Directed Reading Program Notes

Fall 2025

Contents

1 Preliminary Definitions

This section will have many basic definitions and theorems used in our study of basic Quantum theory. These definitions are very formal and are most found in [1].

1.1 Groups and Rings

Definition 1.1: Group.
A group is a set G and a map $\cdot : G \times G \to G$ such that

(a) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ *for all* $a, b, c \in G$.

(b) *There exists an element* e *with* $a = a \cdot e = e \cdot a$ *for all* $a \in G$.

(c) *For any* $a \in G$, *there exists an element* $a^{-1} \in G$ *such that* $a \cdot a^{-1} = a^{-1} \cdot a = e$.

A group is considered abelian if it is also commutative. Groups are sets with a single operation, but we can extend this to two operations by considering rings.

Definition 1.2: Ring.
A ring R is a set with two operations, $+, \times : R \times R \to R$ with

(a) $(R, +)$ *is an abelian group with* $e = 0$.

(b) $a \times (b \times c) = (a \times b) \times c$ *for all* $a, b, c \in R$.

(c) *There exists* $1 \in R$ *with* $1 \times a = a \times 1 = a$ *for all* a.

(d) $(a + b) \times c = a \times c + b \times c$ *and* $c \times (a + b) = c \times a + c \times b$ *for all* $a, b, c \in R$.

Rings are considered commutative if their $\times$ operation is commutative.

Definition 1.3: Ideal.
Let R be a commutative ring. A set $I \subseteq R$ is an ideal if

(a) $0 \in I$.

(b) *For all* $a, r \in I$,*we have* $ar \in I$.

(c) *For all* $a, b \in I$, *then* $a + b \in I$.

Definition 1.4: Subring.
Let R be a ring. A set $K \subseteq R$ is an subring if

(a) $0, 1 \in K$.

(b) *For all* $a, r \in K$, *we have* $ar \in K$.

(c) *For all* $a, b \in K$, *we have* $a + b \in K$.

Now that we have definitions for these algebraic objects, we wish to classify functions between them. To this end, we need to classify functions which preserve our desired operations.

Definition 1.5: Homomorphism.
*Let R, R' be rings (or groups, just ignore **(b)**). A function $f : R \to R'$ is a ring (group) homomorphism if*

(a) $f(r_1 r_2) = f(r_1) f(r_2)$ *for all* $r_1, r_2 \in R$.

(b) $f(r_1 + r_2) = f(r_1) + f(r_2)$ *for all* $r_1, r_2 \in R$.

(c) $f(1_R) = 1_{R'}$

Definition 1.6: Isomorphism.
A homomorphism $f : R \to R'$ is an isomorphism if there is an inverse homomorphism $f^{-1} : R' \to R$ such that

$$f^{-1} \circ f(x) = x \tag{1.1}$$

for all $x \in R$. If it exists, we say $R \cong R'$.

1.2 Fields and Vector Spaces

We can further define sets with multiple operations by considering Fields.

Definition 1.7: Field.
A field is a set F and two operations $+, \cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ such that

- ***(a)*** $a + (b + c) = (a + b) + c$ *for all* $a, b, c \in \mathbb{F}$
- ***(b)*** $a + b = b + a$ *for all* $a, b \in \mathbb{F}$
- ***(c)*** *There exists* $0, 1 \in \mathbb{F}$ *with* $0 + a = a$ *and* $1 \cdot a = a$ *for all* $a \in \mathbb{F}$.
- ***(d)*** $a(b + c) = ab + ac$ *and* $(a + b)c = ac + bc$ *for all* $a, b, c \in \mathbb{F}$.
- ***(e)*** $ab = ba$ *for all* $a, b \in \mathbb{F}$.
- ***(f)*** *For all* $a \neq 0$, *there exists an* $a^{-1} \in \mathbb{F}$ *such that* $a \cdot a^{-1} = a \cdot^{-1} a = 1$.

We can use these properties to see the following fact:

Proposition 1.8.
A commutative ring R is a field if and only if $\{0\}, R$ are the only ideals.

We can also define vector spaces over fields, where each element in the vector space is a vector, and all scalars are in a field. This definition is equivalent to that which you see in linear algebra, although we are now using "abstract" vectors which are defined to be "an element in a vector space".

Definition 1.9: Vector Space.
Let $\mathbb{F}$ be a field. An $\mathbb{F}$-vector space is a set V and two operations $+ : V \times V \to V$ and $\cdot : \mathbb{F} \times V \to V$ such that

- ***(a)*** $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ *for all* $\vec{a}, \vec{b}, \vec{c} \in V$.
- ***(b)*** *There exists a* $\vec{0} \in V$ *with* $\vec{0} + \vec{a} = \vec{a}$.
- ***(c)*** *For all* $\vec{a} \in V$, *there exists* $-\vec{a} \in V$ *with* $\vec{a} + (-\vec{a}) = \vec{0}$.
- ***(d)*** $r(\vec{a} + \vec{b}) = r\vec{a} + r\vec{b}$ *for all* $\vec{a}, \vec{b} \in V$ *and* $r \in \mathbb{F}$.
- ***(e)*** $(r + s)\vec{a} = r\vec{a} + s\vec{a}$ *for all* $r, s \in \mathbb{F}$ *and* $\vec{a} \in V$.

Note that a ring as in Definition 1.2 is a set satisfying all axioms of a field in Definition 1.7 except for **(e)** and **(f)**. If it is a commutative ring, then it satisfies **(e)**.

Example 1.1

$(\mathbb{Z}, \cdot, +)$ is a commutative ring, but not a field.

An important aspect of vector spaces is the linearity of functions. We define this as follows:

Definition 1.10.
Let V, W be $\mathbb{F}$-vector spaces. Then $f : V \to W$ is linear if

- ***(a)*** $f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$ *for all* $\vec{x}, \vec{y} \in V$.
- ***(b)*** $f(r\vec{x}) = rf(\vec{x})$ *for all* $\vec{x} \in V$ *and* $r \in \mathbb{F}$.

1.3 Modules and Tensor Products

Now that we have most of our algebraic objects, we can define modules and tensor products. Modules are vector spaces that are defined over rings rather than fields, meaning we both lose and gain important properties with each definition.

Definition 1.11: Left Module.
Let R be a ring. A left R-module is a set with two operations satisfying the conditions of Definition 1.9

Definition 1.12: Tensor Product.
Let R be a ring, M a right R-module and N a left R-module. Then we define

$$M \otimes_R N = F_{\mathbb{Z}}(M \times N)/A \tag{1.2}$$

where

$$A = \langle (mr) \times n - m \times (rn),\ (m_1 + m_2) \times n - (m_1 \times n) - (m_2 \times n),\ m \times (n_1 + n_2) - m \times n_1 - m \times n_2 \rangle \tag{1.3}$$

for some $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r \in R$. We write $m \otimes n$ for the image of $m \times n \in M \times n$ in $F_{\mathbb{Z}}(M \times N)$. We also write for $x \in M \otimes N$

$$x = \sum_i k_i(m_i \otimes n_i), \qquad k_i \in \mathbb{Z}, m_i \in M, n_i \in N \tag{1.4}$$

Finally, we have the following rules for all elements $m \in M$, $n \in N$ and $r \in R$:

$$(mr) \otimes n = m \times (rn) \tag{1.5}$$
$$(m_1 + m_2) \otimes n = (m_1 \otimes n) + (m_2 \otimes n) \tag{1.6}$$
$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2 \tag{1.7}$$

Example 1.2

We want to show that

$$\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$$

We can repeatedly apply (1.5) to $1 \otimes 1$ to see

$$1 \otimes 1 = 1 \cdot 3 \otimes 1 = 1 \otimes 0 = 0 \otimes 0 = 0$$

Definition 1.13: External Direct Sum.
Let M and N be R-Modules. Let $M \oplus N = M \times N$, and define

$$(n_1, n_2) + (m_1, m_2) = (n_1 + m_1, n_2 + m_2) \tag{1.8}$$
$$r(n_1, n_1) = (rn_1, rn_2) \tag{1.9}$$

2 Lie groups and Algebras

We can now begin our study into mathematic physics, specifically starting with Lie groups. All notes are based on Chapter 16 of [2].

2.1 Matrix Lie Groups

Most of our work will be done on the space $M_n(\mathbb{C})$ which is the space of $n \times n$ matrices with entries in $\mathbb{C}$. Note that $M_n(\mathbb{C}) \cong C^{n^2} \cong \mathbb{R}^{2n^2}$, so we can use our knowledge of finite dimensional real vector spaces to create parallels into matrix groups.

Definition 2.1: Convergence in M_n.
We say that a sequence A_m converges to A in $M_n(\mathbb{C})$ if A_m converges elementwise as $m \to \infty$. This means for all $1 \leq j, k \leq n$ each $(A_m)_{jk} \to (A)_{jk}$ in $\mathbb{C}$.

We say that a matrix subgroup G is **closed** if every sequence $\{A_m\} \in G$ converges to an element $A \in G$ or some A which has $\det A = 0$. We define a **matrix Lie group** $G \subset GL_n(\mathbb{C})$ as a closed subset. Now that we have defined convergences and closure, we can begin thinking about the **connectedness** of matrix groups.

Definition 2.2: Connected.
*A matrix Lie group G is **connected** if for all $A, B \in G$ there is a continuous path $r : [0,1] \to M_n(\mathbb{C})$ such that $r(0) = A$ and $r(1) = B$ and such that $r(t)$ in contained in G for all t.*

Definition 2.3: Simply Connected.
A matrix Lie group G is simply connected if its is connected for every $r(t)$ that is a closed loop, i.e $r(0) = r(1)$ can be shrunk continuously to a point in G.

We can also define compactness in the same way as $\mathbb{R}^{2n^2}$ since it is isomorphic to $M_n(\mathbb{C})$.

2.2 Lie Algebras

Definition 2.4: Lie Algebra.
A Lie algebra over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space $\mathcal{G}$ together with a map $[\cdot,\cdot]$ which satisfies the following:

(a) *$[\cdot,\cdot]$ is bilinear. This means that it satisfies Definition **??** in both inputs.*

(b) *$[X, Y] = -[Y, X]$ for all $X, Y \in \mathcal{G}$.*

(c) *$[X, X] = 0$ for all $X \in \mathcal{G}$.*

(d) *We have the following identity:*

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \tag{2.1}$$

Note that **(d)** is sometimes referred to as the Jacobi identity. We can also define Lie Algebra homomorphisms and isomorphisms similarly to Definitions 1.5 and 1.6, the only difference being is that an isomorphism need only be a bijective homomorphism. We also define subalgebras similarly to subrings and subgroups.

Definition 2.5: Subalgebra.
For a Lie algebra $\mathcal{G}$, we define $\mathcal{H} \subseteq \mathcal{G}$ as a subalgebra if $[X, Y] \in \mathcal{H}$ for all $X, Y \in \mathcal{H}$.

We again have ideals in lie algebras, seeing

Definition 2.6: Lie Algebra Ideal.
We say that $\mathcal{I} \subseteq \mathcal{G}$ is an ideal if $[X, Y] \in \mathcal{I}$ for all $X \in \mathcal{G}$ and $Y \in \mathcal{I}$.

Note that by definition, an ideal is automatically a subalgebra. This definition of ideal is equivalent to saying that $\mathcal{I} = (Y)$, that is, $\mathcal{I}$ is generated by an element Y.

2.3 Matrix Exponential

Definition 2.7: Matrix Exponential.
Given a matrix $X \in M_n(\mathbb{C})$ we define by the usual power series

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!} \tag{2.2}$$

By definition of power series, this converges absolutely on all of $M_n(\mathbb{C})$. This construction lends itself nicely to defining the following properties:

Lemma 2.8.
The matrix exponential has the following properties:

(a) $e^0 = I_n$

(b) $e^{X^T} = \left(e^X\right)^T$ *where X^{tr} is the matrix transpose of X.*

(c) $e^{X^*} = \left(e^X\right)^*$ *where X^* is the adjoint of X.*

(d) *If $A, A^{-1} \in GL_n(\mathbb{F})$ for some field $\mathbb{F}$, then*

$$e^{AXA^{-1}} = Ae^X A^{-1} \tag{2.3}$$

and $e^{A^{-1}} = \left(e^A\right)^{-1}$

(e) $\det e^X = e^{\mathrm{tr}\,(X)}$

(f) *We have*

$$e^{X+Y} = \lim_{m\to\infty} \left(e^{X/m} e^{Y/m}\right)^m \tag{2.4}$$

Specifically, if $XY = YX$, then $e^{X+Y} = e^X e^Y$.

After defining the exponential, it is natural to see how we can correspond its value to some sort of "inverse". To do this, we first need to define single parameter subgroups.

Definition 2.9: Single Parameter Subgroups.
We say that $H \subseteq GL_n(\mathbb{C})$ is a single parameter subgroup if it is a continuous homomorphism $A : \mathbb{R} \to GL_n(\mathbb{C})$ such that $A(0) = I_n$ and $A(s+t) = A(s)A(t)$ for all $s, t \in \mathbb{R}$.

Now, we can correlate every single parameter subgroup to an exponential map with the following theorem:

Theorem 2.10.
If $A(\cdot)$ is a single parameter subgroup of $GL_n(\mathbb{C})$, there exists a unique $X \in M_n(\mathbb{C})$ such that

$$A(t) = e^{tX} \tag{2.5}$$

This allows us to say that every matrix is the exponential of another, which ultimately lets us prove Theorem **??**.

2.4 Lie Algebras and Matrix Lie Groups

Now that we have defined the matrix exponential, we can use it to redefine Matrix Lie groups as follows:

Definition 2.11.
If $G \in GL_n(\mathbb{C})$ is a matrix Lie group, then the Lie algebra $\mathcal{G}$ is defined by

$$\mathcal{G} = \left\{X \in M_n(\mathbb{C}) : e^{tX} \in G \quad \forall t \in \mathbb{R}\right\} \tag{2.6}$$

This is equivalent to the following proposition:

Proposition 2.12.
X belongs to a $\mathcal{G}$ if and only the one parameter subgroup generated by X lies entirely in G. That is, $(X) \subseteq G$.

This new definition also gives us some new, nice properties of matrix Lie algebras.

Lemma 2.13.
For any matrix Lie group G, the Lie algebra $\mathcal{G}$ of G satisfies the following:

(a) $0 \in \mathcal{G}$

(b) *For all $X \in \mathcal{G}$, $tX \in \mathcal{G}$ for all t.*

(c) *For all $X, Y \in \mathcal{G}$, $X + Y \in \mathcal{G}$.*

(d) *For all $A \in G$ and $X \in \mathcal{G}$ we have $AXA^{-1} \in \mathcal{G}$.*

(e) *For all $X, Y \in \mathcal{G}$, we have $[X, Y] \in \mathcal{G}$.*

We now wish to look deeper into the connection between the Lie algebras generated by matrix lie groups and we arrive at the following theorem:

Theorem 2.14.
Suppose G_1 and G_2 are matrix Lie groups with corresponding Lie algebras $\mathcal{G}_1$, and $\mathcal{G}_2$ respectively. Now, suppose $\Phi : G_1 \to G_2$ is a Lie group homomorphism. Then there exists a unique linear map $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ such that

$$\Phi(e^{tX}) = e^{t\varphi(X)} \tag{2.7}$$

for all $t \in R$ and $X \in \mathcal{G}$

We can go further, noting that this map has the following properties:

Lemma 2.15.
The linear map φ has the following properties:

(a) $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$ *for all $X, Y \in \mathcal{G}$.*

(b) $\varphi(AXA^{-1}) = \Phi(A)\varphi(X)\Phi(A)^{-1}$ *for all $A \in G$ and $X \in \mathcal{G}$*

(c) *We can compute $\varphi(X)$ directly from the following identity:*

$$\varphi(X) = \frac{d}{dt}\Phi(e^{tX})\bigg|_{t=0} \tag{2.8}$$

This lemma gives us the following proposition for φ:

Proposition 2.16.
φ is a Lie group Homomorphism and $\varphi(e^{tX})$ is a smooth function of t for all X.

This proposition can the be generalized to another corollary:

Corollary 2.17.
Suppose that G_1 and G_2 are matrix Lie groups with corresponding Lie algebras $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. If $G_1 \cong G_2$, then $\mathcal{G}_1 \cong \mathcal{G}_2$.

There are many important results which use $M_n(\mathbb{C})$ as a manifold, but we will not use them in these notes. The most important result we glean from these is as follows:

Theorem 2.18.
If a matrix Lie group G is connected, then for all $A \in G$ there exists a finite sequence $X_1, X_2, \ldots X_n$ of elements in $\mathcal{G}$ such that

$$A = e^{X_1}e^{X_2}\ldots e^{X_n} \tag{2.9}$$

This is an extremely powerful result, as it allows us to use the exponential map to correspond any element in a lie Group to a product of elements in the corresponding Lie algebra. We can now use this to extend a homomorphism between Lie groups to a Lie algebra homomorphism via the following theorem:

Theorem 2.19.
Suppose that G_1 and G_2 are matrix Lie groups with corresponding Lie algebras $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. Suppose $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ is a Lie algebra homomorphism. If G_1 is simply connected, then there exists a unique Lie group homomorphism $\Phi : G_1 \to G_2$ such that φ and Φ are related as in Lemma 2.15.

Example 2.1

The Lie algebras generated by $SU_2(\mathbb{C})$ and $SO_3(\mathbb{C})$ are isomorphic, but $SU_2(\mathbb{C}) \ncong SO_3(\mathbb{C})$. This is because $SU_2(\mathbb{C})$ is not simply connected.

2.5 Representations of Lie Groups and Lie Algebras

Definition 2.20: Finite-Dimensional Representation of a Lie Group.
Let $G \subset GL_n(\mathbb{C})$ be a matrix Lie group. We say that Π is a finite-dimensional representation of G if Π is a continuous homomorphism such that $\Pi : G \to GL(V)$, the group of invertible linear transformations over some vector space V.

Note that in this definition, we do not specify what V is. This is intentional! Determining a suitable V for our purpose will be a main challenge for this topic. We can also define representations for Lie Algebras.

Definition 2.21: Finite-Dimensional Representation of a Lie Algebra.
A finite-dimensional representation of a Lie Algebra $\mathcal{G}$ is a Lie Algebra homomorphism $\Gamma : \mathcal{G} \to GL(V)$ where we consider $GL(V)$ as a Lie algebra equipped with $[X, Y] = XY - YX$.

Note that we write the representation Π with vector space V as (Π, V).

Definition 2.22: Invariant Subspace.
If $\Pi : G \to GL(V)$ is a representation of a matrix Lie group G, then a subspace $W \subset V$ is called an invariant subspace if $\Pi(g)w \in W$ for all $g \in G$ and $w \in W$. This definition also works for Lie algebra representations.

Definition 2.23: Irreducible Representation.
We call a representation of a group or Lie Algebra if the only invariant subspaces are $W = V$ and $W = \{0\}$

Definition 2.24: Intertwining Map.
If (Π, V_1) and (Σ, V_2) are representations of a Lie group, a map $\Phi : V_1 \to V_2$ is an intertwining map if

$$\Phi(\Pi(g)v) = \Sigma(g)\Phi(v) \tag{2.10}$$

for all $v \in V_1$.

Theorem 2.25: Schur's Lemma.
If V_1 and V_2 are irreducible representations of a group or Lie Algebra, then the following hold:

(a) *If $\Phi : V_1 \to V_2$ is an intertwining map, then either $\Phi \equiv 0$ or Φ is an isomorphism.*

(b) *If $\Phi : V_1 \to V_2$ and $\Psi : V_1 \to V_2$ are nonzero intertwining maps, then there exists a nonzero constant $c \in \mathbb{C}$ such that $\Phi = c\Psi$. In particular, if Φ is an intertwining map from V_1 to itself then $\Phi = c\mathbb{I}$.*

We will now move on to unitary representations of Lie groups.

Definition 2.26.
Suppose V us a finite-dimensional Hilbert Space over $\mathbb{C}$. Denote the group of invertible linear transformations of V which preserves the inner product. A unitary representation of a Lie group G is a continuous homomorphism $\Pi : G \to U(V)$ for some finite-dimensional Hilbert space V.

Proposition 2.27.
Let $\Pi : G \to GL(V)$ be a finite-dimensional representation of a connected matrix Lie group G, and let π be the associated representation of the Lie Algebra $\mathcal{G}$ of G. Let $\langle \cdot, \cdot \rangle$ be an inner product on V. Then Π is unitary with respect to this inner product if and only if $\pi(X)$ is skew self-adjoint for all $X \in \mathcal{G}$, meaning

$$\pi(X)^* = -\pi(X) \tag{2.11}$$

Definition 2.28.
Suppose V is a finite-dimensional Hilbert space over $\mathbb{C}$. The projective unitary group over V, is the quotient group

$$PU(V) = U(V)/\{e^{i\theta} I\} \tag{2.12}$$

where we are modding out the unitary group of matrices by the trivial rotations.

Proposition 2.29.
If V is a finite-dimensional Hilbert space over $\mathbb{C}$, then $PU(V)$ is isomorphic to a matrix Lie group.

We will now consider basic mechanisms for combining representations, namely, tensor products, direct sums, and dual representations. The first two are seen in Section 1 of these notes, and duals will be defined below. We see

Definition 2.30.
The dual of a representation $\Pi^{tr} : G \to GL(V^*)$, Π is given by

$$\Pi^{tr}(A) = \Pi(A^{-1})^{tr} = (\Pi(A)^{tr})^{-1} \tag{2.13}$$

Definition 2.31.
A finite-dimensional representation of a group or Lie algebra is said to be completely reducible if it is isomorphic to a direct sum of irreducible representations.

Proposition 2.32.
Every finite dimensional unitary representation if a group or Lie algebra is completely reducible.

References

[1] David S. Dummit and Richard M. Foote. *Abstract Algebra*. 3rd ed. New York: Wiley, 2004.

[2] Brian C. Hall. *Quantum Theory for Mathematicians*. Graduate Texts in Mathematics 267. Springer New York, 2013. ISBN: 978-1-4614-7115-8.