Homework 3

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1. A man does not live by bread alone

Since the carbohydrate, protein and fat values need to stay in certain intervals, I wrote a system Ax $\leq$ b. This system can be rewritten into Ax = b by adding slack variables into x, A is concatenated with an identity matrix, cost vector c is concatenated with zeros and b stays the same. The new system can be solved using interior point. Using my own implementation of the interior point I get feasible solutions. Unfortunately those solutions are not optimal. Solutions vary depending on the used on the used δ. The optimal solution for the problem is 6.23 bread and 0.59 veg.oil, which implies a cost of 148.83. By trying different deltas the best solution I got is 7.69 bread and 0.88 eggs for a cost function of 208.94.

2. Analytic center

2.1 There exists at least one feasible solution

If $|I| = 0$ or $|I| = 1$, then the statements holds directly. If $|I| > 1$, then we can use x_i and x_j, where $s(x_i)_i > 0$ and $s(x_j)_j > 0$. Let $x_{i,j} = \frac{x_i + x_j}{2}$, where $s(x_{i,j})_i > 0$ and $s(x_{i,j})_j > 0$. We know that $x_{i,j}$ is a feasible solution, because the set of feasible solutions is a convex set. If n = $|I|$, we can take n feasible solutions (one for each index) for which $s(x_i)_i > 0$ and then take the average x_{AVG}. The average x_{AVG} is again a solution for which $s(x_{AVG})_i > 0$ for all i $\in$ I.

2.2 Analytic center is a strictly convex optimization problem

Instead of maximaizing f = $\prod_{i \in I} s(x)_i$ we can minimize g = $-\prod_{i \in I} s(x)_i$. Again we can simplify the function by applying a logarithm h = log(g). We will show that the hessian of h is PD, which implies that h is a strictly convex function.

$$h = -\sum_{i \in I} \log(b_i - Ax)$$

$$\frac{\partial h}{\partial x} = -\sum_{i \in I} \frac{A_i^T}{b_i - A_i x}$$

$$\frac{\partial^2 h}{\partial^2 x} = \sum_{i \in I} \frac{A_i^T A_i}{(b_i - A_i x)^2}$$

$$x^T A_i^T A_i x = (A_i x)^T A_i x = (A_i x)^2 \geq 0$$

We notice that function h is PD iff $(A_i x)^2 > 0$ for all i. If $A_i x \neq 0$ for all i $\in$ I, then $(A_i x)^2 > 0$. We observe that the rank of the hessian is exactly n = $|I| \leq$ m = number of variables. Hessian id PD iff n = m. The current system $Ax \leq b$ can be rewritten into a new system $A'x \leq b'$, where we remove the equations which are not present in I. The analytic center for both systems is the same. For the new system n = m, which implies that also the hessian is PD. Since h is strictly convex, then g and f are strictly convex.

2.3 Uniqness of the analytic center

Since function h is strictly convex it has at most one global maximum. If I is a not empty set, then there exists at least one solution for the analytic center, implying that there is an unique solution for minimizing h.

2.4 Solution to 6. problem

We can construct the function f which is represents the cost function for the analytic center. We can calculate the derivatives and find the maximum of function f.

$$f(x_1, x_2) = -x_1 \cdot -x_2 \cdot (1 - ax_1 - x_2)$$

$$f_{x_1} = x_2 - 2ax_1x_2 - x_2^2$$

$$f_{x_2} = x_1 - ax_1^2 - 2x_1x_2$$

We want $f_{x_1} = f_{x_2} = 0$, which implies that $x_1^* = \frac{1}{3a}$ and $x_2^* = \frac{1}{3}$. The analytic center is at $(\frac{1}{3a}, \frac{1}{3})$.

2.5 Solution to 7. problem

We can use the solution from the previous subsection, where a = 1. The analytic center is $(\frac{1}{3}, \frac{1}{3})$.