MATH 530: Complex Analysis Qualifying Exam Prep

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Contents

Part I
Notes

1 Preliminaries

1.1 Continuity in the Complex Plane

Much like in $\mathbb{R}$, we wish to describe the continuity of functions, and we can defined them in two ways:

Definition 1.1 (Continuity at a point). *For a point $z_0 \in S \subset \mathbb{C}$, we say a function f is continuous if*

$$\lim_{z \to z_0} f(z) = f(z_0) \tag{1.1}$$

Definition 1.2 (Continuity on a set). *For a set $S \subset \mathbb{C}$, we say a function f is continuous on S if f is continuous for all $z \in S$.*

We can also divide a function f into its real and imaginary parts in the following ways:

$$f(z) = u(z) + iv(z) \tag{1.2}$$
$$f(x, y) = u(x, y) + iv(x, y) \tag{1.3}$$
$$f(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} \tag{1.4}$$

1.2 Differentiation in the Complex Plane

Definition 1.3 (Complex Differentiability). *Let $\Omega \subseteq \mathbb{C}$ be open, and $f : \Omega \to \mathbb{C}$. If f is complex differentiable at $z_0 \in \Omega$, this means that*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1.5}$$

exists.

Theorem 1.1. *If f is differentiable at z_0 then f is continuous at z_0 and*

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + R(z)(z - z_0) \tag{1.6}$$

With

$$\lim_{z \to z_0} R(z) = 0 \tag{1.7}$$

Proof. □

Theorem 1.2. *If $f = u + iv$ is holomorphic, then it satisfies the Cauchy-Riemann equations:*

$$u_x = v_y \tag{1.8}$$
$$u_y = -v_x \tag{1.9}$$

Corollary 1.2.1. *If $f = u + iv$ is holomorphic, then*

$$f' = u_x + iv_x = v_y - iu_y \tag{1.10}$$

Theorem 1.3. *Suppose u and v are C^1 on an open set $\Omega \subset \mathbb{C}$. If u, v satisfy the Cauchy-Riemann equations, then $f = u + iv$ is holomorphic.*

Proof. □

Definition 1.4. *Define the following derivatives with respect to a complex variable:*

$$\frac{\partial}{\partial z} = \frac{1}{z} \left(\frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right) \tag{1.11}$$
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{z} \left(\frac{\partial}{\partial x} - \frac{1}{i} \frac{\partial}{\partial y} \right) \tag{1.12}$$

Proposition 1.1. *If f is holomorphic at z_0 then*

$$\frac{\partial f}{\partial z}(z_0) = f'(z_0) \tag{1.13}$$
$$\frac{\partial f}{\partial \overline{z}}(z_0) = 0 \tag{1.14}$$

Also if $F(x, y) = f(x + iy)$ then

$$\det J_f(x_0, y_0) = |f'(z_0)|^2 \tag{1.15}$$

Proof. Check using Cauchy-Riemann equations. □

1.3 Power Series

Recall from real analysis that polynomials in z are holomorphic by the real sum and product ruls, so if we define a polynomial by

$$P(z) = \sum_{n=0}^{N} a_n (z - z_0)^n \tag{1.16}$$

then we can approximate a function f by a power series given by

$$f(x) \approx \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{1.17}$$

Definition 1.5. *Let $\{r_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$. Then*

$$\limsup_{n\to\infty} r_n = \lim_{n\to\infty} \left(\sup\{r_k : k > n\}\right) \tag{1.18}$$

Note that this limit always exists since the supremum is nonincreasing.

Theorem 1.4 (Absolute Convergence of Power Series)**.** *Given $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, let*

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} \tag{1.19}$$

(a) If $|z - z_0| < R$, the series converges absolutely.

(b) If $|z - z| > R$ the series diverges.

(c) If $|z - z_0| = R$, there is no way of knowing if the series converges or diverges.

Definition 1.6. *R defined in (1.19) is called the radius of convergence and $D_R(z_0)$ is the disk of convergence.*

Theorem 1.5 (Uniform Convergence of Power Series)**.** *Assume $R > 0$ as in Theorem 1.4, and choose some $r \in (0, R)$. Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.20}$$

in $D_R(0)$. Then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely and uniformly in $D_r(0)$ to f.

Proposition 1.2 (Ratio Test)**.** *Let a_n be coefficients of a power series. If*

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L \tag{1.21}$$

then

$$\lim_{n\to\infty} |a_n^{1/n} = L \tag{1.22}$$

Using power series, we can define certain properties of complex exponentials, seeing

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{1.23}$$

where the radius of convergence is $R = 1/0 = \infty$, so it converges everywhere. All of the usual properties of the real exponential function apply to the complex exponential, which can be proven by the use of power series.

Theorem 1.6. *The function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in its disk of convergence $D_R(0)$. Also $f'(z)$ is a power series with the same disk of convergence and*

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1} = g(z) \tag{1.24}$$

Proof. We need to show that $g(z)$ has the same disk of convergence $D_R(0)$. Let $\varepsilon > 0$ and

$$S_N(z) = \sum_{n=0}^{N} a_n z^n$$
$$\Sigma_N = \sum_{n=N+1}^{\infty} a_n z^n$$

Choose some $z_0 \in D_r(0)$ and $r > 0$ such that $|z_0| < r < R$ and h such that $|z_0 + h| < r$. Then

$$\begin{aligned}\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) &= \left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)\right) + (S_N'(z_0) - g(z_0)) + \left(\frac{\Sigma_N(z_0+h) - \Sigma_N(z_0)}{h}\right) \\ &= (*) + (**) + (***)\end{aligned}$$

Then we can bound each part to show that the limit goes to zero. For $(***)$ we see that

$$(***) \leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1}$$

So then choose some M that for all $N \geq M$ that $|(***)| < \varepsilon$. We know that $S_N'(z_0) \to g(z_0)$ as $N \to \infty$, so we can choose M_2 such that $|(**)| < \varepsilon$ for all $N > M_2$. Then pick $M = \max(M, M_2)$ and there exists some δ such that $|h| < \delta$ implies $|(*)| < \varepsilon$, so $|(*) + (**) + (***)| < \varepsilon$ so then the limit goes to zero and we are finished. □

Corollary 1.6.1. *A power series is infinitely differentiable in its disk of convergence and the derivatives are also power series with the same disk of convergence and given by term-by-term differentiation.*

Proof. Apply Theorem 1.6 repeatedly. □

2 Complex Integration

Much like with integration in $\mathbb{R}^2$, integration over $\mathbb{C}$ also has a notion of a path integral. We begin by defining the length of a contour as follow:

Definition 2.1. *Let $z : [a, b] \to \mathbb{C}$ be a piecewise C^1 parametrization of γ. Then*

$$\text{length}(\gamma) = \sum_{a}^{b} |z'(t)|\, dt \tag{2.1}$$

We also have the same estimate as we do in $\mathbb{R}^2$, seeing

Proposition 2.1. *Let $z : [a, b] \to \mathbb{C}$ be a piecewise C^1 parametrization of γ, then we have the following estimate:*

$$\int_\gamma f(z)\, dz \leq \max_{z \in \gamma} |f(z)| \cdot \int_a^b |z'(t)|\, dt \tag{2.2}$$

Now that we can define an integral over $\mathbb{C}$, it is natural to extend the notion of the fundamental theorem of calculus.

Theorem 2.1 (Fundamental Theorem of Calculus)**.** *If f is continuous and has a primitive F in some domain $\Omega \subset \mathbb{C}$ and $z : [a, b] \to \mathbb{C}$ is a parametrization of a curve γ, then we see*

$$\int_\gamma f(z)\, dz = F(b) - F(a) \tag{2.3}$$

Corollary 2.1.1. *If f is holomorphic in a region Ω and $f' \equiv 0$ on Ω, then $f \equiv const.$*

2.1 Cauchy's Theorem and the Cauchy Integral Formula

Theorem 2.2 (Goursat's Theorem)**.** *If you have a triangle T with its interior contained inside an open set Ω and f is holomorphic on Ω, then*

$$\int_T f(z)\, dz = 0 \tag{2.4}$$

From here, we can generalize to

Theorem 2.3 (Cauchy's Theorem on a Convex Open Set)**.** *If Ω is a convex open set and f is continuous on Ω, and f is analytic on $\Omega \setminus \{p\}$ for any $p \in \Omega$, then*

$$\int_\gamma f(z)\, dz = 0 \tag{2.5}$$

for any closed curve γ such that the trace of γ is in Ω.

Lemma 2.1. *If f is holomorphic on a convex open set Ω, then it has a holomorphic antiderivative given by*

$$F(z) = \int_{L_z^z} f(w)\, dw \tag{2.6}$$

Where L_a^z is the line from any $a \in \Omega \setminus \{z\}$ to z.

This allows us to use the notion of so-called "toy contours". These are closed curves with an interior and we can join any two points in the interior of a simply connected domain with finitely many straight lines. These toy contours will be useful when combined with other theorems about integrals over specific contours, namely the Cauchy Integral formula.

Theorem 2.4 (Cauchy Integral Formula)**.** *If f is analytic on an open set Ω, with some fixed $a \in D_r(z_0) \subseteq \overline{D_r(z_0)} \subseteq \Omega$ then*

$$f(a) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{z - a}\, dz \tag{2.7}$$

From here, we arrive at a way to define the nth derivative of f using Theorem 2.4.

Corollary 2.4.1.

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{(z - a)^{n-1}}\, dz \tag{2.8}$$

This also means that if f is analytic, then f' is also analytic.

Corollary 2.4.2 (Cauchy Estimate)**.** *Using Proposition 2.1, we arrive at*

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \max_{z \in C_r(z_0)} |f(z)| \tag{2.9}$$

Theorem 2.5. *If f is holomorphic on an open set Ω and $\overline{D_r(z_0)} \subseteq \Omega$, the for all $z \in D_r(z_0)$ we have*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \tag{2.10}$$

Theorem 2.6 (Liouville)**.** *If a function f is entire and bounded, then f is constant.*

Theorem 2.7 (Fundamental Theorem of Algebra)**.** *If $P(z) = a_n z^n + \cdots + a_0$ is a polynomial in $\mathbb{C}$ of degree $n \geq 1$, then p has exactly n roots with multiplicity and*

$$P(z) = a_n (z - w_1)^{k_1} \ldots (z - w_n)^{k_n} \tag{2.11}$$

where w_n are the roots of P and $\sum_i k_i = n$.

Theorem 2.8 (Identity Theorem)**.** *if f is holomorphic on a domain Ω and*

$$z_f = \{z \in \Omega | f(z) = 0\} \tag{2.12}$$

then if $z_f = \Omega$ or z_f has no limit points in Ω.

Corollary 2.8.1. *Holomorphic functions that agree on a subset with a limit pt in a domain agree on that domain.*

Theorem 2.9. *If f is holomorphic and $f' \equiv 0$ on a domain then f is constant.*

Theorem 2.10 (Morera)**.** *If f is continuous on an open set Ω and for all triangles T such that the interior and boundary are in Ωm then*

$$\int_T f(z)\, dz = 0 \tag{2.13}$$

implies that f is holomorphic on Ω.

Part II
Past Quals

Exam 1: January 2024 - Bell

Problem 1.1: Contour Integration

Calculate

$$\int_0^\infty \frac{1}{x^n+1}\,dx \tag{1.1.1}$$

for positive integers $n \geq 2$ by integrating a complex function around the closed contour that follows the real axis from the origin to $R > 0$, then follows the circular arc $Re^i\theta$ as θ ranges from zero to $2\pi/n$, then returns to the origin via the line segment joining $Re^{2\pi i/n}$ to the origin, and let $R \to \infty$. Show all your calculations and explain all limits.

Solution to Problem 1.1:

□

Problem 1.2: Image under Conformal Map

Describe the image of the half-strip $\{z = x + iy : -1 < x < 1, 0 < y < \infty\}$ under the mapping $f(z) = \frac{z-1}{z+1}$.

Solution to Problem 1.2:

□

Problem 1.3: Existence of Complex Antiderivative

(a) Prove that $f(z) = \frac{1}{z}$ does not have a complex antiderivative in $\mathbb{C} \setminus \{0\}$.

(b) Find all integers n such that the function $g(z) = z^n e^{1/z}$ has a complex antiderivative in $\mathbb{C} \setminus \{0\}$.

Solution to Problem 1.3:

(a) If f was to have an antiderivative, then there must exist a primitive F such that $F' = f$ on $\mathbb{C} \setminus \{0\}$. For the sake of contradiction, suppose f indeed has an antiderivative F. Take some parametrization γ with endpoints $a, b \in \mathbb{C} \setminus \{0\}$. Then by the complex Fundamental Theorem of Calculus, we have

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)) \tag{1.3.1}$$

Which we can let $\gamma : [0, 2\pi] \to \mathbb{C} \setminus \{0\}$ such that $t \mapsto e^{it}$, we then see

$$\int_\gamma f(z)\,dz = F(\gamma(2\pi)) - F(\gamma(0)) = 0 \tag{1.3.2}$$

from the Complex FTC. But from Cauchy's Integral formula, we see

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} e^{-it}\left(ie^{it}\right)\,dt \tag{1.3.3}$$

$$= \int_0^{2\pi} i\,dt = 2\pi i \tag{1.3.4}$$

which is a contradiction. Thus, $f(z) = 1/z$ cannot have an antiderivative.

(b) We can use a power series expansion to see that

$$e^{1/z} = \sum_{k=1}^{\infty} \frac{1}{k!} \frac{1}{z^k} \tag{1.3.5}$$

So then

$$g(z) = z^n \sum_{k=1}^{\infty} \frac{1}{k!} \frac{1}{z^k} \tag{1.3.6}$$

$$= \sum_{k=1}^{\infty} \frac{z^{n-k}}{k!} \tag{1.3.7}$$

Now we will use the residue theorem to reason that the residue is zero for $z = 0$ if and only if it has an antiderivative on $\mathbb{C} \setminus \{0\}$. The residue of $g(z)$ at 0 is given by the $k = -1$ term in the power series expansion. This happens when $n - k = -1$, so $k = n + 1$. Then the residue at $z = 0$ is given by $\frac{1}{(n+1)!}$ which $n + 1 \geq 0$ and 0 otherwise. Thus, for the residue to be zero, we need $n < -1$. Thus, $g(z)$ has an antiderivative for all $n < -1$.

□

Problem 1.4: An application of Rouché's theorem

Let f be an analytic function with a zero of order 2 at z_0. Prove that there exists $\varepsilon > 0$ and $\delta > 0$ such that for every w in $D_\varepsilon(0) \setminus \{0\}$, the equation $f(z) = w$ has exactly 2 distinct roots in the set $D_\delta(z_0) \setminus \{z_0\}$.

Solution to Problem 1.4: Since f has a zero of order 2 at z_0, we can express it as

$$f(z) = w + (z - z_0)^2 g(z) \tag{1.4.1}$$

for some $g(z)$ such that $g(z_0) \neq 0$. □

Problem 1.5: No analytic function from the punctured disk to annulus

Prove that there is no analytic function that maps the punctured disk $\{z \in \mathbb{C} : 0 < |z| < 1\}$ one-to-one onto the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

Solution to Problem 1.5:

□

Exam 2: August 2024 - Eremenko

Problem 2.1: Entire function is even

Let f be an entire function which takes real values on the real and imaginary axes. Prove that f is even.

Solution to Problem 2.1:

$\square$

Problem 2.2: Residue

Find the residue

$$\operatorname{Res}_{z=0} \frac{1}{(e^z - 1)^2} \tag{2.2.1}$$

Solution to Problem 2.2:

$\square$

Problem 2.3: Radius of Convergence

Consider the series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{2.3.1}$$

where $a_0 \neq 0$ and $a_n = a_{n-1} - 2a_{n-2}$ for $n \geq 2$. Find the radius of convergence of this series.

Solution to Problem 2.3:

$\square$

Problem 2.4: Contour Integral

Evaluate the integral

$$\int_{|z|=2} \frac{z^4}{z^5 + 15z + z}, dz \tag{2.4.1}$$

Where the circle is parametrized counterclockwise.

Solution to Problem 2.4:

$\square$

Problem 2.5: Area of injective map

Consider the polynomial

$$f(z) = z + z^2/2 \tag{2.5.1}$$

(a) Prove that f is injective in the unit disk $U = \{z : |z| < 1\}$.

(b) Find the area of the image $f(U)$

Solution to Problem 2.5:

(a)

(b)

□

Problem 2.6: Finding solutions

Find all solutions of the equation

$$\tan z = 2i \tag{2.6.1}$$

and make a picture of them

Solution to Problem 2.6:

□

Problem 2.7: Real part is Function of Imaginary Part

Let $f = u + iv$ be a non-constant analytic function in some region where u and v are real valued harmonic functions. Is it possible that $u = F \circ v$ where F is some continuously differentiable function mapping the real line onto itself? If yes, give an example, if no, give a proof.

Solution to Problem 2.7:

□

Exam 3: August 2023 - Datchev

Problem 3.1: Fixed Points

Let f be an unbounded entire function and $\Omega \subset \mathbb{C}$ be a nonempty open set. Show that there exists $p \in \mathbb{C}$ such that $f(p) \in \Omega$.

Solution to Problem 3.1:

$\square$

Problem 3.2: Harmonic functions are surjective or constant

Let $u : \mathbb{C} \to \mathbb{R}$ be a harmonic function. Prove that u is either surjective or constant.

Solution to Problem 3.2:

$\square$

Problem 3.3: Find all Contraction Mappings

Find all entire functions f such that $|f(z)| \leq |z|$ for all z and $f(i) = 1$.

Solution to Problem 3.3:

$\square$

Problem 3.4: Contour Integral

Evaluate

$$\int_\gamma f(z)\, dz \tag{3.4.1}$$

where $f(z) = \tan((1+i)z)$ and γ is the circle $|z| = 2$ oriented clockwise.

Solution to Problem 3.4:

$\square$

Problem 3.5: Conformal Mapping

Let $\Omega = \mathbb{C} \setminus \{z \in \mathbb{R} : z \leq 0\}$. Find a bijective holomorphic function $f : \Omega \to \Omega$ such that $f(1) = i$.

Solution to Problem 3.5: We know that linear fractional transformations of the form

$$z \mapsto \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \tag{3.5.1}$$

are biholomorphic, so we are looking for a transformation that maps the plane minus the negative real line to itself such that $1 \to i$. We can choose 2 other points to determine the value, so we choose

$$1 \mapsto i \tag{3.5.2}$$
$$2i \mapsto \infty \tag{3.5.3}$$
$$0 \mapsto 0 \tag{3.5.4}$$

So then

$$\frac{i-2i}{i-0}\frac{z-0}{z-2i} \tag{3.5.5}$$

So then the inverse of this

$$\frac{-z}{z-2i} = w \implies wz - 2iw = -z \tag{3.5.6}$$
$$(w+1)z - 2iw = 0 \tag{3.5.7}$$
$$z = \frac{2iw}{w+1} \tag{3.5.8}$$
$$\tag{3.5.9}$$

□

Problem 3.6: Zeroes of Complex Polynomial

Let $f(z) = z^{1000} + z^{100} + z^{10} + 1$. Find an $R > 0$ such that if $f(z) = 0$ then $R < |z| < R + 1$.

Solution to Problem 3.6:

□

Problem 3.7: Derivative is zero implies Polynomial

Let $\Omega \subset \mathbb{C}$ be a nonempty set, and let $f : \Omega \to \mathbb{C}$ be holomorphic. Suppose that for every $z \in \Omega$ there is a positive integer n such that $f^{(n!)}(z) = 0$. Prove that f is a polynomial.

Solution to Problem 3.7:

□

Exam 4: January 2023 - Lempert

Problem 4.1

Compute

$$\int_{|z|=2} \frac{e^{iz}\, dz}{4z^2 - \pi^2} \tag{4.1.1}$$

where the path of integration is oriented counterclockwise.

Solution to Problem 4.1:

□

Problem 4.2: F

r a natural number n, let T_n denote the polynomial

$$T_n(z) = 1 - \frac{z^2}{3} + \frac{z^4}{5!} + \cdots + (-1)^n \frac{z^{2n}}{(2n+1)!} \tag{4.2.1}$$

Prove that there is no such n_0 such that T_n has exactly 6 roots in the disk $\{z \in \mathbb{C} : |z| < 10\}$ when $n > n_0$.

Solution to Problem 4.2:

□

Problem 4.3

If $\cos z = \cos w$ for some complex numbers z, w, prove that there is an integer k such that $z = w + 2k\pi i$ or $z = -w + 2k\pi i$.

Solution to Problem 4.3: Note that $\cos z = \frac{e^{iz}+e^{-iz}}{2}$. Trivially, this is true when $k = 0$. So then since $\cos z = \cos w$, we have

$$\frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iw} + e^{-iw}}{2} \tag{4.3.1}$$

Note that for any $k \in \mathbb{Z}$, $e^{2\pi ki} = 1$, so $\cos z = e^{2\pi ki} \cos z$. So then

$$e^{2\pi ki} \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iz+2\pi ki} + e^{-iz+2\pi i}}{2} = \cos(z + 2\pi ki) = \cos(w) \tag{4.3.2}$$

□

Problem 4.4

Is there a harmonic function $u : \mathbb{D} \to \mathbb{R}$ such that $\lim_{z \to \zeta} u(z) = \infty$ for every $\zeta \in \partial D$.

Solution to Problem 4.4: □

Problem 4.5

Let $Q = \{z \in \mathbb{C} : \operatorname{Re} z, \operatorname{Im} z > 0\}$ stand for the first quadrant. Find a biholomorphic map $F : Q \to Q$ such that $F(2+i) = 1+2i$.

Solution to Problem 4.5:

□

Problem 4.6

Suppose g is a holomorphic function on some domain, and $1/\overline{g}$ is also holomorphic there. Prove that g is constant.

Solution to Problem 4.6: Since $\frac{1}{\overline{g}}$ is holomorphic, $\frac{\partial}{\partial \overline{z}} \frac{1}{\overline{g}} = 0$. Now, writing the derivative out explicitly we see

$$0 = \frac{\partial}{\partial \overline{z}} \frac{1}{\overline{g}} = -\frac{\partial \overline{g}}{\partial z} \frac{1}{(\overline{g})^2} \tag{4.6.1}$$

In order for this to be true we need $\frac{\partial \overline{g}}{\partial \overline{z}} \equiv 0$, meaning that $\overline{g}$ is holomorphic. Since both g and $\overline{g}$ are holomorphic, g must be constant. This claim is easily proved by the Cauchy-Riemann equations. We see for $g = u + iv$, $\overline{g} = u - iv$ if both are holomorphic we have the following four CR equations

$$u_x = v_y, \quad u_y = -v_x \tag{4.6.2}$$
$$u_x = -v_y, \quad u_y = v_x \tag{4.6.3}$$
$$\implies u_y = -v_x = v_x, \quad u_x = v_y = -v_y \tag{4.6.4}$$

And the only way for all of these to be satisfied are if u, v are constant. Thus, we are finished and g is constant. □

Exam 5: August 2022 - Bell

Problem 5.1

Solution to Problem 5.1:

□

Problem 5.2

Solution to Problem 5.2:

□

Problem 5.3

Solution to Problem 5.3:

□

Problem 5.4

Solution to Problem 5.4:

□

Problem 5.5

Solution to Problem 5.5:

□

Problem 5.6

Solution to Problem 5.6:

□

Problem 5.7

Suppose that u us a non-constant real valued harmonic function on the complex plane such that $u(0) = 0$. Prove that there is at least one point on each circle centered at the origin where u vanishes.

Solution to Problem 5.7:

□