Homework 2

1\.

b) $f_1(p) = \int_{-1}^{1} p(x)\,dx = \frac{2a}{3} + 2c$

$f_2(p) = \int_{0}^{1} p(x)\,dx = \frac{a}{3} + \frac{b}{2} + c$

$f_3(p) = \int_{0}^{2} p(x)\,dx = \frac{8a}{3} + 2b + 2c$

$$\left[\begin{array}{ccc|ccc} 2/3 & 0 & 2 & 1 & 0 & 0 \\ 1/3 & 1/2 & 1 & 0 & 1 & 0 \\ 8/3 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 0 & 6 & 3 & 0 & 0 \\ 2 & 3 & 6 & 0 & 6 & 0 \\ 8 & 6 & 6 & 0 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 0 & 6 & 3 & 0 & 0 \\ 0 & 3 & 0 & -3 & 6 & 0 \\ 0 & 6 & -18 & -12 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 3/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & -18 & -6 & -12 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -2 & 1/2 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1/3 & 2/3 & -1/6 \end{array}\right]$$

$\beta_1(x, y, z) = \cancel{\frac{9}{14}x - y + \frac{2}{7}z} = \frac{1}{2}x - y + \frac{1}{3}z$

$\beta_2(x, y, z) = \cancel{-\frac{12}{7}x + 2y + \frac{4}{7}z} = -2x + 2y + \frac{2}{3}z$

$\beta_3(x, y, z) = \cancel{\frac{3}{2}x - \frac{1}{7}z} = \frac{1}{2}x - \frac{1}{6}z$

c) $$I = \begin{bmatrix} b^{1T} \\ b^{2T} \\ b^{3T} \end{bmatrix} \cdot G \cdot \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} =$$

$$= \begin{bmatrix} b^{1T} \\ b^{2T} \\ b^{3T} \end{bmatrix} \cdot \begin{bmatrix} 2/3 & 0 & 4/3 \\ 0 & 2/3 & 0 \\ 4/3 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} +1/2 & -2 & 1/2 \\ -1 & 2 & 0 \\ 1/3 & 2/3 & -1/6 \end{bmatrix} =$$

$$= \begin{bmatrix} b^{1T} \\ b^{2T} \\ b^{3T} \end{bmatrix} \cdot \begin{bmatrix} \frac{13}{45} & -\frac{16}{45} & \frac{4}{45} \\ -2/3 & 4/3 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} b^{1T} \\ b^{2T} \\ b^{3T} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3/4 & 1/2 \\ 45/4 & 3 & -11/4 \end{bmatrix}$$

$p_1(x) = 1$

$p_2(x) = \frac{3}{4}x + \frac{1}{2}$

$p_3(x) = \frac{45}{4}x^2 + 3x - \frac{11}{4}$

3. Dual basis of the standard basis is the standard basis.

$A = 3$
$B = 6$
$C = 8$

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}, \begin{bmatrix}8\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}, \begin{bmatrix}8\\0\end{bmatrix}\right) + T\left(\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}, \begin{bmatrix}8\\0\end{bmatrix}\right) =$$

$$= 3 \cdot 8\, T\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right) + 6 \cdot 8 \cdot T\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right) +$$

$$+ 3 \cdot 8 \cdot T\left(\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right) + 6 \cdot 8 \cdot T\left(\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right) =$$

$$= 48 - 24 = 24$$

2. $\mathrm{Bil}(V \times V) \longmapsto (V \longmapsto V^*)$ — φ takes a bilinear form and maps it to a linear map.

$\varphi : \mathrm{Bil}(V \times V) \longmapsto (V \longmapsto V^*)$

$\gamma \longmapsto \alpha_\gamma$ where $\alpha_\gamma : \vec{v} \longrightarrow \gamma(\vec{v}, \cdot)$

$\boxed{1}$ φ is linear

$$\varphi(a\gamma_1 + b\gamma_2) = \alpha_{a\gamma_1 + b\gamma_2}(\vec{v}) = (a\gamma_1 + b\gamma_2)(\vec{v}, \cdot) = a\gamma_1(\vec{v}, \cdot) + b\gamma_2(\vec{v}, \cdot) =$$

$$= a\varphi(\gamma_1) + b\varphi(\gamma_2)$$

$\boxed{2}$ φ is injective $\quad \gamma \in \ker \varphi$

$\varphi(\gamma) = \alpha_\gamma(\vec{v}) = \gamma(\vec{v}, \cdot) = 0$ for all $\vec{v}$

Kernel is trivial iff γ is zero. $\Rightarrow$ φ is injective.

$\boxed{3}$ $\dim(\mathrm{Bil}(V \times V)) = \dim(V \longrightarrow V^*) = n^2$

φ is bijective and $\mathrm{Bil}(V \times V) \cong V \longmapsto V^*$