MATH 530: Complex Analysis Qualifying Exam Prep

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Contents

Part I
Notes

These notes are a brief summary of Dr. Buzzard's Spring 2025 MA530 class. I will assume the reader has a basic understanding of real analysis and will not include superfluous definitions or theorems. The class was based on Stein and Shakarchi's *Complex Analysis*, so if anything is missing look there. Unless specified otherwise, suppose $\mathbb{D}$ is the unit disk of radius 1 centered at 0, Ω is a proper subset of the complex plane, z is a complex number or variable, and $f = u + iv$ is a complex function composed of real functions u and v. These conventions are not always true, but use context clues and it will hopefully make sense.

1 Preliminaries

1.1 Continuity in the Complex Plane

Much like in $\mathbb{R}$, we wish to describe the continuity of functions, and we can defined them in two ways:

Definition 1.1.
For a point $z_0 \in S \subset \mathbb{C}$, we say a function f is ***continuous*** *if*

$$\lim_{z \to z_0} f(z) = f(z_0) \tag{1.1}$$

Definition 1.2.
For a set $S \subset \mathbb{C}$, we say a function f is ***continuous*** *on S if f is continuous for all $z \in S$.*

We can also divide a function f into its real and imaginary parts in the following ways:

$$f(z) = u(z) + iv(z) \tag{1.2}$$
$$f(x, y) = u(x, y) + iv(x, y) \tag{1.3}$$
$$f(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} \tag{1.4}$$

1.2 Differentiation in the Complex Plane

Definition 1.3.
Let $\Omega \subseteq \mathbb{C}$ be open, and $f : \Omega \to \mathbb{C}$. If f is ***complex differentiable*** *at $z_0 \in \Omega$, this means that*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1.5}$$

exists.

Theorem 1.4.
If f is differentiable at z_0 then f is continuous at z_0 and

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + R(z)(z - z_0) \tag{1.6}$$

With

$$\lim_{z \to z_0} R(z) = 0 \tag{1.7}$$

Proof. □

Theorem 1.5.
If $f = u + iv$ is holomorphic, then it satisfies the Cauchy-Riemann equations:

$$u_x = v_y \tag{1.8}$$
$$u_y = -v_x \tag{1.9}$$

Corollary 1.6.
If $f = u + iv$ is holomorphic, then

$$f' = u_x + iv_x = v_y - iu_y \tag{1.10}$$

Theorem 1.7.
Suppose u and v are C^1 on an open set $\Omega \subset \mathbb{C}$. If u, v satisfy the Cauchy-Riemann equations, then $f = u + iv$ is holomorphic.

Proof. □

Definition 1.8.
Define the following derivatives with respect to a complex variable:

$$\frac{\partial}{\partial z} = \frac{1}{z}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \tag{1.11}$$

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{z}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) \tag{1.12}$$

Proposition 1.9.
If f is holomorphic at z_0 then

$$\frac{\partial f}{\partial z}(z_0) = f'(z_0) \tag{1.13}$$

$$\frac{\partial f}{\partial \overline{z}}(z_0) = 0 \tag{1.14}$$

Also if $F(x, y) = f(x + iy)$ then

$$\det J_f(x_0, y_0) = |f'(z_0)|^2 \tag{1.15}$$

Proof. Check using Cauchy-Riemann equations. □

1.3 Power Series

Recall from real analysis that polynomials in z are holomorphic by the real sum and product rules, so if we define a polynomial by

$$P(z) = \sum_{n=0}^{N} a_n(z - z_0)^n \tag{1.16}$$

then we can approximate a function f by a power series given by

$$f(x) \approx \sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{1.17}$$

Definition 1.10.
Let $\{r_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$. Then

$$\limsup_{n\to\infty} r_n = \lim_{n\to\infty} \left(\sup\{r_k : k > n\}\right) \tag{1.18}$$

Note that this limit always exists since the supremum is non-increasing.

Theorem 1.11: Absolute Convergence of Power Series.
Given $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, let

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} \tag{1.19}$$

(a) *If $|z - z_0| < R$, the series converges absolutely.*

(b) If $|z - z| > R$ the series diverges.

(c) If $|z - z_0| = R$, there is no way of knowing if the series converges or diverges.

Theorem 1.12: Uniform Convergence of Power Series.
Assume $R > 0$ as in Theorem 1.11, and choose some $r \in (0, R)$. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.20}$$

in $D_R(0)$. Then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely and uniformly in $D_r(0)$ to f.

Proposition 1.13: Ratio Test.
Let a_n be coefficients of a power series. If

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L \tag{1.21}$$

then

$$\lim_{n\to\infty} |a_n^{1/n} = L \tag{1.22}$$

Using power series, we can define certain properties of complex exponentials, seeing

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{1.23}$$

where the radius of convergence is $R = 1/0 = \infty$, so it converges everywhere. All of the usual properties of the real exponential function apply to the complex exponential, which can be proven by the use of power series.

Theorem 1.14.
The function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in its disk of convergence $D_R(0)$. Also $f'(z)$ is a power series with the same disk of convergence and

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1} = g(z) \tag{1.24}$$

Proof. We need to show that $g(z)$ has the same disk of convergence $D_R(0)$. Let $\varepsilon > 0$ and

$$S_N(z) = \sum_{n=0}^{N} a_n z^n$$

$$\Sigma_N = \sum_{n=N+1}^{\infty} a_n z^n$$

Choose some $z_0 \in D_r(0)$ and $r > 0$ such that $|z_0| < r < R$ and h such that $|z_0 + h| < r$. Then

$$\begin{aligned} \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) &= \left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)\right) + (S_N'(z_0) - g(z_0)) + \left(\frac{\Sigma_N(z_0+h) - \Sigma_N(z_0)}{h}\right) \\ &= (*) + (**) + (***) \end{aligned}$$

Then we can bound each part to show that the limit goes to zero. For $(***)$ we see that

$$(***) \leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1}$$

So then choose some M that for all $N \geq M$ that $|(***)| < \varepsilon$. We know that $S_N'(z_0) \to g(z_0)$ as $N \to \infty$, so we can choose M_2 such that $|(**)| < \varepsilon$ for all $N > M_2$. Then pick $M = \max(M, M_2)$ and there exists some δ such that $|h| < \delta$ implies $|(*)| < \varepsilon$, so $|(*) + (**) + (***)| < \varepsilon$ so then the limit goes to zero and we are finished. □

Corollary 1.15.
A power series is infinitely differentiable in its disk of convergence and the derivatives are also power series with the same disk of convergence and given by term-by-term differentiation.

Proof. Apply Theorem 1.14 repeatedly. □

2 Complex Integration

Much like with integration in $\mathbb{R}^2$, integration over $\mathbb{C}$ also has a notion of a path integral. We begin by defining the length of a contour as follow:

Definition 2.1.
Let $z : [a,b] \to \mathbb{C}$ be a piecewise C^1 parametrization of γ. Then

$$\text{length}(\gamma) = \sum_a^b |z'(t)|\, dt \tag{2.1}$$

We also have the same estimate as we do in $\mathbb{R}^2$, seeing

Proposition 2.2.
Let $z : [a,b] \to \mathbb{C}$ be a piecewise C^1 parametrization of γ, then we have the following estimate:

$$\int_\gamma f(z)\, dz \leq \max_{z\in\gamma} |f(z)| \cdot \int_a^b |z'(t)|\, dt \tag{2.2}$$

Now that we can define an integral over $\mathbb{C}$, it is natural to extend the notion of the fundamental theorem of calculus.

Theorem 2.3: Fundamental Theorem of Calculus.
If f is continuous and has a primitive F in some domain $\Omega \subset \mathbb{C}$ and $z : [a,b] \to \mathbb{C}$ is a parametrization of a curve γ, then we see

$$\int_\gamma f(z)\, dz = F(b) - F(a) \tag{2.3}$$

Corollary 2.4.
If f is holomorphic in a region Ω and $f' \equiv 0$ on Ω, then $f \equiv const.$

2.1 Cauchy's Theorem and the Cauchy Integral Formula

Theorem 2.5.
A holomorphic function in an open disk has a primitive in that disk.

Theorem 2.6: Goursat's Theorem.
If you have a triangle T with its interior contained inside an open set Ω and f is holomorphic on Ω, then

$$\int_T f(z)\, dz = 0 \tag{2.4}$$

From here, we can generalize to

Theorem 2.7: Cauchy's Theorem on a Convex Open Set.
If Ω is a convex open set and f is continuous on Ω, and f is analytic on $\Omega \setminus \{p\}$ for any $p \in \Omega$, then

$$\int_\gamma f(z)\, dz = 0 \tag{2.5}$$

for any closed curve γ such that the trace of γ is in Ω.

Lemma 2.8.
If f is holomorphic on a convex open set Ω, then it has a holomorphic antiderivative given by

$$F(z) = \int_{L_z^z} f(w)\, dw \tag{2.6}$$

Where L_a^z is the line from any $a \in \Omega \setminus \{z\}$ to z.

This allows us to use the notion of so-called ”toy contours”. These are closed curves with an interior and we can join any two points in the interior of a simply connected domain with finitely many straight lines. These toy contours will be useful when combined with other theorems about integrals over specific contours, namely the Cauchy Integral formula.

Theorem 2.9: Cauchy Integral Formula.
If f is analytic on an open set Ω, with some fixed $a \in D_r(z_0) \subseteq \overline{D_r(z_0)} \subseteq \Omega$ then

$$f(a) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{z-a}\, dz \tag{2.7}$$

From here, we arrive at a way to define the nth derivative of f using Theorem 2.9.

Corollary 2.10.

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{C_r(a)} \frac{f(z)}{(z-a)^{n-1}}\, dz \tag{2.8}$$

This also means that if f is analytic, then f' is also analytic.

Corollary 2.11: Cauchy Estimate.
Using Proposition 2.2, we arrive at

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \max_{z \in C_r(z_0)} |f(z)| \tag{2.9}$$

Proof. We can apply Corollary 2.10 to $f^{(n)}(z_0)$, we see

$$\left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta\right| \tag{2.10}$$

$$= \frac{n!}{2\pi} \left|\int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{(Re^{i\theta})^{n+1}} Rie^{i\theta}\, d\theta\right| \tag{2.11}$$

$$\leq \frac{n!}{r^n} \max_{z \in C_r(z_0)} |f(z)| \tag{2.12}$$

□

Theorem 2.12.
If f is holomorphic on an open set Ω and $\overline{D_r(z_0)} \subseteq \Omega$, the for all $z \in D_r(z_0)$ we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \tag{2.13}$$

Proof. Fix some $z \in \mathbb{D}$ by Theorem 2.9, we have

$$f(z) = \frac{1}{2\pi i} \int_{C(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \tag{2.14}$$

We wish to do some manipulations on the integrand to write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)} \tag{2.15}$$

Since $\zeta \in C(z_0)$ and $z \in \mathbb{D}$ is fixed, there is some $0 < r < 1$ such that

$$\left|\frac{z - z_0}{\zeta - z_0}\right| < r \tag{2.16}$$

Therefore we can expand in a geometric series, seeing

$$\frac{1}{1-\left(\frac{z-z_0}{\zeta-z_0}\right)}=\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^n \tag{2.17}$$

which converges uniformly for all $\zeta \in C(z_0)$, which allows us to freely interchange the sum and integral, seeing

$$f(z)=\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{c(z_0)}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta\right)(z-z_0)^n \tag{2.18}$$

which is a power series with

$$a_n=\frac{1}{2\pi i}\int_{c(z_0)}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta=\frac{f^{(n)}(z_0)}{n!} \tag{2.19}$$

by Corollary 2.10, so we are finished. □

2.2 Holomorphic Functions

Theorem 2.13: Liouville.
If a function f is entire and bounded, then f is constant.

Proof. Since $\mathbb{C}$ is a connected domain, we only need to prove that $f' \equiv 0$ so then we can apply Corollary 2.4. For each $z_0 \in \mathbb{C}$ and $R > 0$, by Corollary 2.11 we arrive at a bound of

$$|f'(z_0)| \leq \frac{C}{R} \tag{2.20}$$

For some constant $C \in \mathbb{C}$ which is a bound for f. If we let $R \to \infty$, then $|f'(z_0)| \to 0$ □

Theorem 2.14: Fundamental Theorem of Algebra.
If $P(z) = a_n z^n + \cdots + a_0$ is a nonconstant polynomial in $\mathbb{C}$ of degree $n \geq 1$, then p has exactly n roots with multiplicity and

$$P(z)=a_n(z-w_1)^{k_1}\dots(z-w_n)^{k_n} \tag{2.21}$$

where w_n are the roots of P and $\sum_i k_i = n$.

Proof. We first need to prove that P indeed has a root. Suppose for the sake of contradiction that it doesn't. Then $1/P(z)$ will be a bounded holomorphic function. By Theorem 2.13, $1/P(z)$ must be constant, which implies $P(z)$ is constant, which is a contradiction. Thereby, $P(z)$ has at least one root. Call this root w_1. Then we can write $z = (z - w_1) + w_1$. Substituting this into P we see

$$\begin{aligned}
P(z)&=a_n((z-w_1)+w_1)^n+a_{n-1}((z-w_1)+w_1)^{n-1}\cdots+a_1((z-w_1)+w_1)+a_0 &(2.22)\\
&=a_n\left(\sum_{k=1}^{n}\binom{n}{k}(z-w_1)^k w_1^{n-k}\right)+a_{n-1}\left(\sum_{k=1}^{n-1}\binom{n-1}{k}(z-w_1)^k w_1^{n-1-k}\right)+\cdots+a_0 &(2.23)\\
&=b_n(z-w_1)^n+\cdots+b_1(z-w_1)+b_0 &(2.24)
\end{aligned}$$

for some constants b_i with $b_n = a_n$. Since $P(w_1) = 0$, we know that $b_0 = 0$, and we have

$$P(z)=(z-w_1)\left(b_n(z-w_1)^{n-1}+\cdots+b_1\right)=(z-w_1)Q(z) \tag{2.25}$$

We can again prove that $Q(z)$ has a root and continue the process to see that $P(z)$ has exactly n roots, and can be written as

$$P(z)=C(z-w_1)(z-w_2)\dots(z-w_n) \tag{2.26}$$

for some constant $C \in \mathbb{C}$. Since each term in $P(z)$ is monic, the coefficient in front of z^n will be C, so then $C = a_n$ and we are finished. □

Theorem 2.15: Identity Theorem.
if f is holomorphic on a domain Ω and

$$z_f = \{z \in \Omega | f(z) = 0\} \tag{2.27}$$

then if $z_f = \Omega$ or z_f has no limit points in Ω.

Corollary 2.16.
Holomorphic functions that agree on a subset with a limit pt in a domain agree on that domain.

Theorem 2.17.
If f is holomorphic and $f' \equiv 0$ on a domain then f is constant.

Theorem 2.18: Morera.
If f is continuous on an open set Ω and for all triangles T such that the interior and boundary are in Ω then

$$\int_T f(z)\,dz = 0 \tag{2.28}$$

implies that f is holomorphic on Ω.

Proof. By an extension of Theorem 2.5 to Ω, f has a primitive F in Ω that satifies $F' = f$. Since F is differentiable once, it must be infinitely differentiable, so then f is holomorphic. □

Definition 2.19.
*A sequence $\{f_n\}$ **converges uniformly** on compact subsets of Ω to a function f if given some compact $k \subset \Omega$ and $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that when $n \geq N$ and $z \in K$ then $|f_n(z) - f(z)| < \varepsilon$. We say that $f_n \to f$.*

Theorem 2.20.
If f_n is holomorphic, then $f_n \to f$ implies that f is holomorphic.

Theorem 2.21: Schwarz Reflection Principle.
Let Ω^+ be a symmetric domain about the real line such that

1. *f is continuous on Ω^+ up to $\mathbb{R}$,*
2. *$f(x) \in \mathbb{R}$ for all $x \in \Omega \cap \mathbb{R}$,*
3. *f analytic in Ω^+*

Then

$$F(z) = \begin{cases} f(z) & z \in \Omega^+ \\ f(x) & x \in \Omega \cap \mathbb{R} \\ \overline{f(z)} & z \in \Omega^- \end{cases} \tag{2.29}$$

is analytic on Ω.

Theorem 2.22: Symmetry Principle.
If Ω is a symmetric domain about the real line and if f^+ and f^- are holomorphic on Ω^+ and Ω^- respectively, and they extend continuously to $\Omega \cap \mathbb{R}$ with $f^+(x) = f^-(x)$ for all $x \in \Omega \cap \mathbb{R}$, then

$$F(z) = \begin{cases} f^+(z) & z \in \Omega^+ \\ f^+(x) = f^-(x) & x \in \Omega \cap \mathbb{R} \\ \overline{f^-(z)} & z \in \Omega^- \end{cases} \tag{2.30}$$

is holomorphic on Ω.

Theorem 2.23: Runge's Approximation Theorem.
Any function that is holomorphic in a neighborhood of a compact set can be approximated uniformly on K by rational functions whose singularities lie in K^c. Moreover, if K^c is connected, we can approximate it by polynomials.

3 Holes and Poles

Theorem 3.1.
If f is holomorphic on a connected open set Ω with a zero at a point z_0 and $f \equiv 0$ on Ω, then there is a neighborhood of z_0, $U \subset \Omega$ and a unique $n \in \mathbb{N}$ such that for all $z \in U$

$$f(z) = (z - z_0)^n g(z) \tag{3.1}$$

for some non-vanishing holomorphic function g on U.

Theorem 3.2.
If f has a pole at z_0, then there exists a neighborhood of z_0, $U \in \Omega$ and a unique $n \in \mathbb{N}$ such that for any $z \in U$

$$f(z) = (z - z_0)^{-n} h(z) \tag{3.2}$$

for some non-vanishing holomorphic function h on U.

Theorem 3.3.
If f has a pole at z_0, then in a neighborhood of z_0 we have

$$f(z) = a_{-n}(z - z_0)^n + a_{-n+1}(z - z_0)^{n-1} + \cdots + a_{-1}(z - z_0) + g(z) \tag{3.3}$$

Where g is a holomorphic function in a neighborhood of z.

3.1 Residues

Definition 3.4.
*We define the **residue** of f at z_0 as*

$$\operatorname{Res}_{z=z_0} f = a_{-1} \tag{3.4}$$

Theorem 3.5.
If f is holomorphic on an open set Ω containing a toy contour γ and its interior except at a countable number of poles at $z_1, \dots z_n$ inside of γ, we have

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z_k} f \tag{3.5}$$

Theorem 3.6.
If f has a pole of order n at z_0 then

$$\operatorname{Res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n f(z)\right) \tag{3.6}$$

Lemma 3.7.
If f and g are holomorphic on $D_r(z_0)$ and f and g have zeroes at z_0 of order m and $m+1$ respectively then

$$\operatorname{Res}_{z_0} \frac{f}{g} = (m+1) \frac{f^{(m)}(z_0)}{g^{(m+1)}(z_0)} \tag{3.7}$$

Lemma 3.8.
If p and q are polynomials and $\deg(q) \geq \deg(p) + 2$ then

$$\lim_{r \to \infty} \int_{S_r} \frac{p(z)}{q(z)}\,dz = 0 \tag{3.8}$$

for any segment of a circle of radius r, S_r.

Lemma 3.9: Jordan.
If C_R is a semicircle centered at 0 with radius R lying in the upper half plane and $s > 0$ then

$$\left| \int_{C_R} e^{isz} f(z)\,dz \right| \leq \frac{\pi}{s} \max_{z \in C_R} |f(z)| \tag{3.9}$$

Theorem 3.10: Laurent Series.
Suppose f is analytic on $A(z_0, \rho_1, \rho_2) = \{z \in \mathbb{C} | \rho_1 < |z - z_0| < p_2\}$ where $\rho_1 < r < R < \rho_2$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n} \tag{3.10}$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_{\tilde{r}}} \frac{f(z)}{(z - z_0)^{n+1}} \, dz \tag{3.11}$$

when $\rho_1 < \tilde{r} < \rho_2$.

3.2 Classifying Singularities and Zeroes

Definition 3.11.
An isolated singularity of f exists at z_0 if f is defined in a deleted neighborhood of z_0, but not at z_0 itself. This singularity is

- ***removable*** *if* $\lim_{z \to z_0} |f(z)| < \infty$,
- *a **pole** if* $\lim_{z \to z_0} |f(z)| = \infty$,
- ***essential*** *otherwise.*

Theorem 3.12.
If f has an isolated singularity at z_0 then

(a) *$a_{-n} = 0$ for all $n \in \mathbb{N}$ if and only if z_0 is removable.*

(b) *there are finitely many $a_{-n} \neq 0$ if and only if z_0 a pole.*

(c) *There are infinitely many $a_{-n} \neq 0$ if and only if z_0 is essential.*

Theorem 3.13: Riemann's Theorem on Removable Singularities.
Suppose that f is holomorphic on an open set Ω except possibly at a point $z_0 \in \Omega$. If f is bounded on $\Omega \setminus \{z_0\}$, then z_0 is a removable singularity.

Definition 3.14.
*A **Jordan Curve** or a simple closed curve is a continuous map $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma(0) = \gamma(1)$ and $\gamma|_{[0,1)}$ is injective.*

Theorem 3.15: Jordan Curve Theorem.
For a Jordan curve γ, $\mathbb{C} \setminus \operatorname{Im}(\gamma)$ consists of exactly two connected components, one of which is bounded by $\operatorname{Im}(\gamma)$.

Definition 3.16.
*An open and connected domain Ω is **simply connected** if the interior of every Jordan curve contained in Ω is also contained in Ω. Equivalently, Ω is simply connected if any two curves in Ω with the same endpoints are homotopic in Ω.*

Theorem 3.17.
If f is holomorphic in Ω, and γ_1 and γ_2 are homotopic in Ω, then

$$\int_{\gamma_1} f(z) \, dz = \int_{\gamma_2} f(z) \, dz \tag{3.12}$$

In particular if Ω is simply connected then for any closed curve γ in Ω, we have $\int_\gamma f(z) \, dz = 0$.

Theorem 3.18.
Let f be holomorphic on a simply connected domain Ω, then for any curve $\gamma_{z_0}^z \subseteq \Omega$ from z_0 to z,

$$F(z) = \int_{\gamma_{z_0}^z} f(w) \, dw \tag{3.13}$$

is a primitive of $f(z)$.

Theorem 3.19: Casorati-Weierstrass.
If $f : D_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic with an essential singularity at z_0, then the image of $D_r(z_0) \setminus \{z_0\}$ is dense in $\mathbb{C}$.

Proof. We will proceed by contradiction. Assume that $f(D_r(z_0) \setminus \{z_0\})$ is not dense in $\mathbb{C}$. This means that there exists some $w \in \mathbb{C}$ and $\delta > 0$ such that $|f(z) - w| > \delta$ for all $z \in D_r(z_0) \setminus \{z_0\}$. We now define a new function on the punctured disk

$$g(z) = \frac{1}{f(z) - w} \tag{3.14}$$

which is holomorphic and bounded by $\frac{1}{\delta}$. By Theorem 3.13, g has a removable singularity at z_0. If $g(z_0) \neq 0$, then $f(z) - w$ is holomorphic at z_0, contradicting the fact that z_0 is an essential singularity. If $g(z_0) = 0$, then $f(z) - w$ must have a pole at z_0, which also contradicts the fact that z_0 is an essential singularity. Thus, $f(D_r(z_0) \setminus \{z_0\})$ is dense in $\mathbb{C}$. ☐

Definition 3.20.
*A **meromorphic** function is one that is holomorphic except on a set of points which are poles.*

Theorem 3.21.
A function f is meromorphic in $\overline{\mathbb{C}}$ if and only if it is rational.

Theorem 3.22: Argument Principle.
If f is a meromorphic on an open set containing the closed interior of a simple closed curve γ and f has no poles or zeroes on γ, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \#_{zeroes} - \#_{poles} \tag{3.15}$$

Where $\#_{zeroes}$is the number of zeroes inside of γ with multiplicity and $\#_{poles}$is the number of poles inside of γ with multiplicity

Theorem 3.23: Rouché's Theorem.
Suppose f and g are meromorphic on an open set containing the closed interior of a simple closed curve γ and f has no poles on γ and $\forall z \in \gamma$, $|f(z) > |g(z)|$ Then

$$\#_z^{f+g} - \#_p^{f+g} = \#_z^f - \#_p^f \tag{3.16}$$

Where $\#_z$ and $\#_p$ are the number of zeroes and poles of f or $f + g$ inside of γ.

Proof. For some $t \in [0, 1]$ define

$$f_t(z) = f(z) + tg(z) \tag{3.17}$$

so that $f_0 = f$ and $f_1 = f + g$. Let n_t denote the number of zeroes of f_t inside the circle counted with multiplicities so that n_t is an integer. Clearly f_t has no zeroes on the circle, so then Theorem 3.22 implies

$$n_t = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f_t(z)}\, dz \tag{3.18}$$

We wish to prove that n_t is constant, so we simply need to show that it is a continuous function of t since if it were not, the intermediate value theorem would give us a point where n_t is not an integer. We observe that $\frac{f'(z)}{f_t(z)}$ is continuous for all $t \in [0, 1]$ and $z \in C$, and $f_t(z)$ never vanishes on C. Since n_t is both continuous and an integer, it must be constant, meaning $n_0 = n_1$ which is our theorem. ☐

Theorem 3.24: Hurwitz.
Suppose $\{f_n\}$ is a sequence of holomorphic functions on a domain Ω such that $f_n \to f$. If all f_n are non-vanishing on Ω, then

- *$f \equiv 0$ on Ω, or*
- *f is non-vanishing on Ω.*

Theorem 3.25: Open Mapping Theorem.
All non-constant holomorphic functions map open sets to open sets.

Proof. Let w_0 belong to the image of a non-constant holomorphic f, and say $w_0 = f(z_0)$. Define $g(z) = f(z - w)$ and write

$$g(z) = (f(z) - w_0) + (w_0 - w) = F(z) + G(z) \tag{3.19}$$

Now choose $\delta > 0$ such that the disk $|z - z_0| \leq \delta$ is contained in Ω and $f(z) \neq w$ on the circle $|z - z_0| = \delta$. We can then choose some $\varepsilon > 0$ such taht we have $|f(z) - w_0| \geq \varepsilon$ on the circle $|z - z_0| = \delta$. Now if $|w - w_0| < \varepsilon$ we have $|F(z) > |G(z)|$ on $|z - z_0| = \delta$, and by Theorem 3.23, we know that $g = F + G$ has a zero inside of the circle since F does. Therefore any point w near w_0 also belong to the image of f. □

Theorem 3.26: Maximum Modulus Principle.
If f is holomorphic and non-constant on a domain Ω, then $|f|$ foes not attain a local maximum in Ω.

Theorem 3.27: Maximum Principle.
If f is holomorphic on a bounded domain Ω and is continuous on $\overline{\Omega}$, then $|f|$ attains its maximum on $\partial\Omega$.

Theorem 3.28.
If Ω is open, simply connected, $1 \in \Omega$, and $0 \notin \Omega$, then there exists a branch of the logarithm $F(z) = \log_\Omega(z)$ such that

- ***(a)*** *F is holomorphic in Ω,*
- ***(b)*** *$e^{F(z)} = z$,*
- ***(c)*** *$F(z) = \ln(x)$ whenever $x \in \mathbb{R}$ and x is in some fixed neighborhood of 1,*
- ***(d)*** *$F'(z) = \frac{1}{z}$*

Theorem 3.29.
Let Ω be open and simply connected and f be a holomorphic function that is nonvanishing on Ω, then there exists some holomorphic $g(z)$ on Ω such that

$$f(z) = e^{g(z)} \tag{3.20}$$

for any $z \in \Omega$.

Theorem 3.30: Super Inverse Function Theorem.
Suppose f is holomorphic on a domain Ω_1 and 1-1 on Ω_1. Then $\Omega_2 = f(\Omega_1)$ is a domain and $f^{-1} : \Omega_2 \to \Omega_1$ is holomorphic. ALso f' is non-vanishing on Ω_1 and

$$\left(f^{-1}\right)'(w) = \frac{1}{f'\left(f^{-1}(w)\right)} \tag{3.21}$$

4 Harmonic Functions

Definition 4.1.
*A **harmonic function** is a function u where*

$$\Delta u = 0 \tag{4.1}$$

where Δ is the complex Laplacian.

Corollary 4.2.
A function being holomorphic implies that it is harmonic. A function being harmonic implies that it is analytic.

Corollary 4.3.
If u is harmonic on a domain Ω, then it is locally equivalent to a real part of a holomorphic function. If Ω is simply connected, then it is globally equivalent.

Corollary 4.4.
If a function $f = u + iv$ is harmonic, then both u and v are harmonic.

Definition 4.5.
*The **harmonic conjugate** of a function u is the function v that satisfies*

$$\Delta(u+iv) = 0 \tag{4.2}$$

Definition 4.6.
*A **trigonometric polynomial** is a finite sum of the form*

$$f(x) = \sum_{n=-N}^{N} c_n e^{inx} \tag{4.3}$$

where $c_n \in \mathbb{C}$.

Theorem 4.7.
The coefficients of the power series expansion of a function f which is holomorphic in a disk $D_R(z_0)$ are given by

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta}\, d\theta \tag{4.4}$$

for all $n \geq 0$ and $0 < r < R$. Moreover, $a_n = 0$ for all $n < 0$.

Proof. Since $f^{(n)}(z_0) = a_n n!$, Theorem 2.10 gives

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \tag{4.5}$$

for γ the circle of radius r centered at z_0 with positive orientation. We can choose $\zeta = z_0 + re^{i\theta}$, and see

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z_0 + re^{i\theta})}{((z_0 + re^{i\theta}) - z_0)^{n+1}} rie^{i\theta}\, d\theta \tag{4.6}$$

$$= \frac{1}{2\pi r^n} \int_\gamma f(z_0 + re^{i\theta}) e^{-i(n+1)\theta} e^{i\theta}\, d\theta \tag{4.7}$$

$$= \frac{1}{2\pi r^n} \int_\gamma f(z_0 + re^{i\theta}) e^{-in\theta}\, d\theta \tag{4.8}$$

When $n < 0$, the function $\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}$ is holomorphic in the disk, so by Theorem 2.7, it must be 0. ☐

Theorem 4.8: Holomorphic Mean Value Property.
If f is holomorphic in $D_r(z_0)$ and $r \in (0, R)$ then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\, d\theta \tag{4.9}$$

Proof. Use Theorem 4.7 with $n = 0$, so then $a_0 = f(z_0)$. ☐

Theorem 4.9: Harmonic Mean Value Property.
If u is harmonic in $D_r(z_0)$ and $r \in (0, R)$ then

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})\, d\theta \tag{4.10}$$

Proof. Since harmonic functions are the real part of a holomorphic function, define $u = \operatorname{Re} f$ and take the real part of both sides of (4.9). ☐

Theorem 4.10.
If u is harmonic in a neighborhood of $\overline{\mathbb{D}}$, then for any $r \in [0, 1)$ and $\theta \in \mathbb{R}$

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) P_r(\theta - t)\, dt \tag{4.11}$$

where

$$P_r(s) = \sum_{m=-\infty}^{\infty} r^{|m|} e^{ims} \tag{4.12}$$

is the Poisson kernel.

Proposition 4.11.
When $z = re^{i\theta}$,

(a)

$$P_r(\theta - t) = \operatorname{Re}\left(\frac{e^{it} + z}{e^{it} - z}\right) \tag{4.13}$$

(b)

$$P_r(t) = \frac{1 - r^2}{1 - 2r\cos t + r^2} \tag{4.14}$$

4.1 Dirichlet Problem

Given a continuous and real-valued function f on $\partial\mathbb{D}$, we want to find a real-valued u such that $\Delta u = 0$ in $\mathbb{D}$ and $u|_{\partial\mathbb{D}} = f$. This is a popular starting problem in the study of PDE.

Theorem 4.12: Existence.
The solution to the Dirichlet problem on the unit disk is given by

$$u(re^{i\theta}) = \begin{cases} f(e^{i\theta}) & r = 1 \\ P_r * f(\theta) & r \in [0, 1) \end{cases} \tag{4.15}$$

Theorem 4.13: Uniqueness.
The solution given in Theorem 4.12 is unique.

Theorem 4.14.
If u is continuous on Ω such that for all $z \in \Omega$ there exists some $\{r_n\}_{n\in\mathbb{N}}$ such that $r_n > 0$ and $\lim_{n\to\infty} r_n = 0$ and

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + r_n e^{it})\,dt \tag{4.16}$$

then u is harmonic in Ω.

5 Conformal Mapping

Definition 5.1.
*A holomorphic function that is locally injective is **conformal**. That is, it preserves angles.*

Lemma 5.2: Schwarz Lemma.
Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic and $f(0) = 0$. Then

- ***(a)*** $|f(z)| \leq |z|$
- ***(b)*** $|f'(0)| \leq 1$
- ***(c)*** *If equality holds in either case for $z \neq 0$, then f is a rotation.*

Proof. We can expand f in a power series centered at 0, seeing

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots \tag{5.1}$$

Since $f(0) = 0$, we have $a_0 = 0$ and therefore $f(z)/z$ is holomorphic in $\mathbb{D}$. If $|z| = r < 1$ then since $|f(z)| \leq 1$ we have

$$\left|\frac{f(z)}{z}\right| \leq \frac{1}{r} \tag{5.2}$$

by Theorem 3.26, we know that this is true whenever $|z| \leq r$. Letting $r \to 1$ gives us **(a)**. For **(b)**, observe that if $g(z) = f(z)/z$, then $|g(z)| \leq 1$ throughout $\mathbb{D}$ and

$$g(0) = \lim_{z\to 0}\frac{f(z) - f(0)}{z} = f'(0) \tag{5.3}$$

so then $|f'(0)| \leq 1$. Moreover, if $|f'(0)| = 1$, then $g(0) = 1$, and by Theorem 3.27, g is constant, meaning $f(z) = cz$ with $|c| = 1$. We know that $f(z)/z$ attains its maximum in the interior of $\mathbb{D}$ and must therefore be constant, say $f(z) = cz$. Evaluating this at z_0 we see that $|c| = 1$, meaning $c = e^{i\theta}$ for some θ meaning that f is a rotation and proving **(c)**. ☐

Definition 5.3.
The set of automorphisms is defined as

$$\mathrm{Aut}(\Omega) := \{f : \Omega \to \Omega : f \textit{ is biholomorphic}\} \tag{5.4}$$

Corollary 5.4.
$\mathrm{Aut}(\Omega)$ *is group under composition.*

Theorem 5.5.
The set of automorphisms of the disk are given by

$$\mathrm{Aut}(\mathbb{D}) = \{\lambda\varphi_a : a \in \mathbb{D}, |\lambda| = 1\} \tag{5.5}$$

where

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z} \tag{5.6}$$

Proposition 5.6.
$\mathrm{Aut}(\mathbb{D})$ *is transitive on* $\mathbb{D}$.

5.1 Linear Fractional Transformations

Definition 5.7.
*A **linear fractional transformation** (or Möbius transformation) is a conformal map of the form*

$$f(z) = \frac{az + b}{cz + d} \tag{5.7}$$

where $ad - bc \neq 0$ *and* $a, b, c, d \in \mathbb{C}$.

Proposition 5.8.
The class of linear fractional transformations are generated by

- ***(a)*** $z \mapsto rz$, *for real* $r > 0$
- ***(b)*** $z \mapsto e^{i\theta}z$ *for some* $\theta \in \mathbb{R}$
- ***(c)*** $z \mapsto z + b$ *for* $b \in \mathbb{C}$
- ***(d)*** $z \mapsto 1/z$

Corollary 5.9.
Linear fractional transformation map the set of all lines and circles to itself. Note that lines do not necessarily go to lines, nor circles to circles.

Proposition 5.10.
Linear fractional transformations map the extended complex plane onto itself and are biholomorphic.

A useful formula for concocting linear fractional transformations is to pick three points and map one to 0, one to ∞ and one to 1, which gives the following form

$$f(z) = \frac{c - b}{c - a}\frac{z - a}{z - b} \tag{5.8}$$

where $a \mapsto 0$, $b \mapsto \infty$ and $c \mapsto 1$.

Lemma 5.11.
A linear fractional transformation that fixes three points must be the identity.

Ω_1	Ω_2	$f(z)$
Unit Disk ($\mathbb{D}$)	Upper Half Plane ($\mathbb{H}_+$)	$\frac{iz+i}{-z+1}$
Horizontal Strip of width π	Upper half plane	e^z
Right Half Plane	Plane minus the negative reals	z^2
A sector with angle β from the positive reals	A sector with angle α from the positive reals	$z^{\alpha/\beta}$

Figure 1: Table of useful conformal maps.

Corollary 5.12.
If two linear fractional transformations agree at three points, then they are equivalent.

Since linear fractional transformations are transitive, it is useful to use a composition of maps when concocting a map from one domain to another. A short list of useful linear fractional transformations $f : \Omega_1 \to \Omega_2$ can be found below in Figure 1.

Definition 5.13.
*An **exhaustion** is a sequence of compact sets $\{K_n\} \subseteq \Omega$ such that*

- ***(a)*** $K_n \subseteq \text{int}(K_{n+1})$
- ***(b)*** *Any compact $K \subseteq \Omega$ satisfies $K \subseteq K_N$ for some $N \in \mathbb{N}$*

Lemma 5.14.
Any open set $\Omega \subseteq \mathbb{C}$ has an exhaustion.

Theorem 5.15: Riemann Mapping Theorem.
Let $\Omega \subset \mathbb{C}$ be open and simply connected. For any point $z_0 \in \Omega$ there exists a unique biholomorphic map $f : \Omega \to \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.

6 Normal Families

Definition 6.1.
*Let $\mathcal{F}$ be a family of holomorphic functions on an open set Ω. $\mathcal{F}$ is **normal** if every sequence in $\mathcal{F}$ has a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ for some holomorphic function f.*

Definition 6.2.
*A set of holomorphic functions $\mathcal{F}$ is **equicontinuous** if for all compact $K \subset \Omega$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $|z - w| < \delta$ implies that $|f(z) - f(w)| < \varepsilon$ for all $f \in \mathcal{F}$.*

Definition 6.3.
*A family $\mathcal{F}$ is **uniformly bounded** if for all compact $K \subseteq \Omega$ there is a $B > 0$ such that $|f(z)| \leq B$ for all $z \in K$ and all $f \in \mathcal{F}$.*

Theorem 6.4: Montel.
If a family of functions is uniformly bounded, then it is equicontinuous and normal.

Part II
Past Qualifying Exams

These are problems from past Purdue qualifying exams which can be found here: `https://www.math.purdue.edu/academic/grad/qualexams.html`. Note that all solutions are either mine or sourced from other resources, so the validity of them should not be assumed. I have tried to reference relevant theorems in the notes section above, but some results may be assumed as true when they were proved in class or homework.

Exam 1: January 2024 - Bell

Problem 1.1

Calculate

$$\int_0^\infty \frac{1}{x^n+1}\,dx \tag{1.1.1}$$

for positive integers $n \geq 2$ by integrating a complex function around the closed contour that follows the real axis from the origin to $R > 0$, then follows the circular arc $Re^i\theta$ as θ ranges from zero to $2\pi/n$, then returns to the origin via the line segment joining $Re^{2\pi i/n}$ to the origin, and let $R \to \infty$. Show all your calculations and explain all limits.

Solution to Problem 1.1:

□

Problem 1.2

Describe the image of the half-strip $\{z = x + iy : -1 < x < 1, 0 < y < \infty\}$ under the mapping $f(z) = \frac{z-1}{z+1}$.

Solution to Problem 1.2:

□

Problem 1.3

(a) Prove that $f(z) = \frac{1}{z}$ does not have a complex antiderivative in $\mathbb{C} \setminus \{0\}$.

(b) Find all integers n such that the function $g(z) = z^n e^{1/z}$ has a complex antiderivative in $\mathbb{C} \setminus \{0\}$.

Solution to Problem 1.3:

(a) If f was to have an antiderivative, then there must exist a primitive F such that $F' = f$ on $\mathbb{C} \setminus \{0\}$. For the sake of contradiction, suppose f indeed has an antiderivative F. Take some parametrization γ with endpoints $a, b \in \mathbb{C} \setminus \{0\}$. Then by the complex Fundamental Theorem of Calculus, we have

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)) \tag{1.3.1}$$

Which we can let $\gamma : [0, 2\pi] \to \mathbb{C} \setminus \{0\}$ such that $t \mapsto e^{it}$, we then see

$$\int_\gamma f(z)\,dz = F(\gamma(2\pi)) - F(\gamma(0)) = 0 \tag{1.3.2}$$

from the Complex FTC. But from Cauchy's Integral formula, we see

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} e^{-it}\left(ie^{it}\right)\,dt \tag{1.3.3}$$

$$= \int_0^{2\pi} i\,dt = 2\pi i \tag{1.3.4}$$

which is a contradiction. Thus, $f(z) = 1/z$ cannot have an antiderivative.

(b) We can use a power series expansion to see that

$$e^{1/z} = \sum_{k=1}^{\infty} \frac{1}{k!}\frac{1}{z^k} \tag{1.3.5}$$

So then

$$g(z) = z^n \sum_{k=1}^{\infty} \frac{1}{k!}\frac{1}{z^k} \tag{1.3.6}$$

$$= \sum_{k=1}^{\infty} \frac{z^{n-k}}{k!} \tag{1.3.7}$$

Now we will use the residue theorem to reason that the residue is zero for $z = 0$ if and only if it has an antiderivative on $\mathbb{C} \setminus \{0\}$. The residue of $g(z)$ at 0 is given by the $k = -1$ term in the power series expansion. This happens when $n - k = -1$, so $k = n + 1$. Then the residue at $z = 0$ is given by $\frac{1}{(n+1)!}$ which $n + 1 \geq 0$ and 0 otherwise. Thus, for the residue to be zero, we need $n < -1$. Thus, $g(z)$ has an antiderivative for all $n < -1$.

□

Problem 1.4

Let f be an analytic function with a zero of order 2 at z_0. Prove that there exists $\varepsilon > 0$ and $\delta > 0$ such that for every w in $D_\varepsilon(0) \setminus \{0\}$, the equation $f(z) = w$ has exactly 2 distinct roots in the set $D_\delta(z_0) \setminus \{z_0\}$.

Solution to Problem 1.4: Since f has a zero of order 2 at z_0, we can express it as

$$f(z) = w + (z - z_0)^2 g(z) \tag{1.4.1}$$

for some $g(z)$ such that $g(z_0) \neq 0$. □

Problem 1.5

Prove that there is no analytic function that maps the punctured disk $\{z \in \mathbb{C} : 0 < |z| < 1\}$ one-to-one onto the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

Solution to Problem 1.5:

□

Exam 2: August 2024 - Eremenko

Problem 2.1

Let f be an entire function which takes real values on the real and imaginary axes. Prove that f is even.

Solution to Problem 2.1:

□

Problem 2.2

Find the residue

$$\operatorname{Res}_{z=0} \frac{1}{(e^z - 1)^2} \tag{2.2.1}$$

Solution to Problem 2.2:

□

Problem 2.3

Consider the series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{2.3.1}$$

where $a_0 \neq 0$ and $a_n = a_{n-1} - 2a_{n-2}$ for $n \geq 2$. Find the radius of convergence of this series.

Solution to Problem 2.3:

□

Problem 2.4

Evaluate the integral

$$\int_{|z|=2} \frac{z^4}{z^5 + 15z + z}, dz \tag{2.4.1}$$

Where the circle is parametrized counterclockwise.

Solution to Problem 2.4:

□

Problem 2.5

Consider the polynomial

$$f(z) = z + z^2/2 \tag{2.5.1}$$

(a) Prove that f is injective in the unit disk $U = \{z : |z| < 1\}$.

(b) Find the area of the image $f(U)$

Solution to Problem 2.5:

(a)

(b)

□

Problem 2.6

Find all solutions of the equation

$$\tan z = 2i \tag{2.6.1}$$

and make a picture of them

Solution to Problem 2.6:

□

Problem 2.7

Let $f = u + iv$ be a non-constant analytic function in some region where u and v are real valued harmonic functions. Is it possible that $u = F \circ v$ where F is some continuously differentiable function mapping the real line onto itself? If yes, give an example, if no, give a proof.

Solution to Problem 2.7:

□

Exam 3: August 2023 - Datchev

Problem 3.1

Let f be an unbounded entire function and $\Omega \subset \mathbb{C}$ be a nonempty open set. Show that there exists $p \in \mathbb{C}$ such that $f(p) \in \Omega$.

Solution to Problem 3.1:

□

Problem 3.2

Let $u : \mathbb{C} \to \mathbb{R}$ be a harmonic function. Prove that u is either surjective or constant.

Solution to Problem 3.2:

□

Problem 3.3

Find all entire functions f such that $|f(z)| \leq |z|$ for all z and $f(i) = 1$.

Solution to Problem 3.3:

□

Problem 3.4

Evaluate

$$\int_\gamma f(z)\, dz \tag{3.4.1}$$

where $f(z) = \tan((1+i)z)$ and γ is the circle $|z| = 2$ oriented clockwise.

Solution to Problem 3.4:

□

Problem 3.5

Let $\Omega = \mathbb{C} \setminus \{z \in \mathbb{R} : z \leq 0\}$. Find a bijective holomorphic function $f : \Omega \to \Omega$ such that $f(1) = i$.

Solution to Problem 3.5: We know that linear fractional transformations of the form

$$z \mapsto \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \tag{3.5.1}$$

are biholomorphic, so we are looking for a transformation that maps the plane minus the negative real line to itself such that $1 \to i$. We can choose 2 other points to determine the value, so we choose

$$1 \mapsto i \tag{3.5.2}$$
$$2i \mapsto \infty \tag{3.5.3}$$
$$0 \mapsto 0 \tag{3.5.4}$$

So then

$$\frac{i-2i}{i-0}\frac{z-0}{z-2i} \tag{3.5.5}$$

So then the inverse of this

$$\frac{-z}{z-2i} = w \implies wz - 2iw = -z \tag{3.5.6}$$
$$(w+1)z - 2iw = 0 \tag{3.5.7}$$
$$z = \frac{2iw}{w+1} \tag{3.5.8}$$
$$\tag{3.5.9}$$

□

Problem 3.6

Let $f(z) = z^{1000} + z^{100} + z^{10} + 1$. Find an $R > 0$ such that if $f(z) = 0$ then $R < |z| < R + 1$.

Solution to Problem 3.6:

□

Problem 3.7

Let $\Omega \subset \mathbb{C}$ be a nonempty set, and let $f : \Omega \to \mathbb{C}$ be holomorphic. Suppose that for every $z \in \Omega$ there is a positive integer n such that $f^{(n!)}(z) = 0$. Prove that f is a polynomial.

Solution to Problem 3.7:

□

Exam 4: January 2023 - Lempert

Problem 4.1

Compute

$$\int_{|z|=2} \frac{e^{iz}\,dz}{4z^2-\pi^2} \tag{4.1.1}$$

where the path of integration is oriented counterclockwise.

Solution to Problem 4.1:

□

Problem 4.2

For a natural number n, let T_n denote the polynomial

$$T_n(z) = 1 - \frac{z^2}{3} + \frac{z^4}{5!} + \cdots + (-1)^n \frac{z^{2n}}{(2n+1)!} \tag{4.2.1}$$

Prove that there is no such n_0 such that T_n has exactly 6 roots in the disk $\{z \in \mathbb{C} : |z| < 10\}$ when $n > n_0$.

Solution to Problem 4.2:

□

Problem 4.3

If $\cos z = \cos w$ for some complex numbers z, w, prove that there is an integer k such that $z = w + 2k\pi i$ or $z = -w + 2k\pi i$.

Solution to Problem 4.3: Note that $\cos z = \frac{e^{iz}+e^{-iz}}{2}$. Trivially, this is true when $k = 0$. So then since $\cos z = \cos w$, we have

$$\frac{e^{iz}+e^{-iz}}{2} = \frac{e^{iw}+e^{-iw}}{2} \tag{4.3.1}$$

Note that for any $k \in \mathbb{Z}$, $e^{2\pi ki} = 1$, so $\cos z = e^{2\pi ki}\cos z$. So then

$$e^{2\pi ki}\frac{e^{iz}+e^{-iz}}{2} = \frac{e^{iz+2\pi ki}+e^{-iz+2\pi i}}{2} = \cos(z+2\pi ki) = \cos(w) \tag{4.3.2}$$

□

Problem 4.4

Is there a harmonic function $u : \mathbb{D} \to \mathbb{R}$ such that $\lim_{z\to\zeta} u(z) = \infty$ for every $\zeta \in \partial D$.

Solution to Problem 4.4: □

Problem 4.5

Let $Q = \{z \in \mathbb{C} : \operatorname{Re} z, \operatorname{Im} z > 0\}$ stand for the first quadrant. Find a biholomorphic map $F : Q \to Q$ such that $F(2+i) = 1+2i$.

Solution to Problem 4.5: $\square$

Problem 4.6

Suppose g is a holomorphic function on some domain, and $1/\overline{g}$ is also holomorphic there. Prove that g is constant.

Solution to Problem 4.6: Since $\frac{1}{\overline{g}}$ is holomorphic, $\frac{\partial}{\partial \overline{z}} \frac{1}{\overline{g}} = 0$. Now, writing the derivative out explicitly we see

$$0 = \frac{\partial}{\partial \overline{z}} \frac{1}{\overline{g}} = -\frac{\partial \overline{g}}{\partial z} \frac{1}{(\overline{g})^2} \tag{4.6.1}$$

In order for this to be true we need $\frac{\partial \overline{g}}{\partial \overline{z}} \equiv 0$, meaning that $\overline{g}$ is holomorphic. Since both g and $\overline{g}$ are holomorphic, g must be constant. This claim is easily proved by the Cauchy-Riemann equations. We see for $g = u + iv$, $\overline{g} = u - iv$ if both are holomorphic we have the following four CR equations

$$u_x = v_y, \quad u_y = -v_x \tag{4.6.2}$$
$$u_x = -v_y, \quad u_y = v_x \tag{4.6.3}$$
$$\implies u_y = -v_x = v_x, \quad u_x = v_y = -v_y \tag{4.6.4}$$

And the only way for all of these to be satisfied are if u, v are constant. Thus, we are finished and g is constant. $\square$

Exam 5: August 2022 - Bell

Problem 5.1

Solution to Problem 5.1:

□

Problem 5.2

Solution to Problem 5.2:

□

Problem 5.3

Solution to Problem 5.3:

□

Problem 5.4

Solution to Problem 5.4:

□

Problem 5.5

Solution to Problem 5.5:

□

Problem 5.6

Solution to Problem 5.6:

□

Problem 5.7

Suppose that u us a non-constant real valued harmonic function on the complex plane such that $u(0) = 0$. Prove that there is at least one point on each circle centered at the origin where u vanishes.

Solution to Problem 5.7: Suppose for the sake of contradiction that there is not a point in which u vanishes
$\square$

Exam 6: August 2021 - Lempert

Problem 6.1

Compute

$$\int_{-\infty}^{\infty} \frac{e^{3ix} - 3e^{ix} + 2}{x^2}\, dx \tag{6.1.1}$$

Solution to Problem 6.1:

□

Problem 6.2: C

nstruct a biholomorphic function between the strip and

Solution to Problem 6.2:

□

Problem 6.3

If φ is a holomorphic function on D that vanishes at 0, prove that there is no holomorphic function ψ on $\mathbb{D} \setminus \{0\}$ such that $\varphi = e^{\psi}$ on $\mathbb{D} \setminus \{0\}$.

Solution to Problem 6.3: For the sake of contradiction, suppose there is such a holomorphic ψ. Then by the argument principle

$$\frac{1}{2\pi i} \int_{C_r} \frac{\varphi'(z)}{\varphi(z)}\, dz = 1 \tag{6.3.1}$$

Where C_r is the closed loop with radius $r \to 0$. If ψ is indeed $\varphi = e^{\psi}$, then

$$\frac{1}{2\pi i} \int_C \frac{\psi' e^{\psi}}{e^{\psi}}\, dz = \frac{1}{2\pi i} \int_C \psi'\, dz = 1 \tag{6.3.2}$$

Since ψ is holomorphic, so it ψ', so then its integral is 0, a contradiction with our previous use of the argument principle. Thus, there is no holomoprhic ψ such that $\varphi = e^{\psi}$. □

Exam 7: January 2020 - Bell & Lempert

Problem 7.1

Solution to Problem 7.1:

(a) We begin by expressing

$$I = \int_0^\infty \frac{\ln z}{z^3+1}\,dz - e^{2\pi i/3}\int_0^\infty \frac{\ln z}{z^3+1} + \frac{2\pi I/3}{z^3+1}\,dz \tag{7.1.1}$$

(b)

(c) The residue is given by

$$\operatorname{Res}_{z=e^{i\pi/3}} f(z) = \lim_{z\to e^{i\pi/3}} (z - e^{i\pi/3})\frac{\log z}{z^3-1} \tag{7.1.2}$$

$$= \lim_{z\to e^{i\pi/3}} \frac{\log z + \frac{1}{z}z - \frac{e^{i\pi/3}}{z}}{3z^2} \tag{7.1.3}$$

$$= \frac{\pi e^{5\pi i/6}}{9} \tag{7.1.4}$$

(d)

□

Exam 8: August 2017 - Bell

Problem 8.1

Suppose the power series centered about zero for an entire function converges uniformly on the whole complex plane. What can you say about the entire function? Explain.

Solution to Problem 8.1: Define our dunction to be

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{8.1.1}$$

Then for any $\varepsilon > 0$ we have some $N \in \mathbb{N}$ such that if $k > N$ then

$$\left| f(z) - \sum_{n=0}^{k} a_n z^n \right| < \varepsilon \tag{8.1.2}$$

for any $z \in \mathbb{C}$. Then by Liouville's Theorem (Theorem 2.13), we know that $f(z) - \sum_{n=0}^{N} \equiv const$ for $z \in \mathbb{C}$, which means it must be $f(z)$ must be a polynomial. $\square$

Problem 8.2

Suppose that $u(z, s)$ is a continuous real values function on $\mathbb{C} \times \mathbb{R}$ usch that $u(z, s)$ is harmonic in z for each fixed s. Define

$$U(z) = \int_{-1}^{1} u(z, s)\, ds \tag{8.2.1}$$

(a) Give an $\varepsilon - \delta$ proof that U is continuous on $\mathbb{C}$.

(b) Prove that U is harmonic on $\mathbb{C}$ without taking derivatives.

Solution to Problem 8.2:

(a) Consider some $z_0 \in \mathbb{C}$ and $s \in [-1, 1]$ and pick some $\varepsilon > 0$. We know that u is uniformly continuous on the compact set $\overline{\mathbb{D}(z_0)} \times [-1, 1]$. By definition, this means that there exists some $\delta > 0$ such that if $(z, s) \in \overline{\mathbb{D}(z_0)} \times [-1, 1]$ and

$$|(z, s) - (z_0, s)| = \sqrt{|z - z_0|^2 + |s - s_0|^2} < \delta \tag{8.2.2}$$

then

$$|u(z, s) - u(z_0, s_0)| < \varepsilon/2 \tag{8.2.3}$$

We can pick $|z - z_0| < \delta$ and $s = s_0$, which means that if $|z - z_0| < \min(1, \delta)$ then

$$|U(z) - U(z_0)| = \left| \int_{-1}^{1} (u(z, s) - u(z_0, s))\, ds \right| \tag{8.2.4}$$

$$\leq \left| \int_{-1}^{1} |u(z, s) - u(z_0, s)|\, ds \right| \tag{8.2.5}$$

$$\leq \int_{-1}^{1} \varepsilon/2\, ds = \varepsilon \tag{8.2.6}$$

(b) Since u is harmonic in z for each fixed $s_0 \in \mathbb{R}$, it satisfies the Mean Value Property in Theorem 4.9 in

$\mathbb{C}$. meaning for any $s_0 \in \mathbb{R}$ and $r > 0$ we have

$$u(z, s_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}, s_0)\, d\theta \tag{8.2.7}$$

for any $z \in \mathbb{C}$. We then see

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} U(z + re^{i\theta}, s_0)\, d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^{1} u(z + re^{i\theta}, s_0)\, ds\, d\theta & (8.2.8)\\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^{1} u(z + re^{i\theta}, s_0)\, d\theta\, ds & (8.2.9)\\
&= \frac{1}{2\pi} (u_z, s)\, ds & (8.2.10)\\
&= U(z) & (8.2.11)
\end{aligned}$$

Which means that $U(z)$ is harmonic.

□

Problem 8.3

Suppose that $f(z)$ is an entire function such that $f(z + \pi) = f(z)$ for all z and $f(z + i\pi) = f(z)$ for all z. Prove that f must be a constant function.

Solution to Problem 8.3: Consider the square S with side pi with a vertex at 0 in lower right quadrant. The function f is continuous, so for any $z \in S$, we have that $|f(z)| \leq M$ for some $M \geq 0$. But since $f(S) = F(\mathbb{C})$ since the function repeats itself for every square of side π, then $|f(z)| \leq M$ for all z, meaning by Liouville's Theorem (Theorem 2.13), f is constant. □

Problem 8.4

Suppose that $R(z) = P(z)/Q(z)$ where P and Q are complex polynomials and the degree of $Q(z)$ is at least two greater that the degree of $P(z)$, Show that the sum of the residues of $R(z)$ in complex plane must be zero.

Solution to Problem 8.4: We know that there is some disk of radius $M > 0$ such that all of the roots $z_1, \ldots, z_n$ of $Q(z)$ are contained in $D_M(0)$. So then by the residue theorem, for $r \geq M$ we have

$$\begin{aligned}
2\pi i \sum_{n=1}^{N} \operatorname{Res}_{z_i} f(z) &= \left| \int_{C_r(0)} R(z)\, dz \right| & (8.4.1)\\
&\leq 2\pi r \max_{|z|=r} |R(x)| & (8.4.2)\\
&= 2\pi r \max_{|z|=r} \frac{a_n z^n + \cdots + a_0|}{|b_{n+2+k} z^{n+2+k} + \cdots + b_0|} & (8.4.3)\\
&\approx 2\pi r \max_{|z|=r} \left| \frac{a_n z^n}{b_{n+2+k} z^{n+2+k}} \right| & (8.4.4)\\
&= \frac{2\pi}{r^{1+k}} \frac{a_n}{b_{n+2+k}} \to 0 & (8.4.5)
\end{aligned}$$

as $r \to \infty$, so we are finished. □

Problem 8.5

Show that the family of one-to-one conformal mapping of the horizontal strip $\Omega = \{z : 0 < \text{Im } < 1\}$ onto itself such that given any two points z_1 and z_2 in the strip, there is a mapping in the family that maps z_1 to z_2.

Solution to Problem 8.5: We first note that the map $z \to e^{\pi z}$ takes Ω to the upper half plane, and then the Cayley transform $f(z) = \frac{-z+i}{-z-i}$ take the upper half plane to the unit disk, and they both have holomorphic inverses. Since we can construct a biholomorphic map $F : \Omega \to \mathbb{D}$, we can use the automorphisms of the disk that takes 0 to a,

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z} \tag{8.5.1}$$

and compose this with F to swap any two points. Let $w_1 = F(z_1)$ and $w_2 = F(z_2)$. Then we can define $\psi(z) = F^{-1} \circ \varphi_{w_1} \circ \varphi_{w_2} \circ F(z)$ to be our map that swaps z_1 and z_2. $\square$

Problem 8.6

Explain why

$$\frac{\sin z^2}{(z-1)(z+1)} \tag{8.6.1}$$

has an analytic antiderivative on $\mathbb{C} \setminus [-1, 1]$.

Solution to Problem 8.6: Pick some simple closed curve γin $\mathbb{C} \setminus [-1, 1]$. Then the line $[-1, 1]$ is either in the interior of γ or the complement of its interior. In the latter case, by Theorem 2.9, we know that

$$\int_\gamma f(z)\, dz = 0 \tag{8.6.2}$$

In the former case, we have by the Residue Theorem

$$\int_\gamma f(z)\, dz = 2\pi i \left(\text{Res}_{\,1} f + \text{Res}_{\,-1} f\right) \tag{8.6.3}$$

Since both poles are simple, we can directly compute the residues with the limit formula. In either case, the function has an antiderivative. $\square$

Problem 8.7

Compute

$$\int_\gamma \frac{\sin z}{z^{10}}\, dz \tag{8.7.1}$$

where γ denotes an ellipse with one focus at the origin parameterized in the clockwise direction.

Solution to Problem 8.7: We can use power series expansion of $f(z) = \frac{\sin z}{z^{10}}$ to see

$$f(z) = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \dots}{z^{10}} \tag{8.7.2}$$

$$= \frac{1}{z^9} - \frac{1}{3!z^7} + \frac{1}{5!z^5} - \frac{1}{7!z^3} + \frac{1}{9!z} - \dots \tag{8.7.3}$$

So then $\operatorname{Res}_0 f = \frac{1}{9!}$. Since the only singularity of f is at 0, we then see that

$$\int_\gamma \frac{\sin z}{z^{10}}\, dz = 2\pi i \operatorname{Res}_0 f = \frac{-2\pi i}{9!} \tag{8.7.4}$$

□

Problem 8.8

Prove that every harmonic function on a simply connected domain Ω can be expressed as $u(z) = \ln|f(z)|$ where $f(z)$ is a nonvanishing analytic function on Ω. Is the function $f(z)$ unique? Explain.

Solution to Problem 8.8: Let v be a harmonic conjugate of u on Ω. Then

$$g(z) = u(z) + iv(z) \tag{8.8.1}$$

is holomorphic on Ω. Let $f(z) = e^{g(z)}$ on Ω. Then we see

$$\ln|f(z)| = \ln|e^{u(z)+iv(z)}| \tag{8.8.2}$$

$$= \ln|e^{u(z)}e^{iv(z)}| \tag{8.8.3}$$

$$= \ln(e^{u(z)}) = u(z) \tag{8.8.4}$$

This function is not unique since any $g(z) + ik$ for any constant k also works. □