Mathematical Theory of Scattering Resonances

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Theorem 2.9

In this section, I will prove Theorem 2.9 from Dyatlev and Sworski using a more general case seen in Robert Hintz' notes.

Theorem 1. *Let $V \in L^\infty(\mathbb{R};\mathbb{R})$ and suppose that $w(t,x)$ is the solution of*

$$\begin{cases} (D_t^2 - P_V)u(t,x) = 0 \\ u(0,x) = \psi(x) \in H^1_{comp}(\mathbb{R}) \\ \partial_t u(0,x) = \varphi(x) \in L^2_{comp}(\mathbb{R}) \end{cases} \tag{1}$$

then for any $A > 0$

$$u(t,x) = \sum_{\operatorname{Im}\lambda_j > -A} \sum_{\ell=0}^{m_R(\lambda_j)-1} t^\ell e^{-i\lambda_j t} f_{j,\ell}(x) + E_A(t) \tag{2}$$

where the sum is finite.

$$\sum_{\ell=0}^{m_R(\lambda_j)-1} t^\ell e^{-i\lambda_j t} f_{j,\ell}(x) - \operatorname{Res}_{\mu=\lambda_j}\left((iR_V(\mu)\varphi + \mu R_V(\mu)\psi)\, e^{-i\mu t}\right) \tag{3}$$

$$(P_V - \lambda_j^2)^{\ell+1} f_{j,\ell} = 0 \tag{4}$$

and for any $K > 0$ for which $\operatorname{supp} w_j \subset (-K,K)$*, there exist constants $C_{K,A}$ and $T_{K,A}$ such that*

$$\|E_A(t)\|_{H^2([-K,K])} \le C_{K,A} e^{-tA\left(\|\psi\|_{H^1} + \|\varphi\|_{L^2}\right)}, \qquad t \ge T_{K,A}$$

Theorem 1. We can begin by rephrasing the problem. We note that is V is a real-valued function, then P_V must be self-adjoint on $L^2(\mathbb{R})$ and the solution to the initial value problem in (1) is

$$u(t,x) = \cos\left(t\sqrt{P_V}\right) + \frac{\sin\left(t\sqrt{P_V}\right)}{\sqrt{P_V}}\varphi(x)$$

which is found by using the functional calculus for P_V. We note that $u \in \mathcal{C}^0(\mathbb{R}_t; H^1_c(\mathbb{R}^3)) \cap \mathcal{C}^1(\mathbb{R}_t; L^2_c(\mathbb{R}^3))$. For simplicity we shall first consider the case that $\psi \equiv 0$ and $\varphi \in L^2_c(\mathbb{R}^3)$. We can rephrase (1) as a forcing problem: that is, we can set $\tilde{u}(t,x) = H(t)u(t,x)$ where $H(t)$ is the Heaviside function. We then compute that

$$(\partial_t^2 - \Delta + V)\tilde{u} = \delta(t)\varphi(x)$$

This equation vanishes for $t < 0$. We will be lazy and now denote $u(t,x) = \tilde{u}(t,x)$. We now have the system

$$\left(\partial_t^2 - \Delta + V\right) u(t,x) = \delta(t)\varphi(x) \tag{5}$$

$$u(t,x) = 0, \qquad t < 0 \tag{6}$$

Which if we take the Fourier transform in t, we have

$$(P_V - \lambda^2)\int_{\mathbb{R}} e^{i\lambda t} u(t,x)\, dt \tag{7}$$

Which helps point to attempting to construct the solution to (5) using the inverse Fourier transform. We see

$$\widehat{u}(\lambda,x) = R_V(\lambda)\varphi(x)$$

$$u(t,x) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda = M} e^{-i\lambda t} R_V(\lambda)\varphi(x)\, d\lambda \tag{8}$$

Where M is chosen appropriately such that $R_V(\lambda)$ doesn't have any poles in the integration contour. We can verify that (8) does indeed satisfy (5). We see

$$\begin{aligned}\left(\partial_t^2 - \Delta + V\right) u(t,x) &= \left(\partial_t^2 - \Delta + V\right) \frac{1}{2\pi} \int_{\operatorname{Im}\lambda = M} e^{-i\lambda t} R_V(\lambda)\varphi(x)\, d\lambda \\ &= \end{aligned}$$

Now that we know that this is what we are looking for, we can begin to make sense of the integral. If we look towards the following Lemma:

Lemma 1. *For $M > 0$ sufficiently large, the formula*

$$u(t,x) = \frac{1}{2\pi} \int_{\operatorname{Im}\lambda = M} e^{-i\lambda t} R_V(\lambda)\varphi\, d\lambda \tag{9}$$

defines an element of $L^2_{\text{loc}}\left(\mathbb{R}_t; L^2(\mathbb{R}^3_x)\right)$ which vanishes for $t < 0$ and solves

$$\left(\partial_t^2 - \Delta + V\right) u(t,x) = \delta\varphi(x) \tag{10}$$

$$u(t,x) = 0, \qquad t < 0 \tag{11}$$

Proof of Lemma 1. We can use our estimate previously, that

$$\|R_V(\lambda)\|_{L^2 \to L^2} \leq \frac{C}{|\lambda| \operatorname{Im}\lambda}$$

to see that

$$\|R_V(\lambda)\varphi\|_{L^2(\mathbb{R}^3)} \leq \frac{C}{|\lambda| \operatorname{Im}\lambda} \|\varphi\|_{L^2(\mathbb{R}^3)} \tag{12}$$

for some $\operatorname{Im}\lambda \geq M > 0$ with M large enough. We can write $\lambda = \sigma + iM$, we notice that this is square integrable in σ. We see

$$u(x,t) = \frac{1}{2\pi} e^{Mt} \int_{\mathbb{R}} e^{-i\lambda t} R_V(\sigma + iM)\varphi(x)\, d\sigma \tag{13}$$

Which is in $L^2_{\text{loc}}(\mathbb{R}_t; L^2(\mathbb{R}^3_x))$. For $t < 0$, we have that $|e^{-i\lambda t}| = e^{-\operatorname{Im}\lambda |t|}$ which goes to 0 as $\operatorname{Im}\lambda \to \infty$. Then by Cauchy's Theorem, we can shift the contour of integration to $\operatorname{Im}\lambda = M'$ for any $M' \geq M$ since the integrand has no poles for $\operatorname{Im}\lambda \geq M$. Whihc gives us

$$u(t,x) = \frac{1}{2\pi} \int_{\operatorname{Im}\lambda = M'} e^{-i\lambda t} R_V(\lambda) \tag{14}$$

$\square$

With this Lemma, we now wish to shift our contour to the lower half plane so that we see the exponentially decaying contribution from the resonances. We have the following propositions:

Proposition 1. *For $R > 0$ and $\rho \in \mathcal{C}^\infty_c(B(0,R))$, we have*

$$\|\rho R_0(\lambda)\rho\|_{L^2 \to H^j} \leq C\langle\lambda\rangle^{j-1} e^{2R(\operatorname{Im}\lambda)_-}, \qquad j = 0, 1, 2$$

$\square$

2.7 Complex Scaling in One Dimension

BIG IDEA: We want to restrict the complex second derivative to the real axis and then deform it away from the support of the potential so that P can be restricted to it. When we do this, the operator becomes elliptic at $\pm\infty$, but loses self-adjointness. First, we want to define our curve $\Gamma \subset \mathbb{C}$ be a C^1 simple curve. Let γ be a parametrization from $\mathbb{R} \to \Gamma$ and let $f \in C^1(\Gamma)$ in the sense that $f \circ \gamma \in C^1(\mathbb{R})$. Then we define differentiation by

$$\partial_z^\Gamma f(z_0) = \gamma'(t_0)^{-1}\partial_t(f \circ \gamma)(t_0), \qquad \gamma(t_0) = z_0 \tag{15}$$

and we can define

$$D_z^\Gamma = \frac{1}{i}\partial_z^\Gamma \tag{16}$$

By the chain rule, this is independent of γ. When integrating, we use the measure

$$dz = \gamma'(t)\,dt, \qquad |dz| = |\gamma'(t)|\,dt \tag{17}$$

For a bounded, compactly supported potential V we can assume

$$\Gamma \cap \mathbb{R} \supset [-L, L], \qquad \operatorname{supp} V \subset (-L, L) \tag{18}$$

Which means that V is a well defined function on Γ, so that putting

$$P_{V,\Gamma} := (D_z^\Gamma)^2 + V(z) \tag{19}$$

makes some sense. We can then make some assumptions on the behavior of Γ at ∞.

$\exists\theta \in (0,\pi), a_\pm \in \mathbb{C}, K$*align For* $\lambda \in \mathbb{C} \setminus \{0\}$ *and* $f \in C_0^1(\Gamma)$*, define*

$R_{0,\Gamma}(\lambda)f(z) := \dfrac{i}{2\lambda}\displaystyle\int_\Gamma e^{i\lambda\varphi(z,w)}f(w)\,dw$

and

$$\varphi(\gamma(t),\gamma(s)) := \pm(\gamma(t) - \gamma(s)), \qquad \pm(t-s) \geq 0 \tag{20}$$

For $\lambda \in \Lambda_\Gamma$, $R_{0,\Gamma}(\lambda)$ extends to an operator $L^2(\Gamma) \mapsto H^2(\Gamma)$ which is a two-sided inverse of $\left(\left(D_z^\Gamma\right)^2 - \lambda^2\right) : H^2(\Gamma) \mapsto L^2(\Gamma)$.

Proof. We can directly calculate for the case when $f \in C_c^2(\Gamma)$. We see

$$R_{0,\Gamma}(\lambda)\left(\left(D_z^\Gamma\right)^2 - \lambda^2\right)f(z) = \frac{i}{2\lambda}\int_\Gamma e^{i\lambda\varphi(z,w)}\left(\left(D_z^\Gamma\right)^2 - \lambda^2\right)f(w)\,dw \tag{21}$$

$$\tag{22}$$

$\square$