University of Ljubljana, Faculty of Computer and Information Science

Probabilistic Analysis and Randomized Algorithms

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Analysis of Algorithms and Heuristic Problem Solving, October 2019

Finding maximum

```
findMax(n) {
   fbest = -∞ ;
   for (i=1 ; i <= n ; i++) {
      fi = check(i) ;
      if (fi > fbest) {
          fbest = fi ;
          process(i) ;
       }
   }
}
```

* $O(n \cdot c_{check} + m \cdot c_{process})$
* worst case analysis
* probabilistic analysis
* randomization

Probabilistic analysis

* assumptions about the input distributions
* indicator random variables

Randomization

* to avoid “bad” input sequences, we intentionally randomize the input

```
void findMax(n) {
   randomly shuffle elements
   fbest = 0 ;
   for (i=1 ; i <= n ; i++) {
      fi = check(i) ;
      if (fi > fbest) {
          fbest = fi ;
          process(i) ;
      }
   }
}
```

Randomize the input

PERMUTE-BY-SORTING(A)

$n = A.length$
let $P[1 \,..\, n]$ be a new array
for $i = 1$ **to** n
    $P[i] = \text{RANDOM}(1, n^3)$
sort A, using P as sort keys

Randomize the input

RANDOMIZE-IN-PLACE(A)

$n = A.length$
for $i = 1$ **to** n
    swap $A[i]$ with $A[\text{RANDOM}(i, n)]$

On-line maximum

* on-line maximum: elements arrive one by one, randomly shuffled; we can check them but we can select only one

Find online maximum

```
findMaxOnline(k, n) {
   fbest = -∞ ;
   for (i=1 ; i <= k ; i++) {
      if (score(i) > fbest)
          fbest = fi ;
   }
  for (i=k+1 ; i <= n ; i++) {
      if (score(i) > fbest)
          return(i) ;
   }
   return(n) ;
}
```

- How to select k, that we shall select the best one with the largest probability?
- What is the probability that we select the best one using this strategy?

Graph min-cut

Contraction algorithm:

repeat {
    select random edge *e=(u,v)*
    contract *e*:
        replace *u* and *v* with super-node *w*
        keep connections of *u* and *v* also for *w*
        keep parallel edges, but not loop
}
until (graph has only two nodes v_1 and v_2)
return cut defined by v_1

- randomized algorithm
- probabilistic analysis

Introduction to pseudo-random numbers

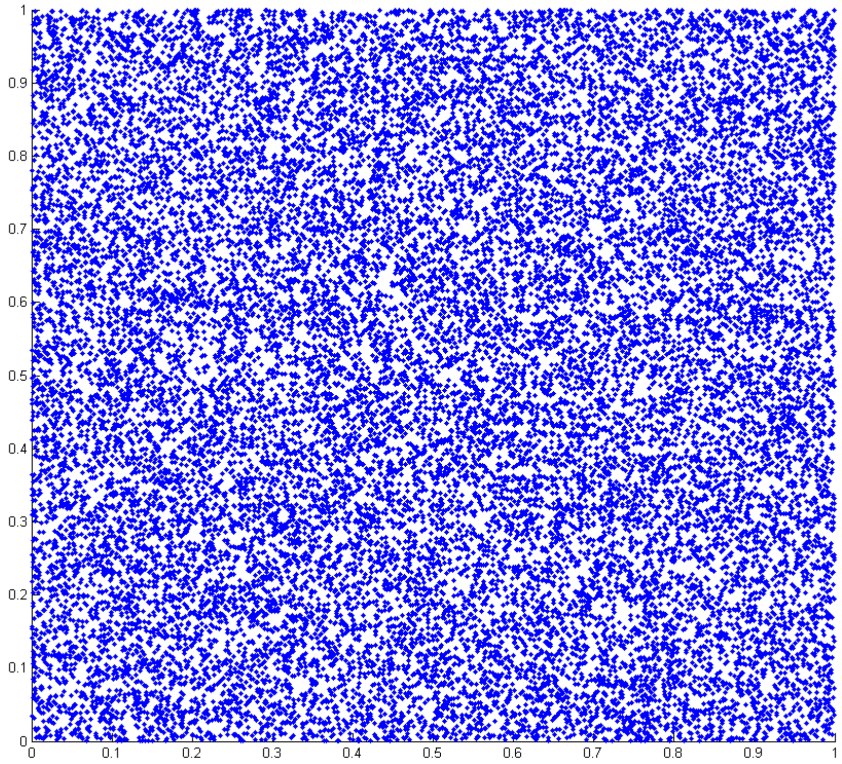
Applications of pseudo random numbers

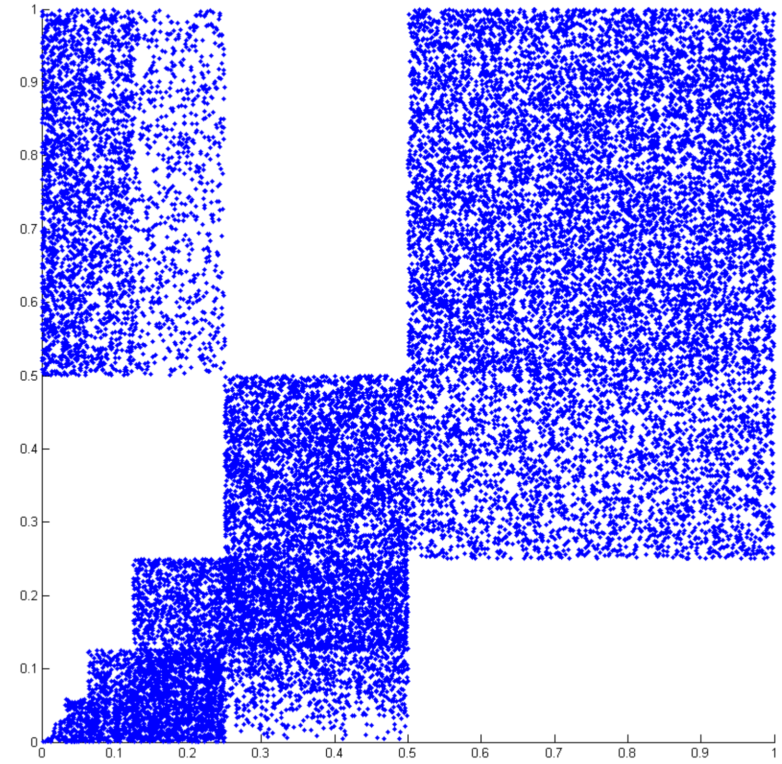
* computer simulations
* cryptography
* statistical sampling and estimation
* Monte Carlo methods
* data analysis and modelling
* computer games
* games of chance

* hardware and software generators
* quality of (pseudo)random numbers: speed and randomness

Matlab example

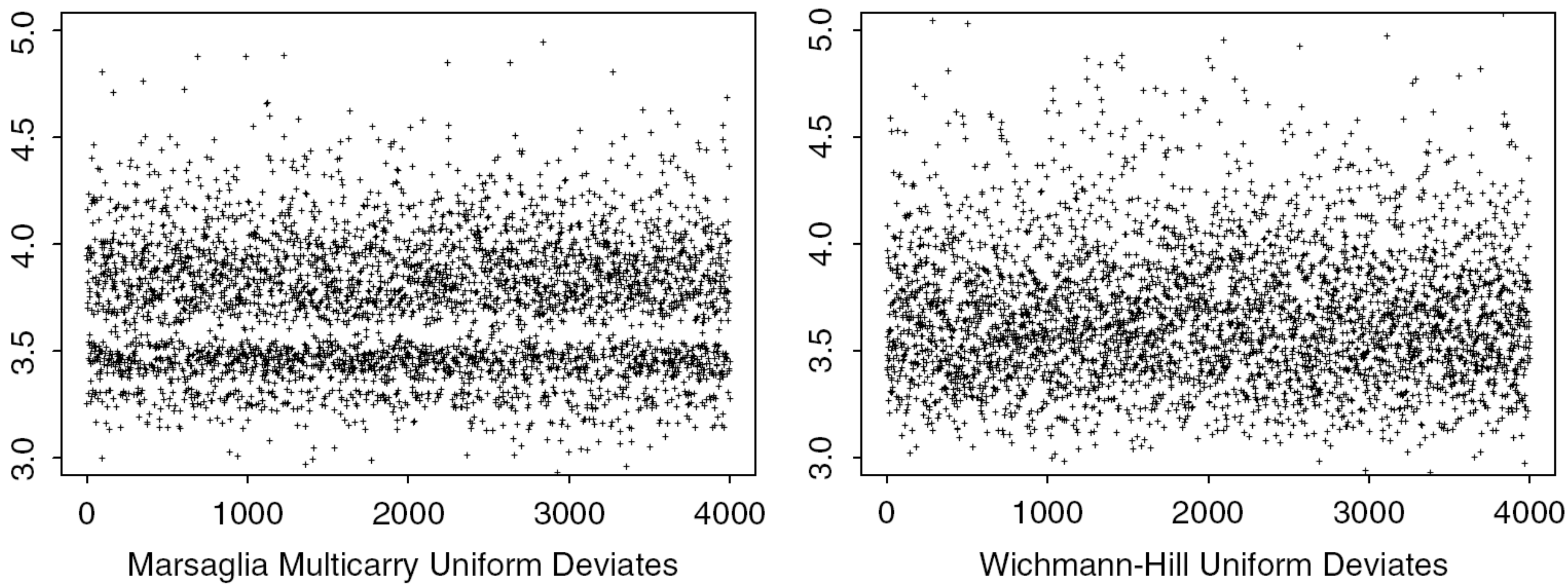
```
Z = rand(28,100000);
condition = Z(1,:) < 1/4;
scatter(Z(16,condition),Z(28,condition),'.');
```





* P. Savicky: A strong nonrandom pattern in Matlab default random number generator. Technical Report, Institute of Computer Science, Academy of Sciences of Czech Republic (2006)

Example



* Value-at-Risk (financial analysis)
  B. D. McCullough: A Review of TESTU01.
  Journal of Applied Econometrics, 21: 677–682 (2006)

Quality criteria

* randomness
* speed of generator
* period

Linear congruential generators

* simplest and most common
  $x_i = (a \cdot x_{i-1} + c) \bmod m \quad u_i = x_i / m$

* A notorious example:
  RANDU:
  $x_i = 65539 \cdot x_{i-1} \bmod 2^{31}$
* simple but bad

MINSTD

* used as a standard for a long time
  $x_i = 16807 \cdot x_{i-1} \bmod (2^{31}-1)$

i	x_i decimal	x_i binary
1	1	1
2	16807	100000110100111
3	282475249	10000110101100011101011110001
4	1622650073	1100000101101111010110011011001
5	984943658	111010101101010000110000101010
6	...	...

Combined linear congruential generator

* combinations of linear congruential generators
* improvements: addition, subtraction, bit mixing
* better randomness, small period

Multiple recursive generators

* higher order recursions
  $x_i = (a_1 \cdot x_{i-1} + \ldots + a_k \cdot x_{i-k}) \bmod m$
  $u_i = x_i / m$

* e.g., (Knuth, 1998):
  $x_i = (271828183 \cdot x_{i-1} + 314159269 \cdot x_{i-2}) \bmod (2^{31}-1)$

* combined multiple recursive generators

Other generators

* combinations
* non-linear generators (quadratic, multiplicative, floating point generators, inverse generators)
* (linear) recursive bit generators (modulo 2, operators)
* cryptographic (ISAAC, AES, BBS,…)
* AES http://en.wikipedia.org/wiki/Advanced_Encryption_Standard

BBS (Blum-Blum-Shrub)

* bit generator
* select two large prime integers p and q (e.g., at least 40 decimal places)
* let m = pq
* $X_i = X_{i-1}^2 \bmod m$
* b_i = parity(X_i) (0 if even, 1 if odd)
* finding dependency is equivalent to factorization of m (finding multipliers p and q).
* Currently there is no polynomial non-quantum algorithm for integer factorization
* the numbers are therefore provably random enough for most uses

Criteria of randomness

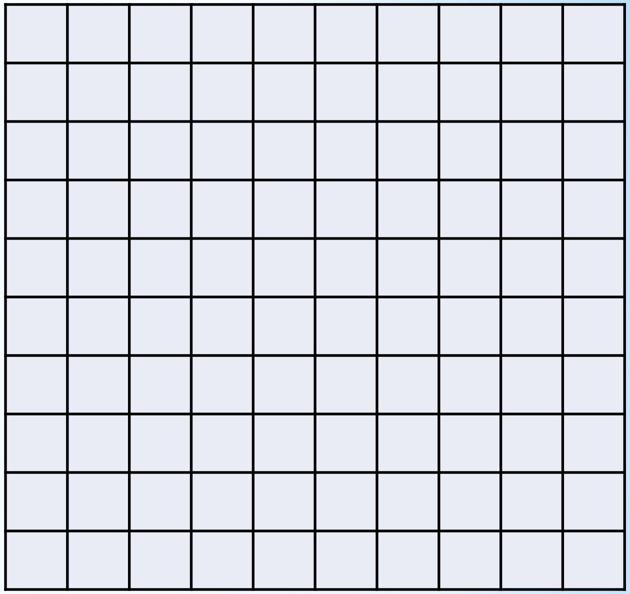
* generate a sequence of t numbers, $u_i \in (0, 1)$
* hypothesis
  $u_0, u_1, \ldots u_{t-1}$ are independent uniformly distributed random variables $U(0,1)$
* equivalent:
  vector $(u_0, u_1, \ldots u_{t-1})$
  is uniformly randomly distributed in unit hypercube $(0,1)^t$
* equivalent: sequence of independent random bits

Statistical tests for randomness

* infinitely many possible tests
* only show dependencies, cannot prove that dependencies do not exists
* increase of trust

* “*The difference between the good and bad RNGs, in a nutshell, is that the bad ones fail very simple tests whereas the good ones fail only very complicated tests that are hard to figure out or impractical to run*.”
  L’Ecuyer and Simard, 2007. TestU01: A C Library for Empirical Testing of Random Number Generators. *ACM Transactions on Mathematical Software*.

An example of test



* Pearson's X^2 goodness-of-fit test
* put generated numbers into k cells (e.g., two-dimensional grid)
* for each cell we know the expected number of elements E_i
* let O_i be the observed number of samples from each cell
* statistics

$$X_0^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

* if hypothesis of uniform distribution of numbers is true, the statistics X_0^2 is chi-squared distributed with k-1 degrees of freedom
* we reject the hypothesis if $X_0^2 > X^2_{\alpha,k-p-1}$

Ideas of statistical tests

- one sequence of numbers:
  - tests of groups,
  - gaps,
  - increasing subsequences
- several sequences, hypercube partitioning
  - statistics on partitions
  - statistics on distances
- one sequence of bits
  - cryptographic tests,
  - compressiveness,
  - spectral tests (Fourier),
  - autocorrelation
- several bit sequences

A toolbox of tests

* L'Ecuyer and Simard, 2007. TestU01: A C Library for Empirical Testing of Random Number Generators. *ACM Transactions on Mathematical Software*.
* results: not many generators pass all tests
* poor results for some popular software (Excel, MATLAB, Mathematica, Java)
* improvements in recent years and advent of hardware generators