ &= \emptyset^0 \cup \bigcup_{i=1}^{\infty} \emptyset^i \\ &= \{\varepsilon\} \cup \bigcup_{i=1}^{\infty} \emptyset \\ &= \{\varepsilon\} \end{aligned}$$

Similarly, if $L = \{\varepsilon\}$, then $L^i = \{\varepsilon\}$ for all i, and

$$\begin{aligned} \{\varepsilon\}^* &= \bigcup_{i=0}^{\infty} \{\varepsilon\}^i \\ &= \bigcup_{i=1}^{\infty} \{\varepsilon\} \\ &= \{\varepsilon\} \end{aligned}$$

However, if L is neither $\emptyset$ nor $\{\varepsilon\}$, then there exists a string $s \in L$ with length at least 1. Then $s, ss, sss, \ldots$, are in L^*, hence L^* is infinite. □

1.2 Regular Languages

1.2.1 Finite Automata

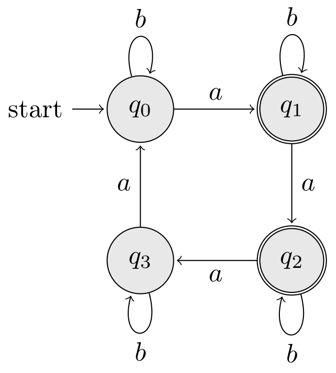
Definition 1.2.1. A **Deterministic Finite-State Automata** (DFA) or **Finite-State Machine** is a quintuple $(A, Q, \tau, q_0, \mathcal{F})$ where

A is the **alphabet**
Q is a finite, non-empty **set of states**
$\tau : Q \times A \to Q$ is the **transition function**
q_0 is the **initial state**
$\mathcal{F} \subseteq Q$ is the set of **final states**

We can extend τ as follows:
$\tau^* : Q \times A^* \to Q$

$$\tau^*(q, s) = \begin{cases} q & \text{if } s = \varepsilon \\ \tau^*(\tau(q, s_0), s') & \text{if } s = s_0 \cdot s' \end{cases}$$

We proceed informally and use τ to refer to τ^*.
Consider the following DFA:



The figure indicates that we begin at state q_0. The double-circles for states q_1 and q_2 indicate that they are accepting or final states. An arrow indicates the state to move to after receiving an input. For example, if we receive the input string *abba*, we begin at state q_0 and receive a, so we move to state q_1. We then receive b and stay in q_1. We repeat this for the next symbol, b, and then move to q_2 upon receiving the final a. Since q_2 is a final state, we say that this DFA **accepts** the string *abba*.

We can represent the above DFA using a table, as follows:

		a	b	
$\rightarrow$	q_0	q_1	q_0	0
	q_1	q_2	q_1	1
	q_2	q_3	q_2	1
	q_3	q_0	q_3	0

The first column indicates the states, while the first row indicates the symbols. The final column indicates whether a state is accepting: 0 refers to a non-final state, 1 to a final state. The remaining values indicate the transition function τ, e.g. $\tau(q_0, a) = q_1$, indicated by the entry corresponding to row q_0 and column a. Finally, the arrow pointing to q_0 indicates that it is the starting position.

Definition 1.2.2. Let $\underset{\sim}{D}$ be some DFA. Then $L(\underset{\sim}{D})$, the language accepted by the DFA, is

$$\{s \in A^* | \tau(q_0, s) \in \mathcal{F}\}$$

Definition 1.2.3. A language is **regular** if and only if there exists a DFA that accepts it.

Definition 1.2.4. A **Non-Deterministic Finite-State Automata** (NFA) is a quintuple $(A, Q, \tau, q_0, \mathcal{F})$ where

A is the **alphabet**

Q is a finite, non-empty **set of states**

$\tau : Q \times A \to 2^Q$ is the **transition function**

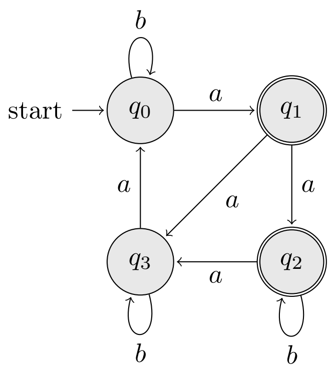
q_0 is the **initial state**

$\mathcal{F} \subseteq Q$ is the set of **final states**

We can extend τ as follows:
$\tau^* : 2^Q \times A^* \to 2^Q$

$$\tau^*(P, s) = \begin{cases} P & \text{if } s = \varepsilon \\ \tau^*\left(\bigcup_{q \in P} \tau(q, s_0), s'\right) & \text{if } s = s_0 \cdot s' \end{cases}$$

We proceed informally and use τ to refer to τ^*.
Consider the following NFA:



The diagrams for an NFA and DFA follow the same notation. However, the notation for the table differs slightly:

		a	b	
$\rightarrow$	q_0	q_1	q_0	0
	q_1	q_2q_3	$\emptyset$	1
	q_2	q_3	q_2	1
	q_3	q_0	q_3	0

The values of the transition function are now sets. We informally refer to the set $\{q_0\}$ by q_0, and similarly the set $\{q_2, q_3\}$ by q_2q_3. In some cases, to avoid ambiguity, we will use commas, e.g. we may represent $\{q_2, q_3\}$ as q_2, q_3. We similarly say, given a string s, if there exists a path through an NFA that ends in a final state, we say that the NFA **accepts** s.

Similarly, we define the set of languages accepted by an NFA $\underset{\sim}{N}$, $L(\underset{\sim}{N})$, as

$$L(\underset{\sim}{N}) = \{s \in A^* | \tau(q_0, s) \cap \mathcal{F} \neq \emptyset\}$$

We can convert an NFA to a DFA on the powerset 2^Q by using the **subset construction**: begin with the initial state and traverse the NFA, adding unseen states to the left-most column until all paths have been exhausted. For example, with our NFA above, we begin with:

		a	b	
$\rightarrow$	q_0	q_1	q_0	0

q_0 has already been seen, so we ignore it. q_1 is new, so we add it to the table:

		a	b
$\rightarrow$	q_0	q_1	q_0
	q_1		

We now visit the corresponding states of q_1, which are q_2q_3 and $\emptyset$, both of which have not yet been visited.

		a	b
$\rightarrow$	q_0	q_1	q_0
	q_1	q_2q_3	$\emptyset$
	q_2q_3		
	$\emptyset$		

When q_2 receives a, it transitions to state q_3. When q_3 receives a, it transitions to state q_0, so q_2q_3 transitions to q_0q_3. Similarly, q_2q_3 transitions to state q_2q_3 when it receives b.

		a	b
$\rightarrow$	q_0	q_1	q_0
	q_1	q_2q_3	$\emptyset$
	q_2q_3	q_0q_3	q_2q_3
	$\emptyset$		

The empty set transitions to the empty set, by definition.

		a	b
$\rightarrow$	q_0	q_1	q_0
	q_1	q_2q_3	$\emptyset$
	q_2q_3	q_0q_3	q_2q_3
	$\emptyset$	$\emptyset$	$\emptyset$

q_0q_3 has not yet been visited, so we add it to the left-most column:

		a	b
$\rightarrow$	q_0	q_1	q_0
	q_1	q_2q_3	$\emptyset$
	q_2q_3	q_0q_3	q_2q_3
	$\emptyset$	$\emptyset$	$\emptyset$
	q_0q_3		

Then we visit its corresponding states:

		a	b
$\rightarrow$	q_0	q_1	q_0
	q_1	q_2q_3	$\emptyset$
	q_2q_3	q_0q_3	q_2q_3
	$\emptyset$	$\emptyset$	$\emptyset$
	q_0q_3	q_0q_1	q_0q_3

Continuing, we end with the following DFA:

		a	b
$\rightarrow$	q_0	q_1	q_0
	q_1	q_2q_3	$\emptyset$
	q_2q_3	q_0q_3	q_2q_3
	$\emptyset$	$\emptyset$	$\emptyset$
	q_0q_3	q_0q_1	q_0q_3
	q_0q_1	$q_1q_2q_3$	q_0
	$q_1q_2q_3$	$q_0q_2q_3$	q_2q_3
	$q_0q_2q_3$	$q_0q_1q_3$	$q_0q_2q_3$
	$q_0q_1q_3$	$q_0q_1q_2q_3$	q_0q_3
	$q_0q_1q_2q_3$	$q_0q_1q_2q_3$	$q_0q_2q_3$

However, we need to include the accepting states. The accepting states of the NFA are q_1 and q_2, and thus any state including either state is accepting:

		a	b	
$\rightarrow$	q_0	q_1	q_0	0
	q_1	q_2q_3	$\emptyset$	1
	q_2q_3	q_0q_3	q_2q_3	1
	$\emptyset$	$\emptyset$	$\emptyset$	0
	q_0q_3	q_0q_1	q_0q_3	0
	q_0q_1	$q_1q_2q_3$	q_0	1
	$q_1q_2q_3$	$q_0q_2q_3$	q_2q_3	1
	$q_0q_2q_3$	$q_0q_1q_3$	$q_0q_2q_3$	1
	$q_0q_1q_3$	$q_0q_1q_2q_3$	q_0q_3	1
	$q_0q_1q_2q_3$	$q_0q_1q_2q_3$	$q_0q_2q_3$	1

Note that an NFA does not necessarily admit a DFA with as many states. Consider the following example:

		a	b	
$\to$	0	$\{1,2,\ldots,n\}$	0	0
	1	2	1	0
	2	3	2	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	i	$i+1$	i	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$n-1$	n	$n-1$	0
	n	1	n	1

The NFA above admits the following DFA:

		a	b	
$\to$	0	$\{1,2,\ldots,n\}$	0	0
	$\{1,2,\ldots,n\}$	$\{1,2,\ldots,n\}$	$\{1,2,\ldots,n\}$	1

The above DFA contains only 2 states, despite the NFA containing $n+1$ states.

For an NFA, there is no guarantee of a unique smallest NFA which accepts the same strings. However, for a DFA, such a notion exists.

Consider two states, p and q, and corresponding L_p and L_q, where L_p has initial state p and L_q has initial state q. We say that p and q are distinguishable if there exists a string s such that s is in L_p and not in L_q, or vice-versa. We use this notion to **reduce** a DFA.

Begin with a partition of Q into subsets $\mathcal{F}$ and $Q - \mathcal{F}$, i.e., the accepting and rejecting states. For a pair of states p, q if the result of transitioning p and q falls into different partitions, we partition the subset and continue.

For example, given the following DFA:

		a	b	
$\to$	0	1	2	0
	1	2	3	1
	2	3	4	0
	3	0	5	1
	4	5	6	0
	5	6	7	1
	6	7	0	0
	7	4	1	1

We have two partitions:

Rejecting	Accepting
0, 2, 4, 6	1, 3, 5, 7

Now, 0 gets sent to the accepting partition by a and to the rejecting partition by b. Similarly, 2, 4, and 6 get sent to the accepting partition by a and to the rejecting partition by b. Thus, they belong to the same partition.

In the same vein, 1 gets sent to the rejecting partition by a and to the accepting partition by b. Similarly, 3, 5, and 7 get sent to the rejecting partition by a and to the accepting partition by b. Thus, our next partition is

Rejecting	Accepting
0, 2, 4, 6	1, 3, 5, 7
0, 2, 4, 6	1, 3, 5, 7

That our row is the same as the preceding one indicates that we have finished, and now have a minimal DFA. Call the first subset p and the second q. When an element in p receives a, it is sent to q. When it receives b, it is sent to p. Similar logic for q gives our new DFA:

		a	b	
$\to$	p	q	p	0
	q	p	q	1

Recall that p began as a subset of the rejecting elements and q the accepting elements, which informs the last column of the above table.

Not all DFAs can be reduced. An obvious example is the above reduced DFA. For a less trivial example, consider the following DFA:

		a	b	
$\rightarrow$	0	1	2	0
	1	2	3	1
	2	3	4	0
	3	0	5	1
	4	5	6	0
	5	6	7	1
	6	7	0	0
	7	4	2	1

Begin, as in the previous problem, with two partitions:

Rejecting	Accepting
0, 2, 4, 6	1, 3, 5, 7

As in the previous problem, 0, 2, 4, and 6 get sent to the same partition under a and b, respectively. Under a, 1, 3, 5, and 7 go to the rejecting partition. However, under b, 7 goes to the rejecting partition while 1, 3, and 5 go to the accepting partition, which means we must create a new partition for 7.

Rejecting	Accepting	
0, 2, 4, 6	1, 3, 5, 7	
0, 2, 4, 6	1, 3, 5	7

We continue the process, noting that there is no need to consider singletones, i.e., the partition $\{7\}$ is already in its finale state. Under a, 0, 2, and 4 get sent to the $\{1, 3, 5\}$ partition. Under b, they get sent to the $\{0, 2, 4, 6\}$ partition. However, 6 gets sent to the $\{7\}$ partition, and so it must be partitioned separately. Similarly, 1 and 3 get sent to the $\{0, 2, 4, 6\}$ partition under a, and to the $\{1, 3, 5\}$ partition under b. 5, on the other hand, gets sent to the $\{7\}$ partition, and must be partitioned separately. In total, we have:

Rejecting		Accepting		
0, 2, 4, 6		1, 3, 5, 7		
0, 2, 4, 6		1, 3, 5		7
0, 2, 4	6	1, 3	5	7

We continue:

Rejecting				Accepting			
0, 2, 4, 6				1, 3, 5, 7			
0, 2, 4, 6				1, 3, 5			7
0, 2, 4			6	1, 3		5	7
0, 2		4	6	1	3	5	7
0	2	4	6	1	3	5	7

Notice that the reduced DFA has 8 states, like the original! This means that the original DFA is already reduced, and cannot be reduced further.

1.2.2 Regular Expressions

Definition 1.2.5. Given an alphabet A, we define a **regular expression**

(a)
- $a \in A$ is a regular expression denoting the language $\{a\}$
- ε is a regular expression denoting $\{\varepsilon\}$
- $\emptyset$ is a regular expression denoting $\emptyset$

(b) If α and β are regular expressions denoting the languages $L(\alpha)$ and $L(\beta)$, respectively, then
- $\alpha \cup \beta$ denotes $L(\alpha) \cup L(\beta)$
- $\alpha \cdot \beta$ denotes $L(\alpha) \cdot L(\beta)$
- α^* denotes $L(\alpha)^*$

1.2.3 Regular Grammars

1.2.4 Solutions of Certain Language Equations

1.3 Accepted by Turing Machines

1.4 Exercise Set 1

Exercise 1: Construct DFAs for the following NFAs using the subset construction:

(a)

		a	
$\rightarrow$	1	2	0
	2	3	0
	3	4	0
	4	5	0
	5	6	0
	6	7	0
	7	1, 2	1

(b)

		a	b	c	
$\rightarrow$	1	2	2	2	1
	2	3	1	1, 2	1
	3	4	3	$\emptyset$	1
	4	5	4	4	1
	5	1	5	5	1

(c)

		a	b	c	
$\rightarrow$	1	2	2	2	1
	2	3	1	2, 3	1
	3	4	3	$\emptyset$	1
	4	5	4	4	1
	5	1	5	5	1

Solution. (a)

		a	
$\rightarrow$	1	2	0
	2	3	0
	3	4	0
	4	5	0
	5	6	0
	6	7	0
	7	1, 2	1
	1, 2	2, 3	0
	2, 3	3, 4	0
	3, 4	4, 5	0
	4, 5	5, 6	0
	5, 6	6, 7	1
	6, 7	1, 2, 7	1
	1, 2, 7	1, 2, 3	0
	1, 2, 3	2, 3, 4	0
	2, 3, 4	3, 4, 5	0
	3, 4, 5	4, 5, 6	0
	4, 5, 6	5, 6, 7	1
	5, 6, 7	1, 2, 6, 7	1
	1, 2, 6, 7	1, 2, 3, 7	1
	1, 2, 3, 7	1, 2, 3, 4	0
	1, 2, 3, 4	2, 3, 4, 5	0
	2, 3, 4, 5	3, 4, 5, 6	0
	3, 4, 5, 6	4, 5, 6, 7	1
	4, 5, 6, 7	1, 2, 5, 6, 7	1
	1, 2, 5, 6, 7	1, 2, 3, 6, 7	1
	1, 2, 3, 6, 7,	1, 2, 3, 4, 7	1
	1, 2, 3, 4, 7	1, 2, 3, 4, 5	0
	1, 2, 3, 4, 5	2, 3, 4, 5, 6	0
	2, 3, 4, 5, 6	3, 4, 5, 6, 7	1
	3, 4, 5, 6, 7	1, 2, 4, 5, 6, 7	1
	1, 2, 4, 5, 6, 7	1, 2, 3, 5, 6, 7	1
	1, 2, 3, 5, 6, 7	1, 2, 3, 4, 6, 7	1
	1, 2, 3, 4, 6, 7	1, 2, 3, 5, 7	1
	1, 2, 3, 5, 7	1, 2, 3, 4, 6	0
	1, 2, 3, 4, 6	2, 3, 4, 5, 7	1
	2, 3, 4, 5, 7	1, 2, 3, 4, 5, 6	0
	1, 2, 3, 4, 5, 6	2, 3, 4, 5, 6, 7	1
	2, 3, 4, 5, 6, 7	1, 2, 3, 4, 5, 6, 7	1
	1, 2, 3, 4, 5, 6, 7	1, 2, 3, 4, 5, 6, 7	1

$\square$

Exercise 2: Reduce the following DFAs:

(a)

		a	b	
$\rightarrow$	1	2	3	0
	2	3	2	1
	3	4	5	0
	4	1	8	1
	5	6	7	0
	6	7	6	1
	7	8	1	0
	8	5	4	1

(b)

		a	b	
$\rightarrow$	1	2	3	0
	2	3	2	1
	3	4	5	0
	4	1	8	1
	5	6	7	0
	6	7	6	1
	7	8	1	0
	8	5	5	1

(c) Your result of 1(b).

(d) Your result of 1(c).

Exercise 3: Construct NFAs for the following regular expressions using the construction given in class; then find the corresponding DFAs; then reduce them:

(a) $(a^2 \cup a^3 \cup a^5)^*$ over $\{a\}$

(b) $(a^2)^*(a^3)^*(a^5)^*$ over $\{a\}$

(c) $(abc \cup ab)^* aa^*(ab)^*$ over $\{a, b, c\}$

(d) $0^*(00 \cup 11)^*(01 \cup 10)^*1^*$ over $\{0, 1\}$

Exercise 4: Construct regular expressions for the languages accepted by the following automata:

(a)

		a	b	c	
$\rightarrow$	1	2	2	2	1
	2	3	1	2, 3	1
	3	4	3	$\emptyset$	1
	4	1	4	4	1

(b)

		a	b	
$\rightarrow$	A	B	C	0
	B	A	C	0
	C	B	A	1