MATH 825 Final Presentation: Bipolar Green's Functions

Kale Stahl

These lecture notes will follow Gamelin's "Complex Analysis" [1] Chapter 16. We will first define Bipolar Green's functions, and then prove tehir existence on Riemann surfaces. Afterwords we will state the Uniformization Theorem and sketch a proof using Bipolar Green's Functions.

1 Bipolar Green's Functions

Not every Riemann surface has an associated Green's function. However, every surface has a bipolar Green's function. Here, we will prove this fact. First, we need to define what we mean by a bipolar Green's function.

Definition 1. *Let q_1 and q_2 be distinct points of a Riemann surface R. Let D_1 and D_2 be disjoint coordinate disks containing q_1 and q_2 respectively., with coordinate maps $z_1(p)$ and $z_2(p)$ satisfying $z_1(q_1) = z_2(q_2) = 0$. A* ***bipolar Green's function*** *with poles at q_1 and q_2 is a harmonic function $G(p, q_1, q_2)$ on $R \setminus \{q_1, q_2\}$ such that*

- ***(a)*** *$G(p, q_1, q_2) + \log|z_1(p)|$ is harmonic at q_1.*
- ***(b)*** *$G(p, q_1, q_2) - \log|z_1(p)|$ is harmonic at q_2.*
- ***(c)*** *$G(p, q_1, q_2)$ is bounded on $R \setminus (D_1 \cup D_2)$.*

Note that $G(p, q_1, q_2)$ is not uniquely determined, though it is unique up to the addition of a bounded harmonic function. b

Theorem 1. *For each pair of distinct points q_1 and q_2 on a Riemann surface, there is a bipolar Green's function.*

This theorem is nice, but we can prove a slightly simpler theorem that will generalize to this. WE seek to prove the following Lemma instead:

Lemma 1. *Let S be a finite bordered Riemann surface, and let q_1 and q_2 distinct points on S. Let $B_1 = \{|z_1(p)| \le \sigma\}$ and $B_2 = \{|z_2(p)| \le \sigma\}$ be disjoint closed coordinate disks, where $z_1(q_1) = z_2(q_2) = 0$. Then there is a constant $C > 0$ such that*

$$|g_r(p, q_1) - g_R(p, q_2)| \le C, \qquad p \in R \setminus (B_1 \cup B_2) \tag{1}$$

for all Riemann surfaces R containing $S \cup \partial S$ for which Green's function $G_R(p, q)$ exists.

Proof. Let some $\rho > 0$ satisfy $\rho \le \sigma$. For the sake of simplicity, take some $j = 1, 2$ and define A_j to be the closed coordinate disk $\{|z_j(p)| \le \rho\}$ and let $M_j = M_j(R)$ be the maximum of $g_r(p, q_j)$ on ∂A_j. For some $p \in B_j$ we have

$$g_R(p, q_j) + \log|z_j(p)| \le \max\{g_R(q, q_j) : q \in B_j\} + \log \sigma \tag{2}$$

We know that $g_R(p, q_j) + \log|z_j(p)|$ is harmonic on B_j. By the maximum principle the estimate still works for all $p \in B_j$. If we take the supremum over all $p \in \partial A_j$, we then have the following estimate:

$$M_j + \log \rho \le \max\{g_R(q, q_j) : q \in \partial B_j\} + \log \sigma \tag{3}$$

This can also be interpreted as there must exist some $p_j \in \partial B_j$ such that $M_j + \log \rho \le g_R(p_j, q_j) + \log \sigma$. WE can then arrive at the following:

$$M_j - g_R(p_j, q_j) \le \log\left(\frac{\sigma}{\rho}\right) \tag{4}$$

We know that $M_j - g_R(p,q_j)$ must be a harmonic function harmonic function on $S \setminus (A_1 \cup A_2)$. If we apply the Harnack estimate to the surface $S \setminus (A_1 \cup A_2)$ and the compact subset $\partial B_1 \cup \partial B_2$, we can obtain some a constant C_0 such that $M - g_R(p,q_j) \leq C_0$ for $p \in \partial B_1 \cup \partial B_2$. Thus,

$$M_j - C_0 \leq g_R(p,q_j) \leq M_j, \qquad p \in \partial B_j \cup \partial B_2 \tag{5}$$

Now, consider the case of $j = 1$. Since $g_R(p,q_1)$ is harmonic for all $p \in B_2$ and satisfies (5) on ∂B_2, it must satisfy (5) for all $p \in B_2$. In particular, if we choose $p = q_2$ we have

$$M_1 - C_0 \leq g_R(q_2,q_1) \leq M_1 \tag{6}$$

Using the same thinking for $j = 2$, we have

$$M_2 - C_0 \leq g_R(q_1,q_2) \leq M_2 \tag{7}$$

Since by definition, $g_R(q_2,q_1) = g_R(q_1,q_2)$, we can see subtract these two estimates to see that $|M_1 - M_2| \leq C_0$. Applying (5) to this estimate we then see

$$|g_R(p,q_1) - g_R(p,q_2)| \leq 2C_0, \qquad p \in \partial B_1 \cup \partial B_2 \tag{8}$$

Since the Green's Function vanishes on the the boundary of R, (8) holds for $R \setminus (B_1 \cup B_2)$ by the maximum principle. Thus, if we define $C = C_0$, the lemma is proved. □

Now that we have established this lemma, we can begin to generalize it to the statement seen in Theorem 1. We can utilize a trick by approximating R by surfaces $R \setminus B_\epsilon$ for B_ϵ is a closed coordinate disk $\{|z_0(p)| \leq \epsilon\}$ centered at some $p_0 \in R$ The Green's function $g_\epsilon(p,q)$ exists for $R \setminus B_\epsilon$, and we can form a bipolar Green's function by $g_\epsilon(p,q_1) - g_\epsilon(p,q_2)$. We can use the lemma to note that this function is bounded, and by the compactness of $R \setminus B_\epsilon$ in R, we can take the limit as some sequence $\epsilon_j \to 0$ to obtain a bipolar Green's function that is defined on all of R.

2 Uniformization Theorem

Since these notes are meant for a 25 minute presentation, we will state the Uniformization Theorem and outline a proof, rather than proving the entire theorem.

Theorem 2. *(Uniformization Theorem) Each simply connected Riemann surface is conformally equivalent to either the open unit disk $\mathbb{D}$, the complex plane $\mathbb{C}$, or the Riemann Sphere $\mathbb{C}^*$.*

Proof Sketch. We want to split this proof into two parts. First we will show that if a Green's function exists for R, it can be used to map R conformally to an open unit disk. Then, if the Green's function for R does not exists, we can use the bipolar Green's function to map R to the punctured plane or the Riemann sphere. □

References

[1] Theodore W. Gamelin. *Complex Analysis.* Springer, 2001.