MATH 823 Lecture: Sobolev Inequalities

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1 Essential Definitions

In this lecture, we will study sobolev functions leading to the Sobolev inequality and the compactness theorem as a direct conseuqence. In order to do this, we must begin by defining what exactly is a Sobolev function. We first define a **Sobolev Space** as

Definition 1 (Sobolev Space)**.** *We say the function f belongs to the Sobolev Space $W^{1,p}(U)$ if $f \in L^p(U)$ and all its weak partial derivatives exist and also belong to $L^P(U)$*

where we define a **Weak Partial Derivitive** as

Definition 2 (Weak Partial Derivative)**.** *For a function $f \in L^1_{loc}(U)$ we say that g_i is the weak partial derivative of f with respect to x_i in U if*

$$\int_U f \frac{\partial \varphi}{\partial x_i} = -\int_U g_i \varphi \, dx$$

for all $\varphi \in C^1_c(U)$.

For the sake of convenience we will define

$$Df := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

assuming each weak partial derivative exists.

If a function f is in a Sobolev Space $W^{1,p}(U)$ for some $1 \leq p \leq \infty$ then we call f a **Sobolev Function**.

Definition 3 (Convergence in a Sobolev Space)**.** *We say that $f_k \to f$ provided $\|f_k - f\|_{W^{1,p}(U)} \to 0$ and $f_k \to f$ in $W^{1,p}_{loc}(U)$ provided $\|f_k - f\|_{W^{1,p}(V)} \to 0$ for each $V \subset\subset U$.*

2 Sobolev Inequalities

Lemma 1. *Assume $f \in W^{1,p}(U)$ for some $1 \leq p < \infty$. Then there exists a sequence $\{f_k\}_{k=1}^{\infty} \subset W^{1,p}(U) \cap C^{\infty}(U)$ such that $f_k \to f$ in $W^{1,p}(U)$*

Theorem 1 (Gagliardo-Nirenerbg-Sobolev Inequality)**.** *Assume $1 \leq p < n$. There exists a constant C_1 depending only on p and n such that*

$$\left(\int_{\mathbb{R}^n} |f|^{p^*} \right)^{1/p^*} \leq C_1 \left(\int_{\mathbb{R}^n} |Df|^p \right)^{1/p}$$

for all $f \in W^{1,p}(\mathbb{R}^n)$.

Proof. We will proceed with this proof via induction. For the base case of $p = 1$, we see that by Lemma 1, without loss of generality we may assume that $f \in C_c^1(\mathbb{R}^n)$ since there is some sequence of functions of compact support that approximate it. Then for $1 \leq i \leq n$

$$f(x_1, \ldots, x_i, \ldots, x_n) = \int_{-\infty}^{x_i} \frac{\partial f}{\partial x_i}(x1, \ldots, t_i, \ldots, x_n)\, dt$$

and thus by Hölder's inequality we have

$$|f(x)| \leq \int_{-\infty}^{\infty} |Df(x_1, \ldots, t_i, \ldots, x_n)|\, dt_i$$

Thus

$$|f(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Df(x_1, \ldots, t_i, \ldots, x_n)|\, dt_i \right)^{\frac{1}{n-1}}$$

Note that $\frac{n}{n-1}$ is the Sobolev Conjugate of 1. If we integrate both side with respect to x_1 we see

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f|^{1^*}\, dx_1 \leq \left(\int_{-\infty}^{\infty} |Df|\, dt_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Df|\, dt_i \right)^{\frac{1}{n-1}} \, dx_1$$

We can continue integrating on x_i to eventually find

$$\begin{aligned}
\int_{\mathbb{R}^n} |f|^{1^*}\, dx &\leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |Df|\, dx_1 \ldots dt_i \ldots dx_n \right)^{\frac{1}{n-1}} \\
&= \left(\int_{\mathbb{R}^n} |Df|\, dx \right)^{\frac{n}{n-1}}
\end{aligned}$$

Thus, we clearly have

$$\left(\int_{\mathbb{R}^n} |f|^{1^*}\, dx \right)^{\frac{1}{1^*}} \leq \int_{\mathbb{R}^n} |Df|\, dx \tag{$*$}$$

and so we have our base case of $p = 1$. For our inductive hypothesis, suppose $(*)$. For $1 < p < n$, set $g = |f|^{\gamma}$ with $\gamma > 0$. Applying $(*)$ to g, we see

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |f|^{\frac{\gamma n}{n-1}}\, dx \right)^{\frac{n-1}{n}} &\leq \gamma \int_{\mathbb{R}^n} |f|^{\gamma - 1} |Df|\, dx \\
&\leq \gamma \left(\int_{\mathbb{R}^n} |f|^{\frac{(\gamma-1)p}{p-1}}\, dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Df|^p\, dx \right)^{\frac{1}{p}}
\end{aligned}$$

Now, we can choose our γ such that

$$\frac{\gamma n}{n-1} = \frac{(\gamma - 1)p}{p-1}$$

Then we have

$$\frac{\gamma n}{n-1} = \frac{(\gamma - 1)p}{p-1} = \frac{np}{n-p} = p^*$$

Thus we

$$\left(\int_{\mathbb{R}^n} |f|^{p^*}\, dx \right)^{\frac{n-1}{n}} \leq C \left(\int_{\mathbb{R}^n} |f|^{p^*}\, dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Df|^p\, dx \right)^{\frac{1}{p}}$$

so we have

$$\left(\int_{\mathbb{R}^n} |f|^{p^*} \right)^{1/p^*} \leq C_1 \left(\int_{\mathbb{R}^n} |Df|^p \right)^{1/p}$$

where C depends only on n and p. $\square$

Lemma 2. *For each $1 \le p < \infty$ there exists a constant C depending only on n and p such that*

$$\int_{B(x,r)} |f(y) - f(z)|^p \, dy \le Cr^{n+p-1} \int_{B(x,r)} |Df(y)|^p |y - z|^{1-n} \, dy$$

for all $B(x,r) \subset \mathbb{R}^n$, $f \in C^1(B(x,r))$ and $z \in B(x,r)$.

Theorem 2 (Morrey's Inequality). ***(i)*** *For each $n < p < \infty$ there exists a constant C depending only on p and n such taht*

$$|f(y) - f(z)| \le Cr \left(\fint_{B(x,r)} |Df|^p \, dw \right)^{1/p}$$

for all $B(x,r) \subset \mathbb{R}^n$, $f \in W^{1,p}(U(x,r))$, and $\mathcal{L}^n$ a.e. $y, z \in U(x,r)$.

(ii) *In particular, if $f \in W^{1,p}(\mathbb{R}^n)$, then the limit*

$$\lim_{r \to 0} (f)_{x,r} := f^*(x)$$

exists for all $x \in \mathbb{R}^n$ and is Hölder continuous with exponent $1 - n/p$.

Proof. First, we will assume that $f \in C^1$ and use Lemma 2 with $p = 1$ seeing

$$\begin{aligned}
|f(y) - f(z)| &= \fint_{B(x,r)} |f(y) - f(z)| \, dw \\
&= \fint_{B(x,r)} |f(y) - f(z) - f(w) + f(w)| \, dw \\
&\le \fint_{B(x,r)} |f(y) - f(w)| + |f(w) - f(z)| \, dw && \text{Triangle Inequality} \\
&= \fint_{B(x,r)} |f(y) - f(w)| \, dw + \fint_{B(x,r)} |f(w) - f(z)| \, dw \\
&= \frac{1}{r^n} \int_{B(x,r)} |f(y) - f(w)| \, dw + \frac{1}{r^n} \int_{B(x,r)} |f(w) - f(z)| \, dw \\
&\le C \int_{B(x,r)} |Df(w)||y - w|^{1-n} \, dw + Cr^n \int_{B(x,r)} |Df(w)||z - w|^{1-n} \, dw && \text{Lemma 2} \\
&= C \int_{B(x,r)} |Df(w)| \left(|y - w|^{1-n} + |z - w|^{1-n} \right) \, dw \\
&\le C \left(\int_{B(x,r)} \left(|y - w|^{1-n} + |z - w|^{1-n} \right)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left(\int_{B(x,r)} |Df|^p \, dw \right)^{\frac{1}{p}} && \text{Hölder's Inequality} \\
&\le Cr^{\left(n - (n-1)\frac{p}{p-1}\right)\left(\frac{p-1}{p}\right)} \left(\int_{B(x,r)} |Df|^p \, dw \right)^{\frac{1}{p}} \\
&= Cr^{1 - \frac{n}{p}} \left(\int_{B(x,r)} |Df|^p \, dw \right)^{\frac{1}{p}}
\end{aligned}$$

By Lemma 1, we know that C^1 is dense in L^n and thus we can construct a sequence of functions approximating f in L^n and apply dominated convergence theorem to finish the proof of **(i)**.
To prove **(ii)**, we now suppose that $W^{1,p}(\mathbb{R}^n)$. Then for $\mathcal{L}^n$ we can apply the estimate seen in **(i)** with $r = |x - y|$ to see

$$\begin{aligned}
|f(y) - f(z)| &\le Cr^{1 - \frac{n}{p}} \left(\int_{B(x,r)} |Df|^p \, dw \right)^{\frac{1}{p}} \\
&\le C\|Df\|_{L^p(\mathbb{R}^n)} |x - y|^{1 - \frac{n}{p}}
\end{aligned}$$

Thus f is equal to a Hölder Continuous function with exponent $1 - \frac{n}{p}$ $\square$