# Recommendations for practical use of numerical methods in linear and nonlinear dynamics

Mirza M. Irfan Baig[1] and Thomas Grätsch[2]

[1]Simpson Gumpertz & Heger, Inc.
Waltham, MA 02453, USA

[2]Hamburg University of Applied Sciences
Faculty of Technology and Computer Science
Berliner Tor 21, 20099 Hamburg, Germany

**Abstract**
In this paper we give some recommendations for the use of practical numerical methods for the solution of single degree of freedom linear and nonlinear systems. The equations we consider are first or second order ordinary differential equations. Besides our recommendations for single degree of freedom systems, we also give suggestions about using these methods for multi-degree of freedom systems like those arising in finite element analysis.

## 1 Introduction

Many important physical phenomena that evolve over time can be modeled by single degree of freedom systems. Typically the governing equations are first or second order differential equations. The exact solution for these initial-value problems may be available for some special set of initial conditions and/or forcing functions. But more generally an exact solution, although existing, may not be available in closed form. In such cases, recourse must be made to numerical techniques for computation of an *approximate* solution. However, the method used must be stable and accurate enough to give a reasonable degree of confidence in the solution obtained. Moreover, convergence to the exact solution should be assured as more effort (finer discretization) is made toward obtaining the approximate solution.

We start with giving a short overview of the Newton-Raphson method for solving nonlinear equations. This is followed by a brief introduction to numerical solution of initial value problems; we discuss some popular techniques and give their formulation. The use of these techniques is then demonstrated in some typical numerical examples. Finally, we give recommendations on the use of these techniques for practical problems.

## 2 The Newton-Raphson method

The Newton-Raphson method is a linearization technique that iteratively calculates the solution of a given nonlinear equation. Any nonlinear equation can be written as

$$f(u) = 0 \tag{1}$$

where $u$ is the solution to be calculated. Consider a point $a$ close to the solution $u$. Assuming the function to be smooth enough near the solution, the above equation can be written as a truncated Taylor series expansion about $a$ approximately as

$$f(a) + f'(a)\Delta u \approx 0 \tag{2}$$

The above equation forms the basis for the iterative procedure to calculate the solution. Assuming an initial guess $u^{(0)}$ for the solution is known, the increment $\Delta u$ to be added to this approximation to get an improved value for the solution is calculated as

$$\Delta u = -\frac{f(u^{(0)})}{f'(u^{(0)})} \tag{3}$$

The improved approximation after this first iteration is then obtained as

$$u^{(1)} = u^{(0)} + \Delta u \tag{4}$$

The iteration is repeated until the increment $\Delta u$ is smaller than a pre-defined tolerance. Assuming that $i$ iterations have been carried out, the next approximation is then given by

$$u^{(i+1)} = u^{(i)} - \frac{f(u^{(i)})}{f'(u^{(i)})}$$

The success of the method depends on the regularity of the function near the solution: the derivative of the function should be non-zero and finite at the solution for the method to converge. Moreover, if the derivative is Lipschitz continuous (does not show sharp needle-like spikes), the method converges quadratically near the solution (error after each successive approximation is some constant times the square of the error in the previous approximation). For more details and proofs of convergence properties, see [3].

## 3 Numerical solution of initial value problems

Typically, for the numerical solution of initial value problems, the independent variable time is discretized into steps of finite size $\Delta t$. The time derivatives appearing in the ordinary differential equation are then usually approximated by using finite difference expressions, (see [1]). The finite difference approximations, when substituted for time derivatives in the differential equation, result in an expression involving known solution values at a given time step and unknown solution values at the next time step. The solution at the

next time step is then calculated by simple algebraic manipulation, or in the case of nonlinear systems, when such algebraic rearrangement is not possible, by the Newton-Raphson method. The accuracy of the finite difference approximation influences the quality and behavior of the numerical solution.

Numerical integration schemes are broadly classified as either *explicit* or *implicit*. In explicit schemes the governing equation is written at time $t$ for which all the solution variables are already known, and the difference equation is then solved for the solution at the next time step $t + \Delta t$. The finite difference approximations, depending on their order and accuracy, may use solution at times $t - \Delta t$, $t - 2\Delta t$ or even earlier.

Explicit methods are only conditionally stable, i.e. the time step size $\Delta t$ should be lower than a *critical step size* in order to obtain stable solutions. The solution "blows up" if a time step $\Delta t$ larger than the critical limit is used. Explicit methods are generally preferred for solution of problems where the interesting part of the solution is changing rapidly in time like in wave propagation problems or crash analyses.

In implicit schemes, the governing equation is written at time $t + \Delta t$, while the solution is known for time $t$. The difference equation resulting from the use of an appropriate finite difference approximation is then solved for solution variable at $t+\Delta t$. Implicit schemes can be either conditionally or unconditionally stable. Unconditionally stability (no upper limit on $\Delta t$) is usually guaranteed for linear problems only, and may be lost in nonlinear cases. However it is usually preferable to use in nonlinear analyses also, those methods which are stable in linear analyses. Implicit methods are better suited for problems where the solution variation over time is less rapid, and relatively larger time steps can adequately resolve the solution.

### 3.1 The Euler forward difference method

The Euler forward difference scheme, commonly referred to as the Euler-Cauchy method, is an *explicit*, conditionally stable time integration method. We present this scheme here only for academic interest as it is only first order accurate and a very small time step size needs to be used to get reasonable accuracy. It is almost always used, if ever, for solution of first order differential equations. We seek to solve the equation

$$\dot{u} = f(u, t) \tag{5}$$

The derivative of the solution is approximated as the forward difference equation

$${}^{t}\dot{u} = \frac{{}^{t+\Delta t}u - {}^{t}u}{\Delta t} \tag{6}$$

The solution at time $t + \Delta t$ is then calculated from the expression

$${}^{t+\Delta t}u = {}^{t}u + f({}^{t}u, t)\Delta t \tag{7}$$

### 3.2 The Euler backward difference method

The Euler backward difference method is an *implicit*, first order accurate method. It shows good stability, and the solution in fact tends to decay rather than blow-up as in Euler forward difference method. The method uses the following finite difference approximation

$$ {}^{t+\Delta t}\dot{u} = \frac{{}^{t+\Delta t}u - {}^{t}u}{\Delta t} \tag{8} $$

Writing equation (5) at time $t + \Delta t$, and substituting equation (8), we obtain

$$ {}^{t+\Delta t}u = {}^{t}u + f({}^{t+\Delta t}u, t + \Delta t)\Delta t \tag{9} $$

If the function $f$ is linear in $u$, the computation of ${}^{t+\Delta t}u$ can be carried out by simple algebraic re-ordering. However if $f$ is nonlinear in $u$, simple algebraic manipulation may not be possible. In such a case one may employ the Newton-Raphson method for solving the equation, since an exact value of ${}^{t+\Delta t}u$ is not available for calculating $f({}^{t+\Delta t}u, t+\Delta t)$.

Euler methods are only first order accurate (the error reduces linearly as the time step size is reduced), and very small time steps need to be employed to get a reasonably accurate solution. Better methods than these are available for practical problems.

### 3.3 The central difference method

The central difference method is an *explicit*, second order accurate method. The method uses finite difference approximations which are centered at time $t$. The first and second derivative of the solution $u$ are approximated as

$$ {}^{t}\dot{u} = \frac{{}^{t+\Delta t}u - {}^{t-\Delta t}u}{2\Delta t} \tag{10} $$

$$ {}^{t}\ddot{u} = \frac{{}^{t+\Delta t}u - 2\,{}^{t}u + {}^{t-\Delta t}u}{\Delta t^2} \tag{11} $$

The central difference method is most commonly used in single degree of freedom vibration problems. The second order differential equation governing a mass-spring-damper system is

$$ m\ddot{u} + c\dot{u} + f(u) = R(t) \tag{12} $$

where $m$ is the mass, $c\dot{u}$ is viscous damping force, and $f(u)$ is the internal force in the spring. $R(t)$ is the externally applied force as a function of time. $f(u)$ can be either a linear or nonlinear function, e.g. for linear problems, $f(u) = ku$, where $k$ is the spring constant. We write equation (12) at time $t$ and use the finite difference approximations from equations (10) and (11)

$$ m\frac{{}^{t+\Delta t}u - 2\,{}^{t}u + {}^{t-\Delta t}u}{\Delta t^2} + c\frac{{}^{t+\Delta t}u - {}^{t-\Delta t}u}{2\Delta t} + f({}^{t}u) = R(t) \tag{13} $$

On rearrangement one obtains

$$^{t+\Delta t}u = \Big[\, R(t) - f(\,^{t}u) + \frac{c}{2\Delta t}\,^{t-\Delta t}u + \frac{m}{\Delta t^2}(2\,^{t}u - \,^{t-\Delta t}u)\Big]/(\frac{m}{\Delta t^2} + \frac{c}{2\Delta t}) \tag{14}$$

Even though the central difference method is an explicit method, for single degree of freedom systems the time step constraint is almost never a problem, since time step size needed to resolve the solution is much smaller than the critical step size. The method is second order accurate and is known to perform exceedingly well. Moreover it possesses very attractive conservation properties making it very suitable for long time simulations, for example see [6].

### 3.4 The Runge-Kutta Method

Among the Runge-Kutta family of numerical integrators, (see [1]), the most popular is the fourth order accurate *explicit* Runge-Kutta method. From now on we refer to this method as simply the Runge-Kutta method. The idea behind the scheme is to approximate the slope of the secant of the solution, from one time step to the next. The slope of the secant is approximated as the weighted average of tangents computed at various points within the time step, using successively improved approximations. The Runge-Kutta method is typically presented for first-order differential equation, but can be applied to higher order differential equations by introducing secondary variables. We seek to solve equation (5); then the Runge-Kutta algorithm is given by the expressions

$$\begin{aligned} h_1 &= f(\,^{t}u, t) \\ h_2 &= f(\,^{t}u + \frac{\Delta t}{2}h_1, t + \frac{\Delta t}{2}) \\ h_3 &= f(\,^{t}u + \frac{\Delta t}{2}h_2, t + \frac{\Delta t}{2}) \\ h_4 &= f(\,^{t}u + \Delta t\, h_3, t + \Delta t) \end{aligned} \tag{15}$$

The solution at next time step is then approximated by

$$^{t+\Delta t}u = \,^{t}u + \frac{\Delta t}{6}(h_1 + 2h_2 + 2h_3 + h_4) \tag{16}$$

The Runge-Kutta method is an extremely accurate scheme. However it requires the function $f(\,^{t}u, t)$ to be evaluated four times for each time step. In some cases, e.g. in elasto-plastic or visco-plastic spring systems, the function evaluation itself can be quite expensive, and the Runge-Kutta method may turn out to be computationally costly. For a given time step size, the method is four times as expensive as the central difference method, and for second order differential equations like vibration problems, it may be more practical to use central difference method with about half the time step size and get similar solution accuracy. Nevertheless, the Runge-Kutta method is widely used for its accuracy and the fact that reliable codes are available that carry out adaptive time step adjustment, making the method even more accurate (e.g. MATLAB's ode45 function).

### 3.5 The Newmark method

The Newmark method was originally introduced for solution of second order differential equations arising from dynamic equilibrium in structural dynamics, see [4]. The Newmark method essentially approximates the velocity and displacement at time $t + \Delta t$ using the expressions

$$ {}^{t+\Delta t}\dot{u} = {}^{t}\dot{u} + [(1-\delta)\,{}^{t}\ddot{u} + \delta\,{}^{t+\Delta t}\ddot{u}]\Delta t \tag{17} $$

$$ {}^{t+\Delta t}u = {}^{t}u + {}^{t}\dot{u}\Delta t + [(1/2-\alpha)\,{}^{t}\ddot{u} + \alpha\,{}^{t+\Delta t}\ddot{u}]\Delta t^2 \tag{18} $$

where $\delta$ and $\alpha$ are parameters which can be chosen independent of each other. There are however limitations on the choice of these parameters if an accurate and unconditionally stable (in linear analyses) integration scheme is desired. Making use of equations (17) and (18) in equation (12) written at time $t+\Delta t$, an equation with only ${}^{t+\Delta t}u$ as the unknown is obtained

$$ \begin{aligned} (\frac{m}{\alpha\Delta t^2} + c\frac{\delta}{\alpha\Delta t})\,{}^{t+\Delta t}u = {} & R(t+\Delta t) - f({}^{t+\Delta t}u) + m\big[(\frac{1}{2\alpha}-1)\,{}^{t}\ddot{u} + \frac{1}{\alpha\Delta t^2}({}^{t}u + {}^{t}\dot{u}\Delta t)\big] \\ & + c\big[\frac{\delta}{\alpha\Delta t}({}^{t}u + {}^{t}\dot{u}\Delta t) - {}^{t}\dot{u} \\ & - {}^{t}\ddot{u}\Delta t(1-\frac{\delta}{2\alpha})\big] \end{aligned} \tag{19} $$

If $f(u)$ is linear, the solution ${}^{t+\Delta t}u$ can be directly calculated from the above expression. However if $f(u)$ is nonlinear, Newton-Raphson method may be used to solve the above equation for ${}^{t+\Delta t}u$.

The Newmark method is unconditionally stable as long as (see [5])

$$ \begin{aligned} & \delta \geq 0.5 \\ & \alpha \geq 0.25(0.5+\delta)^2 \end{aligned} \tag{20} $$

For $\delta = 0.5$ and $\alpha = 0.25$ the method is unconditionally stable (for linear problems) and produces no numerical damping (reduction in solution amplitude due to numerical errors, even in the absence of a damper) in the solution. We consider only this choice of parameters in the subsequent examples.

### 3.6 The *predictor-corrector* methods

There is a another class of solution methods called the *predictor-corrector* methods. The solution step comprises of two parts. In the first part, the predictor step, the governing equation is written at time $t$ for which the solution ${}^{t}u$ is known, and using an explicit method a solution value ${}^{t+\Delta t}u^*$ at time $t+\Delta t$ is *predicted*. In the second part of the procedure, the corrector step, the governing equation is written at time $t+\Delta t$ (using expressions

for an implicit scheme), and the predicted solution ${}^{t+\Delta t}u^*$ is used in the calculation of $f$. The corrected solution ${}^{t+\Delta t}u$ is then calculated. The corrector step in practice can be repeated a number of times until the difference in the successively computed corrected values becomes very small. Predictor-corrector methods can give very high order of accuracy if the solution is very smooth. However we do not consider these methods further in this discussion.

# 4 Numerical examples

## 4.1 A linear spring-mass-damper system

We consider a simple spring-mass-damper system subject to a step load. The governing equation is

$$\begin{aligned} &m\ddot{u} + c\dot{u} + ku = R(t), \\ &\dot{u}(0) = u(0) = 0, \\ &R(t) = \begin{cases} 0, & t = 0 \\ 10, & t > 0 \end{cases} \end{aligned} \tag{21}$$

We assumed $m = 1$, $k = 10$ and $c = 2\xi w_n m$ (all units according to SI system), where $w_n = \sqrt{k/m} = 3.16\,\text{rad/sec}$ and $\xi = 0.02$ (damping ratio), and used the Runge-Kutta method, the central difference method and the Newmark method ($\delta = 0.5$, $\alpha = 0.25$) with time step size $\Delta t = 0.1\,\text{sec}$. For the central difference method the critical time step size for stability is given by $\Delta t_{critical} = T/\pi = 0.63\,\text{sec}$, (see [5]), where $T = 2\pi/w_n = 1.99\,\text{sec}$. This implies that, for this problem, less than four time steps per period of vibration are required for the central difference method to remain stable. For a single degree of freedom system this limitation is not of any practical significance since, in general, about 20 time steps per period of vibration should be used to get a reasonably accurate solution. Figure 1 shows the results obtained along with the exact analytical solution for the problem. As expected, the Runge-Kutta method produces the most accurate solution, and is virtually indistinguishable from the exact solution. The solution obtained by the central difference method is not as accurate, but clearly performs better than the Newmark method. The solution obtained by the Newmark method exhibits *period elongation* with every cycle of vibration. For single degree-of-freedom vibration problems, linear and nonlinear, we recommend using the central difference method with a time step size small enough to resolve the rapidly varying parts of the solution. As mentioned earlier, for elasto-plastic springs, central difference method may be computationally more efficient than the more accurate Runge-Kutta method, as it requires fewer spring force calculations.

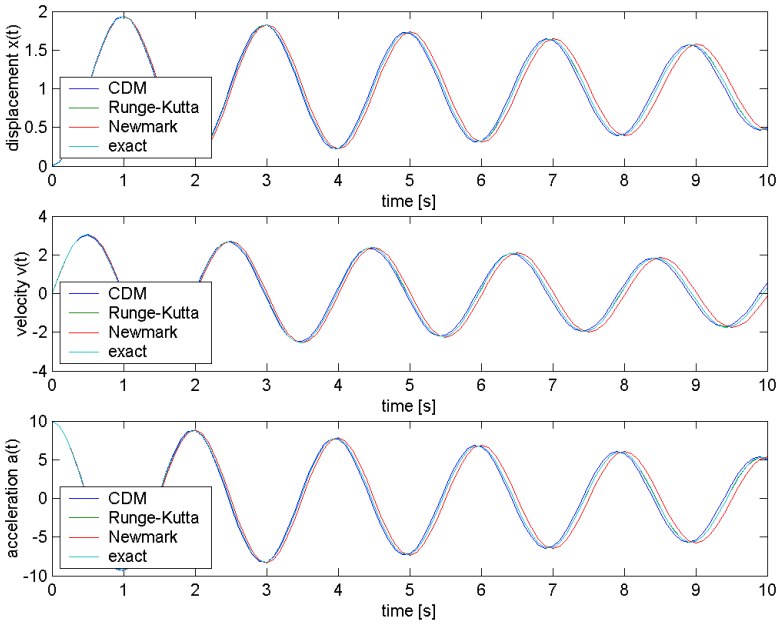



Figure 1: Solution of a linear spring-mass-damper system subject to a step function loading, using various numerical methods; $\Delta t = 0.1\,\text{sec}$; Newmark parameters $\delta = 0.5$, $\alpha = 0.25$.

### 4.2 A nonlinear first order equation

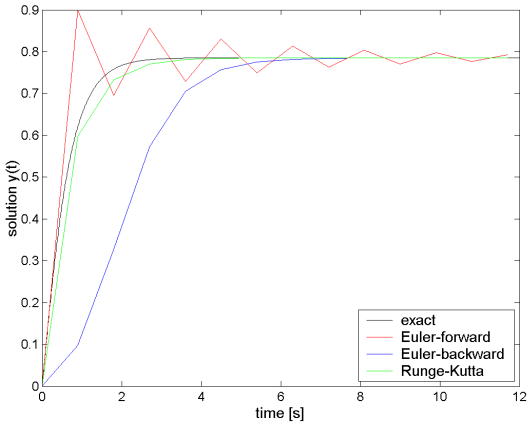
Consider the first order nonlinear differential equation

$$\begin{aligned} \dot{y} &= \cos(2y) \\ y(0) &= 0 \end{aligned} \tag{22}$$

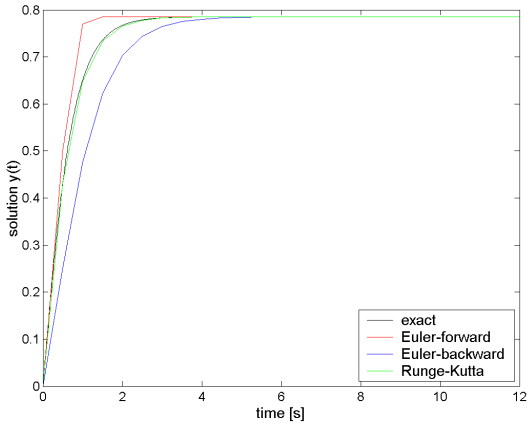
The exact solution for this equation is available in closed form,

$$y = \frac{1}{2} \sin^{-1} \frac{e^{4t} - 1}{e^{4t} + 1} \tag{23}$$

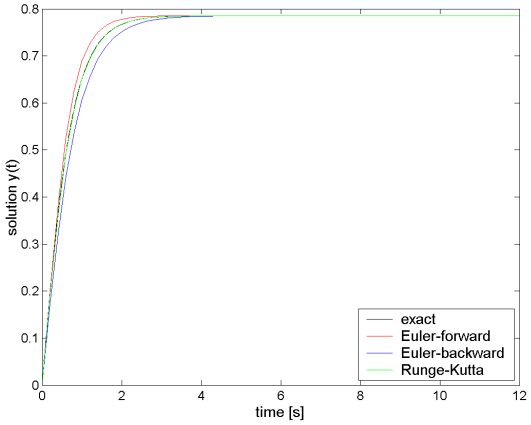
We solved this equation using the Euler forward, Euler backward, and Runge-Kutta methods. Figure 2a shows the solution obtained using a relatively large time step $\Delta t = 0.9\,\text{sec}$. The Euler forward method, clearly shows instability: numerical solution shows oscillations about the exact solution. The Euler backward method, although stable, is highly inaccurate. In contrast, the solution obtained by the Runge-Kutta method is quite accurate, and is very close to the analytical solution at time intervals $\Delta t$ apart. Next we solved the problem using time step size $\Delta t = 0.5\,\text{sec}$, $0.2\,\text{sec}$, and $0.05\,\text{sec}$, see figures 2b, 2c and 2d. Notice that the solution obtained using both Euler forward and backward method, shows an error which is proportional to the time step size, pointing to first order accuracy of these methods. Much smaller time step size is required to make the solution from these methods match closely the exact solution; even with time step $\Delta t = 0.05\,\text{sec}$, the solutions obtained



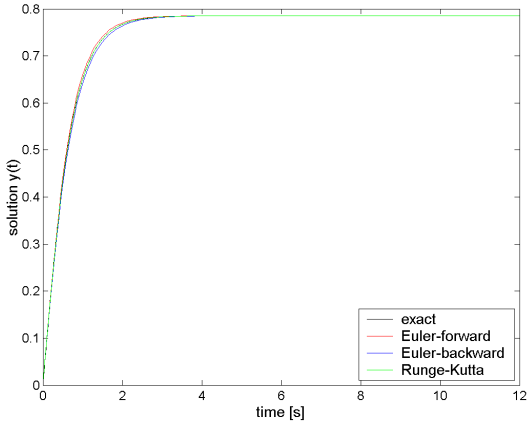

(a) $\Delta t = 0.9\,\text{sec}$




(b) $\Delta t = 0.5\,\text{sec}$




(c) $\Delta t = 0.2\,\text{sec}$




(d) $\Delta t = 0.05\,\text{sec}$

Figure 2: Solution of $\dot{y} = \cos(2y)$; $y(0) = 0$

using the Euler methods are visibly not as accurate as that obtained by the Runge-Kutta method. The Runge-Kutta method however, is extremely accurate even with $\Delta t = 0.5\,\text{sec}$ which is still relatively large. For such first order systems, we recommend Runge-Kutta method over any other methods discussed in this paper.

### 4.3 A nonlinear truss example

Next we solve a nonlinear problem involving the displacement of a truss element. We do not consider dynamics effects, assuming that the loads are applied at such a rate that inertial effects are not induced. Consider a truss element as shown in figure 3. The initial length of the bar is $L_0 = \sqrt{V^2 + H^2}$. For any non-zero applied force $R$, the right end of the truss deflects down by an amount $u$, and the bar shortens to a new length $L$. Assuming

small strains, the engineering strain in the bar is given by

$$\epsilon = \frac{L_0 - L}{L_0} \tag{24}$$

where

$$L = \sqrt{(V-u)^2 + H^2} \tag{25}$$

Here we take compressive strain and stress as positive; $R$ is taken positive acting downwards. The axial force in the bar is then

$$\begin{aligned} F_{axial} &= EA\epsilon \\ &= EA\frac{L_0 - L}{L_0} \end{aligned} \tag{26}$$

assuming linear elastic material behavior. The vertical component $F$ of this axial force $F_{axial}$ is then

$$F = F_{axial}\sin\theta \quad = EA\big(\frac{L_0 - L}{L_0}\big)\big(\frac{V-u}{L}\big) \tag{27}$$

The force $F$ must be always equal to the applied load $R$ to satisfy vertical equilibrium of the joint where load is applied.

$$R - F(u) = 0 \tag{28}$$

From equation (27), the force $F$ is a nonlinear function of the downward displacement $u$ (since $L$ is a nonlinear function of $u$ as well), and Newton-Raphson method may be used to solve this equation of equilibrium

$$f = R - F = 0 \tag{29}$$

Since $R$ does not depend on the displacement $u$ in this example, we have

$$\frac{df}{du} = -\frac{dF}{du} \tag{30}$$

$$\frac{df}{du} = -\Big(\frac{\partial F}{\partial u} + \frac{\partial F}{\partial L}\frac{\partial L}{\partial u}\Big) \tag{31}$$

The solution approximation for $(i+1)^{th}$ iteration, for load $R$, is then calculated by

$$u^{(i+1)} = u^{(i)} + \frac{R - F(u^{(i)})}{K^{(i)}} \quad \text{with} \tag{32}$$

$$K^{(i)} = \frac{dF}{du}\Big|_{u^{(i)}} \tag{33}$$

$K^{(i)}$ is the tangent stiffness of the system computed using the $i^{th}$ estimate during the Newton-Raphson procedure.

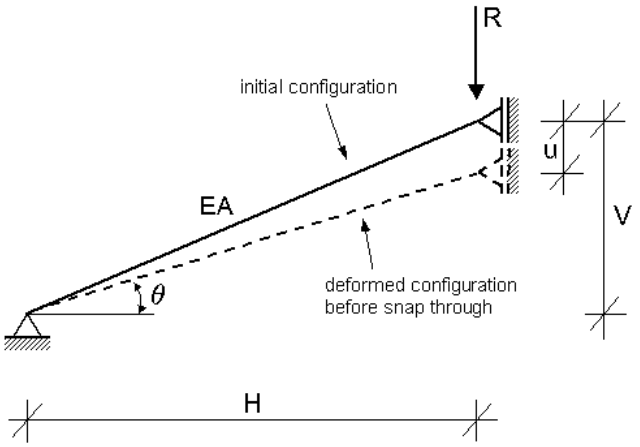



Figure 3: Nonlinear truss example. $EA = 1000$, $V = 3$, $H = 10$

If a displacement controlled experiment (real or numerical) is carried out, one can calculate for a given displacement $u$, the exact load required to keep the joint in equilibrium. However, if a *monotonically* increasing load is applied, the increasing displacement reaches a value where the stiffness of the system becomes zero. This is a bifurcation point i.e. more than one displacement values correspond to equilibrium condition for that particular load. Once this point on the load-deflection curve is reached, the solution jumps from the unstable equilibrium configuration to the stable one: the truss bar suddenly jumps to the configuration corresponding to the displacement $u$ that is actually larger than the value $V$. In this configuration the bar is in tension rather than in compression. Newton-Raphson method captures this quite accurately as is seen in figure 4. Subsequent increase in the load actually causes the system to stiffen, as can be seen in the increasing slope of the load deformation curve.

### 4.4 The Duffing oscillator

Finally, we present an example involving nonlinear dynamic analysis. The model equation we use is the non-homogeneous Duffing equation

$$\begin{aligned} &m\ddot{u} + k(u + u^3) = R(t) \\ &R(t) = \sin(0.1t) + 0.1\cos(0.1t) \\ &u(0) = \dot{u}(0) = 0 \end{aligned} \tag{34}$$

with $m = 1$, $k = 1$. Notice that this equation does not have a damping term, and the internal spring force term is nonlinear i.e. $f(u) = k(u + u^3)$. When solving this term using any of the explicit methods, no change in the algorithm is required. However when solving the equation using the Newmark method ($\delta = 0.5$, $\alpha = 0.25$), Newton-Raphson solution method needs to be applied as well. Figure 5 shows the results obtained using the Runge-Kutta method, the central difference method and the Newmark method, using time step size $\Delta t = 0.5\,\text{sec}$. We also calculated the solution using the Runge-Kutta method with

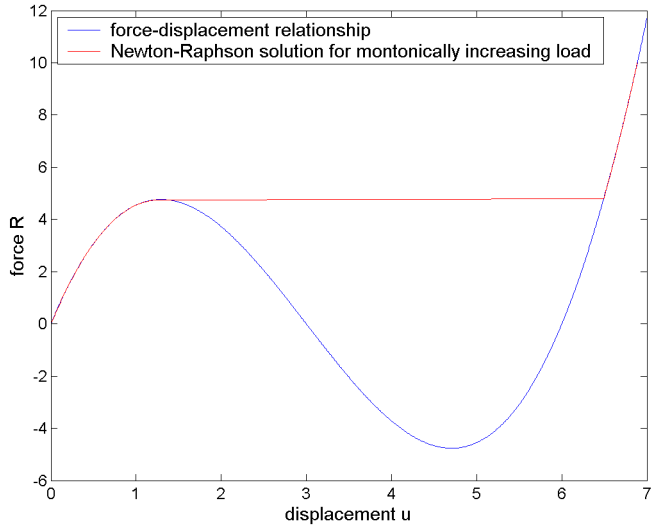
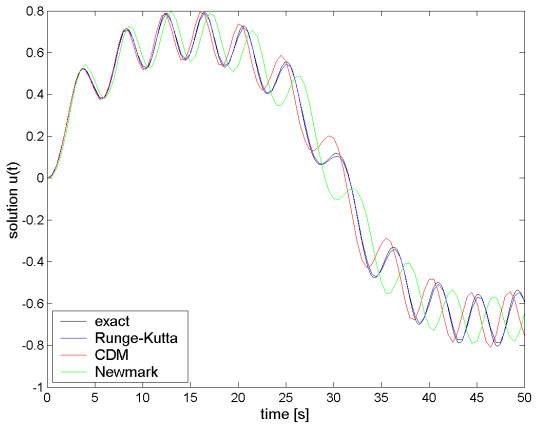



Figure 4: Nonlinear truss example: Load-deflection relationship obtained using a displacement controlled approach, and calculated displacements for varying load $R$.

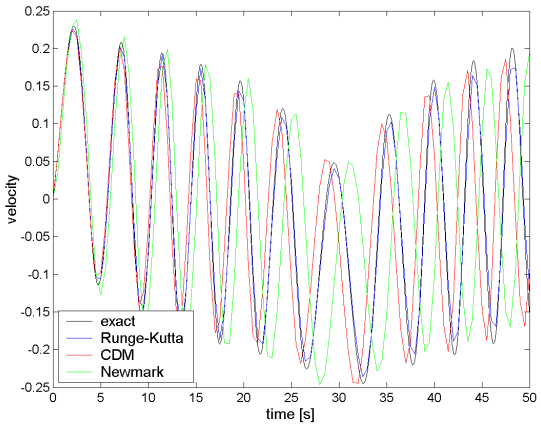
$\Delta t = 0.01\,\text{sec}$ and treat it as the "exact" solution. For the time step size $\Delta t = 0.5\,\text{sec}$, the solution obtained using the Runge-Kutta method is very close to the "exact" solution. The Newmark method produces the least accurate solution, as the errors due to period elongation accumulate over time. Also since the Newmark method is an implicit method, Newton-Raphson iterations are carried out to calculate solution at each time step. This makes Newmark method more expensive than the central difference method, while being less accurate. The central difference method produces less accurate solution than the Runge-Kutta method. However the computational effort for the Runge-Kutta is also four times more than that for the central difference method. Even though the Runge-Kutta method is more accurate, we generally recommend using the central difference method for such single degree of freedom nonlinear problems with a somewhat smaller time step. The Newmark-method used with Newton-Raphson method is only recommended for solution of highly nonlinear problems, the solution of which is less smooth and finite difference approximations like those used in the Runge-Kutta method and the central difference method are not appropriate.

## 5 Recommendations for multi-degree of freedom systems

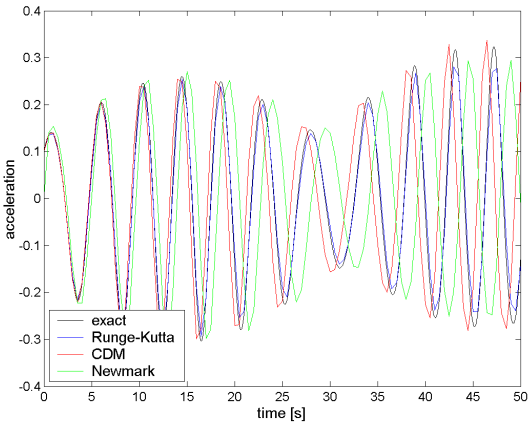
For multi-degree of freedom systems, the recommendations for use of time integration schemes are less straightforward. Such systems generally result from using the finite element method for solution of more complex problems. In such cases, the choice of time integration method depends on the properties of the response expected. For simple structural vibration problems, where only the first few modes are excited, explicit methods like




(a) Displacement




(b) Velocity




(c) Acceleration

Figure 5: The "exact" and calculated solutions for the Duffing oscillator; $\Delta t = 0.5\,\text{sec}$.

the central difference method are not the methods of choice. The critical time step limit for multi-degree of freedom systems is dictated by the highest mode, and consequently a very small time step size must be used in order to obtain a stable solution. As stated above, if most of the interesting part of the solution is contained within a few lowest modes, one generally wants to use a time step size considerably larger than the critical limit. Implicit and unconditionally stable (for linear problems) schemes like the Newmark method with $\delta = 0.5$ and $\alpha = 0.25$ are generally preferred for solution of such problems. However, implicit method that are unconditionally stable for linear problems, may begin to show instability when used for nonlinear problems, especially for long time simulations. The use of large time steps does not allow accurate resolution of the highest modes. These modes, even though not containing the interesting part of the solution, may still contribute to cumulative errors over long times, and ultimately effect the stability of the solution.

There are problems, however, when one needs to resolve the highest modes contained within the multi-degree of freedom systems, requiring a large number of time steps. For such problems explicit methods are very attractive since for each time step the computational cost is relatively low. Wave propagation problems, where one wants to accurately resolve the wavefront moving through a medium, is one such example. Crash analyses are another type which involve high velocity impact, and need a very small time step size. Use of implicit methods, requires the solution of an $N \times N$ system matrix, where $N$ is the number of degrees of freedom of the problem. This factorization of system matrix may be performed once if the system is linear, but may have to be carried out many times if the problem shows nonlinear behavior (for each Newton-Raphson iteration). If the time step size required is very small, the whole procedure may become numerically prohibitively expensive. Explicit schemes on the other hand generally do not require the solution of any system of equations. All that may be needed, for each time step, is the calculation of a vector of internal forces. For the central difference method, this vector of internal forces needs to be calculated only once for each time step. For the Runge-Kutta method, this operation needs to be performed four times. Although much less expensive than the solution of an $N \times N$ system matrix, it is still the most expensive step in explicit methods, and may be considerably so if the calculation of the vector of internal forces involves nonlinear functions. Therefore the Runge-Kutta methods, even though more accurate for a given step size, are four times more expensive than the central difference method. The central difference method performs quite satisfactorily for a large class of problems in solid and structural mechanics where explicit methods are suitable. We recommend using the central difference method for multi-degree of freedom problems when solution of interest lies in the highest modes.

# 6 Conclusion

Based on the examples presented in this paper, we can generally recommend the central difference method for the solution of single degree of freedom second order differential equations governing vibration problems. For similar computational effort, the Newmark

method generally gives less accurate results, due to the period elongation introduced in the numerical results. For general first order differential equations, linear or nonlinear, the Runge-Kutta method is very accurate and should give reliable and accurate results. However the computational cost for the Runge-Kutta method is four times more than the cost of the central difference method.

For multi-degree of freedom systems, the choice of a time integration scheme depends on the nature of the problem. In cases where the interesting and dominant part of the solution lies in the lower modes, the Newmark method is generally preferred, since a large time step size may be used. However for problems like wave propagation and crash analyses, the solution lies in the highest modes, and a small time step size is required to get an acceptable solution accuracy. Explicit methods are very attractive for such problems, and we recommend using the central difference method. The Runge-Kutta method, though more accurate than the central difference method, is about four times more expensive computationally, and just using the central difference method with a smaller time step size may still be less expensive while providing accuracy similar to that obtained using the Runge-Kutta method.

## References


[1] Collatz L. The Numerical Treatment of Differential Equations, 3rd Edition. *Springer–Verlag* 1959.

[2] Strang G. Computational Science and Engineering. *Wellesley–Cambridge Press* 2007.

[3] Atkinson K, Han W. Theoratical Numerical Analysis: A Functional Analysis Framework, 2nd Edition. *Springer* 2005.

[4] Newmark NM. A Method of Computation for Structural Dynamics. *ASCE Journal of Engineering Mechanics Division* 1951; Vol. 85, pp. 67–94.

[5] Bathe KJ. Finite Element Procedures. *Prentice-Hall* 1996.

[6] Kane C, Marsden JE, Ortiz M, West M. Variational Integrators and the Newmark Algorithm for Conservative and Dissipative Mechanical Systems. *International Journal of Numerical Methods in Engineering* 2000; 49:1295–1325.