Biharmonic Scattering

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1 Limits

1.1 Identities

The following are identities used to prove Theorem 1:

$$\left(H_\nu^{(1)}(z)\right)' = H_{\nu-1}^{(1)} - \frac{\nu}{z} H_\nu^{(1)}(z) \tag{1}$$

$$H_0^{(1)}(z) \sim \frac{2i}{\pi} \ln z \qquad k \to 0 \tag{2}$$

$$H_\nu^{(1)}(z) \sim -\frac{i}{\pi} \Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu} \qquad k \to 0 \tag{3}$$

1.2 Proof of Limit

Theorem 1

Let

$$T_{ik} w = \sum_n w_n \gamma_n e^{in\theta} \tag{4}$$

Where

$$\gamma_n = \frac{ik \left(H_n^{(1)}(ikR)\right)'}{H_n^{(1)}(ikR)} \tag{5}$$

Then we have

$$\lim_{k \to 0} \gamma_n = \frac{-|n|}{R} \tag{6}$$

Proof. We first use identities to calculate $\left(H_n^{(1)}(ikR)\right)'$. We see

$$\left(H_n^{(1)}(ikR)\right)' = H_{n-1}^{(1)}(ikR) - \frac{n}{ikR} H_n^{(1)}(ikR) \tag{7}$$

So then

$$\gamma_n = \frac{ik \left(H_{n-1}^{(1)}(ikR) - \frac{n}{ikR} H_n^{(1)}(ikR)\right)}{H_n^{(1)}(ikR)} \tag{8}$$

Taking the limit as $k \to 0$ we see

$$\lim_{k\to 0} \gamma_n = \lim_{k\to 0} ik \frac{\left(H_{n-1}^{(1)}(ikR) - \frac{n}{ikR} H_n^{(1)}(ikR) \right)}{H_n^{(1)}(ikR)} \tag{9}$$

$$= \lim_{k\to 0} \frac{ikH_{n-1}^{(1)}(ikR)}{H_n^{(1)}(ikR)} - \frac{ik}{ik} \frac{-nH_n^{(1)}(ikR)}{RH_n^{(1)}(ikR)} \tag{10}$$

$$= \lim_{k\to 0} \frac{ikH_{n-1}^{(1)}(ikR)}{H_n^{(1)}(ikR)} + \frac{-n}{R} \tag{11}$$

Now, if we compute the limit in the first term we get

$$\lim_{k\to 0} \frac{ikH_{n-1}^{(1)}(ikR)}{H_n^{(1)}(ikR)} \sim \lim_{k\to 0} \frac{ik\left(-\frac{i}{\pi}\Gamma(n-1)\left(\frac{ikR}{2}\right)^{1-n}\right)}{\left(-\frac{i}{\pi}\Gamma(n)\left(\frac{ikR}{2}\right)^{-n}\right)} \tag{12}$$

$$= \lim_{k\to 0} \frac{-k^2 R\Gamma(n-1)}{2\Gamma(n)} \tag{13}$$

$$= 0 \tag{14}$$

For all $n > 1$. When $n = 0$, we have the following:

$$\lim_{k\to 0} \frac{ikH_{-1}^{(1)}(ikR)}{H_0^{(1)}(ikR)} \sim \lim_{k\to 0} \frac{ik\left(-\frac{ie^{-\pi i}}{\pi}\Gamma(-1)\left(\frac{ikR}{2}\right)\right)}{\frac{2i}{\pi}\ln(ikR)} \tag{15}$$

$$= \lim_{k\to 0} \frac{-ik^2 Re^{-\pi i}\Gamma(-1)}{4\ln(ikr)} \tag{16}$$

$$= 0 \tag{17}$$

When $n = 1$ we have

$$\lim_{k\to 0} \frac{ikH_0^{(1)}(ikR)}{H_1^{(1)}(ikR)} \sim \lim_{k\to 0} \frac{ik\frac{2i}{\pi}\ln(ikr)}{-\frac{i}{\pi}\Gamma(1)\left(\frac{2}{ikr}\right)} \tag{18}$$

$$= \lim_{k\to 0} \frac{k^2 R\ln(ikR)}{\Gamma(1)} = \lim_{k\to 0} \frac{R\ln(ikR)}{\Gamma(1)\frac{1}{k^2}} \tag{19}$$

$$= \lim_{k\to 0} \frac{R\frac{iR}{ikR}}{-\Gamma(1)\frac{2}{k^3}} = \lim_{k\to 0} \frac{k^2 R}{2\Gamma(1)} = 0 \tag{20}$$

With this, we have shown that Theorem 1 holds for any $n \geq 0$. $\square$

1.3 Numerics

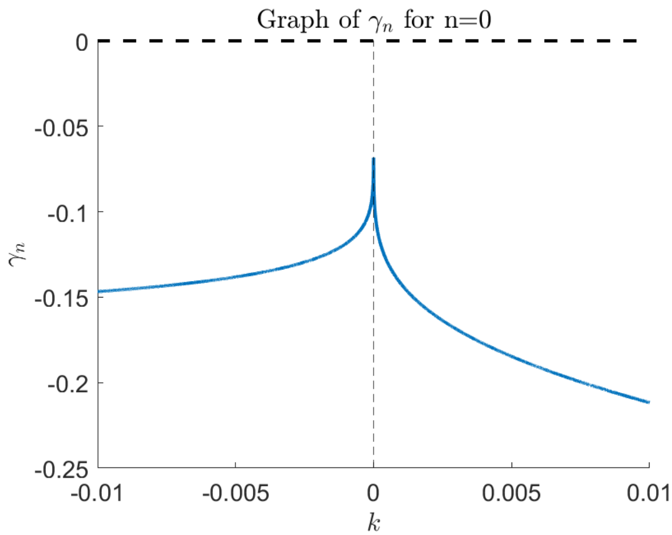

Figure 1: Numerics for $n = 0$.

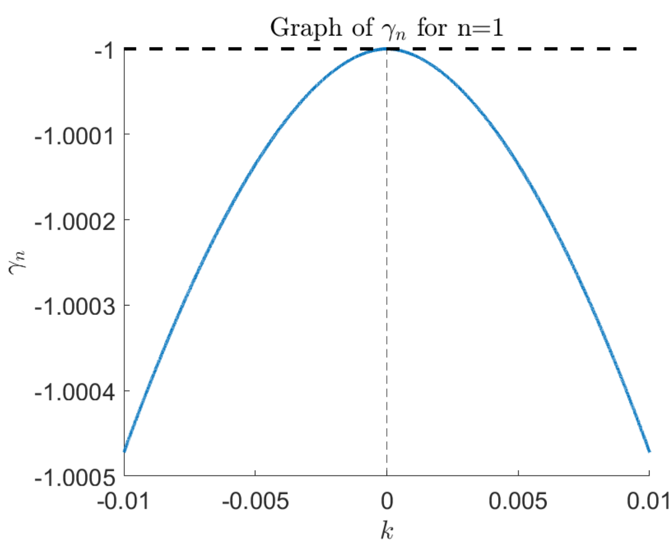

Figure 2: Numerics for $n = 1$.

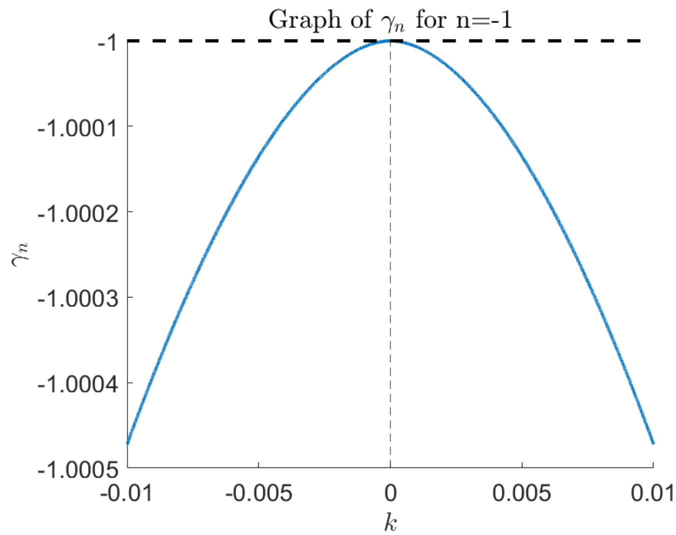

Figure 3: Numerics for $n = -1$.

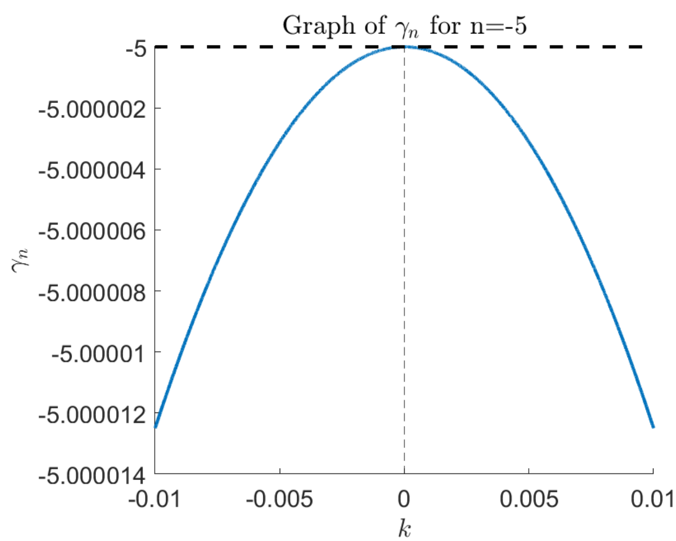

Figure 4: Numerics for $n = -5$.

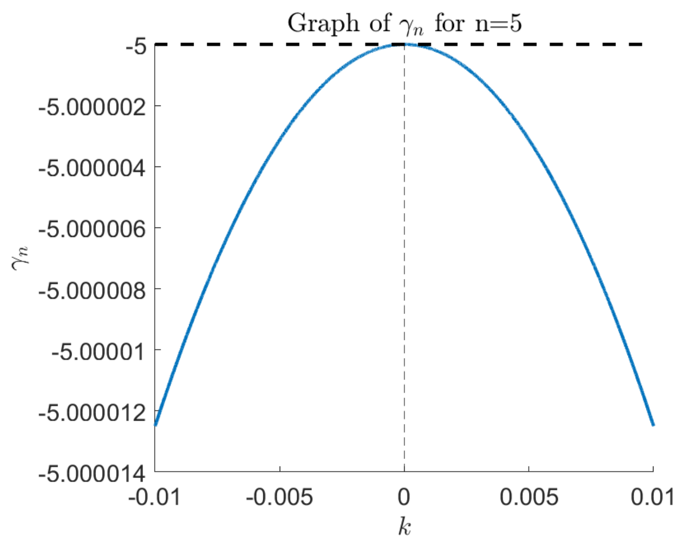

Figure 5: Numerics for $n = 5$.