Additional Proofs to Kirsch and Grinberg's "The Factorization Method for Inverse Problems"

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Abstract

The goal of this project is to provide additional proof and explanation of the factorization of the far-field operator for the inverse scattering problem. We seek to prove that the far field operator can indeed be factored into an auxiliary and Herglotz operator. We will then prove that utilizing this factorization yields a unique solution to the inverse scattering problem and implement this method numerically.

1 Equality of the Far-Field and Solution Operators

Our goal is to prove an auxiliary theorem in the factorization of the far-field operator. Before we do this, we define the far field operator as follows:

Definition 1 (Far-Field Operator). *Define the far-field operator* $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ *as*

$$(Fg)(\hat{x}) = \int\limits_{\mathbb{S}^2} u^\infty(\hat{x}, \theta) g(\theta)\, ds(\theta) \tag{1}$$

where $u^\infty(\hat{x})$ *is the far field pattern of* u *defined as*

$$u^\infty(\hat{x}) = \int\limits_{\Gamma} \left[u(y) \frac{\partial}{\partial u(y)} e^{-ik\hat{x}\cdot y} - \frac{\partial u(y)}{\partial u} e^{-ik\hat{x}\cdot y} \right] ds(y) \qquad \hat{x} \in \mathbb{S}^2 \tag{2}$$

When considering the inverse scattering problem, this operator contains the known data. It is the goal to use this operator to obtain explicit characterizations about the unknown domain D. We then will define an operator to which this far-field operator decomposes into, which is based on the Herglotz operator.

Definition 2. *As an auxiliary operator, define* $H : L^2(\mathbb{S}^2) \to H^{1/2}$ *as*

$$-(Hg)(\hat{x}) = -\int\limits_{\mathbb{S}^2} g(\theta) e^{ikx\cdot\theta}\, ds(\theta) \qquad x \in \Gamma \tag{3}$$

note that this is the trace on Γ *of the Herglotz wave function with density* g*.*

Definition 3. *Let the data-to-pattern operator* $G : H^{1/2}(\Gamma) \to L^2(\mathbb{S}^2)$ *be defined by* $Gf = mv^\infty$ *where* $v^\infty \in L^2(\mathbb{S}^2)$ *is the far field pattern of the solution* v *to the exterior Dirichlet problem with boundary data* $f \in H^{1/2}(\Gamma)$*, meaning that* $v \in H^1_{loc}(\mathbb{R}^3 \setminus \bar{D})$ *solves*

$$\Delta v + k^2 v = 0 \qquad \textit{outside } D \tag{4}$$

$$v = f \qquad \textit{on } \Gamma \tag{5}$$

and

Using these definitions, we will prove the following theorem involving the elementary factorization of the far-field operator into the solution operator G and the auxiliary operator H.

Theorem 1. *Let* F *and* H *be the operators defined above.* $G : H^{1/2} \to L^2(\mathbb{S}^2)$ *be the solution operator. We then have*

$$Fg = -GHg \tag{6}$$

Proof. For some data $f_\theta(x) = -e^{ik\theta\cdot x}$, the G operator

$$Gf_\theta(x) = u^\infty(\hat{x}, \theta) \tag{7}$$

We note that

$$(Hg)(x) = \int g(\theta)e^{ikx\cdot\theta}\, ds(\theta) \tag{8}$$

Applying the solution operator G, we have

$$\begin{align}
(GHg)(x) &= G\int_{\mathbb{S}^2} g(\theta)e^{ikx\cdot\theta}\, ds(\theta) \tag{9}\\
&= \int_{\mathbb{S}^2} g(\theta)Gf_\theta\, ds(\theta) \tag{10}\\
&= \int_{\mathbb{S}^2} u^\infty(\hat{x}, \theta)g(\theta)\, ds(\theta) \tag{11}\\
&= -Fg \tag{12}
\end{align}$$

Now, we need to show that we can indeed pull the operator inside of the integral. To do this we first must decompose f into f^+ and f^- where $f = f^+ - f^-$ such that f^+ and f^- are positive semidefinite. This is to say that f^+ is equivalent to f when $f > 0$ and 0 otherwise and f^- is equivalent to f when $f < 0$ and 0 otherwise. Now, by the Approximation Theorem we can pick a sequence $\{f_n^+\}$ of monotone increasing positive simple functions such that $f_n^+ \to f^+$ as $n \to \infty$. We can write each term in this sequence as

$$f_n^+ = \sum_{i=0}^{n} \alpha_i(x)\chi_{E_i}(\theta) \tag{13}$$

Where α_i are some sequence of constants and χ_{E_i} are characteristic functions of some set E_i. Thus, we see by the summation definition of the Lebesgue integral, we see that the G operator can be pulled inside and then by the Monotone Convergence theorem, we can switch the integral and limit for f^+. We can apply the same linearity to f^- by the linearity of the Lebesgue integral. So then with this shown, we indeed have

$$Fg = -GHg \tag{14}$$

□

2 Factorization of the Far-Field Operator

Definition 4.

$$S\varphi(x) := \int_\Gamma \Phi(x, y)\varphi(y)\, ds(y), \qquad x \in \Gamma \tag{15}$$

with

$$\Phi(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \qquad x, y \in \mathbb{R}^3, \quad x \neq y \tag{16}$$

Lemma 1. *We have that $v = S\varphi$ is a solution to the Hemholtz equation. Moreover, $H^*\varphi$ is the far-field pattern for the $v = S\varphi$ function.*

Proof. Define $\Phi(x, y)$ as the Green's function for the three dimensional Hemholtz equation. We begin by noting for a fixed $y \neq x$, we have

$$\Delta_x\Phi(x, y) + k^2\Phi(x, y) = \delta(x - y) \tag{17}$$

We can multiply both sides by $\varphi(y)$ and integrate over ∂D to see

$$\int\limits_{\partial D} \Delta_x \Phi(x,y)\varphi(y) + k^2\Phi(x,y)\varphi(y)\, ds(y) = \int\limits_{\partial D} \delta(x-y)\varphi(y)\, ds(y) \tag{18}$$

Since $x \neq y$, the right-hand side goes to 0 and we have

$$\int\limits_{\partial D} \Delta_x \Phi(x,y)\varphi(y)\, ds(y) + k^2 \int\limits_{\partial D} \Phi(x,y)\varphi(y)\, ds(y) = 0 \tag{19}$$

By linearity of the Lebesgue integral, and the fact that Δ_x is independent of the integral, we are able to pull it out of the integral. To prove this fact, we must first consider the limit definition of the derivatives included in the Δ_x operator. Considering a single derivative with respect to x_1 we have

$$\frac{\partial}{\partial x}\Phi(x,y) = \lim_{(\delta x_1)_n \to 0} \frac{\Phi(x + (\delta x_1)_n, y) - \Phi(x,y)}{(\delta x_1)_n} \tag{20}$$

Define $(\delta x_1)_n = \{\frac{1}{n}\}_{n=1}^{\infty}$ so then $(\delta x_1)_n \to 0$ as $n \to \infty$. We will now apply Lesbesgue's Dominated Convergence theorem. We first note that $\left|\frac{\Phi(x+(\delta x_1)_n,y)-\Phi(x,y)}{(\delta x_1)_n}\right|$ is bounded from above by $|\Phi(x,y)|$ which is in $L^2(\partial D)$. Thus, by the Dominated convergence Theorem we have

$$\int\limits_{\partial D} \frac{\partial}{\partial x_1}\Phi(x,y)\, ds(y) = \int\limits_{\partial D} \lim_{(\delta x_1)_n \to 0} \frac{\Phi(x + \delta x_n, y) - \Phi(x,y)}{\delta x_n}\, ds(y) = \frac{\partial}{\partial x_1}\int\limits_{\partial D} \Phi(x,y)\, ds(y) \tag{21}$$

The same argument can be repeated for both the second derivative of x_1 and the two derivatives of x_2, meaning the entire Δ_x operator can be pulled outside the integral. Finally, by applying Definition 1, we see that

$$\Delta_x S\varphi(x) + k^2 S\varphi(x) = 0 \tag{22}$$

Meaning that $v = S\varphi(x)$ is indeed a solution to the Hemholtz equation.

Next we will show that $v = S\varphi(x)$ satisfies the necessary radiation condition. We note that our radiation condition is given by

$$\frac{\partial v}{\partial r} - ikv = \mathcal{O}(r^{-2}) \qquad r = |x| \to \infty \tag{23}$$

We will begin by applying the radiation condition to $\Phi(x,y)$, seeing

$$\frac{\partial}{\partial |x|}\Phi(x,y) - ik\Phi(x,y) = \mathcal{O}(|x|^{-2}) \tag{24}$$

multiply both side by $\varphi(y)$ and integrating over ∂D, we have

$$\int_{\partial D} \frac{\partial}{\partial |x|}\Phi(x,y)\varphi(y)\, ds(y) - \int_{\partial D} ik\Phi(x,y)\varphi(y)\, ds(y) = \int_{\partial D} \mathcal{O}(|x|^{-2})\, ds(y) \tag{25}$$

By the same argument as previous, we can pull out the derivative, seeing

$$\frac{\partial}{\partial r} S\varphi(x) - ikS\varphi(x) = \mathcal{O}(|x|^{-2}) \tag{26}$$

We see that the right-hand side will go to zero as the integral is over the boundary and any small terms will vanish. Thus, $v = S\varphi(y)$ satisfies our radiation condition. Finally we will show that $H^*\varphi$ is the far-field pattern. We start by noting

$$\Phi^{\infty}(\hat{x}, y) = e^{-ik\hat{x}\cdot y} \tag{27}$$

and in general we have

$$\varphi(\hat{x}, y, k) = \frac{e^{ik|x|}}{|x|}\left[\varphi^{\infty}(\hat{x}, y, k) + \mathcal{O}(|x|^{-2})\right] \tag{28}$$

We can then see

$$S\varphi(\hat{x}) = \frac{e^{ik|x|}}{|x|}\left[(S\varphi)^{\infty}(\hat{x}) + \mathcal{O}(|x|^{-2})\right] \tag{29}$$
$$= \int\limits_{\partial D} \Phi(\hat{x}, y)\varphi(y)\, ds(y) \tag{30}$$
$$= \int\limits_{\partial D} \frac{e^{ik|x|}}{|x|}\left[\Phi^{\infty}(\hat{x}, y) + \mathcal{O}(|x|^{-2})\right]\varphi(y)\, ds(y) \tag{31}$$
$$= \frac{e^{ik|x|}}{|x|}\left[\int\limits_{\partial D} \Phi^{\infty}(\hat{x}, y)\varphi(y)\, ds(y) + \mathcal{O}(|x|^{-2})\int\limits_{\partial D} \varphi(y)\, ds(y)\right] \tag{32}$$
$$= \frac{e^{ik|x|}}{|x|}\left[\int\limits_{\partial D} e^{-ik\hat{x}\cdot y}\varphi(y)\, ds(y) + \mathcal{O}(|x|^{-2})\right] \tag{33}$$
$$= \frac{e^{ik|x|}}{|x|}\left[H^*\varphi(y) + \mathcal{O}(|x|^{-2})\right] \tag{34}$$
$$\tag{35}$$

From here we can conclude that indeed $(S\varphi)^{\infty}(\hat{x}) = H^*\varphi(\hat{x})$ and we are finished. □

Theorem 2. *If k^2 is not a Dirichlet Eigenvalue of $-\Delta$ in D we arrive at the factorization*

$$F = H^*S^{-1}H \tag{36}$$

Proof. We will first calculate H^* explicitly. We see for some inner product $\langle \cdot\, , \cdot \rangle$ and some $f \in L^2(\mathbb{S}^2)$ and $g \in H^{1/2}(\partial D)$ we have

$$\langle Hf\, , g\rangle_{H^{1/2}} = \langle f\, , H^*g\rangle_{L^2} \tag{37}$$

Define

$$\langle f(x)\, , g(x)\rangle_{H^{1/2}(\partial D)} = \int\limits_{\partial D} \overline{g(x)}f(x)\, ds(\theta) \tag{38}$$

and

$$\langle f(x)\, , g(x)\rangle_{L^2(\mathbb{S}^2)} = \int\limits_{\mathbb{S}^2} \overline{g(x)}f(x)\, ds(\theta) \tag{39}$$

Using (37) we can calculate H^*, seeing

$$\langle Hf\, , g\rangle_{H^{1/2}} = \int\limits_{\partial D} \overline{g(x)}Hf(x)\, ds(\theta) \tag{40}$$
$$= \int\limits_{\partial D} \overline{g(x)}\int\limits_{\mathbb{S}^2} f(\theta)e^{ikx\cdot\theta}\, ds(\theta)\, ds(\theta) \tag{41}$$
$$= \int\limits_{\partial D}\int\limits_{\mathbb{S}^2} \overline{g(x)e^{-ikx\cdot\theta}}f(\theta)\, ds(\theta)\, ds(\theta) \tag{42}$$
$$= \int\limits_{\mathbb{S}^2}\int\limits_{\partial D} \overline{g(x)e^{-ikx\cdot\theta}\, ds(\theta)}f(\theta)\, ds(\theta) \tag{43}$$
$$= \langle f\, , H^*g\rangle_{L^2} \tag{44}$$

So then we have

$$H^*g(\theta) = \int\limits_{\partial D} g(x)e^{-ikx\cdot\theta}\, ds(\theta) \tag{45}$$

By Lemma 1, we know that $H^* = GS$ since GS yields the far field pattern of S. Thus we have,

$$-H^*S^{-1}Hg = -GSS^{-1}H = -GIH = -GH \tag{46}$$

So then by Theorem 1 we know that $-GH = F$ so then $-H^*S^{-1}H = F$ and we are finished.

$\square$

3 Uniqueness of the Solution to the Factorization Method

4 Numeric Simulations

MATLAB code samples for implementing the factorization method are found in [1]

References

[1] Roland Potthast and Gen Nakamura. *Inverse Modeling an introduction to the theory and method of inverse problems and data assimilation.* IOP Publishing, 2015.