Mathematics 2, Homework 3 (part 2/1) **March 24th 2022**

Preamble

The material on interior point methods stems from two manuscripts [1, 2]. The similarities are striking, the differences are subtly hidden. Why not choose a single source?

It turns out that the analysis of invariants, in particular (I3), is much nicer in [1], where I particularly liked the alternative approach by A. Karrenbauer. On the other hand the extraction of optimal solution (stopping criterion for the iterative improvement) is much more concise in [2].

The caveat is that the authors switch the meaning of parameters, most notably m and n. These appear also in related quantities, which consequently leads to confusion.

Our take was to use notation from [1], and the so the stopping criterion in [2] had to be recast.

Program that runs in the interior

1. *Write a computer program that implements the interior point method.*

You can use a programming language of your choice. Using library routines for computing lower lever operations (such as solving systems of linear equations) is permissible.

$$\begin{aligned} \min\ & c^T x \\ & Ax = b \\ & x \geq 0, \end{aligned} \tag{P}$$

The input to your program is a LP program in canonical form (as (P)) above with all data coordinates being integers. Proceed along the following steps:

(a) First remove unnecessary equations so that A has full row rank.

(b) Construct a related LP (P') as in (8)[1].

(c) Construct its dual LP (D') as in (9)MS1.

(d) Find an initial iterate (x^0, y^0, s^0, μ^0) satisfying invariants (I1), (I2) and (I3). See [1, page 11].

(e) Proceed with iterative improvements solving system (S), see [1, page 4].

(f) The iterative process can be stopped as described in [2, page 11], and then the optimal solution x^* (and also the optimal of dual (y^*, s^*)) can be computed. This may be tricky, the stopping parameters may be impossible to achieve if computing in floating-point arithmetic. Be prepared to invent an ad-hoc way out.

Your implementation should allow for rich logging, so that you can document an optimization run.

A man does not live by bread alone

2. Solve the following LP diet problem.

The nutritional values as well as prices of a selection of foods is given in the table[1].

	potatoes	**bread**	**milk**	**eggs**	**yoghurt**	**veg.oil**	**beef**	**strawbrs.**
cost	10	22	15	45	40	20	87	21
CH	18	48	5	1	5	0	0	8
PR	2	11	3	13	3	0	15	1
FT	0	5	3	10	3	100	30	1
EN	77	270	60	140	61	880	330	32

The *cost* row is price in Euro-cents per 100g. The subsequent rows *CH, PR, FT* contain the amount of carbohydrates, protein and fat, respectively, again measured in grams (per 100g). Finally *EN* is the energy value of 100g of said foods in kcal. Note that beef is relatively cheap, but it is a consequence of its relatively high fat content — we have chosen a relatively cheap cut.

Find the optimal diet (in terms of being as cheap as possible) which satisfies the daily needs of an average male: the daily energy consumption should be between 2200 and 240 kcal, the intake of CH between 250 and 370, the intake of PR between 50 and 170, and the intake of FT between 50 and 90, all in grams.

Use both your own program and a commercial application of your choice to verify the correctness of your solution.

Analytic center

A LP can have many optimal solutions. For example, the set of points on a playing die that lies furthest from the top of a table is a facet (and not a single vertex).

A combinatorial approach (simplex algorithm) for finding an optimum will terminate in a vertex solution. The interior point method converges to the analytic center of the set of optimal solutions. Let us give the proper definition, see also[2] [3].

Let

$$Ax \leq b \tag{1}$$

be a system of m linear inequalities, and let $\Phi = \{x; Ax \leq b\}$ be the set of its *feasible solutions.* We denote $s(x) = b - Ax$. Let $I \subseteq \{1, \dots, n\}$ be the set of coordinates/indices, for which there exists $x \in \Phi$, so that $(Ax)_i < b_i$ or equivalently $s(x)_i > 0$.[3]

The *unique*[4] vector $x \in \Phi$ which maximizes

$$\prod_{i \in I} s(x)_i$$

[1]The data is provided in file hw.dat, there is no need to retype the numbers.
[2]Also available here.
[3]x_i is the i-th coordinate of $x \in \mathbb{R}^n$.
[4]You have to show this in the first place.

is called the *analytic center* of a system of linear inequalities (1).

3. *Show that there exists $x \in \Phi$ so that for all $i \in I$ we have $s(x)_i > 0$.*
4. *Show that the analytic center optimization problem is equivalent to a strictly convex optimization problem.*
5. *Show that the analytic center is unique.*
6. *Let a be a positive real. Find the analytic centre of*

$$
\begin{aligned}
-x_1 &\leq 0 \\
-x_2 &\leq 0 \\
ax_1 + x_2 &\leq 1
\end{aligned}
$$

7. *Find the analytic center of the system containing the following* four *linear inequalities:*

$$
\begin{aligned}
-x_1 &\leq 0 \\
-x_2 &\leq 0 \\
x_1 + x_2 &\leq 1 \\
x_1 + x_2 &\leq 1
\end{aligned}
$$

References

[1] Kurt Mehlhorn and Sanjeev Saxena. A still simpler way of introducing the interior-point method for linear programming (ver. 16oct18). *CoRR*, abs/1510.03339, 2015.

[2] Kurt Mehlhorn and Sanjeev Saxena. A still simpler way of introducing the interior-point method for linear programming (ver. 8dec21). *CoRR*, abs/1510.03339, 2015.

[3] Imre Pólik and Tamás Terlaky. *Interior Point Methods for Nonlinear Optimization*, pages 215–276. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

Upload your solution in a single .zip archive which contains the source-code and a .pdf.