# Homework 1

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(1) (a)

$p_1(x) = x^2 - 1$

$p_2(x) = x^2 + x + 1$

$p_3(x) = x^2 + x$

$a \cdot p_1(x) + b\, p_2(x) + c\, p_3(x) = 0 \iff a = b = c = 0$

$a(x^2 - 1) + b(x^2 + x + 1) + c(x^2 + x) = 0$

$\underline{a x^2} - a + \underline{b x^2} + b x + b + \underline{c x^2} + c x = 0$

$(a + b + c)\, x^2 + (b + c)\, x + (-a + b) = 0$

$a + b + c = 0 \Rightarrow 2b + c = 0 \Rightarrow c = -2b$

$b + c = 0 \Rightarrow -b = 0 \Rightarrow b = 0 \Rightarrow c = 0 \Rightarrow a = 0$

$b - a = 0 \Rightarrow a = b$

$a = b = c = 0 \Rightarrow \{p_1, p_2, p_3\}$ is a basis for $\mathbb{R}_2[x]$.

(b)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$\beta_1(x, y, z) = x - y$

$\beta_2(x, y, z) = x - y + z$

$\beta_3(x, y, z) = -x + 2y - z$

(2) (a)

$$\int_{-1}^{1} ax^2+bx+c\,dx = a\frac{x^3}{3}\Big]_{-1}^{1} + b\frac{x^2}{2}\Big]_{-1}^{1} + cx\Big]_{-1}^{1} = \frac{2a}{3} + 2c$$

$$\int_{0}^{1} ax^2+bx+c\,dx = a\frac{x^3}{3} + b\frac{x^2}{2} + cx\Big]_{0}^{1} = \frac{a}{3} + \frac{b}{2} + c$$

$$\int_{0}^{2} ax^2+bx+c\,dx = a\frac{x^3}{3} + b\frac{x^2}{2} + cx\Big]_{0}^{2} = \frac{8a}{3} + 2b + 2c$$

$$\alpha_1\left(\frac{2a}{3}+2c\right) + \alpha_2\left(\frac{a}{3}+b/2+c\right) + \alpha_3\left(\frac{8a}{3}+2b+2c\right) = 0$$

$$\left[\begin{array}{ccc|c} \frac{2}{3} & \frac{1}{3} & \frac{8}{3} & 0 \\ 0 & \frac{1}{2} & 2 & 0 \\ \cancel{2} & \cancel{1} & 1 & 0 \end{array}\right] \sim \begin{matrix} 2 & 1 & 8 \\ 0 & 1 & 4 \\ 2 & 1 & 2 \end{matrix} \sim \begin{matrix} 2 & 1 & 8 \\ 0 & 1 & 4 \\ 0 & 0 & -6 \end{matrix} \sim \begin{matrix} 2 & 1 & 8 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{matrix} \sim$$

$$\sim \begin{matrix} 2 & 1 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Rank $= 3 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

(b) $\begin{bmatrix} 2/3 & 0 & 2 \end{bmatrix} \cdot b_1 = 1$ , $\begin{bmatrix} 2/3 & 0 & 2 \end{bmatrix} \cdot b_2 = 0$ , ...

$$\begin{bmatrix} 2/3 & 0 & 2 \\ 1/3 & 1/2 & 1 \\ 8/3 & 2 & 2 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2/3 & 0 & 2 & 1 & 0 & 0 \\ 1/3 & 1/2 & 1 & 0 & 1 & 0 \\ 8/3 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3/2 & 3 & 0 & 3 & 0 \\ 2 & 0 & 6 & 3 & 0 & 0 \\ 8 & 6 & 6 & 0 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 3 & 6 & 0 & 6 & 0 \\ 2 & 0 & 6 & 3 & 0 & 0 \\ 8 & 6 & 6 & 0 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 0 & 6 & 3 & 0 & 0 \\ 0 & 3 & 0 & -3 & 6 & 0 \\ 8 & 6 & 0 & 3 & 0 & 3 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 6 & 3 & 0 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 2 & 1 & 0 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 4 & -4 & 1 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 2 & 0 & 6 & 3 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & +1/2 & -2 & 1/2 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 6 & 2 & 4 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & +1/2 & -2 & 1/2 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1/3 & 2/3 & -1/6 \end{array}\right]$$

$$\mathcal{B} = \left\{ \begin{bmatrix} -1/2 \\ -1 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ -1/6 \end{bmatrix} \right\}$$

(3) (a)

Gramian matrices

$$G_g = \begin{bmatrix} 2 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 2/5 \\ 2/3 & 0 & 2/5 & 0 \\ 0 & 2/5 & 0 & 2/7 \end{bmatrix}, \quad G_g \text{ is PD} \quad \Rightarrow g \text{ is inner product}$$

$$G_h = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}, \quad G_h \text{ is PD} \quad \Rightarrow h \text{ is inner product}$$

(b) Let $p(x) = ax^3 + bx^2 + cx + d$ and $q = ex^3 + fx^2 + ix + j$

$$g(p, q) = 0 \quad \wedge \quad h(p, q) \neq 0$$

$$\begin{cases} \begin{bmatrix} j & i & f & e \end{bmatrix} \cdot G_g \cdot \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} = 0 \\ \begin{bmatrix} j & i & f & e \end{bmatrix} \cdot G_h \cdot \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} \neq 0 \end{cases}$$

$$\rightarrow \begin{bmatrix} j & i & f & e \end{bmatrix} \left( G_g - G_h \right) \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} \neq 0$$

Examples are $(x, 1)$, $(x^3, 1)$, $(x^2, x^3)$.

(c) $u_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad e_1 = \dfrac{u_1}{\sqrt{g(u_1, u_1)}} = u_1 \cdot \dfrac{1}{\sqrt{2}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - u_1 \cdot \dfrac{\overbrace{g(\cancel{x}, 1)}^{0}}{g(1, 1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \dfrac{1}{\sqrt{2/3}} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{3/2} \\ 0 \end{bmatrix}$

$u_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \underbrace{u_2 \dfrac{g(x^2, x)}{g(x, x)}}_{0} - \underbrace{u_1 \dfrac{g(x^2, 1)}{g(1, 1)}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \dfrac{2}{3} \cdot \dfrac{1}{2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/3 \end{bmatrix}$

(d)

(d) $g(v, w) = 0 \qquad v \in V,\ w \in V^{\perp}$

$$g(1, w) = 0, \quad g(x, w) = 0, \quad g(x^2, w) = 0$$

$$\int_{-1}^{1} 1 \cdot (a x^3 + b x^2 + c x + d)\, dx = b \cdot \frac{2}{3} + d \cdot 2 = 0$$

$$\int_{-1}^{1} x (a x^3 + b x^2 + c x + d)\, dx = a \cdot \frac{2}{5} + c \frac{2}{3} = 0$$

$$\int_{-1}^{1} x^2 (a x^3 + b x^2 + c x + d)\, dx = b \cdot \frac{2}{5} + d \frac{2}{3} = 0$$

$$\begin{array}{cccc|c} a & b & c & d & \\ 0 & 2/3 & 0 & 2 & 0 \\ 2/5 & 0 & 2/3 & 0 & 0 \\ 0 & 2/5 & 0 & 2/3 & 0 \end{array} \sim \begin{matrix} 6 & 0 & 10 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 6 & 0 & 10 \end{matrix} \sim \begin{matrix} 1 & 0 & 10/6 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & -8 \end{matrix} \Rightarrow \begin{matrix} b = d = 0 \\ a = -\frac{5}{3} c \end{matrix}$$

$$V^{\perp} = \mathcal{L}\left\{ c \cdot \begin{bmatrix} -5/3 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ c \in \mathbb{R} \right\}$$