

**MATH 553: Abstract Algebra Qualifying Exam Prep**

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# Part I

## Notes

These are a brief summary of Dr. Basu's Spring 2025 MA553 class notes. Modules were originally not in the syllabus, but he decided to cover them so some things we proved are here. Also note that we "covered" Galois theory on the literal last day of class, so everything that is here are my own notes. I will also assume the reader has a basic understanding of group and ring theory and will not include superfluous definitions or theorems here. The class was based on Dummit and Foote's *Algebra*, so if anything is missing look there. Unless specified otherwise, suppose  $G$  is a group,  $R$  a ring, and  $M$  is a module, and  $p$  is a prime integer. These conventions are not always true, but use context clues and it will hopefully make sense.

## 1 Groups

### 1.1 Simple Groups

**Theorem 1.1: Cauchy.**

If  $G$  is a finite group,  $p$  is prime, then if  $p \mid |G|$ , then  $G$  contains an element of order  $p$ .

**Definition 1.2.**

A group is simple if  $H \leq G$  implies that  $H = G$ .

**Definition 1.3.**

If  $1 = N_0 \leq N_1 \leq \dots \leq N_k = G$  is a composition series if

- (a)  $N_i \trianglelefteq N_{i+1}$
- (b)  $N_{i+1}/N_i$  is simple.

Note that the notion of group normality is not transitive.

**Theorem 1.4: Jordan-Hölder.** (a)  $G$  has a composition series.

- (b) The composition factors for  $G$  are unique.

**Definition 1.5.**

A group is solvable if there exists a chain of subgroups  $1 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_k = G$  such that  $G_{i+1}/G_i$  is abelian. Note that this is not necessarily simple.

### 1.2 Group Actions

**Definition 1.6.**

Let  $A \neq \emptyset$  where  $A$  is a group action of  $G$  on  $A$ .  $G(A)$  is a map from  $G \rightarrow \text{Aut}(A)$ , meaning for each  $g \in G$

- (a)  $e \cdot a = a$  for all  $a \in A$
- (b)  $g_1 \cdot (g_2 \cdot a) = (g_1 \cdot g_2) \cdot a$

**Definition 1.7.** (a) The kernel of  $G(A)$  is the kernel of  $G \rightarrow \text{Aut}(A)$  which is defined as

$$\ker G(A) = \{g \in G \mid g \cdot \varphi = \varphi \quad \forall \varphi \in \text{Aut}(A)\} \quad (1.1)$$

- (b) For each  $a \in A$  the stabilizer of  $a$ ,  $G_a$  is given by

$$G_a = \{\varphi \in G \mid \varphi \cdot a = a\} \quad (1.2)$$

- (c) An action is faithful if and only if it is injective as a map.

**Proposition 1.8.**

Let  $G(A)$  define  $a \sim b$  if and only if  $a = g \cdot b$  for some  $g \in G$ . This is an equivalence class and

$$|[a]| = |G : G_a| \quad (1.3)$$

**Definition 1.9.** (a) The equivalence class of  $a$  is called the orbit of  $a \in G$ .

(b) The action of  $G(A)$  is transitive if and only if it has only one orbit.

**Theorem 1.10.**

Let  $G = \{g_1, \dots, g_n\}$ ,  $H \leq G$ , and define  $G(G/H)$  by  $g \cdot aH = (ga)H$ .

(a)  $G$  acts transitively on  $G/H$ .

(b) The stabilizer of  $H$  in  $G$  is  $H$ .

(c) The kernel is  $\bigcap_{g \in G} gHg^{-1}$  which is the largest normal subgroup of  $G$  contains in  $H$ .

**Corollary 1.11: Cayley.**

If  $|G| = n$ , then  $G \leq S_n$ .

**Proposition 1.12.**

If  $|G| = n$  and  $p$  is the smallest prime divisor of  $n$  and  $H \leq G$ ,  $|G : H| = p$  then  $H \trianglelefteq G$ .

*Proof.* Let  $H$  be as above, let  $G(G/H)$  by left multiplication. Let  $K$  be the action of  $G$  onto itself and  $k = |H : K|$ . Then since  $|G : K| = |G : H||H : K| = pk$ , which implies that  $p_k |p|$  and  $k|(p-1)!$ . This means that  $k = 1$  since there is always a prime less than  $p$ , so then  $H = K$ .  $\square$

**Definition 1.13.**

Define  $G(G)$  by  $g \cdot a = gag^{-1}$ . We say that  $a, b \in G$  are conjugate if and only if there are in the same orbit of the group action, i.e. there exists some  $g$  such that  $a = gb g^{-1}$ .

**Proposition 1.14.**

The number of conjugates of  $S \subseteq G$  is equal to  $|G : N_G(S)|$ .

*Proof.*  $|G : G_S| = \{g \in G : g|gSg^{-1} = S\} = \{g \in G | g \cdot S = S\}$   $\square$

Note that if  $S = \{s\}$  then  $N_G(S) = C_G(S)$  the number of conjugates of  $S$  is  $|G : C_G(S)|$ .

**Theorem 1.15.**

Let  $|G| < \infty$  and let  $g_1, \dots, g_n$  be representation of the distinct conjugate cosets not contained in  $Z(G)$ . Then

$$|G| = |Z(G)| + \sum_{i=1}^n |D : G_G(g_i)| \quad (1.4)$$

**Theorem 1.16.**

If  $p$  is prime and  $P$  is a group of order  $p^\alpha$ , then  $|Z(P)| \neq 1$ .

**Corollary 1.17.**

If  $|P| = p^2$  with  $p$  prime, then

(a)  $P$  is abelian.

(b)  $P \cong \mathbb{Z}_{p^2}$ .

**Proposition 1.18.**

Let  $H \trianglelefteq G$  and let  $G(H)$  by conjugation, then the kernel of this action is  $C_G \subset H$ . and  $G/C_G(H) \mapsto \text{Aut}(H)$ .

**Corollary 1.19.**

$K \leq G$  and  $K \cong gKg^{-1}$  in  $|K| = |gKg^{-1}|$ .

**Corollary 1.20.**

$H \leq G$ , and  $N_G(H)/C_G(H)$  injects into to  $\text{Aut}(H)$  then  $G/Z(G) \rightarrow \text{Aut}(G)$ .

**Definition 1.21.**

We define the group of inner automorphism of  $G$  by

$$\text{Inn}(G) = \{G_g | G_g(a) = gag^{-1}\} \quad (1.5)$$

**Corollary 1.22.**

$\text{Inn}(G) \cong G/Z(G)$ .

Note that  $H \trianglelefteq G$  if and only if  $gHg^{-1} = H$  for all  $g \in G$  which is true if and only if  $g \in \text{Inn}(G)$  restricts to  $\text{Aut}(H)$  which is not necessarily in  $\text{Inn}(H)$ .

**Definition 1.23.**

$H \leq G$  and we say that  $H \text{ char } G$  if any  $\varphi \in \text{Aut}(G)$  has  $\varphi(H) = H$ .

**Proposition 1.24.** (a) Characteristic subgroups are normal.

(b) If  $H$  is the unique subgroup of  $G$  with order  $h$  then  $H \text{ char } G$

(c)  $K \text{ char } H$  then  $H \trianglelefteq G$ ,  $K \not\trianglelefteq G$ .

**Definition 1.25.**

Let  $p$  a prime, and  $G$  a group. Then

(a) A subgroup of order  $p^\beta$  is a  $p$ -subgroup.

(b) If  $|G| = p^\alpha m$  where  $p \nmid m$  then a subgroup of order  $p^\alpha$  is a Sylow  $p$ -subgroup.

(c) The set of Sylow  $p$ -subgroups is  $\text{Syl}_p(G)$  and  $n_p = |\text{Syl}_p(G)|$ .

**Theorem 1.26: Sylow.**

Let  $|G| = p^\alpha m$ .

(a)  $\text{Syl}_p(G) \neq \emptyset$ .

(b) If  $P \in \text{Syl}_p(G)$  and  $Q$  is a  $p$ -subgroup of  $G$  then there is a  $g \in G$  such that  $Q \leq gPg^{-1}$ .

(c)  $n_p \equiv 1 \pmod{p}$ . Also if  $n_p = |G : N_G(P)|$  for any  $P \in \text{Syl}_p(G)$  then  $n_p | m$ .

**Corollary 1.27.**

Let  $P \in \text{Syl}_p(G)$ . The following are equivalent:

(a)  $n_p = 1$ ,

(b)  $P \trianglelefteq G$ ,

(c)  $P \text{ char } G$

(d) If  $X$  is a set of element with  $p$  order, then  $\langle x \rangle$  is a  $p$ -group.

**Corollary 1.28.**

All groups of order 15 are cyclic.

**Theorem 1.29.**

$A_n$  is simple for  $n \geq 5$ .

**Theorem 1.30: Fundamental Theorem of Finitely Generated Abelian Groups.**

Let  $g$  be a finitely generated abelian group.

(a)  $G \cong \mathbb{Z}^n \times Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k}$

(i)  $n \geq 0$ ,  $n \geq 2$ , for all  $i$

(ii)  $n_{i+1} | n_i$

(b) This composition is unique.

**Definition 1.31.**

$r$  is called the free rank or Betti number of  $G$ , and  $\{n_i\}$  are the invariant functions of  $G$ , and if  $n = 0$ , then  $G$  is type  $(n_1, \dots, n_k)$ .

**Corollary 1.32.**

If  $n$  is square free and  $|G| = n$ , and if  $G$  is abelian, then  $G$  is cyclic.

**Theorem 1.33: Primary Decomposition Theorem.**

$G$  is abelian of order  $|G| = p_1^{\alpha_1} \dots p_k^{\alpha_k}$

(a)  $G \cong A_1 \times \dots \times A_k$  where each  $|A_i| = p_i^{\alpha_i}$

(b) For one such are with order  $|A| = p^\alpha$  we have

$$A \cong Z_{p^{\beta_1}} \times \dots \times Z_{p^{\beta_n}} \quad (1.6)$$

where  $\sum \beta_i = \alpha$  and  $\beta_1 \geq \dots \geq \beta_n \geq 1$ .

(c) This decomposition is unique.

**Definition 1.34.** (a)  $[x, y] = x^{-1}y^{-1}xy$

(b)  $A, B \subset G$  then

$$[A, B] = \langle \{[a, b] \mid a \in A, b \in B\} \rangle \quad (1.7)$$

(c)  $G' = [G, G]$  is the commutator subgroup of  $G$ .

**Proposition 1.35.** (a)  $xy = yx[x, y]$

(b)  $H \trianglelefteq G$  if and only if  $[H, G] \subset H$ .

(c)  $G \in \text{Aut}(G)$ , then  $G([x, y]) = [G(x), G(y)]$  and  $G' \text{ char } G$

(d)  $G/G'$  is the largest abelian group of order  $G$ .

(e) If  $\varphi : G \rightarrow A$  and  $A$  is abelian, then  $\varphi$  factors  $G/G'$ .

**Proposition 1.36.**

Let  $H, K \leq G$ . The number of distinct ways of uniquely writing elements of  $HK$  as  $hk$  for  $h \in H, k \in K$  is  $|H \cap K|$ . If  $|H \cap K| = 1$  then each element of  $HK$  can be uniquely written.

**Theorem 1.37.**

Let  $H, K$  be a subgroup of  $G$ . If  $H, K \trianglelefteq G$  and  $|H \cap K| = 1$  then  $HK \cong H \times K \leq G$  where  $|HK| = |H||K|$ .

**Theorem 1.38.**

Let  $H$  and  $K$  be groups, and let  $\varphi : K \rightarrow \text{Aut}(H)$ . Let  $\cdot$  be the action of  $K$  on  $H$  and let  $G$  be the set of pairs  $(h, k)$  and form multiplication as

$$(h_1, k_1)(h_2, k_2) = (h_1, k_1 \cdot h_2 / k_1 k_2) \quad (1.8)$$

**Proposition 1.39.**

If  $G$  is a simple group and  $H \leq G$ , then  $|G| \mid [G : H]!$ .

**Lemma 1.40.**

$N \trianglelefteq G$  if  $N$  and  $G/N$  are solvable, then  $G$  is solvable

**Lemma 1.41.**

Every  $p$ -group is solvable.

## 2 Rings

### 2.1 Integral Domains

**Definition 2.1.**

A ring  $R$  is an integral domain if

(a)  $R$  is commutative

(b)  $1 \in R$ , and  $1 \neq 0$

(c)  $ab = 0$  implies that  $a = 0$  or  $b = 0$

**Definition 2.2.**

A ring  $R$  is a field if

(a)  $R$  is commutative

(b)  $1 \in R$ , and  $1 \neq 0$

(c) For all  $a \neq 0 \in R$ , there exists  $b \in R$  such that  $ab = 1$ .

**Proposition 2.3.**

Every finite integral domain is a field.

*Proof.* Take some  $a \in R$ . Consider  $x \mapsto ax$  and  $ax = ay$ , which implies  $a(x - y) = 0$  meaning either  $a = 0$  or  $x - y = 0$ , and since  $a \neq 0$  we have that  $x = y$  and  $1 = ab$  for some  $b$   $\square$

**Proposition 2.4.**

If  $R$  is a domain, then

(a)  $\deg p(x)q(x) = \deg p(x) + \deg q(x)$

(b)  $R[x]$  is a domain.

(c) The units of  $R[x]$  are the units of  $R$ .

**Definition 2.5.**

If  $R$  is an integral domain, we denote by  $Q(R)$  (Field of fractions of  $R$ ) the field

$$Q(R) = \{(a, b) \in R \times R \setminus \{0\} / \sim\} \quad (2.1)$$

where  $(a, b) \sim (c, d) \leftrightarrow ad = bc$ . We also have  $\overline{(a, b)} + \overline{(c, d)} = \overline{(ad + bc, bd)}$  and  $\overline{(a, b)} + \overline{(c, d)} = \overline{(ac, bd)}$ . Then  $\overline{(a, b)}^{-1} = \overline{(b, a)}$  for  $\overline{(a, b)} \neq (0, 1)$ .

**Lemma 2.6.**

Suppose  $R$  is an integral domain. Then  $R[x_1, \dots, x_n]$  is also an integral domain.

*Proof.* It suffices to prove that  $R[x]$  is also an integral domain. If

$$f = a_m x^m + \dots + a_0, \quad a_m \neq 0 \quad (2.2)$$

$$g = b_n x^n + \dots + b_0, \quad b_n \neq 0 \quad (2.3)$$

Then  $fg = a_m b_n x^{m+n} + \dots$  but  $a_m b_n \neq 0$  since  $R$  is an integral domain. So  $R[x]$  has no zero divisors.  $\square$

**Proposition 2.7.**

$R \rightarrow Q(R)$  is an example of a ring homomorphism which is an epimorphism without being surjective as a map.

## 2.2 Euclidean Domains

**Definition 2.8.**

If  $I, J$  are ideals of a commutative ring  $R$  then

- $I \cap J$  is an ideal.
- $I + J = \{a + b : a \in I, b \in J\}$  is an ideal.
- $IJ = \{\sum_i a_i b_i : a_i \in I, b_i \in J\}$  is an ideal and  $IJ \subset I \cap J$ .

**Proposition 2.9.**

Let  $I \subseteq R$ . Then

(a)  $I = R$  if and only if  $I$  contains a unit.

(b) If  $R$  is commutative then  $R$  is a field if and only if  $0, R$  are the only ideals.

(c) If  $R$  is a field and  $S$  is a ring and there is some ring homomorphism  $f : R \rightarrow S$ , then  $f = 0$  or  $f$  is injective.

**Definition 2.10.**

If  $S \subset R$  is a subset  $(S) = \{\sum_{s \in S} a_s s : a_s \in R\}$  is an ideal and is called the ideal generated by  $S$ . If  $S = \{s\}$  then  $(S) = (s)$  and is called a principal ideal.

**Proposition 2.11.**

In a unital ring, every proper ideal is contained in a maximal ideal.

**Definition 2.12.**

An ideal  $p \subset R$  is a prime ideal if it satisfies the property

$$xy \in P \implies x \in P \text{ or } y \in P \quad (2.4)$$

An ideal  $M \subset R$  is a maximal ideal if  $M \neq R$  and satisfies for every ideal  $I$   $M \subset I \implies I = M$  or  $R$ .

**Theorem 2.13.** (a)  $P$  is prime if and only if  $R/P$  is an integral domain.

(b)  $M$  is a maximal ideal if and only if  $R/M$  is a field.

**Corollary 2.14.**

If  $R$  is a unital domain, then maximal ideals are prime.

**Theorem 2.15.**

Let  $R$  be commutative (not necessarily unit). Let  $D \subset R$  which is closed under multiplication, does not contain a zero divisor, and does not contain 0. Then there is a commutative ring  $Q$  such that  $R \subset Q$  and every element of  $D$  is a unit in  $Q$ . Moreover,

(a) Every element of  $Q$  is of the form  $\frac{r}{d}$  for  $r \in R$  and  $d \in D$ .

(b)  $Q$  is the smallest ring containing  $R$  in which all elements of  $D$  are units, i.e. if  $S$  is a commutative unital ring with identity and  $\varphi : R \rightarrow S$  is an injective map such that  $\varphi(d)$  is a unit for each  $d \in D$ , then there is an injective  $\Phi : Q \rightarrow S$  such that

$$\begin{array}{ccc} R & \xrightarrow{\quad} & Q \\ & \searrow \varphi & \swarrow \Phi \\ & S & \end{array}$$

**Definition 2.16.**

$I, J \subset R$  we say that  $I, J$  are coprime or comaximal if and only if  $I + J = R$

**Theorem 2.17: Chinese Remainder Theorem.**

Let  $A_1, \dots, A_k \subset R$  such that  $A_i, A_j$  are comaximal for  $i \neq j$ . The map

$$R \rightarrow R/A_1 \times R/A_2 \times \dots \times R/A_k \quad (2.5)$$

has kernel  $A_1 \cap \dots \cap A_k$ .

**Definition 2.18.**

A function  $N : R \rightarrow \mathbb{N}$  with  $N(0) = 0$  is called a norm on  $R$ . If  $N(a) > 0$  for all  $a \neq 0$ , then  $N$  is a positive norm.

**Definition 2.19.**

The integral domain  $R$  is a euclidean domain if it admits a norm  $N$  such that for all  $a, b$  with  $b \neq 0$  there are  $q$  and  $r$  such that

$$a = bq + r \quad (2.6)$$

Where  $r = 0$  or  $N(r) < N(b)$ .

**Proposition 2.20.**

Every Ideal in a Euclidean domain is principal. More specifically,  $I \subset R$  then  $I = (d)$  for  $d \in I$  with minimal norm.

*Proof.* If  $I = 0$ , then we are done, so let  $I \neq 0$  and take  $d$  as above.  $(d) \subseteq I$ , so let  $a \in I$ . There exists some  $q, r$ , such that  $a = qd + r$ ,  $r = a - qd$  and  $r = 0$ . Then we are finished.  $\square$



**Definition 2.21.**

Let  $b \neq 0$ .

- (a)  $b|a$  implies that  $a = bx$  for some  $x$ .
- (b)  $d$  is a GCD for  $a, b$  written as  $d = (a, b)$  if and only if
  - (i)  $d|a$  and  $d|b$
  - (ii) If  $d'|a$  and  $d'|b$  then  $d'|d$ .

**Proposition 2.22.**

$((a, b))$  is the unique smallest principal ideal containing  $a$  and  $b$ .

**Proposition 2.23.**

Let  $R$  be a domain. Then if  $(d) = (d')$  for some  $d, d' \neq 0$ , then  $d = ud'$  for  $u$  a unit.

**Theorem 2.24.**

Let  $R$  be a Euclidean domain,  $a, b \in R$  be nonzero. Then the Euclidean Algorithm yields

$$a = q_0b + r_0 \tag{2.7}$$

$$b = q_1r_0 + r_1 \tag{2.8}$$

$$\vdots$$

$$r_{n-1} = q_{n+1}r_n \tag{2.9}$$

Then

- (a)  $r_n = (a, b)$ .
- (b)  $r_n = ax + by$

## 2.3 Principal Ideal Domains

**Theorem 2.25.**

If  $R$  is a Principal Ideal Domain, then every irreducible element is prime.

**Proposition 2.26.**

Let  $R$  be a PID and  $a, b \in R \setminus \{0\}$ . Then  $(a, b) = (d)$  for some  $d$ .

- (a)  $d = (a, b)$
- (b)  $d = ax + by$  for  $x, y \in R$ .
- (c)  $d$  is unique up to unit multiplication.

**Proposition 2.27.**

Every nonzero prime ideal in a PID  $R$ , is a maximal ideal. Moreover, If  $R[x]$  is a PID, then  $R$  is a field.

**Theorem 2.28.**

Suppose  $R$  is a PID and  $a \in R$ . Then the following are equivalent:

- (i)  $a$  is irreducible
- (ii)  $a$  is prime
- (iii)  $(a)$  is prime
- (iv)  $(a)$  is maximal

## 2.4 Unique Factorization Domains

### Theorem 2.29.

Every PID is a Unique Factorization Domain.

*Proof.* If  $r$  is a unit, then  $r$  is irreducible and we are finished. We claim that  $(r) \subset (p_1) \subseteq (p_{11}) \subset (p_{111}) \subset \dots$  cannot happen. This is because if  $I_1 \subseteq I_2 \subseteq \dots$ , then  $I = \bigcup I_n$  meaning  $(a) = I$  and  $a_k \in I_k$  implying  $(a) \subseteq I_k \subseteq I$ . For uniqueness we proceed by induction on  $n$ .  $\square$

### Definition 2.30.

Let  $R$  be a domain.

(a) Suppose  $r \in R$  is not a unit and is nonzero. Then  $r$  is irreducible if and only if  $r = ab$  implies  $a$  or  $b$  is a unit. Otherwise, it is reducible.

(b) An element  $p \in R$  is prime if and only if  $(p)$  is prime if and only if  $p|ab$  implies  $p|a$  or  $p|b$ .

(c)  $a$  and  $b$  are associates if and only if they differ by a unit.

### Theorem 2.31.

If  $R$  is a UFD, then  $R[x]$  is also a UFD.

### Corollary 2.32.

$R$  is a UFD implies that  $R[x_1, x_2, \dots, x_n]$  is also a UFD.

### Definition 2.33.

For  $R = \mathbb{Z}$ ,  $f \in \mathbb{Z}[x]$ ,  $f \neq 0$  is a primitive if  $\gcd(\text{coefficients of } f) = 1$

### Proposition 2.34.

Suppose  $f \in \mathbb{Q}[x]$ ,  $f \neq 0$ . Then there exists  $c \in \mathbb{Q}$ ,  $f_0 \in \mathbb{Z}[x]$  such that  $f = cf_0$  with  $f_0$  primitive. Up to multiplication with units,  $f_0$  and  $c$  are uniquely defined. Moreover,  $f \in \mathbb{Z}[x]$  if and only if  $c \in \mathbb{Z}$ . We call  $c$  the content of  $f$ .

### Theorem 2.35: Gauss Lemma.

Let  $R$  be a UFD,  $F$  the field of fractions of  $R$ . Let  $P(x) \in R[x]$ . If  $P(x)$  is reducible in  $F[x]$  then it is reducible in  $R[x]$ .

### Corollary 2.36.

Suppose  $f \in \mathbb{Z}[x]$  is primitive and  $g \in \mathbb{Z}[x]$ . If  $f|g$  in  $\mathbb{Q}[x]$ , then  $f|g$  in  $\mathbb{Z}[x]$

### Corollary 2.37.

$f \in \mathbb{Z}[x]$  primitive and  $f$  is irreducible over  $\mathbb{Z}[x]$  then  $f$  is irreducible over  $\mathbb{Q}[x]$ .

### Corollary 2.38.

If the GCD of the coefficients of  $P(x)$  is 1, then  $P(x)$  is reducible in  $F[x]$  if and only if  $P(x)$  is reducible in  $R[x]$

### Theorem 2.39: Rational Root Theorem.

Let

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0 \quad (2.10)$$

with  $a_i \in \mathbb{Z}$  and  $a_n, a_0 \neq 0$ . Each rational root of the polynomial written  $x = p/q$  satisfies

(a)  $P$  is an integer factor of  $a_0$

(b)  $q$  is an integer factor of the leading coefficient  $a_n$ .

### Theorem 2.40.

$R[x]$  is a UFD if and only if  $R$  is a UFD.

### Proposition 2.41.

Let  $I \subset R$ ,  $P(x) \in R[x]$  be nonconstant and monic. Then if  $P(x)$  is reducible in  $R[x]$ ,  $\overline{P(x)}$  is reducible in  $(R/I)[x]$ .

### Theorem 2.42.

$f \in \mathbb{Z}[x]$  is irreducible if and only if either

(a)  $f = c$  where  $c \in \mathbb{Z}$  is prime.

(b)  $f$  is primitive and irreducible in  $\mathbb{Q}[x]$ .

### Theorem 2.43.

Every irreducible in  $\mathbb{Z}[x]$  is a prime.

### 3 Modules

**Definition 3.1.**

Let  $A$  be a commutative ring. An  $A$ -module  $M$  is an abelian group  $M = (M, \oplus, 0)$  along with a map  $\odot : A \times M \rightarrow M$  satisfying

- (i)  $c \odot (d \odot \alpha) = (cd) \odot \alpha$
- (ii)  $c \odot (\alpha \oplus \beta) = c \odot \alpha \oplus c \odot \beta$
- (iii)  $c + d \odot \alpha = c \odot \alpha \oplus d \odot \alpha$
- (iv)  $1 \odot \alpha = \alpha$

**Definition 3.2.**

If  $M$  and  $N$  are  $A$ -modules then  $f : M \rightarrow N$  is a homomorphism.

**Definition 3.3.**

$N$  submodule of  $M$  and  $B \subset N$ . We say that  $B$  is a basis of  $N$  if

- (i)  $B$  is linearly independent.
- (ii)  $\text{span}(B) = N$

#### 3.1 Free Modules

**Definition 3.4.**

$M$  is a free  $A$ -module if it has a basis.

**Theorem 3.5.**

Any two bases of a free  $A$ -module have the same cardinality.

**Definition 3.6.**

We say that  $M$  is a direct sum of the submodules  $N_i$  if  $M \cong \bigoplus_i N_i$ . In particular if  $N_1, N_2$  are submodules of  $M$ ,  $M = N_1 \oplus N_2$  if and only if  $M = N_1 + N_2$ , and  $N_1 \cap N_2 = 0$ .

Unlike in vector spaces, if  $N \subset M$  is a submodule then there might not exist  $N' \subset M$  such that  $M = N \oplus N'$ . If such an  $N'$  exists, we call  $N'$  a complementary submodule of  $N$ .

**Definition 3.7.**

If  $N$  is a submodule of  $M$  such that there exists  $N' \subset M$  such that  $M = N \oplus N'$ , then  $N$  is called a direct factor of  $M$ .

**Definition 3.8.**

We say that  $p \in \text{End}(M) = \text{hom}(M, M)$  is a projector if  $p \circ p = p$ .

**Theorem 3.9.**

If  $N$  is a submodule of  $M$ , then  $N$  is a direct factor of  $M$  if and only if there exists a projector  $p \in \text{End}(M)$  such that  $N = p(M)$ .

*Proof.* Suppose  $p \in \text{End}(M)$  is a projector such that  $N = p(M)$ . Then  $M = p(M) + (1-p)M$ . Suppose  $\alpha \in p(M) \cap (1-p)M$ . Then

$$\alpha = p(\beta) = (1-p)(\gamma) = \gamma - p(\gamma) \quad (3.1)$$

Applying  $p$  to both sides we get

$$p(\beta) = p(\gamma) - p(\gamma) = 0 \implies \alpha = 0 \quad (3.2)$$

So  $M = p(M) \oplus (1-p)M$ . Conversely if  $N$  is a direct factor built complementary  $N$  then every  $\alpha \in M$  can be expressed uniquely as  $\alpha = \beta + \gamma$ ,  $\beta \in N, \gamma \in N'$ .  $\square$

**Theorem 3.10.**

If  $A$  is a PID. Then every submodule of a finite  $A$ -module is free.

*Proof.* Suppose  $L$  is a free  $A$ -module and  $\{\alpha_1, \dots, \alpha_n\}$  is a basis of  $L$  and  $M$  submodule of  $L$ . Let  $M_i = M \cap \text{Span}(\alpha_1, \dots, \alpha_n)$ . Let  $P_i : L \rightarrow A$  denote the coordinate function.  $P_i(M_i)$  is an ideal of  $A$ , and  $P_i(M_i) = (d_i)$ ,  $d_i \in A$ . So then there exists  $\beta_i \in M_i$  such that  $P_i(\beta_i) = d_i$ . Let  $N_i = \text{Span}(\beta_i)$ . We claim that for  $i = 1, \dots, n$ ,  $M_i = \sum N_i$  and the sum is direct. Suppose we know this for  $M_h = \sum_{j \leq h} N_j$  for all  $h < k$ .  $\alpha \in M_k$ . From definition of  $N_k$ , there exists  $\beta \in N_k$  such that  $P_k(\alpha) = P_k(\beta)$  since  $N_k \cong P_k(M_k)$  so  $P_k(\alpha - \beta) = 0$  which implies  $\alpha - \beta \in M_{k-1} = M \cap \text{Span}(\alpha_1, \dots, \alpha_{k-1})$ . By induction  $\alpha - \beta \in \sum_{j \leq k} N_j$ ,  $\beta \in N_k$ . So,  $\alpha \in \sum_{j \leq k} N_j$  and  $M_k = \sum_{j \leq k} N_j$ . You get the idea.  $\square$

**Theorem 3.11.**

Suppose  $A$  is a PID and  $L$  a free  $A$ -module and  $M$  submodule of  $L$  (necessarily free) of rank  $n$ . Then there exists a basis  $B$  of  $L$  and  $\beta_1, \dots, \beta_n \in B$  and  $a_1, \dots, a_n$  such that  $\{\beta_1, \dots, \beta_n\}$  is a basis of  $M$  and  $a_1 | a_2 | \dots | a_n$ . Moreover,  $M' = \text{Span}(\beta_1, \dots, \beta_n)$  and  $(a_1) \supset (a_2) \supset \dots \supset (a_n)$  and uniquely determined by  $L, M$ . Moreover, if  $(L/M)_{\text{tor}} \cong \bigoplus_i^n A/(a_i)$  and  $L/M \cong (L/M)_{\text{tor}} \oplus (\text{Free module})$ .

## 4 Fields

**Theorem 4.1.**

Suppose  $K$  is a field and  $G$  is a finite subgroup of  $K^\times$ . Then  $G$  is cyclic.

**Definition 4.2.**

$k \subset K$  is a field.  $K$  is an **extension** of  $k$ .  $K$  is a  $k$ -vector space.  $\dim_k K = [K : k]$  the degree of the extension  $K/k$ .

**Proposition 4.3.**

$k \subset K \subset L$ . Then

$$[L : k] = [L : K][K : k] \quad (4.1)$$

**Definition 4.4.**

Suppose  $K/k$  is a field extension and  $\alpha \in K$ . We say that  $\alpha$  is **algebraic** over  $k$  if there exists some  $f \in k[x]$  such that  $f(\alpha) = 0$ . Otherwise  $\alpha$  is **transcendental** over  $k$ .

**Definition 4.5.**

Let  $K_1, K_2 \subset K$ . Then the **composite field** of  $K_1$  and  $K_2$ ,  $K_1 K_2$  is the smallest subfield of  $K$  containing  $K_1$  and  $K_2$ .

**Proposition 4.6.**

Every finite field extension is algebraic.

**Definition 4.7.**

$K/k$  is an **algebraic extension** if every  $\alpha \in K$  is algebraic over  $k$ .

**Theorem 4.8.**

$[K : k] < \infty$ , then  $K/k$  is algebraic.

**Lemma 4.9.**

Suppose  $k/K$  is a field extension and  $\alpha \in K$  is algebraic over  $k$ . Let  $\varphi_\alpha : k[x] \rightarrow K$  to be the evaluation homomorphism at  $\alpha$ ,  $f \mapsto f(\alpha)$ . Then

$$\text{im } \varphi_\alpha = k(\alpha) \quad (4.2)$$

$$\ker \varphi_\alpha = (\text{Irr}(\alpha, k)) \quad (4.3)$$

Where  $\text{Irr}(\alpha, k)$  is the unique monic polynomial generating  $\ker \varphi_\alpha$ .

**Theorem 4.10.**

If  $k$  is a field and  $f \in k[x]$ ,  $f$  is not a non-zero constant. Then there exists  $K/k$  and  $\alpha \in K$  such that  $f(\alpha) = 0$  in  $K$ .

**Theorem 4.11.**

If  $k$  is a field and  $f \in k[x]$ , then any two splitting fields are isomorphic.

**Definition 4.12.**

Suppose  $S \subset k[x]$  is a set of polynomials and  $\bar{k}$  on algebraic closure of  $k$ . Then  $k \subset K \subset \bar{k}$  and is a **splitting field** of  $S$  if  $K$  contains all the roots of each  $f \in S$ , and  $K$  is generated by these roots.

**Theorem 4.13.**

Suppose  $\bar{k}$  is an algebraic closure of  $K$  and  $K, k \subset K \subset \bar{k}$  is a splitting field. Then any embedding

$$\sigma : K \rightarrow \bar{k}, \quad \sigma|_K = I_k \quad (4.4)$$

induces an automorphism of  $K$ .

**Theorem 4.14.**

$f \in k[x]$  has simple roots of and only if  $(f, f') = (1)$ .

**Corollary 4.15.**

If  $\text{char } k = 0$  and  $f \neq 0$  is irreducible, then  $f$  has simple roots.

**Definition 4.16.**

Suppose  $k$  is a field  $\text{char } k = p$

$$\varphi_p : k \rightarrow k, \quad \alpha \mapsto \alpha^p \quad (\text{Frobenius Map}) \quad (4.5)$$

**Proposition 4.17.**

Suppose  $k$  is a field with  $\text{char } k = p$ .

$$(\alpha + \beta)^p = \alpha^p + \beta^p \quad (4.6)$$

**Definition 4.18.**

An irreducible polynomial  $f \in k[x]$  is **separable** if all its roots are simple.

**Proposition 4.19.**

Suppose  $\text{char } k = p$  and  $f$  is irreducible and not separable. Then there exists irreducible  $g \in k[x]$  such that  $f = g(x^p)$ .

**Proposition 4.20.**

Let  $\text{char } k = p$  and  $a \in k$ . Then  $x^p - a$  is either irreducible or a  $p$ th power.

**Definition 4.21.**

$k$  is **perfect** if every irreducible is separable.

**Proposition 4.22.**

$\text{char } k = 0$ , then  $k$  is perfect.

**Proposition 4.23.**

$\text{char } k = p$ . Then  $k$  is perfect if and only if  $\varphi_p(k) = k^p = k$ .

**Corollary 4.24.**

All finite fields are perfect.

**Definition 4.25.**

$K/k$  is a field extension.  $\alpha \in K$  is separable over  $k$ , if  $\text{Irr}(\alpha, k)$  is separable.  $K/k$  is separable if every  $\alpha \in K$  is separable.

## 4.1 Galois Theory

**Definition 4.26: Galois Extension.**

$K/k$  is a **Galois Extension** if it is normal and separable.

**Proposition 4.27.**

Let  $K/F$  be a field extension and  $\alpha \in K$  be algebraic over  $F$ . Then for any  $\sigma \in \text{Aut}(K/F)$ ,  $\sigma\alpha$  is a root of the minimal polynomial for  $\alpha$  over  $F$  i.e.,  $\text{Aut}(K/F)$  permutes the roots of irreducible polynomials.

**Definition 4.28.**

If  $H$  is a subgroup of the group of automorphisms of  $K$ , the subfield of  $K$  fixed by all the elements of  $H$  is called the **fixed field** of  $H$

**Definition 4.29.**

If  $f(x)$  is a separable polynomial over  $F$ , then the **Galois group** of  $f(x)$  is the Galois group of the splitting field of  $f(x)$  over  $F$ .

**Theorem 4.30: Fundamental Theorem of Galois Theory.**

Let  $K/F$  be a Galois extension and set  $G = \text{Gal}(K/F)$ . Then there is a bijection between the subfields of  $K$  containing  $F$  and the subgroups of  $G$  given by the correspondences

$$E \rightarrow \{\text{the elements of } G \text{ fixing } E\} \quad (4.7)$$

$$\{\text{the fixed field of } H\} \leftarrow H \quad (4.8)$$

which are inverse to each other. Under this correspondence

(a) If  $E_1, E_2$  correspond to  $H_1, H_2$  respectively, then  $E_1 \subseteq E_2$  if and only if  $H_2 \leq H_1$ .

(b)  $[K : H] = |H|$  and  $[E : F] = |G : H|$ , the index of  $H$  in  $G$

$$\begin{array}{c} K \\ \Big|_{|H|} \\ E \\ \Big|_{|G:H|} \\ F \end{array}$$

(c)  $K/E$  is always Galois, with Galois group  $\text{Gal}(K/E) = H$

## Part II

## Past Qualifying Exams

These are problems from past Purdue qualifying exams which can be found <https://www.math.purdue.edu/academic/grad/qaalexams.html>. Note that all solutions are either mine or sourced from other resources, so the validity of them should not be assumed. I have tried to reference relevant theorems in the notes section above, but some results may be assumed as true when they were proved in class or homework.

## Exam 1: January 2024 - Shahidi

## Problem 1.1: Solvability

Let  $p, q$  and  $r$  be three distinct prime numbers with  $p > qr$ . Let  $n$  be a positive integer. Show that every group  $G$  of order  $O(G) = p^n qr$  is solvable. Conclude that every group of order 294 or 1210 is solvable.

## Solution to Problem 1.1:

□

## Problem 1.2: Polynomials of prime order

Let  $q$  be a prime number and let

$$f_q(x) = x^{q-1} + x^{q-2} + \cdots + 1 \quad (1.2.1)$$

- (a) Suppose a prime number  $p$  divides  $f_q(a)$  for some integer  $a$ . Prove that either  $p = q$  or  $p \equiv 1 \pmod{q}$ .
- (b) Prove there are infinitely many primes of the form  $qb + 1$ , where  $b$  is an integer.

## Solution to Problem 1.2:

- (a)
- (b)

□

## Problem 1.3: Irreducibles on Euclidean Domain

- (a) Prove that  $A = \mathbb{Z}[\sqrt{-2}]$  is a Euclidean domain.
- (b) Show that

$$A/(3 + 2\sqrt{-2}) \cong \mathbb{Z}/17\mathbb{Z} : \mathbb{F}_{17} \quad (1.3.1)$$

- (c) Show that  $x^4 + 3$  is irreducible over  $\mathbb{F}_{17}$  and conclude that

$$f(x) = x^4 - 170x + 9 + 4\sqrt{-2} \in A[x] \quad (1.3.2)$$

is irreducible over  $A[x]$ .

**Solution to Problem 1.3:**

- (a)
- (b)
- (c)

□

**Problem 1.4**

Let  $n$  be an integer and let

$$f(x) = x^3 - (n-3)x^2 - nx - 1 \quad (1.4.1)$$

- (a) Show that  $f(x)$  is irreducible over  $\mathbb{Q}[x]$ .
- (b) Show that if  $a$  is a root of  $f(x)$  then  $-1/(a+1)$  is also a root of  $f(x)$ .
- (c) Let  $K$  be the splitting field of  $f(x)$  over  $\mathbb{Q}$ . Show that the Galois extension over  $K/\mathbb{Q}$  is cyclic of order 3. Conclude that all the roots of  $f(x) = 0$  are real.

**Solution to Problem 1.4:**

□

**Problem 1.5****Solution to Problem 1.5:**

□

**Problem 1.6****Solution to Problem 1.6:**

□

**Problem 1.7****Solution to Problem 1.7:**

□



## Exam 2: August 2024

### Problem 2.1

- (a) Prove that  $x^3 - x - 1$  is irreducible over  $\mathbb{Z}$ .
- (b) Prove that  $x + 1$  and  $x^3 - x - 1$  are relatively prime in  $\mathbb{Z}[x]$  i.e. they generate the whole ring.
- (c) Give a simpler interpretation of  $\mathbb{Z}[x]/(x + 1)(x^3 - x - 1)$

**Solution to Problem 2.1:**

□

### Problem 2.2

If  $n \in \mathbb{N}$ ,  $n \geq 1$ . Prove that  $f_n(x) = (x - 1)(x - 2) \dots (x - n) - 1$  is irreducible over  $\mathbb{Z}$ . Is it irreducible over  $\mathbb{Q}$ ?

**Solution to Problem 2.2:**

□

## Exam 3: August 2022 - Shahidi

### Problem 3.1

- (a) Show that every solvable group has a non-trivial normal abelian subgroup.
- (b) Let  $G$  be a group and denote  $\text{Aut}(G)$  the group of its automorphisms. Assume  $\text{Aut}(G)$  is solvable. Prove that  $G$  is solvable.

**Solution to Problem 3.1:**

□

### Problem 3.2: Classifying all groups of order $pq$

Let  $p$  and  $q$  be two prime numbers with  $p < q$ . Let  $G$  be a group of order  $pq$ .

- (a) Assume  $p$  does not divide  $q - 1$ . Show that  $G$  is cyclic which is a direct product of a  $q$ -Sylow subgroup  $Q$  and a  $p$ -Sylow subgroup  $P$  of  $G$ .
- (b) Assume  $p|q - 1$  and  $G$  is not cyclic. Conclude that in this case  $G$  is non-abelian and is a semi-direct product of a  $q$ -Sylow subgroup  $Q$  and a  $p$ -Sylow subgroup  $P$  of  $G$ , but not their direct product.
- (c) Let  $p$  and  $q$  be two primes as above with  $p|q - 1$ . Let  $P$  and  $Q$  be the cyclic groups of orders  $p$  and  $q$  respectively. Show that all the semi-direct products  $Q \rtimes_{\varphi} P$  where  $\varphi : P \rightarrow \text{Aut}(Q)$  and non-trivial homomorphisms, are isomorphic. You may assume the fact that finite subgroups of the multiplication group of a field are cyclic.

**Solution to Problem 3.2:**

□

### Problem 3.3

Let  $\alpha$  be a real root of

$$f(x) = x^{23} - 5x^{19} + 25x^{11} - 30x^8 + 35x^5 + 10 \quad (3.3.1)$$

Is  $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^{10})$ ?

**Solution to Problem 3.3:**  $f(x)$  is irreducible by Eisenstein's criterion, therefore

$$[\mathbb{Q}(\alpha) : \mathbb{Q}] = \deg f = 23 \quad (3.3.2)$$

Since  $\mathbb{Q} \subset \mathbb{Q}(\alpha^{10}) \subset \mathbb{Q}(\alpha)$ , we know that

$$23 = [\mathbb{Q}(\alpha^{10}) : \mathbb{Q}][\mathbb{Q}(\alpha) : \mathbb{Q}(\alpha^{10})] \quad (3.3.3)$$

If  $[\mathbb{Q}(\alpha^{10}) : \mathbb{Q}] = 1$ , then  $\mathbb{Q}(\alpha^{10}) = \mathbb{Q}$  which implies that  $\alpha^{10} \in \mathbb{Q}$ . If this is true, then  $\alpha$  is a root of  $x^{10} - \alpha^{10} \in \mathbb{Q}[x]$  which implies that  $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leq 10$  which is a contradiction. Thus  $[\mathbb{Q}(\alpha) : \mathbb{Q}(\alpha^{10})] = 1$  which implies that  $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^{10})$ . □

## Exam 4: January 2017

### Problem 4.1

Let  $r < q$  be primes and  $G$  be a group of order  $p^2q$ .

- (a) Prove that  $n_q = 1$  or  $p^2$ .
- (b) Prove that if  $n_q = p^2$  then  $G$  contains a unique Sylow  $p$  subgroup.
- (c) Prove that  $G$  is solvable.

### Solution to Problem 4.1:

- (a) If  $n_q | p^2$ , then  $n_q \equiv 1 \pmod q$  by Theorem 1.26(c). So then, we have that either  $n_q = 1$  or  $n_q = p$  or  $n_q = p^2$ . Since  $n_q \not\equiv 1 \pmod q$ , this means that  $n_q$  is either 1 or  $p^2$ .
- (b) If  $n_q = p^2$  any two  $q$ -Sylow groups have trivial intersection since they have prime order. This means that there are at least  $p^2(q-1)$  elements of order  $q$ . None of these elements can be contained in a  $p$ -Sylow subgroup, meaning that only  $p^2$  elements which are in a  $p$ -Sylow subgroup meaning that  $n_q = 1$ .
- (c) If  $n_q = 1$ , let  $H \in \text{Syl}_q(G)$ . This means that  $H \trianglelefteq G$ ,  $|H| = q$ , and  $|G/H| = p^2$ . By Lemma 1.40, we know that since  $H$  and  $G/H$  are solvable, then so is  $G$ . Now, If  $n_q = p^2$  and  $n_p = 1$ , let  $K \in \text{Syl}_p(G)$ . Again, we know that  $K \trianglelefteq G$ ,  $|K| = p^2$ , and  $|G/K| = q$ . Since  $K$  and  $G/K$  are solvable, then  $G$  is solvable too by Lemma 1.40. Thus,  $G$  is always solvable.

□

### Problem 4.2

Let  $H \trianglelefteq G$ . Suppose  $P \in \text{Syl}_p(H)$ . Prove that the following are equivalent:

- (a)  $P \trianglelefteq G$
- (b)  $P \trianglelefteq H$

**Solution to Problem 4.2:** The first part is trivial, so we begin by proving the reverse implication. Let  $g \in G$ . Then  $gPg^{-1} \subset gHg^{-1} = H$ , meaning  $|gPg^{-1}| = |P|$ . Since  $P \trianglelefteq H$ , we have that  $n_p(H) = 1$  by Corollary 1.27. Since  $P$  and  $gPg^{-1}$  are both  $p$ -Sylow subgroups of  $H$ , we have that  $gPg^{-1} = P$  which implies that  $P \trianglelefteq G$ . □

### Problem 4.3

Let  $|G| = 66$ .

- (a) Prove that  $G$  has a normal subgroup of order 11.
- (b) Prove that  $G$  has a normal subgroup of order 33.
- (c) Prove that  $G$  has an element of order 33.
- (d) Prove that  $G$  cannot be embedded into  $S_{12}$ .

### Solution to Problem 4.3:

- (a)  $n_{11} | 6$ , and  $n_{11} \equiv 1 \pmod{11}$  by Theorem 1.26, so then  $n_{11} = 1$ . If  $P \in \text{Syl}_{11}(G)$ , then by Corollary 1.27,  $P \trianglelefteq G$ .

(b)

□

**Problem 4.4**

Let  $R$  be a commutative ring with  $1 \neq 0$ , and  $M$  a maximal ideal of  $R$ . Prove the following are equivalent:

- (a)  $x \notin M$
- (b) There exists  $a \in R$  such that  $1 + ax \in M$ .

**Solution to Problem 4.4:**

□

**Problem 4.5**

- (a) Prove that  $\mathbb{Z}[x]$  is not a PID.
- (b) Prove that  $\mathbb{Z}[x]/(x^2 + 5)$  is not a UFD.

**Solution to Problem 4.5:**

□

## Exam 5: August 2013

### Problem 5.1

In which of the following rings is every ideal principal?

- (a)  $\mathbb{Z}/4\mathbb{Z}$
- (b)  $\mathbb{Z} \oplus \mathbb{Z}$
- (c)  $\mathbb{Z}/4\mathbb{Z}[x]$
- (d)  $\mathbb{Z}/6\mathbb{Z}[x]$

### Solution to Problem 5.1:

□

### Problem 5.2: L

Let  $k$  be field of characteristic that is not 2 or 3. Prove that the following are equivalent:

- (a) Any sum of squares in  $k$  is itself a square.
- (b) Whenever a cubic polynomial of  $f$  factors completely in  $k$ , so does its derivative.

**Solution to Problem 5.2:** For the forward implication, we know that

$$f(x) = (x - a)(x - b)(x - c), \quad \forall a, b, c \in k \quad (5.2.1)$$

and

$$f'(x) = (x - a)(x - b) + (x - a)(x - c) + (x - b)(x - c) \quad (5.2.2)$$

$$= 3x^2 - (2a + 2b + 2c)x + ab + ac + bc \quad (5.2.3)$$

so then the discriminant of  $f'$  is given by

$$\delta(f') = (2a + 2b + 2c)^2 - 12(ab + ac + bc) \quad (5.2.4)$$

$$= 4a^2 + 4b^2 + 4c^2 - 4ab - 4ac - 4bc \quad (5.2.5)$$

$$= 2(a - b)^2 + 2(b - c)^2 + 2(a - c)^2 \quad (5.2.6)$$

which is a sum of squares, and since every sum of squares is a square, then  $\Delta(f')$  is a square so it factors completely and we are finished.

For the reverse implication, we know that  $a^2 + b^2 + c^2 - ab - ac - bc$  is a square. so then  $c = a + b$  means that  $a^2 + b^2 - ab$  is a square, and □

## Part III

## Extra Problems

## Exam 6: Basu Practice Final Spring 2025

This is the practice final given to us to be "representative" of Basu's final exam in Spring 2025. Many of the final problems were either word for word used, or very slightly changed to be easier. Again, validity of solutions should not be assumed.

**Problem 6.1: Conjugacy Classes**

Let  $G$  be a finite group.

- (a) What is the conjugacy class of an element  $g \in G$ ?
- (b) Prove that the number of elements in a conjugacy class divides the order  $|G|$ .
- (c) If  $G$  has only 2 conjugacy classes, prove that  $G$  has order 2.

**Solution to Problem 6.1:**

- (a) The conjugacy class  $C_g$  of  $g$  is defined as

$$C_g = \{x \in G : xgx^{-1} = g\} \quad (6.1.1)$$

- (b) We note that

$$|C_g| = [G : C_G(g)] \quad (6.1.2)$$

where  $C_G(g)$  is the centralizer of  $g$ . Since the centralizer is a subgroup of  $G$ , we can apply Lagrange's Theorem to see

$$|G| = |C_G(g)|[G : C_G(g)] = |C_G(g)||C_g| \quad (6.1.3)$$

So then the order of a conjugacy class divides the order of  $G$ .

- (c) Let  $x \in G$  such that  $x \neq e$ , and let  $C_x$  be the conjugacy class of  $x$ . Trivially we must have that  $C_e = \{e\}$ , which implies that if  $|G| = n$ , then  $|C_x| = n - 1$ . By the previous problem, we can apply Lagrange's theorem to see that  $n - 1 | n$ , which means that only  $n = 2$ .

□

**Problem 6.2: Group is abelian if there is an automorphism for every element**

Let  $G$  be a finite group. Suppose that for every  $a, b \in G$  distinct from the identity, there is an automorphism of  $G$  taking  $a$  to  $b$ . Prove that  $G$  is abelian.

**Solution to Problem 6.2:** Since  $G$  is finite, every element of  $G$  has finite order. Since any two elements of  $G \setminus \{e\}$  are related by an automorphism of  $G$ , all elements must have the same order, say  $q$ . Since all powers of an element of  $G \setminus \{e\}$  have either order  $q$  or 1, then  $q$  must be prime. By Sylow's Theorem, the order of  $G$  is a power of  $q$ . Thus,  $Z(G)$  contains an element other than  $e$ , meaning  $Z(G) = G$  which means  $G$  is abelian. □

**Problem 6.3: Intersection of subgroups has finite index**

Let  $G$  be a group and  $H, K$  subgroups of  $G$  such that  $H$  has a finite index in  $G$ . Prove that  $K \cap H$  has a finite index in  $K$ .

**Solution to Problem 6.3:** Since  $H \cap K$  is a subgroup of both  $H$  and  $K$ , both  $[K : H \cap K]$  and  $[H : H \cap K]$  are well defined. We see

$$[G : H] = [G : K][K : H \cap K][K \cap H : H] \quad (6.3.1)$$

Since  $[G : H]$  is finite, and  $[K \cap H : H]$  is finite, then  $[K : K \cap H]$  is also finite.  $\square$

**Problem 6.4: Cyclic if only one subgroup shares order**

Let  $G$  be a finite group of order  $n$  with the property that for each  $d$  such that  $d|n$ , there is at most one subgroup of  $G$  of order  $d$ . Prove that  $G$  is cyclic.

**Solution to Problem 6.4:** Let  $D$  be the set of all orders of elements of  $G$ . If  $a \in G$  has order  $|a| = d$ , then  $\langle a \rangle$  is the unique subgroup in  $G$  of order  $d$  and so all elements of order  $d$  must be in  $\langle a \rangle$ . It follows that there are exactly  $\varphi(d)$  elements in  $G$  of order  $d \in D$ . We then see

$$n = \sum_{d \in D} \varphi(d) \leq \sum_{d|n} \varphi(d) = n \quad (6.4.1)$$

So then  $n \in D$  and  $G$  is cyclic.  $\square$

**Problem 6.5: Group is cyclic if it has subgroup of order 2**

Let  $p$  be an odd prime and  $G$  a group of order  $2p$ . Suppose that  $G$  has a normal subgroup of order 2. Prove that  $G$  is cyclic.

**Solution to Problem 6.5:** We know that  $[N : G] = p$ , so then  $|G/N| = p$  and is cyclic. Let  $gN$  generate  $G/N$ . In particular, we have  $g^p N = (gN)^p = N$ . Suppose  $G$  is not cyclic. Then since any abelian group of order 2 is cyclic, then the order of  $g$  must be  $p$ . Since  $N$  is normal in  $G$ ,  $gxg^{-1}$ , the order of  $gx$  is  $2p$  so  $G$  is cyclic.  $\square$

**Problem 6.6: Finding Galois Group**

Let  $p$  be an odd prime number and  $\varphi_p = X^{p-1} + \cdots + 1 \in \mathbb{Q}[X]$ . Prove that  $K = \mathbb{Q}[X]/(\varphi_p)$  is a splitting field of  $\varphi_p$  and  $K/\mathbb{Q}$  is a Galois extension. What is the Galois group of the extension  $K/\mathbb{Q}$ ?

**Solution to Problem 6.6:****1. Roots of  $\varphi_p(X)$** 

Let  $\zeta_p$  be a primitive  $p$ -th root of unity, i.e.,  $\zeta_p = e^{2\pi i/p}$ . Then the roots of the polynomial  $\varphi_p(X)$  are exactly the primitive  $p$ -th roots of unity:

$$\zeta_p, \zeta_p^2, \dots, \zeta_p^{p-1}.$$

Therefore, the minimal polynomial of  $\zeta_p$  over  $\mathbb{Q}$  is  $\varphi_p(X)$ , which is irreducible in  $\mathbb{Q}[X]$ .

## 2. The field $K$

The quotient ring

$$K = \mathbb{Q}[X]/(\varphi_p)$$

is a field because  $\varphi_p(X)$  is irreducible in  $\mathbb{Q}[X]$ . Moreover,  $K$  is isomorphic to the number field  $\mathbb{Q}(\zeta_p)$  via the isomorphism

$$\mathbb{Q}[X]/(\varphi_p) \cong \mathbb{Q}(\zeta_p),$$

sending the class of  $X$  to  $\zeta_p$ .

## 3. $K$ is a splitting field of $\varphi_p$

Since  $\varphi_p(X)$  splits completely in  $\mathbb{Q}(\zeta_p)$ , and all of its roots are in  $\mathbb{Q}(\zeta_p)$ , the field  $K = \mathbb{Q}(\zeta_p)$  is the splitting field of  $\varphi_p(X)$  over  $\mathbb{Q}$ .

## 4. $K/\mathbb{Q}$ is a Galois extension

An extension is Galois if it is both normal and separable. Since  $\mathbb{Q}$  has characteristic 0, all field extensions are separable. Also,  $K = \mathbb{Q}(\zeta_p)$  is the splitting field of a separable polynomial  $\varphi_p(X)$ , hence  $K/\mathbb{Q}$  is normal. Therefore,  $K/\mathbb{Q}$  is Galois.

## 5. The Galois group

The Galois group  $\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$  is isomorphic to the group of units of the ring  $\mathbb{Z}/p\mathbb{Z}$ , i.e.,

$$\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times.$$

This group has order  $\varphi(p) = p - 1$  and is cyclic, since  $(\mathbb{Z}/p\mathbb{Z})^\times$  is cyclic for any prime  $p$ .

The isomorphism is given by sending a Galois automorphism  $\sigma$  to the integer  $a \in (\mathbb{Z}/p\mathbb{Z})^\times$  such that  $\sigma(\zeta_p) = \zeta_p^a$ .

## Conclusion

The field  $K = \mathbb{Q}[X]/(\varphi_p) \cong \mathbb{Q}(\zeta_p)$  is the splitting field of  $\varphi_p(X)$  over  $\mathbb{Q}$ . The extension  $K/\mathbb{Q}$  is Galois, and the Galois group is

$$\text{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times \cong \mathbb{Z}_{p-1}.$$

□

### Problem 6.7: Every subgroup of a field is cyclic

Prove that every finite subgroup of the multiplicative group of a field is cyclic.

**Solution to Problem 6.7:** Let  $G$  be the multiplicative group of a finite field with order  $n$ . Let  $x \in G$  such that  $|x| = d$ . Let  $H$  be the subgroup of  $G$  generated by  $x$ . If  $G = H$ , we are done, so suppose otherwise. Then there exists  $y \in G \setminus H$  with  $|y| = m$  and  $\ell = \text{lcm}(d, m)$ . Suppose  $d = \ell$ . Then  $m|d$  and  $y^d = 1$ . This contradicts the fact that the number of solutions to  $X^d - 1 = 0$  is less than or equal to  $d$ , meaning  $d < \ell$ . By a theorem in the book, there exists an element  $z \in G$  such that  $|z| = \ell$ . We can repeat this process until we find a generator of  $G$ , which means that  $H$  is cyclic. □

### Problem 6.8: $\mathbb{Z}[X]$ is a UFD

Prove that the ring  $\mathbb{Z}[X]$  is a unique factorization domain.



**Solution to Problem 6.8:** We prove that  $\mathbb{Z}[X]$  is a UFD by using the following facts:

1. A principal ideal domain (PID) is a UFD.
2. The ring  $\mathbb{Z}$  is a principal ideal domain, hence a UFD.
3. If  $R$  is a UFD, then  $R[X]$  is also a UFD.

Now, let us apply these facts:

- Since  $\mathbb{Z}$  is a principal ideal domain, it is a UFD.
- By a standard result in commutative algebra, if  $R$  is a UFD, then the polynomial ring  $R[X]$  is also a UFD.
- Therefore, since  $\mathbb{Z}$  is a UFD, the ring  $\mathbb{Z}[X]$  is also a UFD.

Hence, every non-zero, non-unit element in  $\mathbb{Z}[X]$  can be written as a product of irreducible elements, and this factorization is unique up to order and units.  $\square$

### Problem 6.9: Splitting fields

Let  $k$  be a field and  $f \in k[X]$  and let  $\deg(f) = n$ . Prove that if  $K$  is a splitting field of  $f$  then  $[K : k]$  divides  $n!$ .

**Solution to Problem 6.9:** Let  $n_1, n_2, \dots, n_k$  be the degrees of the irreducible factors of  $f$ . Then  $\sum_i n_i = n$ . We know this holds for all irreducibles, so

$$[K : k] \mid n_1! n_2! \dots n_k! \quad (6.9.1)$$

However, the coefficient

$$\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1! n_2! \dots n_k!} \quad (6.9.2)$$

Is the number of ways to choose to separate  $n$  things into  $k$  groups of sizes  $n_1, \dots, n_k$  and as such must be an integer. So,

$$n! = j(n_1! n_2! \dots n_k!) \quad (6.9.3)$$

for some  $j \in \mathbb{Z}$ , so then  $[K : k] \mid n!$  and we are done.  $\square$

### Problem 6.10: Counting subgroups of $S_p$

Let  $p$  be a prime and  $S_p$  denote the symmetric group on  $p$  elements.

- (a) What is the order of a  $p$ -Sylow subgroup of  $S_p$ ?
- (b) What is the number of  $p$ -Sylow subgroups in  $S_p$ ?
- (c) Deduce that  $(p-1)! \equiv -1 \pmod{p}$ .

**Solution to Problem 6.10:**

- (a) The order of any  $p$  subgroup is  $p$ . Since  $p$  is defined to be the largest prime power that divides the order of  $S_p$ , which is always  $p$  since  $|S_p| = p!$ .
- (b) All of the elements of order  $p$  consist of a  $p$ -cycle of the first  $p$  natural numbers, so there are exactly  $(p-1)!$

elements of order  $p$ . Each subgroup of order  $p$  contains  $p - 1$  elements of order  $p$  (the non-identity elements), so the intersection of any two subgroups is trivial, so the number of subgroups of order  $p$  is

$$\frac{(p-1)!}{p-1} = (p-2)! \quad (6.10.1)$$

(c) By Sylow's Third theorem, part (b) implies that

$$(p-2)! \equiv 1 \pmod{p} \quad (6.10.2)$$

multiplying both sides by  $(p-1)$  gives

$$(p-1)! \equiv p-1 \equiv 1 \pmod{p} \quad (6.10.3)$$

□

### Problem 6.11: Composition of normal field extensions

Prove or disprove: The composition of any two normal extension of a field  $k$  is normal.

**Solution to Problem 6.11:** This is not true, a counterexample would be

$$\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt[4]{2}) \quad (6.11.1)$$

□

### Problem 6.12: Group with order 2 is abelian

Prove that a group  $G$  in which every element is of order 2 is abelian.

**Solution to Problem 6.12:** Since every element has order 2,  $a^{-1} = a$  for all  $a \in G$ , so

$$[a, b] = aba^{-1}b^{-1} = abab = (ab)^2 = e \quad (6.12.1)$$

So the group is abelian.

□

### Problem 6.13: Union of conjugates is a subgroup

Prove that if  $H$  is a proper subgroup of a finite group  $G$ , then  $\bigcup_{x \in G} x^{-1}Hx \neq G$ .

**Solution to Problem 6.13:** Let  $G$  have order  $n$ , and since  $H$  is a proper subgroup, let  $[G : H] = m > 1$ . Let  $N(H)$  be the normalizer of  $H$  in  $G$ , which contains  $H$ . As such,  $[G : N(H)] \leq [G : H]$ . We can let  $G$  act by conjugation, so then the orbit of  $G$  is the set of all conjugate subgroups. So the stabilizer of  $G$  is exactly the normalizer  $N(H)$ , so then by the Orbit-Stabilizer Theorem, the number of all conjugate subgroups is equal to  $[G : N(H)]$ . Each of these subgroups has cardinality  $|H|$ , and each has the identity  $e$ , so the union has at most  $1 + [G : N(H)](|H| - 1)$  elements

in the union. So

$$1 + [G : N(H)](|H| - 1) \leq 1 + [G : H](|H| - 1) \quad (6.13.1)$$

$$= 1 + |G| - m \quad (6.13.2)$$

$$= |G| + (1 - m) \quad (6.13.3)$$

$$< |G| \quad (6.13.4)$$

since  $m > 1$ , so the union of conjugate subgroups is a proper subset and not the whole of  $G$ .  $\square$

#### Problem 6.14: Semidirect Product

Prove that if  $H$  and  $K$  are subgroups of finite index in a group  $G$ , and  $[G : H]$  and  $[G : K]$  are relatively prime, then  $G = HK$ .

**Solution to Problem 6.14:** From the textbook, we know that if  $[G : K]$  is finite, then  $[H : H \cap K] = [G : K]$  if and only if  $G = KH$ . So then we know that

$$[G : H][H : H \cap K] = \frac{|G|}{|H|} \frac{|H|}{|H \cap K|} = \frac{|G|}{|H \cap K|} \frac{|K|}{|K|} = [G : K][K : H \cap K] \quad (6.14.1)$$

Since  $[G : K]$  and  $[G : H]$  are relatively prime, we know that  $[G : H]$  must divide  $[K : H \cap K]$ . So then we get that  $[G : H] = [K : H \cap K]$  and by our proposition we are finished.  $\square$

#### Problem 6.15: Not all elements are in the conjugate

Prove that if  $H$  is a proper subgroup of finite index in a group  $G$  (possibly infinite), then there exists  $x \in G$  not belonging to any conjugate of  $H$ .

**Solution to Problem 6.15:** Let  $H$  be a proper subgroup of  $G$  with finite index  $n = [G : H]$ . We aim to show that there exists an element  $x \in G$  such that  $x \notin gHg^{-1}$  for any  $g \in G$ .

Let  $X$  be the set of left cosets of  $H$  in  $G$ , so  $|X| = n < \infty$ . Consider the action of  $G$  on  $X$  by left multiplication:

$$g \cdot aH = gaH \quad \text{for } g, a \in G.$$

This defines a group homomorphism

$$\varphi : G \rightarrow \text{Sym}(X) \cong S_n,$$

where  $S_n$  is the symmetric group on  $n$  letters.

Let  $K = \ker(\varphi)$ . Then  $K$  is a normal subgroup of  $G$  contained in the intersection of all conjugates of  $H$ :

$$K \subseteq \bigcap_{g \in G} gHg^{-1}.$$

Moreover, since  $\varphi(G) \leq S_n$ , the image is finite, and thus the kernel  $K$  is of finite index in  $G$  (as the kernel of a homomorphism to a finite group).

Now, since  $H$  is a proper subgroup, the image  $\varphi(G)$  is a nontrivial subgroup of  $S_n$ , hence  $K$  is a proper subgroup of  $G$ .

Assume for contradiction that every  $x \in G$  lies in some conjugate of  $H$ , i.e.,

$$G = \bigcup_{g \in G} gHg^{-1}.$$

But there are only finitely many distinct conjugates of  $H$  (since  $[G : H] < \infty$ ), say

$$G = \bigcup_{i=1}^m g_i H g_i^{-1}.$$

This expresses  $G$  as a finite union of proper subgroups.

However, a standard result in group theory (e.g., B.H. Neumann's theorem) states that a group cannot be expressed as a finite union of proper subgroups unless one of them is equal to the whole group. Therefore, this leads to a contradiction.

Hence, there must exist some element  $x \in G$  such that  $x \notin gHg^{-1}$  for any  $g \in G$ .  $\square$

### Problem 6.16: $G/H$ is cyclic and abelian

Prove that if  $H$  is a subgroup contained in the center of a group  $G$ , then  $H$  is a normal subgroup. Moreover, if  $G/H$  is cyclic, prove that  $G$  is abelian.

**Solution to Problem 6.16:** If  $H \subset Z(G)$  and  $h \in H$ , then  $h \in Z(G)$ , so for every  $g \in G$ , we have

$$g^{-1}hg = g^{-1}gh = eh = h \quad (6.16.1)$$

Since this works for any  $h \in H$ , we have that  $g^{-1}Hg = H$  and  $H$  is normal.

Since  $G/H$  is cyclic, say it is generated by  $\langle gH \rangle$ . Then for some  $a, b \in G$ ,  $a \in g^i H$  and  $b \in g^j H$ . Then for some  $h_1, h_2 \in H$  we have

$$ab = (g^i h_1)(g^j h_2) \quad (6.16.2)$$

$$= g^{i+j} h_1 h_2 \quad (6.16.3)$$

$$= g^j h_2 g^i h_1 \quad (6.16.4)$$

$$= ba \quad (6.16.5)$$

so then it is commutative.  $\square$

### Problem 6.17: Indexes of Finite Groups

Let  $G$  be a finite group and  $p$  the smallest prime that divides the order of  $G$ . Prove that a subgroup of index  $p$  in  $G$  is normal.

**Solution to Problem 6.17:** Let  $H$  be a subgroup of index  $p$ . Then  $G$  acts on the set of left cosets of  $H$  by left multiplication,  $x(gH) = xgH$ . This action induces a homomorphism from  $G \rightarrow S_p$ , of which whose image,  $K$  is in  $H$ . Then  $G/K$  is isomorphic to a subgroup of  $S_p$ , and has order dividing  $p!$ . But it also has order dividing  $|G|$ , and since  $p$  is the smallest prime which does this, then  $|G/K| = p$ . We see

$$|G/K| = [G : K] = [G : H][H : K] = p[H : K] \quad (6.17.1)$$

so then  $[H : K] = 1$ , so  $K = H$  and since  $K$  is normal,  $H$  is thus normal.  $\square$

**Problem 6.18: Orbit Stabilizer**

Let  $G$  be a finite group acting on a finite set  $X$ . Prove that the number of orbits equals

$$\frac{1}{|G|} \sum_{g \in G} |Fix(g)| \quad (6.18.1)$$

where  $Fix(g)$  is the set of elements of  $X$  which are fixed by the action of  $g$ .

**Solution to Problem 6.18:**

$$\frac{1}{|G|} \sum_{g \in G} |Fix(g)| = \frac{1}{|G|} \sum_{g \in G} |\{x \in X : gx = x\}| \quad (6.18.2)$$

$$= \frac{1}{|G|} \sum_{x \in X} |\{g \in G : gx = x\}| \quad (6.18.3)$$

$$= \frac{1}{|G|} \sum_{x \in X} |Stab(x)| \quad (6.18.4)$$

$$= \frac{1}{|G|} \sum_{x \in X} \frac{|G|}{|Orb(x)|} \quad (6.18.5)$$

$$= \sum_{x \in X} \frac{1}{|Orb(x)|} \quad (6.18.6)$$

$$= \sum_{Orb(x) \in X/G} \left( \sum_{x \in Orb(x)} \frac{1}{|Orb(x)|} \right) \quad (6.18.7)$$

$$= \sum_{Orb(x) \in X/G} 1 \quad (6.18.8)$$

$$= |X/G| \quad (6.18.9)$$

□

**Problem 6.19: Proof of Burnside's Lemma**

Let  $G$  be a finite group acting transitively on a set of cardinality at least 2. Prove that there exists  $g \in G$  such that  $Fix(g) = \emptyset$ .

**Solution to Problem 6.19:** By Burnside's Lemma we have

$$|orb(G)| = \frac{1}{|G|} \sum_{g \in G} |Fix(g)| = 1 \quad (6.19.1)$$

Since it is transitive, so it has exactly 1 orbit. However, since the number of fixed points for each element must be an integer, only one can be one. So then there exists at least 1  $g \in G$  that fixes no points. □

**Problem 6.20: Proof of First Sylow Theorem**

Define  $p$ -Sylow subgroups of finite group  $G$  and prove they always exist.

**Solution to Problem 6.20:** A  $p$ -Sylow subgroup is a subgroup of  $G$  in which all elements have an order of  $p^n$  for some  $n$ , and is maximal among all  $p$ -subgroups of  $G$ .

To prove they always exist, let  $|G| = kp^n$  such that  $p \nmid k$ . Let  $\mathcal{S} = \{S \subseteq G : |S| = p^n\}$  which is the set of all subsets of  $G$  which have exactly  $p^n$  elements. Let  $N = |\mathcal{S}|$ . We know that

$$N = \binom{p^n k}{p^n} \equiv k \pmod{p} \quad (6.20.1)$$

Let  $G$  act on  $\mathcal{S}$  by the following:

$$\forall S \in \mathcal{S} : g * S = gS = \{x \in G : x = gs : s \in S\} \quad (6.20.2)$$

which means  $g * S$  is the left coset of  $S$  by  $g$  which is a group action. Now, let  $\mathcal{S}$  have  $r$  orbits under this action which partition  $\mathcal{S}$ , meaning

$$|\mathcal{S}| = |\text{Orb}(S_1)| + |\text{Orb}(S_2)| + \cdots + |\text{Orb}(S_r)| \quad (6.20.3)$$

If each orbit had length divisible by  $p$ , then  $p|N$ . But this cannot be the case, as  $N \equiv k \pmod{p}$ , so at least one orbit has length which is not divisible by  $p$ . So then for some  $S$ , there is  $|\text{Orb}(S)| = m : p \nmid m$ . Let  $s \in S$ . Then  $\text{Stab}(S)s = S$  meaning  $|\text{Stab}(S)| = |S| = p^n$  and since the stabilizer is a subgroup, they must always exist.  $\square$

### Problem 6.21: Abelian if every Sylow is normal and abelian

Suppose that  $G$  is a finite group such that every Sylow subgroup is normal and abelian. Show that  $G$  is abelian.

**Solution to Problem 6.21:** Let  $x, y \in G$ . We split into multiple cases.

Case 1: If  $x$  and  $y$  are in the same Sylow subgroup, and since they are all abelian, we clearly have  $xy = yx$ .

Case 2: If  $x, y$  are not in the same Sylow subgroup, then suppose that  $x \in P$  and  $y \in Q$ , Sylow  $p$  and  $q$  subgroups respectively. Since  $P$  and  $Q$  are normal, we have

$$xyx^{-1}y^{-1} \in P \cap Q = \{e\} \quad (6.21.1)$$

Which implies  $xy = yx$  meaning it is abelian.

Case 3:  $x, y$  are in no Sylow  $p$  subgroups. Then they are both the identity and clearly are abelian.  $\square$

### Problem 6.22: Product Ideals

Let  $R$  be a commutative ring and  $I, J$  ideals of  $R$ .

- (a) Define the ideals  $IJ$ ,  $I \cap J$ , and  $I + J$  and prove in case that they are ideals.
- (b) Prove that  $IJ \subset I \cap J$ .
- (c) Suppose  $R$  is a PID. Show that  $IJ = I \cap J$  if and only if  $I + J = R$ .

**Solution to Problem 6.22:**

(a) We see that

$$IJ = \{a_1b_1 + \cdots + a_nb_n : n \in \mathbb{N}, a_i \in I, b_j \in J\} \quad (6.22.1)$$

We need to show first that  $IJ$  is a subring of  $R$ . After this, suppose that  $r \in R$  and  $a \in IJ$  where  $a = i_1j_1 + \cdots + i_nj_n$ . Note that

$$ra = r(i_1j_1 + i_2j_2 + \cdots + i_nj_n) = ri_1j_1 + ri_2j_2 + \cdots + ri_nj_n \quad (6.22.2)$$

Since  $I$  is an ideal, then  $ri_k \in I$ , so then  $ra \in IJ$  and  $IJ$  is a left ideal. Since  $R$  is commutative, it is also a right ideal.

(b) Clearly  $IJ \subset I$  and  $IJ \subset J$ ,

(c) We begin with the reverse direction. Suppose  $I + J = R$ . Then

$$I \cap J = (I \cap J) \cdot R \quad (6.22.3)$$

$$= (I \cap J) \cdot (I + J) \quad (6.22.4)$$

$$= (I \cap J) \cdot I + (I \cap J) \cdot J \quad (6.22.5)$$

$$\subset IJ + IJ \quad (6.22.6)$$

$$= IJ \quad (6.22.7)$$

This, combined with part (b), gives us that  $IJ = I \cap J$ .

The reverse direction can be seen by supposing  $IJ = I \cap J$ . For the sake of contradiction, suppose  $I + J \neq R$ . Then there exists a maximal ideal  $m \subset R$  with  $I + J \subset m$ .

□

### Problem 6.23: Irreducibles in an Integral Domain

Let  $R$  be an integral domain.

- (a) Define irreducible elements and prime elements of  $R$ .
- (b) Prove that every prime element of  $R$  is irreducible.

### Solution to Problem 6.23:

- (a) An irreducible element in an integral domain is a non-zero element that is not invertible (not a unit) and is not the product of two non-invertible elements. An element is prime if it is not zero or unit and whenever  $P$  divides  $ab$ , for some  $a, b \in R$  then  $p$  divides  $a$  or  $p$  divides  $b$ .
- (b) Let  $(p)$  be prime in  $R$ . Let  $p = ab$ . Clearly  $ab \in (p)$  but  $(p)$  is prime, so either  $a \in (p)$  or  $b \in (p)$ . Suppose WLOG,  $a \in (p)$  then there is some  $r \in R$  in  $a = pr \implies p = prb$ . By cancellation,  $1 = rb$  thus since  $p$  is irreducible,  $b$  is a unit and  $p$  is prime.

□

### Problem 6.24: Irreducibles in a PID

Let  $R$  be a PID and  $a \in R$  such that  $a \neq 0$  and  $a$  is not a unit. Prove that the following are equivalent:

- (a)  $a$  is irreducible.
- (b)  $a$  is prime.
- (c)  $(a)$  is a prime ideal.
- (d)  $(a)$  is a maximal ideal.

**Solution to Problem 6.24:** We prove the equivalence in a cycle: (a)  $\Rightarrow$  (d)  $\Rightarrow$  (c)  $\Rightarrow$  (b)  $\Rightarrow$  (a).

(a)  $\Rightarrow$  (d): Assume  $a$  is irreducible. Since  $R$  is a PID, all ideals are principal. We claim that  $(a)$  is maximal.

Let  $(a) \subseteq (b) \subseteq R$  for some ideal  $(b)$ . Then  $a \in (b)$ , so  $a = br$  for some  $r \in R$ . Since  $a$  is irreducible, either  $b$  or  $r$  is a unit.

- If  $r$  is a unit, then  $a$  and  $b$  are associates, so  $(a) = (b)$ .
- If  $b$  is a unit, then  $(b) = R$ , so  $(a) \subseteq R$ .

Therefore, there are no ideals strictly between  $(a)$  and  $R$ , so  $(a)$  is maximal.

**(d)  $\Rightarrow$  (c):** Every maximal ideal in a commutative ring is a prime ideal. Therefore, if  $(a)$  is maximal, then it is also prime.

**(c)  $\Rightarrow$  (b):** Suppose  $(a)$  is a prime ideal. We want to show that  $a$  is a prime element. Let  $a \mid bc$  for some  $b, c \in R$ . Then  $bc \in (a)$ , so by primality of the ideal, either  $b \in (a)$  or  $c \in (a)$ . Hence,  $a \mid b$  or  $a \mid c$ . Thus,  $a$  is a prime element.

**(b)  $\Rightarrow$  (a):** Suppose  $a$  is prime. We show  $a$  is irreducible.

Assume  $a = bc$  for some  $b, c \in R$ . Since  $a$  is prime, it must divide  $b$  or  $c$ . Without loss of generality, assume  $a \mid b$ . Then  $b = ad$  for some  $d \in R$ , and so:

$$a = bc = (ad)c = a(dc).$$

Canceling  $a$  (which is nonzero and not a zero divisor in an integral domain), we get  $1 = dc$ , so  $c$  is a unit. Therefore,  $a$  is irreducible.

Hence, all four statements are equivalent. □

### Problem 6.25: Every PID is a UFD

Prove that every PID is a UFD.

**Solution to Problem 6.25:** Let  $R$  be a PID and suppose that a nonzero element  $a \in R$  can be written two separate ways as products of irreducibles such that

$$a = p_1 p_2 \dots p_r = q_1 q_2 \dots q_s \tag{6.25.1}$$

where each  $p_i$  and  $q_j$  are irreducible in  $R$  and  $s \geq r$ . Then  $p_1$  divides the product  $q_1 \dots q_s$  and so  $p_1$  divides some  $q_j$  for some  $j$  since  $p$  is prime. By reordering, we can suppose that  $p_1 \mid q_1$ , meaning  $p_1 = u_1 q_1$  for some unit  $u_1$  of  $R$ . Since  $q_1$  and  $p_1$  are both irreducible. Thus,

$$p_1 p_2 \dots p_r = u_1 p_1 q_2 \dots q_s \tag{6.25.2}$$

So then we get

$$1 = u_1 u_2 \dots u_r q_{r+1} \dots q_s \tag{6.25.3}$$

Since none of the  $q_j$  are a unit, then  $r = s$  and  $p_j$  is associated with  $q_j$  in some permutation. Thus,  $R$  is a unique factorization domain. □



## Exam 7: Misc. Book Problems

### Problem 7.1: No Simple Groups

Prove that there are no simple groups of order

- (a) 30
- (b) 105
- (c) 56

### Solution to Problem 7.1:

(a) Let  $G$  be a simple group of order  $30 = 2 \cdot 3 \cdot 5$ . So By the Sylow theorems, we have

$$n_2 \equiv 1 \pmod{2} \quad (7.1.1)$$

$$n_3 \equiv 1 \pmod{3} \quad (7.1.2)$$

$$n_5 \equiv 1 \pmod{5} \quad (7.1.3)$$

meaning

$$n_2 \in \{1, 3, 5, 15\} \quad (7.1.4)$$

$$n_3 \in \{1, 10\} \quad (7.1.5)$$

$$n_5 \in \{1, 6\} \quad (7.1.6)$$

We will proceed case by case. If  $n_2 = 1$ , then let  $P \in \text{Syl}_2(G)$ . Since  $n_2 = 1$ , this implies that  $P \trianglelefteq G$  meaning that  $|P| = 2$  which is a contradiction as  $G$  is simple. Thus,  $n_2 \neq 1$ . The same argument holds for  $n_3 = 1$  and  $n_5 = 1$ . So we are left with

$$n_2 \in \{3, 5, 15\} \quad (7.1.7)$$

$$n_3 = 10 \quad (7.1.8)$$

$$n_5 = 6 \quad (7.1.9)$$

Let  $H_1, H_2, \dots, H_6 \in \text{Syl}_5(G)$ . We know that  $|H_i| = 5$ . Clearly  $H_i$  has 4 elements of order 5, so

$$|H_i \cap H_j| \mid |H_i| \implies |H_i \cap H_j| \mid 5 \quad (7.1.10)$$

If  $|H_i \cap H_j| = 5$ , then  $H_i \cap H_j = H_i$  which is a contradiction, meaning  $H_i \cap H_j = 1$  for all  $i \neq j$ . This means that  $G$  has at least 24 elements of order 5, but since  $n_3 + 24 = 34 > 30 = |G|$  this is impossible. Thus, there are no simple groups of order 30.

- (b)
- (c)
- (d)

□

### Problem 7.2

Let  $\alpha$  be a real root of  $x^3 - 3x - 1$ . Does  $\sqrt{2} \in \mathbb{Q}(\alpha)$ ?

**Solution to Problem 7.2:**  $x^3 - 3x - 1$  is irreducible by the rational root theorem (show this more), so

$$[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3 \quad (7.2.1)$$

$$[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2 \quad (7.2.2)$$

Since 2 does not divide 3, if  $\sqrt{2} \in \mathbb{Q}$ , then  $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\alpha)$  which is a contradiction.  $\square$

### Problem 7.3: F

nd the minimal polynomial of  $\sqrt[6]{2}$  over  $\mathbb{Q}(\sqrt{2})$ .

**Solution to Problem 7.3:** We see that  $[\mathbb{Q}(\sqrt[6]{2}) : \mathbb{Q}] = 6$ , since  $x^6 - 2$  is irreducible by Eisenstein's criterion, so  $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ , meaning  $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt[6]{2})$ . Therefore,

$$[\mathbb{Q}(\sqrt[6]{2}) : \mathbb{Q}(\sqrt{2})] = 3 \quad (7.3.1)$$

We know that  $\sqrt[6]{2}$  is a root of  $x^3 - \sqrt{2}$  over  $\mathbb{Q}(\sqrt{2})$ . Since the degree is equal to the degree of the extension,  $x^3 - \sqrt{2}$  must be the minimal polynomial.  $\square$