Smoothed complexity of local max-cut for special graphs

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Abstract

Local search heuristics is often used to deal with NP-hard optimization problems. However, if some of them work well in practice, they behave poorly on the worst instances. This is the case with the Maximum-Cut Problem, where the simple algorithm FLIP takes exponential time in finding a local max-cut on some graphs. It has been found that adding a small amount of noise on the edges could improve greatly the running time of FLIP. This motivates the study of the smoothed complexity of local max-cut where a small perturbation factor ϕ is added to the edges. We show that for several classes of graphs the smoothed complexity of FLIP is polynomial in n and ϕ where n is the number of nodes.

1 Introduction

Let $G = (V, E)$ be a graph with n vertices and m edges. Let $w : E \to [-1, 1]$ be an edge weight function. A cut is a partition of the vertices $\sigma : V \to \{-1, 1\}$. The cut weight is defined as :

$$h(\sigma) = \frac{1}{2} \sum_{uv \in E} (1 - \sigma(u)\sigma(v))w(uv)$$

The hamming distance between two cuts is defined as :

$$d(\sigma, \sigma') = \#\{v : \sigma(v) \neq \sigma'(v)\}$$

The local max-cut problem consists in finding partition σ such that :

$$\neg\exists\ \sigma' \in \{-1,1\}^V, h(\sigma') > h(\sigma) \text{ and } d(\sigma,\sigma') = 1$$

In other words, there exists no vertex v such that, if the vertex is flipped, i.e. change the sign of $\sigma(v)$, then $h(\sigma)$ increases. Such a state σ is called locally optimal.

A really naïve algorithm called FLIP solves this problem. It finds a vertex which if flipped leads to an amelioration of the total weight cut, flips it and repeats until no such vertex exists. Some instances of graphs have been found to force FLIP to perform an exponential number of steps before terminating. However, for most of the graphs, FLIP terminates in a reasonable time. Moreover, when adding a small amount of noise to the graphs for which the number of steps is exponential, FLIP's running time seem to behave as polynomial. This leads to the study of the smoothed complexity of FLIP, in which noise is added to the edge weights. From now on, complexity will stand for smoothed complexity in this paper.

Etscheid and Röglin [1] proved that the complexity was at most quasi-polynomial in n for arbitrary graphs, with the insight that it may be polynomial. Angel et al. (2016) [2] proved that the complexity was polynomial for complete graphs.
We study here other special classes of graphs, for which the complexity is polynomial.

2 Notation and preliminary lemmas

Let $X = (X_e)_{e\in E} \in [-1,1]^E$ a random vector with independent entries, corresponding to the edge weights. We assume that X_e has density f_e with respect to the Lebesgue mesasure, and we denote $\phi = \max_{e\in E} ||f_e||_\infty$.

Lemma 2.1 *(Lemma 2.1 [2])*
Let $\gamma_1, ..., \gamma_k$ be k linearly independent vectors in $\mathbb{Z}^E$. Then the joint density of $(\langle\gamma_i, X\rangle)_{i\leq k}$ is bounded by ϕ^k. In particular, if sets $J_i \subseteq \mathbb{R}$ have measure at most ϵ each, then

$$\mathcal{P}(\forall i \in [k], \langle\gamma_i, X\rangle \in J_i) \leq (\phi\epsilon)^k$$

We do not give here a proof for this lemma, see the reference [2] for a detailed proof.
Define a move vector $\alpha(v)$ as a vector indexed by E whose entries are :

$$\alpha(v)_{uw} = \begin{cases} \sigma(u)\sigma(w) & \text{if } uw \in E \text{ and } ((u = v) \text{ or } (w = v)) \\ 0 & \text{otherwise} \end{cases}$$

For a sequence $L = (v_1, ..., v_l)$ a word of vertices and initial state σ_0, let $\alpha(v_i), i \in [l]$ be the corresponding move vectors. Let σ_t be the state just after flip of vertex v_t, remark that it depends only on the initial state σ_0 and the vertices in the sequence. The improvement made by a move is defined as :

$$h(\sigma_t) - h(\sigma_{t-1}) = \langle \alpha(v_t), X \rangle$$

Where X is the edge weight vector.
A matrix A_L is defined as the concatenation of the move vectors as columns. We call a sequence ϵ-slowly improving if all moves yield an improvement of at most ϵ.

We introduce here the concept of critical block. A block B is defined as a substring of a sequence L.

Let $S(L)$ be the set containing the distinct vertices in L and $s(L)$ be its cardinality, $s_1(L)$ the number of distinct vertices in L that appear only once, $s_2(L)$ the number of distinct vertices in L that appear more than once. Let $l(B)$ be the length of the block, i.e. the number of moves. A block B is critical if: for $l(B) \geq 2s(B)$ and every block B' strictly contained in B has $l(B') < 2s(B')$.
We use the following abuse of notation, if B is a sequence, a block, or a subsequence : rank(B) stands as the rank of the matrix obtained by the concatenation of the move vectors in B.

Lemma 2.2 *(Lemma 4.1 [2])*
For complete graphs with n vertices: fix any positive integer $n \geq 2$. Given a sequence consisting of $s(L) < n$ letters and with length $l(L) \geq 2s(L)$, there exists a critical block B in L. Moreover, a critical block satisfies $l(B) = 2s(B)$ and rank$(B) \geq 1.25s(B)$

This is the result on which [2] based their proof for the complexity of complete graph.

Lemma 2.3

Given $J_1, ...J_l$ sets composed of at most k closed intervals each, with J_t having at most measure ϵ, $\forall t \leq l$. If $l(L) = l, s(L) < n$, and rank$(L) \geq s(L)$ then

$$P(\langle \alpha(v_t), X\rangle \in J_t \quad \forall t \leq l) \leq (2n/\epsilon)^{s_0}(16(2k+1)\phi\epsilon)^{rank(A_L)}$$

Where s_0 is the number of vertices that have at least one neighbour which is not in L.

Proof. Let I be the set of the edges corresponding to independent rows in A_L. They do not depend on σ_0 since the starting configuration only multiplies each row by 1 or -1.
Define $T = \{v \in V : vw \in I \text{ or } v \in S(L)\}$. $|T| \leq 2\text{rank}(A_L) + s(L)$.
We split $h(\sigma)$ in three part $h_0(\sigma), h_1(\sigma), h_2(\sigma)$ Where h_i is the restriction of h to the edges that have i endpoints in T.
$h_0(\sigma_t) - h_0(\sigma_{t-1}) = 0$. Since the edges whose both endpoints are not flipped do not provoke a change in the total weight cut.

$$h_1(\sigma_t) - h_1(\sigma_{t-1}) = -\sigma_t(v_t) \sum_{u \notin T, uv_t \in E} \sigma_0(u) X_{uv_t}$$

$$= \sigma_t(v_t) Q(v_t).$$

where $Q(v_t) = -\sum_{u \notin T, uv_t \in E} \sigma_0(u) X_{uv_t}$. Since $|X_e| \leq 1$ and the maximum neighbours of a vertex is n, $Q(v_t) \in [-n, n]$. By defining $D = 2\epsilon\mathbb{Z} \cap [-n, n]$, there exists some $d(v_t) \in D$ such that $|Q(v_t) - d(v_t)| \leq \epsilon$.

$h_2(\sigma_t) - h_2(\sigma_{t-1}) = \langle \alpha'(t), X\rangle$ where $\alpha'(t)$ is a vector indexed by E whose coordinates are

$$\alpha'_{uw}(t) = \begin{cases} -\sigma_t(u)\sigma_t(w) & uw \in E, v_t \in \{u, w\}, \{u, w\} \subseteq T \\ 0 & \text{otherwise} \end{cases}$$

Since $\alpha'(t)$ concerns only the rows linearly independant from A_L, $\text{rank}([\alpha'(t)]_{t \leq l(L)}) = \text{rank}(A_L)$.

$$h(\sigma_t) - h(\sigma_{t-1}) = \langle \alpha', X\rangle + \sigma_t(v_t)d_t + \delta_t \text{ where } |\delta_t| \leq \epsilon$$

Since $|\delta_t| \leq \epsilon$, if $h(\sigma_t) - h(\sigma_{t-1}) \in J_t$ then :

$$\exists j \in J_t, |j - \langle \alpha', X\rangle - \sigma_t(v_t)d_t| \leq \epsilon$$

Since $\bigcup_{i\leq k} J_t^i = J_t$ is of measure at most epsilon, the set $S : \bigcup_{i\leq k} J_t'^i$, with $J_k'^i = [\min(J_t^i) - \epsilon, \max J_t^i + \epsilon]$, is of measure at most $(2k+1)\epsilon$.
As $\sigma_t(v_t)$ is -1 or 1, $\langle \alpha', X\rangle$ lies in a set defined as :

$$S_{\text{final}} = \{p : p - d_t \in S\} \cup \{q : q + d_t \in S\}$$

This set is of measure at most $2(2k+1)\epsilon$. Using lemma 2.1, the probability that $\langle \alpha', X\rangle$ belong to S_{final} is at most $(2(2k+1)\phi\epsilon)^{\text{rank}(A_L)}$. Using union bound over $\sigma_{t\in T}$ and d :

$$P(\langle \alpha(v_t), X\rangle \in J_t) \quad \forall t \leq l) \leq 2^{2\text{rank}(A_L)+s} \left(\frac{2n}{\epsilon}\right)^{s_0} (2(2k+1)\phi\epsilon)^{\text{rank}(A_L)}$$

Remark the s_0 instead of s in exponent, since vertices that have no-non flipped neighbors need not be taken in this bound.
By using $\text{rank}(L) \geq s(L)$, we get the desired bound.

Corollary 2.3.1 *If $l(L) = l, s(L) < n$, and rank$(L) \geq s(L)$ then*
P(L is ϵ-slowly improving from some state $\sigma_0 \leq (2n/\epsilon)^{s_0}(48\phi\epsilon)^{rank(A_L)}$

It follows from the previous lemma, by replacing all J_t by $[0, \epsilon]$.

3 Proof for graphs with one clique and low degrees vertices

Let $G = (V, E)$ be a graph with n vertices and m edges. Assume that this graph contains a clique H of r vertices and that the degree of vertices in the set $G \setminus H$ is at most $\log(n)$.

Proposition 3.1 *With high probability, there exists no ϵ-slowly improving sequence of length $2n$ from any starting configuration σ_0, for ϵ being O(1/poly(n)).*

Proving this proposition implies that the smoothed complexity is $O(\text{poly}(n))$. If there exists no such sequence, then $2n^2/\epsilon$ sequence of $2n$ moves yield an improvement of at least $2n^2$ which is the maximum improvement possible since $h(\sigma) \in [-n^2, n^2]$.

Thus number of steps is $O(n^3/\epsilon)$ which is $O(poly(n))$.

We will prove the proposition by considering different sequences of size $2n$.
Let l(L) $= 2n$:
D is the event corresponding to $\exists L, \sigma_0,$ s.t. L is ϵ-slowly improving from σ_0
D_1 is the event corresponding to $\exists L, \sigma_0,$ s.t. L is ϵ-slowly improving from σ_0 and $S(L) \nsubseteq H$.
D_2 is the event corresponding to $\exists L, \sigma_0,$ s.t. L is ϵ-slowly improving from σ_0 and $S(L) \subseteq H$ and $s(L) < r$.
D_3 is the event corresponding to $\exists$ critical block $B, \sigma_0,$ s.t. B is ϵ-slowly improving from σ_0 and $S(L) \subseteq H$, $s(B) = r$
By the lemma 2.2 on existence of critical blocks and the fact that if $s(L) = n$ implies that some vertex with degree at most $\log(n)$ is chosen we can have this bound:

$$P(D) \leq P(D_1) + P(D_2) + P(D_3)$$

Consider D_1. Fix L and σ_0 then there must be a vertex $v \in S(L)$ whose degree is at most $\log(n)$. By lemma 2.1, the probability that $\alpha_v \in [0, \epsilon] = \phi\epsilon$. Using union bound on the number of vertices and the starting configuration of those vertices we have:

$$P(D_1) \leq 2^{\log(n)} n\phi\epsilon \tag{1}$$

By taking $\epsilon = n^{-2-\eta}\phi^{-1}$ with $\eta > 0$ there exists no such sequence with high probability.

Now consider D_2. We consider the subgraph G' induced by the restriction to the clique H. Denote $A_L^{G'}$ the matrix formed by the concatenation of the move vectors of L restricted to G'. We observe that rank$(A_L^{G'}) \leq$ rank$_(A_L)$ Since $A_L^{G'}$ is a submatrix of A_L.
By lemma 2.2, there exists a critical block B whose rank is at least $1.25s(B)$.We now can use corollary 2.3.1 to have this bound:

$$P(B \text{ is } \epsilon\text{-slowly improving from some } \sigma) \leq \left(\frac{2n}{\epsilon}\right)^{s(B)} (48\phi\epsilon)^{\frac{5s(B)}{4}}$$

Because the number of blocks using s letters is n^{2s}, we have:

$$P(D_2) \leq 2 \sum_{s<n} (96\phi^{5/4} n^3 \epsilon^{1/4})^s$$

By choosing $\epsilon = n^{-(12+\eta)}\phi^{-5}$ this sums goes to zero.

For D_3, we use a trick to show that the rank$(A_L) \geq 1.25s(B) - (s(B) - s_0(B))$. We choose some $u \in V \backslash H$. For each vertex v which has a non-flipped neighbour, we delete that edge and add uv to the graph. This does not change rank(A_L) since the row added is the same as the row deleted times 1 or -1. Now we add edges from u to the remaining vertices on the clique, increasing the rank by at most $s(B) - s_0(B)$. The subgraph G' determined by $H \cup \{u\}$ is thus complete and we can use lemma 2.2 to have rank$(A_L^{G'}) \geq 1.25s(B)$. Then rank$(A_L) \geq 1.25s(B) - (s(B) - s_0(B))$. By corollary 2.3.1 we have :

$$P(D_3) \leq \sum_{s<n} n^{2s} \left(\frac{2n}{\epsilon}\right)^{s_0} (48\phi\epsilon)^{s/4+s_0} \leq \sum_{s<n} (Cn^3\phi^{5/4}\epsilon^{1/4})^s, \text{ for } C > 0$$

By choosing $\epsilon = n^{-(12+\eta)}\phi^{-5}$ this sums goes to zero, thus concluding the proof.

4 Proof for graphs with multiple distant cliques

Let $G = (V, E)$ be a graph with n vertices and m edges. Assume that this graph contain a family H of cliques $H_1, ...H_h$ of $r_1, ..., r_h$ vertices and that the degree of vertices in the set $G \backslash \bigcup_{i=1}^{h} H_i$ is at most $\log(n)$. Furthermore, forall $1 \leq i < j \leq h$ and any pair of two vertices v and w such that $v \in H_i, w \in H_j$; we have $v \neq w, vw \notin E$

Any pair of cliques is distant, meaning that a path going from a vertex of one clique to the other is of length at least 2.

The precedent proof can be easily extended to those graphs. We will prove a similar proposition, but on longer sequences.

Proposition 4.1

With high probability, there exists no ϵ-slowly improving sequence of length $2n^2$ from any starting configuration σ_0, for ϵ is O(1/poly(n)).

Lemma 4.2 *Let L be a sequence of q moves such that $S(L) \subseteq \bigcup_{i \leq k} A_i \subseteq V$, where $A_1, ..., A_k$ are edge-disjoint sets. Then, there exists a sequence with the same vertices but a different ordering on the moves such that $\forall l < j \leq q$, if $v_l \in A_i$ and $v_j \in A_i$, then $v_d \in A_i \quad \forall l \leq d \leq j$*

Proof. The proof is very straightforward. Suppose we have v_t and v_{t+1} which are distant, the amelioration brought by v_t is equal to :

$$-\sigma_t(v_t) \sum_{u \in V, uv_t \in E} w_{uv_t} \sigma_t(u) = -\sigma_{t+1}(v_t) \sum_{u \in V, uv_t \in E} w_{uv_t} \sigma_{t+1}(u_t)$$

We recall that σ_t is the partition just after the flip of v_t. The equality above holds because σ_t and σ_{t+1} differ only by the value of v_{t+1} and $(v_t v_{t+1})$ does not belong to E. This means that we can flip v_t at time t or $t+1$, this is true also for v_{t+1}. We can then swap them, and both are still improving moves with the same amelioration of total weight. By repeating the swaps, we reach a new sequence where all moves of vertices $\in A_i \quad \forall i \leq k$ are consecutive.

If we consider now sequences of length $2n^2$. Either there is a vertex with logarithmic degree and the whole sequence is not ϵ-slowly improving either we can reorder them with the previous lemma, such that vertices belonging to the same clique are consecutive. Since the number of cliques is upperbounded by n, by pigeonhole argument we have at least one sequence of size at least $2n$, which contains vertices from only a clique. We showed that with high probability such a sequence is not ϵ-slowly improving. The whole sequence is then not ϵ-slowly improving, concluding the proof.

5 Proof for two cliques with a matching

Now that we know how to deal with cliques without connection between them, we want to study a simple case where cliques are connected.
Let $G = (V, E)$ be a graph with n vertices and m edges. Assume that this graph consist of two cliques H_1, H_2 of $n/2$ vertices. Furthermore, each vertex in H_1 has a unique neighbour in H_2, and vice versa.
Here we would like to use critical blocks to find a good lower bound on the rank. However, the proof on the rank [2] highly depends on the following notions:
A Transition block T is a maximal substring of the sequence such that every vertex in T appears more than once in the sequence.
A Singleton block S is a maximal substring of the sequence such that every vertex in S appears only once in the sequence.
Without repeating their proof here, we give a general explanation of it. The

sequence is separated into Transition blocks and Singleton blocks in alternance. We pick a vertex v that appears in T_i and in T_j for $j > i$ and a vertex w appearing in a Singleton block in between, the row of the edge vw is independent of many other rows. This helps finding a bound on the rank depending of s_1 which is hard for arbitrary graphs. Along with a bound on the rank depending on s_2 which exists for arbitrary graphs and the criticality of the block, they manage to find this lower bound on the rank.
However, in the case of two cliques with a matching, a critical block may contain vertices from both cliques and there is no guarantee that a vertex present in two transition blocks will have an edge to a singleton vertex in between, (e.g $S_1(B) \subset H_1$ and $S_2(B) \subset H_2$).
Since there are edges between the cliques, we cannot use a reordering argument like in the proof in the previous section. Indeed, reordering the moves could lead to illegal moves where the improvement is negative. Proving then than a move is not in $[0, \epsilon]$ would not suffice to say that the whole sequence is not ϵ-slowly improving since it could be cancelled by another move.
Denote $\alpha(v_t)_{E_i}$ the improvement made by a move, considering only the edges in $E_i \subseteq E$, similarly denote X_{E_i} the restriction of X (the matrix of the weights indexed by the rows) to the rows corresponding only to E_i

Lemma 5.1 *Let $\alpha(v_t)$ be a move and let E_1, E_2 be a partition on the edges touching the vertex v.*
The probability that $\alpha(v_t)$ is in $[0, \epsilon]$ is less or equal than the probability that $\alpha(v_t)_{E_1} \in \bigcup_{u \in \{-1,1\}^{E_2}} [\langle u, X_{E_2} \rangle, \langle u, X_{E_2} \rangle + \epsilon]$

The proof follows from the fact that $\alpha(v_t) = \alpha(v_t)_{E_1} + \alpha(v_t)_{E_2}$, and that $\alpha(v_t)_{E_2}$ cannot take values outside $\bigcup_{u \in \{-1,1\}^{E_2}} \{\langle u, X_{E_2} \rangle\}$, with a simple union-bound we get the result of the lemma above.

Proposition 5.2
With high probability, there exists no ϵ-slowly improving sequence of length $4n$ from any starting configuration σ_0, for ϵ is O(1/poly(n)).

Proof. We introduce here the concept of critical subsequence.
A subsequence $CSub$ is critical if $l(CSub) \geq 2s(CSub)$ and every subsequence Sub' strictly contained in $CSub$ has $l(Sub') < 2s(Sub')$.
Consider a sequence L of size $4n$ which contains vertices from both H_1 and

H_2. Other cases are proven in the same way as the proof above.
By pigeonhole argument, there must be a H_i for which there exists a subsequence Sub containing only vertices in H_i is of length $2n$.
Take $CSub$ the critical subsequence with respect to Sub. It can be constructed as the minimal substring of Sub with respect to inclusion in the following way:
Take a vertex from Sub, add it to $CSub$, if $CSub$ is critical stop, otherwise take the next vertex of Sub and add it to $CSub$, repeat. If we reach the right end of Sub we expand to the left instead. By definition of criticality, a critical subsequence must exist, because there can be only n different vertices and $l(Sub) = 2n$.
Denote $N(v_t)$ the neighbours of v_t in the same clique than v_t, denote u_t the neighbour of v_t in the other clique. Using lemma 5.1, for the sequence to be ϵ-slowly improving, we need :

$$\langle \alpha(v_t), X \rangle \in [0, \epsilon] \quad \forall v_t \in l(CSub)$$
$$\langle \alpha(N(v_t)), X_{N(v_t)} \rangle \in [-w(u_t, v_t), -w(u_t, v_t) + \epsilon] \cup [w(u_t, v_t), w(u_t, v_t) + \epsilon]$$

Let call A_1 the event that $s(CSub) < r_i$, the rank of this subsequence is at least $1.25s(Csub)$ since the restriction of the matrix of the moves to the clique, is equal to the matrix of moves of a critical block in a complete graph. Using lemma 2.3, with union-bound on possible subsequence we get :

$$P(A_1) \leq 2 \sum_{s<n} (160\phi^{5/4} n^3 \epsilon^{1/4})^s$$

By choosing $\epsilon = n^{-(12+\eta)}\phi^{-5}$ this sums goes to zero.

Let call A_2 the event that $s(Csub) = r_i$, since we got rid of the influence of the non-moving vertices over the moving vertices with lemma 5.1, we have the following simple union-bound:

$$P(A_2) \leq \sum_{s<n} (80\phi n^3 \epsilon)^s$$

Where we used lemma 2.1 and a trivial lowerbound on the rank. By choosing $\epsilon = n^{-(3+\eta)}\phi^{-5}$ this sum goes to zero.

6 Proof for multiple cliques with connections

Using what we saw so far, we can prove that FLIP is polynomial for a most general case.
Let $G = (V, E)$ be a graph of n vertices and m edges composed of a family $H = \{H_1, H_2, ..., H_h\}$ of cliques and some vertices whose degree is at most $\log(n)$. Assume further that all vertices in H have at most $\log(n)$ edges going to the other cliques.
We will prove that FLIP terminates in a polynomial number of step by proving the following proposition :

Proposition 6.1 *With high probability, there exists no ϵ-slowly improving sequence L of size $2n^2$*

Consider a sequence L of size $2n^2$.
Denote A the event that L is ϵ-slowly improving.
Denote A_1 the event that L is ϵ-slowly improving and $S(L) \not\subseteq \bigcup_{i=1}^{h} H_i$
Denote A_2 the event that L contains a critical subsequence $CSub$ of size at most $2n$ and that there exists $H_i \in H$ s.t. $S(Csub) \subset H_i$
Denote A_3 the event that L contains a critical subsequence $CSub$ of size at most $2n$ and that there exists $H_i \in H$ s.t. $S(Csub) = H_i$

From the proof on the existence of critical subsequence, we see that :

$$P(A) \leq P(A_1) + P(A_2) + P(A_3)$$

Assume A_1, then there exists some vertex v in the sequence that has logarithmic degree. We saw that with high probability there exists no such v that leads to an improvement of less than ϵ for $\epsilon = n^{-2-\eta}\phi^{-1}$, with $\eta > 0$.

We saw that the $CSub$ in A_2 has rank at least $1.25s(CSub)$. We can also use the lemma 5.1 with $E_2(v_t)$ being the edges going from v_t to another clique. Since there are at most $\log(n)$ such edges, the improvement made by the rest of the edges must lie in the union of at most $2^{\log(n)}$ intervals of measure ϵ, we can also say that the improvement made by the other edges must lie in an interval of measure at most $n\epsilon$.
Using lemma 2.3.1 along with an union-bound on possible $CSub$ we get the

following bound :

$$P(A_2) \leq \sum_{s<n} n^{2s} \left(\frac{2n}{n\epsilon}\right)^s (16(n+1)^{5/4}\phi^{5/4}\epsilon^{5/4})^s$$
$$\leq \sum_{s<n} C(n^{13/4}\phi^{5/4}\epsilon^{1/4})^s, C > 0$$

Taking $\epsilon = n^{-13-\eta}\phi^{-5}$, this sums goes to zero.

For A_3, we combine the argument for A_2 and the trick used at the end of section 3. We come to this slightly different bound.

$$P(A_3) \leq \sum_{s<n} n^{2s} \left(\frac{2n}{n\epsilon}\right)^{s_0} (Cn\phi\epsilon)^{s/4+s_0} \leq \sum_{s<n} (Cn^{13/4}\phi^{5/4}\epsilon^{1/4})^s, \text{ for } C > 0$$

Taking $\epsilon = n^{-13-\eta}\phi^{-5}$, this sums goes to zero.

7 Conclusion

Previous work on smoothed complexity of local max-cut has given the insight that it may be polynomial for arbitrary graphs. Actually the bound is quasi-polynomial [1] for arbitrary graphs, and polynomial for complex graphs [2]. Using a simple proof for the complexity of graphs with log degrees, we extended the proof of complete graphs to graphs containing both cliques and log degrees vertices. We have shown different proofs for multiple classes of graphs, the last one at section 6 being the stronger result since it includes the other classes.

The proof relies on the very high connectivity of cliques on one hand, on the very low connectivity between different components on the other hand. The classes of graphs that seem to be troublesome are the ones with high connectivity without being complete, e.g. the expander graphs. On the expander graphs it will be hard to discretize the influence of non-moving vertices, it is thus an interesting class to investigate.

References

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