Smoothed Complexity of Local Search of Max-Cut for graph with logarithmic degree

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1 Introduction

Let $G(V, E)$ be a graph with a set V of n vertices and E of m edges. Forall vertex v that belongs to V, the degree of v is inferior or equal to $c \cdot log(n)$ where c is a constant. Let $w : E \rightarrow]-1, 1[$ be an edge weight function. This restriction is for commodity in the proof, we can easily generalize to other weight function by dividing all weight by $w_{max} = max_e(|w_e|)$ as long as the variance of the weight function is bounded.
Let $\sigma : V \rightarrow -1, 1$ be a partition, $\sigma_0 a starting configuration$ and let the size of the cut corresponding to this partition be :

$$\frac{1}{2} \sum_{uv \in E} w(uv)(1 - \sigma(u)\sigma(v))$$

We call a cut locally optimal, if there exists no vertex v such that if v is moved to the other side of the cut, the size of the cut increases. A simple algorithm called FLIP solves the problem of finding a local-optimal cut. Search for a vertex which, if flipped, improves the size of the cut (we call it an improving move), if there is none stop, else repeat. Let call s such a sequence of move

Formally we now assume that the weight on edge $e \in E$ is given by a random variable $X_e \in [-1, 1]$ which has a density with respect to the Lebesgue measure bounded from above by ϕ (for example this forbids X_e to be too close to a point mass). We assume that these random variables are

independent.
We will prove here that the smoothed complexity is polynomial.

2 Proof

2.1 Bounding a move

Let call t a move. Remark that α_t, the amelioration to the cut provided by the move t, depends only on the vertex which is flipped and its neighbors. Here we use a particular case of the lemma 7 provided by Etscheid and Röglin[2014].

Lemma 2.1 *For* $\epsilon > 0$ *and a fixed move* t $\mathbf{Pr}(\alpha_t \in [0, \epsilon]) \leq \phi\epsilon$

2.2 Bounding all moves using union bound

We use the union bound over the n possible vertices and the $2^{c \cdot log(n)}$ possible starting configuration. This gives us that for any move :
$\mathbf{Pr}(\exists \sigma_0 : \text{an improvement made by a move} \in [0, \epsilon]) \leq n 2^{clog(n)} \phi\epsilon$

2.3 Finding a good epsilon

Now we choose $\epsilon = n^{-(1+c+\eta)}\phi^{-1}$ with $\eta > 0$. Thus, with high probability, each move improves the cut by at least ϵ.

Since the cut is in $]-cnlog(n), cnlog(n)[$. The number of steps is at most $\frac{2cnlog(n)}{\epsilon}$.

The complexity is polynomial with $O(n^{2+c+\eta} log(n)\phi)$