UNIVERSITY OF HOUSTON

NOTES

COSC 3340
Intro. to Automata and Computability

Ernst Leiss

Khalid Hourani

Contents

Chapter 1

Formal Languages

1.1 Regular Languages

1.1.1 Introduction

Definition 1.1.1.1. An **Alphabet** is a finite, non-empty set of atomic symbols.

Definition 1.1.1.2. A **word** or **string** is any finite sequence of symbols from an alphabet.

Definition 1.1.1.3. The **length** of a string, s, denoted $|s|$, is the number of symbols in s.

Definition 1.1.1.4. Given strings $s = s_1 s_2 \dots s_n$ and $t = t_1 t_2 \dots t_m$, their **concatenation** is defined

$$s \cdot t = s_1 s_2 \cdots s_n t_1 t_2 \cdots t_m$$

We denote by ε the **empty string**, the unique string of 0 characters.

Definition 1.1.1.5. Let A be any alphabet. The **Kleene Closure** of A, denoted A^*, is the set of all strings of any length over A.

Theorem 1.1.1.1. Let A be any finite set. Then A^* is countably infinite.

Proof. That A^* is infinite is straightforward: since A is non-empty, take $a \in A$. Then

$$\{a, aa, aaa, \dots\} \subseteq A^*$$

To see that it is countable, we first write $|A| = n$. Now, consider the set of all strings of length 0. This is simply $\{\varepsilon\}$. Moreover, there are n strings of length 1, n^2 strings of length 2, n^3 strings of length 3, and so on. Thus, we map ε to 0, the strings of length 1 to $1, 2, \dots, n$, the strings of length 2 to $n+1, n+2, \dots, n+n^2$, the strings of length 3 to $n+n^2+1, n+n^2+2, \dots, n+n^2+n^3$, and so on. This is a bijection from A^* to $\mathbb{N}$, which completes the proof. $\square$

Definition 1.1.1.6. Given an alphabet A, a **formal language** or simply **language** L is any subset of A^*.

Theorem 1.1.1.2. Given an alphabet A, the set of languages over A is uncountable.

Proof. Suppose, by way of contradiction, that the set of languages were countable, i.e., that we can enumerate the set as $\{L_1, L_2, L_3, \dots\}$. Consider the set of all strings $\{s_1, s_2, s_3, \dots\}$. Let L be the language defined as follows:

$$s_i \in L \text{ if and only if } s_i \notin L_i$$

To see that L is not in the above list, consider s_i. If s_i is in L, then s_i is not in L_i, by construction, and $L \neq L_i$. Similarly, if s_i is not in L, then s_i must be in L_i, by construction, and $L \neq L_i$. In other words, for all i, $L \neq L_i$. Then L is not in the above list, which is a contradiction. Hence, the set of languages is uncountable. $\square$

All set operations, such as union, intersection, complement, set-difference, etc. can be applied to languages, since languages are simply subsets of a Kleene Closure of an alphabet.

Definition 1.1.1.7. Given two languages L_1 and L_2, the concatenation $L_1 \cdot L_2$ is given by

$$L_1 \cdot L_2 = \{s \cdot t \mid s \in L_1 \text{ and } t \in L_2\}$$

Clearly, we have

$$\begin{aligned} L \cdot \emptyset &= \emptyset = \emptyset \cdot L \\ L \cdot \{\varepsilon\} &= L = \{\varepsilon\} \cdot L \end{aligned}$$

Note that $L_1 \cdot L_2$ is not the same as $L_1 \times L_2$. Let $L_1 = L_2 = \{\varepsilon, 0, 00\}$. Then

$$L_1 \times L_2 = \{(\varepsilon,\varepsilon),(\varepsilon,0),(\varepsilon,00),(0,\varepsilon),(0,0),(0,00),(00,\varepsilon),(00,0),(00,00)\}$$

whereas

$$L_1 \cdot L_2 = \{\varepsilon, 0, 00, 000, 0000\}$$

Definition 1.1.1.8. Given a language L, the **Kleene Closure** of L, L^*, is

$$L^* = \bigcup_{i=0}^{\infty} L^i$$

where

$$L^i = \begin{cases} \{\varepsilon\} & \text{if } i = 0 \\ L \cdot L^{i-1} & \text{otherwise} \end{cases}$$

Note that, while 0^0 is normally left undefined, we define $\emptyset^0 = \{\varepsilon\}$.

Theorem 1.1.1.3. L^* is finite if and only if $L = \emptyset$ or $L = \{\varepsilon\}$.

Proof. If $L = \emptyset$, then $L^i = \emptyset^i = \emptyset$ for $i > 0$. Then

$$\begin{aligned} \emptyset^* &= \bigcup_{i=0}^{\infty} \emptyset^i \\ &= \emptyset^0 \cup \bigcup_{i=1}^{\infty} \emptyset^i \\ &= \{\varepsilon\} \cup \bigcup_{i=1}^{\infty} \emptyset \\ &= \{\varepsilon\} \end{aligned}$$

Similarly, if $L = \{\varepsilon\}$, then $L^i = \{\varepsilon\}$ for all i, and

$$\begin{aligned} \{\varepsilon\}^* &= \bigcup_{i=0}^{\infty} \{\varepsilon\}^i \\ &= \bigcup_{i=1}^{\infty} \{\varepsilon\} \\ &= \{\varepsilon\} \end{aligned}$$

However, if L is neither $\emptyset$ nor $\{\varepsilon\}$, then there exists a string $s \in L$ with length at least 1. Then $s, ss, sss, \ldots$, are in L^*, hence L^* is infinite. □

1.1.2 Finite Automata

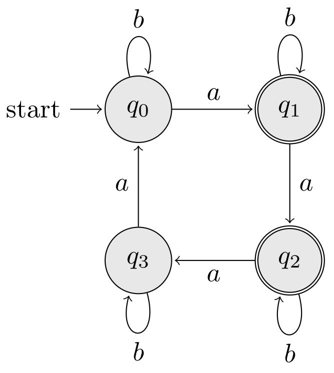
Definition 1.1.2.1. A **Deterministic Finite-State Automata** (DFA) or **Finite-State Machine** is a quintuple $(A, Q, \tau, q_0, \mathcal{F})$ where

A is the **alphabet**
Q is a finite, non-empty **set of states**
$\tau : Q \times A \to Q$ is the **transition function**
q_0 is the **initial state**
$\mathcal{F} \subseteq Q$ is the set of **final states**

We can extend τ as follows:
$\tau^* : Q \times A^* \to Q$

$$\tau^*(q, s) = \begin{cases} q & \text{if } s = \varepsilon \\ \tau^*(\tau(q, s_0), s') & \text{if } s = s_0 \cdot s' \end{cases}$$

We proceed informally and use τ to refer to τ^*.
Consider the following DFA:



The figure indicates that we begin at state q_0. The double-circles for states q_1 and q_2 indicate that they are accepting or final states. An arrow indicates the state to move to after receiving an input. For example, if we receive the input string *abba*, we begin at state q_0 and receive a, so we move to state q_1. We then receive b and stay in q_1. We repeat this for the next symbol, b, and then move to q_2 upon receiving the final a. Since q_2 is a final state, we say that this DFA **accepts** the string *abba*.

We can represent the above DFA using a table, as follows:

	a	b	
$\to q_0$	q_1	q_0	0
q_1	q_2	q_1	1
q_2	q_3	q_2	1
q_3	q_0	q_3	0

The first column indicates the states, while the first row indicates the symbols. The final column indicates whether a state is accepting: 0 refers to a non-final state, 1 to a final state. The remaining values indicate the transition function τ, e.g. $\tau(q_0, a) = q_1$, indicated by the entry corresponding to row q_0 and column a. Finally, the arrow pointing to q_0 indicates that it is the starting position.

Definition 1.1.2.2. Let $\underset{\sim}{D}$ be some DFA. Then $L(\underset{\sim}{D})$, the language accepted by the DFA, is

$$\{s \in A^* \mid \tau(q_0, s) \in \mathcal{F}\}$$

Definition 1.1.2.3. A language is **regular** if and only if there exists a DFA that accepts it.

Definition 1.1.2.4. A **Non-Deterministic Finite-State Automata** (NFA) is a quintuple

$$(A, Q, \tau, q_0, \mathcal{F})$$

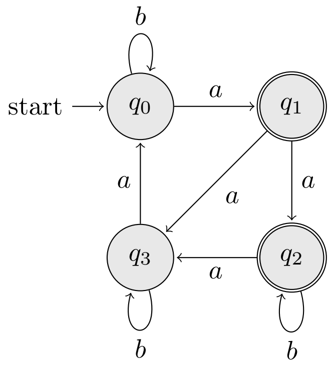
where

A is the **alphabet**
Q is a finite, non-empty **set of states**
$\tau : Q \times A \to 2^Q$ is the **transition function**
q_0 is the **initial state**
$\mathcal{F} \subseteq Q$ is the set of **final states**

We can extend τ as follows:
$\tau^* : 2^Q \times A^* \to 2^Q$

$$\tau^*(P, s) = \begin{cases} P & \text{if } s = \varepsilon \\ \tau^*\left(\bigcup_{q \in P} \tau(q, s_0), s'\right) & \text{if } s = s_0 \cdot s' \end{cases}$$

We proceed informally and use τ to refer to τ^*.
Consider the following NFA:



The diagrams for an NFA and DFA follow the same notation. However, the notation for the table differs slightly:

	a	b	
$\to q_0$	q_1	q_0	0
q_1	q_2q_3	$\emptyset$	1
q_2	q_3	q_2	1
q_3	q_0	q_3	0

The values of the transition function are now sets. We informally refer to the set $\{q_0\}$ by q_0, and similarly the set $\{q_2, q_3\}$ by q_2q_3. In some cases, to avoid ambiguity, we will use commas, e.g. we may represent $\{q_2, q_3\}$ as q_2, q_3. We similarly say, given a string s, if there exists a path through an NFA that ends in a final state, we say that the NFA **accepts** s.
Similarly, we define the set of languages accepted by an NFA $\underset{\sim}{N}$, $L(\underset{\sim}{N})$, as

$$L(\underset{\sim}{N}) = \{s \in A^* \mid \tau(q_0, s) \cap \mathcal{F} \neq \emptyset\}$$

It should be clear that each DFA is an NFA, but the reverse is not true. However, we can convert an NFA to a DFA on the powerset 2^Q by using the **subset construction**: begin with the initial state and traverse the NFA, adding unseen states to the left-most column until all paths have been exhausted. For example, with our NFA above, we begin with:

	a	b	
$\rightarrow q_0$	q_1	q_0	0

q_0 has already been seen, so we ignore it. q_1 is new, so we add it to the table:

	a	b
$\rightarrow q_0$	q_1	q_0
q_1		

We now visit the corresponding states of q_1, which are q_2q_3 and $\emptyset$, both of which have not yet been visited.

	a	b
$\rightarrow q_0$	q_1	q_0
q_1	q_2q_3	$\emptyset$
q_2q_3		
$\emptyset$		

When q_2 receives a, it transitions to state q_3. When q_3 receives a, it transitions to state q_0, so q_2q_3 transitions to q_0q_3. Similarly, q_2q_3 transitions to state q_2q_3 when it receives b.

	a	b
$\rightarrow q_0$	q_1	q_0
q_1	q_2q_3	$\emptyset$
q_2q_3	q_0q_3	q_2q_3
$\emptyset$		

The empty set transitions to the empty set, by definition.

	a	b
$\rightarrow q_0$	q_1	q_0
q_1	q_2q_3	$\emptyset$
q_2q_3	q_0q_3	q_2q_3
$\emptyset$	$\emptyset$	$\emptyset$

q_0q_3 has not yet been visited, so we add it to the left-most column:

	a	b
$\rightarrow q_0$	q_1	q_0
q_1	q_2q_3	$\emptyset$
q_2q_3	q_0q_3	q_2q_3
$\emptyset$	$\emptyset$	$\emptyset$
q_0q_3		

Then we visit its corresponding states:

	a	b
$\rightarrow q_0$	q_1	q_0
q_1	q_2q_3	$\emptyset$
q_2q_3	q_0q_3	q_2q_3
$\emptyset$	$\emptyset$	$\emptyset$
q_0q_3	q_0q_1	q_0q_3

Continuing, we end with the following DFA:

	a	b
$\rightarrow q_0$	q_1	q_0
q_1	q_2q_3	$\emptyset$
q_2q_3	q_0q_3	q_2q_3
$\emptyset$	$\emptyset$	$\emptyset$
q_0q_3	q_0q_1	q_0q_3
q_0q_1	$q_1q_2q_3$	q_0
$q_1q_2q_3$	$q_0q_2q_3$	q_2q_3
$q_0q_2q_3$	$q_0q_1q_3$	$q_0q_2q_3$
$q_0q_1q_3$	$q_0q_1q_2q_3$	q_0q_3
$q_0q_1q_2q_3$	$q_0q_1q_2q_3$	$q_0q_2q_3$

However, we need to include the accepting states. The accepting states of the NFA are q_1 and q_2, and thus any state including either state is accepting:

	a	b	
$\rightarrow q_0$	q_1	q_0	0
q_1	q_2q_3	$\emptyset$	1
q_2q_3	q_0q_3	q_2q_3	1
$\emptyset$	$\emptyset$	$\emptyset$	0
q_0q_3	q_0q_1	q_0q_3	0
q_0q_1	$q_1q_2q_3$	q_0	1
$q_1q_2q_3$	$q_0q_2q_3$	q_2q_3	1
$q_0q_2q_3$	$q_0q_1q_3$	$q_0q_2q_3$	1
$q_0q_1q_3$	$q_0q_1q_2q_3$	q_0q_3	1
$q_0q_1q_2q_3$	$q_0q_1q_2q_3$	$q_0q_2q_3$	1

Note that an NFA does not necessarily admit a DFA with as many states. Consider the following example:

	a	b	
$\rightarrow 0$	$\{1,2,\ldots,n\}$	0	0
1	2	1	0
2	3	2	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$i+1$	i	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	n	$n-1$	0
n	1	n	1

The NFA above admits the following DFA:

	a	b	
$\rightarrow 0$	$\{1,2,\ldots,n\}$	0	0
$\{1,2,\ldots,n\}$	$\{1,2,\ldots,n\}$	$\{1,2,\ldots,n\}$	1

The above DFA contains only 2 states, despite the NFA containing $n+1$ states.

That every NFA admits a DFA which accepts the same language shows that the class of languages denoted by DFAs, $\mathcal{L}_{\text{DFA}}$, is the same as the class of languages denoted by NFAs, $\mathcal{L}_{\text{NFA}}$, i.e, that

$$\mathcal{L}_{\text{DFA}} = \mathcal{L}_{\text{NFA}}$$

For an NFA, there is no guarantee of a unique smallest NFA which accepts the same strings. However, for a DFA, such a notion exists.

Consider two states, p and q, and corresponding L_p and L_q, where L_p has initial state p and L_q has initial state q. We say that p and q are distinguishable if there exists a string s such that s is in L_p and not in L_q, or vice-versa. We use this notion to **reduce** a DFA.

Begin with a partition of Q into subsets $\mathcal{F}$ and $Q - \mathcal{F}$, i.e., the accepting and rejecting states. For a pair of states p, q if the result of transitioning p and q falls into different partitions, we partition the subset and continue.

For example, given the following DFA:

	a	b	
$\rightarrow 0$	1	2	0
1	2	3	1
2	3	4	0
3	0	5	1
4	5	6	0
5	6	7	1
6	7	0	0
7	4	1	1

We have two partitions:

Rejecting	Accepting
0, 2, 4, 6	1, 3, 5, 7

Now, 0 gets sent to the accepting partition by a and to the rejecting partition by b. Similarly, 2, 4, and 6 get sent to the accepting partition by a and to the rejecting partition by b. Thus, they belong to the same partition.

In the same vein, 1 gets sent to the rejecting partition by a and to the accepting partition by b. Similarly, 3, 5, and 7 get sent to the rejecting partition by a and to the accepting partition by b. Thus, our next partition is

Rejecting	Accepting
0, 2, 4, 6	1, 3, 5, 7
0, 2, 4, 6	1, 3, 5, 7

That our row is the same as the preceding one indicates that we have finished, and now have a minimal DFA. Call the first subset p and the second q. When an element in p receives a, it is sent to q. When it receives b, it is sent to p. Similar logic for q gives our new DFA:

	a	b	
$\rightarrow p$	q	p	0
q	p	q	1

Recall that p began as a subset of the rejecting elements and q the accepting elements, which informs the last column of the above table.

Not all DFAs can be reduced. An obvious example is the above reduced DFA. For a less trivial example, consider the following DFA:

	a	b	
$\rightarrow 0$	1	2	0
1	2	3	1
2	3	4	0
3	0	5	1
4	5	6	0
5	6	7	1
6	7	0	0
7	4	2	1

Begin, as in the previous problem, with two partitions:

Rejecting	Accepting
0, 2, 4, 6	1, 3, 5, 7

As in the previous problem, 0, 2, 4, and 6 get sent to the same partition under a and b, respectively. Under a, 1, 3, 5, and 7 go to the rejecting partition. However, under b, 7 goes to the rejecting partition while 1, 3, and 5 go to the accepting partition, which means we must create a new partition for 7.

Rejecting	Accepting
0, 2, 4, 6	1, 3, 5, 7
0, 2, 4, 6	1, 3, 5 7

We continue the process, noting that there is no need to consider singletones, i.e., the partition $\{7\}$ is already in its finale state. Under a, 0, 2, and 4 get sent to the $\{1, 3, 5\}$ partition. Under b, they get sent to the $\{0, 2, 4, 6\}$ partition. However, 6 gets sent to the $\{7\}$ partition, and so it must be partitioned separately. Similarly, 1 and 3 get sent to the $\{0, 2, 4, 6\}$ partition under a, and to the $\{1, 3, 5\}$ partition under b. 5, on the other hand, gets sent to the $\{7\}$ partition, and must be partitioned separately. In total, we have:

Rejecting		Accepting		
0, 2, 4, 6		1, 3, 5, 7		
0, 2, 4, 6		1, 3, 5		7
0, 2, 4	6	1, 3	5	7

We continue:

Rejecting				Accepting			
0, 2, 4, 6				1, 3, 5, 7			
0, 2, 4, 6				1, 3, 5			7
0, 2, 4			6	1, 3		5	7
0, 2		4	6	1	3	5	7
0	2	4	6	1	3	5	7

Notice that the reduced DFA has 8 states, like the original! This means that the original DFA is already reduced, and cannot be reduced further.

1.1.3 Regular Expressions

Definition 1.1.3.1. Given an alphabet A, we define a **regular expression**

(a)
- $a \in A$ is a regular expression denoting the language $\{a\}$
- ε is a regular expression denoting $\{\varepsilon\}$
- $\emptyset$ is a regular expression denoting $\emptyset$

(b) If α and β are regular expressions denoting the languages $L(\alpha)$ and $L(\beta)$, respectively, then
- $\alpha \cup \beta$ denotes $L(\alpha) \cup L(\beta)$
- $\alpha \cdot \beta$ denotes $L(\alpha) \cdot L(\beta)$
- α^* denotes $L(\alpha)^*$

By convention, we define precedence of the operations $\cup$, $\cdot$, and * in that order. Thus,

$$b \cdot a^* \cup c = (b \cdot (a^*)) \cup c$$

A regular expression α over an alphabet A denotes the set of languages which accept α. Thus, we would like to construct an NFA $\underset{\sim}{N}$ such that $L(\underset{\sim}{N}) = L(\alpha)$.

The following NFA rejects all strings but a:

	a	$b \neq a$	
$\to q_0$	q_1	$\emptyset$	0
q_1	$\emptyset$	$\emptyset$	1

An NFA for only ε would appear as:

	$c \in A$	
$\to q_0$	$\emptyset$	1

And finally, an NFA for only $\emptyset$ is:

	$c \in A$	
$\to q_0$	$\emptyset$	0

Now, suppose we have an NFA for α and β. We wish to determine NFAs for $\alpha \cup \beta$, $\alpha \cdot \beta$, and α^*.

We define

$$\begin{aligned}\underset{\sim}{N}_\alpha &= (A, Q_\alpha, \tau_\alpha, q_0, \mathcal{F}_\alpha)\\ \underset{\sim}{N}_\beta &= (A, Q_\beta, \tau_\beta, q_0, \mathcal{F}_\beta)\end{aligned}$$

such that

$$\begin{aligned}L(\underset{\sim}{N}_\alpha) &= L(\alpha)\\ L(\underset{\sim}{N}_\beta) &= L(\beta)\\ Q_\alpha \cap Q_\beta &= \{q_0\}\end{aligned}$$

and clarify that these automata are non-returning, i.e., that $q_0 \notin \tau(q_0, s)$ for any s of length 1 or greater.

We construct the **Union**

$$\underset{\sim}{N}_{\alpha\cup\beta} = (A, Q_{\alpha\cup\beta}, \tau_{\alpha\cup\beta}, q_0, \mathcal{F}_{\alpha\cup\beta})$$

where $Q_{\alpha\cup\beta} = Q_\alpha \cup Q_\beta$, $\mathcal{F}_{\alpha\cup\beta} = \mathcal{F}_\alpha \cup F_\beta$ and, for all $q \in Q_{\alpha\cup\beta}$ and $a \in A$

$$\tau_{\alpha\cup\beta}(q, a) = \begin{cases} \tau_\alpha(q_0, a) \cup \tau_\beta(q_0, a) & \text{if } q = q_0 \\ \tau_\alpha(q, a) & \text{if } q \in Q_\alpha - \{q_0\} \\ \tau_\beta(q, a) & \text{if } q \in Q_\beta - \{q_0\} \end{cases}$$

The **Concatenation** is constructed

$$\underset{\sim}{N}_{\alpha\beta} = (A, Q_{\alpha\beta}, \tau_{\alpha\beta}, q_0, \mathcal{F}_{\alpha\beta})$$

where $Q_{\alpha\beta} = Q_\alpha \cup Q_\beta$,

$$\mathcal{F}_{\alpha\beta} = \begin{cases} \mathcal{F}_\beta & \text{if } q_0 \notin \mathcal{F}_\beta \\ \mathcal{F}_\alpha \cup (\mathcal{F}_\beta - \{q_0\}) & \text{if } q_0 \in \mathcal{F}_\beta \end{cases}$$

and, for all $q \in Q_{\alpha\beta}$ and $a \in A$

$$\tau_{\alpha\beta}(q, a) = \begin{cases} \tau_\alpha(q, a) \cup \tau_\beta(q_0, a) & \text{if } q \in \mathcal{F}_\alpha \\ \tau_\alpha(q, a) & \text{if } q \in Q_\alpha - \mathcal{F}_\alpha \\ \tau_\beta(q, a) & \text{if } q \in Q_\beta - \{q_0\} \end{cases}$$

Finally, the **Kleene Closure** is constructed

$$\underset{\sim}{N}_{\alpha^*} = (A, Q_{\alpha^*}, \tau_{\alpha^*}, q_0, \mathcal{F}_{\alpha^*})$$

where $Q_{\alpha^*} = Q_\alpha$, $\mathcal{F}_{\alpha^*} = \mathcal{F}_\alpha \cup \{q_0\}$ and, for all $q \in Q_{\alpha^*}$ and $a \in A$

$$\tau_{\alpha^*}(q, a) = \begin{cases} \tau_\alpha(q, a) \cup \tau_\alpha(q_0, a) & \text{if } q \in \mathcal{F}_\alpha \\ \tau_\alpha(q, a) & \text{if } q \in Q_\alpha - \mathcal{F}_\alpha \end{cases}$$

This allows us to construct NFAs from a regular expression. Suppose we have a regular expression ab over $\{a, b\}$. Then we have

NFA for a

	a	b	
$\rightarrow q_0$	q_1	$\emptyset$	0
q_1	$\emptyset$	$\emptyset$	1

NFA for b

	a	b	
$\rightarrow q_0$	$\emptyset$	q_2	0
q_2	$\emptyset$	$\emptyset$	1

Applying the above construction for concatenation gives

NFA for ab

	a	b	
$\rightarrow q_0$	q_1	$\emptyset$	0
q_1	$\emptyset$	q_2	0
q_2	$\emptyset$	$\emptyset$	1

1.1.4 Solutions of Certain Language Equations

Given a regular expression, we can form an NFA which admits the same language by solving **Language Equations**. We show the following lemma before proceeding to examples:

Lemma 1. If $X = L \cdot X \cup M$ then $X = L^* \cdot M$ is a solution, and is unique if $\varepsilon \notin L$.

Proof. Clearly, $L^* \cdot M$ is a solution, since

$$L^* \cdot M = L \cdot (L^* \cdot M) \cup M$$

To prove uniqueness, suppose s_1 and s_2 are distinct solutions. There must exist a shortest-length string in s_1, say s. □

Consider the following NFA:

	a	b	
$\rightarrow 1$	2	1, 3	0
2	$\emptyset$	3	0
3	2, 3	1	1

This admits the following set of equations

$$X_1 = aX_2 \cup bX_1 \cup bX_3 \tag{1.1}$$
$$X_2 = bX_3 \tag{1.2}$$
$$X_3 = aX_2 \cup aX_3 \cup bX_1 \cup \varepsilon \tag{1.3}$$

We substitute (2) into (1) and (3):

$$X_1 = abX_3 \cup bX_1 \cup bX_3$$
$$X_3 = abX_3 \cup aX_3 \cup bX_1 \cup \varepsilon$$

which we rewrite as

$$X_1 = (ab \cup b)X_3 \cup bX_1$$
$$X_3 = (ab \cup a)X_3 \cup bX_1 \cup \varepsilon$$

We now apply our lemma to the equation for X_3

$$X_1 = (ab \cup b)X_3 \cup bX_1$$
$$X_3 = (ab \cup a)^*(bX_1 \cup \varepsilon)$$

We substitute X_3 into the equation for X_1

$$\begin{aligned}
X_1 &= (ab \cup b)(ab \cup a)^*(bX_1 \cup \varepsilon) \cup bX_1 \\
&= \big((ab \cup b)(ab \cup a)^* \cup b\big)\, X_1 \cup (ab \cup b)(ab \cup a)^* \cup bX_1 \\
&= \big((ab \cup b)(ab \cup a)^* \cup b\big)\, X_1 \cup (ab \cup b)(ab \cup a)^* \\
&= \big((ab \cup b)(ab \cup a)^* \cup b\big)^*(ab \cup b)(ab \cup a)^*
\end{aligned}$$

Consider the example:

	a	b	
$\rightarrow 1$	2	3	0
2	2	3	0
3	2	3	1

This admits the following system of equations:

$$\begin{aligned}
X_1 &= aX_2 \cup bX_3 \\
X_2 &= aX_2 \cup bX_3 \\
X_3 &= aX_2 \cup bX_3 \cup \varepsilon
\end{aligned}$$

From our lemma, we have $X_2 = a^*bX_3$:

$$\begin{aligned}
X_1 &= aa^*bX_3 \cup bX_3 \\
X_3 &= aa^*bX_3 \cup bX_3 \cup \varepsilon
\end{aligned}$$

which can be simplified:

$$\begin{aligned}
X_1 &= (aa^*b \cup b)X_3 \\
X_3 &= (aa^*b \cup b)X_3 \cup \varepsilon
\end{aligned}$$

Applying our lemma to X_3, we have

$$X_3 = (aa^*b \cup b)^*$$

Subsituting into X_1 gives

$$X_1 = (aa^*b \cup b)(aa^*b \cup b)^*$$

One final example:

	a	b	
$\rightarrow 1$	$\emptyset$	1, 2	1
2	1	$\emptyset$	1

$$\begin{aligned}
X_1 &= bX_1 \cup bX_2 \cup \varepsilon \\
X_2 &= aX_1 \cup \varepsilon
\end{aligned}$$

Substituting our equation for X_2 into X_1 gives

$$\begin{aligned}
X_1 &= bX_1 \cup b(aX_1 \cup \varepsilon) \cup \varepsilon \\
&= (b \cup ba)X_1 \cup b \cup \varepsilon \\
&= (b \cup ba)^*(b \cup \varepsilon)
\end{aligned}$$

1.1.5 Extended Regular Expressions

The languages we have discussed so far are **regular languages**. That is,

- Deterministic Finite Automaton
- Non-Deterministic Finite Automaton
- Regular Expression
- Solution of Languages Equations

are all regular languages. The following are **Closure Properties** of a regular language:

Theorem 1.1.5.1. Let $\mathcal{L}_\infty$ and $\mathcal{L}_\in$ be regular languages in some alphabet A. Then

1. $\mathcal{L}_1 \cup \mathcal{L}_2$
2. $\mathcal{L}_1 \cdot \mathcal{L}_2$
3. $\mathcal{L}_1^*$
4. $\overline{\mathcal{L}_1}$

are all regular languages in A.

Proof. 1, 2, and 3 follow from the definitions of regular expressions. For 4, consider a DFA $\underset{\sim}{D} = (A, Q, \tau, q_0, \mathcal{F})$ and consider any word $s \in A^*$. Further, let $\underset{\sim}{D}' = (A, Q, \tau, q_0, Q - \mathcal{F})$. If $w \in L(\underset{\sim}{D})$, then $w \notin L(\underset{\sim}{D}')$. On the other hand, if $w \notin L(\underset{\sim}{D})$, then $w \in L(\underset{\sim}{D}')$. Then $L(\underset{\sim}{D}') = \overline{L(\underset{\sim}{D})}$. □

This allows us to define the regular expression $\overline{\alpha}$:

Definition 1.1.5.1. Let α be any regular expression in some alphabet A. Then the regular expression $\overline{\alpha}$ is defined by

$$\overline{\alpha} = \overline{L(\alpha)}$$

If a regular expression contains a complement, it is an **extended regular expression**.

We can construct the DFA of the complement of a regular expression by finding the corresponding DFA and swapping the accepting and rejecting states. For example, consider the regular expression $\overline{01^*} \cap \overline{10^*}$ over $\{0, 1\}$.

$$\overline{01^*} \cap \overline{10^*} = \overline{\overline{\overline{01^*} \cup \overline{10^*}}}$$

Similarly, we consider the example $\overline{(\overline{01^*}0)}^*$ over $\{0, 1, 2\}$.

It should be noted that the above process of swapping accepting and rejecting states *only works on a DFA*. Thus, if you wish to take the complement of an NFA, you must first convert it to a DFA.

1.1.6 Non-Regular Languages

Suppose we have a DFA

$$\underset{\sim}{D} = (A, Q, \tau, q_0, \mathcal{F})$$

with $|Q| = n$. Consider the following set of equations

$$\begin{aligned}
q_1 &= \tau(q_0, a_1) \\
q_2 &= \tau(q_1, a_2) \\
&\vdots \\
q_i &= \tau(q_{i-1}, a_i) \\
&\vdots \\
q_{n-1} &= \tau(q_{n-2}, a_{n-1}) \\
q_n &= \tau(q_{n-1}, a_n)
\end{aligned}$$

Notice that we have $n+1$ states above, but only n states in Q. Then, by the Pidgeonhole Principle, there must be some state q_i that is visited twice. In other words, there exist an i, j with $i < j$ such that $q_i = q_j$. We then consider a string $s = a_1 a_2 \dots a_n$. Let $v_1 = a_1 a_2 \dots a_i$, $v_2 = a_{i+1} a_{i+2} \dots a_j$, and $v_3 = a_{j+1} a_{j+2} \dots a_n$. We see that

$$\tau(q_0, s) = \tau(q_0, v_1 v_2^k v_3)$$

for all $k \geq 0$.

$$\begin{tikzcd}
\text{start} \arrow[r] & q_0 \arrow[r, "v_1"] & q_i \arrow[loop above, "v_2"] \arrow[r, "v_3"] & q_n
\end{tikzcd}$$

Thus, we state **The Pumping Lemma** and provide a proof:

Theorem 1.1.6.1 (The Pumping Lemma)**.** Let L be any reguar language with corresponding DFA $(A, Q, \tau, q_0, \mathcal{F})$. Then there exists a $p > 0$ (called the **pumping length**) such that, for any string s of length p or longer, we can write $s = s_1 s_2 s_3$ and

- $|s_2| \geq 1$
- $|s_1 s_2| \leq p$
- $\tau(q_0, s) = \tau(q_0, s_1 s_2^n s_3)$ for all $n \geq 0$

We can use the above theorem to prove that certain languages are not regular.

Theorem 1.1.6.2. The language given by

$$L = \{a^i b^i \mid i \geq 0\}$$

is not regular.

Proof. Suppose, by way of contradiction, that L is regular with pumping length p. Consider the string $s = a^p b^p$. By the pumping lemma, we can write $s = s_1 s_2 s_3$ with $|s_1 s_2| \leq p$ and $|s_2| \geq 1$. Then $s_1 = a^p - k$ and $s_2 = a^k$ for some $1 \leq k \leq p$. Further, we have

$$\begin{aligned}
s_1 s_2^n s_3 &= a^{p-k} (a^k)^n b^p \\
&= a^{p-k} a^{kn} b^p \\
&= a^{p+(n-1)k} b^p \in L
\end{aligned}$$

Taking $n = 2$ gives $a^{p+k} b^p \in L$, a contradiction. Thus, L is not regular. □

1.2 Context-Free Languages

1.2.1 Context-Free Grammars

Definition 1.2.1.1. A **Context-Free Grammar** is a quartuple $G = (N, T, P, S)$ where

N is a finite, non-empty set of variables (also called non-terminals)
T is an alphabet of terminals
$P \subseteq N \times (N \cup T)^*$ is a finite set of productions
$S \in N$ is the starting symbol

For any $(A, \gamma) \in P$, we write $A \to \gamma$, and say A **produces** γ.

By convention, we use upper-case letters to denote variables, lower-case to denote terminals and strings over the terminals, and Greek letters to denote strings over variables and terminals.

Definition 1.2.1.2. Given strings α and β, we say α **derives** β if there exist $A, \alpha_1, \alpha_2, \gamma$ such that

$$\begin{aligned} \alpha &= \alpha_1 A \alpha_2 \\ \beta &= \alpha_1 \gamma \alpha_2 \\ A &\to \gamma \in P \end{aligned}$$

and we write this $\alpha \Rightarrow \beta$.

We can define the language of a context-free grammar:

Definition 1.2.1.3. Given a context-free grammar G, the corresponding **context-free language** is

$$L(G) = \{w \mid S \Rightarrow w\}$$

Theorem 1.2.1.1. Every regular language is a context-free language.

Proof. Let L be a regular language and let $N = (Q, A, \tau, q_0, \mathcal{F})$ be its corresponding minimum DFA. □

However, not every context-free language is regular. For example, the language $\{a^n b^n \mid n \geq 0\}$ is not regular, but is a context-free language given by

1.2.2 Preprocessing a CFG

For any CFG G, we preprocess the language:

- Eliminate useless symbols
- Eliminate ε productions: $A \to \varepsilon$
- Eliminate unit productions: $A \to B$

Eliminating Useless Symbols

For example, suppose G is a context-free grammar given by:

$$\begin{aligned} S &\to aSb \mid cAd \\ A &\to aSc \mid bAd \end{aligned}$$

Then $L(G) = \emptyset$, since every terminal produces a string with a terminal. Thus, we can elminate S and A. In general, for any context-free grammar G, if no string s exists such that $A \Rightarrow s$, then we can eliminate A.

Consider the following example:

$$\begin{aligned} S &\to aS \mid bA \mid \varepsilon \\ A &\to cAA \mid dBB \\ B &\to aBA \mid bAA \mid cAC \\ C &\to aCb \mid S \end{aligned}$$

Notice that S produces ε, and so we cannot eliminate it. Similarly, C produces S, so we cannot eliminate it. At this point, it should be apparent that A and B do not produce terminals, and therefore can be eliminated. Further, we can eliminate terminals c and d since they are not involved in the productions of C or S. Graphically, we have

S

A

B

C

Now note that C cannot be reached from S, the starting state. Thus, we can eliminate C, and similarly eliminate b. Thus, our grammar can be reduced to

$$S \to aS \mid \varepsilon$$

To summarize: if a terminal string cannot be reached from a variable, or a variable cannot be reached from the starting symbol, it can be eliminated.

Eliminating ε Productions

To remove ε productions, we find the nullable non-terminals (variables). These are the non-terminals from which ε can be derived. Specifically, a variable A is nullable if it is of the form

$$A \to \varepsilon$$

or

$$A \to A_1A_2A_3 \dots A_n$$

where each A_i is nullable. Then, simply replace each *combination of nullable variables* with ε and eliminate ε from the right-hand side.

For example, consider the grammar

$$\begin{aligned} S &\to ABCd \\ A &\to BC \\ B &\to bB \mid \varepsilon \\ C &\to cC \mid \varepsilon \end{aligned}$$

Clearly, B and C are nullable. Then A is nullable because it produces BC. We then have

$$\begin{aligned} S &\to ABCd \mid \cancel{A}BCd \mid A\cancel{B}Cd \mid AB\cancel{C}d \mid \cancel{A}\cancel{B}Cd \mid \cancel{A}B\cancel{C}d \mid A\cancel{B}\cancel{C}d \mid \cancel{A}\cancel{B}\cancel{C}d \\ A &\to BC \mid \cancel{B}C \mid B\cancel{C} \mid \cancel{B}\cancel{C} \\ B &\to bB \mid b\cancel{B} \\ C &\to cC \mid c\cancel{C} \end{aligned}$$

which becomes

$$\begin{aligned} S &\to ABCd \mid BCd \mid ACd \mid ABd \mid Cd \mid Bd \mid Ad \mid d \\ A &\to BC \mid C \mid B \\ B &\to bB \mid b \\ C &\to cC \mid c \end{aligned}$$

Eliminating Unit Productions

To eliminate a unit production, simply replace any unit production

$$A \to B$$

with the productions for B. For example, consider the following grammar:

$$\begin{aligned} S &\to Aa \mid B \\ A &\to b \mid B \\ B &\to A \mid a \end{aligned}$$

We see that $S \to B$, $A \to B$, and $B \to A$ are unit rules, and replace them

$$\begin{aligned} S &\to Aa \mid A \mid a \\ A &\to b \mid A \mid a \\ B &\to b \mid B \mid a \end{aligned}$$

Similarly, consider the following example:

$$\begin{aligned} S &\to A \mid SS \\ A &\to B \mid AA \\ B &\to S \mid a \end{aligned}$$

1.2.3 Normal Forms

Chomsky Normal Form

Definition 1.2.3.1. A context-free language G is in **Chomsky Normal Form**[1]if all of its productions are of the form

$$\begin{aligned} A &\to BC \\ A &\to a \end{aligned}$$

Theorem 1.2.3.1. If G is a context-free grammar, there exists a Chomsky Normal Form grammar for $L(G) - \{\varepsilon\}$

Proof. □

To convert to Chomsky Normal Form, first preprocess the CFG. Then, for any production containing a non-solitary terminal, a, replace a with X_a, where $X_a \to a$. After this, we replace every production

$$A \to X_1 X_2 \dots X_n$$

with the following productions

$$\begin{aligned} A &\to X_1 A_1 \\ A_1 &\to X_2 A_2 \\ A_2 &\to X_3 A_3 \\ &\vdots \\ A_{n-2} &\to X_{n-1} X_n \end{aligned}$$

Greibach Normal Form

Definition 1.2.3.2. A context-free language is in **Greibach Normal Form**[2] if all of its productions are of the form

$$A \to aA_1 A_2 \dots A_n$$

To convert a CFG to Greibach Normal Form, you must remove left-recursion. First, preprocess the CFG. We can remove any *direct recursion* of the form

$$A \to A\alpha_1 \mid A\alpha_2 \mid A\alpha_3 \mid \dots \mid A\alpha_m \mid \beta_1 \mid \beta_2 \mid \beta_3 \mid \dots \mid \beta_n$$

where no β_i begins with A, by replacing the productions for A with

$$\begin{aligned} A &\to \beta_1 \mid \beta_2 \mid \beta_3 \mid \dots \mid \beta_n \mid \beta_1 A' \mid \beta_2 A' \mid \beta_3 A' \mid \dots \mid \beta_n A' \\ A' &\to \alpha_1 \mid \alpha_2 \mid \alpha_3 \mid \dots \mid \alpha_m \mid \alpha_1 A' \mid \alpha_2 A' \mid \alpha_3 A' \mid \dots \mid \alpha_m A' \end{aligned}$$

More generally, we eliminate *indirect recursion* by ordering the non-terminals and performing the following algorithm:

[1]This is also called **Chomsky Reduced Form**. Sometimes, Chomsky Normal Form includes the production $S \to \varepsilon$. In our usage, CNF and CRF are identical and do not include a ε production.

[2]Sometimes, the production $S \to \varepsilon$ is incuded. In our use, there is no ε production.

Algorithm Returns CFG with left-recursion removed

```
procedure S(G)
    for each non-terminal $A_i$ do
        for each production $A_i \to \alpha_i$ do
            if $\alpha_i$ begins with $A_j$ for $j < i$ then
                let $\beta_i$ be $\alpha_i$ without the leading $A_j$
                remove the production $A_i \to \alpha_i$
                for each production $A_j \to \alpha_j$ do
                    Add rule $A_i \to \alpha_j\beta_i$
                end for
            end if
        end for
        remove direct left recursion for $A_i$
    end for
end procedure
```

Consider the following example:

$$\begin{aligned} S &\to AS \mid a \\ A &\to BS \mid b \\ B &\to CS \mid c \\ C &\to SS \mid d \end{aligned}$$

There is no preprocessing to do, since there are no useless or nullable variables and no unit productions. Notice that C includes indirect left-recursion. Thus, we write

$$\begin{aligned} C &\to ASS \mid aS \mid d \\ &\to BSSS \mid bSS \mid aS \mid d \\ &\to CSSSS \mid cSSS \mid bSS \mid aS \mid d \end{aligned}$$

Now, we eliminate the direct left-recursion from C

$$\begin{aligned} C &\to cSSS \mid bSS \mid aS \mid d \mid cSSSC' \mid bSSC' \mid aSC' \mid dC' \\ C' &\to SSSS \mid SSSSC' \end{aligned}$$

Next, we substitute the productions for C into B

$$B \to cSSSS \mid bSSS \mid aSS \mid dS \mid cSSSC'S \mid bSSC'S \mid aSC'S \mid dC'S \mid c$$

And similarly for A and S

$$\begin{aligned} A &\to cSSSSS \mid bSSSS \mid aSSS \mid dSS \mid cSSSC'SS \mid bSSC'SS \mid aSC'SS \mid dC'SS \mid cS \mid b \\ S &\to cSSSSSS \mid bSSSSS \mid aSSSS \mid dSSS \mid cSSSC'SSS \mid bSSC'SSS \mid aSC'SSS \mid dC'SSS \mid cSS \mid bS \mid a \end{aligned}$$

And finally for C'

$$\begin{aligned} C' \to\ & cSSSSSSSSS \mid bSSSSSSSS \mid aSSSSSSS \mid dSSSSSS \mid cSSSC'SSSSSS \\ & \mid bSSC'SSSSSS \mid aSC'SSSSSS \mid dC'SSSSSS \mid cSSSSS \mid bSSSS \\ & \mid aSSSC' \mid cSSSSSSSSSC' \mid bSSSSSSSSC' \mid aSSSSSSSC' \mid dSSSSSSC' \mid cSSSC'SSSSSSC' \\ & \mid bSSC'SSSSSSC' \mid aSC'SSSSSSC' \mid dC'SSSSSSC' \mid cSSSSSC' \mid bSSSSC' \mid aSSSC' \end{aligned}$$

Consider similarly the example:

$$\begin{aligned} S &\to AA \\ A &\to BaA \mid a \\ B &\to SBA \mid b \end{aligned}$$

Once again, there is no pre-processing to do. We see that B involves indirect left-recursion. Thus, we write

$$\begin{aligned} B &\to AABA \mid b \\ &\to BaAABA \mid aABA \mid b \end{aligned}$$

This involves direct left-recursion, so we write

$$\begin{aligned} B &\to aABA \mid b \mid aABAB' \mid bB' \\ B' &\to aAABA \mid aAABAB' \end{aligned}$$

We then substitute the productions for B into A

$$A \to aABAaA \mid baA \mid aABAB'aA \mid bB'aA \mid a$$

and finally substitute the productions for A into S

$$S \to aABAaAA \mid baAA \mid aABAB'aAA \mid bB'aAA \mid aA$$

In total, our GNF is

$$\begin{aligned} S &\to aABAaAA \mid baAA \mid aABAB'aAA \mid bB'aAA \mid aA \\ A &\to aABAaA \mid baA \mid aABAB'aA \mid bB'aA \mid a \\ B &\to aABA \mid b \mid aABAB' \mid bB' \\ B' &\to aAABA \mid aAABAB' \end{aligned}$$

1.2.4 Pumping Lemma for Context-Free Languages

Like with regular languages, there exists a pumping lemma for context-free languages.

Theorem 1.2.4.1. Let L be a context-free language. Then there exists a p such that for all strings s such that $|s| \geq p$, there exist strings s_1, s_2, s_3, s_4, s_5 such that

- $|s_2 s_4| \geq 1$
- $|s_2 s_3 s_4| \leq p$
- $s_1 s_2^n s_3 s_4^n s_5 \in L$ for all $n \geq 0$

Proof. □

We can use this to show that certain languages are not context-free. For example,

Theorem 1.2.4.2. $L = \{a^n b^n c^n \mid n \geq 1\}$ is not a context-free language.

Proof. Suppose, by way of contradiction, that L is a context-free language with pumping length p. Without loss of generality, suppose it is in Chomsky Normal Form. Consider the string $s = a^p b^p c^p$. Write $s = s_1 s_2 s_3 s_4 s_5$ with $|s_2 s_4| \geq 1$ and $|s_2 s_3 s_4| \leq p$. We consider $s_2 s_3 s_4$: clearly, it cannot contain a, b, *and* c, for then it would have length greater than p. Thus, $s_2 s_3 s_4$ must be one of the following

1. $s_2 s_3 s_4 = a^i$

2. $s_2s_3s_4 = a^ib^j$
3. $s_2s_3s_4 = b^i$
4. $s_2s_3s_4 = b^ic^j$
5. $s_2s_3s_4 = c^i$

We prove each case separately:

1. Write $s_1 = a^x$, $s_2 = a^y$, $s_3 = a^z$, $s_4 = a^w$. Then $s_5 = a^{p-(x+y+z+w)}b^pc^p$ At least one of y and w is non-zero. Then, by the pumping lemma,

$$\begin{aligned} s_1s_2^ns_3s_4^ns_5 &= a^xa^{ny}a^za^{nw}a^{p-(x+y+z+w)}b^pc^p \\ &= a^{p+(n-1)(y+w)}b^pc^p \end{aligned}$$

The above is clearly not in L for any $n \geq 2$.

2. There are two cases:

   (a) s_2 contains b

   (b) s_2 does not contain b

   In case (a), we have $s_1 = a^x$, $s_2 = a^{p-x}b^y$, $s_3 = b^z$, $s_4 = b^w$, and $s_5 = b^{p-(y+z+w)}c^p$. Pumping gives

$$\begin{aligned} s_1s_2^ns_3s_4^ns_5 &= a^xa^{n(p-x)}b^{ny}b^zb^{nw}b^{p-(y+z+w)}c^p \\ &= a^{n(p-x)+x}b^{p+(n-1)(y+w)}c^p \end{aligned}$$

   For $n \geq 2$, there are more bs than cs, so the above string is not in L. A contradiction.

   In case (b), we have $s_1 = a^x$, $s_2 = a^y$, $s_3 = a^zb^w$, $s_4 =$

□

1.2.5 Closure Properties

Like regular languages, there are closure properties for context-free languages.

Theorem 1.2.5.1. If L_1 and L_2 are context-free languages with grammars $G_1 = (N_1, T, P_1, S_1)$ and $G_2 = (N_2, T, P_2, S_2)$, respectively, then

- $L_1 \cup L_2$
- L_1L_2
- L_1^*

are all context-free languages.

1.2.6 Pushdown Automata

Definition 1.2.6.1. A **Pushdown Automaton** is a septuple

$$\underset{\sim}{P} = (Q, T, \Gamma, \delta, q_0, Z_0, F)$$

where

Q is a finite non-empty set of states
T is an alphabet of input symbols
Γ is an alphabet of stack symbols
$\delta : Q \times (T \cup \{\varepsilon\}) \times \Gamma \to Q \times \Gamma^*$ is the move function
$q_0 \in Q$ is the initial state
$Z_0 \in \Gamma$ is the initial stack symbol
$F \subseteq Q$ is the set of accepting states

We define $(q, a, \gamma) \vdash (q', a', \gamma')$ when $\delta(q, a, \gamma) = (q', a', \gamma')$.

Informally, a pushdown automaton differs from a finite state machine in that it has a stack that allows it to:

1. Choose a transition based on the top of the stack
2. Push to or pop from the stack during a transition

Definition 1.2.6.2. Let $\underset{\sim}{P} = (Q, T, \Gamma, \delta, q_0, Z_0, F)$ be a pushdown automaton. The **language accepted by final state** is given by

$$L(\underset{\sim}{P}) = \{w \in T^* \mid (q_0, w, z_0) \vdash^* (f, \varepsilon, \gamma),\ \text{where } f \in F \text{ and } \gamma \in \Gamma^*\}$$

The **language accepted by empty stack** is given by

$$N(\underset{\sim}{P}) = \{w \in T^* \mid (q_0, w, z_0) \vdash^* (q, \varepsilon, \varepsilon),\ \text{where } q \in Q\}$$

where $\vdash^*$ denotes 0 or more transitions.

Consider the following example:

		a	b	ε
q_0	z_0	$(q_0, z_0 z)$	$\emptyset$	$\emptyset$
	z	(q_0, zz)	(q_1, ε)	$\emptyset$
q_1	z_0	$\emptyset$	$\emptyset$	(q_f, ε)
	z	$\emptyset$	(q_1, ε)	$\emptyset$
q_f	z_0		accept	
	z			

This describes the language $L = \{a^n b^n \mid n \geq 1\}$
Consider the following language on $\{0, 1\}$:

$$L = \{w2w^R \mid w \in \{0, 1\}^*\}$$

where w^R is the reflection of w. This is given by the context-free grammar

$$S \to 0S0 \mid 1S1 \mid 2$$

This can also be defined by the following pushdown automaton:

		0	1	2	ε
q_0	z_0	$(q_0, z_0 z)$	$\emptyset$	$\emptyset$	
	z	(q_0, zz)	(q_1, ε)	$\emptyset$	
q_1	z_0	$\emptyset$	$\emptyset$	(q_f, ε)	
	z	$\emptyset$	(q_1, ε)	$\emptyset$	
q_f	z_0		accept		
	z				

Theorem 1.2.6.1. For any pushdown automaton M, there exist pushdown automata P and Q such that $L(M) = N(P)$ and $N(L) = L(Q)$.

Chapter 2

Exercise Sets

2.1 Exercise Set 1

Exercise 1: Construct DFAs for the following NFAs using the subset construction:

(a)

	a	
$\to 1$	2	0
2	3	0
3	4	0
4	5	0
5	6	0
6	7	0
7	1, 2	1

(b)

	a	b	c	
$\to 1$	2	2	2	1
2	3	1	1, 2	1
3	4	3	$\emptyset$	1
4	5	4	4	1
5	1	5	5	1

(c)

	a	b	c	
$\to 1$	2	2	2	1
2	3	1	2, 3	1
3	4	3	$\emptyset$	1
4	5	4	4	1
5	1	5	5	1

Solution.

(a)

	a	
$\rightarrow$ 1	2	0
2	3	0
3	4	0
4	5	0
5	6	0
6	7	0
7	1, 2	1
1, 2	2, 3	0
2, 3	3, 4	0
3, 4	4, 5	0
4, 5	5, 6	0
5, 6	6, 7	0
6, 7	1, 2, 7	1
1, 2, 7	1, 2, 3	1
1, 2, 3	2, 3, 4	0
2, 3, 4	3, 4, 5	0
3, 4, 5	4, 5, 6	0
4, 5, 6	5, 6, 7	0
5, 6, 7	1, 2, 6, 7	1
1, 2, 6, 7	1, 2, 3, 7	1
1, 2, 3, 7	1, 2, 3, 4	1
1, 2, 3, 4	2, 3, 4, 5	0
2, 3, 4, 5	3, 4, 5, 6	0
3, 4, 5, 6	4, 5, 6, 7	0
4, 5, 6, 7	1, 2, 5, 6, 7	1
1, 2, 5, 6, 7	1, 2, 3, 6, 7	1
1, 2, 3, 6, 7	1, 2, 3, 4, 7	1
1, 2, 3, 4, 7	1, 2, 3, 4, 5	1
1, 2, 3, 4, 5	2, 3, 4, 5, 6	0
2, 3, 4, 5, 6	3, 4, 5, 6, 7	0
3, 4, 5, 6, 7	1, 2, 4, 5, 6, 7	1
1, 2, 4, 5, 6, 7	1, 2, 3, 5, 6, 7	1
1, 2, 3, 5, 6, 7	1, 2, 3, 4, 6, 7	1
1, 2, 3, 4, 6, 7	1, 2, 3, 4, 5, 7	1
1, 2, 3, 4, 5, 7	1, 2, 3, 4, 5, 6	1
1, 2, 3, 4, 5, 6	2, 3, 4, 5, 6, 7	0
2, 3, 4, 5, 6, 7	1, 2, 3, 4, 5, 6 7	1
1, 2, 3, 4, 5, 6, 7	1, 2, 3, 4, 5, 6, 7	1

(b)

	a	b	c	
$\rightarrow 1$	2	2	2	1
2	3	1	1, 2	1
3	4	3	$\emptyset$	1
1, 2	2, 3	1, 2	1, 2	1
4	5	4	4	1
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0
2, 3	3, 4	1, 3	1, 2	1
5	1	5	5	1
3, 4	4, 5	3, 4	4	1
1, 3	2, 4	2, 3	2	1
4, 5	1, 5	4, 5	4, 5	1
2, 4	3, 5	1, 4	1, 2, 4	1
1, 5	1, 2	2, 5	2, 5	1
3, 5	1, 4	3, 5	5	1
1, 4	2, 5	2, 4	2, 4	1
1, 2, 4	2, 3, 5	1, 2, 4	1, 2, 4	1
2, 5	1, 3	1, 5	1, 2, 5	1
2, 3, 5	1, 3, 4	1, 3, 5	1, 2, 5	1
1, 2, 5	1, 2, 3	1, 2, 5	1, 2, 5	1
1, 3, 4	2, 4, 5	2, 3, 4	2, 4	1
1, 3, 5	1, 2, 4	2, 3, 5	2, 5	1
1, 2, 3	2, 3, 4	1, 2, 3	1, 2	1
2, 4, 5	1, 3, 5	1, 4, 5	1, 2, 4, 5	1
2, 3, 4	3, 4, 5	1, 3, 4	1, 2, 4	1
1, 4, 5	1, 2, 5	2, 4, 5	2, 4, 5	1
1, 2, 4, 5	1, 2, 3, 5	1, 2, 4, 5	1, 2, 4, 5	1
3, 4, 5	1, 4, 5	3, 4, 5	4, 5	1
1, 2, 3, 5	1, 2, 3, 4	1, 2, 3, 5	1, 2, 5	1
1, 2, 3, 4	2, 3, 4, 5	1, 2, 3, 4	1, 2, 4	1
2, 3, 4, 5	1, 3, 4, 5	1, 3, 4, 5	1, 2, 4, 5	1
1, 3, 4, 5	1, 2, 4, 5	2, 3, 4, 5	2, 4, 5	1

(c)

	a	b	c	
$\to$ 1	2	2	2	1
2	3	1	2, 3	1
3	4	3	$\emptyset$	1
2, 3	3, 4	1, 3	2, 3	1
4	5	4	4	1
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0
3, 4	4, 5	3, 4	4	1
1, 3	2, 4	2, 3	2	1
5	1	5	5	1
4, 5	1, 5	4, 5	4, 5	1
2, 4	3, 5	1, 4	2, 3	1
1, 5	6	2, 5	2, 5	1
3, 5	1, 4	3, 5	5	1
1, 4	2, 5	2, 4	2, 4	1
6	2, 3	1, 2	2, 3	1
2, 5	1, 3	1, 5	2, 3, 5	1
2, 3, 5	1, 3, 4	1, 3, 5	2, 3, 5	1
1, 3, 4	2, 4, 5	2, 3, 4	2, 4	1
1, 4, 5	1, 2, 4	2, 4, 5	2, 4, 5	1
2, 4, 5	1, 3, 5	1, 4, 5	2, 3, 4, 5	1
2, 3, 4	3, 4, 5	1, 3, 4	2, 3, 4	1
1, 2, 4	2, 3, 5	1, 2, 4	2, 3, 4	1
1, 3, 5	1, 2, 4	2, 3, 5	2, 5	1
2, 3, 4, 5	1, 3, 4, 5	1, 3, 4, 5	2, 3, 4, 5	1
3, 4, 5	1, 4, 5	3, 4, 5	4, 5	1
1, 3, 4, 5	1, 2, 4, 5	2, 3, 4, 5	2, 4, 5	1
1, 2, 4, 5	1, 2, 3, 5	1, 2, 4, 5	2, 3, 4, 5	1
1, 2, 3, 5	1, 2, 3, 4	1, 2, 3, 5	2, 3, 5	1
1, 2, 3, 4	2, 3, 4, 5	1, 2, 3, 4	2, 3, 4	1

□

Exercise 2: Reduce the following DFAs:

(a)

	a	b	
$\to$ 1	2	3	0
2	3	2	1
3	4	5	0
4	1	8	1
5	6	7	0
6	7	6	1
7	8	1	0
8	5	4	1

(b)

	a	b	
$\to$ 1	2	3	0
2	3	2	1
3	4	5	0
4	1	8	1
5	6	7	0
6	7	6	1
7	8	1	0
8	5	5	1

(c) Your result of 1(b).

(d) Your result of 1(c).

Solution.

(a)

Rejecting	Accepting
1, 3, 5, 7	2, 4, 6, 8
1, 3, 5, 7	2, 4, 6, 8

Setting $p = \{1, 3, 5, 7\}$ and $q = \{2, 4, 6, 8\}$:

	a	b	
$\to p$	q	p	0
q	p	q	1

(b)

Rejecting		Accepting			
1, 3, 5, 7		2, 4, 6, 8			
1, 3, 5, 7		2, 4, 6			8
1, 3, 5	7	2, 6		4	8
1 3 5	7	2	6	4	8

The DFA is already reduced.

□

Exercise 3: Construct NFAs for the following regular expressions using the construction given in class; then find the corresponding DFAs; then reduce them:

(a) $(a^2 \cup a^3 \cup a^5)^*$ over $\{a\}$

(b) $(a^2)^*(a^3)^*(a^5)^*$ over $\{a\}$

(c) $(abc \cup ab)^* aa^*(ab)^*$ over $\{a, b, c\}$

(d) $0^*(00 \cup 11)^*(01 \cup 10)^*1^*$ over $\{0, 1\}$

Solution.

(a) NFA for a

	a	
$\to q_0$	q_1	0
q_1	$\emptyset$	1

NFA for a

	a	
$\to q_0$	q_2	0
q_2	$\emptyset$	1

Concatenate these to get a^2

NFA for a^2

	a	
$\to q_0$	q_1	0
q_1	q_2	0
q_2	$\emptyset$	1

Similarly, we have

NFA for a^3

	a	
$\to q_0$	q_3	0
q_3	q_4	0
q_4	q_5	0
q_5	$\emptyset$	1

NFA for a^5

	a	
$\to q_0$	q_6	0
q_6	q_7	0
q_7	q_8	0
q_8	q_9	0
q_9	$\emptyset$	1

(b)

□

Exercise 4: Construct regular expressions for the languages accepted by the following automata:

(a)

	a	b	c	
$\to 1$	2	2	2	1
2	3	1	2, 3	1
3	4	3	$\emptyset$	1
4	1	4	4	1

(b)

	a	b	
$\to A$	B	C	0
B	A	C	0
C	B	A	1

Solution. (a)

$$\begin{aligned}X_1 &= aX_2 \cup bX_2 \cup cX_2 \cup \varepsilon\\ X_2 &= aX_3 \cup bX_1 \cup c(X_2 \cup X_3) \cup \varepsilon\\ X_3 &= aX_4 \cup bX_3 \cup \varepsilon\\ X_4 &= aX_1 \cup bX_4 \cup cX_4 \cup \varepsilon\end{aligned}$$

Solving for X_4

$$\begin{aligned}X_4 &= aX_1 \cup bX_4 \cup cX_4 \cup \varepsilon\\ &= aX_1 \cup (b \cup c)X_4 \cup \varepsilon\\ &= (b \cup c)X_4 \cup aX_1 \cup \varepsilon\\ &= (b \cup c)^*(aX_1 \cup \varepsilon)\\ &= (b \cup c)^*aX_1 \cup (b \cup c)^*\end{aligned}$$

Similarly,

$$\begin{aligned}X_3 &= aX_4 \cup bX_3 \cup \varepsilon\\ &= a\left((b \cup c)^*aX_1 \cup (b \cup c)^*\right) \cup bX_3 \cup \varepsilon\\ &= a(b \cup c)^*aX_1 \cup a(b \cup c)^* \cup bX_3 \cup \varepsilon\\ &= bX_3 \cup a(b \cup c)^*aX_1 \cup a(b \cup c)^* \cup \varepsilon\\ &= b^*(a(b \cup c)^*aX_1 \cup a(b \cup c)^* \cup \varepsilon)\\ &= b^*a(b \cup c)^*aX_1 \cup b^*a(b \cup c)^* \cup b^*\end{aligned}$$

Solving X_2:

$$\begin{aligned}X_2 &= aX_3 \cup bX_1 \cup c(X_2 \cup X_3) \cup \varepsilon\\ &= (a \cup c)X_3 \cup bX_1 \cup cX_2 \cup \varepsilon\\ &= (a \cup c)(b^*a(b \cup c)^*aX_1 \cup b^*a(b \cup c)^* \cup b^*) \cup bX_1 \cup cX_2 \cup \varepsilon\\ &= cX_2 \cup (a \cup c)(b^*a(b \cup c)^*aX_1 \cup b^*a(b \cup c)^* \cup b^*) \cup bX_1 \cup \varepsilon\\ &= c^*((a \cup c)(b^*a(b \cup c)^*aX_1 \cup b^*a(b \cup c)^* \cup b^*) \cup bX_1 \cup \varepsilon)\\ &= c^*(a \cup c)(b^*a(b \cup c)^*aX_1 \cup c^*b^*a(b \cup c)^* \cup c^*b^*) \cup c^*bX_1 \cup c^*\\ &= c^*(a \cup c)b^*a(b \cup c)^*aX_1 \cup c^*(a \cup c)(c^*b^*a(b \cup c)^* \cup c^*b^*) \cup c^*bX_1 \cup c^*\\ &= (c^*(a \cup c)b^*a(b \cup c)^*a \cup c^*b)X_1 \cup c^*(a \cup c)(c^*b^*a(b \cup c)^* \cup c^*b^*) \cup c^*\end{aligned}$$

Finally, solving for X_1:

$$\begin{aligned}X_1 &= aX_2 \cup bX_2 \cup cX_2 \cup \varepsilon\\ &= (a \cup b \cup c)X_2 \cup \varepsilon\\ &= (a \cup b \cup c)((c^*(a \cup c)b^*a(b \cup c)^*a \cup c^*b)X_1 \cup c^*(a \cup c)(c^*b^*a(b \cup c)^* \cup c^*b^*) \cup c^*) \cup \varepsilon\\ &= (a \cup b \cup c)(c^*(a \cup c)b^*a(b \cup c)^*a \cup c^*b)X_1 \cup (a \cup b \cup c)c^*(a \cup c)(c^*b^*a(b \cup c)^* \cup c^*b^*) \cup c^* \cup \varepsilon\\ &= \left((a \cup b \cup c)(c^*(a \cup c)b^*a(b \cup c)^*a \cup c^*b)\right)^*((a \cup b \cup c)c^*(a \cup c)(c^*b^*a(b \cup c)^* \cup c^*b^*) \cup c^*) \cup \varepsilon\end{aligned}$$

(b)

$$\begin{aligned}X_A &= aX_B \cup bX_C\\ X_B &= aX_A \cup bX_C\\ X_C &= aX_B \cup bX_A \cup \varepsilon\end{aligned}$$

Plugging in the equation for X_C into X_B

$$\begin{aligned}X_B &= aX_A \cup b(aX_B \cup bX_A \cup \varepsilon)\\ &= aX_A \cup baX_B \cup b^2X_A \cup b\\ &= baX_B \cup (ba \cup b^2)X_A \cup b\\ &= (ba)^*((ba \cup b^2)X_A \cup b)\\ &= (ba)^*(ba \cup b^2)X_A \cup (ba)^*b\end{aligned}$$

We substitute back into X_C:

$$\begin{aligned}X_C &= aX_B \cup bX_A \cup \varepsilon\\ &= a((ba)^*(ba \cup b^2)X_A \cup (ba)^*b) \cup bX_A \cup \varepsilon\end{aligned}$$

We now substitute the new equations for X_B and X_C into the equation for X_A

$$\begin{aligned}X_A &= aX_B \cup bX_C\\ &= a((ba)^*(ba \cup b^2)X_A \cup (ba)^*b) \cup b(a((ba)^*(ba \cup b^2)X_A \cup (ba)^*b) \cup bX_A \cup \varepsilon)\\ &= a(ba)^*(ba \cup b^2)X_A \cup a(ba)^*b \cup ba((ba)^*(ba \cup b^2)X_A \cup b(ba)^*b) \cup b^2X_A \cup b\\ &= (a(ba)^*(ba \cup b^2) \cup ba(ba)^*(ba \cup b^2) \cup b^2)X_A \cup bab(ba)^*b \cup b^2X_A \cup b\\ &= (a(ba)^*(ba \cup b^2) \cup ba(ba)^*(ba \cup b^2) \cup b^2)^*bab(ba)^*b \cup b\end{aligned}$$

□

2.2 Exercise Set 2

Exercise 1: Prove that the following languages are not regular:

(a) $L = \{x \in (0 \cup 1)^*2(0 \cup 1)^* \mid \text{number of 0s before 2} = \text{number of 1s after 2}\}$

(b) $L = \{x \in (0 \cup 1)^*2(0 \cup 1)^* \mid \text{number of 0s before 2} \neq \text{number of 1s after 2}\}$

(c) $L = \{a^{i^2} \mid i \geq 1\}$

(d) $L = \{a^{2^i} \mid i \geq 1\}$

Solution.

(a) Suppose, by way of contradiction, that L is regular and that p is its pumping length. Consider the string $s = 0^p21^p$. Clearly, $|s| \geq p$. Thus, by the **Pumping Lemma**, there exist strings s_1, s_2, s_3 such that $s = s_1s_2s_3$ with $|s_1s_2| \leq p$ and $|s_2| \geq 1$ and, for all $n \geq 0$, $s_1s_2^ns_3 \in L$. Observe that $s_1s_2 = 0^k$ for some $k \leq p$ (for otherwise $|s_1s_2| > p$), hence $s_3 = 0^{p-k}21^p$. Thus, we write $s_1 = 0^{k-q}$ and $s_2 = 0^q$ for some $q \geq 1$. By the pumping lemma,

$$\begin{aligned}s_1s_2^ns_3 &= 0^{k-q}(0^q)^n0^{p-k}21^p\\ &= 0^{k-q}0^{qn}0^{p-k}21^p\\ &= 0^{p+q(n-1)}21^p\end{aligned}$$

is in L. However, for $n \geq 2$, there are more 0s before the 2 than 1s after, hence $s_1s_2^ns_3 \notin L$. A contradiction. Thus, L is not regular.

(b) Suppose, by way of contradiction, that L is regular and that p is its pumping length. Consider the string $s = 0^p21^{p+p!}$. Clearly, $|s| \geq p$. Thus, by the **Pumping Lemma**, there exist strings s_1, s_2, s_3 such that $s = s_1s_2s_3$ with $|s_1s_2| \leq p$ and $|s_2| \geq 1$ and, for all $n \geq 0$, $s_1s_2^ns_3 \in L$. Observe that $s_1s_2 = 0^k$ for some $k \leq p$ (for otherwise $|s_1s_2| > p$), hence $s_3 = 0^{p-k}21^{p+p!}$. Thus, we write $s_1 = 0^{k-q}$ and $s_2 = 0^q$ for some $q \geq 1$. By the pumping lemma,

$$\begin{aligned}s_1s_2^ns_3 &= 0^{k-q}(0^q)^n0^{p-k}21^{p+p!}\\ &= 0^{k-q}0^{qn}0^{p-k}21^{p+p!}\\ &= 0^{p+q(n-1)}21^{p+p!}\end{aligned}$$

is in L. Now, since $q \leq p$, $q \mid p!$. Thus, taking $n = \frac{p!}{q} + 1$, we have

$$\begin{aligned}s_1s_2^ns_3 &= 0^{p+q(\frac{p!}{q}+1-1)}21^{p+p!}\\ &= 0^{p+q(\frac{p!}{q})}21^{p+p!}\\ &= 0^{p+p!}21^{p+p!}\end{aligned}$$

Thus, $s_1s_2^ns_3 \notin L$, a contradiction. Therefore L is not regular.

(c) In order to reach a contradiction, suppose L is regular and that p is its pumping length. Consider the string $s = a^{p^2}$. By the pumping lemma, we have $s = s_1s_2s_3$. Since $|s_1s_2| \leq p$, this forces $s_1s_2 = a^k$ for some $0 < k \leq p$ and $s_3 = a^{p^2-k}$. Then $s_1 = a^{k-r}$ and $s_2 = a^r$ for some $0 < r \leq k$. Then

$$\begin{aligned} s_1s_2^ns_3 &= a^{k-r}(a^r)^n a^{p^2-k} \\ &= a^{k-r}a^{rn}a^{p^2-k} \\ &= a^{p^2+rn-r} \\ &= a^{p^2+r(n-1)} \end{aligned}$$

Take $n = 2$. Then $s_1s_2^ns_3 = a^{p^2+r} \in L$. However, $p^2 + r$ cannot be a perfect square: since $r \leq p$, we have

$$\begin{aligned} p^2 + r &\leq p^2 + p \\ &< p^2 + p + 1 \\ &= (p+1)^2 \end{aligned}$$

Thus, $s_1s_2^ns_3 \notin L$, a contradiction. Therefore L is not regular.

(d) In order to reach a contradiction, suppose L is regular and that p is its pumping length. Consider the string $s = a^{2^p}$. By the pumping lemma, we have $s = s_1s_2s_3$. Since $|s_1s_2| \leq p$, this forces $s_1s_2 = a^k$ for some $0 < k \leq p$ and $s_3 = a^{2^p-k}$. Then $s_1 = a^{k-r}$ and $s_2 = a^r$ for some $0 < r \leq k$. Then

$$\begin{aligned} s_1s_2^ns_3 &= a^{k-r}(a^r)^n a^{2^p-k} \\ &= a^{k-r}a^{rn}a^{2^p-k} \\ &= a^{2^p+rn-r} \\ &= a^{2^p+r(n-1)} \end{aligned}$$

Take $n = 2$. Then $s_1s_2^ns_3 = a^{2^p+r} \in L$. However, $2^p + r$ cannot be a power of 2: since $r \leq p$, we have

$$\begin{aligned} 2^p + r &\leq 2^p + p \\ &< 2^p + 2^p \\ &= 2^{p+1} \end{aligned}$$

Thus, $s_1s_2^ns_3 \notin L$, a contradiction. Therefore L is not regular.

$\square$

Exercise 2: Construct DFAs for the following extended regular expressions:

(a) $\left[\overline{(000)^*} \cap \left((01)^* \cup (10)^*\right)\right] \cap \overline{(11)^*}$ over $\{0, 1\}$

(b) $\overline{(0 \cup 1)^*01^*10^*01^*1(0 \cup 1)^*} - (10 \cup 01)^*$ over $\{0, 1, 2\}$

Exercise 3: Construct Chomsky Normal Form grammars for $L(G) - \varepsilon$ for the following cfgs G:

(a) $G = (\{a, b\}, \{S, A\}, S, \{S \to aAAaS \mid a, A \to bAA \mid aSSSA \mid \varepsilon\})$

(b) $G = (\{a, b, c\}, \{S, A, B\}, S, \{S \to aAA \mid A, A \to bBBB \mid B \mid \varepsilon, B \to bSSS \mid S \mid \varepsilon\})$

Solution.

(a)

$$\begin{aligned} S &\to aAAaS \mid a \\ A &\to bAA \mid aSSSA \mid \varepsilon \end{aligned}$$

There are no useless productions, since $S \Rightarrow a$ and $A \Rightarrow b$. A is clearly nullable, so we get

$$\begin{aligned} S &\to aAAaS \mid a\cancel{A}AaS \mid aA\cancel{A}aS \mid a\cancel{A}\cancel{A}aS \mid a \\ A &\to bAA \mid b\cancel{A}A \mid bA\cancel{A} \mid b\cancel{A}\cancel{A} \mid aSSSA \mid aSSS\cancel{A} \end{aligned}$$

which reduces to

$$\begin{aligned} S &\to aAAaS \mid aAaS \mid aaS \mid a \\ A &\to bAA \mid bA \mid b \mid aSSSA \mid aSSS \end{aligned}$$

There are no unit productions. Setting $X_a \to a$ and $X_b \to b$, we have

$$\begin{aligned} S &\to X_aAAX_aS \mid X_aAX_aS \mid X_aX_aS \mid a \\ A &\to X_bAA \mid X_bA \mid b \mid X_aSSSA \mid X_aSSS \end{aligned}$$

Finally, we decompose:

$$\begin{aligned} S &\to X_aS_1 \mid X_aS_2 \mid X_aS_3 \mid a \\ A &\to X_bA_1 \mid X_bA \mid b \mid X_aA_2 \mid X_aA_5 \\ S_1 &\to AS_2 \\ S_2 &\to AS_3 \\ S_3 &\to X_aS \\ A_1 &\to AA \\ A_2 &\to SA_3 \\ A_3 &\to SA_4 \\ A_4 &\to SA \\ A_5 &\to SA_6 \\ A_6 &\to SS \end{aligned}$$

(b)

$$\begin{aligned} S &\to aAA \mid A \\ A &\to bBBB \mid B \mid \varepsilon \\ B &\to bSSS \mid S \mid \varepsilon \end{aligned}$$

There are no useless productions, since $S \Rightarrow a$, $A \Rightarrow b$ and $B \Rightarrow b$. Clearly, S, A, and B are all nullable. Thus, we have

$$\begin{aligned} S &\to aAA \mid a\cancel{A} \mid aA\cancel{A} \mid a\cancel{A}\cancel{A} \mid A \mid \cancel{A} \\ A &\to bBBB \mid b\cancel{B}BB \mid bB\cancel{B}B \mid bBB\cancel{B} \mid b\cancel{B}\cancel{B}B \mid b\cancel{B}B\cancel{B} \mid bB\cancel{B}\cancel{B} \mid B \mid \cancel{B} \\ B &\to bSSS \mid b\cancel{S}SS \mid bS\cancel{S}S \mid bSS\cancel{S} \mid b\cancel{S}\cancel{S}S \mid b\cancel{S}S\cancel{S} \mid bS\cancel{S}\cancel{S} \mid b\cancel{S}\cancel{S}\cancel{S} \mid S \mid \cancel{S} \end{aligned}$$

which reduces to

$$\begin{aligned}
S &\to aAA \mid a \mid aA \mid A \\
A &\to bBBB \mid bBB \mid bB \mid B \\
B &\to bSSS \mid bSS \mid bS \mid b \mid S
\end{aligned}$$

We eliminate the unit productions $S \to A$, $A \to B$ and $B \to S$.

$$\begin{aligned}
S &\to aAA \mid a \mid aA \mid bBBB \mid bBB \mid bB \\
A &\to bBBB \mid bBB \mid bB \mid bSSS \mid bSS \mid bS \mid b \\
B &\to bSSS \mid bSS \mid bS \mid b \mid aAA \mid a \mid aA \mid bBBB \mid bBB \mid bB
\end{aligned}$$

We now set $X_a \to a$ and $X_b \to b$:

$$\begin{aligned}
S &\to X_aAA \mid a \mid X_aA \mid X_bBBB \mid X_bBB \mid X_bB \\
A &\to X_bBBB \mid X_bBB \mid X_bB \mid X_bSSS \mid X_bSS \mid X_bS \mid b \\
B &\to X_bSSS \mid X_bSS \mid X_bS \mid b \mid X_aAA \mid a \mid X_aA \mid X_bBBB \mid X_bBB \mid X_bB
\end{aligned}$$

Finally, we decompose:

$$\begin{aligned}
S &\to X_aS_1 \mid a \mid X_aA \mid X_bS_2 \mid X_bS_3 \mid X_bB \\
A &\to X_bS_2 \mid X_bS_3 \mid X_bB \mid X_bA_1 \mid X_bA_2 \mid X_bS \mid b \\
B &\to X_bA_1 \mid X_bA_2 \mid X_bS \mid b \mid X_aS_1 \mid a \mid X_aA \mid X_bS_2 \mid X_bS_3 \mid X_bB \\
S_1 &\to AA \\
S_2 &\to BS_3 \\
S_3 &\to BB \\
A_1 &\to SA_2 \\
A_2 &\to SS
\end{aligned}$$

□

Exercise 4: Construct Greibach Normal Form grammars for $L(G) - \varepsilon$ for the following cfgs G:

(a) $G = (\{a, b\}, \{S, A, B\}, S, \{S \to SaS \mid A, A \to AAAb \mid B \mid \varepsilon, B \to SSS \mid a\})$

(b) $G = (\{a, b\}, \{S, A, B, C\}, S, \{S \to ASS \mid a, A \to bBBB \mid BAA \mid \varepsilon, B \to CSS \mid SSC, C \to SS \mid b\})$

Solution.

(a)

(b)

□

2.3 Exercise Set 3

Exercise 1: Prove that the following languages are not context-free:

(a) $L = \{0^i1^j2^k \mid 0 \leq i < j < k\}$

(b) $L = \{0^{n^2}1^n \mid n \geq 0\}$

(c) $L = \{0^n1^n2^n \mid n \geq 0\}$

(d) $L = \{0^i1^j2^k \mid i > 2j > 3k \geq 1\}$

Solution.

(a) Suppose, by way of contradiction, that L is context-free with pumping length p. Then, by the pumping lemma, there exist strings s_1, s_2, s_3, s_4, and s_5 such that

(b) In order to reach a contradiction, suppose that L is context-free with pumping length p. Then, by the pumping lemma, there exist strings s_1, s_2, s_3, s_4, and s_5 such that

$$s = 0^{p^2}1^p = s_1s_2s_3s_4s_5$$

with $|s_2s_3s_4| \leq p$ and $|s_2s_4| \geq 1$. There are five cases, corresponding to the first occurrence of a 1 in s

Case 1: If the first 1 occurs in s_1, then we write $s_1 = 0^{p^2}1^i$, $s_2 = 1^j$, $s_3 = 1^k$, $s_4 = 1^l$, and $s_5 = 1^{p-(i+j+k+l)}$. Then

$$\begin{aligned} s_1s_2^ns_3s_4^ns_5 &= 0^{p^2}1^i1^{nj}1^k1^{nl}1^{p-(i+j+k+l)} \\ &= 0^{p^2}1^{p+(n-1)(j+l)} \end{aligned}$$

Taking $n = 2$ yields a string with p^2 0s but $p + j + l > p$ 1s, a contradiction.

Case 2: If the first 1 occurs in s_2, then we write $s_1 = 0^i$, $s_2 = 0^{p^2-i}1^j$, $s_3 = 1^k$, $s_4 = 1^l$, and $s_5 = 1^{p-(j+k+l)}$. Then

$$\begin{aligned} s_1s_2^ns_3s_4^ns_5 &= 0^i0^{n(p^2-i)}1^{nj}1^k1^{nl}1^{p-(j+k+l)} \\ &= 0^{np^2-(n-1)i}1^{p+(n-1)(j+l)} \end{aligned}$$

Taking $n = 0$ yields a string

□

Exercise 2: Construct PDAs P such that

i. $L = L(P)$

ii. $L = N(P)$

for the following languages:

(a) $L = \{0^i1^i \mid i \geq 0\} \cup \{0^i1^{2i} \mid i \geq 0\}$

(b) $L = L(G)$ with G given by $S \to aSd \mid aAd, A \to bAc \mid bc$

(c) $L = L(G)$ with G given by $S \to aSd \mid aAd, A \to bAc \mid bSc \mid \varepsilon$

(d) $L = L(G)$ with G given by $E \to +EE \mid *EE \mid id$

Exercise 3: Construct a grammar for the language $N(P)$:

$$P = (\{p, q\}, \{0, 1\}, \{Z, X\}, \delta, p, Z, \emptyset)$$

where

$$\begin{array}{lll} \delta(p, 1, Z) = \{(p, XZ)\} & \delta(p, \varepsilon, Z) = \{(p, \varepsilon)\} & \delta(p, 1, X) = \{(p, XX)\} \\ \delta(a, 1, X) = \{(q, \varepsilon)\} & \delta(p, 0, X) = \{(q, X)\} & \delta(q, 0, Z) = \{(p, Z)\} \end{array}$$

Exercise 4: Construct Turing Machines for each of the languages in exercise 1.

Chapter 3

Exam Cheatsheets

3.1 Exam 1

3.1.1 Converting an NFA to a DFA

Begin with the initial state. Apply the transitions and add any newly visited states. Stop when no new states can be visited.

3.1.2 Reducing a DFA

Partition all states into accepting vs rejecting. Apply all transitions to a "representative" from a partition, then apply the transitions to the remaining members of the partition. If a member differs, move it to a new partition.

3.1.3 Converting a Regular Expression to an NFA

NFA for a:

	a	$b \neq a$	
$\rightarrow q_0$	q_1	$\emptyset$	0
q_1	$\emptyset$	$\emptyset$	1

NFA for ε:

	$c \in A$	
$\rightarrow q_0$	$\emptyset$	1

NFA for $\emptyset$:

	$c \in A$	
$\rightarrow q_0$	$\emptyset$	0

Union

Union initial states, copy remaining states from α and β. Final states are final states from α and β.

Concatenation

For final states of α, union the state from α with the initial state from β. Copy non-final states from α and non-initial states from β. Final states:

If the initial state is rejecting in β, final states are final states from β.

If the initial state is accepting in β, final states are final states from α and β, except the initial state in β.

Kleene Closure

Union final states with the initial state. Copy non-final states. Final states are final states from α and the initial state.

3.1.4 Converting an NFA to a Regular Expression

Set up system of equations. Solve for equation corresponding to initial state. Remember the lemma: if

$$X = LX \cup M$$

then

$$X = L^* M$$

3.2 Exam 2

3.2.1 Proving that a Language is not Regular

Pick a word w in the language. Write $w = xyz$ and pump: show that $xy^n z$ is not in the language. This is a contradiction.

3.2.2 Eliminating ε Productions

Find the nullable non-terminals (variables). These are the non-terminals from which ε can be derived. Specifically, a variable A is nullable if it is of the form

$$A \to \varepsilon$$

or

$$A \to A_1 A_2 A_3 \dots A_n$$

where each A_i is nullable. Then, simply replace each *combination of nullable variables* with ε and eliminate ε from the right-hand side.

3.2.3 Finding a DFA for an Extended Regular Expression

This is the same as finding a DFA for a regular expression, except when finding the complement. To find the complement, turn the NFA into a DFA and flip the accepting and rejecting states. Recall De Morgan's Laws

$$\begin{aligned}\overline{A \cup B} &= \overline{A} \cap \overline{B}\\ \overline{A \cap B} &= \overline{A} \cup \overline{B}\\ A - B &= A \cap \overline{B}\end{aligned}$$

3.2.4 Converting to Chomsky Normal Form

Eliminate useless variables, ε-productions, and unit productions. Then replace non-solitary terminals a with X_a and add $X_a \to a$ to your grammar. Finally, decompose any $X \to X_1 X_2 \dots X_n$ as

$$\begin{aligned}X &\to X_1 A_1\\ A_1 &\to X_2 A_2\\ A_2 &\to X_3 A_3\\ &\vdots\\ A_{n-2} &\to X_{n-1} X_n\end{aligned}$$

3.2.5 Converting to Greibach Normal Form

Eliminate useless variables, ε-productions, and unit productions. Order the variables. Then, where any higher-listed variable produces a lower-listed variable through left-recursion, replace the lower-listed variable. Remove all direct left-recursions with the following rule:

$$A \to A\alpha_1 \mid A\alpha_2 \mid A\alpha_3 \mid \dots \mid A\alpha_m \mid \beta_1 \mid \beta_2 \mid \beta_3 \mid \dots \mid \beta_n$$

where no β_i begins with A, by replacing the productions for A with

$$\begin{aligned} A &\to \beta_1 \mid \beta_2 \mid \beta_3 \mid \dots \mid \beta_n \mid \beta_1 A' \mid \beta_2 A' \mid \beta_3 A' \mid \dots \mid \beta_n A' \\ A' &\to \alpha_1 \mid \alpha_2 \mid \alpha_3 \mid \dots \mid \alpha_m \mid \alpha_1 A' \mid \alpha_2 A' \mid \alpha_3 A' \mid \dots \mid \alpha_m A' \end{aligned}$$

Finally, substitute productions until they are all of the form

$$X \to a$$

or

$$X \to aX_1X_2\dots X_n$$

3.3 Exam 3

3.3.1 Proving a Language is not Context-Free

In order to reach a contradiction, assume the language is context-free with pumping length p. Then, choose a string s in the language and write $s = uvwxy$. Pump s as uv^nwx^ny and show that the pumped string is not in the language for some $n \geq 0$. Usually, this requires division into cases, e.g. "the string contains only the letter a".

3.3.2 Finding a Pushdown Automaton for a Language

For all PDAs, left is top of stack.

PDA for 0^i1^{ni} accepting by Empty Stack

		0	1	ε
q_0	Z_0	(q_0, Z^nZ_0)	$\emptyset$	(q_1, ε)
	Z	(q_0, Z^{n+1})	(q_1, ε)	$\emptyset$
q_1	Z_0	$\emptyset$	$\emptyset$	(q_1, ε)
	Z	$\emptyset$	(q_1, ε)	$\emptyset$

PDA for $0^{ni}1^i$ accepting by Empty Stack

		0	1	ε
q_0	Z_0	(q_1, Z_0)	$\emptyset$	(q_0, ε)
	Z	(q_1, Z)	(q_n, ε)	$\emptyset$
q_1	Z_0	(q_2, Z_0)	$\emptyset$	$\emptyset$
	Z	(q_2, Z)	$\emptyset$	$\emptyset$
		$\vdots$		
q_i	Z_0	(q_{i+1}, Z_0)	$\emptyset$	$\emptyset$
	Z	(q_{i+1}, Z)	$\emptyset$	$\emptyset$
		$\vdots$		
q_{n-1}	Z_0	(q_0, ZZ_0)	$\emptyset$	$\emptyset$
	Z	(q_0, Z^2)	$\emptyset$	$\emptyset$
q_n	Z_0	$\emptyset$	$\emptyset$	(q_n, ε)
	Z	$\emptyset$	(q_n, ε)	$\emptyset$

This PDA accepts by empty stack. To change to accepting by final state, add auxhiliary states q_0', q_f and stack letter Z_0':

PDA for 0^i1^{ni} accepting by Final State

		0	1	ε
q_0'	Z_0'	$\emptyset$	$\emptyset$	(q_0, Z_0Z_0')
	Z_0	$\emptyset$	$\emptyset$	$\emptyset$
	Z	$\emptyset$	$\emptyset$	$\emptyset$
q_0	Z_0'	$\emptyset$	$\emptyset$	$\emptyset$
	Z_0	(q_0, Z^nZ_0)	$\emptyset$	(q_1, ε)
	Z	(q_0, Z^{n+1})	(q_1, ε)	$\emptyset$
q_1	Z_0'	$\emptyset$	$\emptyset$	(q_f, ε)
	Z_0	$\emptyset$	$\emptyset$	(q_1, ε)
	Z	$\emptyset$	(q_1, ε)	$\emptyset$
q_f	Z_0'			
	Z_0		accepting	
	Z			

PDA for $0^{ni}1^i$ accepting by Final State

		0	1	ε
q_0'	Z_0'	$\emptyset$	$\emptyset$	(q_0, Z_0Z_0')
	Z_0	$\emptyset$	$\emptyset$	$\emptyset$
	Z	$\emptyset$	$\emptyset$	$\emptyset$
q_0	Z_0'	$\emptyset$	$\emptyset$	(q_f, ε)
	Z_0	(q_1, Z_0)	$\emptyset$	(q_0, ε)
	Z	(q_1, Z)	(q_n, ε)	$\emptyset$
q_1	Z_0'	$\emptyset$	$\emptyset$	$\emptyset$
	Z_0	(q_2, Z_0)	$\emptyset$	$\emptyset$
	Z	(q_2, Z)	$\emptyset$	$\emptyset$
			$\vdots$	
q_i	Z_0'	$\emptyset$	$\emptyset$	$\emptyset$
	Z_0	(q_{i+1}, Z_0)	$\emptyset$	$\emptyset$
	Z	(q_{i+1}, Z)	$\emptyset$	$\emptyset$
			$\vdots$	
q_{n-1}	Z_0'	$\emptyset$	$\emptyset$	$\emptyset$
	Z_0	(q_0, ZZ_0)	$\emptyset$	$\emptyset$
	Z	(q_0, Z^2)	$\emptyset$	$\emptyset$
q_n	Z_0'	$\emptyset$	$\emptyset$	(q_f, ε)
	Z_0	$\emptyset$	$\emptyset$	(q_n, ε)
	Z	$\emptyset$	(q_n, ε)	$\emptyset$
q_f	Z_0'			
	Z_0		accepting	
	Z			

3.3.3 Converting a CFG into a PDA

First, convert the CFG into Greibach Normal Form. Then, the non-terminals (variables) are the rows (stack symbols) of your PDA, with a single state q. The columns are just the terminals. For any production $X \to aX_1X_2\dots X_n$, add $(q, X_1X_2\dots X_n)$ to the row, column (X, A). If $X \to a$, add (q, ε).

3.3.4 Converting a PDA into a CFG

Suppose there are n states $q_1, q_2, \dots, q_n$. First, add the start productions:

$$S \to (q_1, Z, q_1) \mid (q_1, Z, q_2) \mid \dots \mid (q_1, Z, q_n)$$

For every $(p, XZ) \in \delta(p, b, Z)$, add the productions

$$\begin{aligned}
(p, Z, q_1) &\to b(p, X, q_1)(q_1, Z, q_1) \mid b(p, X, q_2)(q_2, Z, q_1) \mid \dots \mid b(p, X, q_n)(q_n, Z, q_1)\\
(p, Z, q_2) &\to b(p, X, q_1)(q_1, Z, q_2) \mid b(p, X, q_2)(q_2, Z, q_2) \mid \dots \mid b(p, X, q_n)(q_n, Z, q_2)\\
&\vdots\\
(p, Z, q_n) &\to b(p, X, q_1)(q_1, Z, q_n) \mid b(p, X, q_2)(q_2, Z, q_n) \mid \dots \mid b(p, X, q_n)(q_n, Z, q_n)
\end{aligned}$$

If $(p, \varepsilon) \in \delta(p, a, X)$, add

$$(p, X, p) \to a$$

3.3.5 Constructing a Turing Machine

3.4 Exam 4

3.4.1 Recursively Enumerable Languages

If L_1 and L_2 are recursive, then

- L_1^*
- $L_1 \cdot L_2$
- $L_1 \cup L_2$
- $L_1 \cap L_2$
- $\overline{L_1}$
- $L_1 - L_2$

are recursive. However, if L_1 and L_2 are recursively enumerable, then

- L_1^*
- $L_1 \cdot L_2$
- $L_1 \cup L_2$
- $L_1 \cap L_2$

are also recursively enumerable. If L is recursively enumerable, then $\overline{L}$ is recursively enumerable if and only if L is recursive.

For all of the below cases, we answer six questions:

i. Does the language contain a given fixed word w?

ii. Does the language not contain a given fixed word w?

iii. Is the language empty?

iv. Is the language non-empty?

v. Does the language contain all words?

vi. Are there any words not in the language?

1. L_1 and L_2 are both recursive.

   a. $L_1 \cup L_2$

      i. We construct a Turing Machine which always halts and accepts if and only if w is in $L_1 \cup L_2$: simply accept if either Turing Machine for L_1 or L_2 accepts w, otherwise reject. Hence this problem is recursive.

      ii. We construct a Turing Machine which always halts and accepts if and only if w is not in $L_1 \cup L_2$: simply accept w if both Turing Machines for L_1 and L_2 reject w, otherwise reject. Hence this problem is recursive.

      iii. This problem is not recursively enumerable; if $L = \emptyset$, then we must verify all words are not in L. However, there are infinitely many words, and thus any Turing Machine which determines if L is empty will never halt.

      iv. Construct a Turing Machine which accepts if and only if there exists a word w in $L_1 \cup L_2$ (i.e., it is non-empty): enumerate all words as $w_1, w_2, \ldots$. For $i = 0$ to ∞, accept if either Turing Machine for L_1 or L_2 accepts w_i. If no such word exists, then the Turing Machine will not halt. Thus, this problem is recursively enumerable but not recursive.

      v. This problem is not recursively enumerable; if L contains all words, then we must verify all words are in L. However, there are infinitely many words, and thus any Turing Machine which determines if L contains all words will never halt.

      vi. Construct a Turing Machine which accepts if and only if there exists a word w not in the language: enumerate all words as $w_1, w_2, \ldots$. For $i = 0$ to ∞, accept if both Turing Machines for L_1 and L_2 reject w_i. If no such word exists, then the Turing Machine will not halt. Thus, this problem is recursively enumerable but not recursive.

   b. $L_1 \cap L_2$

      i. We construct a Turing Machine which always halts and accepts if and only if w is in $L_1 \cap L_2$: simply accept if both Turing Machines for L_1 and L_2 accept w, otherwise reject. Hence this problem is recursive.

      ii. We construct a Turing Machine which always halts and accepts if and only if w is not in $L_1 \cap L_2$: simply accept w if either Turing Machine for L_1 or L_2 reject w, otherwise reject. Hence this problem is recursive.

      iii. This problem is not recursively enumerable; if $L = \emptyset$, then we must verify all words are not in L. However, there are infinitely many words, and thus any Turing Machine which determines if L is empty will never halt.

      iv. Construct a Turing Machine which accepts if and only if there exists a word w in $L_1 \cap L_2$ (i.e., it is non-empty): enumerate all words as $w_1, w_2, \ldots$. For $i = 0$ to ∞, accept if both Turing Machines for L_1 and L_2 accept w_i. If no such word exists, then the Turing Machine will not halt. Thus, this problem is recursively enumerable but not recursive.

      v. This problem is not recursively enumerable; if L contains all words, then we must verify all words are in L. However, there are infinitely many words, and thus any Turing Machine which determines if L contains all words will never halt.

      vi. Construct a Turing Machine which accepts if and only if there exists a word w not in the language: enumerate all words as $w_1, w_2, \ldots$. For $i = 0$ to ∞, accept if either Turing Machines for L_1 or L_2 rejects w_i. If no such word exists, then the Turing Machine will not halt. Thus, this problem is recursively enumerable but not recursive.

   c. $L_1 - L_2$

      i. We construct a Turing Machine which always halts and accepts if and only if w is in $L_1 - L_2$: simply accept if the Turing Machine for L_1 accepts w and the Turing Machine for L_2 rejects w, otherwise reject. Hence this problem is recursive.

      ii. We construct a Turing Machine which always halts and accepts if and only if w is not in $L_1 - L_2$: simply accept if the Turing Machine for L_1 rejects w or the Turing Machine for L_2 accepts w, otherwise reject. Hence this problem is recursive.

      iii. This problem is not recursively enumerable; if $L = \emptyset$, then we must verify all words are not in L. However, there are infinitely many words, and thus any Turing Machine which determines if L is empty will never halt.

iv. Construct a Turing Machine which accepts if and only if there exists a word w in $L_1 - L_2$ (i.e., it is non-empty): enumerate all words as $w_1, w_2, \ldots$. For $i = 0$ to ∞, accept if the Turing Machine for L_1 accepts w_i and the Turing Machine for L_2 does not. If no such word exists, then the Turing Machine will not halt. Thus, this problem is recursively enumerable but not recursive.

v. This problem is not recursively enumerable; if L contains all words, then we must verify all words are in L. However, there are infinitely many words, and thus any Turing Machine which determines if L contains all words will never halt.

vi. Construct a Turing Machine which accepts if and only if there exists a word w not in the language: enumerate all words as $w_1, w_2, \ldots$. For $i = 0$ to ∞, accept if
   - the Turing Machines for L_1 rejects
   - the Turing Machine for L_2 accepts w_i.

   If no such word exists, then the Turing Machine will not halt. Thus, this problem is recursively enumerable but not recursive.

2. L_1 is recursively enumerable but not recursive and L_2 is recursive.

a. $L_1 \cup L_2$

i. We construct a Turing Machine which accepts if and only if w is in $L_1 \cup L_2$: simply accept if either Turing Machine for L_1 or L_2 accepts w, otherwise reject. Note, we run these machines in parallel: perform move 1 on L_1, then L_2, then move 2, and so on. If w is not in the language, then the Turing Machine may not halt. Hence this problem is recursively enumerable but not recursive.

ii. This problem is not recursively enumerable; if the word is not in $L_1 \cup L_2$, then it is not in L_1 or L_2. Then any Turing Machine for this problem must determine if $w \notin L_1$, which may not halt.

iii. This problem is not recursively enumerable; if $L = \emptyset$, then we must verify all words are not in L. However, there are infinitely many words, and thus any Turing Machine which determines if L is empty will never halt.

iv. Construct a Turing Machine which accepts if and only if there exists a word w in $L_1 \cup L_2$ (i.e., it is non-empty): enumerate all words as $w_1, w_2, \ldots$. For $i = 0$ to ∞, accept if either Turing Machine for L_1 or L_2 accepts w_i. Note, we run these machines in parallel: perform move 1 on w_1, then move 2 on w_1, then move 1 on w_2, then move 3 on w_1, and so on. If no such word exists, then the Turing Machine will not halt. Thus, this problem is recursively enumerable but not recursive.

v. This problem is not recursively enumerable; if L contains all words, then we must verify all words are in L. However, there are infinitely many words, and thus any Turing Machine which determines if L contains all words will never halt.

vi. Construct a Turing Machine which accepts if and only if there exists a word w not in the language: enumerate all words as $w_1, w_2, \ldots$. For $i = 0$ to ∞, accept if both Turing Machines for L_1 and L_2 reject w_i. Note, we run these machines in parallel: perform move 1 on w_1, then move 2 on w_1, then move 1 on w_2, then move 3 on w_1, and so on. If no such word exists, then the Turing Machine will not halt. Thus, this problem is recursively enumerable but not recursive.

b. $L_1 \cap L_2$

i. We construct a Turing Machine which always halts and accepts if and only if w is in $L_1 \cap L_2$: simply accept if both Turing Machines for L_1 and L_2 accepts w, otherwise reject. If w is in the language, then it is in both L_1 and L_2, and both machines will halt and accept on w. Otherwise, w is not in L_1, so the machine for L_1 will never halt. Thus, the problem is recursively enumerable but not recursive.

ii. This problem is not recursively enumerable; if the word is not in $L_1 \cap L_2$, then it is not in L_1 nor L_2. Then any Turing Machine for this problem must determine if $w \notin L_1$, which may not halt.

iii. This problem is not recursively enumerable; if $L = \emptyset$, then we must verify all words are not in L. However, there are infinitely many words, and thus any Turing Machine which determines if L is empty will never halt.

iv. Construct a Turing Machine which accepts if and only if there exists a word w in $L_1 \cap L_2$ (i.e., it is non-empty): enumerate all words as $w_1, w_2, \ldots$. For $i = 0$ to ∞, accept if both Turing Machines for L_1 and L_2 accept w_i. Note, we run these machines in parallel: perform move 1 on w_1, then move 2 on w_1, then move 1 on w_2, then move 3 on w_1, and so on. If no such word exists, then the Turing Machine will not halt. Thus, this problem is recursively enumerable but not recursive.

v. This problem is not recursively enumerable; if L contains all words, then we must verify all words are in L. However, there are infinitely many words, and thus any Turing Machine which determines if L contains all words will never halt.

vi. Construct a Turing Machine which accepts if and only if there exists a word w not in the language: enumerate all words as $w_1, w_2, \ldots$. For $i = 0$ to ∞, accept if both Turing Machines for L_1 and L_2 reject w_i. Note, we run these machines in parallel: perform move 1 on w_1, then move 2 on w_1, then move 1 on w_2, then move 3 on w_1, and so on. If no such word exists, then the Turing Machine will not halt. Thus, this problem is recursively enumerable but not recursive.

c. $L_1 - L_2$

3. L_1 is recursive and L_2 is recursively enumerable but not recursive.

4. L_1 and L_2 are both recursively enumerable.

We write "r.e." to indicate "recursively enumerable but not recursive".

L_1 and L_2 are both recursive			
	$L_1 \cup L_2$	$L_1 \cap L_2$	$L_1 - L_2$
Does the language contain a given fixed word w?	recursive	recursive	recursive
Does the language not contain a given fixed word w?	recursive	recursive	recursive
Is the language empty?	not r.e.	not r.e.	not r.e.
Is the language non-empty?	r.e.	r.e.	r.e.
Does the language contain all words?	not r.e.	not r.e.	not r.e.
Are there any words not in the language?	r.e.	r.e.	r.e.

L_1 is recursively enumerable but not recursive and L_2 is recursive			
	$L_1 \cup L_2$	$L_1 \cap L_2$	$L_1 - L_2$
Does the language contain a given fixed word w?	r.e.	r.e.	r.e.
Does the language not contain a given fixed word w?	not r.e.	not r.e.	not r.e.
Is the language empty?	not r.e.	not r.e.	not r.e.
Is the language non-empty?	r.e.	r.e.	r.e.
Does the language contain all words?	not r.e.	not r.e.	not r.e.
Are there any words not in the language?	not r.e.	not r.e.	not r.e.