

Report

Local Max-Cut in Smoothed Polynomial time

Abhinav Goyal

Shriram S B

Guide: Prof. Ola Svensson

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Abstract

We study the problem of Local Max-Cut. An algorithm for finding Local Max-Cut - FLIP, has been shown to have smoothed polynomial complexity for complete graphs. We show the limitations of extending the same idea applied to complete graphs to arbitrary graphs. We prove that FLIP has smoothed polynomial complexity for any graph G which has a clique and every vertex outside the clique having degree at most $\log(|V(G)|)$.

1 INTRODUCTION

Let $G = (V, E)$ be a connected graph with n vertices and $w : E \rightarrow [-1, 1]$ be an edge weight function. The local max-cut problem asks to find a partition of the vertices $\sigma : V \rightarrow \{-1, 1\}$ whose total cut weight, defined as following:

$$h(\sigma) = \frac{1}{2} \sum_{uv \in E} w(uv)(1 - \sigma(u)\sigma(v)) \quad (1)$$

is locally maximum, in the sense that one cannot increase the cut weight by changing the value of σ at a single vertex. There is a natural algorithm to find such a local maximum, referred to as FLIP algorithm: Start from some initial partition σ , and until reaching a local maximum, repeatedly find a vertex v for which flipping the sign of $\sigma(v)$ would increase the cut weight and carry out this flip. To implement this algorithm, the initial partition and the specific vertex to flip at each step (there can be more than one vertex whose flipping will increase the cut weight) has to be specified. There are edge weight functions for which FLIP takes exponential number of steps before reaching a local maximum. But practically, for most of the edge weights, FLIP terminates in a reasonable time. It is found that the edge weights for which FLIP takes exponential time have some special structure and adding a small amount of noise to the edge weights destroys the structure and makes the instance much easier for FLIP. This encourages the study of smoothed complexity of FLIP, in which we add noise to the edge weights and find the expected number of steps (number of vertex flips made) taken by FLIP with respect to distribution function of noise on edge weights.

1.1 Formally defining the problem

Let $X = (X_e)_{e \in E} \in [-1, 1]^E$ be a random vector with independent entries, where each entry is seen as original weight plus some small independent noise. We assume that X_e has density f_e with respect to the Lebesgue measure, and we denote $\phi = \max_{e \in E} \|f_e\|_\infty$. We use the phrase **with high probability** as probability at least $1 - o_n(1)$ with respect to X . We call $\sigma \in \{-1, 1\}^V$ as a state (describes the partition) and denote by $\sigma(v)$, the value of σ at vertex v . Now, our objective is to find a local maximum of the cut weight which is equivalent to finding the local maximum for $H(\sigma)$ defined as follows:

$$H(\sigma) = -\frac{1}{2} \sum_{uv \in E} X_{uv} \sigma(u) \sigma(v)$$

Here, local maximum is with respect to the hamming distance, $d(\sigma, \sigma') = \#\{v : \sigma(v) \neq \sigma'(v)\}$. We say that change of state from σ to σ' by flipping some vertex v is an **improving move** if $d(\sigma, \sigma') = 1$ and

$H(\sigma') > H(\sigma)$. Given an initial partition σ_0 and a sequence of vertices $L = (v_1, v_2, \dots, v_l)$ of length l which are flipped in the same order from initial state σ_0 , it is said to be an **improving sequence** if the value of H increases after each flip of vertex starting from state σ_0 . FLIP algorithm iteratively performs an improving move until it reaches a local maximum which gives us an improving sequence whose length is the number of steps taken by FLIP. An implementation of FLIP specifies how to choose the initial configuration and how to choose among the improving moves available at each step.

1.2 Earlier known results

Paper [1] showed that for graphs with degree at most $O(\log(n))$, FLIP has smoothed polynomial complexity. Paper [2] showed that for any graph, FLIP has smoothed quasi-polynomial complexity. Paper [3] showed that for complete graphs, FLIP has smoothed polynomial complexity. We use the results and ideas from these papers and try to extend them for arbitrary graphs.

1.3 Analysis for complete graphs

One of the main results of [3], which proves smoothed polynomial complexity of FLIP for complete graphs is as follows.

Theorem 1.1. (Theorem 1.1, [3]). *Let G be a complete graph on n vertices and assume the edge weights $X = (X_e)_{e \in E}$ are independent random variables with $|X_e| \leq 1$ and density bounded above by ϕ . For any $\eta > 0$, with high probability any implementation of FLIP terminates in at most $O(n^{15+\eta}\phi^5)$ steps, with implicit constant depending on η .*

We say that a sequence L is ϵ -slowly improving from an initial state σ_0 if each step of L increases H by at most ϵ (and more than 0).

For proving theorem 1.1, proving the following proposition is sufficient.

Proposition 1.1. (Proposition 1.6, [3]) *Fix $\eta > 0$ and let $\epsilon = n^{-(12+\eta)}\phi^{-5}$. Then with high probability, there is no ϵ -slowly improving sequence of length $2n$ from any σ_0 .*

Proposition 1.1 implies theorem 1.1. This can be seen as follows. Suppose there doesn't exist an ϵ -slowly improving sequence of length $2n$ from any σ_0 . Then, any improving sequence of length $2n * \frac{1}{\epsilon} * n^2$ improves equation (1) by atleast n^2 . But equation (1) can improve by at most n^2 due to bounded weights. So, any improving sequence is of length at most $2n^3/\epsilon = n^{15+\eta}\phi^5$. Since, each execution of FLIP gives an improving sequence equal to the number of steps taken by FLIP, FLIP terminate is at most $n^{15+\eta}\phi^5$ steps.

Each move (flip of vertex $v \in V$) can be viewed as a linear operator which we define now. For any $\sigma \in \{-1, 1\}^V$ and $v \in V$, we denote by σ^{-v} the state equal to σ except for the value corresponding to v which is flipped. For such σ and v , there exist a vector $\alpha = \alpha(\sigma, v) \in \{-1, 0, 1\}^E$ such that $H(\sigma^{-v}) - H(\sigma) = \langle \alpha, X \rangle$. If $v \in V$ is flipped, $\alpha = (\alpha_{uw})_{uw \in E}$ (independent of X) is defined as follows:

$$\alpha_{uw} = \begin{cases} \sigma(u)\sigma(w) & uw \in E \text{ \& } (u = v \text{ or } w = v) \\ 0 & \text{otherwise} \end{cases}$$

Clearly, flipping v is an improving move from a state σ if $\langle \alpha, X \rangle > 0$ (in this case we say α is an improving vector). Similarly, we say that certain moves are linearly independent if the corresponding vectors are.

For non-zero $\alpha \in Z^E$, the random variable $\langle \alpha, X \rangle$ also has density bounded by ϕ , which gives the probability that sequence of length 1 increases H by at most ϵ , $P(\langle \alpha, X \rangle \in (0, \epsilon]) \leq \phi\epsilon$. For a sequence L of length l , we use the following result.

Lemma 1.1. (Lemma 2.1, [3]) *For k linearly independent vectors, $\alpha_1, \dots, \alpha_k \in Z^E$, the joint density of $(\langle \alpha_i, X \rangle)_{i \leq k}$ is bounded by ϕ^k i.e.,*

$$P(\forall i \in [k], \langle \alpha_i, X \rangle \in (0, \epsilon]) \leq (\phi\epsilon)^k$$

For a sequence $L = (v_1, v_2, \dots, v_l)$ of l moves and initial state σ_0 , let $\alpha_i, i \in [l]$ be the corresponding move vectors. Let σ_t be the state just after flip of vertex v_t . We define matrix $A_L = [\alpha_i]_{i \leq l}$. A_L depends on L and σ_0 . Also, we define matrix $A = [-\sigma_t(v_t)\alpha_t]_{t \leq l}$. Since each column of A_L is multiplied by 1 or -1 to obtain A , $\text{rank}(A) = \text{rank}(A_L)$. Using Lemma 1.1, we can say that if there are k independent moves in L i.e., $k = \text{rank}(A)$, the probability that L is ϵ -slowly improving from a given initial state σ_0 is given by,

$$P(\forall i \in [l], \langle \alpha_i, X \rangle \in (0, \epsilon]) \leq (\phi\epsilon)^k$$

By Lemma 2.3, [3], $\text{rank}(A_L)$ depends only on L (sequence of vertices) and not on σ_0 . This is because for two different initial state, any row in the matrix of one is 1 or -1 times the corresponding row of the other. Taking a naive union bound, we get

$$P(L \text{ is } \epsilon\text{-slowly improving from some } \sigma_0) \leq 2^n (\phi\epsilon)^{\text{rank}(L)} \text{ for a given sequence } L \quad (2)$$

$$P(\exists L, \sigma_0 \text{ such that } L \text{ is } \epsilon\text{-slowly improving from } \sigma_0 \text{ and } l(L) = l) \leq 2^n n^l (\phi\epsilon)^k \quad (3)$$

where $l(L)$ is length of sequence L and $k = \min_{l(L)=l} \text{rank}(L)$.

For $l(L) = 2n$, if $\text{rank}(L) = \Omega(n)$ for all sequences L , we can choose a suitable $\epsilon (= 1/\text{poly}(n))$ such that equation (3) is $o_n(1)$. But $\text{rank}(L)$ is not $\Omega(n)$ for all sequences and the exponent 2^n in equation (2) makes it impossible to get $\epsilon (= 1/\text{poly}(n))$ such that equation (2) is $o_n(1)$. Getting a lower bound on rank and reducing exponent in equation (2) helps in getting better bound on probability.

For any sequence L , we use the following notations:

- $S(L)$ = set of vertices flipped in L , $|S(L)| = s(L)$
- $S_1(L)$ = set of vertices flipped only once in L , $|S_1(L)| = s_1(L)$
- $S_2(L)$ = set of vertices flipped more than once in L , $|S_2(L)| = s_2(L)$
- **Singleton block**: Maximal sub-sequence of L having all its vertices in $S_1(L)$
- **Transition block**: Maximal sub-sequence of L having all its vertices in $S_2(L)$

Lemma 1.2. (Lemma 3.1, [3]) For any sequence of moves L one has

(i) $\text{rank}(L) \geq \min(s(L), n-1)$

Furthermore, if $s(L) < n$ and L doesn't visit any state more than once, then

(ii) $\text{rank}(L) \geq s(L) + s_2(L)/2$

(iii) $\text{rank}(L) \geq s_1(L) + \sum_i s(T_i)$ Here, $\sum_i s(T_i) = \sum_v b(v)$ where $b(v)$ is the number of transition blocks in which a vertex appears.

From the above lemma, we only know that rank is at least $s(L)$ and so we need to get a better (less than exponential in n) union bound.

Lemma 1.3. (Lemma 4.4, [3]) Suppose the random weights satisfy $|X_e| \leq 1 \forall e \in E$ and $S(L) < n$, then

$$P(L \text{ is } \epsilon\text{-slowly improving from some } \sigma_0) \leq 2 \left(\frac{4n}{\epsilon} \right)^s (8\phi\epsilon)^{\text{rank}(L)}$$

For using this union bound we need $\text{rank}(L) \geq (1 + \delta)s(L)$, where $\delta > 0$.

Fix $\beta > 0$. A sub-sequence B of a sequence L is called a **critical block** if $l(B) \geq (1 + \beta)s(B)$ and for every sub-sequence B' of B , $l(B') < (1 + \beta)s(B')$. It is easy to see that every sequence L which has $l(L) \geq (1 + \beta)s(L)$ has a critical block. Also, any critical block B has $l(B) = \lceil (1 + \beta)s(B) \rceil$ (Lemma 4.1, [3]).

Lemma 1.4. In complete graph, for a critical block B which doesn't visit any state more than once and $s(B) < n$ and $\beta = 1$, we have $\text{rank}(B) \geq 1.25s(B)$.

The above lemma follows from Lemma 4.1, [3] and Corollary 4.3, [3] (which uses Lemma 1.2) (use $\beta = 1$). From Lemma 1.4, in complete graph, for every sequence L of length $2n$ which doesn't visit any state more than once and $s(L) < n$, we have a sub-sequence B which is a critical block (so, $l(B) = 2s(B)$) such that $\text{rank}(B) \geq 1.25s(B)$. Using the above results, Proposition 1.1 can be proved (proof in Section 4.3, [3]).

2 ANALYSIS FOR ARBITRARY GRAPHS

2.1 Extension of the same idea to arbitrary graphs

As already shown, there doesn't exist an ϵ -slowly improving sequence L of length l from any initial state σ_0 with $l = O(\text{poly}(n))$ and $\epsilon = \Omega(1/\text{poly}(n, \phi))$ implies that any implementation of FLIP terminates in at most $O(\text{poly}(n, \phi))$ steps. So, we try to prove something similar to Proposition 1.1 and see the limitations in extending the idea of proof for complete graphs to arbitrary graphs. What we try to prove is that "with high probability, there doesn't exist an ϵ -slowly improving sequence of length $2n$ from any σ_0 , with $\epsilon = \Omega(1/\text{poly}(n, \phi))$ ".

Definitions for move vector α , matrix A_L for a sequence of moves L and initial state σ_0 and $S(L), S_1(L), S_2(L)$, singleton and transition block defined for a sequence of moves L can be extended to arbitrary graphs as well. $\text{Rank}(A_L)$ is still independent of initial state σ_0 . So, equations (2) and (3) holds. The only thing which need to be done is to extend Lemma 1.2 and Lemma 1.3 for arbitrary graphs.

Lemma 2.1. *For any sequence of moves L , $\text{rank}(L) \geq \min(s(L), n - 1)$*

Proof. Assume $s < n$. Construct a spanning tree T of graph G (G is connected). We construct a lower triangular sub-matrix M of A_L of size s as follows. In T , make a non-flipped vertex as its root. Remove a leaf vertex and the edge attached to it and if the leaf vertex is in $S(L)$, then add the row corresponding to that edge in A_L to M and the column corresponding to some time at which that vertex is flipped to M . Repeat this process till all the vertices in H are removed. While adding a new row and column to H , insert the new row at the top and the new column to the left-most end. Consider some t^{th} row and t^{th} column of M . Suppose the column corresponds to flip of vertex v . Now, the edge in t^{th} row has vertex v as one of its end points and none of the edges above it has v as one of its end point. So, the entries above the entry (t, t) are zero and $(t, t)^{\text{th}}$ entry is non-zero for $k \in [s]$. So, M is lower triangular sub-matrix of A_L with s rows and non-zero diagonal entries. So, $\text{rank}(A_L) \geq s$. For the case when $s = n$, the same proof can be applied by taking some flipped vertex as root of T . \square

Also, for a sequence L , lemma 1.2, (ii) can be proved if every vertex in S has a vertex adjacent to it in $V - S$.

Lemma 2.2. *For a sequence of moves L such that L doesn't visit any state more than once and every vertex in $S(L)$ is adjacent to some vertex in $V - S(L)$, one has $\text{rank}(L) \geq s(L) + s_2(L)/2$*

Proof. For this, we define an auxiliary directed graph H as follows. Vertex set of H is the set of n vertices in G . Since L doesn't visit any state more than once, between two successive occurrences of a repeated vertex v in L , we have a vertex u adjacent to v occurring between them odd number of times. For every repeated vertex v , find one such a vertex u , occurring between the first two appearances of v and add (v, u) to $E(H)$. So, $|E(H)| = s_2$. Since each vertex in H has out-degree atmost 1, the cycles in H are vertex-disjoint. Also, there are no self-loops in H . So, H can be made acyclic by removing at most $s_2/2$ edges, giving a graph with at least $s_2/2$ edges. Define set P' as follows. For each vertex in S , we have an edge incident to it whose other end point is not in S . For each vertex in S , take one such edge and add to P' .

Claim: *For any acyclic sub-graph H' of H , the rows in matrix A of L corresponding to edges in H' (interpreted as undirected) combined with P' are linearly independent.*

$E(H')$ and P' are disjoint as both end points of edges in $E(H')$ are flipped while only one of the end points of edges in P' is flipped. We have an acyclic subgraph of H with $s_2/2$ edges and so the lemma follows from this claim. This claim can be proved by induction on number of edges in H' . If there are no edges in H' , then the edges are linearly independent (consider a column at which vertex v is flipped. It is non-zero only at the row of edge with one end point as v). Now, suppose H' has k edges where $k > 0$. Since H' is acyclic, we have a vertex v which has no incoming edge. Let $(v, u) \in E(H')$ and $\{v, w\}$ in P' . Consider the linear combination of the considered edges: $\sum_{e \in P'} \lambda_e A[e] + \sum_{e \in H'} \mu_e A[e] = 0$. Consider the first two times at which v flips. At the corresponding columns, $A[\{v, w\}]$ will have same entry at both places and $A[\{v, u\}]$ will have opposite entries and all other rows will have 0 entry. So, $\mu_{\{v, u\}} = 0$. Inductive hypothesis applied to the graph with the edge from v removed gives that all coefficient of the linear combination are zero. So, the claim is proved by induction. \square

Lemma 2.3. Suppose the random weights satisfy $|X_e| \leq 1 \ \forall e \in E$ and $s(L) < n$, then

$$P(L \text{ is } \epsilon - \text{slowly improving from some } \sigma_0) \leq \left(\frac{2n}{\epsilon}\right)^{s_0} (64\phi\epsilon)^{\text{rank}(L)}$$

where $s_0(L)$ is number of vertices in L which has at least one adjacent non-flipped vertex.

Proof. For a fixed σ_0 , consider set of all edges (say I) in G whose corresponding rows in A_L are linearly independent. The same set of rows are linearly independent for a different σ_0 because changing σ_0 just multiplies each row by 1 or -1. Define $T = \{v \in V : vw \in I \text{ for } w \in V\} \cup S(L)$. $|T| \leq 2 * \text{rank}(L) + s$. $H(\sigma)$ can be split as $H(\sigma) = H_0(\sigma) + H_1(\sigma) + H_2(\sigma)$ where $H_i(\sigma)$ is sum over edges with exactly i end points in T for $i \in \{0, 1, 2\}$.

$H_0(\sigma_t) - H_0(\sigma_{t-1}) = 0$ as $\sigma_t(v) = \sigma_{t-1}(v)$ for non-flipped vertex v .

$$H_1(\sigma_t) - H_1(\sigma_{t-1}) = -\sigma_t(v_t) \sum_{u \notin T} X_{uv_t} \sigma_0(u) = \sigma_t(v_t) Q(v_t)$$

Define $D = 2\epsilon Z \cap [-n, n]$. $|D| \leq n/\epsilon + 1 \leq 2n/\epsilon$. Since $Q(v_t) \in [-n, n]$, $Q(v_t)$ is within a distance of ϵ for some $d(v_t)$ in D .

$H_2(\sigma_t) - H_2(\sigma_{t-1}) = \langle \alpha'_t, X \rangle$ where

$$\alpha'_t = \begin{cases} -\sigma_t(u)\sigma_t(w) & uw \in E, v_t \in \{u, w\}, \{u, w\} \subseteq T \\ 0 & \text{otherwise} \end{cases}$$

It can be seen that α_t restricted to the edges with both end points in T gives α'_t . $\text{rank}([\alpha'_t]_{t \leq l}) = \text{rank}(L)$ as all rows corresponding to edges in A_L which are linearly independent are also in $[\alpha'_t]$.

$$H(\sigma_t) - H(\sigma_{t-1}) = \langle \alpha'_t, X \rangle + \sigma_t(v_t)d(v_t) + \delta_t, \text{ where } |\delta_t| \leq \epsilon$$

L is ϵ -slowly improving implies,

$$\forall t \in [l], |\langle \alpha'_t, X \rangle + \sigma_t(v_t)d(v_t)| \leq 2\epsilon$$

So, $\langle \alpha'_t, X \rangle$ lies in a 4ϵ range centered around $-d(v_t)$ and $d(v_t)$. α'_t depends only on the vertices in T . Taking union bound over $\sigma_{v \in T}$ and d , we get

$$P(L \text{ is } \epsilon\text{-slowly improving from some } \sigma_0) \leq 2^{2*\text{rank}(L)+s} \left(\frac{2n}{\epsilon}\right)^{s_0} (8\phi\epsilon)^{\text{rank}([\alpha'_t]_{t \leq l})}$$

□

So, we need $\text{rank}(L) \geq (1 + \delta)s_0(L)$ so as to get better bound.

2.2 Problems in case of arbitrary graphs

It has been shown by [3] that there are graphs which have sequences such that each sub-sequence of it has a small rank.

These kind of sequences of such graphs makes it impossible to extend the same idea of rank for complete graphs to arbitrary graphs.

Theorem 2.1. Given $C_1 > C_2 > 0$ and $\delta > 0$, there exists infinitely many graphs G with some initial configuration σ_0 and sequence L such that

(i) $l(L) \geq C_1 s(L)$

(ii) $l(L) \geq C_2 |V(G)|$

(iii) for each block B of L , $\text{rank}(B) \leq (1 + \delta)s_0(B)$

Proof. Let L' be a sequence of moves from initial state σ'_0 using at most n letters, having at least $n/2$ letters, with $l(L') \geq C_1 n$, such that any sub-sequence $B' \subseteq L'$ has $s_2(B')/s(B') \leq \delta/2$ (By Theorem 5.2, [3], there exists a large enough n such that it holds). Let H be a graph with n vertices such that each vertex in $S(L')$ has an adjacent vertex in $V(H) - S(L')$. Construct graph G from H as follows. For each vertex v in H , add k new vertices (k has to be determined) and join them to v . Add a new vertex w and join it to all the newly added vertices. Let L be a sequence in which vertices of L' are flipped in order but after each flip of a vertex in L' , the flip of k newly added vertices joined only to it follows. $l(L) = (1+k)l(L')$ and $s(L) = (1+k)s(L')$. So, $l(L) \geq C_1 s(L)$. $|V(G)| = (1+k)n + 1$. So, we can choose k large enough such that $l(L) \geq C_2 |V(G)|$. For an expanded block B of G from B' of H (B is obtained by appending the k newly joined vertices after each vertex in B'), $rank(B) = rank(B') + s(B') + s_2(B')$. For any sub-sequence B of L , let B' be smallest subsequence of L' such that B is subsequence of expanded sequence of B' . We have

$$\frac{rank(B)}{s_0(B)} - 1 \leq \frac{rank(B') + ks(B') + ks_2(B')}{(k+1)(s(B') - 2)} \quad (4)$$

From $rank(B') \leq n^2$, $s_2(B')/s(B') \leq \delta/2$, we can get a lower bound on k (in terms of n and δ) such that equation (4) is at most δ . This gives infinitely many graphs G satisfying the conditions of the theorem. \square

By theorem 2.1, it is impossible to extend the exact same idea applied to complete graphs to arbitrary graphs. But, it is still possible to deal with the sequences which don't have a sub-sequence B with $rank(B) \geq ks(B)$ where $k > 1$ separately.

2.3 Some Useful Ideas

Lemma 2.4. Fix $\eta > 0$. For $\epsilon = n^{-(2+\eta)}\phi^{-1}$, the probability that there exists a vertex with degree at most $\log(n)$ whose flip changes H in $(0, \epsilon]$ is $o_n(1)$.

Proof. For a given vertex v with degree at most $\log(n)$ and initial state σ , let move vector be α .

$$P(H(\sigma^{-v}) - H(\sigma) \in (0, \epsilon]) = P(\langle \alpha, X \rangle \in (0, \epsilon]) \leq (\phi\epsilon)$$

For fixed vertex v , the number of possible move vectors α for different initial states σ is at most $2^{\log(n)}$. Taking union bound, we get

$$P(\exists \sigma \text{ such that } H(\sigma^{-v}) - H(\sigma) \in (0, \epsilon]) \leq 2^{\log(n)}(\phi\epsilon)$$

$$P(\exists \sigma, \text{ vertex } v \text{ of degree at most } \log(n) \text{ such that } H(\sigma^{-v}) - H(\sigma) \in (0, \epsilon]) \leq n 2^{\log(n)}(\phi\epsilon) = n^2(\phi\epsilon)$$

For $\epsilon = \phi^{-1}n^{-(2+\eta)}$ ($\eta > 0$), the above expression is $o_n(1)$. \square

We can partition the event R : there exists an ϵ -slowly improving sequence of length $2n$ from some initial state σ_0 into two events, R_1, R_2 where

- (i) R_1 : $\exists L, \sigma_0$ such that L is ϵ -slowly improving from σ_0 , $l(L) = 2n$ and $S(L)$ has at least one vertex with degree at most $O(\log(n))$.
- (ii) R_2 : $\exists L, \sigma_0$ such that L is ϵ -slowly improving from σ_0 , $l(L) = 2n$ and all vertices in $S(L)$ have degree at least $\Omega(\log(n))$.

R_1 implies there exists a state and a vertex with degree at most $O(\log(n))$ whose flip from that state improves H by at most ϵ and so, by lemma 2.4, with $\epsilon = n^{-(2+\eta)}$, $\eta > 0$, $P(R_1)$ is $o_n(1)$. For R_2 , we can divide into two, one with $s_0(L) \leq s(L)/p$, $p > 1$ and one with $s_0(L) > s(L)/p$. For $s_0(L) \leq s(L)/p$, by lemma 2.3 and lemma 2.1, we can find suitable $\epsilon (= 1/\text{poly}(n, \phi))$ such that its probability is $o_n(1)$. The only case for which needs to be solved is when all the flipped vertices have degree at least $\Omega(\log(n))$ and $s_0(L) > s(L)/p$ for some $p > 1$.

3 ANALYSIS FOR SPECIAL GRAPHS

3.1 Graphs with a clique and vertices outside clique having maximum degree of $O(\log n)$

We claim that for graphs having an arbitrary size clique, such that all vertices outside the clique have maximum degree $O(\log n)$, with high probability, any implementation of FLIP terminates in $\text{poly}(n)$ number of steps.

Theorem 3.1. *Let G be a graph on n vertices such that G has at least one clique such that all vertices outside that clique have degree at most $\log n$. Assume the edge weights $X = (X_e)_{e \in E}$ are independent random variables with $|X_e| \leq 1$ and density bounded above by ϕ . For any $\eta > 0$, with high probability any implementation of FLIP terminates in at most $O(n^{(3+a+\eta)}\phi^a)$ steps, with implicit constant depending on η , where $a = 2\left(\frac{35+\sqrt{577}}{\sqrt{577}-19}\right)$*

We prove the above using some results given in [1] and [3] and slightly modifying the idea.

Consider a graph G with n vertices which has a clique H of size k such that vertices not in H have degree at most $\log(n)$.

Proposition 3.1. *Fix $\eta > 0$ and let $\epsilon = n^{-(a+\eta)}\phi^{-a}$. Then with high probability, there is no ϵ -slowly improving sequence of length $2n$ from any σ_0 in graph G , where $a = 2\left(\frac{35+\sqrt{577}}{\sqrt{577}-19}\right)$*

As stated in the section of arbitrary graphs, Proposition 3.1 implies Theorem 3.1.

Proof. We divide all the possible sequences L , of length $2n$ into different cases and analyze them separately. We partition the event that there exists an ϵ -slowly improving sequence of length $2n$ from some initial state σ_0 (say R) and calculate probability of each event to get the probability of R . Let $p > 1$. We define events $R, R_0, R_1, R_2, R_3, R_4, R_5$ and R_6 as follows.

- R : $\exists L, \sigma_0$ such that L is ϵ -slowly improving from σ_0 and $l(L) = 2n$.
- R_0 : $\exists L, \sigma_0$ such that L is ϵ -slowly improving from σ_0 , $l(L) = 2n$ and $s(L) = n$
- R_1 : $\exists L, \sigma_0$ such that L is ϵ -slowly improving from σ_0 , $l(L) = 2n$, $s(L) < n$ and $S(L) \not\subseteq H$
- R_2 : $\exists L, \sigma_0$ such that L is ϵ -slowly improving from σ_0 , $l(L) = 2n$, $s(L) < n$ and $S(L) \subseteq H$
- R_3 : \exists critical B, σ_0 such that B is ϵ -slowly improving from σ_0 , $s(B) < n$ and $S(B) \subseteq H$
- R_4 : \exists critical B, σ_0 such that B is ϵ -slowly improving from σ_0 , $s(B) < n$, $S(B) \subseteq H$ and $s(B) < k$.
- R_5 : \exists critical B, σ_0 such that B is ϵ -slowly improving from σ_0 , $s(B) < n$, $S(B) \subseteq H$, $s(B) = k$ and $s_0(B) \leq s(B)/p$.
- R_6 : \exists critical B, σ_0 such that B is ϵ -slowly improving from σ_0 , $s(B) < n$, $S(B) \subseteq H$, $s(B) = k$ and $s_0(B) > s(B)/p$.

Let $\eta > 0$.

For R_0 , since $s(L) = n$, by lemma 2.1, $\text{rank}(L) \geq n - 1$. By naive union bound, we get

$$P(R_0) \leq 2^n n^{2n} (\phi\epsilon)^{n-1}$$

For $\epsilon = \phi^{-1}n^{-(2+\eta)}$, $P(R_0)$ is $o_n(1)$.

In R_1 , since $S(L) \not\subseteq H$, there exists vertex with degree at most $\log(n)$ which is flipped in L . So,

$$P(R_1) \leq P(\exists \sigma, \text{vertex } v \text{ of degree at most } \log(n) \text{ such that } H(\sigma^{-v}) - H(\sigma) \in (0, \epsilon]) \leq n^2(\phi\epsilon)$$

The last inequality follows from lemma 2.4. For $\epsilon = \phi^{-1}n^{-(2+\eta)}$, $P(R_1)$ is $o_n(1)$.

R_2 implies R_3 because as claimed in the proof of complete graphs, any sequence L of length $2n$ has a critical block B and the definition of critical block doesn't depend on the graph. Only the rank property of critical block depends on the graph. $P(R_2) \leq P(R_3)$.

It is clear that $R_3 = R_4 \cup R_5 \cup R_6$.

For R_4 , we claim that for a critical block B such that $S(B) \subseteq H$ and $s(B) < k$, $\text{rank}(B) \geq 1.25s(B)$. This can be seen as follows. Consider the induced sub-graph K of G induced by H . It is a complete graph and B is a sequence such that $s(B) < |V(K)|$. Also, consider the matrix A for the sequence B in graphs G and K (say $A_G(B)$ and $A_K(B)$ respectively). By Lemma 1.4, $\text{rank}(A_K(B)) \geq 1.25s(B)$. If vertex v is flipped and u and w are non-flipped vertices, then rows corresponding to uv and vw are linearly dependent. So, the rows corresponding to edges with no end point or exactly one end point in H in matrix $A_G(B)$ can be expressed as linear combination of edges with both end points in H . So, $\text{rank}(A_G(B)) = \text{rank}(A_K(B)) \geq 1.25s(B)$. So,

$$P(R_4) \leq \sum_{s=1}^{n-1} n^{2s} \left(\frac{2n}{\epsilon} \right)^s (64\phi\epsilon)^{1.25s} = \sum_{s=1}^{n-1} (Cn^3\phi^{5/4}\epsilon^{1/4})^s$$

The last inequality follows from lemma 2.3 and properties of critical block. n^{2s} is the number of possible blocks with s distinct vertices (since length is $2s$). For $\epsilon = \phi^{-5}\epsilon^{-(12+\eta)}$, $P(R_4)$ is $o_n(1)$.

For R_5 , using Lemma 2.3 and $s_0(B) \leq s(B)/k$, we get

$$P(R_5) \leq \sum_{s=1}^{n-1} n^{2s} \left(\frac{2n}{\epsilon} \right)^{s/p} (64\phi\epsilon)^s = \sum_{s=1}^{n-1} (Cn^{2+1/p}\phi\epsilon^{(1-1/p)})^s$$

For R_6 , we claim that for a critical block B such that $s(B) < n$, $S(B) \subseteq H$, $s(B) = k$ and $s_0(B) > s(B)/p$, $\text{rank}(B) \geq 1.25s(B) - s(B)(1 - 1/p)$. This can be seen as follows. Let v be a flipped vertex and u and w be non-flipped vertex. Suppose $uv \in E(G)$, then removing uv and adding vw to $E(G)$ doesn't change rank of matrix A as the row corresponding one will be 1 or -1 times the other. Now, $s(B) = k$ and $s(B) < n$. Choose a vertex w in $V - H$. For each vertex v in $S(B)$ which has a non-flipped neighbour, remove the edge with that non-flipped neighbour and add the edge vw . So far, $\text{rank}(A)$ hasn't changed. Now, for vertices v which don't have a non-flipped adjacent vertex, add edge vw . So, $\text{rank}(A)$ increases by at most the number of newly added edges with is at most $s(B)(1 - 1/p)$. The obtained graph G' and the sequence B is similar to the one in R_4 . So, $\text{rank}_{G'}(B) \geq 1.25s(B)$. So, for the original sequence, we have $\text{rank}_G(B) \geq 1.25s(B) - s(B)(1 - 1/p)$. By Lemma 2.3, we have

$$P(R_6) \leq \sum_{s=1}^{n-1} n^{2s} \left(\frac{2n}{\epsilon} \right)^s (64\phi\epsilon)^{5s/4 - s(1-1/p)} = \sum_{s=1}^{n-1} (Cn^3\phi^{(1/4+1/p)}\epsilon^{(1/p-3/4)})^s$$

Probability of all partition of the event R has been calculated.

$$P(R) = P(R_0) + P(R_1) + P(R_4) + P(R_5) + P(R_6)$$

Choosing p optimally such that exponent of n in ϵ is high and $P(R) = o_n(1)$, we get $p = \frac{17+\sqrt{577}}{36}$ and exponent of n in ϵ as $2\left(\frac{35+\sqrt{577}}{\sqrt{577}-19}\right) + \eta$. So, we get $\epsilon = \phi^{-a}n^{-(a+\eta)}$ where $a = 2\left(\frac{35+\sqrt{577}}{\sqrt{577}-19}\right)$ for which $P(R)$ is $o_n(1)$. □

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