Smoothed Complexity of Local Search of Max-Cut for graph with cliques and other vertices have logarithmic degree

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1 Introduction

Let $G = (V, E)$ be a graph with n vertices and m edges. Assume that this graph contains cliques $H_1, ...H_o$ of $r_1, ..., r_o$ vertices and that the degree of vertices in the set $G \setminus H$ is at most $log(n)$. Furthermore, there exists no edge going from one clique to another.
Let $w : E \to [-1, 1]$ be an edge weight function. The local max-cut problem consists in finding a partition of the vertices σ such that the total cut weight, defined as :

$$h(\sigma) = \frac{1}{2} \sum_{uv \in E} (1 - \sigma(u)\sigma(v))w(uv)$$

is locally optimal. By locally optimal, we mean that there exists no vertex v such that, if we flip the vertex, i.e. change the sign of $\sigma(v)$, then $h(\sigma)$ increases.

A really naive algorithm called FLIP solves this problem. It finds a vertex which if flipped leads to an amelioration of the total weight cut, flips it and repeat until no such vertex exists. Some instances of graphs have been found to have an exponential number of steps before terminating. However, for most of the graphs, FLIP terminates in a reasonable time. Moreover, when adding a small amount of noise to the complicated graphs, FLIP's running time improved greatly. This lead to the study of the smoothed complexity of FLIP, in which noise is added to the edge weights.

2 Notation and preliminary lemmas

Let $X = (X_e)_{e \in E} \in [-1,1]^E$ a random vector with independent entries, corresponding to the edge weights. We assume that X_e has density f_e with respect to the Lebesgue mesasure, and we denote $\phi = max_{e \in E} ||f_e||_\infty$.

Lemma 2.1 *(Lemma 2.1 [1])*
Let $\alpha_1, ..., \alpha_k$ be k linearly independent vectors in $\mathbb{Z}^E$. Then the joint density of$(\langle \alpha_i, X \rangle)_{i \leq k}$ is bounded by ϕ^k. In particular, if sets $J_i \in \mathbb{R}$ have measure at most ϵ each, then

$$\mathcal{P}(\forall i \in [k], \langle \alpha_i, X \rangle \in J_i) \leq (\phi\epsilon)^k$$

This lemma will prove determinant for the proof.

We define a move vector α_v as a vector indexed by E whose entries are :

$$\alpha_{uw} = \begin{cases} \sigma(u)\sigma(w) & \text{if } uw \in E \text{ and } ((u = v) \text{ or } (w = v)) \\ 0 & \text{otherwise} \end{cases}$$

For a sequence $L = (v_1, ..., v_l)$ of l moves and initial state σ_0, let $\alpha_i, i \in [l]$ be the corresponding move vectors. Let σ_t be the state just after flip of vertex v_t. We define matrix A_L has the concatenation of the move vectors as columns. We call a sequence ϵ-slowly improving if all moves yield an improvement of at most ϵ.

Proposition 2.2
With high probability, there exists no ϵ-slowly improving sequence of length $2n^2$ from any starting configuration σ_0, for ϵ is O(1/poly(n)).

Proving this proposition implies that the smoothed complexity is polynomial in n. If there exists no such sequence, then $2n^2/\epsilon$ sequence of $2n^2$ moves yield an improvement of at least $2n^2$ which is the maximum improvement possible since $h(\sigma) \in [-n^2, n^2]$. Thus number of steps is $O(n^4/\epsilon)$ which is $O(poly(n))$.

We introduce here the concept of critical block. A block B is defined as a substring of a sequence L.

Let $S(L)$ be the set containing the distinct vertices in L and $s(L)$ be its cardinality, $s_1(L)$ the number of distinct vertices in L that appear only once, $s_2(L)$ the number of distinct vertices in L that appear multiple times. Let

$l(B)$ be the length of the block, i.e. the number of moves. A block B is critical if $l(B) \geq (1+\beta)s(B)$ and every block B' strictly contained in B has $l(B') < (1+\beta)s(B')$.

Lemma 2.3 *(Lemma 4.1 [1])*
For complete graphs with n vertices: fix any positive integer $n \geq 2$ and a constant $\beta > 0$. Given a sequence consisting of $s(L) < n$ letters and with length $l(L) \geq (1+\beta)s$, there exists a critical block B in L. Moreover, a critical block satisfies $l(B) = \lceil (1+\beta)s(B) \rceil$. Moreover $rank(B) \geq \dfrac{1+4\beta}{1+3\beta}s(B)$

3 Proof of Proposition 2.2

3.1 With only one clique

Lemma 3.1
If $s(L) < n$, and $rank(L) \geq (1+\theta)s(L)$ then

$$P(L \textit{ is } \epsilon\textit{-slowly improving from some sigma}_0) \leq (2n/\epsilon)^{s_0}(64\phi\epsilon)^{rank(A_L)}$$

Where s_0 is the number of vertices that have at least one neighbor which is not in L.

Proof. Let I be the set of the edges corresponding to independent rows in A_L. They do not depend on σ_0 since the starting configuration only multiplies each row by 1 or -1.
Define $T = \{v \in V : vw \in I \text{ or } v \in S(L)\}$. $|T| \leq 2rank(A_L) + s(L)$.
We split $h(\sigma)$ in three part $h_0(\sigma), h_1(\sigma), h_2(\sigma)$ Where h_i is the restriction of h to the edges that have i endpoints in T.
$h_0(\sigma_t) - h_0(\sigma_{t-1}) = 0$. Since the edges whose both endpoints are not flipped do not provoke a change in the total weight cut.

$$h_1(\sigma_t) - h_1(\sigma_{t-1}) = -\sigma_t(v_t) \sum_{u \notin T, uv_t \in E} \sigma_0(u) X_{uv_t}$$
$$= \sigma_t(v_t) Q(v_t).$$

where $(Q(v_t) = -\sum_{u \notin T, uv_t \in E} \sigma_0(u) X_{uv_t})$. Since $|X_e| \leq n$ and the maximum neighbours of a vertex is n, $Q(v_t) \in [-n, n]$. By defining $D = 2\epsilon\mathbb{Z} \cap [-n, n]$, there exists some $d(v_t) \in D$ such that $|Q(v_t) - d(v_t)| \leq \epsilon$.

$$h_2(\sigma_t) - h_2(\sigma_{t-1}) = \langle \alpha', X \rangle \text{ where}$$
$$\alpha'_t = \begin{cases} -\sigma_t(u)\sigma_t(w) & uw \in E, v_t \in \{u, w\}, \{u, w\} \subseteq T \\ 0 & \text{otherwise} \end{cases}$$

Since α'_t concerns only the rows linearly independant from A_L, $rank([\alpha'_t]_{t\leq l}) = rank(A_L)$.

$$h(\sigma_t) - h(\sigma_{t-1} = \langle \alpha', X \rangle + \sigma_t(v_t)d_t + \delta_t \text{ where } |\delta_t| \leq \epsilon$$

Since $|\delta_t| \leq \epsilon$, L is ϵ-slowly improving implies that:

$$|\langle \alpha', X \rangle + \sigma_t(v_t)d_t| \leq 2\epsilon \quad \forall t \leq l$$

We need that $\langle \alpha', X \rangle \in [-d_t - 2\epsilon, -d_t + 2\epsilon] \cup [d_t - 2\epsilon, d_t + 2\epsilon]$, Using lemma 2.1, this is at most $8\phi\epsilon^{rank(A_L)}$. Using union bound over $\sigma_{t\in T}$ and d :

$$\text{P(L is } \epsilon\text{-slowly improving from some } \sigma_0 \leq 2^{2rank(A_L)+s}(\frac{2n}{\epsilon})^{s_0}(8\phi\epsilon)^{rank(A_L)}$$

Remark the s_0 instead of s in exponent, since vertices that have no-non flipped neighbors need not be taken in this bound.
By using $rank(L) \geq (1 + \theta)s(L)$, we get the desired bound.

We consider sequences of size 2n where the vertices in the sequence belong only to one clique H of cardinality r or to the set of vertices of logarithmic degrees.
Let $p > 1, l(L) = 2n$:
E is the event corresponding to $\exists L, \sigma_0,$ s.t. L is ϵ-slowly improving from some σ_0
E_1 is the event corresponding to $\exists L, \sigma_0,$ s.t. L is ϵ-slowly improving from some σ_0 and $S(L) \not\subseteq H$.
E_2 is the event corresponding to $\exists L, \sigma_0,$ s.t. L is ϵ-slowly improving from some σ_0 and $S(L) \subseteq H$ and $s(L) < r$.
E_3 is the event corresponding to $\exists$ critical block $B, \sigma_0,$ s.t. B is ϵ-slowly improving from some σ_0 and $S(L) \subseteq H$, $s(B) = r$ and $s_0(B) \leq s(B)/p$
E_4 is the event corresponding to $\exists$ critical block $B, \sigma_0,$ s.t. B is ϵ-slowly improving from some σ_0 and $S(L) \subseteq H$, $s(B) = r$ and $s_0(B) > s(B)/p$
By the lemma 2.3 on existence of critical blocks and the fact that if $s(L) = n$

implies that some vertex with degree at most log(n) is chosen we can have this bound:

$$P(E) \leq P(E_1) + P(E_2) + P(E_3) + P(E_4)$$

Consider E_1. Fix L and σ_0 then there must be a vertex $v \in S(L)$ whose degree is at most $log(n)$. By lemma 2.1, the probability that $\alpha_v \in [0, \epsilon] = \phi\epsilon$. Using union bound on the number of vertices and the starting configuration of those vertices we have:

$$P(E_1) \leq 2^{log(n)} n\phi\epsilon$$

By taking $\epsilon = n^{-2-\eta}\phi^{-1}$ with $\eta > 0$ there exists no such sequence with high probability.

Now consider E_2. We consider the subgraph G' induced by the restriction to the clique H. We observe that $rank'_G(A_L) \leq rank_G(A_L)$ Since the first is a submatrix of the second one.
By lemma 2.3, there exists a critical block B whose rank is at least 1.25 s(B).We now can use lemma 3.1 to have this bound:

$$P(B \text{ is } \epsilon\text{-slowly improving from some } \sigma) \leq (\frac{2n}{\epsilon})^{s(B)}(64\phi\epsilon)^{\frac{5s(B)}{4}}$$

Because the number of blocks using s letters is n^{2s}, we have:

$$P(E_2) \leq \sum_{s<n} (128\phi^{5/4} n^3 \epsilon^{1/4})^s$$

By choosing $\epsilon = n^{-(12+\eta)}\phi^{-5}$ This sums goes to zero.

By lemma 3.1 we have :

$$P(E_3) \leq \sum_{s<n} n^{2s} (\frac{2n}{\epsilon})^{s/p} (64\phi\epsilon)^s \leq \sum_{s<n} (Cn^{2+1/p}\phi\epsilon^{1-1/p})^s$$

For E_4, we use a trick to show that the $rank(A_L) \geq 1.25s(B) - s(B)(1 - 1/p)$. We choose some $w \in V \setminus H$. For each vertex v which has a non-flipped neighbour, we delete that edge and add vw to the graph. This does not change $rank(A_L)$ since the row added is the same as the row deleted times 1 or -1. Now we add edges from w to the remaining vertices on the clique, increasing the rank by at most $s(B)(1-1/p)$. The subgraph G' determined by $H \cup \{w\}$ is thus complete and we can use lemma 2.3 to have $rank_{G'}(A_L) \geq 1.25s(B)$. Then $rank_G(A_L) \geq 1.25s(B) - s(B)(1-1/p)$. By lemma 3.1 we have :

$$P(E_4) \leq \sum_{s<n} n^{2s}(\frac{2n}{\epsilon})^s(64\phi\epsilon)^{5s/4-s(1-1/p)} \leq \sum_{s<n}(Cn^3\phi^{1/4}\epsilon^{(1/p-3/4)})^s$$

By choosing p optimally, P(E) is o(1).

3.2 Extending the idea to multiple cliques

Lemma 3.2 *Let L be a sequence of q moves such that $S(L) \subseteq \bigcup_{i\leq k} A_i \subseteq V$, where $A_1, ..., A_k$ are edge-disjoint sets. Then, there exists a sequence with the same vertices but a different ordering on the moves such that $\forall l < q, l < j \leq q$, if $v_l \in A_i$ and $v_j \in A_i$, then $v_d \in A_i \quad \forall l \leq d \leq j$*

Proof. The proof is very straightforward. Suppose we have v_t and v_{t+1} which are edge-disjoint, the amelioration brought by v_t is equal to :

$$-\sigma(v_t)\sum_{u\in V, uv_t\in E} w_{uv_t}\sigma(u_t) = -\sigma(v_{t+1})\sum_{u\in V, uv_t\in E} w_{uv_{t+1}}\sigma(u_{t+1})$$

Since v_t and v_{t+1} are edge-disjoint. We can then swap them, and both are still improving moves with the same amelioration of total wieght. By repeating the swaps, we reach a sequence where all moves of vertices $\in A_i \quad \forall i \leq k$ are consecutive.

If we consider now sequences of length $2n^2$. Either there is a vertex with logarithmic degree and the whole sequence is not ϵ-slowly improving either we can reorder them with the previous lemma, such vertices belonging to the same clique are consecutive. Since the number of clique is upperbounded by n, by pigeonhole argument we have at least one sequence of size at least 2n, which contains vertices from only a clique. We showed that with high probability such a sequence is not ϵ-slowly improving. The whole sequence is then not ϵ-slowly improving, concluding the proof.

References

[1] Omer Angel, Sébastien Bubeck, Yuval Peres, and Fan Wei. Local max-cut in smoothed polynomial time. *arXiv preprint arXiv:1610.04807*, 2016.